ON THE *Q*-ANALYSIS OF Q-HYPERGEOMETRIC DIFFERENCE EQUATION

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ABSTRACT

ON THE Q-ANALYSIS OF Q-HYPERGEOMETRIC DIFFERENCE EQUATION

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In this thesis, a fairly detailed survey on the q-classical orthogonal polynomials of the Hahn class is presented. Such polynomials appear to be the bounded solutions of the so called q-hypergeometric difference equation having polynomial coefficients of degree at most two. The central idea behind our study is to discuss in a unified sense the orthogonality of all possible polynomial solutions of the q-hypergeometric difference equation by means of a qualitative analysis of the relevant q-Pearson equation. To be more specific, a geometrical approach has been used by taking into account every posssible rational form of the polynomial coefficients, together with various relative positions of their zeros, in the q-Pearson equation to describe a desired q-weight function on a suitable orthogonality interval. Therefore, our method differs from the standard ones which are based on the Favard theorem and the three-term recurrence relation.

Keywords: Special functions, Classical orthogonal polynomials of a discrete variable, q-polynomials, Orthogonal polynomials on q-linear lattices, q-Hahn class.

ÖZ

Q-HİPERGEOMETRİK FARK DENKLEMİNİN Q-ANALİZİ ÜZERİNE

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Bu tezde, Hahn sınıfına ait q-klasik ortogonal polinomlar hakkında oldukça detaylı bir çalışma sunulmaktadır. Bu tip polinomlar, katsayıları en fazla ikinci dereceden polinomlar olan q-hipergeometrik fark denkleminin sınırlı çözümleri olarak ortaya çıkmaktadırlar. Bu çalışmada temel düşünce, ilgili q-Pearson denkleminin kalitatif analizi aracılığıyla q-hipergeometrik fark denkleminin mümkün olan bütün polinom çözümlerinin ortogonalliklerini genel anlamda ele almaktır. Daha açık olarak, uygun bir ortogonalite aralığında istenilen bir q-ağırlık fonksiyonu tanımlamak için, q-Pearson denklemindeki polinom katsayıların sıfırlarının değişik göreceli pozisyonlarıyla birlikte mümkün olan her rasyonel formunu hesaba katarak bir geometriksel yaklaşım kullanılmıştır. Dolayısıyla, bu çalışma Favard teoremine ve 3-terimli rekürans ilişkisine dayanan standart metodlardan farklıdır.

Anahtar Kelimeler: Özel fonksiyonlar, Ayrık değişkenli klasik ortogonal polinomlar, q-polinomlar, q-doğrusal örgülü ortogonal polinomlar, q-Hahn sınıfı.

To my family

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INTRODUCTION

Family of q-classical polynomials in the Hahn sense, which is a part of classical q-polynomials, is first introduced by Wolfgang Hahn in 1949 [32]. They satisfy a q-difference equation of hypergeometric type (q-EHT) [3, 4, 6, 17, 42],

$$\sigma(x;q)D_qD_{q^{-1}}y(x) + \tau(x,q)D_qy(x) + \lambda(q)y(x) = 0 \tag{1.1}$$

where the coefficients $\sigma(x,q)$ and $\tau(x,q)$ are polynomials of at most second and first degree in x, respectively, $\lambda(q)$ is a constant and

$$D_qy(x) = \frac{y(x) - y(qx)}{(1-q)x}, \qquad x \neq 0, \quad D_qy(0) = y'(0) \tag{1.2}$$

stands for the q-Jackson derivative [3, 4, 6, 30, 53].

The so-called q-polynomials have enormous applications in several problems on theoretical and mathematical physics, e.g., in the continued fractions, Eulerian series, [27], algebras and quantum groups [37, 38, 54], discrete mathematics, algebraic combinatorics (coding theory, design theory, various theories of group representation) [20], q-Schrödinger equation and q-harmonic oscillators [14, 15, 16, 18, 19, 40].

The classical q-polynomials are the discrete version of the classical orthogonal polynomials (Hermite, Laguerre, Jacobi, Bessel). The theory of discrete polynomials is rather developed [3, 6, 17, 32, 42, 49, 46, 47, 48]. There are several approaches in the study of these polynomials [3, 4, 5, 6, 11, 17, 36, 42, 43, 46, 47, 48, 49].

1.1 Classical Orthogonal Polynomials

Orthogonal polynomials are particularly useful in the category of special functions since they have considerable superiority in miscellaneous problems [13] most of which lead to the classical hypergeometric differential equation (EHT) [3, 22, 36, 42, 43, 51],

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0 \tag{1.3}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree, respectively, and λ is a constant [3, 46, 48]. It can be shown that EHT in (1.3) has polynomial solutions $P_n(x)$ of exact degree n for particular values of λ of the form [46, 48],

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma'' \tag{1.4}$$

for $n = 0, 1, \ldots$. The famous classical orthogonal polynomials associated with the names Jacobi $P_n^{(\alpha,\beta)}(x)$, Laguerre $L_n^{\alpha}(x)$ and Hermite $H_n(x)$ are all solutions of EHT with λ defined by (1.4). The parameters leading to these polynomials are listed in Table (1.1) [46, 48].

Table 1.1: The classical orthogonal polynomials

	Hermite	Laguerre	Jacobi	Bessel
$P_n(x)$	$H_n(x)$	$L_n^{\alpha}(x)$	$P_n^{(\alpha,\beta)}(x)$	$B_n^{\alpha}(x)$
(a, b)	$(-\infty, \infty)$	$(0, \infty)$	$(-1, 1)$	$\mathbb{T} := \{\lvert z\rvert = 1, z \in \mathbb{C}\}$
$\sigma(x)$	1	x	$1 - x^2$	x^2
$\tau(x)$	$-2x$	$\alpha + 1 - x$	$\beta - \alpha - (\alpha + \beta + 2)x$	$(\alpha + 2)x + 2$
λ_n	$2n$	n	$n(n + \alpha + \beta + 1)$	$-n(n + \alpha + 1)$
$\rho(x)$	e^{-x^2}	$x^{\alpha}e^{-x}$	$(1 - x)^{\alpha}(1 + x)^{\beta}$	$\frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\Gamma(a+2)}{\Gamma(a+k+1)}(-\frac{2}{x})^k$
		$\alpha > -1$	$\alpha, \beta > -1$	$\alpha > -2$

Definition 1.1 *[4] We say that the orthogonal polynomial sequence (OPS) (Table 1.1) $\{P_n\}$ is a classical OPS with respect to the weight function ρ [46, 48] if*

$$\int_a^b y_m(x)y_n(x)\rho(x)dx = d_n^2\delta_{mn} \tag{1.5}$$

where δ_{mn} is the Kronecker delta, d_n is the norm of the polynomial P_n, ρ is a solution of the Pearson equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\sigma(x)\rho(x)\Big] = \tau(x)\rho(x), \tag{1.6}$$

where σ and τ are fixed polynomials of degree at most 2 and exactly 1, respectively, such that the following boundary conditions hold

$$\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0. \tag{1.7}$$

Remark 1.2 *The boundary condition given in (1.7) is not valid for the Bessel polynomials since Bessel polynomials are orthogonal on unit circle (For more details see [3]).*

We remark that Hermite, Laguerre, Jacobi and Bessel families are the only classical orthogonal polynomials satisfying the definition 1.1 [3, 4, 22, 46, 48]. Definition 1.1 is only one of the way to characterize the sequence of classical orthogonal polynomials. There are also other characteristics, one of them is TTRR. In particular, Chihara [22], Freud [28], Nevai [45] and Szegö [51] studied on the orthogonal polynomials starting from the TTRR.

Theorem 1.3 *(TTRR [3, 46, 48]) The orthogonal polynomials $(P_n)_n$ satisfy a three-term recurrence relation of the form*

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \tag{1.8}$$

where $\alpha_n, \beta_n, \gamma_n$ are some numerical sequences and $P_{-1}(x) = 0$, $P_0(x) = 1$.

The converse statement of the theorem for TTRR implies the Favard theorem which is considered by many authors.

Theorem 1.4 *(The Favard Theorem [6, 22, 26]) Let $(P_n)_{n\geq 0}$ be a monic polynomial basis sequence. Then, $(P_n)_{n\geq 0}$ is an MOPS if and only if there exist two sequences of complex numbers $(d_n)_{n\geq 0}$ and $(g_n)_{n\geq 1}$, such that $g_n \neq 0$, $n \geq 1$ and*

$$xP_n(x) = P_{n+1}(x) + d_n P_n(x) + g_n P_{n-1}(x), \quad P_{-1} = 0, \quad P_0 = 1, \quad n \geq 0. \tag{1.9}$$

Moreover, the functional u such that the polynomials $(P_n)_{n\geq 0}$ are orthogonal with respect to it, is positive definite if and only if $(d_n)_{n\geq 0}$ is a real sequence and $(g_n) > 0$ for all $n \geq 1$.

Another point of view for the characterization of classical orthogonal polynomials was developed by Sonine for Hermite, Laguerre and Jacobi polynomials in 1887 and by Hahn [31] in 1939.

Theorem 1.5 *(Sonine-Hahn [4, 31, 41]) A given sequence of orthogonal polynomials* $(P_n)_n$ *is a classical sequence if and only if the sequence of its derivatives* $(P'_n)_n$ *is an orthogonal polynomial sequence.*

Alternatively, in 1885, Routh [50] and in 1929, Bochner [21] deal with the characterization problem in an another way and they propounded that the classical orthogonal polynomials satisfy a second-order differential equation of hypergeometric type (1.3). Derivatives of (1.3) give also a differential equation of hypergeometric type but now for $y^{(n)}$ which is also derived by Nikiforov and Uvarov [48].

Another characterization for the orthogonal polynomials is the well known Rodrigues formula which is derived by Tricomi [52] and Cryer [23]. The Rodrigues formula provides explicit representation for the classical polynomials which satisfies a differential equation of hypergeometric type (1.3) (see [33]).

According to the all discussions above we perform the following theorem extracted from [4].

Theorem 1.6 *[4] Let* $(P_n)_n$ *be an OPS. The following statements are equivalent:*
(1) $(P_n)_n$ *is a classical orthogonal polynomial sequence (COPS) (Hildebrandt [33]).*
(2) The sequence of its derivatives $(P'_n)_{n\geq 1}$ *is an OPS with respect to the weight function* $\rho_1(x) = \sigma(x)\rho(x)$*, where* ρ *satisfies the Pearson equation (Sonine and Hahn [31])*

$$\Big[\sigma(x)\rho(x)\Big]' = \tau(x)\rho(x). \tag{1.10}$$

(3) $(P_n)_n$ *satisfies the second order linear differential equation with polynomial coefficients of the form (1.3) (Bochner [21]).*
(4) $(P_n)_n$ *can be expressed by the Rodrigues formula (Tricomi [52] and Cryer [23])*

$$P_n(x) = \frac{B_n}{\rho(x)}\frac{d^n}{dx^n}\Big[\sigma^n(x)\rho(x)\Big]. \tag{1.11}$$

(5) There exist three sequences of complex numbers $(a_n)_n, (b_n)_n, (c_n)_n$ *and a polynomial* σ*,* $\deg(\sigma) \leq 2$*, such that [2]*

$$\sigma(x)P'_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 1. \tag{1.12}$$

(6) There exist two sequences of complex numbers $(f_n)_n$ and $(g_n)_n$ such that the following relation for the monic polynomials holds (Marcellán et al. [41])

$$P_n(x) = \frac{P'_{n+1}(x)}{n+1} + f_n P'_n(x) + g_n P'_{n-1}(x), \quad g_n \neq \gamma_n, \quad n \geq 1, \tag{1.13}$$

where γ_n is the corresponding coefficient of the TTRR (1.9).

1.2 Classical Orthogonal Polynomials of Discrete Variable

The so-called discrete polynomials (Hahn, Kravtchouk, Meixner and Charlier [22, 44, 46, 48]) and the q-polynomials [3, 36, 49, 46, 48] are both discrete version of the classical continuous polynomials which were first realized by Hahn [32]. Actually, Hahn was motivated by Chebyshev's study done in 1855s on the orthogonal polynomials. In this regard, in 1949, Hahn [32] introduced a linear operator $H_{q,w}$

$$H_{q,w} f(x) = \frac{f(qx+w) - f(x)}{(q-1)x + w}, \quad q,\ w \in \mathbb{R}^+. \tag{1.14}$$

together with the problem of finding all OPS $(P_n)_n$ satisfying one of the following properties [4, 17, 22];

1. $\{H_{q,w} P_n(x)\}$ is an OPS.

2. $P_n(x)$ satisfies a difference equation of the form

$$\sigma(x) H^2_{q,w} P_n(x) + \tau(x) H_{q,w} P_n(x) + \lambda_n P_n(x) = 0, \tag{1.15}$$

where $\sigma(x)$ and $\tau(x)$ are independent of n, and are polynomials of degrees at most 2 and 1, respectively.

3. $P_n(x)$ has a Rodrigues-type representation

$$\rho(x) P_n(x) = H^n_{q,w}\Big[f_1(x) f_2(x) ... f_n(x) \rho(x) \Big], \quad f_i(x) = f_{i+1}(qx+w). \tag{1.16}$$

4. If $P_n(x) = \sum_{k=0}^n a_{n,k} \phi_k(x)$ with $\phi_k(x) x^k$ or $\phi_k(x) = (x;q)_k$, where

$$(x;q)_k = \begin{cases} 1, & k = 0, \\ (1-x)(1-qx)...(1-q^{k-1}x), & k = 1, 2, ... \end{cases} \tag{1.17}$$

are the q-shifted factorials, then $a_{n,k}/a_{n,k-1}$ is a rational function of n and k or q^n and q^k, respectively.

5.The moments M_n associated with $\{P_n(x)\}$, defined by

$$\int_{-\infty}^{\infty} \phi_n(x) d\alpha(x) = M_n, \tag{1.18}$$

satisfy a recurrence relation of the form

$$M_n = \frac{a + bq^n}{c + dq^n} M_{n-1}, \quad ad - bc \neq 0, \tag{1.19}$$

n=1, 2, ... see [1].

1.2.1 Classical Discrete Polynomials

Evaluating (1.14) for $q = w = 1$ leads to $H_{1,1} = \Delta$ and the so-called discrete polynomials (those of Hahn, Meixner, Krawtchouk and Charlier) which are solutions of difference equation of hypergeometric type of the form [46, 48]

$$\sigma(s)\Delta\nabla P_n(s) + \tau(s)\Delta P_n(s) + \lambda_n P_n(s) = 0 \tag{1.20}$$

where $\Delta f(s) = f(s+1) - f(s)$, $\nabla f(s) = f(s) - f(s-1)$ and $\deg(\sigma) \leq 2$, $\deg(\tau) = 1$, λ_n is a constant. Some characteristics of the classical discrete polynomials can be listed in Table 1.2 [46, 48].

Table 1.2: The classical orthogonal polynomials of discrete variables

	Hahn	Meixner	Krawtchouk	Charlier
$P_n(x)$	$h_n^{\alpha,\beta}(s;N)$	$M_n^{\gamma,\mu}(s)$	$K_n^p(s)$	$C_n^\mu(s)$
(a,b)	$[0,N]$	$[0,\infty)$	$[0,N+1]$	$[0,\infty)$
$\sigma(s)$	$s(s-\beta-N-1)$	s	s	s
$\tau(s)$	$-(\alpha+1)N+(\alpha+\beta+2)s$	$(\mu-1)s+\mu\gamma$	$\frac{Np-s}{1-p}$	$\mu-s$
λ_n	$n(n+\alpha+\beta+1)$	$(1-\mu)n$	$\frac{n}{1-p}$	n
$\rho(s)$	$\frac{\Gamma(N+\alpha-s)\Gamma(\beta+s+1)}{\Gamma(N-s)\Gamma(s+1)}$	$\mu^s\frac{\Gamma(\gamma+s)}{\Gamma(\gamma)\Gamma(s+1)}$	$\frac{N!p^s(1-p)^{N-s}}{\Gamma(N+1-s)\Gamma(s+1)}$	$\frac{e^{-\mu}\mu^s}{\Gamma(s+1)}$
	$\alpha,\beta>-1, n\leq N-1$	$\gamma>0, 0<\mu<1$	$0<p<1, n\leq N-1$	$\mu>0$

Definition 1.7 *[4, 46, 48] We say that the discrete polynomial sequence $(P_n)_n$ (Table 1.2) is a classical OPS of discrete variables if they are orthogonal on the integers $[a, b-1]$ with respect to the weight function $\rho(s)$ together with the relation*

$$\sum_{s=a}^{b-1} P_n(s)P_m(s)\rho(s) = d_n^2\delta_{mn} \tag{1.21}$$

provided that the boundary condition $\sigma(s)\rho(s)x^k\Big|_{s=a,b} = 0,\ k = 0,1,...$ *is satisfied. Here,* d_n *is the norm of the polynomial and the weight function* ρ *satisfies the Pearson equation* $\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)$.

There are different aspects for the characterization of discrete polynomials. For instance, definition 1.7 characterize the discrete polynomials. Another characterization of the discrete polynomials is the Rodrigues formula which is considered by Erdélyi and Weber [25] in 1952. Lesky stated in 1962 that discrete orthogonal polynomials are classical if and only if its differences ΔP_n is an discrete OPS [29, 39].

1.2.2 Classical q-Polynomials

A q-analog of the Chebychev polynomials is due to Markov in 1884 [22] which can be regarded as the first example of q-polynomial family. In 1949, Hahn introduced the q-Hahn class [32] and obtained the most general orthogonal polynomial on the exponential lattice, the so-called big q-Jacobi polynomials, by taking $w = 0$ and $q \in (0,1)$ in the linear operator $H_{q,w}$. In the Hahn case, $H_{q,0} = D_q$ where D_q is the q-Jackson derivative defined in (1.2). Hahn studied on the q-polynomials included in the q-Hahn scheme which are the solutions of the q-difference equation of hypergeometric type (1.1) and their q-derivatives are also orthogonal [32, 43].

Afterwards, around 1980s they have been considered by several authors with different aspects. Most popular ones are; G. Andrews, R. Askey and A. Nikiforov, V. Uvarov who generate the Askey scheme and the Nikiforov-Uvarov scheme, respectively. G. Andrews and R. Askey [11] have only considered particular cases [36] based on the basic hypergeometric series [30]

$$ {}_r\varphi_p\left(\begin{matrix} a_1,...,a_r \\ b_1,...,b_p \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k...(a_r;q)_k}{(b_1;q)_k...(b_p;q)_k} \frac{z^k}{(q;q)_k} [(-1)^k q^{k(k-1)/2}]^{p-r+1} \tag{1.22} $$

where $(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i)$, $(a;q)_0 = 1$ denotes the q-shifted factorial. And the idea for the Nikiforov-Uvarov approach is grounded on the second order hypergeometric type difference equation on non-uniform lattices [3, 49, 46, 48],

$$ \sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n[x(s)]}{\nabla x(s)} + \tau(s)\frac{\Delta P_n[x(s)]}{\Delta x(s)} + \lambda_n P_n[x(s)] = 0. \tag{1.23} $$

Atakishiyev, Rahman and Suslov have proved that (1.23) has polynomial solutions of hypergeometric type if and only if the lattice $x(s)$ is a linear, q-linear (exponential), quadratic or q-quadratic (exponential) of the form [17]

$$x(s) = \begin{cases} C_1 q^{-s} + C_2 q^s & if \quad q \neq 1 \\ C_3 s^2 + C_4 s & if \quad q = 1 \end{cases} \tag{1.24}$$

where $q \in \mathbb{C}$ and C_1, C_2, C_3, C_4 are constants s.t. $(C_1, C_2) \neq (0, 0)$, $(C_3, C_4) \neq (0, 0)$. The lattice is linear if $C_3 = 0$; otherwise it is quadratic and it is q-linear if one of C_1, C_2 is zero; otherwise it is q-quadratic. Actually, the sufficient part of this statement has been proved by Nikiforov-Uvarov.

We remark that polynomial solutions of the difference equation of hypergeometric type (1.23) with linear lattice $x(s) = s$ leads to the classical discrete polynomials (Hahn, Meixner, Krawtchouk and Charlier) which are discrete polynomials on uniform lattice and difference equation (1.23) with non-uniform lattices such as q-linear lattice of the form $x(s) = q^s$ or $x(s) = q^{-s}$ enable us the q-Hahn scheme [3, 4, 5, 36, 46, 48] (see [3, 17, 49, 46, 48] for quadratic and q-quadratic lattices).

Another approach based on the functional analysis has been considered by R. Álvarez Nodarse, F. Marcellán and J. C. Medem [43] where the authors have proved several characterizations of such orthogonal polynomials (see also [4]) starting from the so-called distributional q-Pearson equation. In particular, in [43] a classification of all possible families of orthogonal polynomials on the exponential lattice has been established, and latter on in [6] the comparison with the q-Askey and the Nikiforov-Uvarov scheme has been done, obtaining two new families of orthogonal polynomials. For more details on the q-polynomials on the linear exponential lattice we refer the readers to the works [3, 4, 5, 6, 24, 35, 42, 49, 46, 47, 48], and references therein.

Some important characterizations for the classical orthogonal polynomials of discrete variable analogue to the classical continuous and discrete ones have been done by Atakishiyev, Rahman and Suslov [17] and Álvarez-Nodarse [4] as in the following manner.

Definition 1.8 *[17] An OPS $\{P_n[x(s)]\}_{n=0}^{\infty}$ on a real interval $(x(a), x(b))$ is classical if and only if:*

(i) $P_n[x(s)]$ satisfies a difference equation of the form (1.23) with $x(s)$ given by (1.24).

(ii) A positive weight function $\rho(s)$ satisfying the Pearson-type difference equation

$$\frac{\nabla}{\nabla x_1(s)}[\sigma(s+1)\rho(s+1)] = \tau(s)\rho(s), \quad x_1(s) = x(s+1/2) \tag{1.25}$$

exists.

(iii) The boundary conditions $\sigma(s)\rho(s)x^k(s-\frac{1}{2})\Big|_{a,b} = 0$ hold for $k = 0, 1, 2, \ldots.$

Theorem 1.9 *[4] Let $(P_n)_n$ be an OPS on a linear type lattice $x(s)$ satisfying*

$$\sum_{s=a}^{b-1} P_n(s)P_m(s)\rho(s)\Delta x(s-\frac{1}{2}) = \delta_{mn}d_n^2 \tag{1.26}$$

and let $\sigma(s)$ and $\rho(s)$ be two functions such that the boundary condition $\sigma(s)\rho(s)x^k(s-\frac{1}{2})\Big|_{a,b} = 0$ holds for $k = 0, 1, 2, \ldots.$ Then the following statements are equivalent

(1) $(P_n)_n$ is a classical OPS.

(2) The sequence of its differences $(\Delta P_n/\Delta x(s))_n$ also is an OPS with respect to the weight function $\rho_1(s) = \sigma(s+1)\rho(s+1)$, where ρ satisfy

$$\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\Delta x(s-\frac{1}{2}). \tag{1.27}$$

(3) $(P_n)_n$ satisfies the second order linear difference equation with polynomial coefficients (1.23).

(4) $(P_n)_n$ can be expressed by Rodrigues-type formula

$$P_n(s) = \frac{B_n}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}[\rho_n(s)] \tag{1.28}$$

where $\rho_n(s) = \rho(s+n)\prod_{m=1}^{n}\sigma(s+m)$, $x_k(s) = x(s+\frac{k}{2})$ and B_n is a constant with $B_n \neq 0$.

(5) There exist three sequences of complex numbers $(a_n)_n, (b_n)_n, (c_n)_n$, and a polynomial ϕ, $\deg(\phi) \leq 2$, s. t.

$$\phi(s)\frac{\Delta P_n(s)}{\Delta x(s)} = a_nP_{n+1}(s) + b_nP_n(s) + c_nP_{n-1}(s), \quad n \geq 1. \tag{1.29}$$

(6) There exist three sequences of complex numbers $(e_n)_n, (f_n)_n, (g_n)_n$, such that the following relation holds for all $n \geq 1$

$$P_n(s) = e_n\frac{\Delta P_{n+1}(s)}{\Delta x(s)} + f_n\frac{\Delta P_n(s)}{\Delta x(s)} + g_n\frac{\Delta P_{n-1}(s)}{\Delta x(s)}, \tag{1.30}$$

where $e_n \neq 0$, $g_n \neq \gamma_n$, for all $n \in \mathbb{N}$, and γ_n is the corresponding coefficient of the TTRR.

An important contribution to the theory of the q-polynomials of the exponential lattice (the so-called $q-$Hahn tableau, named after the work of Koornwinder [38]) has been done in the very recent book [35], which has a complete analysis of the orthogonal polynomial solutions of the difference equation of hypergeometric type with the help of the Favard theorem. On the other hand, in [24] the authors introduced the q-Hahn scheme by using the difference calculus on the linear lattice as well as a very simple geometrical analysis based on the behavior of the polynomial coefficients of the difference equation of hypergeometric type. In this thesis, we deal with the orthogonality properties of the q-polynomials of the $q-$Hahn tableau but from a point of view different from the one used in [35]. In fact, we make a unified treatment of the orthogonality following an idea by Nikiforov and partially published in [24]. Our main aim here is going further in the analysis started in [24] and study all possible families of orthogonal polynomials which are orthogonal with respect to a weight function satisfying the q-Pearson equation as well as certain boundary conditions.

We introduce the statement of our approach hereinbelow.

Statement of the Problem

In this section, we state the problem in the thesis. The thesis includes the survey on characterization of polynomial solutions of the q-difference equation (1.1) in the following aspect:

Definition 1.10 *An OPS $(P_n)_n$ on a real interval (a, b) is classical if and only if*
(i) $P_n(x, q)$ satisfies a q-difference equation of the form

$$\sigma_1(x; q)D_{q^{-1}}D_qP_n(x, q) + \tau(x, q)D_qP_n(x, q) + \lambda_n(q)P_n(x, q) = 0$$

and equivalently,

$$\sigma_2(x; q)D_qD_{q^{-1}}P_n(x, q) + \tau(x, q)D_{q^{-1}}P_n(x, q) + \lambda_n(q)P_n(x, q) = 0$$

where the coefficients $\sigma_1(x, q)$ and $\sigma_2(x, q)$ have the following relation

$$\sigma_2(x, q) := q\left[\sigma_1(x, q) + (1 - q^{-1})x\tau(x, q)\right].$$

Here, $\deg(\sigma_1) \leq 2$, $\deg(\sigma_2) \leq 2$, $\deg(\tau) = 1$ *and $\lambda_n(q)$ is a constant.*
(ii) $\{D_q^nP_n(x, q)\} = \underbrace{D_q...D_q}_{n} P_n(x, q)$ *and* $\{D_{q^{-1}}^nP_n(x, q)\} = \underbrace{D_{q^{-1}}...D_{q^{-1}}}_{n} P_n(x, q)$, $n \geq 0$ *are orthogonal polynomials.*

(iii) $P_n(x,q)$ *has the Rodrigues representation as follows*

$$\rho(x,q)P_n(x,q) = B_{1_n}(q)D_q^n[\sigma_1(x,q)\sigma_1(q^{-1}x,q)...\sigma_1(q^{-n+1}x,q)\rho(x,q)],$$

or equivalently

$$\rho(x,q)P_n(x,q) = B_{2_n}(q)D_{q^{-1}}^n[\sigma_2(x,q)\sigma_2(qx,q)...\sigma_2(q^{n-1}x,q)\rho(x,q)].$$

(iv) $P_n(x,q)$ *is orthogonal on the real line* (a,b) *with respect to the* $\rho(x,q) > 0$ *satisfying the q-Pearson equation*

$$D_q[\sigma_1(x,q)\rho(x,q)] = q^{-1}\tau(x,q)\rho(x,q) \quad \text{or} \quad \mathrm{D}_{q^{-1}}[\sigma_2(\mathrm{x},\mathrm{q})\rho(\mathrm{x},\mathrm{q})] = \mathrm{q}\tau(\mathrm{x},\mathrm{q})\rho(\mathrm{x},\mathrm{q})$$

in the following sense

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2\delta_{mn}$$

provided that the boundary condition $\sigma_1(x,q)\rho(x,q)x^k\Big|_{a,b} = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k\Big|_{a,b} = 0$ *holds or in another sense*

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = s_n^2\delta_{mn}$$

if boundary condition $\sigma_1(qx,q)\rho(qx,q)x^k\Big|_{a,b} = \sigma_2(x,q)\rho(x,q)x^k\Big|_{a,b} = 0$ *is satisfied.*

The principal results of the approach in definition 1.10 provide the q-polynomials in the Hahn sense.

Our main purpose in the thesis is to develop the orthogonality of all possible polynomial solutions of the q-difference equation by use of the qualitative analysis of the q-Pearson equation. For each family, we obtain all possible orthogonality intervals (that depend on the range of the parameters of each family) as well as the corresponding orthogonality relations. In fact, for all those intervals we determine the corresponding q-weight functions $\rho(x,q)$ satisfying the q-Pearson equation

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}$$

such that $\rho > 0$ and certain boundary condition holds. The main idea behind the analysis of the q-Pearson equation is to study the graphs of $\rho(qx,q)/\rho(x,q)$ which leads to the shape of the graphs of $\rho(x,q)$. In particular, by the analysis of $\rho(qx,q)/\rho(x,q)$ we obtain the behavior of $\rho(x,q)$, e.g., the interval where $\rho(qx,q)/\rho(x,q) < 1$, $\rho(qx,q)/\rho(x,q) > 1$, where $x > 0$, $0 < q < 1$, lead us to the intervals where ρ is increasing and decreasing, respectively.

In the analysis of $\rho(qx,q)/\rho(x,q)$ we consider all possible degrees of the polynomial coefficients $\sigma_1(x,q)$ and $\sigma_2(x,q)$ as well as various relative positions of their zeros (of $\sigma_1(x,q)$ and $\sigma_2(x,q)$). By the study of every possible rational form of the polynomial coefficients we obtain all suitable intervals of orthogonality as well as the kind of orthogonality relations.

Organization of the Thesis

The thesis is arranged as follows:

In Chapter 2, we establish some basic definitions related with q-calculus for our purpose in the thesis.

In Chapter 3, we study some known characterizations [3, 6, 46, 48] of the q-difference equation of hypergeometric type along the same line with definition 1.10.

In Chapter 4, which is the main part of this thesis, we discuss the orthogonality of all possible polynomial solutions of the q-difference equation of hypergeometric type (q-EHT) by use of qualitative analysis of the q-Pearson equation. We mainly concentrate on the q-Pearson equation in accordance with zeros of the polynomial coefficients of the q-EHT considering discrete orthogonality with some certain properties. First of all, we construct a theorem which shows the determination of the end points of the orthogonality intervals according to the zeros of the polynomial coefficients of the q-difference equation. Next, we concentrate on the main results of the geometrical approach of the q-Pearson equation considering all possible degrees of the polynomial coefficients $\sigma_1(x,q)$ and $\sigma_2(x,q)$ as well as every possible relative position of their zeros. In this way, we obtain all suitable intervals of orthogonality as well as the kind of orthogonality relation that can take place in dependence of the zeros of $\sigma_1(x,q)$ and $\sigma_2(x,q)$.

In Chapter 5, we compare each family that we have obtained in Chapter 4 with the q-Askey scheme [6, 42] . Later, we introduce some known limit relations between the identified q-polynomials of the Hahn class and the classical continuous and discrete ones [35, 36].

Chapter 6 concludes the thesis with remarks.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some basic definitions concerning with q-calculus which we use in the thesis and we consider the functions belonging to the following set

$$Q := Q[J] = \{f : J \to \mathbb{R}; J \subset \mathbb{R} \text{ s.t. } f'(0) \text{ exists}\}.$$

2.1 The q-Derivative

Definition 2.1 *(q and q^{-1}-derivatives [30, 36, 38, 53]) Let $f \in Q$. Then q and q^{-1}-Jackson derivatives $D_q f$, $D_{q^{-1}} f$ of a function f on an open interval are given by*

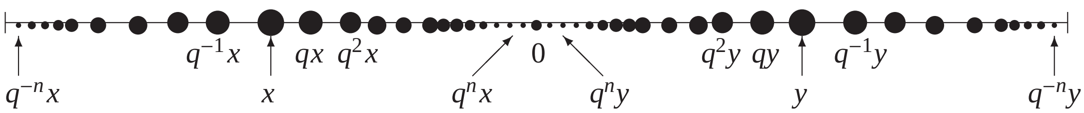
$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases} \tag{2.1}$$

and

$$D_{q^{-1}} f(x) = \begin{cases} \dfrac{f(x) - f(q^{-1}x)}{(1-q^{-1})x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases} \tag{2.2}$$

where $q \in \mathbb{R}$ such that $|q| \neq 0, 1$. Note that $\lim_{q\to 1} D_q f(x) = \lim_{q\to 1} D_{q^{-1}} f(x) = f'(x)$ *if f is differentiable at x.*

Throughout our study we consider $0 < q < 1$ which determines the following diagram.



A diagram of the lattice points.

As a consequence of definition 2.1

$$D_q f(x)\Big|_{x=at} = \frac{f(qat) - f(at)}{(q-1)at} = a^{-1} D_q f(at) \tag{2.3}$$

and more general

$$D_q^n f(x)\Big|_{x=at} = a^{-n} D_q^n f(at). \tag{2.4}$$

Therefore the relation between q and q^{-1}-derivatives can be performed as [36, 53]

$$D_q f(x)\Big|_{x=q^{-1}t} = qD_q f(q^{-1}t) = D_{q^{-1}} f(t). \tag{2.5}$$

Example 2.2 *Let $f(x) = x^n$, where $n \in \mathbb{Z}$, then*

$$D_q f(x) = D_q x^n = \frac{(qx)^n - x^n}{(q-1)x} = [n]_q x^{n-1} \tag{2.6}$$

where

$$[n]_q = \frac{q^n - 1}{q - 1} = q^{n-1} + ... + 1, \quad n \in \mathbb{Z}, \quad [0]_q = 0 \tag{2.7}$$

denotes the q-integer [36, 53].

$$D_{q^{-1}} f(x) = D_{q^{-1}} x^n = \frac{(q^{-1}x)^n - x^n}{(q^{-1}-1)x} = [n]_{q^{-1}} x^{n-1} \tag{2.8}$$

where

$$[n]_{q^{-1}} = \frac{q^{-n} - 1}{q^{-1} - 1} = q^{-(n-1)} + ... + 1, \quad n \in \mathbb{Z} \quad [0]_{q^{-1}} = 0 \tag{2.9}$$

stands for the q^{-1}-integer [30, 36, 53]. One can find the relation between q and q^{-1}-integers, [30, 36, 53]

$$[k]_{q^{-1}} = q^{1-k}[k]_q. \tag{2.10}$$

Definition 2.3 *Let* f and $g \in$ Q. *Then, product rule of q and q^{-1}-differentiations can be derived as follows:*

$$D_q\left[f(x)g(x)\right] = f(x)D_q g(x) + g(qx)D_q f(x) = f(qx)D_q g(x) + g(x)D_q f(x), \tag{2.11}$$

$$D_{q^{-1}}\left[f(x)g(x)\right] = f(x)D_{q^{-1}}g(x) + g(q^{-1}x)D_{q^{-1}} f(x) = f(q^{-1}x)D_{q^{-1}}g(x) + g(x)D_{q^{-1}} f(x). \tag{2.12}$$

Definition 2.4 *(q and q^{-1}-binomial [30, 35, 36]) The q and q^{-1}-binomial are identified with*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{2.13}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \frac{(q;q^{-1})_n}{(q;q^{-1})_k (q;q^{-1})_{n-k}} \tag{2.14}$$

where $(q;q)_n$ and $(q;q^{-1})_n$ are the q and q^{-1}-shifted factorial defined by

$$(a;q)_n = (1-a)(1-qa)(1-q^2a)...(1-q^{n-1}a), \quad (a;q)_0 := 1, \tag{2.15}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-\frac{q}{a})^k q^{\binom{k}{2}-nk}, \tag{2.16}$$

where $a \neq 0$, $k = 0, 1, ..., n$, $n = 1, 2, ...$ and

$$(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-\binom{n}{2}}, \tag{2.17}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$.

We list the following q-shifted factorials that we use in the thesis:

$$(aq^{-n};q)_n = (a^{-1}q;q)_n(-a)^n q^{\binom{n}{2}}, \quad a \neq 0, \tag{2.18}$$

$$(aq^{-n};q)_k = \frac{(a^{-1}q;q)_n}{(a^{-1}q^{1-k};q)_n}(a;q)_k q^{-nk}, \quad a \neq 0, \tag{2.19}$$

$$(a^2;q^2)_n = (a;q)_n(-a;q)_n \tag{2.20}$$

Definition 2.5 *[10, 34, 36] Let $0 < q < 1$. Then the infinite product $(a,q)_\infty$ defined by*

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). \tag{2.21}$$

is convergent.

Remark 2.6 *[10, 34, 36] Note that if*

$$\sum_{n=0}^{\infty} aq^n$$

converges, then the infinite product defined by (2.21) also converges.

The infinite product implies that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2.22}$$

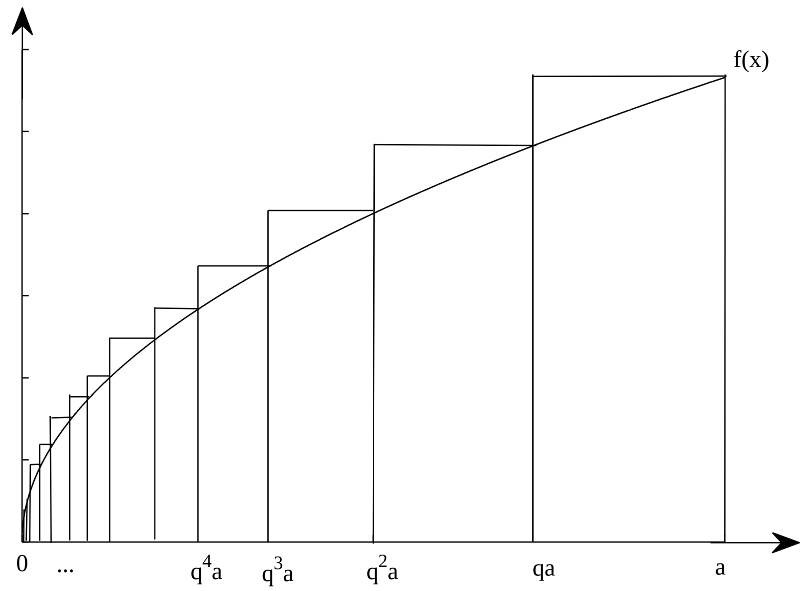
2.2 The q-Integral

The q-integral is introduced by Thomae in 1869 and later on, by Jackson in 1910 which has the definition as the following.

Definition 2.7 *(q-integral [30, 36, 38, 53]) Let $f \in Q[0,a]$. The definite q-integral is defined as*

$$\int_0^a f(x)d_qx = (1-q)a\sum_{j=0}^{\infty} q^j f(q^j a) \tag{2.23}$$

provided that the sum converges absolutely. Here d_qx is called the Fermat measure [10]. The q-integral defined in (2.23) is a Riemann sum on an infinite partition $\{aq^n,\ n \geq 0\}$ on the interval $[0,a]$.



Using this definition we may consider an inner product by setting

$$\langle f, g\rangle = \int_0^a f(t)\overline{g(t)}d_qt\,. \tag{2.24}$$

The resulting Hilbert space is commonly denoted by $L_q^2(0,a)$. The space $L_q^2(0,a)$ is a separable Hilbert space [12]. Then the orthogonality with respect to the weight function $w(t)$ is defined by the relation

$$\langle f, g\rangle = \int_0^a f(t)\,g(t)\,w(t)\,d_qt = 0\,. \tag{2.25}$$

Note that in case of $0 < a < b$,

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.26}$$

On the other hand, in case of $a < 0 < b$,

$$\int_a^b f(x)d_qx := \int_a^0 f(x)d_qx + \int_0^b f(x)d_qx. \tag{2.27}$$

Definition 2.8 *[30, 36, 38, 53] Let $f \in Q[0,\infty)$ and $Q(-\infty,\infty)$, respectively. The improper q-integral of $f(x)$ on $[0,\infty)$ and on $(-\infty,\infty)$ are defined to be*

$$\int_0^\infty f(x)d_qx = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}}^{q^j} f(x)d_qx, \quad 0 < q < 1. \tag{2.28}$$

Notice that

$$\int_0^\infty f(x)d_qx = \lim_{N\to\infty} \int_0^{q^{-N}} f(x)d_qx, \tag{2.29}$$

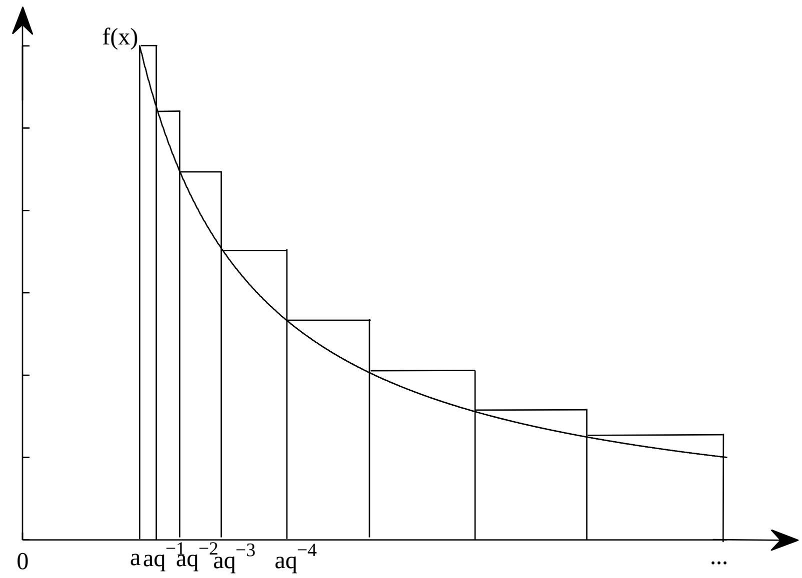
and the bilateral q-integral

$$\int_{-\infty}^\infty f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j[f(q^j) + f(-q^j)]. \tag{2.30}$$

Definition 2.9 *(q^{-1}-integral [30, 36, 38, 53]) Let $f \in Q[a,\infty)$. The following improper q^{-1}-integral is defined by*

$$\int_a^\infty f(x)d_{q^{-1}}x = -(1-q^{-1})a\sum_{j=0}^{\infty} q^{-j} f(q^{-j}a). \tag{2.31}$$

The improper q^{-1}-integral is a Riemann sum on an infinite partition $\{aq^{-n}, n \geq 0\}$ of the interval $[a,\infty)$.



Note that in case of $0 < a < b$,

$$\int_a^b f(x)d_{q^{-1}}x = \int_a^\infty f(x)d_{q^{-1}}x - \int_b^\infty f(x)d_{q^{-1}}x. \tag{2.32}$$

Remark 2.10 *It is clear that, when $f(x)$ is continuous on $(0, a)$*

$$\lim_{q\to 1^-} \int_0^a f(x)d_qx = \int_0^a f(x)dx. \tag{2.33}$$

Proposition 2.11 *(Fundamental theorem of q-calculus [30, 53]) Let $f \in Q[a, b]$. Then*

$$\int_a^b D_q f(x)d_qx = f(b) - f(a) \tag{2.34}$$

where $0 \le a < b \le \infty$.

Proposition 2.12 *[30, 53] Let f and g $\in$ Q[a, b]. Then*

$$f(x)g(x)\Big|_{x=a,b} = \int_a^b f(x)D_qg(x)d_qx + \int_a^b g(qx)D_q f(x)d_qx \tag{2.35}$$

by using product rule defined in (2.11).

2.3 Hypergeometric Series

The function ${}_rF_s$ defined by

$${}_rF_s\left(\begin{matrix} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ...a_r)_k}{(b_1, ...b_s)_k}\frac{z^k}{k!} \tag{2.36}$$

is called the hypergeometric series where $(a_1, .., a_r)_k = (a_1)_k...(a_r)_k$ is the Pochhammer symbol identifed with $(a)_k = a(a + 1)...(a + k - 1)$, $k = 1, 2, ...$, $(a)_0 := 1$. We remark that in case of one of the numerator parameters a_i equals to $-n$, $n = 1, 2, ...$, the hypergeometric series becomes a polynomial of degree n in z. The radius of convergence ρ of the series is [30, 36]

$$\rho = \begin{cases} \infty & if \quad r < s + 1 \\ 1 & if \quad r = s + 1 \\ 0 & if \quad r > s + 1. \end{cases} \tag{2.37}$$

2.4 q-Hypergeometric Series

q-Hypergeometric series ${}_r\phi_s$ is given by

$$ {}_r\phi_s\left(\begin{array}{ccc} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{array}\Big| q; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ...a_r; q)_k}{(b_1, ...b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q; q)_k} \tag{2.38} $$

where $(a_1, .., a_r; q)_k = (a_1; q)_k ... (a_r; q)_k$ is the q-analogue of the Pochhammer symbol defined by (2.15). Analogously, when one of the numerator parameters a_i is q^{-n}, $n = 1, 2, ...$, basic hypergeometric series is a polynomial of degree n in z. The radius of convergence ρ of the series looks like as (2.37) [30, 36].

In particular, assuming $r = s + 1$ in (2.38) leads to

$$ {}_{s+1}\phi_s\left(\begin{array}{ccc} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{array}\Big| q; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ...a_r; q)_k}{(b_1, ...b_s; q)_k} \frac{z^k}{(q; q)_k} \tag{2.39} $$

which was first introduced by Heine in 1846.

We remark that the limit relations [30, 35, 36]

$$ \lim_{q\to 1} \frac{1-q^{\alpha}}{1-q} = \alpha, \quad \lim_{q\to 1} \frac{(q^{\alpha}; q)_k}{(1-q)^k} = (\alpha)_k \tag{2.40} $$

which constitute a crucial role for the theory of the q-analogues of the functions, lead to

$$ \lim_{q\longrightarrow 1^-} {}_r\phi_s\left(\begin{array}{ccc} q^{a_1}, & ..., & q^{a_r} \\ q^{b_1}, & ..., & q^{b_s} \end{array}\Big| q; (q-1)^{1+s-r} z\right) = {}_r F_s\left(\begin{array}{ccc} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{array}; z\right). \tag{2.41} $$

Another important limit properties for q-hypergeometric functions are given in [30, 35, 36]

$$ \lim_{a_r\to\infty} {}_r\phi_s\left(\begin{array}{ccc} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{array}\Big| q; \frac{z}{a_r}\right) = {}_{r-1}\phi_s\left(\begin{array}{ccc} a_1, & ..., & a_{r-1} \\ b_1, & ..., & b_s \end{array}\Big| q; z\right), \tag{2.42} $$

$$ {}_r\phi_s\left(\begin{array}{cccc} a_1, & ..., & a_{r-1}, & \mu \\ b_1, & ..., & b_{s-1}, & \mu \end{array}\Big| q; z\right) = {}_{r-1}\phi_{s-1}\left(\begin{array}{ccc} a_1, & ..., & a_{r-1} \\ b_1, & ..., & b_{s-1} \end{array}\Big| q; z\right). \tag{2.43} $$

2.5 Transformation Formulas

This section includes some essential transformation formulas extracted from [30, 35, 36].

$$ {}_3\phi_2\left(\begin{array}{ccc} q^{-n}, & a, & b \\ & d, & e \end{array}\Big| q; q\right) = \frac{(e/a; q)_n}{(e; q)_n} a^n {}_3\phi_2\left(\begin{array}{ccc} q^{-n}, & a, & d/b \\ & d, & aq^{1-n}/e \end{array}\Big| q; \frac{bq}{e}\right), \tag{2.44} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & b \\ & d, & e \end{matrix}\middle| q; \frac{deq^n}{ab}\right) = \frac{(e/a;q)_n}{(e;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & d/b \\ & d, & aq^{1-n}/e \end{matrix}\middle| q; q\right), \tag{2.45} $$

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, & b \\ & c \end{matrix}\middle| q; z\right) = \frac{(b;q)_n}{(c;q)_n} q^{-n-\binom{n}{2}}(-z)^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, & c^{-1}q^{1-n} \\ & b^{-1}q^{1-n} \end{matrix}\middle| q; \frac{cq^{n+1}}{bz}\right), \tag{2.46} $$

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, & b \\ & c \end{matrix}\middle| q; z\right) = (q^{-n}bz/c;q)_n\, {}_3\phi_2\left(\begin{matrix} q^{-n}, & c/b, & 0 \\ & ccq/bz & \end{matrix}\middle| q; q\right), \tag{2.47} $$

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, & b \\ & c \end{matrix}\middle| q; z\right) = \frac{(b^{-1}c;q)_n}{(c;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, & b, & bc^{-1}q^{-n}z \\ & bc^{-1}q^{1-n}, & 0 \end{matrix}\middle| q; q\right), \tag{2.48} $$

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, & b \\ & c \end{matrix}\middle| q; z\right) = \frac{(b^{-1}c;q)_n}{(c;q)_n}\left(\frac{bz}{q}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, & qz^{-1}, & c^{-1}q^{1-n} \\ & bc^{-1}q^{1-n}, 0 & \end{matrix}\middle| q; q\right), \tag{2.49} $$

$$ {}_1\phi_1\left(\begin{matrix} q^{-n} \\ a \end{matrix}\middle| q; z\right) = \frac{(q^{-1}z)^n}{(a;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, & a^{-1}q^{1-n} \\ & 0 \end{matrix}\middle| q; \frac{aq^{n+1}}{z}\right), \tag{2.50} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & b \\ & 0, & 0 \end{matrix}\middle| q; q\right) = a^n\, {}_2\phi_0\left(\begin{matrix} q^{-n}, & a \\ & - \end{matrix}\middle| q; \frac{bq^n}{a}\right), \tag{2.51} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & b \\ & 0, & 0 \end{matrix}\middle| q; q\right) = (b;q)_n a^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, & 0 \\ & b^{-1}q^{1-n} \end{matrix}\middle| q; \frac{q}{a}\right), \tag{2.52} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & b, & -b \\ & b^2, & q^{-n}z \end{matrix}\middle| q; -z\right) = \frac{1}{(q^{-n}z;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^{-n+1} \\ & qb^2 \end{matrix}\middle| q^2; z^2\right), \tag{2.53} $$

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, & -q^{-n}z \\ & 0 \end{matrix}\middle| q; \frac{q^{n+1}}{z}\right) = (-z)^{-n}q^{\binom{n}{2}}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, & q^{-n+1} \\ & - \end{matrix}\middle| q^2; \frac{z^2}{q}\right), \tag{2.54} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & qz^{-1} \\ & - \end{matrix}\middle| q; -q^n\right) = (z)^{-n}q^{n+\binom{n}{2}}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, & q^{-n+1} \\ & 0 \end{matrix}\middle| q^2; z^2\right). \tag{2.55} $$

CHAPTER 3

THE Q-POLYNOMIALS OF HYPERGEOMETRIC TYPE

There are several ways of introducing and classifying the classical orthogonal polynomials but probably the best one is to start with the differential or difference equations that such polynomials satisfy (see e.g. [3, 6, 35, 46, 48]). We deal here with the so-called q-polynomials of the Hahn class (see e.g. [4, 38]), solution of the so-called q-difference equation of hypergeometric type. In this chapter, we study on the hypergeometric type q-difference equation and we introduce some characteristics along the same line with the definition 1.10. In order to do this, we first pay our attention to the construction of the q-difference equation of hypergeometric type.

3.1 Discrete Version of Differential Equation of Hypergeometric Type: q-Difference Equation of Hypergeometric Type (q-EHT)

In this section, we begin with considering the discretization of the classical differential equation of hypergeometric type (EHT) by use of the Taylor expansion of $y(x)$ about $x = 0$

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + ...$$

By defining the lattice h as $(q-1)x$ with $0 < q < 1$ we have

$$y(qx) = y(x) + (q-1)xy'(x) + \frac{1}{2!}(q-1)^2x^2y''(x) + ...$$

Substituting q^{-1} instead of q at the resulting expression and multiplying the q^{-1}-expression with q^2 and then substracting this form q-expression, we approximate the first derivative

$$y'(x) \sim \frac{1}{1+q}[D_q y(x) + qD_{q^{-1}}y(x)] \quad \text{as} \quad \text{q} \to 1, \tag{3.1}$$

and multiplying the q^{-1} expression with q, then adding this form to q expression gives

$$y''(x) \sim \frac{2q}{1+q} D_q D_{q^{-1}} y(x) \quad \text{as} \quad \text{q} \to 1 \tag{3.2}$$

with order $(q-1)^2$, where the operators denoted by D_q and $D_{q^{-1}}$ imply q and q^{-1} Jackson derivatives of $y(x)$ (2.1) and (2.2), respectively, [30, 36, 38, 53]. By inserting these derivative operators into the classical EHT and using the operational equivalences

$$D_q = D_{q^{-1}} + (q-1)x D_q D_{q^{-1}} \tag{3.3}$$

and

$$D_q D_{q^{-1}} = q^{-1} D_{q^{-1}} D_q, \tag{3.4}$$

we obtain the q-EHT

$$\sigma_1(x;q) D_{q^{-1}} D_q y(x,q) + \tau(x,q) D_q y(x,q) + \lambda(q) y(x,q) = 0 \tag{3.5}$$

where

$$\sigma_1(x;q) := \frac{2}{1+q}\left[\sigma(x) - \frac{1}{2}(q-1)x\tau(x)\right], \tag{3.6}$$

$$\tau(x,q) := \tau(x), \tag{3.7}$$

$$\lambda(q) := \lambda, \tag{3.8}$$

$$y(x,q) := y(x). \tag{3.9}$$

It is clear that, the coefficients $\sigma_1(x;q)$ and $\tau(x,q)$ of the q-EHT are polynomials of at most 2nd and 1st degree in x, respectively. Notice that the q-EHT in (3.5) approaches the classical EHT as $q \to 1$.

The use of the relations between the operators D_q and $D_{q^{-1}}$ in (3.3) and (3.4) makes it possible to find an alternative representation of EHT,

$$\sigma_2(x;q) D_q D_{q^{-1}} y(x,q) + \tau(x,q) D_{q^{-1}} y(x,q) + \lambda(q) y(x,q) = 0 \tag{3.10}$$

where

$$\sigma_2(x,q) := q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right]. \tag{3.11}$$

Notice also that (3.10) becomes the classical EHT as $q \to 1$. Henceforward we call the first equation in (3.5) as the q-EHT of the 1st kind and the one in (3.10) as the q-EHT of the 2nd kind. We note that the q-EHT of the 1st and 2nd kinds are nothing else than the second order linear difference equations of hypergeometric type on the linear exponential lattices

$x(s) = c_1 q^s + c_2$ and $x(s) = c_1 q^{-s} + c_2$, respectively (for further details see e.g. [3, 46]). In the following, we refer to the solutions of (3.5), (3.10) as q-classical orthogonal polynomials (or just q-polynomials). Here and through out the thesis we assume $0 < q < 1$.

By using (3.3), (3.4) and (3.11) in (3.5) or (3.10) an alternative q-difference equation equivalent to (3.5) and (3.10) follows

$$\sigma_2(x,q)D_q y(x,q) - q\sigma_1(x,q)D_{q^{-1}}y(x,q) + (q-1)x\lambda(q)y(x,q) = 0. \tag{3.12}$$

The q-difference equations (3.5), (3.10) and (3.12) can be written as

$$\begin{aligned}&\sigma_2(x,q)y(qx,q) - \left[\sigma_2(x,q) + q^2\sigma_1(x,q)\right]y(x,q) + q^2\sigma_1(x,q)y(q^{-1}x,q)\\&+(q-1)^2x^2\lambda(q)y(x,q) = 0\end{aligned} \tag{3.13}$$

with the help of the operators D_q and $D_{q^{-1}}$ defined by (2.1) and (2.2), respectively. Notice that from (3.12) and (3.13) σ_1 and σ_2 are needed to classify the q-polynomials.

We introduce

$$\sigma_1(x;q^{-1})D_qD_{q^{-1}}y(x,q^{-1}) + \tau(x,q^{-1})D_{q^{-1}}y(x,q^{-1}) + \lambda(q^{-1})y(x,q^{-1}) = 0 \tag{3.14}$$

and

$$\sigma_2(x;q^{-1})D_{q^{-1}}D_qy(x,q^{-1}) + \tau(x,q^{-1})D_qy(x,q^{-1}) + \lambda(q^{-1})y(x,q^{-1}) = 0 \tag{3.15}$$

where

$$\sigma_2(x,q^{-1}) = q^{-1}\sigma_1(x,q^{-1}) + (q^{-1}-1)x\tau(x,q^{-1}) \tag{3.16}$$

which we call these two pairs of equations as q^{-1}-EHT of the 1st kind and q^{-1}-EHT of the 2nd kind, respectively.

Remark 3.1 *Notice that analysing the q-EHT of the 1st and 2nd kinds with* $0 < q < 1$ *is equivalent to* q^{-1}*-EHT with* $s = q^{-1} > 1$.

Remark 3.2 *Throughout the thesis, we define the coefficients of the q-EHT of the 1st and 2nd kinds as the following Taylor polynomials by taking into account that deg*$\sigma_1 \le 2$, *deg*$\sigma_2 \le 2$ *and deg*$\tau = 1$*:*

$$\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q), \tag{3.17}$$

$$\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q), \tag{3.18}$$

$$\tau(x,q) = \tau'(0,q)x + \tau(0,q). \tag{3.19}$$

Remark 3.3 *Notice that by using the relation between $\sigma_1(x,q)$ and $\sigma_2(x,q)$ defined by (3.11), the coefficients have relations*

$$\frac{1}{2}\sigma_2''(0,q) = q\frac{1}{2}\sigma_1''(0,q) + (q-1)\tau'(0,q), \tag{3.20}$$

$$\sigma_2'(0,q) = q\sigma_1'(0,q) + (q-1)\tau(0,q), \tag{3.21}$$

$$\sigma_2(0,q) = q\sigma_1(0,q). \tag{3.22}$$

In accordance with [3, 6, 43, 47], we can determine the degrees of the coefficients σ_1 and σ_2 in (3.12) from the relation in (3.11), using the fact that $\sigma_1(0,q) = 0 \Leftrightarrow \sigma_2(0,q) = 0$ ($\sigma_1(0,q) \neq 0 \Leftrightarrow \sigma_2(0,q) \neq 0$). Therefore we have two classes: the non-zero class which corresponds to the case when $\sigma_1(0,q) \neq 0 \Leftrightarrow \sigma_2(0,q) \neq 0$ and the zero class when $\sigma_1(0,q) = 0 \Leftrightarrow \sigma_2(0,q) = 0$ which lead to the following proposition.

Proposition 3.4 *Let $\rho(x,q)$ be the q-weight function satisfying the q-Pearson equation (3.24) with $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q)$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$.*

If $\sigma_1(0,q) \neq 0$, the following cases arise
(1a) If $\deg[\sigma_1(x,q)] < 2$, then $\deg[\sigma_2(x,q)] = 2$.
(1b) If $\deg[\sigma_1(x,q)] = 2$, then $\deg[\sigma_2(x,q)] \leq 2$.

If $\sigma_1(0,q) = 0$, then:
(2a) If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2$, $\sigma_1''(0,q) \neq 0$, then $\deg[\sigma_2(x,q)] = 2$, or $\deg[\sigma_2(x,q)] = 1$.
(2b) If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x$, $\sigma_1''(0,q) \neq 0$, $\sigma_1'(0,q) \neq 0$, then $\deg[\sigma_2(x,q)] = 2$ or $\deg[\sigma_2(x,q)] = 1$.
(2c) If $\sigma_1(x,q) = \sigma_1'(0,q)x$, $\sigma_1'(0,q) \neq 0$, then $\deg[\sigma_2(x,q)] = 2$.

Proof. (1a) If $\deg[\sigma_1(x,q)] < 2$. Then, $\sigma_2(x,q) = (q-1)\tau'(0,q)x^2 + ...$ It is obvious that $\deg[\sigma_2(x,q)] = 2$ since $\tau'(0,q) \neq 0$.
(1b) If $\deg[\sigma_1(x,q)] = 2$, then $\sigma_1''(0,q) \neq 0$. Using the relations of the coefficients in (3.20), (3.21) and (3.22) with $\tau'(0,q) \neq -\frac{\frac{1}{2}\sigma_1''(0,q)}{1-q^{-1}}$ provide $\sigma_2''(0,q) \neq 0 \Leftrightarrow \deg[\sigma_2(x,q)] = 2$ and if $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{1-q^{-}1}$, then, $\sigma_2''(0,q) = 0$. In case of $\tau(0,q) \neq -\frac{\sigma_1'(0,q)}{1-q^{-1}} \Leftrightarrow \deg[\sigma_2(x,q)] = 1$ and when $\tau(0,q) = -\frac{\sigma_1'(0,q)}{1-q^{-1}} \Leftrightarrow \deg[\sigma_2(x,q)] = 0$.
(2a) Inserting the polynomials $\sigma_1(x,q)$, $\sigma_1(0,q) = 0$ and $\tau(x,q)$ into (3.11) gives,

$$\sigma_2(x,q) = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right]x^2 + (1-q^{-1})\tau(0,q)x.$$

In the case of $\tau'(0,q) \neq -\dfrac{\frac{1}{2}\sigma_1''(0,q)}{1-q^{-1}}$, $\sigma_2''(0,q) \neq 0$ then $\deg[\sigma_2(x,q)] = 2$. If $\tau'(0,q) = -\dfrac{\frac{1}{2}\sigma_1''(0,q)}{1-q^{-1}}$, $\sigma_2''(0,q) = 0$, then $\deg[\sigma_2(x,q)] = 1$.

(2b) This case is obtained in a similar way as in part (a).

(2c) $\sigma_1(x,q) = \sigma_1'(0,q)x \Rightarrow \sigma_2(x,q) = (q-1)\tau'(0,q)x^2 + \left[\sigma_1'(0,q) + (1-q^{-1})\tau(0,q))\right]x$, $\tau'(0,q) \neq 0$. □

3.2 The q-Weight Function

In this section, we discuss the q-weight functions for polynomial solutions of two pairs of the q-EHT. In order to do this, consider the q-EHT of the 1st and 2nd kinds in their self-adjoint forms

$$D_q\left[\sigma_1(x,q)\rho_1(x,q)D_{q^{-1}}y(x)\right] + q^{-1}\lambda(q)\rho_1(x,q)y(x) = 0 \tag{3.23}$$

where $\rho_1(x,q)$ is the q-weight function satisfying the so-called q-Pearson equation

$$D_q\left[\sigma_1(x,q)\rho_1(x,q)\right] = q^{-1}\tau(x,q)\rho_1(x,q) \tag{3.24}$$

and

$$D_{q^{-1}}\left[\sigma_2(x,q)\rho_2(x,q)D_q y(x)\right] + q\lambda(q)\rho_2(x,q)y(x) = 0 \tag{3.25}$$

in which the q-weight function $\rho_2(x,q)$ satisfies the q^{-1}-Pearson equation

$$D_q^{-1}\left[\sigma_2(x,q)\rho_2(x,q)\right] = q\tau(x,q)\rho_2(x,q). \tag{3.26}$$

Remark 3.5 *By use of (3.11), the q-Pearson equation and the q^{-1}-Pearson equation can be rewritten as*

$$\frac{\rho_1(qx,q)}{\rho_1(x,q)} = \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} \tag{3.27}$$

and

$$\frac{\rho_2(q^{-1}x,q)}{\rho_2(x,q)} = \frac{\sigma_2(x,q) + (1-q)x\tau(x,q)}{\sigma_2(q^{-1}x,q)} = \frac{q\sigma_1(x,q)}{\sigma_2(q^{-1}x,q)}, \tag{3.28}$$

respectively. It is clear that $\rho_1(x,q) \equiv \rho_2(x,q)$. Then without loss of generality we define $\rho_1(x,q) = \rho_2(x,q) := \rho(x,q)$.

3.2.1 Computation of the q-Weight Functions

In this part, our aim is to calculate the q-weight functions corresponding to the orthogonal polynomials together with $\left\langle \rho, P_n^2 \right\rangle \neq 0$, $n \geq 0$. The following lemma allows us to find the explicit form of the q-weight function.

Lemma 3.6 *Let f satisfy the relation*

$$\frac{f(qx;q)}{f(x;q)} = \frac{a(x;q)}{b(x;q)}, \tag{3.29}$$

where a and b are given functions, and assume that the limits

$$\lim_{x\to 0} f(x;q) = f(0,q) \quad \text{and} \quad \lim_{x\to\infty} f(x;q) = f(\infty, q)$$

exist. Then, f admits the following two q-integral representations, in the case the corresponding integrals converge,

$$f(x,q) = f(0,q)\exp\left[\int_0^x \frac{1}{(q-1)t}\log\left[\frac{a(t,q)}{b(t,q)}\right]d_qt\right] \tag{3.30}$$

where the q-integral is defined by (2.23), or

$$f(x,q) = f(\infty,q)\exp\left[\int_x^\infty \frac{1}{(1-q^{-1})t}\log\left[\frac{a(t,q)}{b(t,q)}\right]d_{q^{-1}}t\right] \tag{3.31}$$

where the q^{-1}-integral is identified by (2.31).

Proof. By applying the logarithmic function to (3.29), multiplying the obtained expression by $1/(q-1)t$ and taking the q-integral we arrive at

$$\int_0^x \frac{1}{(q-1)t}\log\left[\frac{f(qt,q)}{f(t,q)}\right]d_qt = \int_0^x \frac{1}{(q-1)t}\log\left[\frac{a(t,q)}{b(t,q)}\right]d_qt.$$

But

$$\int_0^x \frac{1}{(q-1)t}\log\left[\frac{f(qt,q)}{f(t,q)}\right]d_qt = \lim_{n\to\infty}\sum_{j=0}^{n}\left[\log\left(f(q^jx,q)\right) - \log\left(f(q^{j+1}x,q)\right)\right] \tag{3.32}$$

$$= \log\left[f(x,q)\right] - \log\left[f(0,q)\right]. \tag{3.33}$$

The last equality follows from the fact that $f(q^{n+1}x,q) \to f(0,q)$ as $n \to \infty$, $0 < q < 1$. The other representation can be proven in a similar way. □

Notice that we can use (3.30) in order to compute ρ satisfying the q-Pearson equation for all possible degrees of the polynomials σ_1 and σ_2 identified by the Proposition 3.4.

At first, we deal with the non-zero case, that is, $(\sigma_1(0,q) \neq 0, \sigma_2(0,q) \neq 0)$ together with Proposition 3.4 which leads to the following well-known results [6, 42].

Theorem 3.7 *Let $(P_n)_{n\geq 0}$ be a solution of the q-EHT in self-adjoint form in (3.23) with the q-weight function ρ. If $a_1(q)$, $b_1(q)$ are the non-zero roots of $\sigma_1(x,q)$ and $a_2(q)$, $b_2(q)$ of $\sigma_2(x,q)$. Then, we obtain the following situations for the q-weight function as Table 3.1 and Table 3.2.*

Table 3.1: The q-weight function for non-zero case according as the degrees of σ_1 and σ_2

	$\sigma_1(x,q)$	$\sigma_2(x,q)$	q-Weight function
(1)	$\frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][(x-b_1(q))]$, $\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$, $\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) \neq 0$	$\dfrac{(a_1^{-1}(q)qx,\ b_1^{-1}(q)qx;q)_\infty}{(a_2^{-1}(q)x,\ b_2^{-1}(q)x;q)_\infty}$
(2)	$\frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][(x-b_1(q))]$, $\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q) \neq 0$	$\sigma_2'(0,q)[x-a_2(q)]$, $\sigma_2'(0,q)a_2(q) \neq 0$	$\dfrac{(a_1^{-1}(q)qx,\ b_1^{-1}(q)qx;q)_\infty}{(a_2^{-1}(q)x;q)_\infty}$
(3)	$\frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][(x-b_1(q))]$, $\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q) \neq 0$	$\sigma_2(0,q)$, $\sigma_2(0,q) \neq 0$	$(a_1^{-1}(q)qx,\ b_1^{-1}(q)qx;q)_\infty$
(4)	$\sigma_1'(0,q)[x-a_1(q)]$, $\sigma_1(0,q)a_1(q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$, $\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) \neq 0$	$\dfrac{(a_1^{-1}(q)qx;q)_\infty}{(a_2^{-1}(q)x,\ b_2^{-1}(q)x;q)_\infty}$
(5)	$\sigma_1(0,q)$, $\sigma_1(0,q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$, $\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) \neq 0$	$\dfrac{1}{(a_2^{-1}(q)x,\ b_2^{-1}(q)x;q)_\infty}$

Table 3.2: Alternative q-weight function for non-zero case according as the degrees of the polynomial coefficients

	q-Weight function	q^α
(1)	$\rho(x,q) = x^\alpha \dfrac{(a_2(q)q/x,\ b_2(q)q/x;q)_\infty}{(a_1(q)/x,\ b_1(q)/x;q)_\infty}$	$\dfrac{\frac{1}{2}\sigma_2''(0,q)q^{-3}}{\frac{1}{2}\sigma_1''(0,q)}$
(2)	$\rho(x,q) = \dfrac{x^\alpha}{\sqrt{x^{\log_q x-1}}} \dfrac{(a_2(q)q/x;q)_\infty}{(a_1(q)/x,\ b_1(q)/x;q)_\infty}$	$\dfrac{\sigma_2'(0,q)q^{-3}}{\frac{1}{2}\sigma_1''(0,q)}$
(3)	$\rho(x,q) = \dfrac{x^\alpha}{x^{\log_q x-1}} \dfrac{1}{(a_1(q)/x,\ b_1(q)/x;q)_\infty}$	$\dfrac{\sigma_2(0,q)q^{-3}}{\frac{1}{2}\sigma_1''(0,q)}$
(4)	$\rho(x,q) = x^\alpha \sqrt{x^{\log_q x-1}} \dfrac{(a_2(q)q/x,\ b_2(q)q/x;q)_\infty}{((a_1(q)/x;q)_\infty}$	$\dfrac{\frac{1}{2}\sigma_2''(0,q)q^{-2}}{\sigma_1'(0,q)}$
(5)	$\rho(x,q) = x^\alpha x^{\log_q x-1}(a_2(q)q/x,\ b_2(q)q/x;q)_\infty$	$\dfrac{\frac{1}{2}\sigma_2''(0,q)q^{-1}}{\sigma_1(0,q)}$

Proof. Proof is based on the q-Pearson equation defined by (3.27). Consider case 1 that is,

$$\begin{aligned}\sigma_1(x,q) &= \frac{1}{2}\sigma_1''(0,q)\left[x-a_1(q)\right]\left[x-b_1(q)\right],\\ \sigma_2(x,q) &= q\left[\sigma_1(x,q)+(1-q^{-1})x\tau(x,q)\right] = \frac{1}{2}\sigma_2''(0,q)\left[x-a_2(q)\right]\left[x-b_2(q)\right],\end{aligned}$$

taking into account $q^{-1}\sigma_2(0,q)=\sigma_1(0,q) \Rightarrow q^{-1}\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q)=\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$, i.e., the polynomials $q^{-1}\sigma_2(x,q)$ and $\sigma_1(x,q)$ have same independent term, we can write the q-Pearson equation as the following form

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)(x-a_2(q))(x-b_2(q))}{\frac{1}{2}\sigma_1''(0,q)(qx-a_1(q))(qx-b_1(q))} = \frac{(1-a_2^{-1}(q)x)(1-b_2^{-1}(q)x)}{(1-a_1^{-1}(q)qx)(1-b_1^{-1}(q)qx)} \tag{3.34}$$

from which ρ follows from (3.30) as

$$\begin{aligned}\rho(x,q) &= \rho(0,q)\exp\left\{\int_0^x \frac{1}{(q-1)t}\Big[\ln(1-a_2^{-1}(q)t)+\ln(1-b_2^{-1}(q)t)\right.\\ &- \left.\ln(1-a_1^{-1}(q)qt)-\ln(1-b_1^{-1}(q)qt)\Big]d_qt\right\}.\end{aligned} \tag{3.35}$$

Now with the definition (2.23) of the q-Jackson integral ρ is equivalent to

$$\begin{aligned}\rho(x,q) &= \rho(0,q)\exp\left\{\sum_{k=0}^{\infty}\ln(1-a_1^{-1}(q)q^{k+1}x)+\ln(1-b_1^{-1}(q)q^{k+1}x)\right.\\ &- \left.\ln(1-a_2^{-1}(q)q^kx)-\ln(1-b_2^{-1}(q)q^kx)\right\}\\ &= \rho(0,q)\exp\left\{\ln\Big[\prod_{k=0}^{\infty}(1-a_1^{-1}(q)q^{k+1}x)(1-b_1^{-1}(q)q^{k+1}x)\Big]\right.\\ &- \left.\ln\Big[\prod_{k=0}^{\infty}(1-a_2^{-1}(q)q^kx)(1-b_2^{-1}(q)q^kx)\Big]\right\}\\ &= \rho(0,q)\frac{(a_1^{-1}(q)qx,b_1^{-1}(q)qx;q)_\infty}{(a_2^{-1}(q)x,b_2^{-1}(q)x;q)_\infty}\end{aligned} \tag{3.36}$$

in which $a_1(q)q^{-1-k}$, $b_1(q)q^{-1-k}$, $k\geq 0$ are zeros and $a_2(q)q^{-j}$, $b_2(q)q^{-j}$, $j\geq 0$ are poles with $a_1(q),b_1(q)\in\mathbb{C}-\{0\}$ and $a_2(q),b_2(q)\in\mathbb{C}-\{0\}$. Notice that the function obtained in (3.36) is meromorphic and thus, it is continuous with $\rho(0,q)\neq 0$, then we can take without loss of generality that $\rho(0,q)=1$.

All the other cases in Table 3.1 can be determined in a similar manner. However, the cases defined in Table 3.2 are not based on the same idea. In order to identify those cases we consider case 2 in Table 3.2 satisfying the q-Pearson equation

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{a(1-a_2(q)/x)}{x(1-q^{-1}a_1(q)/x)(1-q^{-1}b_1(q)/x)} \tag{3.37}$$

where $a = \dfrac{q^{-3}\sigma_2'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}$. We can define ρ as a product of two functions $\rho(x,q) = f(x,q)g(x,q)$

$$\Rightarrow \frac{\rho(qx,q)}{\rho(x,q)} = \frac{f(qx,q)}{f(x,q)}\frac{g(qx,q)}{g(x,q)}$$

where $\dfrac{f(qx,q)}{f(x,q)} = \dfrac{a}{x}$ and $\dfrac{g(qx,q)}{g(x,q)} = \dfrac{(1-a_2(q)/x)}{(1-q^{-1}a_1(q)/x)(1-q^{-1}b_1(q)/x)}$. To find the corresponding function $f(x,q)$, we use the function $h^{(\beta)} : [0,\infty) \to \mathcal{R}$ given in [6] identified by Häcker

$$h^{(\beta)}(x) = \sqrt{x^{\log_q^{x^\beta}} - \beta}, \quad \beta \neq 0.$$

We assume that $f(x,q) = \dfrac{|x|^\alpha}{h^{(1)}(x)}$, $\alpha \in \mathbb{C} - \{0\}$ where $q^\alpha = a$

$$f(qx,q) = \frac{q^\alpha\,|x|^\alpha}{x\sqrt{x^{\log_q x} - 1}} = \frac{a}{x}f(x,q).$$

For the computation of $g(x,q)$, the equation (3.30) does not work since it gives a divergent infinite product. Then, we use (3.31) which leads to

$$\frac{g(qx,q)}{g(x,q)} = \frac{(1-a_2(q)/x)}{(1-q^{-1}a_1(q)/x)(1-q^{-1}b_1(q)/x)} \Leftrightarrow \frac{g(q^{-1}x,q)}{g(x,q)} = \frac{(1-a_1(q)/x)(1-b_1(q)/x)}{(1-qa_2(q)/x)},$$

we attain the desired result

$$\begin{aligned}
\frac{g(x,q)}{g(\infty,q)} &= \exp\left[\int_x^\infty \frac{1}{(1-q^{-1})t}\ln\left[\frac{(1-a_1(q)/t)(1-b_1(q)/t)}{(1-qa_2(q)/t)}\right]d_{q^{-1}}t\right] \\
&= \exp\left[\lim_{n\to\infty}\sum_{j=0}^{n}\left[\ln(1-q^{1+j}a_2(q)/x) - \ln(1-q^ja_1(q)/x) - \ln(1-q^jb_1(q)/x)\right]\right] \\
&= \frac{(qa_2(q)/x;q)_\infty}{(a_1(q)/x, b_1(q)/x;q)_\infty}. \qquad (3.38)
\end{aligned}$$

Obviously, the product in $g(x,q)$ is uniformly convergent in any compact subset of the complex plane that does not include the points $\{q^na_1(q), n \geq 0\}, \{q^nb_1(q), n \geq 0\}$ and $\{0\}$. Moreover, this product is convergent as $x \to \infty$, thus $g(\infty,q) = c \neq 0$ and hence without loss of generality we can take it as 1. Then the q-weight function can be arranged as

$$\rho(x,q) = f(x,q)g(x,q) = \frac{|x|^\alpha}{\sqrt{x^{\log_q x - 1}}}\frac{(qa_2(q)/x;q)_\infty}{(a_1(q)/x, b_1(q)/x;q)_\infty}. \qquad (3.39)$$

The other cases defined in Table 3.2 can be constructed analogously. □

The next step is to compute the q-weight function identified by (3.32) and (3.31) by taking account of the zero case; $\sigma_1(0,q) = \sigma_2(0,q) = 0$ together with all possible degrees identified by the Proposition 3.4. In order to do this, we establish similar framework as follows:

Theorem 3.8 *Let $(P_n)_{n\geq 0}$ be a solution of the q-EHT in self-adjoint form in (3.23) with the q-weight function ρ. If $a_1(q)$ and $b_1(q)$ are the zeros of $\sigma_1(x,q)$ and $a_2(q)$ and $b_2(q)$ of $\sigma_2(x,q)$ with $b_1(q) = 0$ and $b_2(q) = 0$. Then we obtain the Table 3.3 and Table 3.4 [6, 42].*

Table 3.3: The q-weight function for zero case according as the degrees of σ_1 and σ_2

	$\sigma_1(x,q)$	$\sigma_2(x,q)$	q-Weight function	q^{α}
(1)	$\frac{1}{2}\sigma_1''(0,q)x^2$, $\frac{1}{2}\sigma_1''(0,q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)x[x-a_2(q)]$, $\frac{1}{2}\sigma_2''(0,q)a_2(q) \neq 0$	$\frac{\lvert x\rvert^{\alpha}}{\sqrt{x^{\log_q x-1}}}\frac{1}{(a_2^{-1}(q)x;q)_\infty}$	$-\frac{q^{-3}\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\frac{1}{2}\sigma_1''(0,q)}$
(2)	$\frac{1}{2}\sigma_1''(0,q)x^2$, $\frac{1}{2}\sigma_1''(0,q) \neq 0$	$\sigma_2'(0,q)x$, $\sigma_2'(0,q) \neq 0$	$\lvert x\rvert^{\alpha}\sqrt{x^{\log_q \frac{1}{x}+1}}$	$\frac{\sigma_2'(0,q)q^{-3}}{\frac{1}{2}\sigma_1''(0,q)}$
(3)	$\frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$, $\frac{1}{2}\sigma_1''(0,q)a_1(q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)x[x-a_2(q)]$, $\frac{1}{2}\sigma_2''(0,q)a_2(q) \neq 0$	$\lvert x\rvert^{\alpha}\frac{(a_1^{-1}(q)qx;q)_\infty}{(a_2^{-1}(q)x;q)_\infty}$	$\frac{\frac{1}{2}\sigma_2''(0,q)q^{-2}a_2(q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)}$
(4)	$\frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$, $\frac{1}{2}\sigma_1''(0,q)a_1(q) \neq 0$	$\sigma_2'(0,q)x$, $\sigma_2'(0,q) \neq 0$	$\lvert x\rvert^{\alpha}(a_1^{-1}(q)qx;q)_\infty$	$-\frac{q^{-2}\sigma_2'(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)}$
(5)	$\frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$, $\frac{1}{2}\sigma_1''(0,q)a_1(q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)x^2$, $\frac{1}{2}\sigma_2''(0,q) \neq 0$	$\lvert x\rvert^{\alpha}\sqrt{x^{\log_q x-1}}(a_1^{-1}(q)qx;q)_\infty$	$-\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)}$
(6)	$\sigma_1'(0,q)x$, $\sigma_1'(0,q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)x[x-a_2(q)]$, $\frac{1}{2}\sigma_2''(0,q)a_2(q) \neq 0$	$\lvert x\rvert^{\alpha}\frac{1}{(a_2^{-1}(q)x;q)_\infty}$	$-\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\sigma_1'(0,q)}$
(7)	$\sigma_1'(0,q)x$, $\sigma_1'(0,q) \neq 0$	$\frac{1}{2}\sigma_2''(0,q)x^2$, $\frac{1}{2}\sigma_2''(0,q) \neq 0$	$\lvert x\rvert^{\alpha}\sqrt{x^{\log_q x-1}}$	$\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$

Table 3.4: Alternative q-weight function for zero case according as the degrees of the polynomial coefficients

	q-Weight function	q^{α}
(1)	$\rho(x,q) = \lvert x\rvert^{\alpha}(qa_2(q)/x;q)_\infty$	$\frac{q^{-3}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)}$
(3)	$\rho(x,q) = \lvert x\rvert^{\alpha}\frac{(qa_2(q)/x;q)_\infty}{(a_1(q)/x;q)_\infty}$	$\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)}$
(4)	$\rho(x,q) = \frac{\lvert x\rvert^{\alpha}}{\sqrt{x^{\log_q x-1}}}\frac{1}{(a_1(q)/x;q)}_\infty$	$\frac{q^{-2}\sigma_2'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}$
(5)	$\rho(x,q) = \lvert x\rvert^{\alpha}\frac{1}{(a_1(q)/x;q)_\infty}$	$\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)}$
(6)	$\rho(x,q) = \lvert x\rvert^{\alpha}\sqrt{x^{\log_q x-1}}(qa_2(q)/x;q)_\infty$	$\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$

Proof. We compute the q-weight function by taking into account of the zero case identified by the proposition 3.4 as two separate states since the ratio in q- Pearson equation differs [6] whether the polynomials $\sigma_1(x,q)$ and $\sigma_2(x,q)$ have zero roots with multiplicity two or not. Starting with the q-Pearson equation (3.24), we consider

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}x\sigma_{02}(x,q)}{qx\sigma_{01}(qx,q)}$$

where $\sigma_2(x,q) = x\sigma_{02}(x,q)$ and $\sigma_1(qx,q) = x\sigma_{01}(qx,q)$. Notice that constant terms of the polynomials σ_{01} and σ_{02} may not be equivalent. We suppose that the q-weight function can be defined as $\rho(x,q) = |x|^{\alpha}\rho_0(x,q)$, $\alpha \in \mathbb{C} - \{0\}$ where

$$\frac{\rho_0(qx,q)}{\rho_0(x,q)} = \frac{\sigma_{02}(x,q)}{\sigma_{01}(qx,q)}.$$

Then, according to the cases defined in Table 3.3 we use (3.30) to compute $\rho_0(x,q)$. To show how it happens, we consider the 3th case which has the q-weight function of the form $\rho(x,q) = |x|^{\alpha}\rho_0(x,q)$ where $q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)}$ with

$$\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1(0,q)x = \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)],$$

$$\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2(0,q)x = \frac{1}{2}\sigma_2''(0,q)x[x - a_2(q)]$$

and

$$\frac{\rho_0(qx,q)}{\rho_0(x,q)} = \frac{(1 - a_2^{-1}(q)x)}{(1 - a_1^{-1}(q)qx)}. \tag{3.40}$$

By using (3.30) for (3.40) we get

$$\rho_0(x,q) = \frac{(a_1^{-1}(q)qx,q)_\infty}{(a_2^{-1}(q)x,q)_\infty}, \tag{3.41}$$

then

$$\rho(x,q) = |x|^{\alpha}\frac{(a_1^{-1}(q)qx,q)_\infty}{(a_2^{-1}(q)x,q)_\infty}, \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)},$$

which is the q-weight function for 3th case given in Table 3.3. So as to determine the q-weight function defined in Table 3.4 we concern with

$$\rho(x,q) = |x|^{\alpha}\rho_0(x,q), \quad \alpha = \log_q \frac{q^{-2}\sigma_2''(0,q)}{\sigma_1''(0,q)}$$

where

$$\frac{\rho_0(qx,q)}{\rho_0(x,q)} = \frac{(1 - a_2(q)/x)}{(1 - a_1(q)q^{-1}/x)} \Leftrightarrow \frac{\rho_0(q^{-1}x,q)}{\rho_0(x,q)} = \frac{(1 - a_1(q)/x)}{(1 - a_2(q)q/x)}. \tag{3.42}$$

By using (3.31) we arrive at

$$\rho_0(x,q) = \frac{(a_2(q)q/x,q)_\infty}{(a_1(q)/x,q)_\infty}, \tag{3.43}$$

then

$$\rho(x,q) = |x|^{\alpha}\,\frac{(a_2(q)q/x,q)_\infty}{(a_1(q)/x,q)_\infty}, \quad \alpha = \log_q \frac{q^{-2}\sigma_2''(0,q)}{\sigma_1''(0,q)}.$$

At last, we need to calculate the q-weight function for the 2nd and 7th case where the function identified by Häcker $h^{(\beta)} : [0,\infty) \to \mathcal{R}$

$$h^{(\beta)}(x) = \sqrt{x^{\log_q^{x^\beta} - \beta}}, \quad \beta \neq 0 \tag{3.44}$$

is used. Applying the property $h^{(\beta)}(qx) = x^{\beta} h^{(\beta)}(x)$, for all $x \geq 0$ provides $\rho_0(x,q) = h^{(1)}(x)$ for the 2nd case, and $\rho_0(x,q) = h^{(-1)}(x)$ for the 7th [6].

The other cases can be obtained analogously. □

Remark 3.9 *We remark that the q-Pearson equation have solutions which are different from the ones given in the Table 3.1, Table 3.2 and Table 3.3, Table 3.4. This different forms arise from the structure of the q-Pearson equation. The procedure of computing new representations is to define the q-weight function as the product of two or more functions satisfying a q-Pearson equation. Then, in order to determine the solutions we use the suitable identity among (3.30), (3.31) or the function identified by Häcker.*

3.3 Polynomial Solutions of the q-EHT of the 1st kind

Polynomial solutions of the q-EHT of the 1st kind named q-polynomials of the 1st kind are presented by showing that all q-derivatives of the functions of the hypergeometric type are also functions of hypergeometric type which can be proved in the following theorem.

Theorem 3.10 *Let $v_k(x,q) = D^{(k)}_{q^{-1}} y(x,q)$ with $v_0(x,q) = y(x,q)$, then $v_k(x,q)$, $k = 0, 1, \ldots$ are also solutions of a q-EHT.*

Proof. Applying $D_{q^{-1}}$ to the q-EHT in (3.5) and by use of the product rule identified by (2.12) and relation between the operators in (3.4), we have

$$\sigma_{1_1}(x;q) D_{q^{-1}} D_q v_1(x,q) + \tau_{1_1}(x,q) D_q v_1(x,q) + \lambda_{1_1}(q) v_1(x,q) = 0$$

where,

$$\sigma_{1_1}(x;q) = \sigma_1(q^{-1}x,q),$$
$$\tau_{1_1}(x,q) = \tau(x,q) + D_{q^{-1}}\sigma_1(x,q),$$
$$\lambda_{1_1}(q) = q^{-1}[\lambda(q) + D_{q^{-1}}\tau(x,q)].$$

It is seen that $v_1(x,q)$ also satisfies a q-EHT of the 1st kind . By applying the q^{-1}-derivative to the q-EHT of the 1st kind successively, the q-EHT for $v_k(x,q)$, $k = 0, 1, ...$ is determined in the form

$$\sigma_{1_k}(x;q)D_{q^{-1}}D_q v_k(x,q) + \tau_{1_k}(x,q)D_q v_k(x,q) + \lambda_{1_k}(q)v_k(x,q) = 0 \tag{3.45}$$

where,

$$\sigma_{1_k}(x;q) = \sigma_1(q^{-k}x,q), \tag{3.46}$$
$$\tau_{1_k}(x,q) = \tau_{1_{k-1}}(x,q) + D_{q^{-1}}\sigma_{1_{k-1}}(x,q), \tag{3.47}$$
$$\lambda_{1_k}(q) = q^{-1}\left[\lambda_{1_{k-1}}(q) + D_{q^{-1}}\tau_{1_{k-1}}(x,q)\right] \tag{3.48}$$

with $\sigma_{1_0}(x;q) := \sigma_1(x;q)$, $\tau_{1_0}(x,q) = \tau(x,q)$ and $\lambda_{1_0}(q) = \lambda(q)$. It is obvious that $\sigma_{1_k}(x;q)$ and $\tau_{1_k}(x,q)$ are the polynomials of degree at most two and exactly one, respectively and $\lambda_{1_k}(q)$ is a constant. Then, $v_k(x,q)$ also satisfies a q-EHT of the 1st kind. □

Moreover, by solving the coefficients $\tau_{1_k}(x,q)$ and $\lambda_{1_k}(q)$ successively, explicit forms

$$\tau_{1_k}(x,q) = \tau(x,q) + \frac{\sigma_1(q^{-k}x,q) - \sigma_1(x,q)}{(q^{-1}-1)x} \tag{3.49}$$
$$\lambda_{1_k}(q) = q^{-k}\left[\lambda(q) + [k]_q\tau'(0,q) + \frac{1}{2}[k-1]_{q^{-1}}[k]_q\sigma_1''(0,q)\right] \tag{3.50}$$

are determined in which $[k]_q$ and $[k]_{q^{-1}}$ are the q and q^{-1}-analogues of k defined by (2.7) and (2.9), respectively.

Notice that by means of the relation between $\sigma_2''(0,q)$ and $\sigma_1''(0,q)$ identified by (3.20);

$$\sigma_2''(0,q) = q\left[\sigma_1''(0,q) + 2(1-q^{-1})\tau'(0,q)\right],$$

$$\lambda_{1_k}(q) = q^{-k}\left[\lambda(q) + \frac{[k]_q}{2(1-q^{-1})}\left(q^{-1}\sigma_2''(0,q) - q^{-(k-1)}\sigma_1''(0,q)\right)\right] \tag{3.51}$$

is determined in terms of σ_1 and σ_2. Now condition for polynomial solutions of the q-EHT of the 1st kind are defined in the following theorem.

Theorem 3.11 *The q-EHT of the 1st kind have polynomial solutions, say $y(x) = P_{1_n}(x, q)$, of degree n if and only if*

$$\lambda(q) := \lambda_n(q) = -[n]_q \left[\tau'(0, q) + \frac{1}{2}[n-1]_{q^{-1}}\sigma_1''(0, q)\right], \quad n = 0, 1, ... \tag{3.52}$$

Proof. For specific value of $\lambda(q)$ given in (3.52) makes $\lambda_{1_n}(q) = 0$. Then, the q-EHT of the 1st kind for $v_n(x, q)$ has a constant solution c. Since $v_n(x, q) = D^{(n)}_{q^{-1}} y(x)$ where $y(x)$ is the solution of the q-EHT of the 1st kind and $v_n(x, q) = c$, then $D^{(n)}_{q^{-1}} y(x) = c$ from which we obtain $y(x) := P_{1_n}(x, q)$ is a polynomial of degree n by performing q-integral successively [30, 53]. To prove the converse of the theorem, suppose that the q-EHT of the 1st kind has polynomial solution $y(x) := P_{1_n}(x, q)$, then $D^{(n)}_{q^{-1}} P_{1_n}(x, q) = c$ satisfies a q-EHT of the 1st kind if and only if $\lambda_{1_n}(q) = 0$, which completes the proof. □

Proposition 3.12 *Let $\lambda_{1_{nk}}(q) = q^{-k}\left[\lambda_n(q) + [k]_q\tau'(0, q) + \frac{1}{2}[k-1]_{q^{-1}}[k]_q\sigma_1''(0, q)\right]$ be the coefficient of the q-EHT in (3.45) for $D^{(k)}_{q^{-1}} y_n(x) := v_{kn}(x, q)$. Then,*

$$\lambda_{1_{nk}}(q) = -[n-k]_q \left(\tau'(0, q) + [n+k-1]_{q^{-1}} \frac{1}{2}\sigma_1''(0, q)\right), \quad \lambda_{1_{n0}}(q) = \lambda_n(q). \tag{3.53}$$

Proof. Proof is trivial by use of $\lambda_n(q)$ defined by (3.52) and substituting this value in $\lambda_{1_{nk}}$. □

Remark 3.13 *It is possible to write the q-EHT of the 1st kind for $v_{kn}(x, q)$ in self-adjoint form*

$$D_q\left[\sigma_{1_k}(x, q)\rho_{1_k}(x, q)D_{q^{-1}} v_{kn}(x, q)\right] + q^{-1}\lambda_{1_{nk}}(q)\rho_{1_k}(x, q)v_{kn}(x, q) = 0 \tag{3.54}$$

where $v_{kn}(x, q) = \left(D^{(k)}_{q^{-1}} P_n(x; q)\right)$. Here, the q-weight function $\rho_{1_k}(x, q)$ is the solution of the q-Pearson equation

$$D_q\left[\sigma_{1_k}(x, q)\rho_{1_k}(x, q)\right] = q^{-1}\tau_{1_k}(x, q)\rho_{1_k}(x, q). \tag{3.55}$$

Proposition 3.14 *Let $\rho(x, q)$ be a solution of (3.24) and $\rho_{1_n}(x, q)$ a solution of (3.55). Then,*

$$\rho_{1_n}(x, q) = \sigma_{1_{n-1}}(x, q)\rho_{1_{n-1}}(x, q) = ... = \prod_{k=0}^{n-1} \sigma_1(q^{-k}x, q)\rho(x, q), \tag{3.56}$$

$\rho_{1_0}(x, q) = \rho_1(x, q) = \rho(x, q)$.

Proof. Starting from (3.55) and rewriting it in the equivalent form

$$\frac{\rho_{1_n}(qx, q)\sigma_{1_n}(qx, q)}{\rho_{1_n}(x, q)} = \sigma_{1_n}(x, q) + (1 - q^{-1})x\tau_{1_n}(x, q) \tag{3.57}$$

and substituting (3.46) and (3.47) to the right hand side of (3.57)

$$\frac{\sigma_{1_{n-1}}(x,q)\rho_{1_{n-1}}(x,q)}{\rho_{1_n}(x,q)} = \frac{\sigma_{1_{n-1}}(qx,q)\rho_{1_{n-1}}(qx,q)}{\rho_{1_n}(qx,q)} = c_n(x)$$

is obtained where $c_n(x) = c_n(qx)$. Since it is enough to find a particular solution of the q-Pearson equation for $\rho_{1_n}(x,q)$ (3.56) then we may take $c_n(x) = 1$ which makes

$$\rho_{1_n}(x,q) = \sigma_{1_{n-1}}(x,q)\rho_{1_{n-1}}(x,q).$$

Thus, successive solution gives

$$\rho_{1_n}(x,q) = \sigma_1(q^{-(n-1)}x,q)...\sigma_1(q^{-1}x,q)\sigma_1(x,q)\rho(x,q) = \prod_{k=0}^{n-1}\sigma_1(q^{-k}x,q)\rho(x,q).$$

□

3.3.1 The Rodrigues Formula for Polynomial Solutions of the q-EHT of the 1st kind

The representation of the polynomial solutions is characterized by the so-called Rodrigues formula describing in the following theorem.

Theorem 3.15 *Let $\rho(x,q)$ be the q-weight function defined by the q-Pearson equation (3.24) and $\rho_{1_n}(x,q)$ by (3.56). Then,*

$$P_{1_n}(x,q) = q^n B_{1_n}(q)\frac{D_q^n\left[\rho_{1_n}(x,q)\right]}{\rho(x,q)} \tag{3.58}$$

where

$$B_{1_n}(q) = (-1)^n\frac{[1]_{q^{-1}}[2]_{q^{-1}}...[n]_{q^{-1}}}{\lambda_{1_{nn-1}}(q)\lambda_{1_{nn-2}}(q)...\lambda_{1_{n0}}(q)}P_{1_0}^{(n)}(q) \tag{3.59}$$

stands for normalization constant with $P_{1_0}^{(n)}(q) = \frac{1}{[1]_{q^{-1}}^n}D_{q^{-1}}^n P_{1_n}(x,q)$.

Proof. By using (3.56) we consider (3.54) for $P_{1_0}^{(n)}(x,q) := \frac{1}{[1]_{q^{-1}}^n}D_{q^{-1}}^n P_{1_n}(x,q) = \frac{1}{[1]_{q^{-1}}^n}v_{nn}$ where $[1]_{q^{-1}}^{(n)} = [1]_{q^{-1}}[2]_{q^{-1}}...[n]_{q^{-1}}$

$$D_q\left[\rho_{1_n}(x,q)P_{1_0}^{(n)}(x,q)\right] = -\frac{q^{-1}}{[1]_{q^{-1}}}\lambda_{1_{nn-1}}(q)\left[\rho_{1_{n-1}}(x,q)P_{1_1}^{(n-1)}(x,q)\right].$$

Then applying the operator D_q successively. we obtain

$$D_q^2\left[\rho_{1_n}(x,q)P_{1_0}^{(n)}(x,q)\right] = -\frac{q^{-2}}{[1]_{q^{-1}}[2]_{q^{-1}}}\lambda_{1_{nn-1}}(q)\lambda_{1_{nn-2}}(q)\left[\rho_{1_{n-2}}(x,q)P_{1_2}^{(n-2)}(x,q)\right]$$

$$D_q^3\left[\rho_{1_n}(x,q)P_{1_0}^{(n)}(x,q)\right] = \frac{q^{-3}}{[1]_{q^{-1}}[2]_{q^{-1}}[3]_{q^{-1}}}\lambda_{1_{nn-1}}(q)...\lambda_{1_{nn-3}}(q)\left[\rho_{1_{n-3}}(x,q)P_{1_3}^{(n-3)}(x,q)\right]$$

...

$$D_q^n\left[\rho_{1_n}(x,q)P_{1_0}^{(n)}(x,q)\right] = \frac{(-1)^n q^{-n}}{[1]_{q^{-1}}[2]_{q^{-1}}...[n]_{q^{-1}}}\lambda_{1_{nn-1}}(q)...\lambda_{1_{n0}}(q)\left[\rho_{1_0}(x,q)P_{1_n}^{(0)}(x,q)\right]$$

which is the desired result with $\lambda_{1_{n0}}(q) := \lambda_n(q)$, $\rho_{1_0}(x,q) := \rho(x,q)$, $P^{(0)}_{1_n}(x,q) := P_{1_n}(x,q)$ and $P^{(n)}_{1_0}(x,q) := P^{(n)}_{1_0}(q)$ as $P^{(n)}_{1_0}(q)$ is independent of x. □

Remark 3.16 *Notice that the normalization constant defined by (3.59) can be written in the form*

$$B_{1_n}(q) = \frac{q^{-\binom{n}{2}}}{\prod_{k=0}^{n-1}\left[\tau'(0,q) + [n+k-1]_{q^{-1}}\frac{1}{2}\sigma_1''(0,q)\right]} P^{(n)}_{1_0}(q) \tag{3.60}$$

by using the expression for $\lambda_{1_{nk}}$ defined by (3.53) and the relation between $[k]_q$ and $[k]_{q^{-1}}$ in (2.10).

Proposition 3.17 *Normalization constant $B_{1_n}(q)$ presented in the Rodrigues formula (3.59) can be rewritten for monic polynomials ($P^{(n)}_{1_0}(q) = 1$) as*

$$B_{1_n}(q) = \frac{(-1)^n(1-q^{-1})^n q^{2\binom{n}{2}}}{(\frac{1}{2}\sigma_1''(0,q))^n \left(q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q);q\right)_n} \tag{3.61}$$

where $a_1(q), b_1(q)$ are the roots of the polynomial $\sigma_1(x,q)$ and $a_2(q), b_2(q)$ of the polynomial $\sigma_2(x,q)$ and $(a,q)_n$ is the q-shifted factorial identified by (2.15).

Proof. Inserting $\lambda_{1_{nk}}(q)$, for $k = 0,1,...,n-1$ into the product gives

$$\prod_{k=0}^{n-1}\lambda_{1_{nk}}(q) = (-1)^n[1]_q...[n]_q\prod_{k=0}^{n-1}\left(\tau'(0,q) + [n+k-1]_{q^{-1}}\frac{1}{2}\sigma_1''(0,q)\right).$$

Later, we substitute this value into the normalization constant (3.59) with considering the property between q-number and q^{-1}-number (2.10) and we arrive at the following expression

$$B_{1_n}(q) = \frac{q^{-\binom{n}{2}}}{\left(\tau'(0,q) + [2n-2]_{q^{-1}}\frac{1}{2}\sigma_1''(0,q)\right)...\left(\tau'(0,q) + [n-1]_{q^{-1}}\frac{1}{2}\sigma_1''(0,q)\right)}.$$

Then, by using the relation defined by (3.20) and (3.22), we get

$$\prod_{k=0}^{n-1}\left(\tau'(0,q) + \frac{1}{2}[n+k-1]_{q^{-1}}\sigma_1''(0,q)\right) = \frac{q^{-3\binom{n}{2}}\left(-\frac{1}{2}\sigma_1''(0,q)\right)^n}{(1-q^{-1})^n}\left(q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q);q\right)_n$$

where $\binom{n}{2} = n(n-1)/2$. Afterwards, substituting this product into the above equality for $B_{1_n}(q)$, result is obtained. □

Rodrigues formula is particularly useful to identify the explicit expression for the polynomials P_{1_n}. Alternatively, we introduce another representation of the polynomials in the following proposition by means of the following identity

$$D_q^n f(x) = \frac{1}{(1-q)^n x^n}\sum_{k=o}^{n}(-1)^k q^{k(k+1)/2-nk}\begin{bmatrix} n \\ k \end{bmatrix}_q f(q^k x) \tag{3.62}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is q-binomial identified by (2.13).

Proposition 3.18 *Let $D_q^n f(x)$ be given with (3.62). Then,*

$$P_{1_n}(x,q) = \frac{q^n B_{1_n}(q) \prod_{i=0}^{n-1} \sigma_1(q^{-i}x,q)}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{k(k+1)/2-nk-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \times \frac{\prod_{i=0}^{k-1} \sigma_2(q^i x,q)}{\prod_{i=0}^{k-1} \sigma_1(q^{-(n-1-i)}x,q)} \tag{3.63}$$

where $B_{1_n}(q)$ is the normalization constant defined by (3.60).

Proof. Putting $D_q^n f(x)$ given with (3.62) into the Rodrigues formula, we arrive at the following representation of P_{1_n}

$$P_{1_n}(x,q) = \frac{q^n B_{1_n}(q)}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{k(k+1)/2-nk-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\rho_{1_n}(q^k x,q)}{\rho(x,q)}.$$

By use of $\rho_{1_n}(x,q) = \prod_{i=0}^{n-1} \sigma_1(q^{-i}x,q)\rho(x,q)$, and

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} \Leftrightarrow \frac{\rho(q^{-1}x,q)}{\rho(x,q)} = \frac{q\sigma_1(x,q)}{\sigma_2(q^{-1}x,q)},$$

we attain the result given in the proposition.

□

3.4 Polynomial Solutions of the q-EHT of the 2nd kind

We introduce q-polynomials of the 2nd kind analog to q-polynomials of the 1st kind. We begin with modifying the theorem 3.10:

Theorem 3.19 *Let $u_k(x,q) = D_q^{(k)} y(x,q)$ with $u_0(x,q) = y(x,q)$, then $u_n(x,q)$, $n = 0, 1, ...$ are also solutions of a q-EHT of the 2nd kind.*

Proof. Applying D_q to the q-EHT of the 2nd kind and using the product rule defined by (2.11) give q-difference equation for $u_1(x,q)$

$$\sigma_{2_1}(x;q) D_q D_{q^{-1}} u_1(x,q) + \tau_{2_1}(x,q) D_q u_1(x,q) + \lambda_{2_1}(q) u_1(x,q) = 0$$

where,

$$\sigma_{2_1}(x;q) = \sigma_2(qx,q),$$

$$\tau_{2_1}(x,q) = \tau(x,q) + D_q\sigma_2(x,q),$$

$$\lambda_{2_1}(q) = q[\lambda(q) + D_q\tau(x,q)].$$

By applying the q-derivative to the q-EHT of the 2nd kind successively, we arrive at the q-EHT for $u_k(x,q)$, $k = 0, 1, ...$

$$\sigma_{2_k}(x,q)D_qD_{q^{-1}}u_k(x,q) + \tau_{2_k}(x,q)D_qu_k(x,q) + \lambda_{2_k}(q)u_k(x,q) = 0 \tag{3.64}$$

where,

$$\sigma_{2_k}(x,q) = \sigma_2(q^kx,q), \tag{3.65}$$

$$\tau_{2_k}(x,q) = \tau_{2_{k-1}}(x,q) + D_q\sigma_{2_{k-1}}(x,q), \tag{3.66}$$

$$\lambda_{2_k}(q) = q\left[\lambda_{2_{k-1}}(q) + D_q\tau_{2_{k-1}}(x,q)\right] \tag{3.67}$$

with $\sigma_{2_0}(x,q) = \sigma_2(x,q)$, $\tau_{2_0}(x,q) = \tau(x,q)$ and $\lambda_{2_0}(q) = \lambda(q)$. □

We can write the explicit form of those coefficients

$$\tau_{2_k}(x,q) = \tau(x,q) + \frac{\sigma_2(q^kx,q) - \sigma_2(x,q)}{(q-1)x}, \tag{3.68}$$

$$\lambda_{2_k}(q) = q^k\left[\lambda(q) + [k]_{q^{-1}}\tau'(0,q) + \frac{1}{2}[k-1]_q[k]_{q^{-1}}\sigma_2''(0,q)\right], \tag{3.69}$$

in which q-number $[k]_q$ and q^{-1}-number $[k]_{q^{-1}}$ are defined by (2.7) and (2.9). Another representation of $\lambda_{2_k}(q)$ follows from (3.20)

$$\lambda_{2_k}(q) = q^k\left[\lambda(q) + \frac{[k]_{q^{-1}}}{2(1-q)}\left(q\sigma_1''(0,q) - q^{k-1}\sigma_2''(0,q)\right)\right]. \tag{3.70}$$

Remark 3.20 *Notice that the q-EHT for $u_k(x,q)$ (3.64) and for $v_k(x,q)$ (3.45) are not equivalent since while $v_k(x,q) = D^{(k)}_{q^{-1}}y(x)$, $u_k(x,q) = D^{(k)}_q y(x)$.*

We have polynomial solution of the q-EHT of the 2nd kind for specific value of $\lambda(q)$ as in the q-EHT of the 1st kind which can be stated in the following theorem.

Theorem 3.21 *The q-EHT of the 2nd kind has polynomial solutions, say $y(x) = P_{2_n}(x,q)$, of degree n if and only if*

$$\lambda(q) := \lambda_n(q) = -[n]_{q^{-1}}\left[\tau'(0,q) + \frac{1}{2}[n-1]_q\sigma_2''(0,q)\right], \quad n = 0, 1, ... \tag{3.71}$$

Proof. Proof is done analogously as in Theorem 3.10. □

Remark 3.22 *Notice that $\lambda_{2_{nk}}(q)$ defined by (3.69) can be rewritten as*

$$\lambda_{2_{nk}}(q) = -[n-k]_{q^{-1}}\left(\tau'(0,q) + [n+k-1]_q \frac{1}{2}\sigma_2''(0,q)\right). \tag{3.72}$$

by using (3.71).

Remark 3.23 *It is possible to write the q-EHT of the 2nd kind in self-adjoint form*

$$D_{q^{-1}}\left[\sigma_{2_k}(x,q)\rho_{2_k}(x,q)D_q\left(D_q^{(k)}P_n(x;q)\right)\right] + q\lambda_{2_k}(q)\rho_{2_k}(x,q)\left(D_q^{(k)}P_n(x;q)\right) = 0 \tag{3.73}$$

where $P_0^{(n)}(x;q) = \frac{1}{[1]_q^{(n)}}D_q^{(n)}P_n(x;q)$, $[1]_q^{(n)} = [1]_q[2]_q...[n]_q$. Here, the q-weight function $\rho_{2_n}(x,q)$ is the solution of the q-Pearson equation

$$D_{q^{-1}}\left[\sigma_{2_n}(x,q)\rho_{2_n}(x,q)\right] = q\tau_{2_n}(x,q)\rho_{2_n}(x,q). \tag{3.74}$$

Proposition 3.24 *Let $\rho(x,q)$ be a solution of (3.26) and $\rho_{2_n}(x,q)$ the solution of (3.74). Then,*

$$\rho_{2_n}(x,q) = \sigma_{2_{n-1}}(x,q)\rho_{2_{n-1}}(x,q) = ... = \prod_{k=0}^{n-1}\sigma_2(q^k x,q)\rho(x,q). \tag{3.75}$$

3.4.1 The Rodrigues Formula for Polynomial Solutions of the q-EHT of the 2nd kind

Theorem 3.25 *Let $\rho(x,q)$ be the q-weight function defined by the q^{-1}-Pearson equation (3.26) and $\rho_{2_n}(x,q)$ by (3.75). Then,*

$$P_{2_n}(x,q) = q^{-n}B_{2_n}(q)\frac{D_{q^{-1}}^n\left[\rho_{2_n}(x,q)\right]}{\rho(x,q)} \tag{3.76}$$

where

$$B_{2_n}(q) = (-1)^n\frac{[1]_q[2]_q...[n]_q}{\lambda_{2_{nn-1}}(q)\lambda_{2_{nn-2}}(q)...\lambda_{2_{n0}}(q)}P_{2_0}^{(n)}(q) \tag{3.77}$$

stands for normalization constant with $P_{2_0}^{(n)}(q) = \frac{1}{[1]_q^n}D_q^nP_{2_n}(x,q) = \frac{1}{[1]_q^n}u_{nn}$, $[1]_q^{(n)}$=$[1]_q...[n]_q$.

Remark 3.26 *Notice that the normalization constant defined by (3.77) can be written as the following form*

$$B_{2_n}(q) = \frac{q^{\binom{n}{2}}}{\prod_{k=0}^{n-1}\left[\tau'(0,q) + [n+k-1]_q\frac{1}{2}\sigma_2''(0,q)\right]}P_{2_0}^{(n)}(q) \tag{3.78}$$

by means of the expression for $\lambda_{2_{nk}}$ defined by (3.72) and the relation between $[k]_q$ and $[k]_{q^{-1}}$ denoted with (2.10).

Remark 3.27 *Notice that, another representation of $B_{2_n}(q)$ for monic polynomials ($P_{2_0}^{(n)}(q) = 1$)*

$$B_{2_n}(q) = \frac{q^{\binom{n}{2}}(1-q)^n}{q^n[\frac{1}{2}\sigma_1''(0,q)]^n \left(q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q);q\right)_n} \tag{3.79}$$

and by using the property (2.17),

$$B_{2_n}(q) = \frac{(-1)^n q^{-2\binom{n}{2}}(1-q)^n}{[\frac{1}{2}\sigma_2''(0,q)]^n \left(q^{1-n}a_1^{-1}(q)b_1^{-1}(q)a_2(q)b_2(q);q^{-1}\right)_n} \tag{3.80}$$

where $a_1(q), b_1(q)$ are zeros of $\sigma_1(x,q)$ and $a_2(q), b_2(q)$ of $\sigma_2(x,q)$.

Alternative representation for the monic q-polynomials of the 2nd kind can be introduced via the following finite sum

$$D_{q^{-1}}^n f(x) = \frac{1}{(1-q^{-1})^n x^n} \sum_{k=0}^{n} (-1)^k q^{-k(k+1)/2+nk} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} f(q^{-k}x) \tag{3.81}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}}$ is the q^{-1}-binomial defined by (2.14) and $(a,q^{-1})_n$ is the q^{-1}-shifted factorial defined by (2.17). By applying it to the Rodrigues formula defined in (3.76), it is presented in terms of coefficients $\sigma_1(x,q)$ and $\sigma_2(x,q)$ in the following proposition.

Proposition 3.28 *Let $D_{q^{-1}}^n f(x)$ be given with (3.81). Then,*

$$P_{2_n}(x,q) = \frac{q^{-n}B_{2_n}(q)\prod_{i=0}^{n-1}\sigma_2(q^i x,q)}{(1-q^{-1})^n x^n} \sum_{k=0}^{n} (-1)^k q^{-k(k+1)/2+nk+k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \frac{\prod_{i=0}^{k-1}\sigma_1(q^{-i}x,q)}{\prod_{i=0}^{k-1}\sigma_2(q^{n-1-i}x,q)} \tag{3.82}$$

where $B_{2_n}(q)$ is the normalization constant defined by (3.78).

Remark 3.29 *We remark that by using relations between the polynomial coefficients (3.20) and (3.22), and also the fact that the q-EHT of the 1st and 2nd kinds are equivalent to (3.12) and (3.13), the q-polynomials of the 1st kind $P_{1_n}(x,q)$ and the 2nd kind $P_{2_n}(x,q)$ are equivalent.*

Observe from the representation formula identified by sum in (3.63) that it depends on the polynomial coefficients $\sigma_1(x,q)$ and $\sigma_2(x,q)$. On the other hand, this formula allows us to identify the hypergeometric representation of polynomial solutions of the q-EHT. That's why in the current study, hypergeometric representations of polynomial solutions of the q-EHT of the 1st kind $P_{1_n}(x,q)$ are discussed according as all possible degrees of $\sigma_1(x,q)$ and

$\sigma_2(x,q)$ identified by the Proposition 3.4. We remark that hypergeometric representations of q-polynomials of the 2nd kind $P_{2_n}(x,q)$ can also be found by use of the formula (3.82) which have the equivalent form with the 1st kind polynomials. Then, without loss of generality we assume that $P_{1_n}(x;q) = P_{2_n}(x;q) := P_n(x;q)$.

3.5 Hypergeometric Representation of the q-Polynomials

Hypergeometric representation of all kind of monic polynomials identified by Table 4.1 are introduced in the following by studying on the representation formula (3.63) together with all possible degrees of $\sigma_1(x,q)$ and $\sigma_2(x,q)$ given in Proposition 3.4.

1. Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$, then the representation formula given in (3.63) becomes

$$P_n(x;q) = \frac{x^n}{(q^{n-1}\frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)};q)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-n}, & x/a_2(q), & x/b_2(q) \\ q^{1-n}x/a_1(q), & q^{1-n}x/b_1(q) & \end{matrix}\middle|\, q;q\right) \tag{3.83}$$

by computing the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ with the help of (2.13) defined in Definition 2.4 and the normalization constant $B_{1_n}(q)$ of (3.60)

$$B_{1_n}(q) = \frac{q^{2\binom{n}{2}}(1-q^{-1})^n}{(-1)^n[\frac{1}{2}\sigma_1''(0,q)]^n(q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)};q)_n}. \tag{3.84}$$

Remark 3.30 *Note that*

$${}_r\phi_s\left(\begin{matrix} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{matrix}\middle|\, q;z\right) = a_n\,{}_r\varphi_s\left(\begin{matrix} a_1, & ..., & a_r \\ b_1, & ..., & b_s \end{matrix}\middle|\, q;z\right) \tag{3.85}$$

where a_n is leading coefficient of the polynomial and ${}_r\phi_s$ is q-hypergeometric series defined by (2.38).

Observe from the representation of the q-polynomials defined by the formula (3.83), it is not clear to see that P_n is polynomials of degree n in x. That's why, we perform the transformation formulas defined by (2.44) and (2.45), successively to (3.83) by using (3.11), i.e., $\sigma_2(0,q) =$

$q\sigma_1(0,q) \Leftrightarrow \frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$ which lead to

$$P_n(x;q) = \frac{b_2^n(q)(a_1(q)/b_2(q), b_1(q)/b_2(q);q)_n}{(q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q);q)_n} \times {}_3\varphi_2\left(\begin{array}{c} q^{-n}, \quad q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q), \quad x/b_2(q) \\ a_1(q)/b_2(q), \quad b_1(q)/b_2(q) \end{array}\Big| q;q\right) \tag{3.86}$$

and equivalently

$$P_n(x;q) = \frac{a_2^n(q)(a_1(q)/a_2(q), b_1(q)/a_2(q);q)_n}{(q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q);q)_n} \times {}_3\varphi_2\left(\begin{array}{c} q^{-n}, \quad q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q), \quad x/a_2(q) \\ a_1(q)/a_2(q), \quad b_1(q)/a_2(q) \end{array}\Big| q;q\right). \tag{3.87}$$

Note that since $\sigma_1(x,q)$ and $\sigma_2(x,q)$ are invariant with respect to the transformation $a_1(q) \leftrightarrow b_1(q)$ and $a_2(q) \leftrightarrow b_2(q)$, (3.87) is also obtained from (3.86) by using this kind of transformation. It is apparent from hypergeometric representations defined by (3.86) and (3.87) that the q-classical $\emptyset$-Jacobi/Jacobi (see Table 4.1) monic polynomial P_n is a polynomial of degree exactly n in x.

An alternative equivalent form for the q-hypergeometric series is derived by applying the transformation formula (2.44) to (3.87)

$$P_n(x;q) = \frac{q^{\binom{n}{2}}[-b_1(q)]^n(a_1(q)/a_2(q), a_1(q)/b_2(q);q)_n}{(q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q);q)_n} \times {}_3\varphi_2\left(\begin{array}{c} q^{-n}, \quad q^{n-1}a_1(q)b_1(q)a_2^{-1}(q)b_2^{-1}(q), \quad a_1(q)/x \\ a_1(q)/a_2(q), \quad a_1(q)/b_2(q) \end{array}\Big| q; \frac{qx}{b_1(q)}\right). \tag{3.88}$$

2. Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$ and $\sigma_2(x,q) = \sigma_2'(0,q)[x-a_2(q)]$, then the hypergeometric representation of the corresponding q-classical $\emptyset$-Jacobi/Laguerre (see Table 4.1) monic polynomials follows from (3.63) by substituting the polynomial coefficients defined above and the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ in (2.13)

$$P_n(x;q) = \frac{q^n B_{1_n}(q)[\frac{1}{2}\sigma_1''(0,q)]^n q^{n(1-n)}x^n}{(1-q)^n}(a_1(q)/x, b_1(q)/x;q)_n \times {}_3\varphi_2\left(\begin{array}{c} q^{-n}, \quad x/a_2(q), \quad 0 \\ q^{1-n}x/a_1(q), \quad q^{1-n}x/b_1(q) \end{array}\Big| q;q\right) \tag{3.89}$$

where $B_{1_n}(q)$ denotes the normalization constant derived from (3.60)

$$B_{1_n}(q) = \frac{(1-q^{-1})^n}{(-1)^n q^{n(1-n)}[\frac{1}{2}\sigma_1''(0,q)]^n}. \tag{3.90}$$

Since it is not easy to see that the hypergeometric representation of the $\emptyset$-Jacobi/Laguerre type monic polynomials identified by (3.89) is a polynomial of degree n in x, we first use the transformation formula (2.44) as $a \to 0$, secondly, we apply (2.46) to the resulting formula and last, we use (2.47) which allow us to construct the formula

$$P_n(x;q) = a_2^n(q)(a_1(q)/a_2(q), b_1(q)/a_2(q); q)_n {}_3\varphi_2\left(\begin{matrix} q^{-n}, & x/a_2(q), & 0 \\ a_1(q)/a_2(q), & b_1(q)/a_2(q) & \end{matrix}\middle| q; q\right). \tag{3.91}$$

We remark that hypergeometric representation of the $\emptyset$-Jacobi/Laguerre type monic polynomials identified by (3.91) can also be obtained by taking $b_2(q) \to \infty$ in the case of the $\emptyset$-Jacobi/Jacobi type monic polynomials defined by (3.87) with the help of the relation that $\sigma_2(0,q) = q\sigma_1(0,q)$
$\Leftrightarrow \frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$ (for the further details see [6]).

3. Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$ and $\sigma_2(x,q) = \sigma_2(0,q)$, then the hypergeometric representation of the q-classical $\emptyset$-Jacobi/Hermite (see Table 4.1) monic polynomials is derived from (3.63) as

$$P_n(x;q) = x^n(a_1(q)/x, b_1(q)/x; q)_n {}_3\varphi_2\left(\begin{matrix} q^{-n}, & 0, & 0 \\ q^{1-n}x/a_1(q), & q^{1-n}x/b_1(q) & \end{matrix}\middle| q; q\right) \tag{3.92}$$

with the help of the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by (2.13) and the normalization constant $B_{1_n}(q)$ by (3.60)

$$B_{1_n}(q) = \frac{q^{2\binom{n}{2}}(1-q^{-1})^n}{(-1)^n[\frac{1}{2}\sigma_1''(0,q)]^n}. \tag{3.93}$$

In addition, one can also get hypergeometric representation equivalent to (3.92) as the form

$$P_n(x;q) = q^{\binom{n}{2}}[-b_1(q)]^n {}_2\varphi_1\left(\begin{matrix} q^{-n}, & a_1(q)/x \\ & 0 \end{matrix}\middle| q; \frac{qx}{b_1(q)}\right) \tag{3.94}$$

by use of the transformation formula (2.44) with $a \to 0$, $b \to 0$ together with the idea (2.42) and then (2.50). Notice that (3.94) is in more convenient form to figure out that the q-classical $\emptyset$-Jacobi/Hermite polynomials are of degree n in x.

We note that hypergeometric representation of the $\emptyset$-Jacobi/Hermite type monic polynomials defined by (3.94) can also be derived by assuming $a_2(q)$, $b_2(q) \to \infty$ in the case of the $\emptyset$-Jacobi/Jacobi type monic polynomials identified by (3.88) by use of the expression $\sigma_2(0,q)$ $= q\sigma_1(0,q) \Leftrightarrow \frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$ (for the further details see [6]).

4. Let $\sigma_1(x,q) = \sigma_1'(0,q)[x-a_1(q)]$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$, then the hypergeometric representation of the corresponding q-classical $\emptyset$-Laguerre/Jacobi (see Table 4.1) monic polynomials is obtained by (3.63) by use of the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ defined by (2.13)

$$\begin{aligned} P_n(x;q) &= \frac{q^{n-\binom{n}{2}}(-1)^n B_{1_n}(q)[\sigma_1'(0,q)]^n}{(1-q)^n}(a_1(q)/x;q)_n \\ &\quad \times {}_3\varphi_2\left(\begin{matrix} q^{-n}, & x/a_2(q), & x/b_2(q) \\ & q^{1-n}x/a_1(q), & 0 \end{matrix} \middle| q; q \right) \end{aligned} \tag{3.95}$$

where the normalization constant $B_{1_n}(q)$ follows from (3.60) as

$$B_{1_n}(q) = \frac{q^{-\binom{n}{2}}(q-1)^n}{[\frac{1}{2}\sigma_2''(0,q)]^n}. \tag{3.96}$$

In order to make clear that P_n defined by (3.95) is a polynomial of degree exactly n in x, we perform the transformation formula (2.44) with $d \to 0$ leading to

$$\begin{aligned} P_n(x;q) &= q^{n-2\binom{n}{2}}[-\frac{\sigma_1'(0,q)}{\frac{1}{2}\sigma_2''(0,q)}]^n(a_1(q)/a_2(q);q)_n \\ &\quad \times {}_2\varphi_1\left(\begin{matrix} q^{-n}, & x/a_2(q) \\ & a_1(q)/a_2(q) \end{matrix} \middle| q; \frac{a_1(q)q^n}{b_2(q)} \right). \end{aligned} \tag{3.97}$$

Note that hypergeometric representation of the $\emptyset$-Laguerre/Jacobi type monic polynomials identified by (3.97) can also be obtained by setting $b_1(q) \to \infty$ in the $\emptyset$-Jacobi/Jacobi type monic polynomials (3.87) together with $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow \frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$ (for the further details see [6]).

5. Let $\sigma_1(x,q) = \sigma_1(0,q)$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$, then by use of the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ defined by (2.13), in the representation formula (3.63) for q-polynomials of the 1st kind, the hypergeometric representation of the corresponding q-classical $\emptyset$-Hermite/Jacobi (see Table 4.1) monic polynomials follows

$$P_n(x;q) = \frac{q^n B_{1_n}(q)[\sigma_1(0,q)]^n}{(1-q)^n x^n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, & x/a_2(q), & x/b_2(q) \\ & 0, & 0 \end{matrix} \middle| q; q \right) \tag{3.98}$$

where the normalization constant $B_{1_n}(q)$ follows from (3.60) as

$$B_{1_n}(q) = \frac{q^{-\binom{n}{2}}(q-1)^n}{[\frac{1}{2}\sigma_2''(0,q)]^n}. \tag{3.99}$$

Another hypergeometric representation equivalent to (3.98) is derived as

$$P_n(x;q) = [-b_2(q)]^n q^{-\binom{n}{2}} {}_2\varphi_0\left(\begin{matrix} q^{-n}, & x/a_2(q) \\ & - \end{matrix} \middle| q; \frac{q^n a_2(q)}{b_2(q)} \right) \tag{3.100}$$

by applying the transformation formula (2.51) with $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow \frac{1}{2}\sigma_2''(0,q)a_2(q)$ $b_2(q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$.

We remark that hypergeometric representation of the $\emptyset$-Hermite/Jacobi type monic polynomials identified by (3.100) can also be derived by letting $a_1(q)$, $b_1(q) \to \infty$ in the $\emptyset$-Jacobi/Jacobi type monic polynomials given by (3.87) together with $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow$ $\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$ (for the details see [6]).

6. Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)]$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x[x - a_2(q)]$, then the hypergeometric representation of the corresponding q-classical 0-Jacobi/Jacobi (see Table 4.1) monic polynomials follows

$$
\begin{aligned}
P_n(x;q) &= \frac{q^n B_{1_n}(q)[\frac{1}{2}\sigma_1''(0,q)]^n x^n q^{-2\binom{n}{2}}}{(1-q)^n}(a_1(q)/x;q)_n \\
&\quad \times {}_2\varphi_1\left(\begin{matrix} q^{-n}, & x/a_2(q) \\ & q^{1-n}x/a_1(q) \end{matrix}\,\middle|\, q; \frac{\frac{1}{2}\sigma_2''(0,q)a_2(q)}{q^{1-n}\frac{1}{2}\sigma_1''(0,q)a_1(q)}\right)
\end{aligned} \tag{3.101}
$$

by use of the representation formula (3.63). Here, the normalization constant $B_{1_n}(q)$ can be obtained from (3.60) as follows:

$$
B_{1_n}(q) = \frac{q^{2\binom{n}{2}}(1-q^{-1})^n}{(-1)^n[\frac{1}{2}\sigma_1''(0,q)]^n(q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)};q)_n}. \tag{3.102}
$$

In order to represent a nice hypergeometric representation for (3.101) demonstrating a polynomial of degree n in x, we first carry out the transformation formula (2.48) to (3.101) and then we apply (2.44) together with $d \to 0$ to the resulting formula which bring about

$$
\begin{aligned}
P_n(x;q) &= \frac{[-a_1(q)]^n q^{\binom{n}{2}}(\frac{\frac{1}{2}\sigma_2''(0,q)a_2(q)}{q\frac{1}{2}\sigma_1''(0,q)a_1(q)};q)_n}{(q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)};q)_n} \\
&\quad \times {}_2\varphi_1\left(\begin{matrix} q^{-n}, & q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \\ & \frac{\frac{1}{2}\sigma_2''(0,q)a_2(q)}{q\frac{1}{2}\sigma_1''(0,q)a_1(q)} \end{matrix}\,\middle|\, q; \frac{qx}{a_1(q)}\right).
\end{aligned} \tag{3.103}
$$

We note that hypergeometric representation of the 0-Jacobi/Jacobi polynomials identified by (3.103) can also be obtained by letting $b_1(q),\ b_2(q) \to 0$ assuming that $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow$ $b_1(q)/b_2(q) = \frac{1}{2}\sigma_2''(0,q)a_2(q)/q\frac{1}{2}\sigma_1''(0,q)a_1(q)$ in the $\emptyset$-Jacobi/Jacobi type monic polynomials defined by (3.86) (for the further details see [6]).

7. Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2$, then inserting these values with the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ defined by (2.13) into the representation formula identified by (3.63) generates the following hypergeometric representation of the corresponding q-classical 0-Jacobi/Bessel (see Table 4.1) monic polynomials

$$\begin{aligned} P_n(x;q) &= \frac{q^n B_{1_n}(q)[\frac{1}{2}\sigma_1''(0,q)]^n x^n q^{-2\binom{n}{2}}}{(1-q)^n}(a_1(q)/x;q)_n \\ &\times {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{1-n}x/a_1(q) \end{matrix}\middle| q; \frac{x}{q^{1-n}a_1(q)}\right) \end{aligned} \tag{3.104}$$

where the normalization constant $B_{1_n}(q)$ follows from (3.60)

$$B_{1_n}(q) = \frac{q^{2\binom{n}{2}}(1-q^{-1})^n}{(-1)^n[\frac{1}{2}\sigma_1''(0,q)]^n(q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)};q)_n}. \tag{3.105}$$

In order to get a nice representation formula denoting that P_n is exactly of degree n in x, we apply the transformation formulas (2.50) and then (2.48) with $c \to 0$ together with the limit relation used in (2.42) successively to (3.104) which yield

$$P_n(x;q) = \frac{[-a_1(q)]^n q^{\binom{n}{2}}}{(q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)};q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, & q^{n-2}\frac{\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \\ & 0 \end{matrix}\middle| q; \frac{qx}{a_1(q)}\right). \tag{3.106}$$

It is obvious from the definition of ${}_r\varphi_s$ in (3.85) and (2.38) that P_n in (3.106) represents a polynomial of degree n in x.

On the other hand, observe from the hypergeometric representation of the $\emptyset$-Jacobi/Jacobi polynomials given by (3.88) with the transformation $a_1(q) \leftrightarrow b_1(q)$ and $a_2(q) \leftrightarrow b_2(q)$, limit relation $b_1(q)$, $a_2(q)$, $b_2(q) \to 0$ in company with the property $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow b_1(q)/a_2(q)b_2(q) = \frac{1}{2}\sigma_2''(0,q)/q\frac{1}{2}\sigma_1''(0,q)a_1(q)$ also leads to the 0-Jacobi/Bessel polynomials identified by (3.106).

8. Let $\sigma_1(x,q) = \sigma_1'(0,q)x$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x[x-a_2(q)]$, then one can get the following hypergeometric representation of the corresponding q-classical 0-Laguerre/Jacobi (see Table 4.1) monic polynomials starting with the representation formula identified by (3.63)

$$P_n(x;q) = \frac{q^n B_{1_n}(q)[\sigma_1'(0,q)]^n q^{-\binom{n}{2}}}{(1-q)^n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, & x/a_2(q) \\ & 0 \end{matrix}\middle| q; -\frac{\frac{1}{2}\sigma_2''(0,q)a_2(q)}{q^{1-n}\sigma_1'(0,q)}\right) \tag{3.107}$$

where the normalization constant $B_{1_n}(q)$ can be computed by using (3.60)

$$B_{1_n}(q) = \frac{q^{-\binom{n}{2}}(q-1)^n}{[\frac{1}{2}\sigma_2''(0,q)]^n}. \tag{3.108}$$

Observe that inserting $B_{1_n}(q)$ into (3.107) identifies

$$P_n(x;q) = q^{n-2\binom{n}{2}}\left(-\frac{\sigma_1'(0,q)}{\frac{1}{2}\sigma_2''(0,q)}\right)^n {}_2\varphi_1\left(\begin{array}{cc} q^{-n}, & x/a_2(q) \\ & 0 \end{array}\Big| q; -\frac{\frac{1}{2}\sigma_2''(0,q)a_2(q)}{q^{1-n}\sigma_1'(0,q)}\right) \tag{3.109}$$

as a polynomial of degree n in x.

Note that another method to get the hypergeometric representation of the 0-Laguerre/Jacobi polynomials identified by (3.109) is to let $b_1(q), b_2(q) \to 0$ and $a_1(q) \to \infty$ together with $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow b_1(q)/b_2(q) = -\frac{1}{2}\sigma_2''(0,q)a_2(q)/q\sigma_1'(0,q)$ in the $\emptyset$-Jacobi/Jacobi type monic polynomials defined by (3.87) (for the further details see [6]).

9. Letting $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$ and $\sigma_2(x,q) = \sigma_2'(0,q)x$ in the representation formula identified by (3.63) provide the following hypergeometric representation of the corresponding q-classical 0-Jacobi/Laguerre (see Table 4.1) monic polynomials

$$\begin{aligned} P_n(x;q) &= \frac{q^n B_{1_n}(q)[-\frac{1}{2}\sigma_1''(0,q)]^n q^{-2\binom{n}{2}} x^n}{(1-q)^n}(a_1(q)/x;q)_n \\ &\times {}_2\varphi_1\left(\begin{array}{cc} q^{-n}, & 0 \\ q^{1-n}x/a_1(q) & \end{array}\Big| q; -\frac{\sigma_2'(0,q)}{q^{1-n}\frac{1}{2}\sigma_1''(0,q)a_1(q)}\right) \end{aligned} \tag{3.110}$$

where $B_{1_n}(q)$ is calculated by (3.60) as

$$B_{1_n}(q) = \frac{q^{2\binom{n}{2}}(1-q^{-1})^n}{(-1)^n[\frac{1}{2}\sigma_1''(0,q)]^n}. \tag{3.111}$$

In order to find a better representation to make clear that P_n is a polynomial of degree exactly n in x, we first apply the transformation formula (2.49) to (3.110) letting $b \to 0$, afterwards we perform (2.52) to the resulting function by substituting $B_{1_n}(q)$. As a result, we get

$$P_n(x;q) = [a_1(q)]^n q^{\binom{n}{2}}\left(-\frac{\sigma_2'(0,q)}{q\frac{1}{2}\sigma_1''(0,q)a_1(q)};q\right)_n {}_2\varphi_1\left(\begin{array}{cc} q^{-n}, & 0 \\ -\frac{\sigma_2'(0,q)}{q\frac{1}{2}\sigma_1''(0,q)a_1(q)} & \end{array}\Big| q; \frac{qx}{a_1(q)}\right). \tag{3.112}$$

Furthermore, the hypergeometric representation of the 0-Jacobi/Laguerre polynomials defined by (3.112) can also be derived by use of the limit relation $b_1(q), b_2(q) \to 0$, $a_2(q) \to \infty$

together with the property $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow b_1(q)/b_2(q) = \sigma_2'(0,q)/q\frac{1}{2}\sigma_1''(0,q)a_1(q)$ in the $\emptyset$-Jacobi/Jacobi polynomials given by (3.88) with the transformation $a_1(q) \leftrightarrow b_1(q)$ and $a_2(q) \leftrightarrow b_2(q)$ (for the further details see [6]).

10. Setting $\sigma_1(x,q) = \sigma_1'(0,q)x$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2$ in the representation formula identified by (3.63) brings about the following hypergeometric representation of the corresponding q-classical 0-Laguerre/Bessel (see Table 4.1) monic polynomials

$$P_n(x;q) = q^{n-2\binom{n}{2}} \left[-\frac{\sigma_1'(0,q)}{\frac{1}{2}\sigma_2''(0,q)} \right]^n {}_1\varphi_1 \left(\begin{array}{c} q^{-n} \\ 0 \end{array} \middle| q; -\frac{\frac{1}{2}\sigma_2''(0,q)x}{\sigma_1'(0,q)q^{1-n}} \right) \tag{3.113}$$

with the help of $B_{1_n}(q)$ computed from (3.60)

$$B_{1_n}(q) = \frac{q^{-\binom{n}{2}}(q-1)^n}{[\frac{1}{2}\sigma_2''(0,q)]^n}. \tag{3.114}$$

Observe from the representation formula identified by (3.113) that 0-Laguerre/Bessel type q-polynomials P_n is a polynomial of degree n in x.

On the other hand, hypergeometric representation of the 0-Laguerre/Bessel polynomials given with (3.113) can also be constructed with the help of the limit relation $a_1(q), b_2(q), a_2(q) \to 0$, $b_1(q) \to \infty$ together with the property $\sigma_2(0,q) = q\sigma_1(0,q) \Leftrightarrow \frac{a_1(q)}{a_2(q)b_2(q)} = \frac{\frac{1}{2}\sigma_2''(0,q)}{q\sigma_1'(0,q)}$ in the $\emptyset$-Jacobi/Jacobi polynomials given by (3.88) (for the further details see [6]).

3.6 Orthogonality Property of the q-Polynomials of Hypergeometric Type

In this section, we perform the orthogonality conditions of the polynomial solutions of the q-EHT by means of standard method in the theory of orthogonal polynomials [22, 46, 48]. We begin with introducing the orthogonality property as in the following theorem.

Theorem 3.31 *Let ρ be a function satisfying the q-Pearson equation (3.24) and such that the boundary condition*

$$\sigma_1(x,q)\rho(x,q)x^k\Big|_{x=a,b} = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k\Big|_{x=a,b} = 0 \tag{3.115}$$

is satisfied. Then, the polynomial solutions of the q-EHT are orthogonal with respect to $\rho(x, q)$ *(see (2.25)), i.e.,*

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \qquad (3.116)$$

Analogously, if

$$\sigma_2(x,q)\rho(x,q)x^k\Big|_{x=a,b} = \sigma_1(qx,q)\rho(qx,q)x^k\Big|_{x=a,b} = 0 \qquad (3.117)$$

holds, the q-polynomials satisfy the relation

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = s_n^2(q)\delta_{mn}, \qquad (3.118)$$

where $d_n^2(q)$ *and* $s_n^2(q)$ *denote the squared norm of the polynomials* P_n*,* δ_{mn} *is the Kronecker delta.*

Remark 3.32 *Notice that*

$$\sigma_1(x,q)\rho(x,q) = q^{-1}\sigma_2(q^{-1}x,q)\rho(q^{-1}x,q) \Leftrightarrow \sigma_2(x,q)\rho(x,q) = \sigma_1(qx,q)\rho(qx,q) \qquad (3.119)$$

by using (3.27) and (3.28) with $\rho_1(x,q) = \rho_2(x,q) = \rho(x,q)$ *which gives the equivalences of the boundary conditions.*

Proof. Consider the q-EHT of the 1st kind in self-adjoint form for $P_n(x;q)$ and $P_m(x,q)$, respectively,

$$D_q\left[\rho(x,q)\sigma_1(x,q)D_{q^{-1}}P_n(x;q)\right] + q^{-1}\lambda_n(q)\rho(x,q)P_n(x;q) = 0,$$
$$D_q\left[\rho(x,q)\sigma_1(x,q)D_{q^{-1}}P_m(x,q)\right] + q^{-1}\lambda_m(q)\rho(x,q)P_m(x,q) = 0$$

where the q-weight function satisfies the q-Pearson equation

$$D_q\left[\sigma_1(x,q)\rho(x,q)\right] = q^{-1}\tau(x,q)\rho(x,q) \Leftrightarrow D_{q^{-1}}\left[\sigma_2(x,q)\rho(x,q)\right] = q\tau(x,q)\rho(x,q). \qquad (3.120)$$

Multiplying the first equation with $P_m(x,q)$ and the second with $P_n(x;q)$ and subtracting the second from the first, and applying the q-integral over (a,b) to the resulting equation, we get

$$q^{-1}\left[\lambda_n(q) - \lambda_m(q)\right]\int_a^b P_n(x;q)P_m(x;q)\rho(x,q)d_qx$$
$$+\int_a^b P_m(x;q)D_q\left[\rho(x,q)\sigma_1(x,q)D_{q^{-1}}P_n(x;q)\right]d_qx$$
$$-\int_a^b P_n(x;q)D_q\left[\rho(x,q)\sigma_1(x,q)D_{q^{-1}}P_m(x;q)\right]d_qx = 0.$$

By applying the q-integration by parts and fundamental theorem of q-calculus defined by Proposition 2.11 and Proposition 2.12, respectively, we have

$$
\begin{aligned}
& q^{-1}\left[\lambda_n(q)-\lambda_m(q)\right]\int_a^b P_n(x;q)P_m(x;q)\rho(x,q)d_qx \\
& +\rho(x,q)\sigma_1(x,q)W_q\left[P_m,P_n\right]\Big|_{x=a,b} \\
& +\int_a^b \rho(qx,q)\sigma_1(qx,q)D_qP_m(x;q)\left[D_{q^{-1}}P_n(x;q)\big|_{x\to qx}\right]d_qx \\
& -\int_a^b \rho(qx,q)\sigma_1(qx,q)D_qP_n(x;q)\left[D_{q^{-1}}P_m(x;q)\big|_{x\to qx}\right]d_qx=0
\end{aligned}
$$

where $W_q\left[P_m,P_n\right]=\left[P_m(x;q)D_{q^{-1}}P_n(x;q)-P_n(x;q)D_{q^{-1}}P_m(x,q)\right]$ is the q-Wronskian [6]. Since $D_{q^{-1}}P_m(x;q)\big|_{x\to qx}=D_qP_m(x;q)$ and $D_{q^{-1}}P_n(x;q)\big|_{x\to qx}=D_qP_n(x;q)$ the third and fourth terms are vanished and as a result of the boundary conditions, second term

$$\rho(x,q)\sigma_1(x,q)W_q\left[P_m,P_n\right]\Big|_{x=a,b}=0$$

since the Wronskian $W_q\left[P_m,P_n\right]=\left[P_m(x;q)D_{q^{-1}}P_n(x;q)-P_n(x;q)D_{q^{-1}}P_m(x;q)\right]$ is a polynomial of degree $n+m-1$. Then, we arrive at

$$\int_a^b P_n(x;q)P_m(x;q)\rho(x,q)d_qx=0,$$

for all $m\neq n$, if

$$\lambda_n(q)-\lambda_m(q)=-[n-m]_q\left(\tau'(0,q)+[n+m-1]_{q^{-1}}\frac{1}{2}\sigma_1''(0,q)\right)\neq 0$$

$$\Leftrightarrow\left(\tau'(0,q)+[n+m-1]_{q^{-1}}\frac{1}{2}\sigma_1''(0,q)\right)\neq 0.$$

Nevertheless, when $m=n\Rightarrow\lambda_n(q)=\lambda_m(q)$,

$$\int_a^b P_n(x;q)P_n(x;q)\rho(x,q)d_qx$$

remains arbitrary. As a result, we get

$$\int_a^b P_m(x;q)P_n(x;q)\rho(x,q)d_qx=d_n^2(q)\delta_{mn} \qquad (3.121)$$

where $d_n(q)$ is the norm and δ_{mn} Kronecker delta. The proof of the relation in (3.118) can be accomplished analogously. □

Definition 3.33 *The q-polynomial solutions of the q-EHT of the 1st and 2nd kinds are classical provided that* (a,b) *is an interval on the real axis and the* $\rho(x,q)>0$ *satisfies the q-Pearson equation (3.120) and boundary conditions (3.119) or (3.117).*

In order to calculate the norm $d_n(q)$ we look for the orthogonality property of the q^{-1} and q-derivatives of order k ($k = 1, 2, ...$) of the polynomial solutions of the q-EHT of the 1st and 2nd kinds, i.e., $D^k_{q^{-1}} P_{1_{n+k}}$ and $D^k_q P_{2_{n+k}}$.

3.6.1 Orthogonality Property of $D^{(k)}_{q^{-1}} P_{1_{n+k}}(x, q)$

In this section, we generalize the orthogonality of the q-polynomials of the 1st kind to $v_{kn}(x, q) = D^{(k)}_{q^{-1}} P_{1_{n+k}}(x, q)$, $k = 1, 2, ...$ via the standard method [22, 46, 48].

Proposition 3.34 *Let ρ_{1_k} be a function satisfying the q-Pearson equation identified by (3.55) with $n = k$. Then,*

- *If $0 < a < b$, then the polynomials v_{kn} has the following orthogonality relation*

$$\int_a^{q^k b} v_{kn}(x, q) v_{km}(x, q) \rho_{1_k}(x, q) d_q x = d^2_{1_{kn}}(q) \delta_{mn} \tag{3.122}$$

  provided that

$$\sigma_{1_k}(x, q) \rho_{1_k}(x, q) x^k \Big|_{x=a, q^k b} = 0. \tag{3.123}$$

- *If $a < b < 0$, then v_{kn} are orthogonal in the following sense*

$$\int_{q^k a}^{b} v_{kn}(x, q) v_{km}(x, q) \rho_{1_k}(x, q) d_q x = d^2_{1_{kn}}(q) \delta_{mn} \tag{3.124}$$

  on the condition that

$$\sigma_{1_k}(x, q) \rho_{1_k}(x, q) x^k \Big|_{x=q^k a, b} = 0. \tag{3.125}$$

- *If $a < 0 < b$, then v_{kn} has the following representation for orthogonality*

$$\int_{q^k a}^{q^k b} v_{kn}(x, q) v_{km}(x, q) \rho_{1_k}(x, q) d_q x = d^2_{1_{kn}}(q) \delta_{mn} \tag{3.126}$$

  only if

$$\sigma_{1_k}(x, q) \rho_{1_k}(x, q) x^k \Big|_{x=q^k a, q^k b} = 0. \tag{3.127}$$

Here, $d_{1_{kn}}(q)$ is norm and δ_{mn} is Kronecker delta.

Proof. Proof includes the similar steps with Theorem 3.31. □

Proposition 3.35 *Let $d_{1_{kn}}(q)$ be the norm illustrated with*

$$d^2_{1_{kn}}(q) = \int_a^{q^k b} [v_{kn}(x,q)]^2 \rho_{1_k}(x,q) d_q x \tag{3.128}$$

assuming that $0 < a < b$. *Then,*

$$d^2_{1_{kn}}(q) = \frac{1}{\lambda_{1_{nk}}(q)} d^2_{1_{k+1,n}}(q). \tag{3.129}$$

Proof. In order to find a recurrence relation for $d_{1_{kn}}(q)$ defined by (3.129), we concern with the q-EHT for $v_{kn}(x,q) = D^{(k)}_{q^{-1}} y_n(x)$ in self-adjoint form identified by (3.54)

$$D_q \left[\rho_{1_k}(x,q)\sigma_{1_k}(x,q) D_{q^{-1}} v_{kn}(x,q)\right] + q^{-1}\lambda_{1_{nk}}(q)\rho_{1_k}(x,q) v_{kn}(x,q) = 0.$$

Multiplying above equation with $v_{kn}(x,q)$ and applying q-integral over $(a, q^k b)$, we have

$$\int_a^{q^k b} v_{kn}(x,q) D_q \left[\rho_{1_k}(x,q)\sigma_{1_k}(x,q) D_{q^{-1}} v_{kn}(x,q)\right] d_q x + q^{-1}\lambda_{1_{nk}}(q) d^2_{1_{kn}}(q) = 0.$$

Using $x = q^{-1}t$ transformation for the q-integral in the above equation, we can rewrite it as

$$\int_{qa}^{q^{k+1} b} v_{kn}(q^{-1}t,q) q D_q \left[\rho_{1_k}(q^{-1}t,q)\sigma_{1_k}(q^{-1}t,q) D_{q^{-1}} v_{kn}(q^{-1}t,q)\right] q^{-1} d_q t + q^{-1}\lambda_{1_{nk}}(q) d^2_{1_{kn}}(q) = 0.$$

Now by use of the product rule for $x = q^{-1}t$ defined by (2.11), $\rho_{1_{k+1}}(x,q) = \sigma_{1_k}(x,q)\rho_{1_k}(x,q)$, $v_{k+1,n}(x,q) = D_{q^{-1}} v_{kn}(x,q)$ and

$$D_q f(x)\Big|_{x=q^{-1}t} = q D_q f(q^{-1}t) = D_{q^{-1}} f(t),$$

we get

$$\begin{aligned} &\rho_{1_k}(q^{-1}t,q)\sigma_{1_k}(q^{-1}t,q) v_{kn}(q^{-1}t,q) D_{q^{-1}} v_{kn}(q^{-1}t,q)\Big|_{t=qa}^{q^{k+1}b} \\ - \quad & q^{-1}\int_{qa}^{q^{k+1} b} v^2_{k+1,n}(x,q)\rho_{1_{k+1}}(x,q) d_q x + q^{-1}\lambda_{1_{nk}}(q) d^2_{1_{kn}}(q) = 0 \end{aligned}$$

where the first term is vanished from the boundary condition identified by (3.123). For the second term we divide q-integral into two separate parts

$$-q^{-1}\int_{qa}^{a} v^2_{k+1,n}(x,q)\rho_{1_{k+1}}(x,q) d_q x - q^{-1}\int_{a}^{q^{k+1} b} v^2_{k+1,n}(x,q)\rho_{1_{k+1}}(x,q) d_q x + q^{-1}\lambda_{1_{nk}}(q) d^2_{1_{kn}} = 0$$

from which it is seen that the second term is $d_{k+1,n}$. We deal with the first term according to definition of q-integral that leads to

$$\begin{aligned}
\int_{qa}^{a} v_{k+1,n}^2(x,q)\rho_{1_{k+1}}(x,q)d_qx &= \int_0^a v_{k+1,n}^2(x,q)\rho_{1_{k+1}}(x,q)d_qx - \int_0^{qa} v_{k+1,n}^2(x,q)\rho_{1_{k+1}}(x,q)d_qx \\
&= a(1-q)\sum_{i=0}^{\infty} q^n v_{k+1,n}^2(q^i a,q)\rho_{1_{k+1}}(q^i a,q) \\
&- qa(1-q)\sum_{i=0}^{\infty} q^n v_{k+1,n}^2(q^{i+1}a,q)\rho_{1_{k+1}}(q^{i+1}a,q) \\
&= a(1-q)\rho_{1_{k+1}}(a,q)v_{k+1,n}^2(a,q)
\end{aligned}$$

which is vanished since $a\rho_{1_{k+1}}(a,q) = a\rho_{1_k}(a,q)\sigma_{1_k}(a,q) = 0$. Then, we have

$$-q^{-1}d_{1_{k+1,n}}^2(q) + q^{-1}\lambda_{1_{nk}}(q)d_{1_{kn}}^2(q) = 0$$

which provides the desired recurrence relation

$$d_{1_{kn}}^2(q) = \frac{1}{\lambda_{1_{nk}}(q)}d_{1_{k+1,n}}^2(q).$$

□

Corollary 3.36 *Let $d_{1_{kn}}(q)$ with $d_{1_{0n}}(q) = d_n(q)$ be the norm having the recurrence relation given with (3.129). Then,*

$$d_n^2(q) = (-1)^n A_{1_{nn}} B_{1_n}^2(q) K_{1_n} \tag{3.130}$$

where

$$A_{1_{nn}}(q) = (-1)^n \lambda_{1_{nn-1}}(q)\lambda_{1_{nn-2}}(q)...\lambda_{1_{n0}}(q), \tag{3.131}$$

$$B_{1_n}(q) = \frac{1}{A_{1_{nn}}(q)}v_{nn}(q), \tag{3.132}$$

and

$$K_{1_n} = \int_a^{q^n b} \rho_{1_n}(x,q)d_qx. \tag{3.133}$$

Proof. Solving recurrence relation successively for $d_{1_{kn}}(q)$ obtained in proposition 3.35 gives

$$d_n^2(q) := d_{1_{0n}}^2(q) = \frac{1}{\lambda_{1_{n0}}(q)}d_{1_{1n}}^2(q) = \frac{1}{\lambda_{1_{n0}}(q)\lambda_{1_{n1}}(q)}d_{1_{2n}}^2(q) = ...\frac{1}{\prod_{k=0}^{n-1}\lambda_{1_{nk}}(q)}d_{1_{nn}}^2(q) \tag{3.134}$$

where

$$d_{1_{nn}}^2(q) = \int_a^{q^n b} v_{nn}^2(x,q)\rho_{1_n}(x,q)d_qx = v_{nn}^2\int_a^{q^n b}\rho_{1_n}(x,q)d_qx, \quad v_{nn} = D_{q^{-1}}^{(n)}y_n(x). \tag{3.135}$$

According to the normalization constant in (3.59) defined in the Rodrigues formula

$$B_{1_n}(q) = \frac{1}{A_{1_{nn}}(q)} v_{nn}$$

in which

$$A_{1_{nn}}(q) = (-1)^n \frac{1}{\lambda_{1_{nn-1}}(q)\lambda_{1_{nn-2}}(q)...\lambda_{1_{n0}}(q)}.$$

Then,

$$d_n^2(q) = \frac{1}{\prod_{k=0}^{n-1}\lambda_{1_{nk}}(q)} d_{1_{nn}}^2(q) = \frac{1}{\prod_{k=0}^{n-1}\lambda_{1_{nk}}(q)} v_{nn}^2 K_{1_n} = (-1)^n A_{1_{nn}}(q) B_{1_n}^2(q) K_{1_n}$$

where

$$K_{1_n} = \int_a^{q^n b} \rho_{1_n}(x,q) d_q x$$

can be compute in the following proposition. □

Proposition 3.37 *Let K_{1_n} be given with the q-integral identified by (3.133). Then,*

$$\frac{K_{1_n}}{K_{1_{n+1}}} = \frac{1 + \dfrac{\sigma''_{1_n}(x,q)}{2\tau'_{1_n}(x,q)}}{\sigma_{1_n}(x_n^*,q)} \tag{3.136}$$

where x_n^* is zero of $\tau_{1_n}(x,q)$.

Proof. We begin with

$$K_{1_{n+1}} = \int_a^{q^{n+1}b} \rho_{1_{n+1}}(x,q) d_q x.$$

By using $\rho_{1_{n+1}}(x,q) = \sigma_{1_n}(x,q)\rho_{1_n}(x,q)$ and taking account that $a < q^{n+1}b < q^n b < b$ as $0 < a < b$, K_{1_n} should be rewritten

$$K_{1_{n+1}} = \int_a^{q^n b} \sigma_{1_n}(x,q)\rho_{1_n}(x,q) d_q x - \int_{q^{n+1}b}^{q^n b} \sigma_{1_n}(x,q)\rho_{1_n}(x,q) d_q x$$

in which the second term is vanished by using the boundary condition in (3.123) after applying the definition of q-integral in (2.23). Thus,

$$K_{1_{n+1}} = \int_a^{q^n b} \sigma_{1_n}(x,q)\rho_{1_n}(x,q) d_q x.$$

By replacing $\sigma_{1_n}(x,q) = A_1(q)[\tau_{1_n}(x,q)]^2 + B_1(q)\tau_{1_n}(x,q) + C_1(q)$ into $K_{1_{n+1}}$ implies

$$K_{1_{n+1}} = \int_a^{q^n b} [A_1(q)\tau_{1_n}(x,q) + B_1(q)]\, \tau_{1_n}(x,q)\rho_{1_n}(x,q) d_q x + C_1(q) K_{1_n}$$

which is equivalent to

$$K_{1_{n+1}} = q\int_a^{q^n b} \left[A_1(q)\tau_{1_n}(x,q) + B_1(q)\right] D_q\rho_{1_{n+1}}(x,q)d_qx + C_1(q)K_{1_n}$$

by using the q-Pearson equation and recurrence relation for $\rho_{1_n}(x,q)$ identified by (3.55) and (3.56), respectively. Now applying transformation $x = q^{-1}t$ for the first term in $K_{1_{n+1}}$ and using the q-integration by parts defined by (2.35), we obtain

$$\begin{aligned} K_{1_{n+1}} &= q\left[A_1(q)\tau_{1_n}(q^{-1}t,q) + B_1(q)\right]\rho_{1_{n+1}}(q^{-1}t,q)\Big|_{t=qa}^{q^{n+1}b} - \\ &\quad \int_{qa}^{q^{n+1}a} \rho_{1_{n+1}}(t,q)D_{q^{-1}}\left[A_1(q)\tau_{1_n}(t,q) + B_1(q)\right] d_qt + C_1(q)K_{1_n} \end{aligned} \tag{3.137}$$

where the first term is zero considering $\rho_{1_{n+1}}(x,q) = \sigma_{1_n}(x,q)\rho_{1_n}(x,q)$ and the boundary condition (3.123). Taking the second term into consideration as two separate q-integrals leads to

$$\begin{aligned} K_{1_{n+1}} &= -\int_{qa}^{a} \rho_{1_{n+1}}(t,q)D_{q^{-1}}\left[A_1(q)\tau_{1_n}(t,q) + B_1(q)\right] d_qt - \\ &\quad \int_{a}^{q^{n+1}b} \rho_{1_{n+1}}(t,q)D_{q^{-1}}\left[A_1(q)\tau_{1_n}(t,q) + B_1(q)\right] d_qt + C_1(q)K_{1_n} \end{aligned}$$

in which the first term is vanished by using the definition of q-integral in (2.23), then applying the boundary condition (3.123), we arrive at

$$K_{1_{n+1}} = -A_1(q)\tau'_{1_n}(x,q)K_{1_{n+1}} + C_1(q)K_{1_n}$$

since $D_{q^{-1}}\left[A_1(q)\tau_{1_n}(t,q) + B_1(q)\right] = A_1(q)\tau'_{1_n}(x,q)$, which is the desired result. □

Remark 3.38 *Notice that*

$$D_{q^{-1}}\left[A_1(q)\tau_{1_n}(t,q) + B_1(q)\right] = A_1(q)D_{q^{-1}}\tau_{1_n}(x,q) = A_1(q)\tau'_{1_n}(x,q)$$

since $\tau_{1_n}(x,q)$ *is polynomial of degree* 1.

Remark 3.39 *Note that*

$$A_1(q) = \frac{\sigma''_{1_n}(x,q)}{2[\tau'_{1_n}(x,q)]^2} \tag{3.138}$$

$$C_1(q) = \sigma_{1_n}(x_n^*,q) \tag{3.139}$$

where x_n^* *is the root of* $\tau_{1_n}(x,q)$.

Proposition 3.40 *Let K_{1_n} given with (3.133) satisfy the recurrence relation identifed by (3.136). Then*

$$K_{1_n} = K_{1_0} \prod_{k=0}^{n-1} \frac{\sigma_{1_k}(x_k^*, q)}{1 + \dfrac{\sigma''_{1_k}(x, q)}{2\tau'_{1_k}(x, q)}} \tag{3.140}$$

where $K_{1_0} = \int_a^b \rho(x, q) d_q x$.

Proof. The proof is based on the successive solution of the recurrence relation (3.136). □

Alternative representation of K_{1_n} can be computed as in the following proposition.

Proposition 3.41 *Let K_{1_n} in (3.133) satisfy the recurrence relation denoted in (3.136). Then,*

$$K_{1_n} = (1 - q)q^{-1} a\rho_{1_{N-1}}(q^{-1}a, q) \prod_{k=n}^{N-2} \frac{1 + \dfrac{\sigma''_{1_k}(x, q)}{2\tau'_{1_k}(x, q)}}{\sigma_{1_k}(x_k^*, q)} \tag{3.141}$$

with $n < N - 1$ and $\frac{a}{b} = q^N$.

Proof. Consider the product of ratio for K_{1_i} defined by (3.136) for $i = n, n+1, ..., N-1$ which gives

$$K_{1_n} = K_{1_{N-1}} \prod_{k=n}^{N-2} \frac{1 + \dfrac{\sigma''_{1_k}(x, q)}{2\tau'_{1_k}(x, q)}}{\sigma_{1_k}(x_k^*, q)}$$

where

$$\begin{aligned} K_{1_{N-1}} = \int_a^{q^{N-1}b} \rho_{1_{N-1}}(x, q) d_q x &= \int_a^{q^{-1}a} \rho_{1_{N-1}}(x, q) d_q x \\ &= (1 - q)q^{-1} a\rho_{1_{N-1}}(q^{-1}a, q) \end{aligned}$$

by using the fact that $q^{-1}a = q^{N-1}b$ and applying the definition of the q-integral. □

Proposition 3.42 *Note that the norm identified by (3.118) can be accomplished analogously regarding q^{-1}-derivatives of the q-polynomials of the 1st kind with the orthogonality relation associated with q^{-1}-integral*

$$s_n^2(q) = (-1)^n A_{1nn} B_{1_n}^2(q) M_{1_n} \tag{3.142}$$

where A_{1nn}, $B_{1_n}(q)$ are defined by (3.131), (3.132), respectively and

$$M_{1_n} = \int_a^{q^n b} \rho_{1_n}(x, q) d_{q^{-1}} x. \tag{3.143}$$

3.6.2 Orthogonality Property of $D_q^{(k)} P_{2_{n+k}}(x,q)$

Orthogonality property of $u_{kn}(x,q) = D_q^{(k)} P_{2_{n+k}}(x,q)$, $k = 1, 2, ...$ includes the similar analysis as section 3.6.1.

Proposition 3.43 *Let ρ_{2_k} be a function satisfying the q^{-1}-Pearson equation identified by (3.74) with $n = k$. Then,*

- *If $0 < a < b$, then the polynomials u_{kn} have the following orthogonality relation*

$$\int_{q^{-k}a}^{b} u_{kn}(x,q) u_{km}(x,q) \rho_{2_k}(x,q) d_q x = d_{2_{kn}}^2(q) \delta_{mn} \tag{3.144}$$

provided that

$$\sigma_{2_k}(x,q)\rho_{2_k}(x,q)x^k\Big|_{x=q^{-k}a,b} = 0. \tag{3.145}$$

- *If $a < b < 0$, then u_{kn} has the following representation for orthogonality*

$$\int_{a}^{q^{-k}b} u_{kn}(x,q) u_{km}(k)(x,q) \rho_{2_k}(x,q) d_q x = d_{2_{kn}}^2(q) \delta_{mn} \tag{3.146}$$

if

$$\sigma_{2_k}(x,q)\rho_{2_k}(x,q)x^k\Big|_{x=a,q^{-k}b} = 0. \tag{3.147}$$

- *If $a < 0 < b$, then u_{kn} is orthogonal in the following sense*

$$\int_{a}^{b} u_{kn}(x,q) u_{km}(x,q) \rho_{2_k}(x,q) d_q x = d_{2_{kn}}^2(q) \delta_{mn} \tag{3.148}$$

only if

$$\sigma_{2_k}(x,q)\rho_{2_k}(x,q)x^k\Big|_{x=a,b} = 0. \tag{3.149}$$

Here $d_{2_{kn}}(q)$ is norm and δ_{mn} is Kronecker delta.

Remark 3.44 *We remark that the orthogonality condition for the polynomials $D_{q^{-1}}^{(k)} P_{1_{n+k}}(x,q)$ and $D_q^{(k)} P_{2_{n+k}}(x,q)$, $k = 1, 2, ...$ are not equivalent since the q-Pearson equation for ρ_{1_k}*

$$\frac{\rho_{1_k}(qx,q)}{\rho_{1_k}(x,q)} = \frac{\sigma_{1_k}(x,q) + (1-q^{-1})x\tau_{1_k}(x,q)}{\sigma_{1_k}(qx,q)} \neq \frac{\sigma_{2_k}(x,q)}{\sigma_{1_k}(qx,q)}$$

is not equal to the q-Pearson equation for ρ_{2_k}

$$\frac{\rho_{2_k}(q^{-1}x,q)}{\rho_{2_k}(x,q)} = \frac{\sigma_{2_k}(x,q) + (1-q)x\tau_{2_k}(x,q)}{\sigma_{2_k}(q^{-1}x,q)} \neq \frac{\sigma_{1_k}(x,q)}{\sigma_{2_k}(q^{-1}x,q)}.$$

Corollary 3.45 *Let $d_{2_{kn}}(q)$ be the norm illustrated with*

$$d^2_{2_{kn}}(q) = \int_{q^{-k}a}^{b} [u_{kn}(x,q)]^2 \rho_{2_k}(x,q) d_q x \tag{3.150}$$

assuming that $0 < a < b$. Then,

$$d^2_{2_{kn}}(q) = \frac{1}{\lambda_{2_{nk}}(q)} d^2_{2_{k+1,n}}(q). \tag{3.151}$$

Remark 3.46 *Notice that successive solution of $d_{2_{kn}}(q)$ with $d_{2_{0n}}(q) = d_n(q)$ in (3.151) gives*

$$d^2_n(q) = (-1)^n A_{2_{nn}}(q) B^2_{2_n}(q) K_{2_n} \tag{3.152}$$

where

$$A_{2_{nn}}(q) = (-1)^n \frac{1}{\lambda_{2_{nn-1}}(q)\lambda_{2_{nn-2}}(q)...\lambda_{2_{n0}}(q)}, \tag{3.153}$$

$$B_{2_n}(q) = \frac{1}{A_{2_{nn}}(q)} u_{nn}(q), \tag{3.154}$$

and

$$K_{2_n} = \int_{q^{-n}a}^{b} \rho_{2_n}(x,q) d_q x. \tag{3.155}$$

Remark 3.47 *Note that considering similar analysis as in the q-EHT, recurrence relation*

$$\frac{K_{2_n}}{K_{2_{n+1}}} = \frac{1 + \dfrac{\sigma''_{2_n}(x,q)}{2\tau'_{2_n}(x,q)}}{\sigma_{2_n}(x^*_n, q)} \tag{3.156}$$

is obtained. Here, x^*_n is the root of the equation $\tau_{2_n}(x,q) = 0$ and

$$\sigma_{2_n}(x,q) = A_2(q)[\tau_{2_n}(x,q)]^2 + B_2(q)\tau_{2_n}(x,q) + C_2(q)$$

where

$$A_2(q) = \frac{\sigma''_{2_n}(x,q)}{2[\tau'_{2_n}(x,q)]^2}, \tag{3.157}$$

$$C_2(q) = \sigma_{2_n}(x^*_n, q). \tag{3.158}$$

Proposition 3.48 *Let K_{2_n} in (3.155) satisfy the recurrence relation identifed in (3.156). Then,*

$$K_{2_n} = K_{2_0} \prod_{k=0}^{n-1} \frac{\sigma_{2_k}(x^*_k, q)}{1 + \dfrac{\sigma''_{2_k}(x,q)}{2\tau'_{2_k}(x,q)}} \tag{3.159}$$

where $K_{2_0} = \int_a^b \rho(x,q) d_q x$.

Another representation of K_{2_n} can be introduced as in the following proposition.

Proposition 3.49 *Let K_{2_n} given with (3.155) satisfy the recurrence relation identifed by (3.156). Then,*

$$K_{2_n} = (1-q)b\rho_{2_{N-1}}(b,q)\prod_{k=n}^{N-2}\frac{1+\dfrac{\sigma''_{2_k}(x,q)}{2\tau'_{2_k}(x,q)}}{\sigma_{2_k}(x_k^*,q)} \tag{3.160}$$

with $n < N-1$ and $\frac{b}{a} = q^{-N}$.

Proposition 3.50 *Note that the norm identified by (3.118) can be obtained analogously by considering q-derivatives of the q-polynomials of the 2nd kind with the orthogonality relation together with q^{-1}-integral*

$$s_n^2(q) = (-1)^n A_{2nn} B_{2_n}^2(q) M_{2_n} \tag{3.161}$$

where A_{2nn}, $B_{2_n}(q)$ are defined by (3.153), (3.154), respectively, and

$$M_{2_n} = \int_{q^{-n}a}^{b} \rho_{2_n}(x,q) d_{q^{-1}}x. \tag{3.162}$$

CHAPTER 4

ANALYSIS OF THE ORTHOGONALITY OF THE Q-CLASSICAL POLYNOMIALS IN THE HAHN SENSE

In this chapter, which is the main part of the thesis, we discuss the orthogonality of all possible polynomial solutions of the q-difference equation by use of the q-Pearson equation. We are interesting in finding a suitable interval (a, b) where $\rho > 0$ and boundary condition (3.119) holds such that the polynomial solutions of the q-EHT of the 1st kind (3.5) (or equivalently of the q-EHT of the 2nd kind (3.10)) which are orthogonal with respect to ρ, are supported at the points $\alpha q^{\pm k}$ and $\beta q^{\pm k}$, $\alpha, \beta \in \mathbb{R}$, $k = 0, 1, 2, ...$(3.5).

In accordance with Theorem 3.31, it is enough to find a $\rho > 0$ satisfying the q-Pearson equation

$$\frac{\rho(qx, q)}{\rho(x, q)} = \frac{\sigma_1(x, q) + (1 - q^{-1})x\tau(x, q)}{\sigma_1(qx, q)} = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)}, \tag{4.1}$$

or, equivalently,

$$\frac{\rho(q^{-1}x, q)}{\rho(x, q)} = \frac{\sigma_2(x, q) + (1 - q)x\tau(x, q)}{\sigma_2(q^{-1}x, q)} = \frac{q\sigma_1(x, q)}{\sigma_2(q^{-1}x, q)} \tag{4.2}$$

such that the boundary condition (3.115) holds. Notice from the above expressions that

$$\sigma_2(x, q)\rho(x, q) = q\sigma_1(qx, q)\rho(qx, q) \Leftrightarrow \sigma_2(q^{-1}x, q)\rho(q^{-1}x, q) = q^{-1}\sigma_1(x, q)\rho(x, q). \tag{4.3}$$

The idea is to provide a qualitative analysis of equations (4.1) and (4.2) without solving them that gives the interval of orthogonality. In this analysis, a geometrical approach similar to the one partially presented in [24] has been used . Since we are interested in determining all the possible orthogonality intervals for the q-polynomials according to the behavior of the q-weight function $\rho(x, q)$, we study the behavior of $\rho(qx, q)/\rho(x, q)$, where we can obtain the intervals in which $\rho(x, q)$ is increasing (e.g. $x > 0$ and $\rho(qx, q)/\rho(x, q) < 1$) or decreasing (e.g. $x > 0$ and $\rho(qx, q)/\rho(x, q) > 1$).

Before starting the analysis let us classify the q-polynomials according to [3, 6, 43, 47] in terms of the degrees of the polynomials σ_1 and σ_2 in (3.12) and using the fact that $\sigma_1(0,q) = 0 \Leftrightarrow \sigma_2(0,q) = 0$ ($\sigma_1(0,q) \neq 0 \Leftrightarrow \sigma_2(0,q) \neq 0$). Therefore, as we mentioned before we have the non-zero families correspond to the case when $\sigma_1(0,q) \neq 0 \Leftrightarrow \sigma_2(0,q) \neq 0$ and the zero families when $\sigma_1(0,q) = 0 \Leftrightarrow \sigma_2(0,q) = 0$. In every class we consider all possible degrees of the polynomials $\sigma_1(x,q)$ and $\sigma_2(x,q)$. In fact, from the relation between σ_1 and σ_2 in (3.11), we rewrite the next straightforward proposition in order to see the relationship with Table 4.1:

Proposition 4.1 *Let $\rho(x,q)$ be the q-weight function satisfying the q-Pearson equation (3.24) with $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2+\sigma_1'(0,q)x+\sigma_1(0,q)$ and $\tau(x,q) = \tau'(0,q)x+\tau(0,q)$, $\tau'(0,q) \neq 0$.*

If $\sigma_1(0,q) \neq 0$, the following cases arise
(1a) If $\deg[\sigma_1(x,q)] < 2$, then $\deg[\sigma_2(x,q)] = 2$.
(1b) If $\deg[\sigma_1(x,q)] = 2$, then $\deg[\sigma_2(x,q)] \leq 2$.

If $\sigma_1(0,q) = 0$, then:
(2a) If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2$, $\sigma_1''(0,q) \neq 0$, then $\deg[\sigma_2(x,q)] = 2$ or $\deg[\sigma_2(x,q)] = 1$.
(2b) If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x$, $\sigma_1''(0,q) \neq 0$, $\sigma_1'(0,q) \neq 0$, then $\deg[\sigma_2(x,q)] = 2$ or $\deg[\sigma_2(x,q)] = 1$.
(2c) If $\sigma_1(x,q) = \sigma_1'(0,q)x$, $\sigma_1'(0,q) \neq 0$, then $\deg[\sigma_2(x,q)] = 2$.

Table 4.1: Classification of the q-classical polynomials (positive definite cases)

Non-zero families	Zero families
$\deg\sigma_1$ / $\deg\sigma_2$	$\deg\sigma_1$ / $\deg\sigma_2$
q-Jacobi / q-Jacobi	q-Jacobi / q-Jacobi
q-Jacobi / q-Laguerre	q-Jacobi / q-Laguerre
q-Jacobi / q-Hermite	q-Jacobi / q-Bessel
q-Laguerre/q-Jacobi	q-Laguerre / q-Jacobi
	q-Laguerre / q-Bessel
q-Hermite / q-Jacobi	

Remark 4.2 *Observe from the Table 4.1 that, while q-Jacobi / q-Laguerre corresponds the case* $\deg\sigma_1 = 2$ and $\deg\sigma_2 = 1$, q-Hermite / q-Jacobi means $\deg\sigma_1 = 0$ and $\deg\sigma_2 = 2$, etc. (see [42, 43] for details).

Remark 4.3 *Notice that in non-zero families q-Laguerre / q-Laguerre, q-Laguerre / q-Hermite, q-Hermite / q-Laguerre and q-Hermite / q-Hermite can not appear owing to the relation between the coefficients* $\sigma_1(x,q)$ *and* $\sigma_2(x,q)$*. And analogously, zero families has no q-Laguerre / q-Laguerre because of same reason and there are no q-Bessel / q-Jacobi, q-Bessel / q-Laguerre and q-Bessel / q-Bessel since for these cases there is no suitable interval where* $\rho > 0$ *and boundary condition holds.*

In order to find the interval of orthogonality, we make assumption that $a < b$ in the following. We also assume that ρ is a bounded function (in fact it should be q-integrable and/or q^{-1}-integrable, otherwise (3.116) or (3.118) may not have sense).

Let us start with the case when (a,b) is a finite interval. There are several possibilities such that ρ satisfies the boundary condition (3.119).

Case I. The simplest case is when $\sigma_1(a,q) = \sigma_1(b,q) = 0$. Using (4.2) rewritten of the form

$$\rho(q^{-1}x,q) = \frac{q\sigma_1(x,q)}{\sigma_2(q^{-1}x,q)}\rho(x,q), \tag{4.4}$$

we see that the function $\rho(x,q)$ vanishes for all values of $q^{-k}a$ and $q^{-k}b$, $k = 1,2,\ldots$. But now three different situations appear:

1. $a < 0 < b$. In this case $\rho(x,q)$ vanishes out of the interval (a,b) (all the values $q^{-k}a$ and $q^{-k}b$, $k = 1,2,\ldots$ are out of (a,b)) and therefore there could be a family of polynomials defined on (a,b) orthogonal with respect to a measure supported at the points aq^k and bq^k, $k = 0,1,\ldots$.

2. $0 < a < b$. In this case $\rho(x,q)$ vanishes at the points $q^{-k}a$ that belong to (a,b) and also at $q^{-k}b$ that are out (a,b). Then, the only possibility for having an OPS on (a,b) satisfying the boundary condition is that there exists N such that $bq^N = a$. But, this condition implies that $bq^k = aq^{-(N-k)}$, and therefore for all bq^k, $k = 0,1,\ldots,N$ ρ also vanishes i.e., this case has not interest.

3. $a = 0 < b$ (respectively, $a < b = 0$ but this case reduces to the one when $a = 0$). This case deserves more attention. First of all, if $a = 0$ is a zero of $\sigma_1(x,q)$ then it is also a zero of $\sigma_2(x,q)$, as we already pointed out. Then, the q-Pearson equation (4.1) (respectively (4.2)) simplifies and the above reasoning of cases 1 and 2 can not be

applied. In fact, for this case we could have, in general, a family of polynomials defined on (a, b) orthogonal with respect to a measure supported at the points bq^k, $k = 0, 1, \ldots$.

Case II. Taking into account (4.3), there is also another possibility to have an orthogonality relation on (a, b). Namely, if $q^{-1}a$ and $q^{-1}b$ are both the zeros of $\sigma_2(x, q)$. But then from (4.1)

$$\rho(qx, q) = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)}\rho(x, q), \tag{4.5}$$

it follows that $\rho(x, q)$ vanishes for all values of $q^k a$ and $q^k b$, $k = 0, 1, 2, \ldots$. Again two different situations appear in dependence if $a < 0 < b$ or $0 < a < b$. In the first case, all points of the form $q^k a$ and $q^k b$ are both inside (a, b) so this has not any interest. In the second case $q^k a$ are out (a, b) and $\rho(bq^k, q) = 0$ where $bq^k \in (a, b)$, so we could have an OPS if there exists N such that $aq^{-N} = bq^{-1}$. But $aq^{-k} = bq^{N-k-1}$ and ρ vanishes in all bq^k, thus there is not a suitable q-weight function for this case.

Case III. The next choice to get the boundary condition on (a, b) is to choose $q^{-1}a$ as a zero of $\sigma_2(x, q)$ and b of $\sigma_1(x, q)$. Then, from (4.4) and (4.5) it follows that $\rho(x, q)$ vanishes for all values $q^{-k}b$, $k = 1, 2, ...$ and $q^k a$, $k = 0, 1, ...$ Then, if $a < 0 < b$, $q^k a$ are all inside (a, b) and $q^{-k}b$ are out of (a, b), therefore we can not find a q-weight function satisfying the boundary conditions. Nevertheless, as in the Case I, it could be happen when $a = 0$. In this case it is possible to have a q-weight function defined at the points bq^k, $k \geq 0$.

For the case when $0 < a < b$, $q^k a$ and $q^{-k}b$ are all out (a, b), thus there could be a family of orthogonal polynomials defined on (a, b) with a q-weight function supported at the points $q^k b$ and on $(q^{-1}a, q^{-1}b)$ supported at the points $q^{-k}a$, $k \geq 0$, but in order to have the boundary condition (3.119) there should exists $N \in \mathbb{N}$ such that $aq^{-N} = b$. This case could lead to a q-weight function supported on a finite set of points aq^{-k} $k = 0, 1, \ldots, N$. Notice that, since $aq^{-k} = bq^{N-k}$, we can also define the q-weight function ρ at the points bq^k $k = 0, 1, \ldots, N$ that coincide with the previous ones.

Case IV: Finally, we can assume that a is a zero of $\sigma_1(x, q)$ and $q^{-1}b$ of $\sigma_2(x, q)$. Then, from (4.4) and (4.5) it follows that $\rho(x, q)$ vanishes for all values $q^{-k}a$, $k = 1, 2, ...$ and $q^k b$, $k = 0, 1, ...$ This leads to the following cases: In the first case, when $a < 0 < b$, it is not possible to find a q-weight function satisfying the boundary conditions, but, as in the previous case, one could have a q-weight function only if $b = 0$ and it is defined at the points aq^k,

$k \geq 0$. Finally, in the case $0 < a < b$, it is not possible to find a q-weight function satisfying the required conditions.

Let us now go to the infinite case, i.e., when (a, b) is an infinite interval. Assume that a is finite and $b \to \infty$ (the case $a \to -\infty$ is analogous and can be obtained by using the transformation $x = -t \Leftrightarrow x \in (a, \infty) \Leftrightarrow t \in (-\infty, -a)$). Obviously the boundary condition (3.119) at ∞ reads

$$\lim_{b\to\infty} \sigma_1(b, q)\rho(b, q)b^k = 0 \quad \text{or} \quad \lim_{b\to\infty} \sigma_2(b, q)\rho(b, q)b^k = 0, \quad k = 0, 1, ...$$

Let us consider the possible choices for a;

Case V: If a is root of $\sigma_1(x, q)$, then from (4.4) it follows that $\rho(x, q)$ vanishes for all points $q^{-k}a$, $k = 1, 2, ...$ of the interval (a, ∞) in case of $a > 0$, and therefore there is not any OP defined on (a, ∞).

If $a = 0$ is a root of $\sigma_1(x, q)$ then we could have a q-weight function supported on $(0, \infty)$ defined at the points $\alpha q^{\pm k}$ ($\alpha > 0$ arbitrary), $k \geq 0$ where a normal choice for α is $\alpha = 1$.

When $a < 0$, then we can also have a ρ which is supported on (a, ∞) at points of the form aq^k and $\alpha q^{\pm k}$ ($\alpha > 0$ arbitrary), $k \geq 0$ where $\alpha = 1$.

Case VI: If we now choose $q^{-1}a$ as a zero of $\sigma_2(x, q)$, as we already discussed, ρ is zero at $q^k a$, $k = 0, 1, ...$ Therefore, for $a > 0$ we can have a weight function on $(q^{-1}a, \infty)$ supported at the points $q^{-k}a$, $k \geq 1$. The case when $a < 0$ does not lead to an OPS. Finally, if $a = 0$, we could get a ρ defined on $(0, \infty)$ at the points $\alpha q^{\pm k}$ ($\alpha > 0$ arbitrary), $k \geq 0$.

Case VII: The last choice is when $a \to -\infty$, $b \to \infty$, then the boundary condition (3.119) holds if

$$\lim_{a\to-\infty} \sigma_1(a, q)\rho(a, q)a^k = 0, \quad \lim_{b\to\infty} \sigma_1(b, q)\rho(b, q)b^k = 0, \quad k = 0, 1, ...$$

In this case ρ is defined on $\pm\alpha q^{\pm k}$ ($\alpha > 0$ arbitrary), $k \geq 0$.

All the above discussions can be summarized in the following theorem.

Theorem 4.4 *Let ρ be a bounded non negative function and let denote by $a_1(q)$, $b_1(q)$ the zeros of $\sigma_1(x, q)$ and by $a_2(q)$, $b_2(q)$, those of $\sigma_2(x, q)$. The function ρ satisfying the q-Pearson* *equation (3.24) could satisfy the boundary condition (3.115), and therefore could be a suitable q-weight function for the polynomial solutions of (3.5) in the following cases:*

a) $a = a_1(q)$, $b = b_1(q)$; $a_1(q) < b_1(q)$: $a < 0 < b$.

In this situation ρ *is supported at the points* aq^k *and* bq^k, $k = 1, 2, ...$, *which leads to the orthogonality relation on* (a, b) *of polynomial solutions* $(P_n)_n$ *of q-EHT defined by q-integral (2.27)*

$$\int_{a_1(q)}^{b_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \tag{4.6}$$

b) $a = a_1(q)$, $b = b_1(q)$; $a_1(q) < b_1(q)$: $a = 0 < b$.

In this case ρ *is defined at the points* $0 < \cdots < b^k < \cdots < bq < b \in (0, b]$ *and the orthogonality has the form*

$$\int_{0}^{b_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{4.7}$$

where the q-Jackson integral (2.23) is used.

c) $a = a_2(q)$, $b = a_1(q)$; $a_2(q) < a_1(q)$: $0 = a < b$.

In this case ρ *is supported on* $(0, b)$ *and the orthogonality reads*

$$\int_{0}^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{4.8}$$

where the q-Jackson integral (2.23) is used.

d) $a = a_2(q)$, $b = a_1(q)$; $a_2(q) < a_1(q)$: $0 < a < b$.

In this case, ρ *is supported on* $(a, q^{-1}b)$ *at the points* aq^{-k}; $a < aq^{-1} < aq^{-2} < \cdots < aq^{-N} = q^{-1}b$ *(or, equivalently, on* (qa, b) *at the points of the form* bq^k; $qa = bq^N < \cdots < bq^2 < bq < b$*). Therefore, the orthogonality of the polynomials is written in terms of the q-Jackson integral (2.23)*

$$\int_{qa_2(q)=a_1(q)q^N}^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{4.9}$$

which is, in the case, the finite sum

$$\begin{aligned} \int_{a_1(q)q^N}^{a_1(q)} [\cdot]d_qx &= \int_0^{a_1(q)} [\cdot]d_qx - \int_0^{a_1(q)q^N} [\cdot]d_qx \qquad (4.10) \\ &= (1-q)a_1(q)\sum_{k=0}^{N-1} P_n(q^ka_1(q),q)P_m(q^ka_1(q),q)\rho(q^ka_1(q),q). \end{aligned}$$

The above expression can be also written, at least formally, in terms of the q^{-1}-integral (2.32)

$$\int_{a_2(q)}^{q^{-1}a_1(q)=a_2(q)q^{-N}} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}, \tag{4.11}$$

which becomes into the finite sum

$$\begin{aligned} \int_{a_2(q)}^{a_2(q)q^{-N}} [\cdot]d_{q^{-1}}x &= \int_{a_2(q)}^{\infty} - \int_{a_2(q)q^{-N}}^{\infty} [\cdot]d_{q^{-1}}x \qquad (4.12) \\ &= (1-q^{-1})a_2(q)\sum_{k=0}^{N-1} P_n(q^{-k}a_2(q),q)P_m(q^{-k}a_2(q),q)\rho(q^{-k}a_2(q),q). \end{aligned}$$

e) *$a = a_1(q)$, $b = 0$; $a_1(q) < 0$: $a < b = 0$.*

This case is similar to the case b) but here ρ is defined at the points $a < aq < \cdots < aq^k < \cdots < 0 \in [a,0)$ and the orthogonality is given in terms of the q-integral (2.27)

$$\int_{a_1(q)}^{0} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \tag{4.13}$$

f) *$a = a_1(q) = 0$, $b \to \infty$.*

In this case we have an orthogonality in terms of the integral (2.28)

$$\int_{0}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \tag{4.14}$$

g) *$a = a_1(q) < 0$, $b \to \infty$.*

In this case we have ρ supported on (a,∞) at the points aq^k and $q^{\mp k}$, $k = 0,1,\ldots$. Then the polynomials satisfy the orthogonality

$$\int_{a_1(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx := \int_{a_1(q)}^{0}[\cdot]d_qx + \int_{0}^{\infty}[\cdot]d_qx = d_n^2(q)\delta_{mn}, \tag{4.15}$$

where the first integral is given by (2.27) and the second by (2.28), respectively.

h) *$a = a_2(q) > 0$, $b \to \infty$.*

In this case ρ is defined at the points aq^{-k}, $k = 1,2,...$ and the orthogonality can be written in terms of the q^{-1}-integral (2.31)

$$\int_{a_2(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}. \tag{4.16}$$

i) *$a = a_2(q) = 0$, $b \to \infty$.*

In this case ρ is defined on $(0,\infty)$ and we have the orthogonality in terms of the integral (2.28)

$$\int_{0}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \tag{4.17}$$

j) Finally, when $a \to -\infty$, $b \to \infty$.

We have the orthogonality in terms of the bilateral q-integral (2.30)

$$\int_{-\infty}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}. \tag{4.18}$$

Remark 4.5 *Notice that from the above analysis we can conclude that the following cases do not lead to a suitable q-weight function $\rho > 0$ satisfying the q-Pearson equation (3.24) and the boundary conditions:*

1. $a = a_1(q)$, $b = b_1(q)$; $a_1(q) < b_1(q)$, *and* $0 < a < b$,

2. $a = a_2(q)$, $b = b_2(q)$; $a_2(q) < b_2(q)$,

3. $a = a_2(q)$, $b = a_1(q)$; $a_2(q) < a_1(q)$, *and* $a < 0 < b$,

4. $a = a_1(q)$, $b = a_2(q)$; $a_1(q) < 0$, $a < 0 < b$ *or* $0 < a < b$,

5. $a = a_1(q)$, $b \to \infty$, *and* $a > 0$,

6. $a = a_2(q)$, $b \to \infty$, *and* $a < 0$.

We remark that a completely similar analysis can be done for the boundary condition (3.117). In fact, the results follow if we apply the transformation $x = qt$ to the conditions (3.119).

4.1 The Main Results

In this section, we formulate our main results. We are interesting in ρ satisfying the q-Pearson equation such that $\rho > 0$ and the boundary condition holds. In order to determine ρ, we study on the rational function $\rho(qx,q)/\rho(x,q)$ and from the anaysis of the behaviour of such a function we deduce all possible families of orthogonal polynomials as well as the orthogonality relation including the interval of orthogonality.

Notice from the q-Pearson equation that $\rho(qx,q)/\rho(x,q)$ is a rational function consisting of the ratio of two polynomials of at most second degree $\sigma_2(x,q)$ and $\sigma_1(x,q)$ at nominator and denominator, respectively, i.e., it has at most two zeros and two poles. Actually, in the analysis of $\rho(qx,q)/\rho(x,q)$, we consider all possible degrees of the polynomial coefficients

$\sigma_1(x,q)$ and $\sigma_2(x,q)$ and we construct all possible graphs of $\rho(qx,q)/\rho(x,q)$. In particular, we fix the intervals where $\rho(qx,q)/\rho(x,q) < 1$ or $\rho(qx,q)/\rho(x,q) > 1$, that give us information about the monotonicity of $\rho(x,q)$. Another important data is the horizontal asymptote $\rho(qx,q)/\rho(x,q) \to c$, as $x \to \mp\infty$. All these information allows us to determine the suitable intervals for $\rho(x,q)$ without solving the q-Pearson equation. In such a way we have a complementary characterization for the q-polynomials similar to the one done in [35] but starting from the three-term recurrence relation and the Favard Theorem.

4.2 The Non-zero Case

Let start with the non-zero case. i.e., $q\sigma_1(0,q) = \sigma_2(0,q) \neq 0$.

4.2.1 Constant Case: The q-Classical ∅-Hermite/Jacobi Polynomials

Let $\sigma_1(x,q) = \sigma_1(0,q) \neq 0$, i.e., constant and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$. Then, the q-Pearson equation follows from (4.1) as

$$\begin{aligned} \frac{\rho(qx,q)}{\rho(x,q)} &= \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} \\ &= (1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}x^2 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1(0,q)}x + 1. \end{aligned} \tag{4.19}$$

Remark 4.6 *Observe that $\sigma_1(x,q) = \sigma_1(0,q)$ gives*

$$\sigma_2(x,q) = q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right] = (q-1)\tau'(0,q)x^2 + (q-1)\tau(0,q)x + q\sigma_1(0,q)$$

from which it is seen that $\sigma_2(x,q)$ is quadratic since $\tau'(0,q) \neq 0$. Then, the q-Hermite type q-polynomials of the 1st kind are the q-Jacobi type q-polynomials of the 2nd kind (see Table 4.1).

Let denote by Δ_q the constant

$$\Delta_q := \left[(1-q^{-1})\frac{\tau(0,q)}{\sigma_1(0,q)}\right]^2 - 4(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}.$$

Notice that the function at the right hand side of the q-Pearson equation defined in (4.19) is equivalent to

$$(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}[x - a_2(q)][x - b_2(q)], \text{ if } \Delta_q \neq 0.$$

In fact, if $\Delta_q > 0$ then $a_2(q)$ and $b_2(q) \in \mathbb{R}$ and we assume, without losing any generality that $a_2(q) < b_2(q)$. If $\Delta_q < 0$, $a_2(q)$ and $b_2(q) \in \mathbb{C}$.

If $\Delta_q = 0$ then it takes of the form $(1-q^{-1})\dfrac{\tau'(0,q)}{\sigma_1(0,q)}[x-a_2(q)]^2$, where $a_2(q) \in \mathbb{R}$.

In order to determine the graphs of the ratio $\rho(qx,q)/\rho(x,q)$ according to zeros of $\sigma_1(x,q)$ and $\sigma_2(x,q)$, we first consider all possible positions of the zeros of $\sigma_1(x,q)$ and $\sigma_2(x,q)$ in the following lemma.

Lemma 4.7 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.19) and set*

$$\Lambda_q = \frac{\tau'(0,q)}{\sigma_1(0,q)} \neq 0.$$

Then, the roots of the equation $f(x,q) = 0$ has the following properties;

1. *If $\Lambda_q > 0$, there are two real distinct roots with opposite signs.*
2. *If $\Lambda_q < 0$, there exist three possibilities, i.e.,*
   (a) *if $\Delta_q > 0$, there are two real roots with same signs,*
   (b) *if $\Delta_q = 0$, there are equal real roots,*
   (c) *if $\Delta_q < 0$, there are no real roots.*

Proof.

1. $\Delta_q = \left[(1-q^{-1})\frac{\tau(0,q)}{\sigma_1(0,q)}\right]^2 - 4(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)} > 0$ in case of $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1(0,q)} > 0$ which shows that $f(x,q) = 0$ has two real roots and multiplication of these roots is $\frac{\sigma_1(0,q)}{(1-q^{-1})\tau'(0,q)}$ which is negative since $0 < q < 1$.
2. $\Lambda_q < 0$ is not sufficient condition. Then, according as the sign of Δ_q, properties of zeros of the equation $f(x,q) = 0$ are determined.

□

Our next step is to analyse all possible graphs of $\rho(qx,q)/\rho(x,q)$ in (4.19) according to all possible relative positions of the zeros of σ_2. We assume that the conditions of Lemma 4.7 holds.

To obtain the behaviour of the q-weight function ρ from the graphs of $\rho(qx, q)/\rho(x, q)$, we divide the whole real line into the intervals where $\rho(x, q)$ is monotonic decreasing and increasing. Our aim is to find suitable intervals (as the ones described in Theorem 4.4) where ρ is defined and satisfies the required properties, i.e., $\rho > 0$ and that it fulfills the boundary condition (3.119) or (3.117). Obviously for getting a positive ρ we need to consider only those intervals where $\rho(qx, q)/\rho(x, q) > 0$. If $\rho > 0$ at some point of those intervals, then it is positive in the whole interval. By the q-Pearson equation (4.1) the positivity regions of the ratio $\rho(qx, q)/\rho(x, q)$ coincide with the positivity regions of $\sigma_2(x, q)/\sigma_1(qx, q)$.

Before starting the analysis let us point out that $\rho(qx, q)/\rho(x, q)$ always intercepts the y-axis at the point $y = 1$ since $\sigma_2(0, q) = q\sigma_1(0, q)$ (i.e., the constant terms of σ_1 and σ_2 are the same).

Let $f(x, q) = \rho(qx, q)/\rho(x, q)$ be the function defined in (4.19).

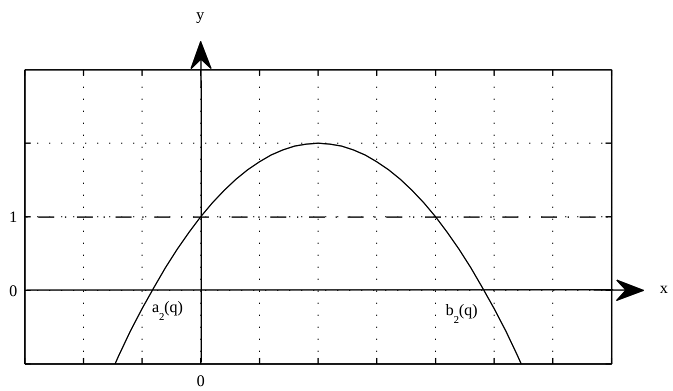


Figure 4.1: Case 1. The function $f(x, q)$ with $\Lambda_q > 0$, $a_2(q) < 0 < b_2(q)$.

Case 1: $a_2(q) < 0 < b_2(q)$, $\Lambda_q > 0$. The graph of f for this case is represented in Figure 4.1. Let us consider the possible intervals in which we can have a suitable q-weight function ρ. As we have already mentioned, they are defined by the zeros of the polynomials σ_1 and σ_2. First of all, notice that since ρ should be a positive weight function and f is negative in the intervals $(-\infty, a_2(q))$ and $(b_2(q), \infty)$, they are not suitable. On the other hand, the interval $(a_2(q), b_2(q))$ is also eliminated due to Remark 4.5.2. As a result, this case does not lead to a suitable q-weight function with the needed properties.

Case 2(a)A: $0 < a_2(q) < b_2(q)$, $\Lambda_q < 0$. This situation appears in Figure 4.2A. In order to find the possible intervals in which we can have a suitable q-weight function ρ we start the analysis by applying the positivity of ρ which allows us to eliminate the interval $(a_2(q), b_2(q))$.

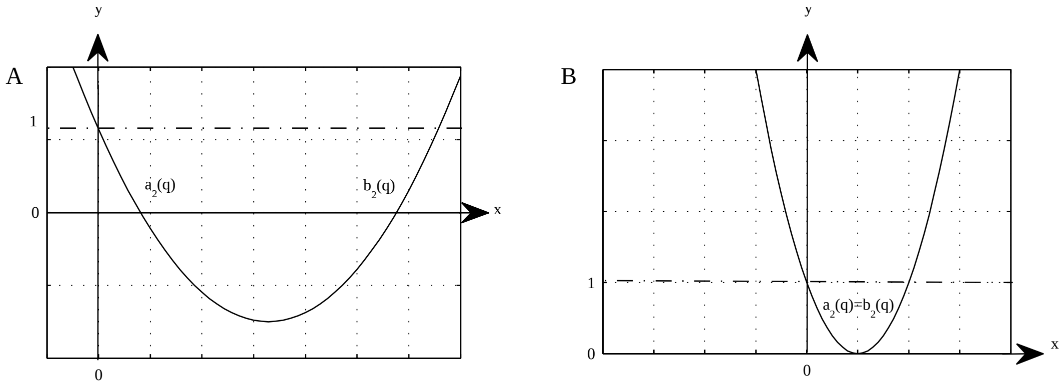


Figure 4.2: Case 2. The function $f(x,q)$ with $\Lambda_q < 0$, Case 2(a)A: $0 < a_2(q) < b_2(q)$, Case 2(b)B: $0 < a_2(q) = b_2(q)$.

On the other hand, Remark 4.5.6 by symmetry property enables us to exclude the interval $(-\infty, b_2(q))$. Let us consider the last interval $(b_2(q), \infty)$. Notice that it coincides with the one described in Theorem 4.4 h), so here it could be possible to have q-weight function ρ. Notice also that since $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = b_2(q)$, then from Figure 4.2A it follows that ρ is decreasing on $(-\tau(0,q)/\tau'(0,q), \infty)$. Since $\rho(qx,q)/\rho(x,q)$ has infinite limit as $x \to +\infty$, then we have $\rho \to 0$ as $x \to \infty$. We can sketch behaviour of ρ according to the above discussion in Figure 4.3.

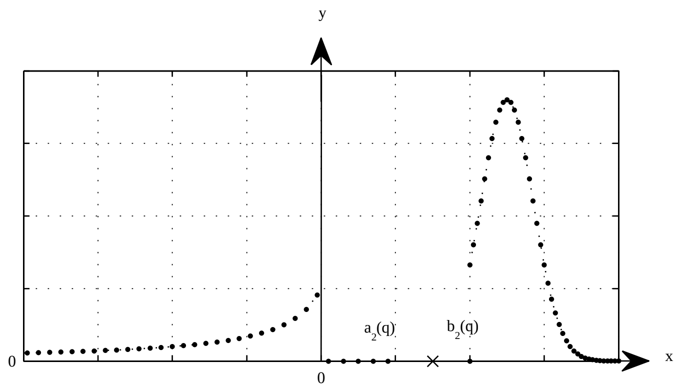


Figure 4.3: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.2.

It is seen from Figure 4.3 that $(b_2(q), \infty)$, supported at the points $b_2(q)q^{-k}$, $k = 0, 1, ...$ (see Theorem 4.4 g)), could be suitable to have ρ. However, it is not enough to assure that ρ satisfies the boundary conditions at $+\infty$. In fact, as it is stated in Theorem 4.4, we should check that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$. To this end, we use, instead of the Pearson equation (4.1), the following *extended* q-Pearson equation:

$$\frac{\sigma_1(qx,q)\rho(qx,q)(qx)^k}{\sigma_1(x,q)\rho(x,q)x^k} = q^k\frac{\sigma_1(x,q)+(1-q^{-1})x\tau(x,q)}{\sigma_1(x,q)} = q^k\frac{q^{-1}\sigma_2(x,q)}{\sigma_1(x,q)}, \tag{4.20}$$

which is a consequence of the identity (see the q-Pearson equation (4.1))

$$\frac{\sigma_1(qx,q)\rho(qx,q)}{\sigma_1(x,q)\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(x,q)}. \tag{4.21}$$

Let define the function g as the left hand side of (4.20)

$$\begin{aligned} g(x,q) = \frac{\sigma_1(qx,q)\rho(qx,q)(qx)^k}{\sigma_1(x,q)\rho(x,q)x^k} &= q^k\frac{\sigma_1(x,q)+(1-q^{-1})x\tau(x,q)}{\sigma_1(x,q)} = \frac{q^{k-1}\sigma_2(x,q)}{\sigma_1(x,q)} \\ &= q^k\Big[(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}x^2 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1(0,q)}x+1\Big] \end{aligned} \tag{4.22}$$

which is represented in Figure 4.4.

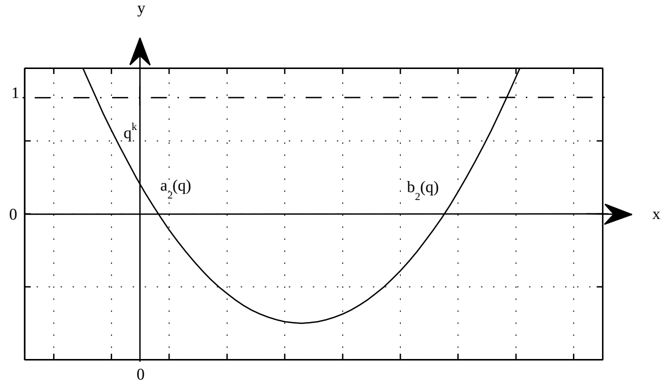


Figure 4.4: A figure of $g(x,q)$ corresponding to Figure 4.2A.

If we now provide a similar anaysis with the function g, we see from Figure 4.4 and (4.20) that, for k large enough, g has the same property with f. Therefore, it is clear from Figure 4.4 that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$.

Case 2(b)B: $0 < a_2(q) = b_2(q)$, $\Lambda_q < 0$. This situation is represented in Figure 4.2B. Note that this case leads to the same interval $(b_2(q),\infty)$ as Case 2(a)A but together with $a_2(q) = b_2(q)$. Notice that Case 2(a)A and Case 2(b)B lead to the following theorem.

Theorem 4.8 *Let $a = b_2(q)$, be the zero of $\sigma_2(x,q)$ and $b = \infty$ and assume that $0 < a_2(q) \leq b_2(q)$, $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1(0,q)} < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.16) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha+\log_q x-1}(qa_2(q)/x, qb_2(q)/x; q)_\infty > 0, \quad x \in (b_2(q),\infty), \tag{4.23}$$

$q^\alpha = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1(0,q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 h)).*

This case corresponds to the case Ia1 in Chapter 11 of [35, pages 335 and 357].

An example of such family is the Al-Salam-Carlitz II polynomials [35] where $a_2(q) = a$, $b_2(q) = 1$,

$$\sigma_1(x,q) = aq^{-1}, \quad \sigma_2(x,q) = (1-x)(a-x),$$
$$\tau(x,q) = \frac{1}{q-1}x - \frac{1+a}{q-1}, \quad \lambda_n(q) = \frac{1}{1-q}[n]_q.$$

Al-Salam-Carlitz II polynomials are orthogonal on $(1,\infty)$ and the conditions $\Lambda_q < 0$ and $0 < a_2(q) \le b_2(q)$ give us the following restriction for the parameters $0 < a \le 1$. By means of Theorem 4.4 h) we can write the orthogonality

$$\int_1^\infty x^{\alpha+\log_q x-1}(q/x, aq/x; q)_\infty V_m^{(\alpha)}(x;q)V_n^{(\alpha)}(x;q)d_{q^{-1}}x = (q^{-1}-1)q^{-\alpha n-n^2}(q;q)_n(q;q)_\infty\delta_{mn} \quad (4.24)$$

together with $0 < a = q^{-\alpha} \le 1$.

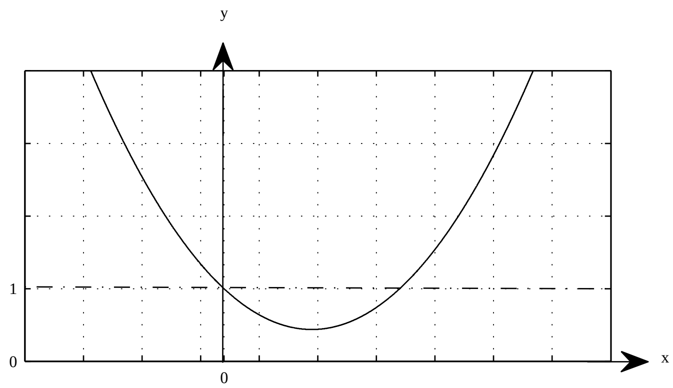


Figure 4.5: Case2(c). The function $f(x,q)$ with $\Lambda_q < 0$, $a_2(q), b_2(q) \in \mathbb{C}$.

Case 2(c): $a_2(q), b_2(q) \in \mathbb{C}$, $\Lambda_q < 0$. This situation is represented in Figure 4.5. It is seen from Figure 4.5 that the only interval is $(-\infty,\infty)$ which is the one described in Theorem 4.4 j). Therefore, it could be possible to have a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, then from Figure 4.5, it follows that ρ is increasing on $(-\infty, x_0)$ and decreasing on (x_0,∞) with $\rho \to 0$ as $x \to \mp\infty$ since $\rho(qx,q)/\rho(x,q) \to \infty$. The previous discussion brings about behaviour of ρ in the following Figure 4.6.

It is also seen from Figure 4.6 that $(-\infty,\infty)$ could be suitable for ρ. But we should analyse the *extended* q-Pearson equation (4.20) to check $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \mp\infty$ which leads

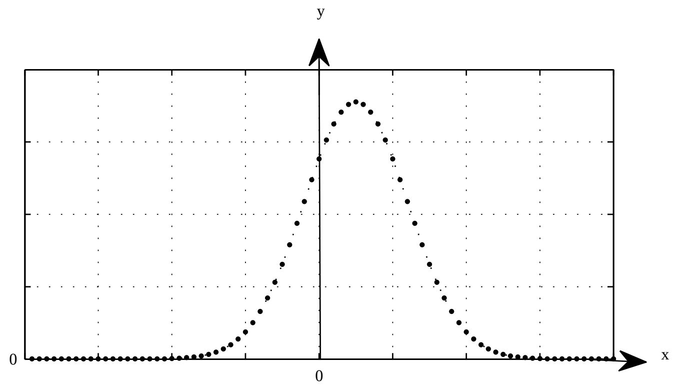


Figure 4.6: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.5.

to Figure 4.7A.

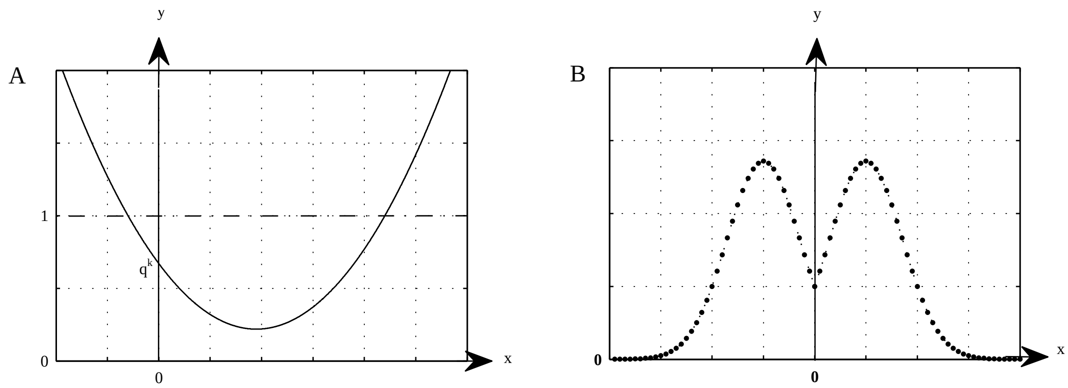


Figure 4.7: A figure of A: $g(x,q)$, B: $\sigma_1(x,q)\rho(x,q)x^k$ related to Figure 4.5.

It is clear from Figure 4.7A that g has the same property with f as $x \to \mp\infty$. Then, $q\sigma_1(x,q)$ $\rho(x,q)x^k = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k \to 0$ as $x \to \mp\infty$, $k = 0, 1, ...$ (see Figure 4.7B). Thus, we have the following theorem.

Theorem 4.9 *Let $a = -\infty$ and $b = \infty$ and assume that $a_2(q), b_2(q) \in \mathbb{C}$, $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1(0,q)} < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.18) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha+\log_q x-1}(qa_2(q)/x, qb_2(q)/x; q)_\infty > 0, \quad x \in (-\infty, \infty), \tag{4.25}$$

$q^\alpha = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1(0,q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 j)).*

This case corresponds to the case Ia1 in Chapter 11 and case Va2 in chapter 10 of [35, pages 335, 357, 283 and 315].

An example of such family is the discrete q^{-1}-Hermite II polynomials [35] where $a_2(q), b_2(q) \in \mathbb{C}$,

$$\sigma_1(x,q) = q^{-1}, \quad \sigma_2(x,q) = 1 + x.^2,$$
$$\tau(x,q) = \frac{1}{q-1}x, \quad \lambda_n(q) = \frac{1}{1-q}[n]_q.$$

Discrete q^{-1}-Hermite II polynomials are orthogonal on $(-\infty,\infty)$ and the conditions $\Lambda_q < 0$ and $0 < a_2(q), b_2(q) \in \mathbb{C}$ hold. By means of Theorem 4.4 j) we can write the orthogonality

$$\int_{-\infty}^{\infty} \frac{1}{(-x^2;q^2)_\infty}\widetilde{h}_m(x;q)\widetilde{h}_n(x;q)d_qx = (1-q)q^{-n^2}(q;q)_n\frac{(q,-q,-1,-1,-q;q)_\infty}{(i,-i,-iq,iq,-i,i,iq,-iq;q)_\infty}\delta_{mn}. \quad (4.26)$$

4.2.2 Linear Case: The q-Classical $\emptyset$-Laguerre/Jacobi Polynomials

Let $\sigma_1(x,q) = \sigma_1'(0,q)x + \sigma_1(0,q) = \sigma_1'(0,q)(x - a_1(q))$, $a_1(q) = -\frac{\sigma_1(0,q)}{\sigma_1'(0,q)}$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$. Then, the q-Pearson equation can be rewritten according to these datas as the following form

$$\begin{aligned}\frac{\rho(qx,q)}{\rho(x,q)} &= \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} \\ &= \frac{(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}x^2 + (1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)})x - a_1(q)}{qx - a_1(q)}. \end{aligned} \quad (4.27)$$

Remark 4.10 *Notice that,* $\sigma_1(x,q) = \sigma_1'(0,q)x + \sigma_1(0,q) = \sigma_1'(0,q)(x - a_1(q))$ *leads to* $\sigma_2(x,q) = q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right] = (q-1)\tau'(0,q)x^2 + (q\sigma_1'(0,q) + (q-1)\tau(0,q))x - q\sigma_1'(0,q)a_1(q)$. *It is seen that* $\sigma_2(x,q)$ *is quadratic since* $\tau'(0,q) \neq 0$. *Hence, the q-Laguerre type q-polynomials of the 1st kind are the q-Jacobi type q-polynomials of the 2nd kind (see Table 4.1).*

Let denote by Δ_q the constant

$$\Delta_q := \left[1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right]^2 + 4a_1(q)(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}.$$

Notice that the nominator in (4.27) can be written as

$$(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x - a_2(q)][x - b_2(q)], \text{ if } \Delta_q \neq 0.$$

In fact, if $\Delta_q > 0$ then $a_2(q)$ and $b_2(q) \in \mathbb{R}$ and we assume, without losing any generality that $a_2(q) < b_2(q)$. If $\Delta_q < 0$, $a_2(q)$ and $b_2(q) \in \mathbb{C}$.

If $\Delta_q = 0$ then the nominator takes the form $(1 - q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x - a_2(q)]^2$, where $a_2(q) \in \mathbb{R}$.

We are interesting in knowing how behave the zeros of the nominator of (4.27) (and so, the zeros of $\rho(qx,q)/\rho(x,q)$). This is given in the following straightforward lemma.

Lemma 4.11 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.27) and set*

$$\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} \neq 0.$$

Then, the roots of the equation $f(x,q) = 0$ have the following properties;

1. *If Λ_q and $a_1(q)$ have opposite signs, then there are two real distinct roots with opposite signs.*
2. *If Λ_q and $a_1(q)$ have same signs, then there exist three possibilities, i.e.,*
   (a) *if $\Delta_q > 0$, there are two real roots with same signs,*
   (b) *if $\Delta_q = 0$, there are equal real roots,*
   (c) *if $\Delta_q < 0$, there are no real roots.*

Proof. The proof, done for constant case, can be suitably modified by taking

$$\Delta_q = \left[1 + (1 - q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right]^2 + 4a_1(q)(1 - q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}$$

so as to obtain each case above. □

Next step is to analyse all possible graphs of $\rho(qx,q)/\rho(x,q)$ in (4.27) according to all possible relative positions of the zeros of σ_1 and σ_2 with the assumption of the conditions of Lemma 4.7. Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.27).

Case 1.A: $a_2(q) < 0 < q^{-1}a_1(q) < b_2(q)$, $\Lambda_q < 0$. This situation is represented in Figure 4.8A. We deal with the possible intervals in which we can have a suitable q-weight function ρ. To this end, we first start with positivity condition of the q-weight function which allows us to exclude the intervals $(-\infty, a_2(q))$ and $(q^{-1}a_1(q), b_2(q))$. Moreover, due to Remark 4.5.3, $(a_2(q), q^{-1}a_1(q))$ can not be used since the boundary condition (3.119) is not satisfied. Let

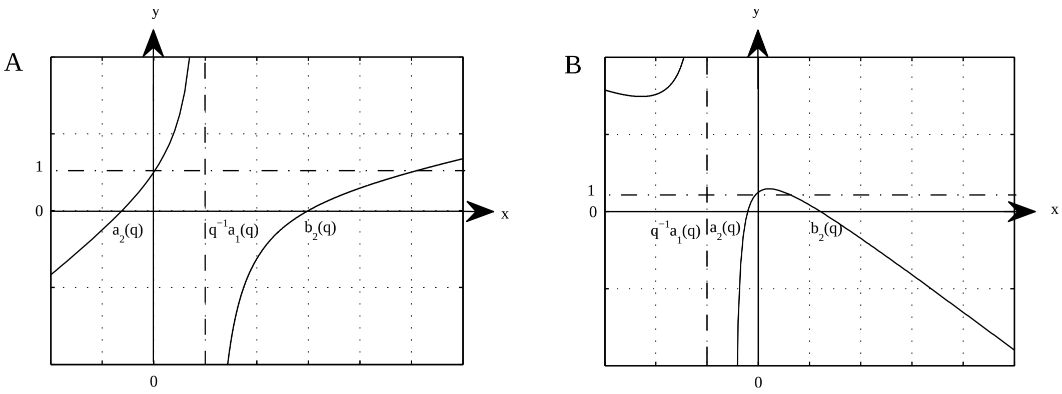


Figure 4.8: Case 1. The function $f(x, q)$ with A: $\Lambda_q < 0, a_2(q) < 0 < q^{-1}a_1(q) < b_2(q)$, B: $\Lambda_q > 0, q^{-1}a_1(q) < a_2(q) < 0 < b_2(q)$.

us consider the last interval $(b_2(q), \infty)$. Notice that it coincides with the one described in Theorem 4.4 h), so here it could be possible to have a suitable q-weight function ρ. Notice also that since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 > x = b_2(q)$, then from Figure 4.8A it follows that ρ is decreasing on $(-\tau(0, q)/\tau'(0, q), \infty)$. Since $\rho(qx, q)/\rho(x, q)$ has an infinite limit as $x \to +\infty$, we have $\rho \to 0$ as $x \to \infty$. We note that according to the information we discussed above, the behaviour of ρ can be sketched as in Figure 4.9 assuming a positive initial value for the q-weight function in each interval.

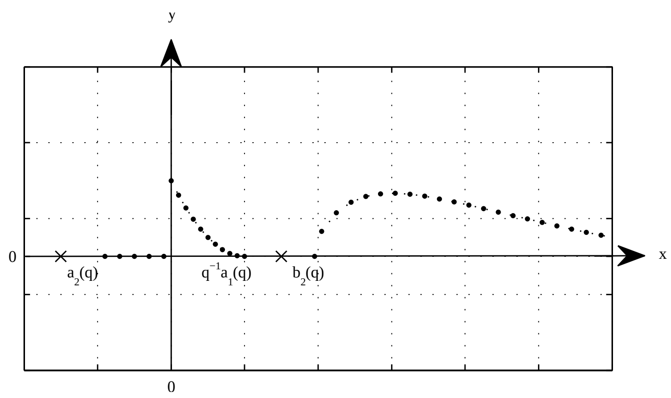


Figure 4.9: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.8A.

It is also apparent from Figure 4.9 that $\rho \to 0$ as $x \to \infty$. However, since it is infinite interval, we should check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to \infty$ by using the *extended* q-Pearson equation (4.20). Performing the same procedure to the *extended* q-Pearson equation leads to Figures 4.10. It is clear that Figure 4.10 is analog to Figure 4.8A. That's why they have the same property as $x \to \infty$ which can be seen in Figure 4.11.

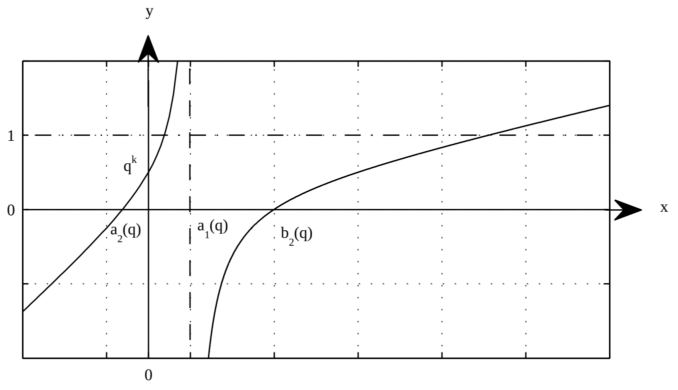


Figure 4.10: A figure of $g_1(x,q)$ corresponding to Figure 4.8a.

As a result, we deduce from Figure 4.11 that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$. Therefore, $(b_2(q),\infty)$ is suitable interval to have ρ with needed property. Thus, we perform the following theorem.

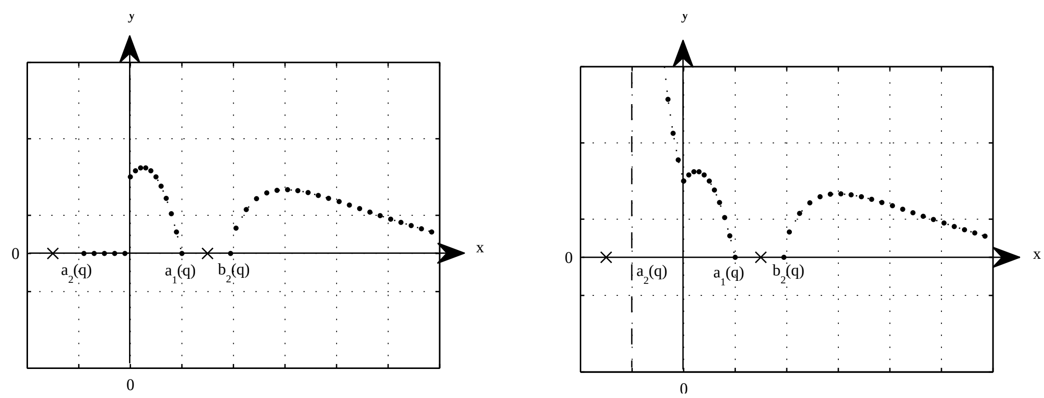


Figure 4.11: A figure of $\sigma_1(x,q)\rho(x,q)x^k$=$\sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k$ related to Figure 4.10.

Theorem 4.12 *Let $a = b_2(q)$ be the zero of $\sigma_2(x,q)$ and $b = \infty$ and assume that $a_2(q) < 0 < q^{-1}a_1(q) < b_2(q)$ and $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.16) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha+\frac{1}{2}\log_q x-1}\frac{(qa_2(q)/x, qb_2(q)/x;q)_\infty}{(a_1(q)/x;q)_\infty} > 0, \quad x \in (b_2(q),\infty), \tag{4.28}$$

$q^\alpha = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 h)).*

This case corresponds to the case IIa2 in Chapter 11 of [35, pages 337 and 358].

An example of such family is the q-Meixner polynomials [35] where $a_1(q) = bq$, $a_2(q) = -bc$, $b_2(q) = 1$,

$$\sigma_1(x,q) = cq^{-2}(x - bq), \quad \sigma_2(x,q) = (x-1)(x+bc),$$
$$\tau(x,q) = -\frac{1}{1-q}x + \frac{cq^{-1} - bc + 1}{1-q}, \quad \lambda_n(q) = \frac{[n]_q}{1-q}.$$

q-Meixner polynomials are orthogonal on $(1,\infty)$ and the conditions $\Lambda_q < 0$ and $a_2(q) < 0 < a_1(q) < b_2(q)$ give us the following restriction for the parameters $c > 0,\ 0 < b < q^{-1}$. By means of Theorem 4.4 h) we can write the orthogonality

$$\int_1^{\infty} x^{\alpha} \sqrt{x^{\log_q x - 1}} \frac{(q/x;q)_\infty(-bcq/x;q)_\infty}{(bq/x;q)_\infty} M_m(x;b,c;q)\ M_n(x;b,c;q) d_{q^{-1}}x = (q^{-1}-1)q^{-n}$$
$$\times \frac{(q,-c^{-1}q;q)_n}{(bq;q)_n} \frac{(q,-c;q)_\infty}{(bq;q)_\infty} \delta_{mn} \qquad (4.29)$$

together with $c = q^{-\alpha} > 0,\ 0 < b < q^{-1}$.

Case 1.B: $q^{-1}a_1(q) < a_2(q) < 0 < b_2(q)$, $\Lambda_q > 0$. This case is represented in Figure 4.8B. Let us examine the possible intervals in which we have a suitable ρ. First of all, the positivity of ρ enables us to skip the intervals $(q^{-1}a_1(q), a_2(q))$, $(b_2(q), \infty)$. On the other hand, the rest two intervals $(-\infty, q^{-1}a_1(q))$ and $(a_2(q), b_2(q))$ are both eliminated due to Remark 4.5.5 (we first need to do the transformation x=-t) as well as Remark 4.5.2. Then, this case does not lead to a suitable ρ with needed properties.

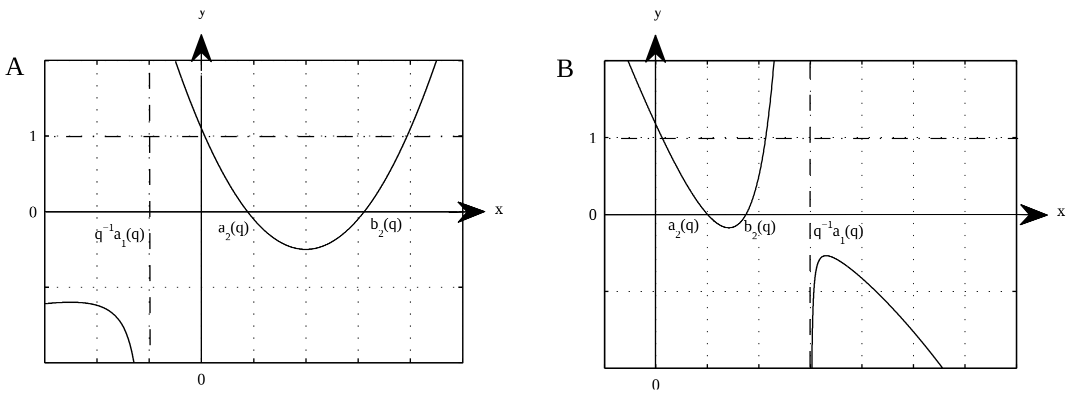


Figure 4.12: Case 2(a). The function $f(x,q)$ with A: $\Lambda_q < 0, q^{-1}a_1(q) < 0 < a_2(q) < b_2(q)$, B: $\Lambda_q > 0, 0 < a_2(q) < b_2(q) < q^{-1}a_1(q)$.

Case 2(a).A: $q^{-1}a_1(q) < 0 < a_2(q) < b_2(q)$, $\Lambda_q < 0$. The representation of this case appears in Figure 4.12. We perform the analogous analysis in order to get the possible intervals in which we have a suitable q-weight function. Thus, we begin with applying the positivity property which allows us to omit the intervals $(-\infty, q^{-1}a_1(q))$ and $(a_2(q), b_2(q))$. Afterwards, Remark 4.5.4 enables us to exclude the interval $(q^{-1}a_1(q), a_2(q))$. On the other hand, If we consider the last interval $(b_2(q), \infty)$, an analogous analysis as the one that has been done in Case 1A yields Figure 4.13.

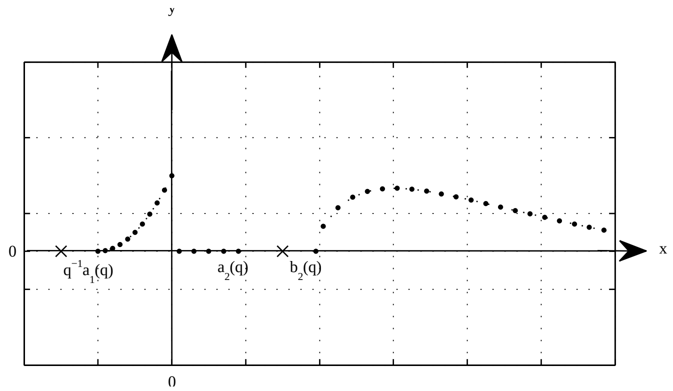


Figure 4.13: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.12A.

It is obvious from Figure 4.13 that $\rho \to 0$ as $x \to \infty$. However, since we should check $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to \infty$, $k = 0, 1, ...$, the analogous procedure as in Case 1A, the *extended* q-Pearson equation (4.20), leads to the Figure 4.14 and therefore Figure 4.15.

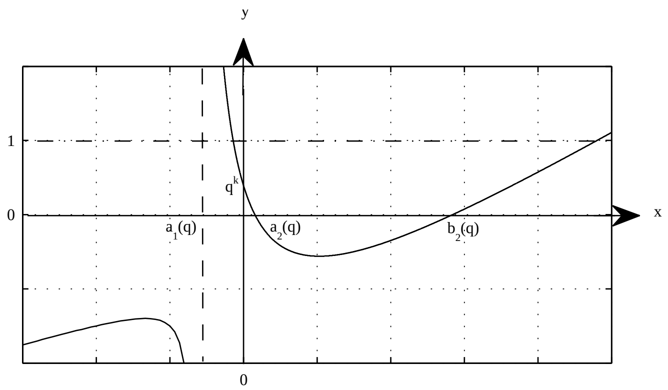


Figure 4.14: A figure of $g(x, q)$ corresponding to Figure 4.12A.

As a result of the Figure 4.15, we arrive at $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to \infty$. Therefore, there exists a q-weight function on $(b_2(q), \infty)$ supported at the points $b_2(q)q^{-k}$, $k = 0, 1, ...$ (see Theorem 4.4 h)).

Notice that Case 2(b) includes the same graphs with Case 2(a).A together with $a_2(q) = b_2(q)$. That's why, case 2(b) also produce the interval $(qb_2(q), \infty)$ associated with $a_2(q) = b_2(q)$.

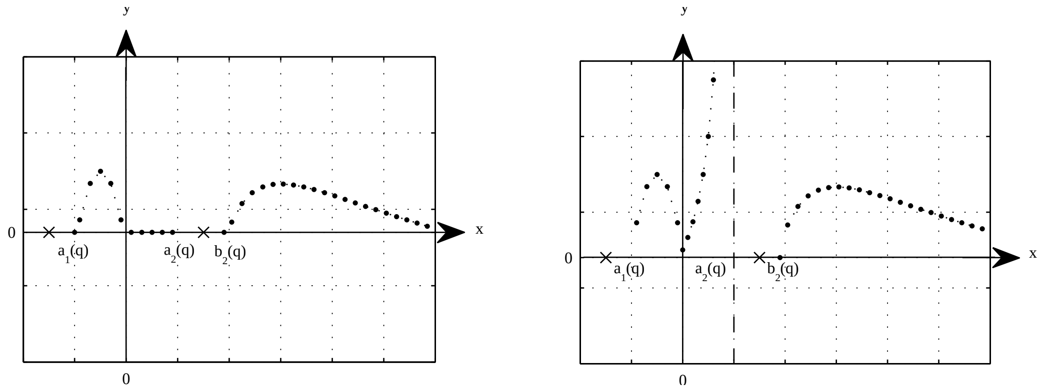


Figure 4.15: A figure of $\sigma_1(x,q)\rho(x,q)x^k$=$\sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k$ related to Figure 4.14.

Therefore, one can obtain the following theorem for this case.

Theorem 4.13 *Let $a = b_2(q)$ be the zero of $\sigma_2(x,q)$ and $b = \infty$ and assume that $q^{-1}a_1(q) < 0 < a_2(q) \le b_2(q)$ and $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.16) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha+\frac{1}{2}\log_q x-1}\frac{(qa_2(q)/x, qb_2(q)/x;q)_\infty}{(a_1(q)/x;q)_\infty} > 0, \quad x \in (b_2(q),\infty), \tag{4.30}$$

$q^\alpha = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 h)).*

This case corresponds to the new orthogonality on the interval $(b_2(q),\infty)$.

An example of such family is the q-Meixner polynomials [35] where $a_1(q) = bq$, $a_2(q) = -bc$, $b_2(q) = 1$,

$$\sigma_1(x,q) = cq^{-2}(x-bq), \quad \sigma_2(x,q) = (x-1)(x+bc),$$
$$\tau(x,q) = -\frac{1}{1-q}x + \frac{cq^{-1}-bc+1}{1-q}, \quad \lambda_n(q) = \frac{[n]_q}{1-q}.$$

q-Meixner polynomials are orthogonal on $(1,\infty)$ and the conditions $\Lambda_q < 0$ and $a_1(q) < 0 < a_2(q) \le b_2(q)$ give us the following restriction for the parameters $c > 0$, $b < 0$, $0 < -bc \le 1$. By means of Theorem 4.4 h) we can write the orthogonality of q-Meixner polynomials

$$\int_1^\infty x^\alpha\sqrt{x^{\log_q x-1}}\frac{(q/x;q)_\infty(-bcq/x;q)_\infty}{(bq/x;q)_\infty}M_m(x;b,c;q)\ M_n(x;b,c;q)d_{q^{-1}}x = (1-q^{-1})q^{-n}$$
$$\times\frac{(q,-c^{-1}q;q)_n}{(bq;q)_n}\frac{(q,-c;q)_\infty}{(bq;q)_\infty}\delta_{mn} \tag{4.31}$$

which coincides with (4.29) but with different choice of parameters $c = q^{-\alpha} > 0$, $b < 0$, $0 < -bc \leq 1$ which is the new orthogonality for q-Meixner polynomials.

Case 2(a).B: $0 < a_2(q) < b_2(q) < q^{-1}a_1(q)$, $\Lambda_q > 0$. This situation is indicated in Figure 4.12B. We perform the analogous procedure to determine a suitable interval where ρ satisfies the certain conditions. Hence, we first consider the positivity of ρ which enables us to remove the intervals $(a_2(q), b_2(q))$ and $(q^{-1}a_1(q), \infty)$. We secondly deal with the interval $(-\infty, a_2(q))$ which is also eliminated due to Remark 4.5.6 (we first need the transformation $x = -t$) since the boundary condition is not satisfied.

We last look at the interval $(b_2(q), q^{-1}a_1(q))$ which concides with the one given in Theorem 4.4 d). Then, here, it could be possible to have a suitable q-weight function. Notice from Figure 4.12B that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = b_2(q) < x_0 < x = q^{-1}a_1(q)$, then it follows that ρ is increasing on $(b_2(q), x_0)$ and decreasing on $(x_0, q^{-1}a_1(q))$ with the property $\rho(qb_2(q), q) = 0$ and $\rho(x, q) \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qb_2(q), q)/\rho(b_2(q), q) = 0$ and $\rho(qx, q)/\rho(x, q) \to \infty$ as $x \to q^{-1}a_1(q)^-$. Therefore, the behaviour of ρ can be determined as in Figure 4.16.

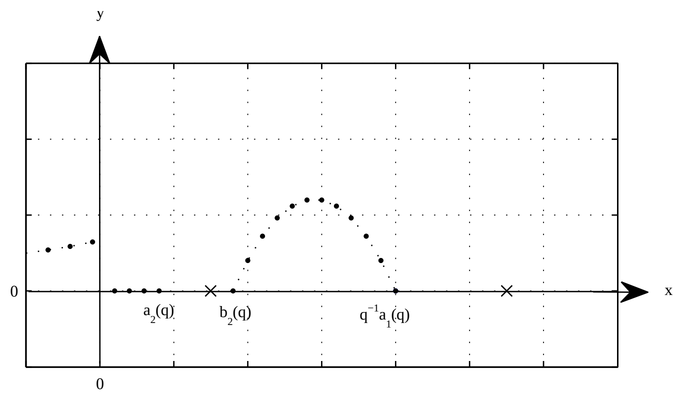


Figure 4.16: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.12B.

Figure 4.16 also displays that $(qb_2(q), a_1(q))$ is the suitable interval in which ρ is defined. Notice that the boundary condition (3.119) holds at $x = qb_2(q)$ and $x = a_1(q)$ since $qb_2(q)$ is the root of $\sigma_2(q^{-1}x, q)$ and $a_1(q)$ of $\sigma_1(x, q)$ (see Theorem 4.4 d)). Observe from Theorem 4.4 d) that for this case while ρ is supported on $(qb_2(q), a_1(q))$ at the points $a_1(q)q^k$, $k = 0, 1, ...$ it could also be supported on $(b_2(q), q^{-1}a_1(q))$ at the points $b_2(q)q^{-k}$, $k = 0, 1, ...$

Notice that Case 2(b) includes the same graphs with Case 2(a).B together with $a_2(q) = b_2(q)$. That's why, Case 2(b) also produces the interval $(b_2(q), q^{-1}a_1(q))$ associated with $a_2(q) = b_2(q)$. Therefore, we perform the following theorem for this case.

Theorem 4.14 *Let $a = qb_2(q)$ be the zero of $\sigma_2(q^{-1}x, q)$ and $b = a_1(q)$ of $\sigma_1(x, q)$ and assume that $0 < a_2(q) \leq b_2(q) < q^{-1}a_1(q)$ and $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} > 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b), i.e., they satisfy the orthogonality (4.9) and (4.11) with respect to the q-weight function*

$$\rho(x, q) = x^{\alpha} x^{\log_q x} (qa_1^{-1}(q)x, qa_2(q)/x, qb_2(q)/x; q)_\infty > 0, \; x \in (a, b) \tag{4.32}$$

$q^{\alpha} = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).*

This case corresponds to the case IIb1 in Chapter 11 of [35, pages 337 and 361].

An example of such family is the quantum q-Kravchuk polynomials [35] where $a_1(q) = q^{-N}, a_2(q) = p^{-1}q^{-N-1}, b_2(q) = 1$,

$$\sigma_1(x, q) = -q^{-2}(x - q^{-N}), \quad \sigma_2(x, q) = (x - 1)(px - q^{-N-1}),$$

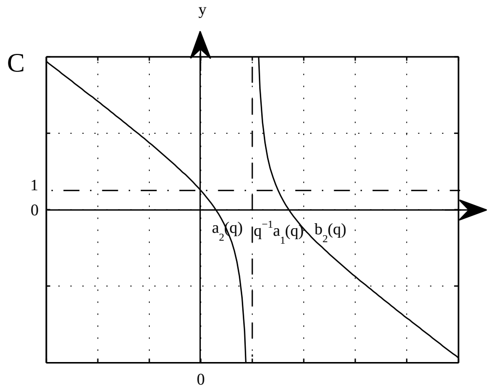
$$\tau(x, q) = -\frac{p}{1-q}x + \frac{p - q^{-1} + q^{-N-1}}{1-q}, \quad \lambda_n(q) = \frac{p}{1-q}[n]_q.$$

Quantum q-Kravchuk polynomials are orthogonal on $(1, q^{-N-1})$ and the conditions $\Lambda_q > 0$ and $0 < a_2(q) \leq b_2(q) < a_1(q)$ give us the following restriction for the parameters $p \geq q^{-N-1}$. By means of Theorem 4.4 d) we can write the orthogonality of quantum q-Kravchuk polynomials

$$\int_1^{q^{-N-1}} x^{\alpha+N+\log_q x}(q^{N+1}x, q/x, p^{-1}q^{-N}/x; q)_\infty K_m^{qtm}(x; p, N; q) K_n^{qtm}(x; p, N; q) d_{q^{-1}}x = (q^{-1} - 1)$$
$$\times (-1)^n p^N q^{\binom{N+1}{2} - \binom{n+1}{2} + Nn} \frac{(q; q)_{N-n}(q, pq; q)_n}{(q, pq; q)_N} (q, p^{-1}q^{-N}, q^{N+1}; q)_\infty \delta_{mn} \tag{4.33}$$

together with $p = q^{\alpha} \geq q^{-N-1}$. Notice from Theorem 4.4 d) that one can also write the orthogonality with finite sum by applying (2.31) to (4.33)

$$\sum_{x=0}^{N} \frac{(pq; q)_{N-x}}{(q; q)_x (q; q)_{N-x}} (-1)^{N-x} q^{\binom{x}{2}} K_m^{qtm}(q^{-x}; p, N; q) K_n^{qtm}(q^{-x}; p, N; q) = (-1)^n p^N$$
$$\times q^{\binom{N+1}{2} - \binom{n+1}{2} + Nn} \frac{(q; q)_{N-n}(q, pq; q)_n}{(q, q; q)_N} \delta_{mn}. \tag{4.34}$$



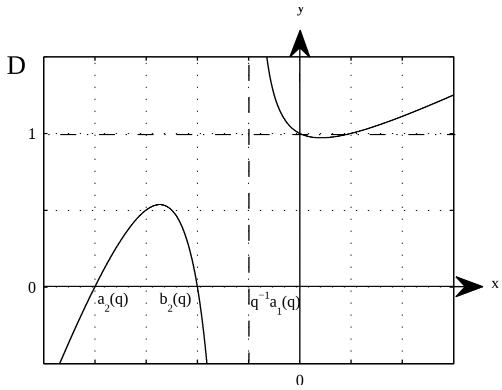


Figure 4.17: Case 2(a). The function $f(x,q)$ with C: $\Lambda_q > 0, 0 < a_2(q) < q^{-1}a_1(q) < b_2(q)$, D: $\Lambda_q < 0, a_2(q) < b_2(q) < q^{-1}a_1(q) < 0$.

Case 2(a).C: $0 < a_2(q) < q^{-1}a_1(q) < b_2(q)$, $\Lambda_q > 0$. This case is placed in Figure 4.17C. Then, according to the graph of f in Figure 4.17C, we analyse the possible intervals in which the q-weight function is defined. Thus, let us start with excluding the negative intervals $(a_2(q), q^{-1}a_1(q))$ and $(b_2(q), \infty)$. Afterwards, one can also eliminate the rest two intervals $(-\infty, a_2(q))$ and $(q^{-1}a_1(q), b_2(q))$ with the help of the Remark 4.5.6 (we first need to use transformation $x = -t$) and Remark 4.5.4, respectively. As a result, this case does not generate a suitable interval where ρ is defined with needed properties.

Case 2(a).D: $a_2(q) < b_2(q) < q^{-1}a_1(q) < 0$, $\Lambda_q < 0$. The graph for this case is placed in Figure 4.17D. We start to skip the negative intervals $(-\infty, a_2(q))$ and $(b_2(q), q^{-1}a_1(q))$. Next, we eliminate the interval $(a_2(q), b_2(q))$ by use of Remark 4.5.2. We last analyse the interval $(q^{-1}a_1(q), \infty)$ which is the one identified in Theorem 4.4 g). Thus, we anticipate that it could be possible to have a suitable ρ on this interval. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = q^{-1}a_1(q)$, then it follows that ρ is increasing on $(q^{-1}a_1(q), x_0)$ and decreasing on (x_0, ∞) which leads to $\rho \to 0$ as $x \to \infty$ since $\rho(qx,q)/\rho(x,q) \to \infty$. As a result of the above discussion we can also construct Figure 4.18 for the q-weight function.

We infer from Figure 4.18 that the boundary conditions (3.119) and (3.117) hold at $x = a_1(q)$ and $\rho \to 0$ as $x \to \infty$, but we still need to ensure $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ by using the *extended* q-Pearson equation (4.20). For this reason, we get Figure 4.19 by applying the same procedure to the *extended* q-Pearson equation (4.20) which helps to construct the Figure 4.20 for the boundary condition (3.119).

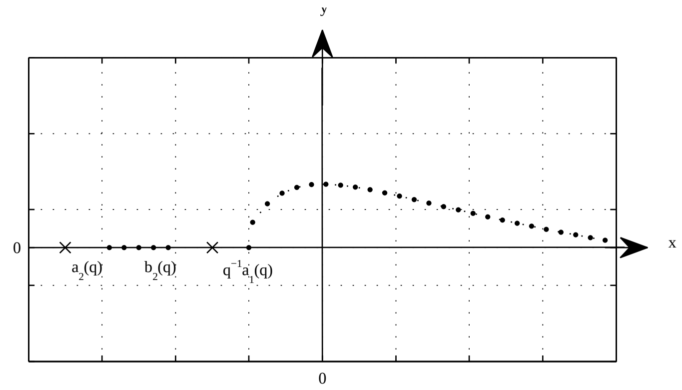


Figure 4.18: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.17D.

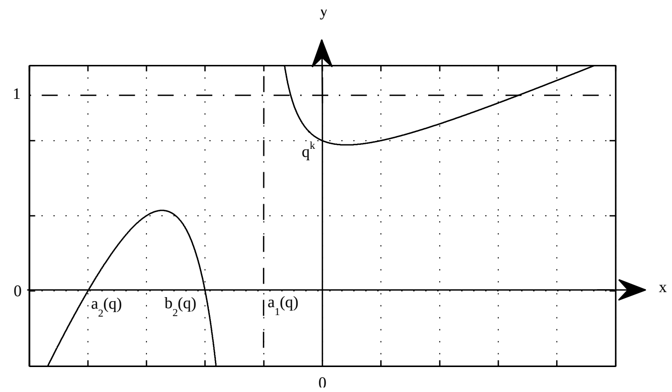


Figure 4.19: A figure of $g(x, q)$ related to Figure 4.17D.

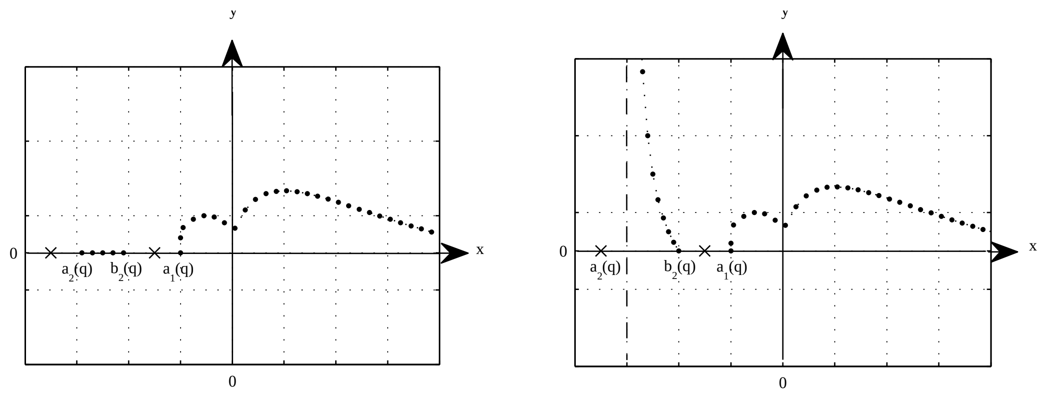


Figure 4.20: A figure of $\sigma_1(x, q)\rho(x, q)x^k = \sigma_2(q^{-1}x, q)\rho(q^{-1}x, q)x^k$ related to Figure 4.19.

Notice from Figure 4.19 that it looks like the one represented in Figure 4.17D. That's why, it leads to the similar properties, i.e., we get $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$. Consequently, we get a suitable ρ on the interval $(a_1(q), \infty)$ supported at the points aq^k and $q^{\mp k}$, $k = 0, 1, ...$ (see Theorem 4.4 g)).

Notice that Case 2(b) includes the same graphs with Case 2(a).D together with $a_2(q) = b_2(q)$. That's why, Case 2(b) also produce the interval $(a_1(q), \infty)$ associated with $a_2(q) = b_2(q)$.

We next construct the following theorem indicating the result discussed in Case 2(a).D.

Theorem 4.15 *Let $a = a_1(q)$ of $\sigma_1(x,q)$ and $b = \infty$ and assume that $a_2(q) \leq b_2(q) < q^{-1}a_1(q) < 0$, $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.15)*

$$\int_{a_1(q)}^{\infty} P_m(x,q)P_n(x,q)\rho(x,q)d_qx = (1-q)\left(a_2(q)b_2(q)a_1^{-1}(q)\right)^{2n}(q, a_2^{-1}(q)a_1(q), b_2^{-1}(q)a_1(q); q)_n$$
$$\times \frac{(q, a_1(q), qa_1^{-1}(q), a_2^{-1}(q)b_2^{-1}(q)a_1(q), qa_2(q)b_2(q)a_1^{-1}(q); q)_\infty}{(a_2^{-1}(q)a_1(q), b_2^{-1}(q)a_1(q), a_2^{-1}(q), b_2^{-1}(q), qa_2(q), qb_2(q); q)_\infty} q^{-n(2n-1)}\delta_{mn} \quad (4.35)$$

with respect to the q-weight function

$$\rho(x,q) = \frac{(a_1^{-1}(q)qx; q)_\infty}{(a_2^{-1}(q)x, b_2^{-1}(q)x; q)_\infty} > 0, \; x \in (a,b) \quad (4.36)$$

$q^\alpha = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q)}{\sigma_1'(0,q)a_1(q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 g)).*

This case corresponds to the case VIa2 in Chapter 10 of [35, pages 285 and 315].

We note that this case leads to the new orthogonality on the interval $(a_1(q), \infty)$ which does not appear in the q-Askey scheme. Actually, this case is analog to the one leading to the q-Meixner polynomials. They differ by the orthogonality interval.

Case 2(c): $a_1(q) < 0, a_2(q), b_2(q) \in \mathbb{C}$, $\Lambda_q < 0$. The situation of this case is represented in Figure 4.21. Notice from Figure 4.21 that $(q^{-1}a_1(q), \infty)$ is the only interval where f is positive. Observe that this interval is exactly same with the one represented in Case 2(a).D. Notice also from the Figure 4.21 and Figure 4.17D that they both have same property on the interval $(q^{-1}a_1(q), \infty)$. Then, the result represented in Figure 4.17D is valid for this case also, i.e., there exists a suitable ρ on $(q^{-1}a_1(q), \infty)$ which is also seen from Figure 4.22 (graph

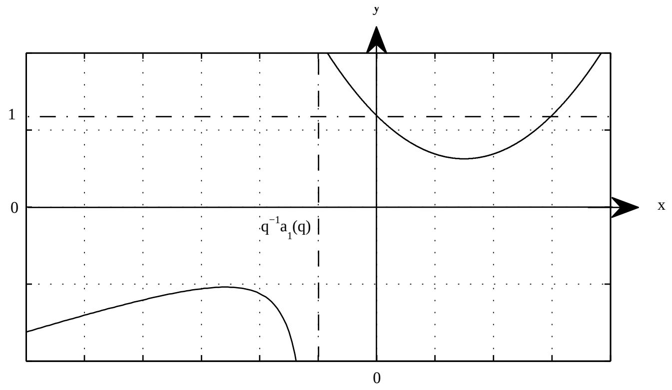


Figure 4.21: Case 2(c). The function $f(x,q)$ with $\Lambda_q < 0, a_1(q) < 0, a_2(q), b_2(q) \in \mathbb{C}$.

of ρ) and Figure 4.23 (graph of the *extended* q-Pearson equation and $\sigma_1(x,q)\rho(x,q)x^k = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k$).

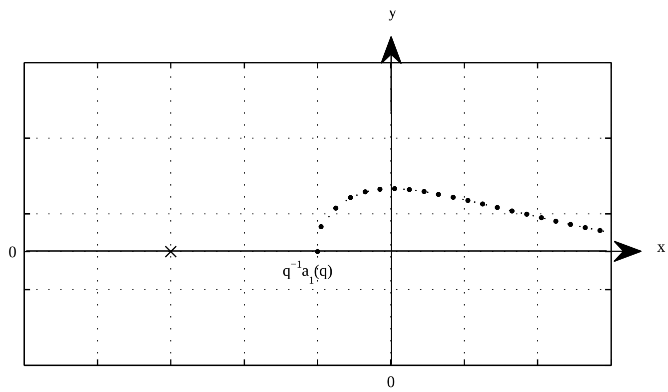


Figure 4.22: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.21.

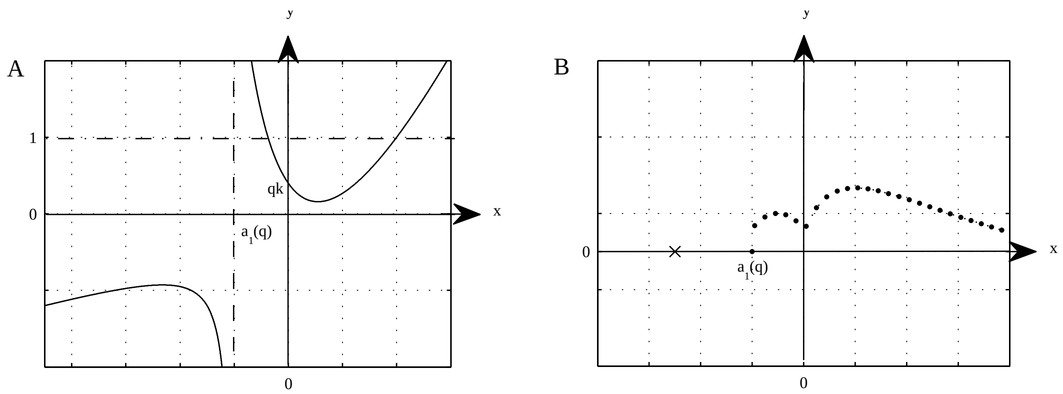


Figure 4.23: A figure of A: $g(x,q)$, B: $\sigma_1(x,q)\rho(x,q)x^k = \sigma_2(q^{-1}x,q)\rho(q^{-1}x,q)x^k$ related to Figure 4.21.

Thus, this case leads to the following theorem.

Theorem 4.16 *Let $a = a_1(q)$ of $\sigma_1(x,q)$ and $b = \infty$ and assume that $a_1(q) < 0, a_2(q), b_2(q) \in \mathbb{C}$, $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.15)*

$$\int_{a_1(q)}^{\infty} P_m(x,q)P_n(x,q)\rho(x,q)d_qx = (1-q)\left(a_2(q)b_2(q)a_1^{-1}(q)\right)^{2n}(q, a_2^{-1}(q)a_1(q), b_2^{-1}(q)a_1(q); q)_n \times \frac{(q, a_1(q), qa_1^{-1}(q), a_2^{-1}(q)b_2^{-1}(q)a_1(q), qa_2(q)b_2(q)a_1^{-1}(q); q)_\infty}{(a_2^{-1}(q)a_1(q), b_2^{-1}(q)a_1(q), a_2^{-1}(q), b_2^{-1}(q), qa_2(q), qb_2(q); q)_\infty} q^{-n(2n-1)}\delta_{mn} \quad (4.37)$$

with respect to the q-weight function

$$\rho(x,q) = \frac{(a_1^{-1}(q)qx; q)_\infty}{(a_2^{-1}(q)x, b_2^{-1}(q)x; q)_\infty} > 0, \; x \in (a,b) \quad (4.38)$$

$q^\alpha = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)a_2(q)b_2(q)}{\sigma_1'(0,q)a_1(q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 g)).*

This case corresponds to the case VIa1 in Chapter 10 of [35, pages 285 and 315].

We note that this case leads to the new orthogonality on the interval $(a_1(q), \infty)$ which does not appear in the q-Askey scheme. Actually, this case is analog to the one leading to the q-Meixner polynomials. They differ by the orthogonality interval.

4.2.3 Quadratic Case

Assume that $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)]$, $a_1(q) < b_1(q)$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$.

Remark 4.17 *We remark that $\sigma_2(x,q)$ follows from (3.11)*

$$\begin{aligned}\sigma_2(x,q) &= q\left\{\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right]x^2 - \left[\frac{1}{2}\sigma_1''(0,q)(a_1(q)+b_1(q))\right.\right.\\ &- \left.\left.(1-q^{-1})\tau(0,q)\right]x + \frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)\right\}. \quad (4.39)\end{aligned}$$

Note that

- $\deg[\sigma_2(x,q)] = 2$ when $\tau'(0,q) \neq -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$,
- $\deg[\sigma_2(x,q)] = 1$ when $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$ and $\tau(0,q) \neq \frac{\frac{1}{2}\sigma_1''(0,q)(a_1(q)+b_1(q))}{(1-q^{-1})}$,

- $\deg[\sigma_2(x,q)] = 0$ if $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$, $\tau(0,q) = \frac{\frac{1}{2}\sigma_1''(0,q)(a_1(q)+b_1(q))}{(1-q^{-1})}$ and $\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q) \neq 0$.

That's why, the q-Jacobi type q-polynomials of the 1st kind are the q-Jacobi, the q-Laguerre and the q-Hermite type q-polynomials of the 2nd kind (see Table 4.1).

4.2.3.1 The q-Classical $\emptyset$-Jacobi/Jacobi Polynomials

We deal with every degree of $\sigma_2(x,q)$ starting with q-Jacobi/q-Jacobi case by letting $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)]$, $a_1(q) < b_1(q)$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$. Then, $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q)$ where

$$\begin{aligned} \frac{1}{2}\sigma_2''(0,q) &= q\Big[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\Big],\ \sigma_2(0,q) = \frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q), \\ \sigma_2'(0,q) &= -q\Big[\frac{1}{2}\sigma_1''(0,q)(a_1(q)+b_1(q)) - (1-q^{-1})\tau(0,q)\Big]. \end{aligned}$$

Thus, the q-Pearson equation follows from (4.1)

$$\begin{aligned} \frac{\rho(qx,q)}{\rho(x,q)} &= \frac{\Big[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]x^2 - \Big[a_1(q)+b_1(q)-(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]x + a_1(q)b_1(q)}{[qx-a_1(q)][qx-b_1(q)]} \\ &= \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}. \end{aligned} \tag{4.40}$$

Let denote by Δ_q the constant

$$\Delta_q := \Big[a_1(q)+b_1(q)-(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]^2 - 4a_1(q)b_1(q)\Big[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big].$$

Notice that the nominator in (4.40) can be writen as

$$\left[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right][x-a_2(q)][x-b_2(q)],\ \text{if } \Delta_q \neq 0.$$

In fact, if $\Delta_q > 0$ then $a_2(q)$ and $b_2(q) \in \mathbb{R}$ and we assume, without losing any generality that $a_2(q) < b_2(q)$. If $\Delta_q < 0$, $a_2(q)$ and $b_2(q) \in \mathbb{C}$.

If $\Delta_q = 0$ then the nominator takes the form $\left[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right][x-a_2(q)]^2$, $a_2(q) \in \mathbb{R}$.

We are interesting in the behavior of zeros of the nominator of (4.40) (and so, the zeros of $\rho(qx,q)/\rho(x,q)$). This is given in the following straightforward lemma.

Lemma 4.18 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.40) and set*

$$\Lambda_q = q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0.$$

Then, the roots of the equation $f(x,q) = 0$ have the following properties;

1. *If $\Lambda_q > 0$ and $a_1(q) < 0 < b_1(q)$, f has two real and distinct roots with opposite signs.*
2. *If $\Lambda_q > 0$ and $0 < a_1(q) < b_1(q)$, there exist three possibilities,*
   (a) *if $\Delta_q > 0$, f has two real roots with the same signs,*
   (b) *if $\Delta_q = 0$, f has two equal real roots,*
   (c) *if $\Delta_q < 0$, f has two complex roots.*
3. *If $\Lambda_q < 0$ and $a_1(q) < 0 < b_1(q)$, there exist three possibilities,*
   (a) *if $\Delta_q > 0$, f has two real roots with the same signs,*
   (b) *if $\Delta_q = 0$, f has two equal real roots,*
   (c) *if $\Delta_q < 0$, f has two complex roots.*
4. *If $\Lambda_q < 0$ and $0 < a_1(q) < b_1(q)$, f has two real distinct roots with opposite signs.*

Remark 4.19 *Notice that $y = \Lambda_q$ is the horizontal asymptote of the function $f(x,q)$.*

Our next step is to analyse all possible graphs of $\rho(qx,q)/\rho(x,q)$ in (4.40) according to all possible relative positions of the zeros of σ_1 and σ_2. We assume that the conditions of Lemma 4.18 hold.

Before starting the analysis notice that $\rho(qx,q)/\rho(x,q)$ always intercepts the y-axis at the point $y = 1$ since $\sigma_2(0,q) = q\sigma_1(0,q)$ (i.e., the constant terms of σ_1 and σ_2 are the same). In addition, to give a full description of the items 1 and 2 of Lemma 4.18 we need to split them in two separate cases: case i) when $\Lambda_q > 1$ and case ii) when $0 < \Lambda_q < 1$.

Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.40).

Case 1.i)A: $q^{-1}a_1(q) < a_2(q) < 0 < b_2(q) < q^{-1}b_1(q)$, $\Lambda_q > 1$. The graph of f for this case is represented in Figure 4.24A. Let us consider now the possible intervals in which we can have

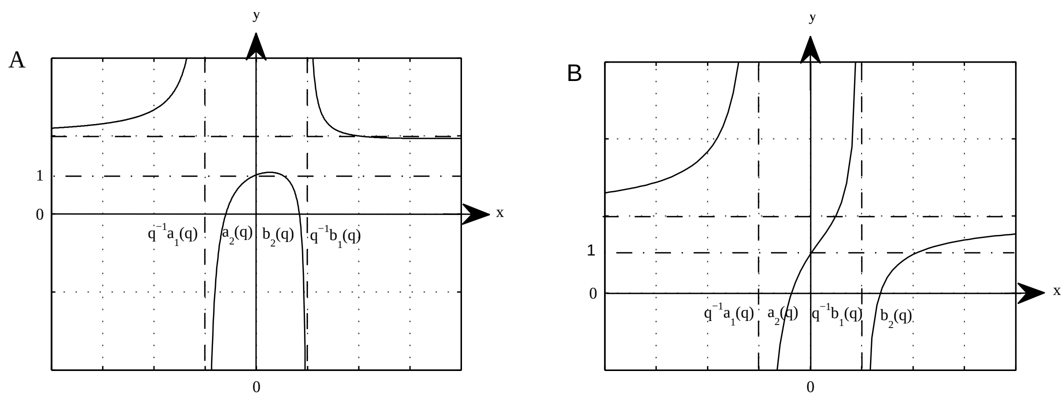


Figure 4.24: Case 1.i) The function $f(x, q)$ with $\Lambda_q > 1$. A: $q^{-1}a_1(q) < a_2(q) < 0 < b_2(q) < q^{-1}b_1(q)$, B: $q^{-1}a_1(q) < a_2(q) < 0 < q^{-1}b_1(q) < b_2(q)$.

a suitable q-weight function ρ. As we already mentioned, they are defined by the zeros of the polynomials σ_1 and σ_2. First of all, notice that since ρ should be a positive weight function and f is negative in the intervals $(q^{-1}a_1(q), a_2(q))$ and $(b_2(q), q^{-1}b_1(q))$ they are not suitable. The interval $(a_2(q), b_2(q))$ can not be used due to Remark 4.5.4 since the boundary conditions can not be satisfied. The same happens with the interval $(q^{-1}b_1(q), \infty)$ (see Remark 4.5.5), and by the symmetry property with $(-\infty, q^{-1}a_1(q))$. Therefore, this case does not lead to a suitable q-weight function with the needed properties.

Case 1.i)B: $q^{-1}a_1(q) < a_2(q) < 0 < q^{-1}b_1(q) < b_2(q)$, $\Lambda_q > 1$. Let us now analyse the situation given in Figure 4.24B. The positivity of ρ allows us to skip the intervals $(q^{-1}a_1(q), a_2(q))$ and $(q^{-1}b_1(q), b_2(q))$. Using Remark 4.5.5 (we first need to do the transformation $x = -t$) as well as Remark 4.5.3 we can eliminate the intervals $(-\infty, q^{-1}a_1(q))$ and $(a_2(q), q^{-1}b_1(q))$, respectively. Let consider now the last interval $(b_2(q), \infty)$. Notice that it coincides with the one described in Theorem 4.4 h), so here it could be possible to have a suitable q-weight function ρ. Notice also that since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 > x = b_2(q)$, then from Figure 4.24B it follows that ρ is decreasing on $(-\tau(0, q)/\tau'(0, q), \infty)$. Since $\rho(qx, q)/\rho(x, q)$ has a finite limit as $x \to +\infty$, we have the chance that $\rho \to 0$ as $x \to \infty$, but it is not enough to assure that ρ satisfies the boundary condition at $+\infty$. In fact, as it is stated in Theorem 3.31, we should check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to \infty$. To this end, we use, instead of the q-Pearson equation (4.1), the following *extended* q-Pearson equation (4.20):

$$
\begin{aligned}
g(x,q) : &= \frac{\sigma_1(qx,q)\rho(qx,q)(qx)^k}{\sigma_1(x,q)\rho(x,q)x^k} \qquad (4.41)\\
&= q^k \frac{\left[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x^2 - \left[a_1(q)+b_1(q)-(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x + a_1(q)b_1(q)}{(x-a_1(q))(x-b_1(q))},
\end{aligned}
$$

which is represented in Figure 4.25.

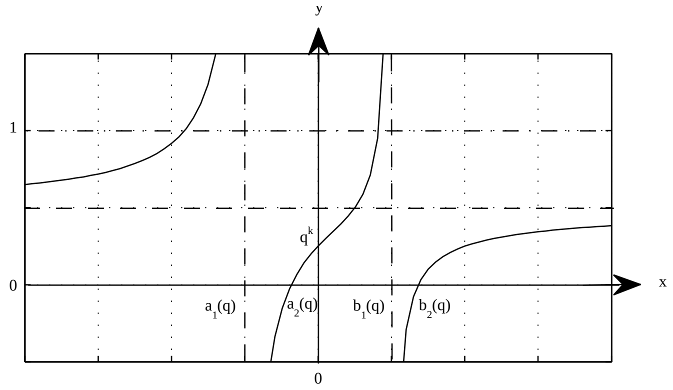


Figure 4.25: A figure of $g(x,q)$ corresponding to Figure 4.24B.

If we now provide a similar anaysis with the function g defined in (4.41), we see from Figure 4.25 that, for k large enough, g is an increasing positive function with a positive limit. Therefore $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$ as $x \to \infty$ and the condition (3.117) does not hold. Thus we can not use this interval for constructing a q-weight function ρ.

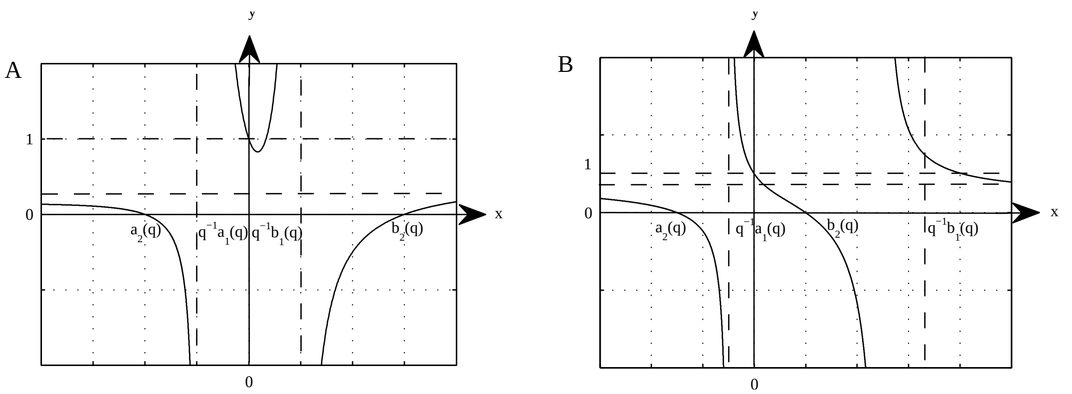


Figure 4.26: Case 1.ii) The function $f(x,q)$ with $0 < \Lambda_q < 1$, A: $a_2(q) < q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < b_2(q)$, B: $a_2(q) < q^{-1}a_1(q) < 0 < b_2(q) < q^{-1}b_1(q)$.

Case 1.ii)A: $a_2(q) < q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < b_2(q)$, $0 < \Lambda_q < 1$. This case is represented in Figure 4.26A. Let us examine all possible intervals in order to find in which ones there could be defined a convenient q-weight function. First of all by the positivity property of ρ we eliminate the intervals $(a_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), b_2(q))$. Notice from Figure 4.26A that the interval $(b_2(q), \infty)$ coincides with the one described in Theorem 4.4 h). However, $f(x, q) < 1$ on this interval thus ρ is increasing on $(b_2(q), \infty)$ which leads to $\rho \not\to 0$ as $x \to \infty$ Therefore $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to \infty$. The same happens for the interval $(-\infty, a_2(q))$ by symmetry properties.

Let us consider the last interval $(q^{-1}a_1(q), q^{-1}b_1(q))$. Observe that this case is described in Theorem 4.4 a), then it could be possible to have a suitable q-weight function ρ. Notice that since $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $q^{-1}a_1(q) < x_0 < q^{-1}b_1(q)$, then ρ is increasing on $(q^{-1}a_1(q), x_0)$ and decreasing on $(x_0, q^{-1}b_1(q))$ where $\rho \to 0$ as $x \to q^{-1}a_1(q)^+$ and $x \to q^{-1}b_1(q)^-$ since $\rho(qx, q)/\rho(x, q) \to \infty$. According to the above discussion we can sketch the graph of ρ which is represented in Figure 4.27 assuming a positive initial value for the q-weight function in each interval.

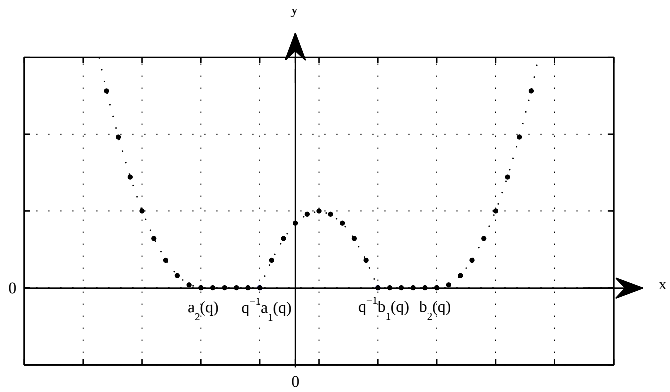


Figure 4.27: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.26A.

We also figure out from Figure 4.27 that, there exists a q-weight function on the interval $(a_1(q), b_1(q))$ supported at the points $a_1(q)q^k$, $b_1(q)q^k$, $k = 0, 1, ...$ (see Theorem 4.4 a)) since the boundary condition (3.115) holds at $x = a_1(q)$ and $x = b_1(q)$. Since this interval is finite, then there is no needed to look at the *extended* q-Pearson equation. Thus, we have the following Theorem.

Theorem 4.20 *Let* $a = a_1(q)$, $b = b_1(q)$, *be the zeros of* $\sigma_1(x, q)$ *and assume that* $a_2(q) < q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < b_2(q)$, *and* $0 < \Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 1$. *Then, there exists a sequence of polynomials* $(P_n)_n$ *orthogonal on* (a, b), *i.e., they satisfy the orthogonality*

(4.6) with respect to the q-weight function

$$\rho(x,q) = \frac{(qa_1^{-1}(q)x, qb_1^{-1}(q)x; q)_\infty}{(a_2^{-1}(q)x, b_2^{-1}(q)x; q)_\infty} > 0, \quad x \in (a_1(q), b_1(q)) \tag{4.42}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 a)).

This case corresponds to the case VIIa1 in Chapter 10 of [35, pages 292 and 318].

An example of such family is the big q-Jacobi polynomials [35] where $a_1(q) = cq$, $b_1(q) = aq$, $a_2(q) = b^{-1}c$, $b_2(q) = 1$,

$$\sigma_1(x,q) = q^{-2}(x-aq)(x-cq), \quad \sigma_2(x,q) = aq(x-1)(bx-c),$$

$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{a(bq-1)+c(aq-1)}{1-q}, \quad \lambda_n(q) = q^{-n}[n]_q \frac{1-abq^{n+1}}{q-1}.$$

Big q-Jacobi polynomials are orthogonal on (cq, aq) and the conditions constructed according to the identity

$$\frac{\sigma_1(qx,q)\rho(qx,q)}{\sigma_1(x,q)\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(x,q)} \tag{4.43}$$

as $0 < q^2\Lambda_q < 1$ and $a_2(q) < a_1(q) < 0 < b_1(q) < b_2(q)$ give us the following restriction for the parameters $c < 0$, $0 < b < q^{-1}$, $0 < a < q^{-1}$. By means of Theorem 4.4 a) we can write the orthogonality of big q-Jacobi polynomials

$$\int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} P_m(x;a,b,c;q)P_n(x;a,b,c;q)d_qx = aq(1-q)\frac{1-abq}{1-abq^{2n+1}}$$
$$\times \frac{(q, abq^2, a^{-1}c, ac^{-1}q; q)_\infty}{(aq, bq, cq, abc^{-1}q; q)_\infty} \frac{(q, bq, abc^{-1}q; q)_n}{(aq, cq, abq; q)_n}(-acq^2)^n q^{\binom{n}{2}}\delta_{mn} \tag{4.44}$$

together with $c < 0$, $0 < b < q^{-1}$, $0 < a < q^{-1}$ [36].

Case 1.ii)B: $a_2(q) < q^{-1}a_1(q) < 0 < b_2(q) < q^{-1}b_1(q)$, $0 < \Lambda_q < 1$. The graph of f for this case is represented in Figure 4.26B. By the positivity of ρ we eliminate the intervals $(a_2(q), q^{-1}a_1(q))$ and $(b_2(q), q^{-1}b_1(q))$. Moreover, $(q^{-1}a_1(q), b_2(q))$ can not be used due to Remark 4.5.4 since the boundary conditions can not be satisfied. The same happens with the interval $(q^{-1}b_1(q), \infty)$ (see Remark 4.5.5). For the last interval $(-\infty, a_2(q))$, an analogous analysis as the one that has been done in Case 1ii)A yields that it is also not suitable for constructing ρ.

Case 2.(a)i)A: $0 < a_2(q) < q^{-1}a_1(q) < b_2(q) < q^{-1}b_1(q)$, $\Lambda_q > 1$. The graph of f for this case is represented in Figure 4.28A. Let us now analyse the situations for this case. We

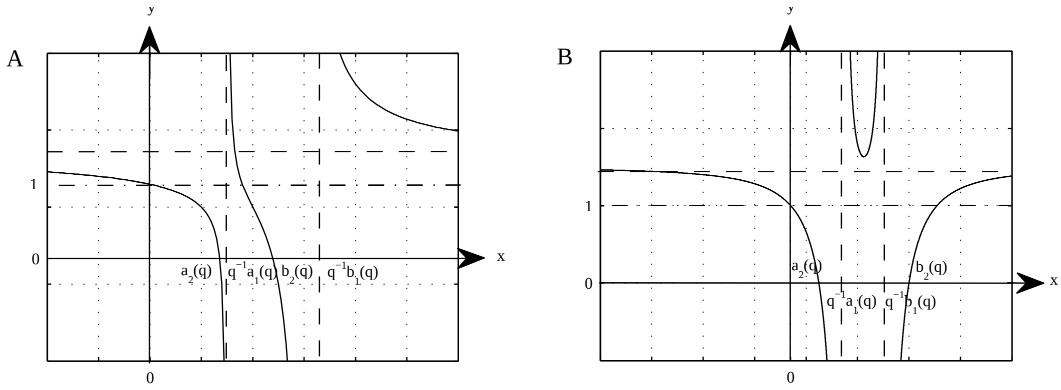


Figure 4.28: Case 2.(a)i) The function $f(x, q)$ with $\Lambda_q > 1$, A: $0 < a_2(q) < q^{-1}a_1(q) < b_2(q) < q^{-1}b_1(q)$, B: $0 < a_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q) < b_2(q)$.

first consider the positivity of ρ which allows to eliminate the intervals $(a_2(q), q^{-1}a_1(q))$ and $(b_2(q), q^{-1}b_1(q))$. The interval $(-\infty, a_2(q))$ is also omitted due to Remark 4.5.6 (by symmetry) since the boundary condition does not hold. The same happens with the intervals $(q^{-1}a_1(q), b_2(q))$ and $(q^{-1}b_1(q), \infty)$ (see Remark 4.5.4 and Remark 4.5.5). As a result, this case does not lead to any convenient ρ.

Case 2.(a)i)B: $0 < a_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q) < b_2(q)$, $\Lambda_q > 1$. This situation appears in Figure 4.28B. Let us examine the possible intervals in which a suitable ρ is defined. We begin with considering positivity of ρ which allows us to eliminate the intervals $(a_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), b_2(q))$. The same happens by the symmetry property with the interval $(-\infty, a_2(q))$ (Remark 4.5.6). $(q^{-1}a_1(q), q^{-1}b_1(q))$ also can not be used due to Remark 4.5.1. For the last interval $(b_2(q), \infty)$, it is seen from Figure 4.28B that $f(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 > x = b_2(q)$, then it follows that ρ is increasing on $(b_2(q), x_0)$ and decreasing on (x_0, ∞). Since ρ is decreasing on (x_0, ∞) and f has a finite limit as $x \to \infty$, then we have a chance that $\rho \to 0$ as $x \to \infty$, but as we mentioned before it is not enough to assure that the boundary condition holds at $+\infty$. That is, we need the analysis of the *extended* q-Pearson equation (4.41) which leads to the analogous result as we obtained in Case 1.i)B that $\sigma_2(x, q)\rho(x, q)x^k \not\to 0$ as $x \to \infty$. That's why we can not obtain a convenient ρ with the needed properties.

Case 2.(a)i)C: $0 < q^{-1}a_1(q) < a_2(q) < b_2(q) < q^{-1}b_1(q)$, $\Lambda_q > 1$. This case is represented in Figure 4.29C. Let us start with performing the positivity property which provides to eliminate the intervals $(q^{-1}a_1(q), a_2(q))$ and $(b_2(q), q^{-1}b_1(q))$. Moreover, the intervals $(a_2(q), b_2(q))$ and $(q^{-1}b_1(q), \infty)$ are both excluded due to Remark 4.5.2 and Remark 4.5.5, respectively. Let us

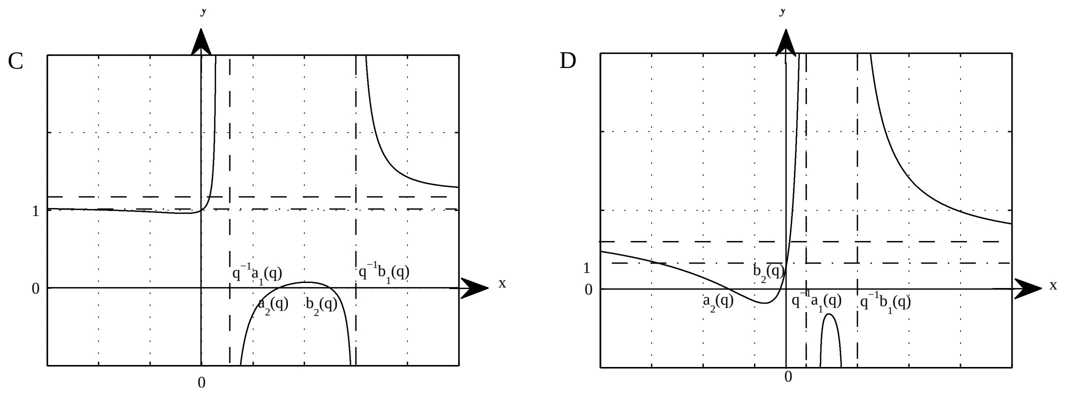


Figure 4.29: Case 2.(a)i) The function $f(x,q)$ with $\Lambda_q > 1$, C: $0 < q^{-1}a_1(q) < a_2(q) < b_2(q) < q^{-1}b_1(q)$, D: $a_2(q) < b_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q)$.

consider now the last interval $(-\infty, q^{-1}a_1(q))$. Notice that it coincides with the one described in Theorem 4.4 g) with the symmetry property. Then, here it could be possible to get a suitable ρ. Notice also that $f(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 < x = q^{-1}a_1(q)$. Then, it follows from Figure 4.29C that ρ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, q^{-1}a_1(q))$. Since ρ is increasing on $(-\infty, x_0)$ and $\rho(qx,q)/\rho(x,q)$ has a finite limit as $x \to -\infty$, we have chance that $\rho \to 0$ as $x \to -\infty$, but it is not enough to assure that ρ satisfies the boundary condition at $-\infty$. In fact, as we said before that we need to apply the analogous analysis to the *extended* q-Pearson equation in order to check that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to -\infty$. One can easily see from the *extended* q-Pearson equation that $\sigma_1(x,q)\rho(x,q)x^k$ is decreasing on $(-\infty, q^{-1}a_1(q))$ for k large enough. Then, $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$ as $x \to -\infty$. As a result, this case does not lead to any suitable ρ.

Case 2.(a)i)D: $a_2(q) < b_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q)$, $\Lambda_q > 1$. This situation appears in Figure 4.29D. We deal with the possible intervals in which we can have a suitable q-weight function ρ. Notice that since ρ should be a positive weight function and f is negative in the intervals $(a_2(q), b_2(q))$ and $(q^{-1}a_1(q), q^{-1}b_1(q))$ they are not suitable. Furthermore, the intervals $(b_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), \infty)$ can not be used due to Remark 4.5.3 and Remark 4.5.5, respectively. As the last interval, let us look at $(-\infty, a_2(q))$. Notice that it coicides with the one described in Theorem 4.4 h) by the symmetry. That's why, it could be possible to have a suitable ρ. However, analogous analysis including the *extended* q-Pearson equation (4.41) as we did before enables us to see that the boundary condition is not satisfied as $x \to -\infty$. Therefore, we can not obtain a convenient q-weight function with the needed properties.

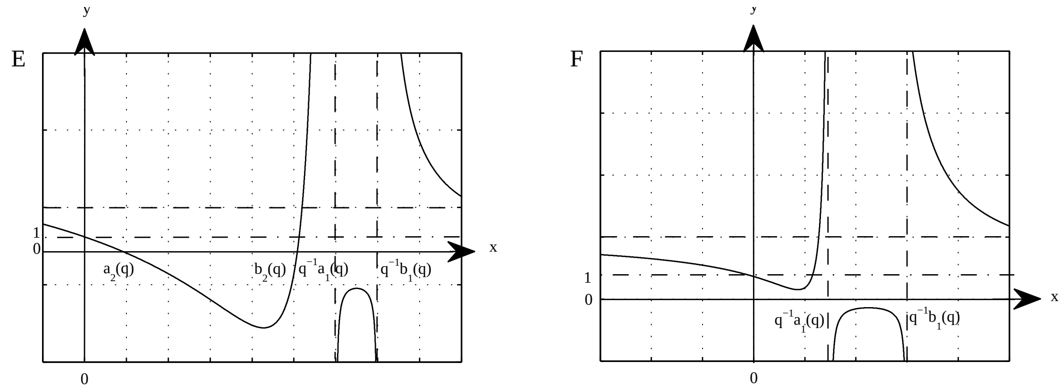


Figure 4.30: The function $f(x,q)$ with $\Lambda_q > 1$, Case 2.(a)i)E: $0 < a_2(q) < b_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q)$, Case 2.(c)i)F: $0 < q^{-1}a_1(q) < q^{-1}b_1(q)$, $a_2(q), b_2(q) \in \mathbb{C}$.

Case 2.(a)i)E: $0 < a_2(q) < b_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q)$, $\Lambda_q > 1$. This situation is identified by Figure 4.30E. Analogously, we begin with excluding the negative intervals $(a_2(q), b_2(q))$ and $(q^{-1}a_1(q), q^{-1}b_1(q))$. In fact, we also eliminate the intervals $(-\infty, a_2(q))$ and $(q^{-1}b_1(q), \infty)$ due to Remark 4.5.6 with the symmetry property and Remark 4.5.5, respectively. Let us now consider the last interval $(b_2(q), q^{-1}a_1(q))$. Notice that it coincides with the one described in Theorem 4.4 d). Then, it could be possible to have a suitable ρ. Notice also from Figure 4.30E that $f(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $b_2(q) < x_0 < q^{-1}a_1(q)$, then it follows that ρ is increasing on $(b_2(q), x_0)$ with $\rho(qb_2(q), q) = 0$ since $\rho(qb_2(q), q)/\rho(b_2(q), q) = 0$ and decreasing on $(x_0, q^{-1}a_1(q))$ with $\rho(x,q) \to 0$ as $x \to q^{-1}a_1(q)^{-}$since $\rho(qx,q)/\rho(x,q) \to \infty$ (see Figure 4.30E). As a result, above discussion leads to Figure 4.31 for ρ starting with positive initial ρ for each interval.

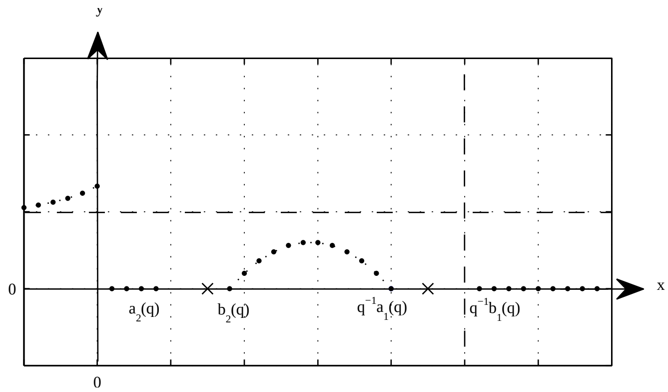


Figure 4.31: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.30E.

Figure 4.31 exhibits that $(qb_2(q), a_1(q))$ is possible interval in which ρ is defined with the needed properties having the supporting points $a_1(q)q^N = qb_2(q)$ or $qb_2(q)q^{-N} = a_1(q)$ (Theorem 4.4 d)). Notice that the boundary condition (3.119) and (3.117) hold since $a_1(q)$ is root of $\sigma_1(x,q)$ and $qb_2(q)$ of $\sigma_2(q^{-1}x,q)$. Observe that interval is finite, then there is no need to look at the *extended* q-Pearson equation. Notice that this case leads to a suitable ρ on $(qb_2(q), a_1(q))$ supported at the points $q^k a_1(q)$, $k = 0, 1, \ldots$.

Remark 4.21 *Note that Case 2(b)i) includes the same graphs with Case 2(a)i) which leads only to the interval $(qb_2(q), a_1(q))$ equivalent to the one in Case 2(a)i)E together with $a_2(q) = b_2(q)$ and we remark that in case of $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)]^2$ and $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)$ $[x-a_2(q)][x-b_2(q)]$, similar figures when $\sigma_1(x,q)$ has two distinct roots are derived together with $a_1(q) = b_1(q)$. However, only Figure 4.30E in company with $a_1(q) = b_1(q)$ leads to a suitable ρ on $(qb_2(q), a_1(q))$ or $(b_2(q), q^{-1}a_1(q))$ by performing the analogous analysis.*

Therefore, according as all results discussed above, we construct the following theorem.

Theorem 4.22 *Let $a = qb_2(q)$ be the zero of $\sigma_2(q^{-1}x,q)$ and $b = a_1(q)$ of $\sigma_1(x,q)$ and assume that $0 < a_2(q) \le b_2(q) < q^{-1}a_1(q) \le q^{-1}b_1(q)$, and $\Lambda_q = q^{-2}[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] > 1$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b) or $(q^{-1}a, q^{-1}b)$, i.e., they satisfy the orthogonality (4.9) and (4.11), respectively with respect to the q-weight function*

$$\rho(x,q) = x^{a}x^{\log_q x}(qa_1^{-1}(q)x, qb_1^{-1}(q)x, qa_2(q)/x, qb_2(q)/x; q)_\infty > 0, \ x \in (a,b) \tag{4.45}$$

$q^a = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).*

This case corresponds to the case IIIb5 in Chapter 11 of [35, page 343].

An example of such family is the q-Hahn polynomials [35] where $a_1(q) = q^{-N}$, $b_1(q) = \alpha q$, $a_2(q) = \beta^{-1}q^{-N-1}$, $b_2(q) = 1$,

$$\sigma_1(x,q) = q^{-2}(x-q^{-N})(x-\alpha q), \quad \sigma_2(x,q) = \alpha q(x-1)(\beta x - q^{-N-1}),$$

$$\tau(x,q) = \frac{1-\alpha\beta q^2}{(1-q)q}x + \frac{\alpha q^{-N}+\alpha\beta q-\alpha-q^{-N-1}}{1-q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-\alpha\beta q^{n+1}}{1-q}.$$

q-Hahn polynomials are orthogonal on $(1, q^{-N-1})$ and the conditions $q^2\Lambda_q > 1$ and $0 < a_2(q) \le b_2(q) < a_1(q) \le b_1(q)$ give us the following restriction for the parameters $\alpha \ge q^{-N-1}$, $\beta \ge$

q^{-N-1}. By means of Theorem 4.4 d) we can also write the orthogonality of q-Hahn polynomials with $q^b = \beta q$

$$\int_1^{q^{-N-1}} x^{b+N+\log_q x}(\alpha^{-1}x, q/x, q^{N+1}x, \beta^{-1}q^{-N}/x; q)_\infty Q_m(x;\alpha,\beta,N|q)Q_n(x;\alpha,\beta,N|q)d_{q^{-1}}x = (q^{-1}-1)$$
$$\times\frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}\frac{(q,\alpha\beta q^{N+2},\beta q;q)_n}{(\alpha q,q^{-N},\alpha\beta q;q)_n}\frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}}\frac{(q,q^{N+1};q)_\infty}{(\alpha q,\beta q^{N+1};q)_\infty}(-\alpha q)^n q^{\binom{n}{2}-Nn}\delta_{mn} \quad (4.46)$$

together with $\alpha \geq q^{-N-1}$, $\beta \geq q^{-N-1}$. Notice from Theorem 4.4 d) that one can also write the orthogonality with finite sum by applying (2.31) to (4.46)

$$\sum_{x=0}^{N}\frac{(\alpha q,q^{-N};q)_x}{(q,\beta^{-1}q^{-N};q)_x}(\alpha\beta q)^{-x}Q_m(q^{-x};\alpha,\beta,N|q)Q_n(q^{-x};\alpha,\beta,N|q) = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}$$
$$\times\frac{(q,\alpha\beta q^{N+2},\beta q;q)_n}{(\alpha q,q^{-N},\alpha\beta q;q)_n}\frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}}(-\alpha q)^n q^{\binom{n}{2}-Nn}\delta_{mn}. \quad (4.47)$$

Case 2.(c)i)F: $0 < q^{-1}a_1(q) < q^{-1}b_1(q)$, $a_2(q), b_2(q) \in \mathbb{C}$, $\Lambda_q > 1$. This situation is represented in Figure 4.30F. Since ρ should be positive, we first start to exclude the negative interval $(q^{-1}a_1(q), q^{-1}b_1(q))$. For the next step we consider the interval $(q^{-1}b_1(q,\infty))$ which is also eliminated due to Remark 4.5.5. Let us now deal with the last interval $(-\infty, q^{-1}a_1(q))$ which coincides with the one described in Theorem 4.4 g) by symmetry. That's why, here, it could be possible to have a convenient ρ. However, an analogous analysis as the one that has been done in Case 2(a)i)C leads to that it is also not suitable interval for defining ρ. As a result, we can not get a suitable ρ for this case.

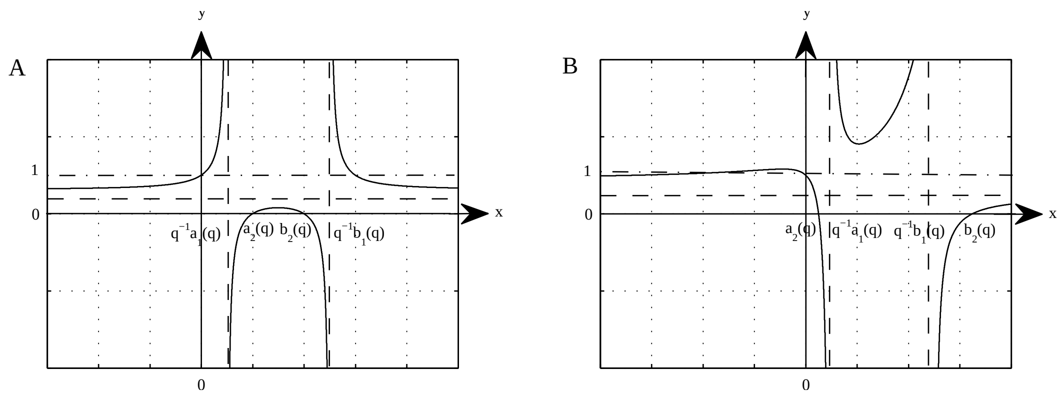


Figure 4.32: Case 2(a)ii) The function $f(x,q)$ with $0 < \Lambda_q < 1$, A: $0 < q^{-1}a_1(q) < a_2(q) < b_2(q) < q^{-1}b_1(q)$, B: $0 < a_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q) < b_2(q)$.

Case 2(a)ii)A: $0 < q^{-1}a_1(q) < a_2(q) < b_2(q) < q^{-1}b_1(q)$, $0 < \Lambda_q < 1$. The representation of this situation appears in Figure 4.32A. We make analogous analysis for all possible intervals in order to find which ones lead to a suitable ρ. Let us start with excluding the negative intervals $(q^{-1}a_1(q), a_2(q))$ and $(b_2(q), q^{-1}b_1(q))$ since ρ should be positive. On the other hand, with the help of Remark 4.5.2 and Remark 4.5.5 we see that $(a_2(q), b_2(q))$ and $(q^{-1}b_1(q), \infty)$ can not be used. Notice that the last interval $(-\infty, q^{-1}a_1(q))$ coincides with the one described in Theorem 4.4 g) by symmetry. However, notice from Figure 4.32A that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 > x = q^{-1}b_1(q)$. Therefore, it follows that ρ is decreasing on $(-\infty, q^{-1}a_1(q))$ which leads to that $\rho \not\to 0$ as $x \to -\infty$. However, as before since it is infinite interval we need to check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to -\infty$ by using the *extended q*-Pearson equation. But, the graph of g looks like the one represented in Figure 4.32A which indicates that $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ as $x \to -\infty$.

Case 2(a)ii)B: $0 < a_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q) < b_2(q)$, $0 < \Lambda_q < 1$. The situation of this case is represented in Figure 4.32B. Analogously, we begin with performing the positivity which leads to eliminate the intervals $(a_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), b_2(q))$. On the other hand, due to the Remark 4.5.6 by symmetry and Remark 4.5.1, we can not use the intervals $(-\infty, a_2(q))$ and $(q^{-1}a_1(q), q^{-1}b_1(q))$. We finish this case by considering the interval $(b_2(q), \infty)$ which is the one described in Theorem 4.4 h). Thus, it could be possible to have a suitable ρ. But, notice from Figure 4.32B that ρ is increasing on this interval. That's why $\rho \not\to 0$ as $x \to \infty$ which leads to $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ $k = 0, 1, 2, ...$ as $x \to \infty$. As a result, it is apparent that this case does not generate a suitable ρ.

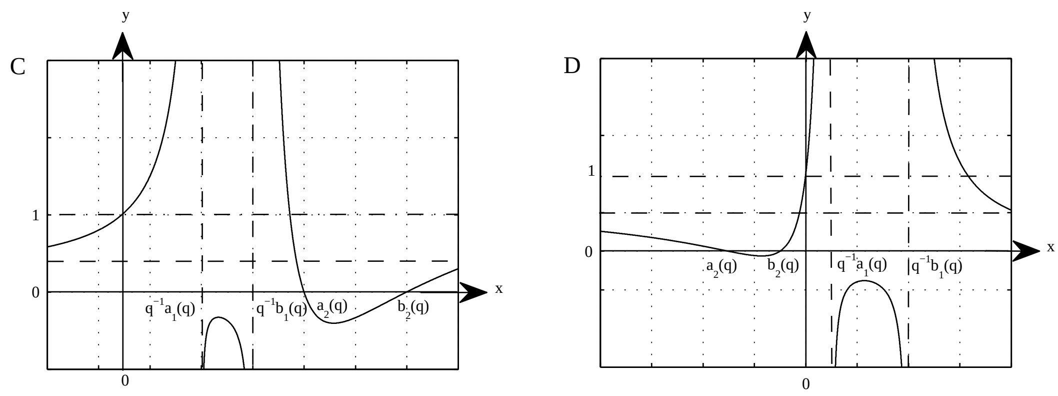


Figure 4.33: Case 2(a)ii) The function $f(x, q)$ with $0 < \Lambda_q < 1$, C: $0 < q^{-1}a_1(q) < q^{-1}b_1(q) < a_2(q) < b_2(q)$, D: $a_2(q) < b_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q)$.

Case 2(a)ii)C: $0 < q^{-1}a_1(q) < q^{-1}b_1(q) < a_2(q) < b_2(q)$, $0 < \Lambda_q < 1$. This case is represented in Figure 4.33C. Positivity of ρ enables us to skip the intervals $(q^{-1}a_1(q), q^{-1}b_1(q))$ and $(a_2(q), b_2(q))$. One can also eliminate the interval $(q^{-1}b_1(q), a_2(q))$ due to Remark 4.5.4. An analogous analysis as the one that has been done in Case 2(a)ii)A and Case 2(a)ii)B for the intervals $(-\infty, q^{-1}a_1(q))$ and $(b_2(q), \infty)$, respectively, yields the same result that these intervals can also not be used to determine a q-weight function. Thus, this case does not generate any intervals where ρ is defined.

Case 2(a)ii)D: $a_2(q) < b_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q)$, $0 < \Lambda_q < 1$. The graph of f is represented in Figure 4.33D. Notice that f is negative on $(q^{-1}a_1(q), q^{-1}b_1(q))$ and $(a_2(q), b_2(q))$. That's why we skip these intervals. Note that the intervals $(b_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), \infty)$ are both eliminated due to Remark 4.5.3 and Remark 4.5.5, respectively. We also exclude the last interval $(-\infty, a_2(q))$ because of the analogous analysis that has been done in Case 2(a)B together with the transformation $x = -t$.

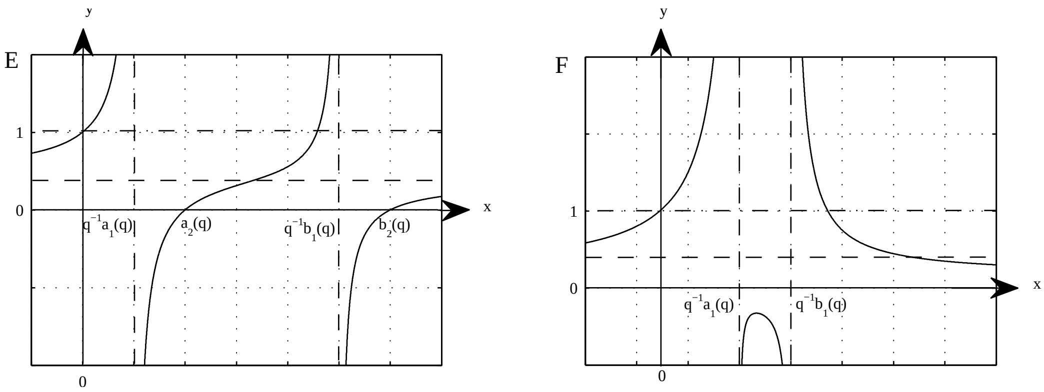


Figure 4.34: The function $f(x, q)$ with $0 < \Lambda_q < 1$, Case 2(a)ii) E: $0 < q^{-1}a_1(q) < a_2(q) < q^{-1}b_1(q) < b_2(q)$, Case 2(c)ii) F: $0 < q^{-1}a_1(q) < q^{-1}b_1(q), a_2(q), b_2(q) \in \mathbb{C}$.

Case 2(a)ii)E: $0 < q^{-1}a_1(q) < a_2(q) < q^{-1}b_1(q) < b_2(q)$, $0 < \Lambda_q < 1$. The graph of this situation appears in Figure 4.34E. The positivity of ρ allows us to skip the intervals $(q^{-1}a_1(q), a_2(q))$ and $(q^{-1}b_1(q), b_2(q))$. Notice that similar analysis as the one that has been done in Case 2(a)A and Case 2(a)B enables us to eliminate the intervals; $(-\infty, q^{-1}a_1(q))$ and $(b_2(q), \infty)$. At last, let us analyse the interval $(a_2(q), q^{-1}b_1(q))$. Notice that this interval is the one that is defined in Theorem 4.4 d). Therefore, there may exist a suitable ρ on this interval. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = a_2(q) < x_0 < q^{-1}b_1(q)$, then from Figure 4.34E it follows that ρ is increasing on $(a_2(q), x_0)$ and decreasing on $(x_0, q^{-1}b_1(q))$ associated with $\rho(qa_2(q), q) = 0$ and $\rho(x, q) \to 0$ as $x \to q^{-1}b_1(q)^-$ (since $\rho(qa_2(q), q)/\rho(a_2(q), q)$=0, $\rho(qx, q)/\rho(x, q) \to \infty$ as $x \to q^{-1}b_1(q)^-$). Thus, according to this discussion, one can easily obtain the behaviour of ρ as in Figure 4.35 by assuming a positive initial value for the q-weight function in each interval.

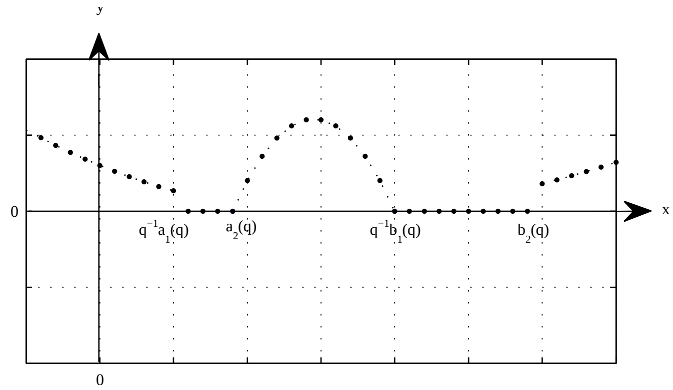


Figure 4.35: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.34E.

It is also clear from Figure 4.35 that the boundary condition holds at $x = a_2(q)$ and $x = q^{-1}b_1(q)$ (see also Theorem 4.5 d)). Thus we can construct a suitable ρ with the needed properties on $(a_2(q), q^{-1}b_1(q))$ supported at the points $a_2(q)q^{-k}$, $k = 0, 1, ...$ and on $(qa_2(q), b_1(q))$ at the points $q^{-1}b_1(q)q^k$, $k = 0, 1, ...$ (see Theorem 4.5 d)). Therefore, we arrive at the following theorem.

Theorem 4.23 *Let $a = qa_2(q)$ be the zero of $\sigma_2(q^{-1}x, q)$ and $b = b_1(q)$ of $\sigma_1(x, q)$ and assume that $0 < q^{-1}a_1(q) < a_2(q) < q^{-1}b_1(q) < b_2(q)$, and $0 < \Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 1$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b) or $(q^{-1}a, q^{-1}b)$, i.e., they satisfy the orthogonality (4.9) and (4.11), respectively with respect to the q-weight function*

$$\rho(x,q) = x^{a}\frac{(qa_2(q)/x, qb_1^{-1}(q)x; q)_\infty}{(a_1(q)/x, b_2^{-1}(q)x; q)_\infty} > 0,\ x \in (a, b) \quad q^{a} = q^{-2}\frac{\frac{1}{2}\sigma_2''(0,q)b_2(q)}{\frac{1}{2}\sigma_1''(0,q)b_1(q)} \tag{4.48}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).

This case corresponds to the case IIIb9 in Chapter 11 of [35, page 366].

An example of such family is the q-Hahn polynomials [35] where $a_1(q) = \alpha q$, $b_1(q) = q^{-N}$, $a_2(q) = 1$, $b_2(q) = \beta^{-1}q^{-N-1}$,

$$\sigma_1(x,q) = q^{-2}(x - q^{-N})(x - \alpha q), \quad \sigma_2(x,q) = \alpha q(x-1)(\beta x - q^{-N-1}),$$

$$\tau(x,q) = \frac{1-\alpha\beta q^2}{(1-q)q}x + \frac{\alpha q^{-N} + \alpha\beta q - \alpha - q^{-N-1}}{1-q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-\alpha\beta q^{n+1}}{1-q}.$$

q-Hahn polynomials are orthogonal on $(1, q^{-N-1})$ and the conditions $0 < q^2\Lambda_q < 1$ and $0 < a_1(q) < a_2(q) < b_1(q) < b_2(q)$ give us the following restriction for the parameters

$0 < \alpha < q^{-1}$, $0 < \beta < q^{-1}$. By means of Theorem 4.4 d) we can also write the orthogonality of q-Hahn polynomials with $q^a = \alpha$

$$\int_1^{q^{-N-1}} x^a \frac{(qa_2(q)/x, qb_1^{-1}(q)x; q)_\infty}{(a_1(q)/x, b_2^{-1}(q)x; q)_\infty} Q_m(x; \alpha, \beta, N|q) Q_n(x; \alpha, \beta, N|q) d_{q^{-1}} x = \frac{(\alpha\beta q^2; q)_N}{(\beta q; q)_N (\alpha q)^N}$$
$$\times \frac{(q, \alpha\beta q^{N+2}, \beta q; q)_n}{(\alpha q, q^{-N}, \alpha\beta q; q)_n} \frac{1 - \alpha\beta q}{1 - \alpha\beta q^{2n+1}} \frac{(q, q^{N+1}; q)_\infty}{(\alpha q, \beta q^{N+1}; q)_\infty} (-\alpha q)^n q^{\binom{n}{2} - Nn} (q^{-1} - 1) \delta_{mn} \quad (4.49)$$

together with $0 < \alpha < q^{-1}$, $0 < \beta < q^{-1}$. Notice from Theorem 4.4 d) that one can write the orthogonality with finite sum by applying (2.31) to (4.49)

$$\sum_{x=0}^{N} \frac{(\alpha q, q^{-N}; q)_x}{(q, \beta^{-1} q^{-N}; q)_x} (\alpha\beta q)^{-x} Q_m(q^{-x}; \alpha, \beta, N|q) Q_n(q^{-x}; \alpha, \beta, N|q) = \frac{(\alpha\beta q^2; q)_N}{(\beta q; q)_N (\alpha q)^N}$$
$$\times \frac{(q, \alpha\beta q^{N+2}, \beta q; q)_n}{(\alpha q, q^{-N}, \alpha\beta q; q)_n} \frac{1 - \alpha\beta q}{1 - \alpha\beta q^{2n+1}} (-\alpha q)^n q^{\binom{n}{2} - Nn} \delta_{mn} \quad (4.50)$$

which coincides with (4.47) but with a different choice of parameters, $0 < \alpha < q^{-1}$, $0 < \beta < q^{-1}$.

Case 2(c)ii) F: $0 < q^{-1}a_1(q) < q^{-1}b_1(q), a_2(q), b_2(q) \in \mathbb{C}$, $0 < \Lambda_q < 1$. The graph of f corresponds to this situation is represented in Figure 4.34F. Notice that we need to eliminate the interval $(q^{-1}a_1(q), q^{-1}b_1(q))$ since ρ should be positive. Notice also that one can exclude the interval $(q^{-1}b_1(q), \infty)$ because of the reason described in Remark 4.5.5. On the other hand, the interval $(-\infty, q^{-1}a_1(q))$ can also be eliminated by use of the analogous analysis as the one described in Case 2(a)ii)A. That's why, we can not obtain any suitable ρ with the needed properties.

Remark 4.24 *We remark that Case 2(b)ii) has similar graphs of f as the ones constructed in Case 2(a)ii) together with $a_2(q) = b_2(q)$. However, any graphs belong to Case 2(b)ii) do not lead to a suitable ρ.*

Case 3(a)A: $q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < a_2(q) < b_2(q)$, $\Lambda_q < 0$. The graph of f corresponds to this case is represented in Figure 4.36A. In order to find the possible intervals where ρ is defined, we carry out the analogous procedure as before cases. Thus, we first consider the positivity property of ρ which allows us to skip the intervals $(-\infty, q^{-1}a_1(q))$, $(q^{-1}b_1(q), a_2(q))$ and $(b_2(q), \infty)$. On the other hand, the interval $(a_2(q), b_2(q))$ is also eliminated by using information given in Remark 4.5.2. If we consider the last interval $(q^{-1}a_1(q), q^{-1}b_1(q))$, it is clear from Theorem 4.4 a) that it could be possible to have a suitable ρ on this interval. Notice that

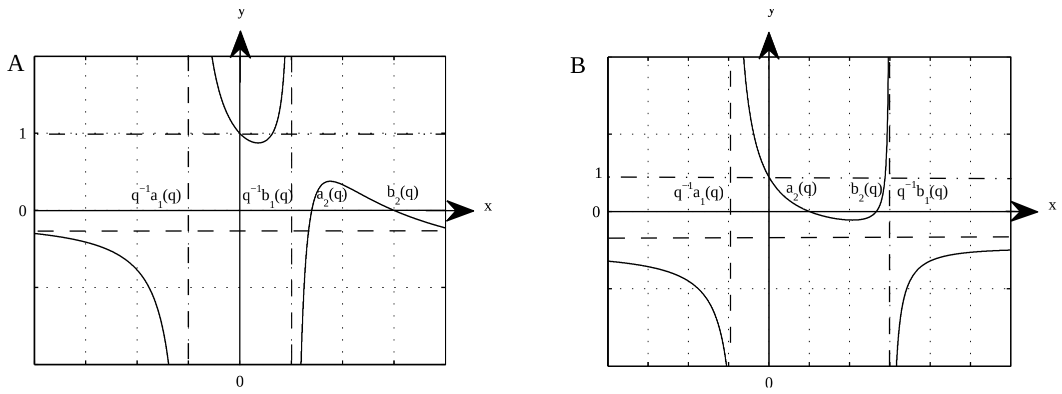


Figure 4.36: Case 3(a). The function $f(x, q)$ with $\Lambda_q < 0$, A: $q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < a_2(q) < b_2(q)$, B: $q^{-1}a_1(q) < 0 < a_2(q) < b_2(q) < q^{-1}b_1(q)$.

$\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = q^{-1}a_1(q) < x_0 < x = q^{-1}b_1(q)$, then from Figure 4.36A, it follows that ρ is increasing on $(q^{-1}a_1(q), x_0)$ and decreasing on $(x_0, q^{-1}b_1(q))$ together with $\rho(x, q) \to 0$ as $x \to q^{-1}a_1(q)^+$ and $x \to q^{-1}b_1(q)^-$ since $\rho(qx, q)/\rho(x, q) \to \infty$. As a result of this information one can easily build the behaviour of ρ as in Figure 4.37.

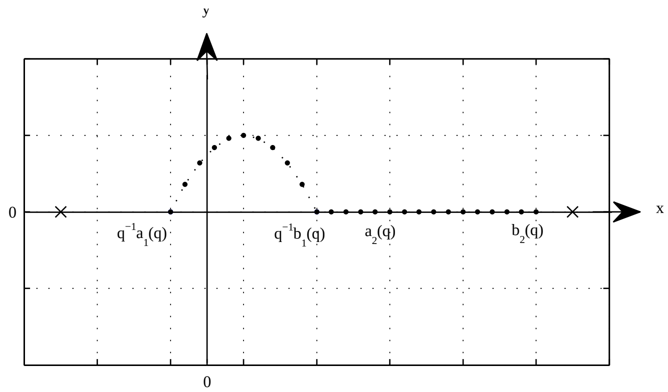


Figure 4.37: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.36A.

It is obvious from Figure 4.37 that $(a_1(q), b_1(q))$ is suitable interval for ρ supported at the points $a_1(q)q^k$ and $b_1(q)q^k$, $k = 0, 1, ...$ (see Theorem 4.4 a)) since the boundary condition (3.115) is satisfied at $x = a_1(q), b_1(q)$.

Remark 4.25 *Note that Case 3(b) includes the same graphs with Case 3(a) which leads to the interval* $(a_1(q), b_1(q))$ *equivalent to the one build in Case 3(a)A together with* $a_2(q) = b_2(q)$

Thus, we construct the following theorem acording to the result of this case.

Theorem 4.26 *Let $a = a_1(q)$ and $b = b_1(q)$ be the zeros of $\sigma_1(x, q)$ and assume that $q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < a_2(q) \le b_2(q)$, and $\Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b), i.e., they satisfy the orthogonality (4.6) with respect to the q-weight function*

$$\rho(x, q) = \frac{(qa_1^{-1}(q)x, qb_1^{-1}(q)x; q)_\infty}{(a_2^{-1}(q)x, b_2^{-1}(q)x; q)_\infty} > 0, \quad x \in (a_1(q), b_1(q)) \tag{4.51}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 a)).

This case leads to the new orthogonality on the interval $(a_1(q), b_1(q))$.

An example of such family is the big q-Jacobi polynomials [35] where $a_1(q) = cq$, $b_1(q) = aq$, $a_2(q) = b^{-1}c$, $b_2(q) = 1$,

$$\sigma_1(x, q) = q^{-2}(x - aq)(x - cq), \quad \sigma_2(x, q) = aq(x - 1)(bx - c),$$

$$\tau(x, q) = \frac{1 - abq^2}{(1 - q)q}x + \frac{a(bq - 1) + c(aq - 1)}{1 - q}, \quad \lambda_n(q) = q^{-n}[n]_q\frac{1 - abq^{n+1}}{q - 1}.$$

Big q-Jacobi polynomials are orthogonal on (cq, aq) and the conditions $q^2\Lambda_q < 0$ and $a_1(q) < 0 < b_1(q) < a_2(q) \le b_2(q)$ give us the following restriction for the parameters $c < 0$, $b < 0$, $abc^{-1}q \le 1$, $0 < a < q^{-1}$. By means of Theorem 4.4 a) we can write the orthogonality

$$\int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} P_m(x; a, b, c; q)P_n(x; a, b, c; q)d_qx = aq(1 - q)\frac{1 - abq}{1 - abq^{2n+1}}q^{\binom{n}{2}}$$
$$\times\frac{(q, abq^2, a^{-1}c, ac^{-1}q; q)_\infty}{(aq, bq, cq, abc^{-1}q; q)_\infty}\frac{(q, bq, abc^{-1}q; q)_n}{(aq, cq, abq; q)_n}(-acq^2)^n\delta_{mn} \tag{4.52}$$

which coincides with (4.44) but with a different choice of parameters, $c < 0$, $b < 0$, $abc^{-1}q \le 1$, $0 < a < q^{-1}$ which is the new orthogonality for big q-Jacobi polynomials.

Case 3(a)B: $q^{-1}a_1(q) < 0 < a_2(q) < b_2(q) < q^{-1}b_1(q)$, $\Lambda_q < 0$. The graph of f is represented in Figure 4.36B. We start with considering the positivity of ρ which allows us to skip the intervals $(-\infty, q^{-1}a_1(q))$, $(a_2(q), b_2(q))$ and $(q^{-1}b_1(q), \infty)$. Notice that the interval $(q^{-1}a_1(q), a_2(q))$ is also excluded by means of Remark 4.5.4. However, since the interval $(b_2(q), q^{-1}b_1(q))$ is the one described in Theorem 4.4 d), it could be possible to have a suitable ρ on this interval. Notice that $f(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = b_2(q) < x_0 < x = q^{-1}b_1(q)$, then from Figure 4.36B, it follows that ρ is increasing on $(b_2(q), x_0)$ and decreasing on $(x_0, q^{-1}b_1(q))$ with $\rho(qb_2(q), q) = 0$ and $\rho(x, q) \to 0$ as $x \to q^{-1}b_1(q)^-$ since $\rho(qb_2(q), q)/\rho(b_2(q), q) = 0$ $\rho(qx, q)/\rho(x, q) \to \infty$ as $x \to q^{-1}b_1(q)^-$. At the end, one can easily construct the behaviour

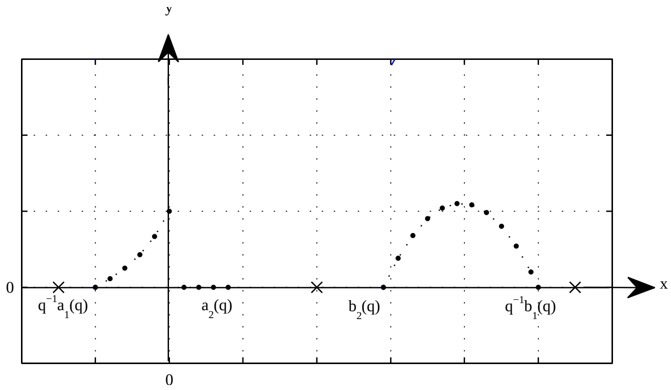


Figure 4.38: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.36B.

of ρ as in Figure 4.38. Note that Figure 4.38 exhibits that there exists a suitable ρ on $(qb_2(q), b_1(q))$ together with $q^2\Lambda_q < 0$, $a_1(q) < 0 < a_2(q) < b_2(q) < b_1(q)$ supported at the points $q^k b_1(q)$, $k = 0, 1, ...$

Remark 4.27 *Note that Case 3(b) includes the same graphs with Case 3(a) which leads to the interval $(qb_2(q), b_1(q))$ equivalent to the one build in Case 3(a)B together with $a_2(q) = b_2(q)$.*

Thus, the following theorem can be stated according to the above discussion.

Theorem 4.28 *Let $a = qb_2(q)$ be the zero of $\sigma_2(q^{-1}x, q)$ and $b = b_1(q)$ of $\sigma_1(x, q)$ and assume that $q^{-1}a_1(q) < 0 < a_2(q) \leq b_2(q) < q^{-1}b_1(q)$, and $\Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b) or $(q^{-1}a, q^{-1}b)$, i.e., they satisfy the orthogonality (4.9) and (4.11), respectively, with respect to the q-weight function*

$$\rho(x, q) = x^a x^{\log_q x}(qa_1^{-1}(q)x, qb_1^{-1}(q)x, qa_2(q)/x, qb_2(q)/x; q)_\infty > 0, \; x \in (a, b) \tag{4.53}$$

$q^a = \frac{q^{-1}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)}$ *which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).*

This case leads to the new orthogonality on the interval $(qb_2(q), b_1(q))$.

An example of such family is the q-Hahn polynomials [35] where $a_1(q) = \alpha q$, $b_1(q) = q^{-N}$, $a_2(q) = \beta^{-1}q^{-N-1}$, $b_2(q) = 1$,

$$\sigma_1(x, q) = q^{-2}(x - q^{-N})(x - \alpha q), \quad \sigma_2(x, q) = \alpha q(x - 1)(\beta x - q^{-N-1}),$$

$$\tau(x,q)=\frac{1-\alpha\beta q^2}{(1-q)q}x+\frac{\alpha q^{-N}+\alpha\beta q-\alpha-q^{-N-1}}{1-q},\quad \lambda_n(q)=-q^{-n}[n]_q\frac{1-\alpha\beta q^{n+1}}{1-q}.$$

q-Hahn polynomials are orthogonal on $(1,q^{-N-1})$ and the conditions $q^2\Lambda_q<0$ and $a_1(q)<0<a_2(q)\leq b_2(q)<b_1(q)$ give us the following restriction for the parameters $\alpha<0,\beta\geq q^{-N-1}$. By means of Theorem 4.4 d) we can write the orthogonality of q-Hahn polynomials

$$\int_1^{q^{-N-1}} x^{b+N}x^{\log_q x}(\alpha^{-1}x,q/x,q^{N+1}x,\beta^{-1}q^{-N}/x;q)_\infty Q_m(x;\alpha,\beta,N|q)Q_n(x;\alpha,\beta,N|q)d_{q^{-1}}x=(q^{-1}-1)$$
$$\times\frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}\frac{(q,\alpha\beta q^{N+2},\beta q;q)_n}{(\alpha q,q^{-N},\alpha\beta q;q)_n}\frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}}\frac{(q,q^{N+1};q)_\infty}{(\alpha q,\beta q^{N+1};q)_\infty}(-\alpha q)^n q^{\binom{n}{2}-Nn}\delta_{mn}\quad(4.54)$$

where $q^b=\beta q$, which coincides with (4.46) but with a different choice of parameters, $\alpha<0$, $\beta\geq q^{-N-1}$. Notice from Theorem 4.4 d) that one can also write the orthogonality with finite sum by applying (2.31) to (4.54)

$$\sum_{x=0}^{N}\frac{(\alpha q,q^{-N};q)_x}{(q,\beta^{-1}q^{-N};q)_x}(\alpha\beta q)^{-x}Q_m(q^{-x};\alpha,\beta,N|q)Q_n(q^{-x};\alpha,\beta,N|q)=\frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}$$
$$\times\frac{(q,\alpha\beta q^{N+2},\beta q;q)_n}{(\alpha q,q^{-N},\alpha\beta q;q)_n}\frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}}(-\alpha q)^n q^{\binom{n}{2}-Nn}\delta_{mn}\quad(4.55)$$

which coincides with (4.47) but with a different choice of parameters, $\alpha<0,\beta\geq q^{-N-1}$ which is the new orthogonality for q-Hahn polynomials.

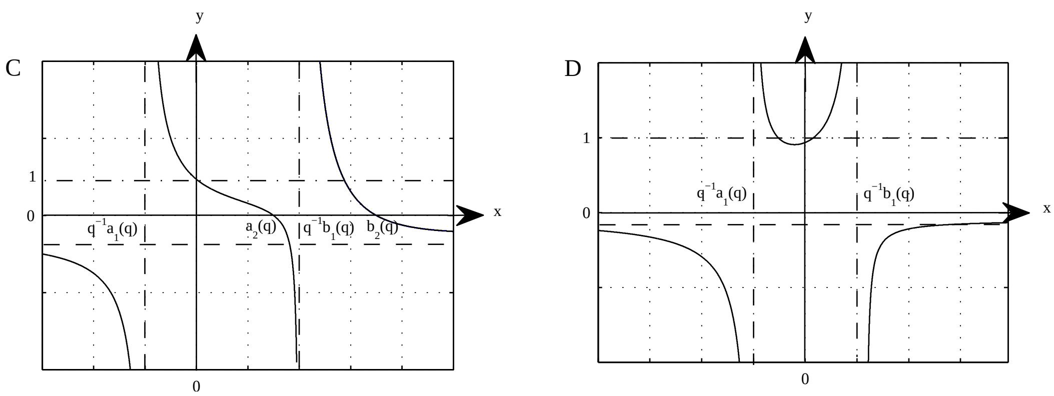


Figure 4.39: The function $f(x,q)$ with $\Lambda_q<0$, Case 3(a)C: $q^{-1}a_1(q)<0<a_2(q)<q^{-1}b_1(q)<b_2(q)$ Case 3(c)D: $q^{-1}a_1(q)<0<q^{-1}b_1(q),a_2(q),b_2(q)\in\mathbb{C}$.

Case 3(a)C: $q^{-1}a_1(q)<0<a_2(q)<q^{-1}b_1(q)<b_2(q)$, $\Lambda_q<0$. The graph of f for this case is represented in Figure 4.39C. We start by applying the analogous procedure. First of all, the positivity of ρ enables us to skip the intervals $(-\infty,q^{-1}a_1(q))$, $(a_2(q),q^{-1}b_1(q))$ and $(b_2(q),\infty)$. Notice also that the intervals $(q^{-1}a_1(q),a_2(q))$ and $(q^{-1}b_1(q),b_2(q))$ are both eliminated due to the Remark 4.5.4. Therefore, this case does not lead to a suitable ρ.

Case 3(c)D: $q^{-1}a_1(q) < 0 < q^{-1}b_1(q), a_2(q), b_2(q) \in \mathbb{C}$, $\Lambda_q < 0$. This case is represented in Figure 4.39D. Notice that f is positive only on the interval $(q^{-1}a_1(q), q^{-1}b_1(q))$. That's why, we eliminate rest two intervals. We remark that the interval $(q^{-1}a_1(q), q^{-1}b_1(q))$ coincides with the one described in Theorem 4.4 a). Then, here it could be possible to have a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x = q^{-1}a_1(q) < x_0 < x = q^{-1}b_1(q)$, then from Figure 4.39D, it follows that ρ is increasing on $(q^{-1}a_1(q), x_0)$ and decreasing on $(x_0, q^{-1}b_1(q))$ with $\rho(x,q) \to 0$ as $x \to q^{-1}a_1(q)^+$ and $x \to q^{-1}b_1(q)^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Thus, it is obvious to construct the behaviour of ρ as in Figure 4.40.

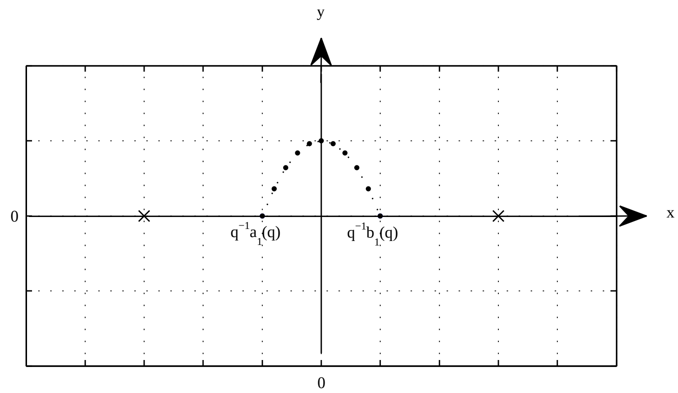


Figure 4.40: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.39D.

We deduce from Figure 4.40 that $(a_1(q), b_1(q))$ is suitable interval for ρ satisfying the needed property supported at the points $a_1(q)q^k$ and $b_1(q)q^k$, $k = 0, 1, ...$ Therefore, we introduce the following theorem related with the result of above discussion.

Theorem 4.29 *Let $a = a_1(q)$ and $b = b_1(q)$ be the zeros of $\sigma_1(x,q)$ and assume that $q^{-1}a_1(q) < 0 < q^{-1}b_1(q)$, $a_2(q), b_2(q) \in \mathbb{C}$, and $\Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.6)*

$$\int_{a_1(q)}^{b_1(q)} P_m(x,q)P_n(x,q)\rho(x,q)d_qx = (b_1(q) - a_1(q))\,(1-q)q^{n(n-1)/2}\,(-a_1(q)b_1(q))^n$$
$$\times\frac{(q, q^{-1}a_2^{-1}(q)b_2^{-1}(q)a_1(q)b_1(q);q)_n}{(q^{-1}a_2^{-1}(q)b_2^{-1}(q)a_1(q)b_1(q), a_2^{-1}(q)b_2^{-1}(q)a_1(q)b_1(q);q)_{2n}} \quad (4.56)$$
$$\times\frac{(q, qb_1(q)a_1^{-1}(q), qa_1(q)b_1^{-1}(q), a_2^{-1}(q)b_2^{-1}(q)a_1(q)b_1(q);q)_\infty}{(a_2^{-1}(q)a_1(q), a_2^{-1}(q)b_1(q), b_2^{-1}(q)a_1(q), b_2^{-1}(q)b_1(q);q)_\infty}$$
$$\times(a_2^{-1}(q)a_1(q), a_2^{-1}(q)b_1(q), b_2^{-1}(q)a_1(q), b_2^{-1}(q)b_1(q);q)_n\delta_{mn}$$

with respect to the q-weight function

$$\rho(x,q) = \frac{(qa_1^{-1}(q)x, qb_1^{-1}(q)x; q)_\infty}{(a_2^{-1}(q)x, b_2^{-1}(q)x; q)_\infty} > 0, \quad x \in (a_1(q), b_1(q)) \tag{4.57}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 a)).

This case leads to the new orthogonality on the interval $(a_1(q), b_1(q))$.

We note that this case leads to the new orthogonality which does not appear in the q-Askey scheme. Actually, this case is analog to the one leading to the big q-Jacobi polynomials. They differ by the properties of the zeros of σ_1 and σ_2.

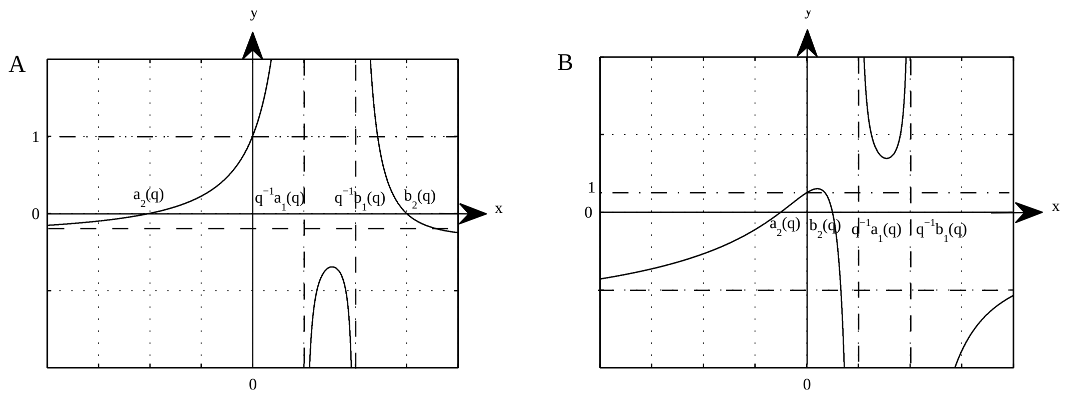


Figure 4.41: Case 4. The function $f(x,q)$ with $\Lambda_q < 0$, A: $a_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q) < b_2(q)$, B: $a_2(q) < 0 < b_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q)$.

Case 4A: $a_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q) < b_2(q)$, $\Lambda_q < 0$. The graph of f is represented in Figure 4.41A. Our aim is to find the possible intervals in which we have a suitable ρ. In this figure, the intervals $(-\infty, a_2(q))$, $(q^{-1}a_1(q), q^{-1}b_1(q))$ and $(b_2(q), \infty)$ are all excluded since f is negative. Moreover, the intervals $(a_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), b_2(q))$ are both eliminated due to Remark 4.5.3 and Remark 4.5.4, respectively. Hence, this case does not lead to a suitable ρ.

Case 4B: $a_2(q) < 0 < b_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q)$, $\Lambda_q < 0$. The graph of f is represented in Figure 4.41B. Since f is negative on the intervals $(-\infty, a_2(q))$, $(b_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), \infty)$, then they can not be used. On the other hand, the intervals $(a_2(q), b_2(q))$, $(q^{-1}a_1(q), q^{-1}b_1(q))$ are both eliminated due to Remark 4.5.2 and Remark 4.5.1, respectively. As a result, we can not get a suitable ρ with needed properties.

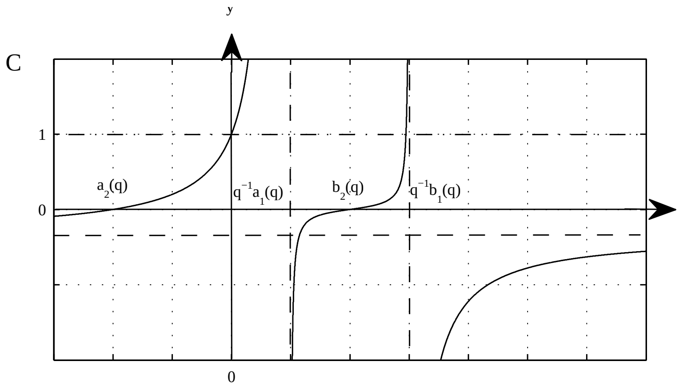


Figure 4.42: Case 4. The function $f(x, q)$ with $\Lambda_q < 0$, $a_2(q) < 0 < q^{-1}a_1(q) < b_2(q) < q^{-1}b_1(q)$.

Case 4C: $a_2(q) < 0 < q^{-1}a_1(q) < b_2(q) < q^{-1}b_1(q)$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.42. The positivity of ρ allows to skip the intervals $(-\infty, a_2(q))$, $(q^{-1}a_1(q), b_2(q))$ and $(q^{-1}b_1(q), \infty)$. Notice that one can also eliminate the interval $(a_2(q), q^{-1}a_1(q))$ due to Remark 4.5.3. Let us deal with the last interval $(b_2(q), q^{-1}b_1(q))$ which is the one described in Theorem 4.4 d). Analogous procedure as the one that has been done in Case 3(a)B allows to build Figure 4.43

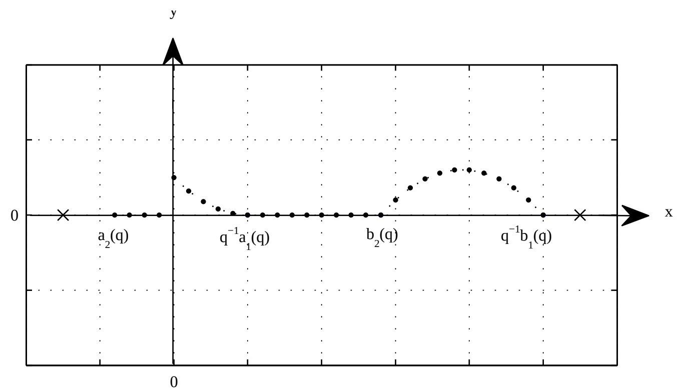


Figure 4.43: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.42C.

It is clear from Figure 4.43 that there exists a q-weight function defined on the interval $(b_2(q), q^{-1}b_1(q))$ supported at the points $q^{-k}b_2(q)$, $k = 0, 1, ...$ or on $(qb_2(q), b_1(q))$ at the points $q^k b_1(q)$, $k = 0, 1, ...$ which lead to the following theorem.

Theorem 4.30 *Let $a = qb_2(q)$ be the zero of $\sigma_2(q^{-1}x, q)$ and $b = b_1(q)$ of $\sigma_1(x, q)$ and assume that $a_2(q) < 0 < q^{-1}a_1(q) < b_2(q) < q^{-1}b_1(q)$, and $\Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then,* *there exists a sequence of polynomials* $(P_n)_n$ *orthogonal on* (a, b) *or* $(q^{-1}a, q^{-1}b)$*, i.e., they satisfy the orthogonality (4.9) and (4.11), respectively with respect to the q-weight function*

$$\rho(x,q) = x^{\mathfrak{a}} \frac{(qa_2(q)/x, qb_1^{-1}(q)x; q)_\infty}{(a_1(q)/x, b_2^{-1}(q)x; q)_\infty} > 0,\ x \in (a,b) \quad q^{\mathfrak{a}} = q^{-2} \frac{\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\frac{1}{2}\sigma_1''(0,q)b_1(q)} \tag{4.58}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).

This case leads to the new orthogonality on the interval $(qb_2(q), b_1(q))$.

An example of such family is the q-Hahn polynomials [35] where $a_1(q) = \alpha q$, $b_1(q) = q^{-N}$, $a_2(q) = \beta^{-1}q^{-N-1}$, $b_2(q) = 1$,

$$\sigma_1(x,q) = q^{-2}(x - q^{-N})(x - \alpha q), \quad \sigma_2(x,q) = \alpha q(x-1)(\beta x - q^{-N-1}),$$

$$\tau(x,q) = \frac{1-\alpha\beta q^2}{(1-q)q} x + \frac{\alpha q^{-N} + \alpha\beta q - \alpha - q^{-N-1}}{1-q}, \quad \lambda_n(q) = -q^{-n}[n]_q \frac{1-\alpha\beta q^{n+1}}{1-q}.$$

q-Hahn polynomials are orthogonal on $(1, q^{-N-1})$ and the conditions $q^2\Lambda_q < 0$ and $a_2(q) < 0 < a_1(q) < b_2(q) < b_1(q)$ give us the following restriction for the parameters $0 < \alpha < q^{-1}$, $\beta < 0$. By means of Theorem 4.4 d) we can write the orthogonality of q-Hahn polynomials

$$\int_1^{q^{-N-1}} x^{\mathfrak{a}} \frac{(qa_2(q)/x, qb_1^{-1}(q)x; q)_\infty}{(a_1(q)/x, b_2^{-1}(q)x; q)_\infty} Q_m(x;\alpha,\beta,N|q) Q_n(x;\alpha,\beta,N|q) d_{q^{-1}}x = (q^{-1}-1) \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}$$
$$\times \frac{(q, \alpha\beta q^{N+2}, \beta q; q)_n}{(\alpha q, q^{-N}, \alpha\beta q; q)_n} \frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}} (-\alpha q)^n q^{\binom{n}{2} - Nn} \delta_{mn} \tag{4.59}$$

where $q^{\mathfrak{a}} = \alpha$, which coincides with the one given in (4.49) but with a different choice of parameters, $0 < \alpha < q^{-1}$, $\beta < 0$. Notice from Theorem 4.4 d) that one can write the orthogonality with finite sum by applying (2.31) to (4.59)

$$\sum_{x=0}^{N} \frac{(\alpha q, q^{-N}; q)_x}{(q, \beta^{-1}q^{-N}; q)_x} (\alpha\beta q)^{-x} Q_m(q^{-x};\alpha,\beta,N|q) Q_n(q^{-x};\alpha,\beta,N|q) = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}$$
$$\times \frac{(q, \alpha\beta q^{N+2}, \beta q; q)_n}{(\alpha q, q^{-N}, \alpha\beta q; q)_n} \frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}} (-\alpha q)^n q^{\binom{n}{2} - Nn} \delta_{mn} \tag{4.60}$$

which coincides with (4.47) but with a different choice of parameters, $0 < \alpha < q^{-1}$, $\beta < 0$ which is the new orthogonality for the q-Hahn polynomials.

Remark 4.31 *We remark that in case of* $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)]^2$ *and* $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)[x - a_2(q)][x - b_2(q)]$*, similar figures when* $\sigma_1(x,q)$ *has two distinct roots are derived together with* $a_1(q) = b_1(q)$*. However, only Figure 4.30E in company with* $a_1(q) = b_1(q)$ *leads to the interval of integration as* $(qb_2(q), a_1(q))$ *or* $(b_2(q), q^{-1}a_1(q))$ *by performing the analogous analysis. On the other hand, these intervals are also valid when* $a_2(q) = b_2(q)$*.*

4.2.3.2 The q-Classical $\emptyset$-Jacobi/Laguerre Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)]$, $a_1(q) < b_1(q)$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$. Then, in case of $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$, it follows from (4.39) that $\sigma_2(x,q) = \sigma_2'(0,q)x + \sigma_2(0,q)$ where

$$\sigma_2'(0,q) = -q\Big[\frac{1}{2}\sigma_1''(0,q)[a_1(q) + b_1(q)] - (1 - q^{-1})\tau(0,q)\Big],\ \sigma_2(0,q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q).$$

Therefore, in this case, the q-Pearson equation follows from (4.40) that

$$\begin{aligned}\frac{\rho(qx,q)}{\rho(x,q)} &= \frac{-\Big[a_1(q) + b_1(q) - (1 - q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]x + a_1(q)b_1(q)}{[qx - a_1(q)][qx - b_1(q)]} \\ &= \frac{-\Big[a_1(q) + b_1(q) - (1 - q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big][x - a_2(q)]}{[qx - a_1(q)][qx - b_1(q)]}\end{aligned} \tag{4.61}$$

where $a_2(q) = \dfrac{a_1(q)b_1(q)}{\Big[a_1(q) + b_1(q) - (1 - q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big]}$.

Let denote by Λ_q the constant

$$\Lambda_q := \Big[a_1(q) + b_1(q) - (1 - q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\Big].$$

Notice from (4.61) that we sketch the graphs for $\rho(qx,q)/\rho(x,q)$ according to the sign of Λ_q concerning with all possible positions of the zeros of $\sigma_1(x,q)$ and $\sigma_2(x,q)$.

Before starting the analysis notice that $\rho(qx,q)/\rho(x,q)$ always intercepts the y-axis at the point $y = 1$ since $\sigma_2(0,q) = q\sigma_1(0,q)$. Notice also that sign of $a_2(q)$ depends of the signs of the zeros of σ_1 and Λ_q. Therefore, in order not to lose any graphs, we split them into three independent cases: Case 1. when $\Lambda_q < 0, a_1(q) < 0 < b_1(q)$ (in the case $\Lambda_q > 0, a_1(q) < 0 < b_1(q)$, the graphs are obtained from Case 1. by the transformation $x = -t$.), Case 2. when $\Lambda_q > 0, 0 < a_1(q) < b_1(q)$ and Case 3. when $\Lambda_q < 0, 0 < a_1(q) < b_1(q)$.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.61).

Case 1.A: $q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < a_2(q)$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.44A. Let us analyse each interval in which we have a q-weight function with needed properties. We first begin with performing the positivity property which allows us to skip the intervals $(-\infty, q^{-1}a_1(q))$ and $(q^{-1}b_1(q), a_2(q))$. Let us consider

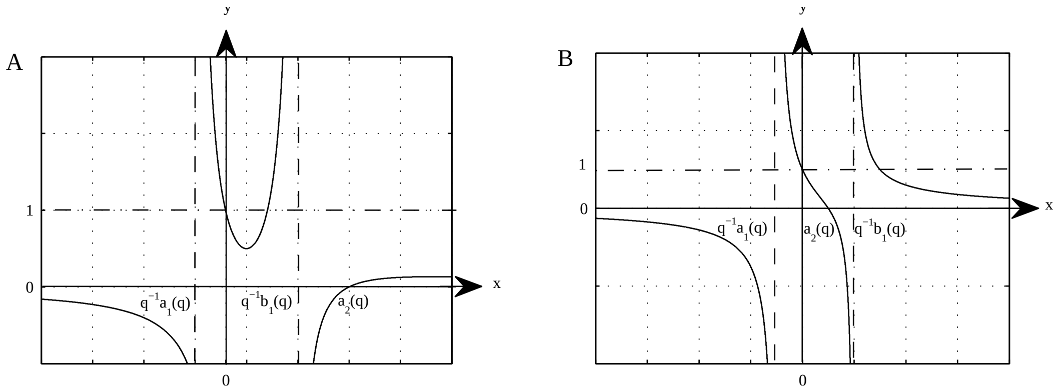


Figure 4.44: Case 1. The function $f(x,q)$ with $\Lambda_q < 0$, A: $q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < a_2(q)$, B: $q^{-1}a_1(q) < 0 < a_2(q) < q^{-1}b_1(q)$.

the interval $(a_2(q), \infty)$. It follows from Figure 4.44A that $\rho(qx,q)/\rho(x,q) < 1$ on $(a_2(q), \infty)$. Thus, ρ is increasing on $(a_2(q), \infty)$ which leads to $\rho \not\to 0$ as $x \to \infty \Longrightarrow \sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to \infty$ which indicates that this interval is not suitable for constructing ρ. Let us now deal with last interval $(q^{-1}a_1(q), q^{-1}b_1(q))$ which is the one described in Theorem 4.4 a). That's why, it could be possible to have a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x = q^{-1}a_1(q) < x_0 < x = q^{-1}b_1(q)$. Then, it follows from Figure 4.44A that ρ is increasing on $(q^{-1}a_1(q), x_0)$ with $\rho \to 0$ as $x \to q^{-1}a_1(q)^+$ since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}a_1(q)^+$ and decreasing on $(x_0, q^{-1}b_1(q))$ with $\rho \to 0$ as $x \to q^{-1}b_1(q)^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}b_1(q)^-$. As a result of this discussion one can easily obtain the following Figure 4.45 for ρ.

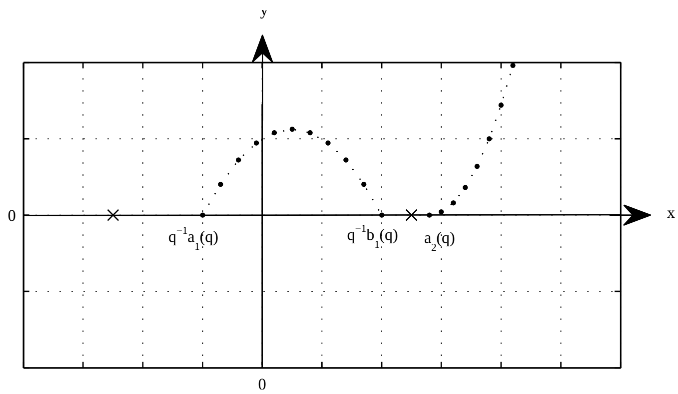


Figure 4.45: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.44A.

It is obvious from Figure 4.45 that there exists a suitable ρ on $(a_1(q), b_1(q))$ supporting at the points $x = q^k a_1(q)$ and $x = q^k b_1(q)$, $k = 0, 1, ...$ (see Theorem 4.4 a)) since the boundary condition (3.119) holds at $x = a_1(q)$ and $x = b_1(q)$. As a result of this case we construct the following theorem.

Theorem 4.32 *Let $a = a_1(q)$ and $b = b_1(q)$ be the zeros of $\sigma_1(x,q)$ and assume that $q^{-1}a_1(q) < 0 < q^{-1}b_1(q) < a_2(q)$, $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$ and $\Lambda_q = [a_1(q) + b_1(q) - (1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.6) with respect to the q-weight function*

$$\rho(x,q) = \frac{(qa_1^{-1}(q)x, qb_1^{-1}(q)x;q)_\infty}{(a_2^{-1}(q)x;q)_\infty} > 0,\ x \in (a,b) \tag{4.62}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 a)).

This case corresponds to the case VIIa1 in Chapter 10 of [35, pages 292 and 318].

An example of such family is the big q-Laguerre polynomials [35] where $a_1(q) = bq$, $b_1(q) = aq$, $a_2(q) = 1$,

$$\sigma_1(x,q) = q^{-2}(x-aq)(x-bq), \quad \sigma_2(x,q) = abq(1-x),$$
$$\tau(x,q) = -\frac{q^{-1}}{q-1}x + \frac{a+b-abq}{q-1}, \quad \lambda_n(q) = \frac{q^{-n}}{q-1}[n]_q.$$

Big q-Laguerre polynomials are orthogonal on (bq, aq) and the conditions $\Lambda_q < 0$ and $a_1(q) < 0 < b_1(q) < a_2(q)$ give us the following restriction for the parameters $b < 0$, $0 < a < q^{-1}$. By means of Theorem 4.4 a) we can write the orthogonality of big q-Laguerre polynomials

$$\int_{bq}^{aq} \frac{(a^{-1}x, b^{-1}x;q)_\infty}{(x;q)_\infty} P_m(x;a,b;q)P_n(x;a,b;q)d_qx = aq(1-q)(-abq^2)^n q^{\binom{n}{2}} \frac{(q;q)_n}{(aq,bq;q)_n} \times \frac{(q,a^{-1}b,ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty}\delta_{mn} \tag{4.63}$$

associated with $b < 0$, $0 < a < q^{-1}$.

Case 1.B: $q^{-1}a_1(q) < 0 < a_2(q) < q^{-1}b_1(q)$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.44B. It is clear from Figure 4.44B that we should skip the intervals $(-\infty, q^{-1}a_1(q))$ $(a_2(q), q^{-1}b_1(q))$ due to the positivity property of ρ. Notice that we eliminate the both intervals $(q^{-1}a_1(q), a_2(q))$ and $(q^{-1}b_1(q), \infty)$ because of the Remark 4.5.4 and Remark 4.5.5, respectively. As a result, this case does not lead to any suitable ρ.

Case 2.C: $0 < q^{-1}a_1(q) < q^{-1}b_1(q) < a_2(q)$, $\Lambda_q > 0$. The graph of f corresponds to this situation is represented in Figure 4.46C. Notice that positivity of ρ allows us to skip

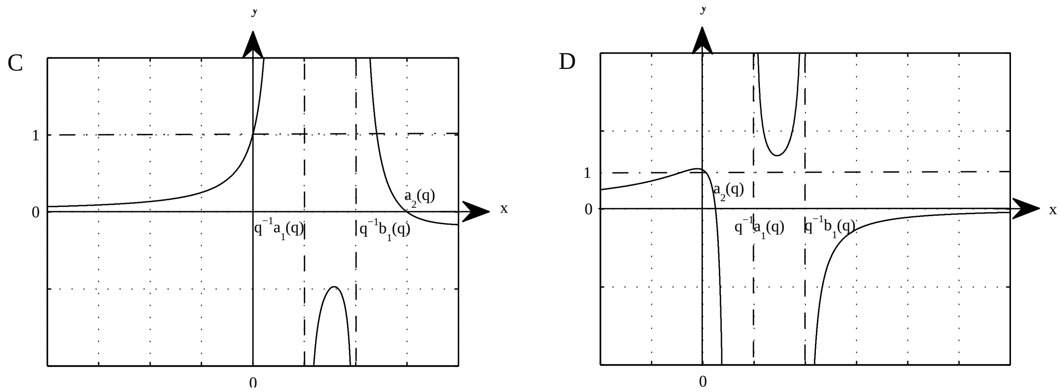


Figure 4.46: Case 2. The function $f(x,q)$ with $\Lambda_q > 0$, C: $0 < q^{-1}a_1(q) < q^{-1}b_1(q) < a_2(q)$, D: $0 < a_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q)$.

the intervals $(q^{-1}a_1(q), q^{-1}b_1(q))$ and $(a_2(q), \infty)$. On the other hand, we exclude the interval $(q^{-1}b_1(q), a_2(q))$ due to Remark 4.5.4. Let us now consider the last interval $(-\infty, q^{-1}a_1(q))$ which coincides with the one described in Theorem 4.4 g) by symmetry. Notice from Figure 4.46C that ρ is decreasing on $(-\infty, q^{-1}a_1(q))$ with $\rho \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}a_1(q)^-$. Thus, $\rho \not\to 0$ as $x \to -\infty$ which implies that $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$ as $x \to -\infty$, $k = 0, 1, ...$ As a result, this case does not lead to any suitable intervals for constructing ρ.

Case 2.D: $0 < a_2(q) < q^{-1}a_1(q) < q^{-1}b_1(q)$, $\Lambda_q > 0$. The graph of f corresponds to this situation is represented in Figure 4.46D. We first start with applying the positivity property which allows us to skip the intervals $(a_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), \infty)$. Notice that we can also not use the intervals $(-\infty, a_2(q))$ and $(q^{-1}a_1(q), q^{-1}b_1(q))$ due to Remark 4.5.6 (by symmetry) and Remark 4.5.1, respectively. That's why, we can not have a suitable ρ with needed properties.

Case 2.E: $0 < q^{-1}a_1(q) < a_2(q) < q^{-1}b_1(q)$, $\Lambda_q > 0$. The graph of f corresponds to this situation is represented in Figure 4.47E. Since ρ should be positive, then $(q^{-1}a_1(q), a_2(q))$ and $(q^{-1}b_1(q), \infty)$ are not suitable. On the other hand, an analogous analysis as the one that has been done in Case 2.C yields that the interval $(-\infty, q^{-1}a_1(q))$ is not suitable for constructing ρ. Let us consider the last interval $(a_2(q), q^{-1}b_1(q))$ which is the one given in Theorem 4.4 d). That's why, it could be suitable for constructing ρ. Notice from Figure 4.47E that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x = a_2(q) < x_0 < x = q^{-1}b_1(q)$. Thus, it follows that ρ is increasing on $(a_2(q), x_0)$ with $\rho(qa_2(q),q) = 0$ (since $\rho(qa_2(q),q)/\rho(a_2(q),q) = 0$)

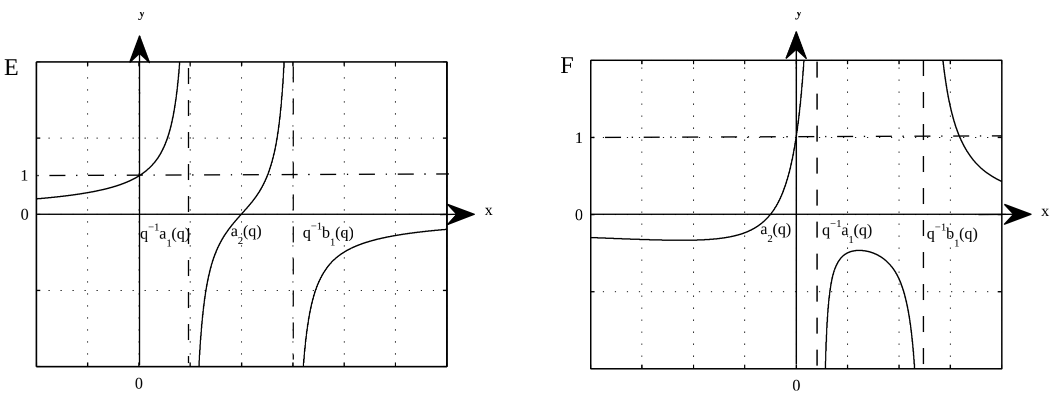


Figure 4.47: The function $f(x,q)$ with Case 2.E: $\Lambda_q > 0$, $0 < q^{-1}a_1(q) < a_2(q) < q^{-1}b_1(q)$, Case 3.F: $\Lambda_q < 0$, $a_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q)$.

and decreasing on $(x_0, q^{-1}b_1(q))$ with $\rho \to 0$ as $x \to q^{-1}b_1(q)^-$ (since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}b_1(q)^-$) which allows us to construct Figure 4.48.

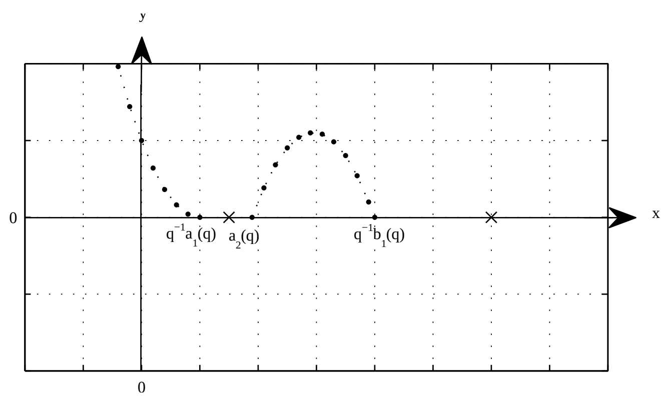


Figure 4.48: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.47E.

It is apparent from Figure 4.48 that $(qa_2(q), b_1(q))$ is suitable interval in which we have a positive q-weight function satisfying the boundary condition (3.119) (see Theorem 4.4 d)) which leads to the following theorem.

Theorem 4.33 *Let* $a = qa_2(q)$ *be the zero of* $\sigma_2(q^{-1}x,q)$ *and* $b = b_1(q)$ *of* $\sigma_1(x,q)$ *and assume that* $0 < q^{-1}a_1(q) < a_2(q) < q^{-1}b_1(q)$, $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$ *and* $\Lambda_q = [a_1(q) + b_1(q) - (1 - q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] > 0$. *Then, there exists a sequence of polynomials* $(P_n)_n$ *orthogonal on* (a,b), *i.e., they satisfy the orthogonality (4.9) or (4.11) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha}\frac{(qa_2(q)/x, qb_1^{-1}(q)x;q)_\infty}{(a_1(q)/x;q)_\infty} > 0,\ x \in (a,b) \quad q^{\alpha} = -\frac{q^{-2}\sigma_2'(0,q)}{\frac{1}{2}\sigma_1''(0,q)b_1(q)} \tag{4.64}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).

This case corresponds to the case IIIb3 in Chapter 11 of [35, pages 343 and 363].

An example of such family is the affine q-Kravchuk polynomials [35] where $a_1(q) = pq, b_1(q) = q^{-N}, a_2(q) = 1$,

$$\sigma_1(x,q) = q^{-1}(x - q^{-N})(x - pq), \quad \sigma_2(x,q) = -pq^{1-N}(x-1),$$
$$\tau(x,q) = \frac{1}{1-q}x - \frac{pq + q^{-N} - pq^{1-N}}{1-q}, \quad \lambda_n(q) = \frac{1}{q-1}[n]_{q^{-1}}.$$

Affine q-Kravchuk polynomials are orthogonal on $(1, q^{-N-1})$ and the conditions $\Lambda_q > 0$ and $0 < a_1(q) < a_2(q) < b_1(q)$ give us the following restriction for the parameters $0 < p < q^{-1}$. By means of Theorem 4.4 d) we can write the orthogonality of affine q-Kravchuk polynomials

$$\int_1^{q^{-N-1}} x^\alpha \frac{(q/x, q^{N+1}x; q)_\infty}{(pq/x; q)_\infty} K_m^{Aff}(x; p, N; q) K_n^{Aff}(x; p, N; q) d_{q^{-1}}x = (pq)^{n-N}(q^{-1} - 1) \times \frac{(q;q)_n (q;q)_{N-n}}{(pq;q)_n (q;q)_N} \frac{(q, q^{N+1}; q)_\infty}{(pq;q)_\infty} \delta_{mn} \quad (4.65)$$

together with $0 < p = q^\alpha < q^{-1}$. Notice from Theorem 4.4 d) that one can also write the orthogonality with finite sum by applying (2.31) to (4.65)

$$\sum_{x=0}^{N} \frac{(pq;q)_x (q;q)_N}{(q;q)_x (q;q)_{N-x}} (pq)^{-x} K_m^{Aff}(q^{-x}; p, N; q) K_n^{Aff}(q^{-x}; p, N; q) = (pq)^{n-N} \frac{(q;q)_n}{(pq;q)_n} \frac{(q;q)_{N-n}}{(q;q)_N} \delta_{mn}. \quad (4.66)$$

Case 3.F: $a_2(q) < 0 < q^{-1}a_1(q) < q^{-1}b_1(q)$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.47F. Note that positivity of ρ enables us to eliminate the intervals $(-\infty, a_2(q))$ and $(q^{-1}a_1(q), q^{-1}b_1(q))$. Moreover, the same happens for the rest intervals $(a_2(q), q^{-1}a_1(q))$ and $(q^{-1}b_1(q), \infty)$ due to Remark 4.5.3 and Remark 4.5.5, respectively.

4.2.3.3 The q-Classical $\emptyset$-Jacobi/Hermite Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)]$, $a_1(q) < b_1(q)$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$. Then, in case of $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$ and $\tau(0,q) = \frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}[a_1(q) + b_1(q)]$, it follows from (4.39) that

$$\sigma_2(x,q) = \sigma_2(0,q) = q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q).$$

Therefore, in this case, the q-Pearson equation follows from (4.40) that

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{a_1(q)b_1(q)}{[qx-a_1(q)][qx-b_1(q)]}. \tag{4.67}$$

We remark that this case leads to the Jacobi/Hermite case (see Table 4.1). To predict the eventual interval in which we have a suitable q-weight function with the needed properties, we start studying with the possible graphs of the ratio $\rho(qx,q)/\rho(x,q)$ given in the q-Pearson equation identified by (4.67). The graphs of the ratio is performed entirely according as the signs of the zeros of $\sigma_1(x,q)$. Thus, we split them into two cases; Case1. $a_1(q) < 0 < b_1(q)$ and Case 2. $0 < a_1(q) < b_1(q)$. Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ always intercepts the y-axis at the point $y = 1$ since $\sigma_2(0,q) = q\sigma_1(0,q)$.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.67).

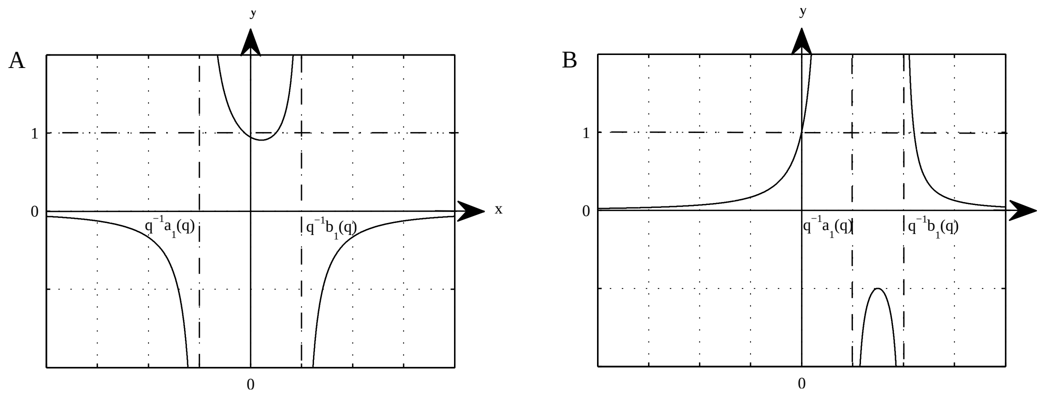


Figure 4.49: The function $f(x,q)$ with Case 1.A: $a_1(q) < 0 < b_1(q)$, Case 2.B: $0 < a_1(q) < b_1(q)$.

Case 1.A: $q^{-1}a_1(q) < 0 < q^{-1}b_1(q)$. The graph of f corresponds to this situation is represented in Figure 4.49A. Let us perfom analogous procedure for each interval. First of all, let us consider the positivity of ρ which enables us to omit the intervals $(-\infty, q^{-1}a_1(q))$ and $(q^{-1}b_1(q), \infty)$. The rest interval $(q^{-1}a_1(q), q^{-1}b_1(q))$ coincides with the one represented in Theorem 4.4 a). That's why, here it could be possible to get a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x = q^{-1}a_1(q) < x_0 < x = q^{-1}b_1(q)$. Then, it follows from Figure 4.49A that ρ is increasing on $(q^{-1}a_1(q), x_0)$ with $\rho \to 0$ as $x \to q^{-1}a_1(q)^+$ (since $\rho(qx,q)/\rho(x,q) \to \infty$) and decreasing on $(x_0, q^{-1}b_1(q))$ with $\rho \to 0$ as $x \to q^{-1}b_1(q)^-$ (since $\rho(qx,q)/\rho(x,q) \to \infty$) which allows us to construct Figure 4.50.

We infer from Figure 4.50 that positive q-weight function exists on $(a_1(q), b_1(q))$. It is obvious that the boundary condition (3.119) holds at $x = a_1(q)$ and $x = b_1(q)$ since they are roots of

$\sigma_1(x,q)$ (see Theorem 4.4 a)). Thus, we construct the following theorem according to this case.

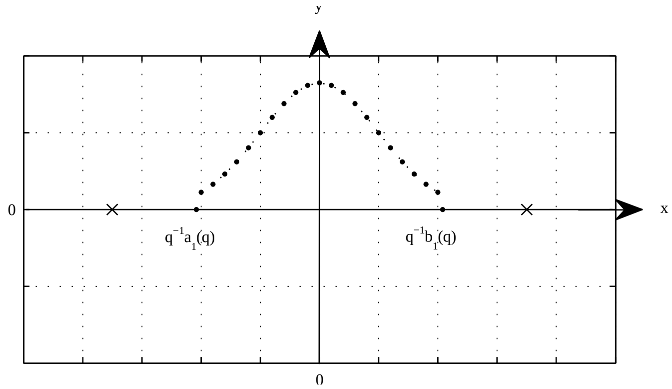


Figure 4.50: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.49A.

Theorem 4.34 *Let $a = a_1(q)$ and $b = b_1(q)$ be the zeros of $\sigma_1(x,q)$ and assume that $q^{-1}a_1(q) < 0 < q^{-1}b_1(q)$ and $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.6) with respect to the q-weight function*

$$\rho(x,q) = (qa_1^{-1}(q)x, qb_1^{-1}(q)x; q)_\infty > 0, \; x \in (a,b) \tag{4.68}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 a)).

This case corresponds to the case VIIa1 in Chapter 10 of [35, pages 292 and 318].

An example of such family is Al-Salam-Carlitz I polynomials [35] where $a_1(q) = a$, $b_1(q) = 1$,

$$\sigma_1(x,q) = q^{-1}(1-x)(a-x), \quad \sigma_2(x,q) = a,$$

$$\tau(x,q) = \frac{1}{1-q}x - \frac{1+a}{1-q}, \quad \lambda_n(q) = \frac{q^{1-n}}{q-1}[n]_q.$$

Al-Salam-Carlitz I polynomials are orthogonal on $(a,1)$ and the conditions $a_1(q) < 0 < b_1(q)$ give us the following restriction for the parameters $a < 0$. By means of Theorem 4.4 a) we can write the orthogonality of Al-Salam-Carlitz I polynomials

$$\int_a^1 (qx, a^{-1}qx; q)_\infty U_m^{(a)}(x;q) U_n^{(a)}(x;q) d_qx = (-a)^n q^{\binom{n}{2}}(1-q)(q;q)_n(q,a,a^{-1}q;q)_\infty \delta_{mn} \tag{4.69}$$

associated with $a < 0$.

Another example of this family is discrete q-Hermite I polynomials which are special case of Al-Salam-Carlitz I polynomials (see [36] for further details).

Case 2.B: $0 < q^{-1}a_1(q) < q^{-1}b_1(q)$. The graph of f corresponds to this situation is represented in Figure 4.49B. We first skip the interval $(q^{-1}a_1(q), q^{-1}b_1(q))$ due to the positivity of ρ. Moreover, we eliminate the interval $(q^{-1}b_1(q), \infty)$ because of Remark 4.5.5 since the boundary condition is not satisfied. The last interval $(-\infty, q^{-1}a_1(q))$ is the one described in Theorem 4.4 g) by symmetry. However, notice from Figure 4.49B that ρ is decreasing on this interval with $\rho \to 0$ as $x \to q^{-1}a_1(q)^-$ (since $\rho(qx,q)/\rho(x,q) \to \infty$) which leads to that $\rho \not\to 0$ as $x \to -\infty \Rightarrow \sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0, 1, ...$ as $x \to -\infty$. Thus, this interval is also not suitable for constructing ρ.

4.3 The Zero Case

We now impose the analogous analysis to the zero case, i.e., we deal with all possible degrees of the polynomial coefficients with additional condition $\sigma_1(0,q) = 0 \Leftrightarrow \sigma_2(0,q) = 0$. Notice that for the zero cases, the q-Pearson equation (3.119) has zero and pole at $x = 0$. Then, when we determine the behaviour of ρ at $x = 0$ we use the following remark.

Remark 4.35 *Behaviour of the q-weight function at $x = 0$ depends on the succesive solution of the q-Pearson equation*

$$\begin{aligned} \rho(qx,q) &= \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}\rho(x,q) \\ \Leftrightarrow \quad \rho(q^k x,q) &= \rho(x,q)\prod_{i=0}^{k-1}\frac{q^{-1}\sigma_2(q^i x,q)}{\sigma_1(q^{i+1}x,q)}. \end{aligned} \tag{4.70}$$

It is apperant that as $k \to \infty$ the behaviour of ρ at $x = 0$ is accomplished which alters according as the degrees of the polynomials $\sigma_1(x,q)$ and $\sigma_2(x,q)$.

4.3.1 Linear Case

Let $\sigma_1(x,q) = \sigma_1'(0,q)x$, i.e., linear with $\sigma_1(0,q) = 0$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$.

Remark 4.36 *Notice that $\sigma_2(x,q)$ is obtained from (3.11) as the form* $\sigma_2(x,q) = q\Big[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\Big] = qx\Big[(1-q^{-1})\tau'(0,q)x + \sigma_1'(0,q) + (1-q^{-1})\tau(0,q)\Big] = qx(1-q^{-1})\tau'(0,q)\Big[x-$

$a_2(q)\Big]$, $\tau'(0,q) \neq 0$. *Note that* $\frac{\tau(0,q)}{\sigma_1'(0,q)} = -\frac{1}{(1-q^{-1})}$ *conduces to* $a_2(q) = 0$. *Thus, it can be easily interpreted that the q-Laguerre type zero family of the 1st kind is the q-Jacobi type zero family of the 2nd kind and the q-Bessel type zero family of the 2nd kind where* $\sigma_2(x,q)$ *has* 0 *as a root with multiplicity two.*

4.3.1.1 The q-Classical 0-Laguerre/Jacobi Polynomials

Let $\sigma_1(x,q) = \sigma_1'(0,q)x$, $\sigma_1(0,q) = 0$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$ and assume that $\frac{\tau(0,q)}{\sigma_1'(0,q)} \neq -\frac{1}{(1-q^{-1})} \Leftrightarrow a_2(q) \neq 0$. Then, $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x$ where

$$\frac{1}{2}\sigma_2''(0,q) = q(1-q^{-1})\tau'(0,q),\ \sigma_2'(0,q) = q[\sigma_1'(0,q) + (1-q^{-1})\tau(0,q)].$$

As a result, the q-Pearson equation folows from (4.1) as

$$\begin{aligned}\frac{\rho(qx,q)}{\rho(x,q)} &= q^{-1}\left[(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}x + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)} + 1\right] \\ &= q^{-1}(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x - a_2(q)]\end{aligned} \tag{4.71}$$

where $a_2(q) = -\frac{1+(1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}}{(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}} \neq 0$.

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ always intercepts y-axis at the point

$$y := y_0 = q^{-1}\left[1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right].$$

Notice for the zero cases that a or b could be zero. That's why, we should know the behaviour of ρ at $x = 0$. To learn this we perform the following remark obtained from Remark 4.35.

Remark 4.37 *Behaviour of the q-weight function at* $x = 0$ *depends on the succesive solution of the q-Pearson equation*

$$\begin{aligned}\rho(qx,q) &= q^{-1}\left[1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right](1 - x/a_2(q))\rho(x,q) \\ \Leftrightarrow \rho(q^k x,q) &= q^{-k}\left[1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right]^k (x/a_2(q);q)_k\rho(x,q).\end{aligned} \tag{4.72}$$

It is apparent that as $k \to \infty$ *the behaviour of* ρ *at* $x = 0$ *is accomplished. Notice that if* $0 < y_0 = q^{-1}\left[1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right] < 1$, $\rho(x,q) \to 0$ *as* $x \to 0$ *otherwise it tends to* $\mp\infty$.

In order not to lose any graphs of $\rho(qx,q)/\rho(x,q)$, we perform independent graphs according to the signs of the zeros of σ_2 and $\Lambda_q := \frac{\tau'(0,q)}{\sigma_1'(0,q)}$ together with $y_0 < 1$, $y_0 > 1$. We note that we have three independent cases: Case 1. when $\Lambda_q > 0$, $a_2(q) > 0$ and $y_0 > 1$, Case 2. when $\Lambda_q < 0$, $a_2(q) < 0$ and $0 < y_0 < 1$ and Case 3. when $\Lambda_q < 0$, $a_2(q) > 0$ and $y_0 < 0$. Then, next step is to execute the graphs of $\rho(qx,q)/\rho(x,q)$ in (4.71).

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.71).

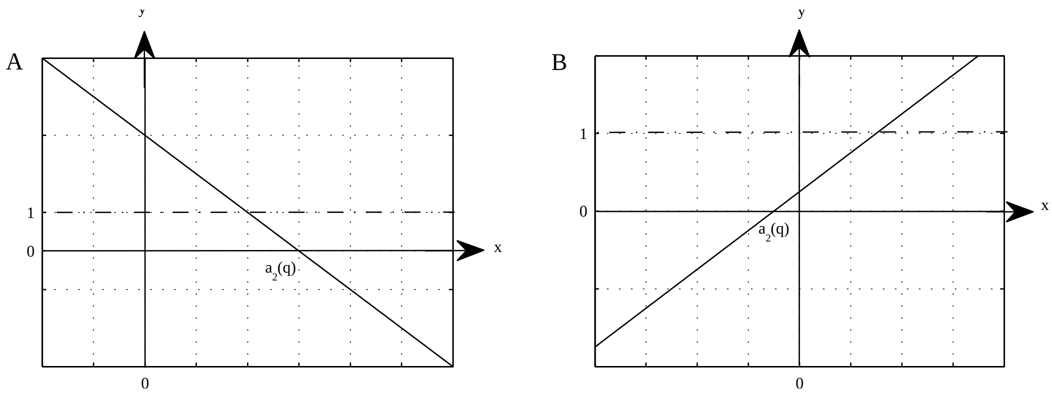


Figure 4.51: The function $f(x,q)$ with Case 1.A: $\Lambda_q > 0$, $a_2(q) > 0$, $y_0 > 1$, Case 2.B: $\Lambda_q < 0$, $a_2(q) < 0$, $0 < y_0 < 1$.

Case 1.A: $a_2(q) > 0$, $y_0 > 1$, $\Lambda_q > 0$. The graph of f corresponds to this situation is represented in Figure 4.51A. By applying the analogous procedure we analyse each interval. First of all, by the positivity of ρ, we skip the interval $(a_2(q),\infty)$. The same happens for the interval $(0, a_2(q))$ due to Remark 4.5.4. Notice that the last interval $(-\infty, 0)$ is the one described in Theorem 4.4 i) by symmetry. Thus, here it could be possible to have a suitable ρ. Notice from 4.51A that $\rho(qx,q)/\rho(x,q) > 1$ on $(-\infty,0)$ which leads to that ρ is increasing on this interval with $\rho \to \infty$ as $x \to 0^-$ since $y_0 > 1$ (see Remark 4.37). Observe that since ρ is increasing and $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to -\infty$, then $\rho \to 0$ as $x \to -\infty$. However, it is seen from the identity identified by (4.21) that the graph of ρ and $\sigma_1\rho$ have the same property, then $\sigma_1\rho \to \infty$ as $x \to 0^-$ which is the boundary condition when $k = 0$. Therefore, there is no suitable ρ for the interval $(-\infty, 0)$.

Case 2.B: $a_2(q) < 0$, $0 < y_0 < 1$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.51B. Positivity of ρ allows us to skip the interval $(-\infty, a_2(q))$. The interval $(a_2(q), 0)$ is also eliminated due to Remark 4.5.4 by symmetry. Let us consider the last interval $(0,\infty)$ which concides with the one described in Theorem 4.4 i). That's why, it could be possible to have a suitable ρ in this interval. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = 0$. Then it follows from Figure 4.51B that ρ is inreasing on $(0, x_0)$ with $\rho \to 0$ as $x \to 0^+$ since $0 < y_0 < 1$ (see Remark 4.37) and decreasing on (x_0, ∞) which leads to $\rho \to 0$ as $x \to \infty$ since $\rho(qx,q)/\rho(x,q) \to \infty$. Then, according to this discussion one can easily get Figure 4.52 by assuming a positive initial value for the q-weight function in each interval.

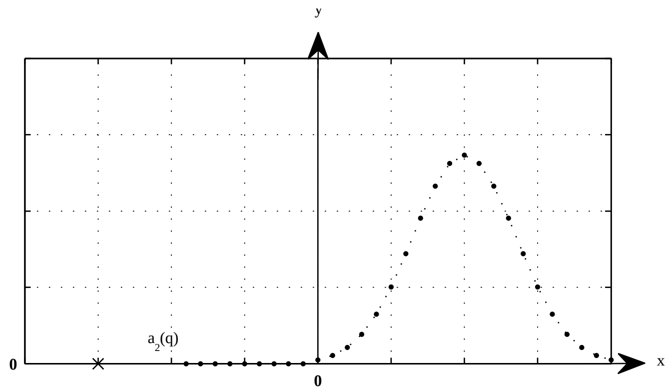


Figure 4.52: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.51B.

It is also obvious from Figure 4.52 that it could be possible to have a suitable ρ on $(0,\infty)$. But we need to check $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$, $k = 0,1,...$ by using the *extended* q-Pearson equation (4.20). It is clear from (4.20) that graph of g looks like the one represented in Figure 4.51B with the y-intercept, $0 < q^{k+1}y_0 < 1$, $k = 0,1,...$ that's why, we have $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$, $k = 0,1,...$ As a result, there exists a convenient ρ on $(0,\infty)$ (see Theorem 4.4 i)) which leads to the following theorem.

Theorem 4.38 *Let $a = 0$ and $b = \infty$ and assume that $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$, $a_2(q) < 0$ and $0 < y_0 = q^{-1}\left[1 + (1-q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right] < 1$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.17) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha}\frac{1}{(x/a_2(q);q)_\infty} > 0,\ x \in (a,b), \quad q^{\alpha} = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\sigma_1'(0,q)} \tag{4.73}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 i)).

This case corresponds to the case IIIa2 in Chapter 10 of [35, pages 272 and 309].

An example of such family is q-Laguerre polynomials [35] where $a_2(q) = -1$,

$$\sigma_1(x,q) = q^{-2}x, \quad \sigma_2(x,q) = q^{\alpha}x(x+1),$$
$$\tau(x,q) = -\frac{q^{\alpha}}{1-q}x + \frac{q^{-1}-q^{\alpha}}{1-q}, \quad \lambda_n(q) = [n]_q\frac{q^{\alpha}}{1-q}.$$

q-Laguerre polynomials are orthogonal on $(0,\infty)$ and the conditions $\Lambda_q < 0$, $a_2(q) < 0$ and $0 < qy_0 < 1$ give us the following restriction for the parameters $\alpha > -1$. By means of Theorem 4.4 i) we can write the orthogonality of q-Laguerre polynomials

$$\int_0^{\infty}\frac{x^{\alpha}}{(-x;q)_{\infty}}L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)d_qx = q^{-n}(1-q)\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_{\infty}}{(q^{\alpha+1},-c,-c^{-1}q;q)_{\infty}}\delta_{mn} \quad (4.74)$$

together with $\alpha > -1$, $c > 0$.

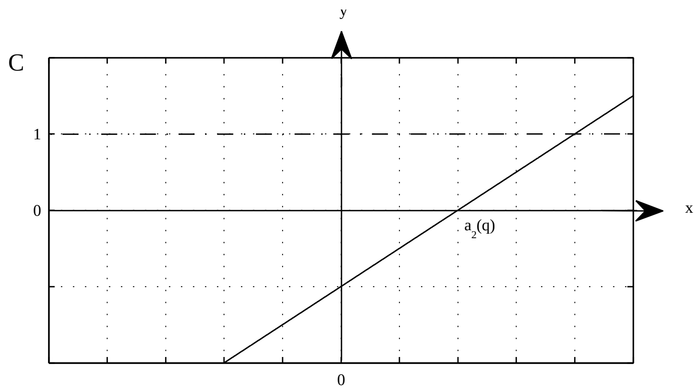


Figure 4.53: The function $f(x,q)$ with Case 3.C: $\Lambda_q < 0$, $a_2(q) > 0$, $y_0 < 0$.

Case 3.C: $a_2(q) > 0$, $y_0 < 0$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.53C. Positivity of ρ enables us to skip the intervals $(-\infty,0)$ and $(0,a_2(q))$. That's why, we deal with only the interval $(a_2(q),\infty)$ which is the one described in Theorem 4.4 h). Then, here it could be possible to have a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = a_2(q)$. Therefore, it follows from Figure 4.53C that ρ is increasing on $(a_2(q),x_0)$ with $\rho(qa_2(q),q) = 0$ (since $\rho(qa_2(q),q)/\rho(a_2(q),q) = 0$) and decreasing on (x_0,∞) which leads to $\rho \to 0$ as $x \to \infty$ (since $\rho(qx,q)/\rho(x,q) \to \infty$). Notice that one can construct the graph of ρ as in Figure 4.54 according as the above discussion by assuming a positive initial value for the q-weight function in each interval.

It is obvious from Figure 4.54 that it could be possible to have a convenient ρ on $(a_2(q),\infty)$. But, it is not enough to assure that ρ satisfies the boundary conditions at $+\infty$. We should check $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$, $k = 0,1,....$ We note that the graph of g looks like

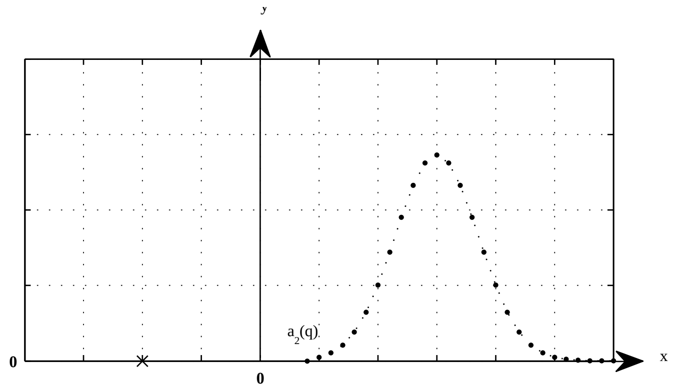


Figure 4.54: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.53.

the one represented in Figure 4.53. That's why, we obtain $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to \infty$, $k = 0, 1, ...$ which leads to the following theorem.

Theorem 4.39 *Let $a = a_2(q)$ be the zero of $\sigma_2(x, q)$ and $b = \infty$ and assume that $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$, $a_2(q) > 0$ and $y_0 = q^{-1}\left[1 + (1 - q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b), i.e., they satisfy the orthogonality (4.16) with respect to the q-weight function*

$$\rho(x, q) = x^{\alpha}\sqrt{x^{\log_q x-1}}(qa_2(q)/x; q)_\infty > 0,\ x \in (a, b), \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\sigma_1'(0, q)} \tag{4.75}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 h)).

This case corresponds to the case IIa2 in Chapter 11 of [35, pages 337 and 358].

An example of such family is q-Charlier polynomials [35] where $a_2(q) = 1$,

$$\sigma_1(x, q) = aq^{-2}x, \quad \sigma_2(x, q) = x(x - 1),$$

$$\tau(x, q) = -\frac{1}{1 - q}x + \frac{a + q}{(1 - q)q}, \quad \lambda_n(q) = [n]_q\frac{1}{1 - q}.$$

q-Charlier polynomials are orthogonal on $(1, \infty)$ and the conditions $\Lambda_q < 0$, $a_2(q) > 0$ and $y_0 < 0$ give us the following restriction for the parameters $a > 0$. By means of Theorem 4.4 h) we can write the orthogonality of q-Charlier polynomials

$$\int_1^{\infty} x^{\alpha}\sqrt{x^{\log_q x-1}}(q/x; q)_\infty C_m(x; a; q)C_n(x; a; q)d_{q^{-1}}x = (q^{-1} - 1)q^{-n}(-a^{-1}q, q; q)_n(-a, q; q)_\infty\delta_{mn} \tag{4.76}$$

with the relation $a > 0$.

4.3.1.2 The q-Classical 0-Laguerre/Bessel Polynomials

Let $\sigma_1(x,q) = \sigma_1'(0,q)x$, $\sigma_1(0,q) = 0$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$ and assume that $\frac{\tau(0,q)}{\sigma_1'(0,q)} = -\frac{1}{(1-q^{-1})} \Leftrightarrow a_2(q) = 0$. Then, $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 = q(1-q^{-1})\tau'(0,q)x^2$. As a result, the q-Pearson equation folows from (4.71)

$$\frac{\rho(qx,q)}{\rho(x,q)} = q^{-1}(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}x. \tag{4.77}$$

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ always intercepts y-axis at the point $y := y_0 = 0$. We perform analogous analysis in order to get the graph of $\rho(qx,q)/\rho(x,q)$ according to sign of $\Lambda_q := \frac{\tau'(0,q)}{\sigma_1'(0,q)}$ which leads to one independent graph as in Figure 4.55A. Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.77).

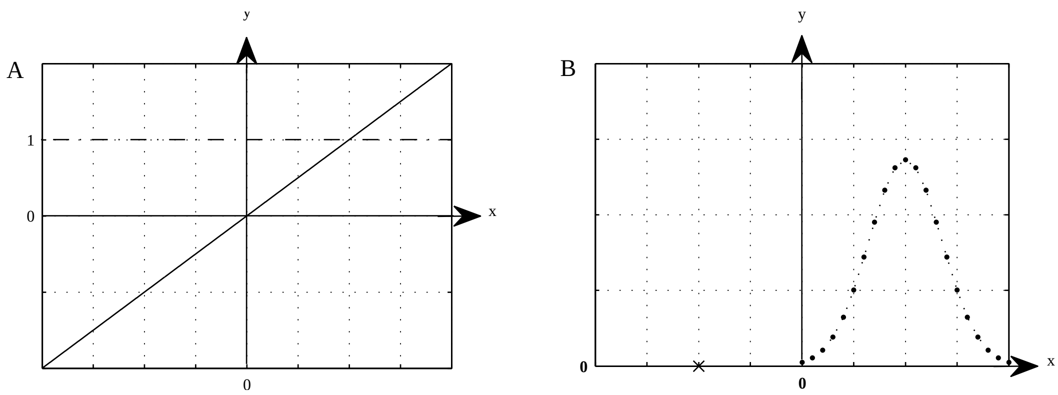


Figure 4.55: Case 1. The function $f(x,q)$ with $\Lambda_q < 0$, $a_2(q) = 0$, B: corresponding positive $\rho(x,q)$.

Case 1.A: $a_2(q) = 0$, $y_0 = 0$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.55A. Notice that f is negative on $(-\infty, 0)$ which enables us to skip it. The other interval $(0,\infty)$ is the one described in Theorem 4.4 i). That's why, here it could be possible to construct a convenient ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = 0$. Then, it follows from Figure 4.55A that ρ is increasing on $(0,x_0)$ and decreasing on (x_0,∞) which leads to that $\rho \to 0$ as $x \to 0^+$ and $x \to +\infty$. It is obvious that one can easily obtain the Figure 4.55B by assuming a positive initial value for the q-weight function in each interval which also indicates that it could be possible to have a suitable ρ on $(0,\infty)$. However, we need to check the boundary condition at $+\infty$ by using the *extended* q-Pearson equation (4.20). We note that the graph of g looks like the one represented in Figure 4.55A. That's why, by the same reason performed for ρ we get $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to +\infty$.

Thus, we build the following theorem.

Theorem 4.40 *Let* $a = 0$ *and* $b = \infty$ *and assume that* $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0$, $a_2(q) = 0$ *and* $y_0 = q^{-1}\left[1 + (1 - q^{-1})\frac{\tau(0,q)}{\sigma_1'(0,q)}\right] = 0$. *Then, there exists a sequence of polynomials* $(P_n)_n$ *orthogonal on* (a, b)*, i.e., they satisfy the orthogonality (4.17) with respect to the q-weight function*

$$\rho(x, q) = x^{\alpha}\sqrt{x^{\log_q x - 1}} > 0,\ x \in (a, b), \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)}{\sigma_1'(0, q)} \tag{4.78}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 i)).

This case corresponds to the case IIIa2 in Chapter 10 of [35, pages 272 and 309].

An example of such family is Stieltjes-Wigert polynomials [35] where $a_2(q) = 0$,

$$\sigma_1(x, q) = q^{-2}x, \quad \sigma_2(x, q) = x^2,$$

$$\tau(x, q) = -\frac{1}{1-q}x + \frac{1}{(1-q)q}, \quad \lambda_n(q) = [n]_q\frac{1}{1-q}.$$

Stieltjes-Wigert polynomials are orthogonal on $(0, \infty)$ and the conditions $\Lambda_q < 0$, $a_2(q) = 0$ and $y_0 = 0$ are satisfied. By means of Theorem 4.4 i) we can write the orthogonality of Stieltjes-Wigert polynomials

$$\int_0^{\infty}\sqrt{x^{\log_q x - 1}}S_m(x; q)S_n(x; q)d_qx = q^{-n}(1-q)\frac{(-tq, -1/t, q; q)_{\infty}}{(q^2; q)_n}\delta_{mn}. \tag{4.79}$$

4.3.2 Quadratic Case

This part consists of two situations. One of them is when zero is the root of $\sigma_1(x, q)$ with miltiplicity two; $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)x^2$ and the other is one; $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)x^2 + \sigma_1'(0, q)x$.

We begin with $\sigma_1(x, q) = \frac{1}{2}\sigma_1''(0, q)x^2$, $\tau(x, q) = \tau'(0, q)x + \tau(0, q)$ which lead to $\sigma_2(x, q) = qx\left\{\left[\frac{1}{2}\sigma_1''(0, q) + (1 - q^{-1})\tau'(0, q)\right]x + (1 - q^{-1})\tau(0, q)\right\}$.

Remark 4.41 *Notice that* $\deg[\sigma_2(x, q)] = 2$ if $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$. Note that if $\tau(0, q) \neq 0$, then $\sigma_2(x, q)$ has simple roots, otherwise it has zero as a root with multiplicity two. On the other hand, $\deg[\sigma_2(x, q)] = 1$ if $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = -\frac{1}{(1-q^{-1})}$. That's why, the q-Bessel type zero family of the 1st kind is the q-Jacobi type, the q-Bessel type and the q-Laguerre type zero family of the 2nd kind.

4.3.2.1 The q-Classical 0-Bessel/Jacobi Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2$, $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$ and assume that $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$ and $\tau(0,q) \neq 0$. Then, it follows that $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x$ where

$$\frac{1}{2}\sigma_2''(0,q) = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right] \neq 0,\ \sigma_2'(0,q) = q(1-q^{-1})\tau(0,q).$$

Thus, the q-Pearson equation takes the form

$$\begin{aligned} \frac{\rho(qx,q)}{\rho(x,q)} &= \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} \\ &= \frac{\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x + (1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}}{q^2 x} \\ &= q^{-1}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{[x - a_2(q)]}{qx} \end{aligned} \tag{4.80}$$

where $a_2(q) = -\frac{(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}}{\left[1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]} \neq 0.$

Remark 4.42 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.80). Then notice that*

$$\Lambda_q := q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0$$

is the horizontal asymptote of the function $f(x,q)$.

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ is discontinues at $x = 0$.

In order not to lose any graphs of $\rho(qx,q)/\rho(x,q)$, we consider every case by taking Case 1. $\Lambda_q > 0$ and Case 2. $\Lambda_q < 0$ concerning with the sign of the zero of σ_2. Furthermore, as before, we need to split 1st case into two separate cases: Case 1.i) when $\Lambda_q > 1$ and Case 1.ii) when $0 < \Lambda_q < 1$.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.80).

Case 1.i)A: $\Lambda_q > 1$, $a_2(q) > 0$. The graph of f corresponds to this situation is represented in Figure 4.56A. We start to analyse each interval in which we have a suitable ρ with certain properties. Thus, we first exclude the interval $(0, a_2(q))$ due to the positivity property of ρ. In the second part, let us look at the interval $(-\infty, 0)$ which is the one described in Theorem 4.4 i) by symmetry. Hence, it could be possible to have a suitable ρ in this interval. Notice from

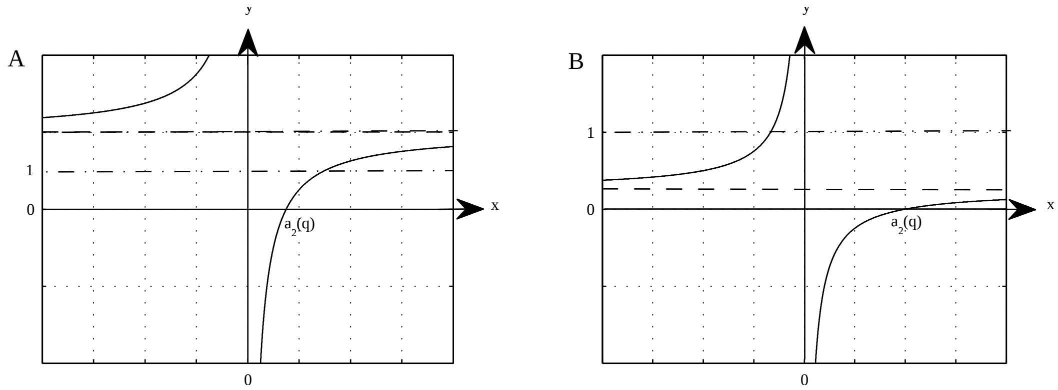


Figure 4.56: The function $f(x,q)$ with Case 1.i)A: $\Lambda_q > 1$, $a_2(q) > 0$, Case 1.ii)B: $0 < \Lambda_q < 1$, $a_2(q) > 0$.

Figure 4.56A that ρ is increasing on $(-\infty, 0)$ with $\rho \to \infty$ as $x \to 0^-$ (since $\rho(qx,q)/\rho(x,q) \to +\infty$) which leads to $\rho \to 0$ as $x \to -\infty$. We note that since the graph of ρ and $\sigma_1\rho$ are similar which is seen from the q-Pearson equation (4.1) and the identity given in (4.21), then $\sigma_1\rho \to \infty$ as $x \to 0^-$ which is the boundary condition when $k = 0$. As a result, the boundary condition is not satisfied at $x = 0$. Then, we can not use the interval $(-\infty, 0)$ for constructing ρ. we lastly consider the interval $(a_2(q), \infty)$ which coincides with the one given in Theorem 4.4 h). Therefore, here it could be possible to get a convenient ρ. Notice from Figure 4.56A that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = a_2(q)$. Hence, it follows that ρ is increasing on $(a_2(q), x_0)$ and decreasing on (x_0, ∞) which leads to $\rho \to 0$ as $x \to \infty$. However, it is not enough to assure that ρ satisfies the boundary condition at $+\infty$. We need to check $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$, $k = 0, 1, ...$ by using the *extended* q-Pearson equation (4.20). We remark that the graph of g looks like the one represented in Figure 4.56A but with the horizontal asymptote $0 < q^{k+2}\Lambda_q < 1$ for k large enough. That's why, $\sigma_1(x,q)\rho(x,q)x^k$ becomes increasing on $(a_2(q), \infty)$ which leads to $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$ as $x \to \infty$ for k large enough. Thus, this case does not lead to any suitable ρ with the needed properties.

Case 1.ii)B: $0 < \Lambda_q < 1$, $a_2(q) > 0$. The graph of f corresponds to this situation is represented in Figure 4.56B. Notice that f is negative on $(0, a_2(q))$ which enables us to skip this interval due to the positivity of ρ. Let us analyse the interval $(-\infty, 0)$. Notice that this interval coincides with the one given in Theorem 4.4 i) by the symmetry. Thus, here it could be possible to have a suitable ρ. Notice from Figure 4.56B that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 < x = 0$. Then, it follows that ρ is decreasing on $(-\infty, x_0)$ and increasing on $(x_0, 0)$ with $\rho \to +\infty$ as $x \to 0^-$ (since $\rho(qx,q)/\rho(x,q) \to +\infty$). Observe that since ρ is decreasing on

$(-\infty, x_0)$, then $\rho \not\to 0$ as $x \to -\infty \Longrightarrow \sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0,1,2,...$ as $x \to -\infty$. Observe also that $\rho \to +\infty$ as $x \to 0^-$. Since the graph of ρ and $\sigma_1\rho$ are similar (see (4.1) and (4.21)), then $\sigma_1\rho \to +\infty$ as $x \to 0^-$ which is the boundary condition when $k = 0$. Therefore, the boundary condition is also not satisfied as $x \to 0^-$. On the other hand, let us consider the last interval $(a_2(q), \infty)$ which is the one described in Theorem 4.4 h). Thus, this interval could be suitable for constructing ρ. Notice from Figure 4.56B that $f(x,q) < 1$ on this interval, thus ρ is increasing on $(a_2(q), \infty)$. Then, $\rho \not\to 0$ as $x \to \infty$ which leads to $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0,1,2,...$ as $x \to \infty$. Hence, this case does not lead to any suitable ρ.

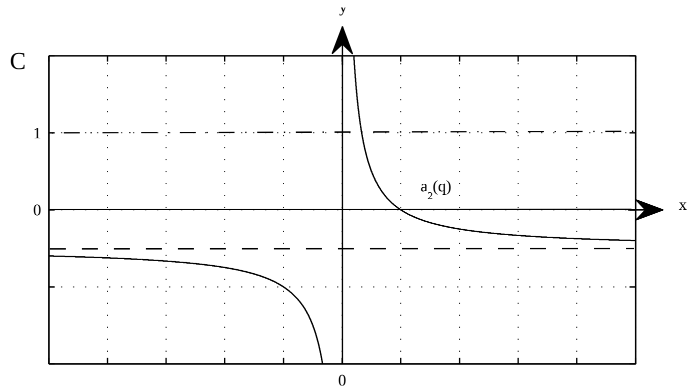


Figure 4.57: The function $f(x,q)$ with Case 2.C: $\Lambda_q < 0$, $a_2(q) > 0$.

Case 2.C: $\Lambda_q < 0$, $a_2(q) > 0$. The graph of f corresponds to this situation is represented in Figure 4.57C. By the positivity of ρ, the intervals $(-\infty, 0)$ and $(a_2(q), \infty)$ are both eliminated. The same happens for the interval $(0, a_2(q))$ due to the Remark 4.5.4. That's why, in this case, there is no suitable interval for constructing ρ with the needed properties.

4.3.2.2 The q-Classical 0-Bessel/Bessel Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2$, $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$ and assume that $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$ and $\tau(0,q) = 0 \Leftrightarrow a_2(q) = 0$. Then, it follows that $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2$ where

$$\frac{1}{2}\sigma_2''(0,q) = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right].$$

Thus, one can perform the q-Pearson equation as the following

$$\frac{\rho(qx,q)}{\rho(x,q)} = q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]. \tag{4.81}$$

Let denote Λ_q by the constant

$$\Lambda_q := q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0.$$

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ is constant.

Notice for the zero cases that a or b could be zero. That's why, we should know the behaviour of ρ at $x = 0$. To learn this we perform the following remark obtained from Remark 4.35.

Remark 4.43 *Behaviour of the q-weight function at* $x = 0$ *depends on the succesive solution of the q-Pearson equation*

$$\begin{aligned}\rho(qx,q) &= q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\rho(x,q)\\ \Leftrightarrow \rho(q^k x,q) &= q^{-2k}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]^k \rho(x,q).\end{aligned} \tag{4.82}$$

It is apparent that as $k \to \infty$ *the behaviour of* ρ *at* $x = 0$ *is accomplished. Notice that if* $0 < \Lambda_q = q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] < 1$, $\rho(x,q) \to 0$ *as* $x \to 0$ *otherwise it tends to* $\mp\infty$.

We introduce analogous analysis in order to obtain graphs of $\rho(qx,q)/\rho(x,q)$ according to the sign of Λ_q by taking Case 1. $\Lambda_q > 0$ and Case 2. $\Lambda_q < 0$. Nevertheless, as before, we need to split 1st case into two separate cases: Case 1.i) when $\Lambda_q > 1$ and Case 1.ii) when $0 < \Lambda_q < 1$.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.81).

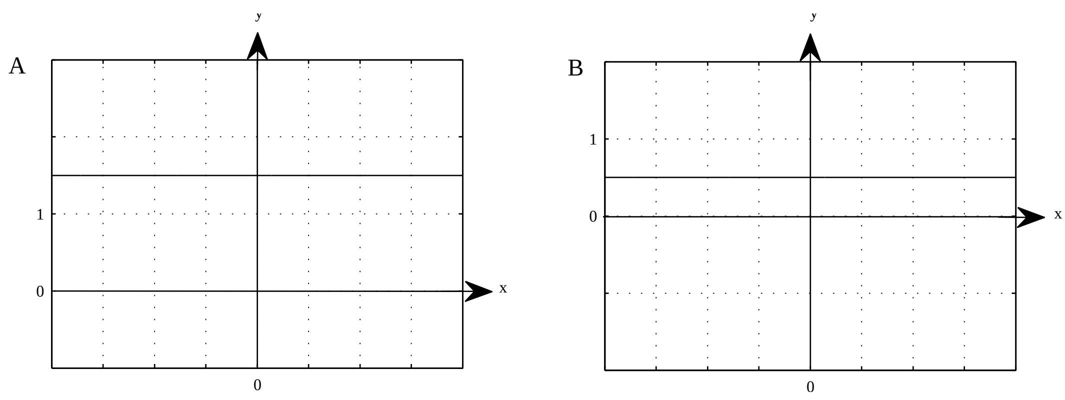


Figure 4.58: The function $f(x,q)$ with Case 1.i)A: $\Lambda_q > 1$, $a_2(q) = 0$, Case 1.ii)B: $0 < \Lambda_q < 1$, $a_2(q) = 0$.

Case 1.i)A: $\Lambda_q > 1$, $a_2(q) = 0$. The graph of f corresponds to this situation is represented in Figure 4.58A. Notice from Figure 4.58A that we have two intervals $(-\infty, 0)$ and $(0, \infty)$. Let us first analyse the interval $(0, \infty)$ which coincides with the one given in Theorem 4.4 i). Then, it could be possible to have a suitable ρ on $(0, \infty)$. Observe that $f > 1$ on this interval. Thus, it follows that ρ is decreasing on $(0, \infty)$ with $\rho \to +\infty$ as $x \to 0^+$ since $\Lambda_q > 1$ (see Remark 4.43). Since $\rho(qx, q)/\rho(x, q)$ has finite limit as $x \to +\infty$, then we have chance that $\rho \to 0$ as $x \to \infty$. But notice that $\rho \to +\infty$ as $x \to 0^+$. Since the graphs of ρ and $\sigma_1\rho$ are similar (see (4.1 and (4.21)), then $\sigma_1\rho \to +\infty$ as $x \to 0^+$ which is the boundary condition when $k = 0$. That's why, this interval does not lead to any suitable ρ with the needed properties. The same happens for the interval $(-\infty, 0)$ by the symmetry.

Case 1.ii)B: $0 < \Lambda_q < 1$, $a_2(q) = 0$. The graph of f corresponds to this situation is represented in Figure 4.58B. Notice that here also we have two intervals; $(-\infty, 0)$ and $(0, \infty)$. Let us deal with $(0, \infty)$ which is the one described in Theorem 4.4 i). Then, it could be possible to have a suitable ρ on $(0, \infty)$. Notice from Figure 4.58B that $f < 1$ on this interval which leads to that ρ is increasing on $(0, \infty)$ with $\rho \to 0$ as $x \to 0^+$ since $0 < \Lambda_q < 1$ (see Remark 4.43) which leads to $\rho \not\to 0$ as $x \to \infty \Rightarrow \sigma_1(x, q)\rho(x, q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to \infty$. Therefore, $(0, \infty)$ is not suitable interval for constructing ρ. The same happens for the interval $(-\infty, 0)$ by the symmetry.

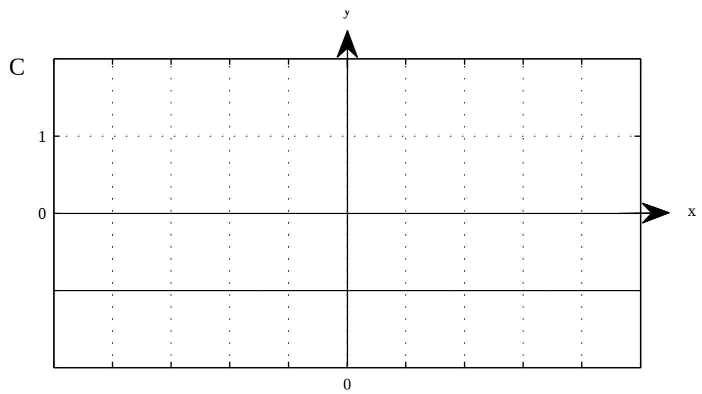


Figure 4.59: The function $f(x, q)$ with Case 2.C: $\Lambda_q < 0$, $a_2(q) = 0$.

Case 2.C: $\Lambda_q < 0$, $a_2(q) = 0$. The graph of f corresponds to this situation is represented in Figure 4.59C. Notice that ρ should be positive and f is negative on $(-\infty, +\infty)$. Hence, there is no interval for constructing positive ρ.

4.3.2.3 The q-Classical 0-Bessel/Laguerre Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2$, $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$ and assume that $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = -\frac{1}{(1-q^{-1})}$. Then, it follows that $\sigma_2(x,q) = \sigma_2'(0,q)x = q(1-q^{-1})\tau(0,q)x$. Thus, the q-Pearson equation follows

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}}{qx}. \tag{4.83}$$

Let denote Λ_q by the constant

$$\Lambda_q := \frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}.$$

By applying the analogous analysis according to the sign of Λ_q we get one independent graph of $\rho(qx,q)/\rho(x,q)$ constructed in Figure 4.60.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.83).

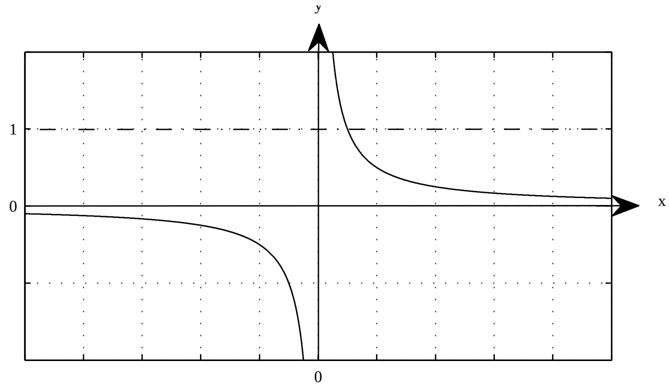


Figure 4.60: The function $f(x,q)$ with Case 1. $\Lambda_q < 0$.

Case 1: $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.60. Notice that Positivity of ρ allows to skip the interval $(-\infty,0)$. Thus, we have only the interval $(0,\infty)$ to analyse if it is possible to have a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 > x = 0$. Then, it follows that ρ is decreasing on $(0,x_0)$ with $\rho \to \infty$ as $x \to 0^+$ since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to \infty$ and increasing on (x_0,∞) which leads to $\rho \to \infty$ as $x \to \infty$ (since $\rho(qx,q)/\rho(x,q) \to 0$ as $x \to \infty$) $\Rightarrow \sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0,1,...$ as $x \to \infty$. As a result, this interval is not convenient for constructing ρ with the needed properties.

We continue with the same analysis to include the situation $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x$

$= \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)]$, $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$. Observe from (3.11) that $\sigma_2(x,q)$ becomes

$$\sigma_2(x,q) = qx\left\{\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right]x + (1-q^{-1})\tau(0,q) - \frac{1}{2}\sigma_1''(0,q)a_1(q)\right\}.$$

Remark 4.44 *We remark that* $\deg[\sigma_2(x,q)] = 2$ in case of $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq, -\frac{1}{(1-q^{-1})}$ otherwise $\deg[\sigma_2(x,q)] = 1$. We note that $\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = \frac{a_1(q)}{(1-q^{-1})}$, leads to $\sigma_2(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2$. Consequently, the q-Jacobi type zero family of the 1st kind is the q-Jacobi, the q-Bessel and the q-Laguerre type zero family of the 2nd kind.

4.3.2.4 The q-Classical 0-Jacobi/Jacobi Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)]$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$ and assume that $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$. Then, observe from (3.11) that $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x$ where

$$\frac{1}{2}\sigma_2''(0,q) = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right],\ \sigma_2'(0,q) = q(1-q^{-1})\tau(0,q) - \frac{1}{2}\sigma_1''(0,q)a_1(q).$$

Hence, the q-Pearson equation follows from (4.1) as the following form

$$\begin{aligned}\frac{\rho(qx,q)}{\rho(x,q)} &= \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}\\ &= \frac{\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x + (1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} - a_1(q)}{q[qx - a_1(q]}\\ &= q^{-1}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{x - a_2(q)}{qx - a_1(q)}\end{aligned} \tag{4.84}$$

where $a_2(q) = -\frac{(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} - a_1(q)}{\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]} \neq 0$.

Remark 4.45 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.84). Then notice that*

$$\Lambda_q := q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0$$

is the horizontal asymptote of the function $f(x,q)$.

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ in (4.84) always intercepts y-axis at the point

$$y := y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

Notice for the zero cases that a or b could be zero. That's why, we should know the behaviour of ρ at $x = 0$. To learn this we perform the following remark.

Remark 4.46 *Notice that behaviour of the q-weight function at $x = 0$ depends on the successive solution of the q-Pearson equation*

$$\begin{aligned}\rho(qx,q) &= \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}\rho(x,q) \Leftrightarrow \rho(q^k x,q) = \rho(x,q)\prod_{i=0}^{k-1}\frac{q^{-1}\sigma_2(q^i x,q)}{\sigma_1(q^{i+1}x,q)} \\ \Leftrightarrow \rho(q^k x,q) &= q^{-k}\left[1-\frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]^k \frac{(x/a_2(q);q)_k}{(qx/a_1(q);q)_k}\rho(x,q). \qquad (4.85)\end{aligned}$$

It is apparent that as $k \to \infty$ the behaviour of ρ at $x = 0$ is accomplished. Notice that if $0 < y_0 = q^{-1}[1-\frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}]<1$, $\rho(x,q) \to 0$ as $x \to 0$ otherwise it tends to $\mp\infty$.

In order not to lose any graphs of $\rho(qx,q)/\rho(x,q)$, we consider every case by taking Case 1. $\Lambda_q > 0$ and Case 2. $\Lambda_q < 0$ together with $y_0 < 1$, $y_0 > 1$. Furthermore, as before, we need to split 1st case into two separate cases: Case 1.i) when $\Lambda_q > 1$ and Case 1.ii) when $0 < \Lambda_q < 1$. Then, our next step is to dispose the order of the zeros of $\sigma_1(x,q)$ and $\sigma_2(x,q)$ according to the knowledge that we discussed above which give all possible graphs for the ratio $\rho(qx,q)/\rho(x,q)$.

Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.84).

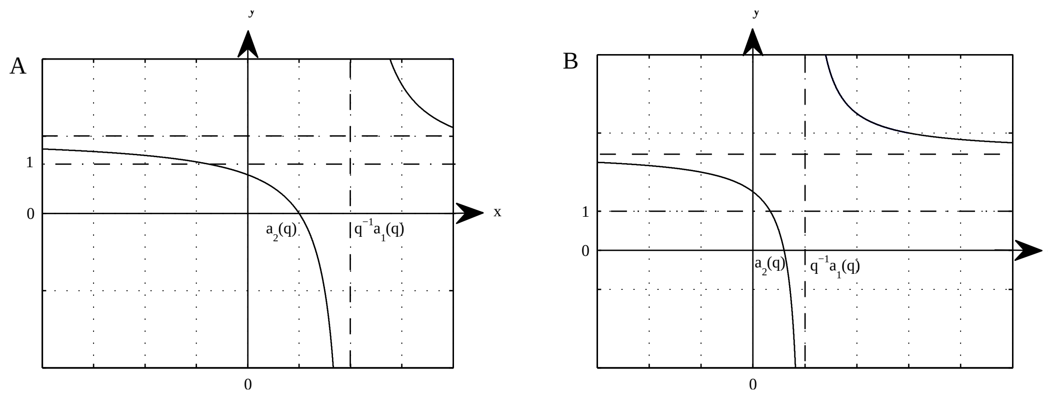


Figure 4.61: Case 1.i) The function $f(x,q)$ with $\Lambda_q > 1$, $0 < a_2(q) < q^{-1}a_1(q)$, A: $0 < y_0 < 1$, B: $y_0 > 1$.

Case 1.i)A: $0 < a_2(q) < q^{-1}a_1(q)$, $0 < y_0 < 1$, $\Lambda_q > 1$. The graph of f for this case is represented in Figure 4.61A. Let us consider now the possible intervals in which we can have a suitable q-weight function ρ. As we've already mentioned, they are defined by the zeros of the polynomials σ_1 and σ_2 and the positions of y_0 according to one. First of all, notice that since ρ should be positive and f is negative in the interval $(a_2(q), q^{-1}a_1(q))$, it is not suitable. The other intervals $(0, a_2(q))$ and $(q^{-1}a_1(q), \infty)$ are both eliminated due to the Remark 4.5.4 and Remark 4.5.5, respectively. The last interval $(-\infty, 0)$ is the one described in Theorem 4.4 i) by the symmetry about y-axis. That's why, it could be possible to have a suitable ρ on this interval. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 < x = 0$, then from Figure 4.61A it follows that ρ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, 0)$ with $\rho \to 0$ as $x \to 0^-$ since $0 < y_0 < 1$ (see Remark 4.46). Since $\rho(qx, q)/\rho(x, q)$ has a finite limit as $x \to -\infty$, we have also the chance that $\rho \to 0$ as $x \to -\infty$, but it is not enough to assure that ρ satisfies the boundary condition at $-\infty$. In fact, as it is stated in Theorem 3.31, we should check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to -\infty$ by using analysis of the *extended* q-Pearson equation (4.20). However, the graph of g looks like the one represented in Figure 4.61A together with the property that $g(x, q) < 1$ on $(-\infty, 0)$ for k large enough, which leads to that $\sigma_1(x, q)\rho(x, q)x^k$ is decreasing on $(-\infty, 0)$ with $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ as $x \to -\infty$. Therefore, this case does not lead to any suitable ρ.

Case 1.i)B: $0 < a_2(q) < q^{-1}a_1(q)$, $y_0 > 1$, $\Lambda_q > 1$. The graph of f corresponds to this situation is represented in Figure 4.61B. Notice that Figure 4.61B is analog to the Figure 4.61A. They differ only for the y-intercept; y_0. That's why, with the similar reason that we perform in Case 1.i)A, we eliminate the intervals $(a_2(q), q^{-1}a_1(q))$, $(0, a_2(q))$ and $(q^{-1}a_1(q), \infty)$. We only need to analyse the interval $(-\infty, 0)$. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = 0 < x_0 < x = a_2(q)$, then from Figure 4.61B it follows that ρ is increasing on $(-\infty, 0)$. Since $y_0 > 1$, then $\rho \to \infty$ as $x \to 0^-$ (see Remark 4.46) and since $\rho(qx, q)/\rho(x, q)$ has a finite limit as $x \to -\infty$, we have also the chance that $\rho \to 0$ as $x \to -\infty$, but it is not enough to assure that ρ satisfies the boundary conditions at $-\infty$. We should check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to -\infty$ by using analysis of the *extended* q-Pearson equation (4.20). However, the graph of g looks like the one discussed in Case 1.i)A, that's why it leads to the same result that $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ as $x \to -\infty$.

Case 1.i)C: $0 < q^{-1}a_1(q) < a_2(q)$, $y_0 > 1$, $\Lambda_q > 1$. The graph of f is represented in Figure 4.62C. The positivity of ρ allows us to skip the interval $(q^{-1}a_1(q), a_2(q))$. Let us consider the intervals $(-\infty, 0)$ and $(0, q^{-1}a_1(q))$ which coincide the ones described in Theorem 4.4 f) (by symmetry) and Theorem 4.4 b). Since $y_0 > 1$ and $\rho(qx, q)/\rho(x, q) > 1$ on $(-\infty, 0)$ and

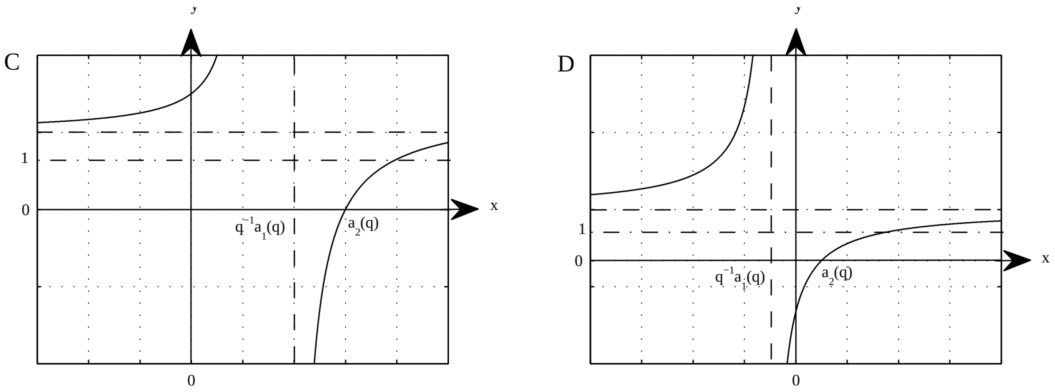


Figure 4.62: Case 1.i) The function $f(x, q)$ with $\Lambda_q > 1$, C: $0 < q^{-1}a_1(q) < a_2(q)$, $y_0 > 1$, D: $q^{-1}a_1(q) < 0 < a_2(q)$, $y_0 < 0$.

$(0, q^{-1}a_1(q))$ it follows from Figure 4.62C that ρ is increasing on $(-\infty, 0)$ and decreasing on $(0, q^{-1}a_1(q))$ with $\rho \to \infty$ as $x \to 0$. Notice from the identity

$$\frac{\sigma_1(qx, q)\rho(qx, q)}{\sigma_1(x, q)\rho(x, q)} = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(x, q)}$$

that the graph of $\sigma_1(x, q)\rho(x, q)$ have the same property with $\rho(x, q)$. That's why, from Figure 4.62C we see that the boundary condition (3.115) does not satisfied at $x = 0$ when $k = 0$ which is the reason for eliminating them. Let us consider the last interval $(a_2(q), \infty)$. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 > x = a_2(q)$, then from Figure 4.62C it follows that ρ is decreasing on $(-\tau(0, q)/\tau'(0, q), \infty)$. Since $\rho(qx, q)/\rho(x, q)$ has a finite limit as $x \to +\infty$, we have the chance that $\rho \to 0$ as $x \to \infty$, but it is not enough to assure that ρ satisfies the boundary conditions at $+\infty$. In fact, we should check that $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to \infty$ by using the *extended* q-Pearson equation (4.20). But by the graph of g we see that $\sigma_1(x, q)\rho(x, q)x^k$ is increasing on $(a_2(q), \infty)$ for k large enough which implies that $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ as $x \to \infty$. Therefore, this case does not lead to any suitable ρ with the needed properties.

Case 1.i)D: $q^{-1}a_1(q) < 0 < a_2(q)$, $y_0 < 0$, $\Lambda_q > 1$. The graph of f corresponds to this situation is represented in Figure 4.62D. Analogous analysis allows us to exclude the negative intervals $(q^{-1}a_1(q), 0)$ and $(0, a_2(q))$. On the other hand, the interval $(-\infty, q^{-1}a_1(q))$ is eliminated due to Remark 4.5.5 with symmetry. The last interval $(a_2(q), \infty)$ is also excluded because of the same reason applied in Case 1.i)C. That's why, we can not have a suitable ρ.

Case 1.i)E: $0 < a_2(q) < q^{-1}a_1(q)$, $0 < y_0 < 1$, $0 < \Lambda_q < 1$. The graph of f corresponds to this situation is represented in Figure 4.63E. Notice that f is negative on $(a_2(q), q^{-1}a_1(q))$. Then,

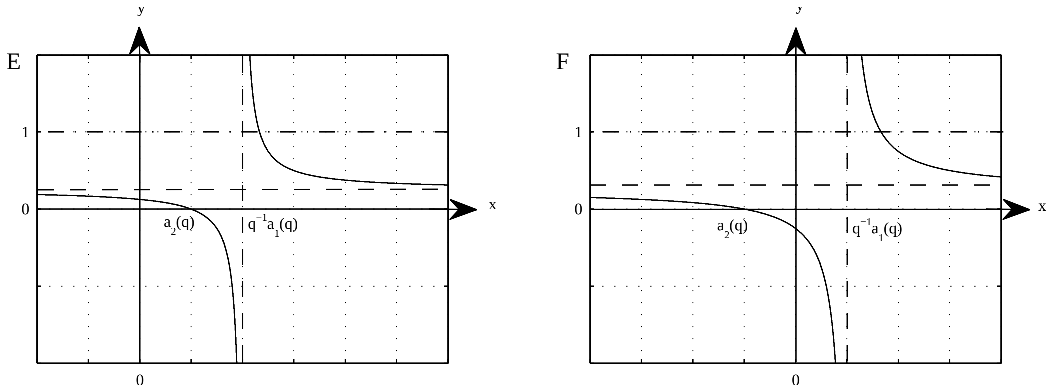


Figure 4.63: Case 1.ii) The function $f(x,q)$ with $0 < \Lambda_q < 1$, E: $0 < a_2(q) < q^{-1}a_1(q)$, $0 < y_0 < 1$, F: $a_2(q) < 0 < q^{-1}a_1(q)$, $y_0 < 0$.

it can not be used. Moreover, Remark 4.5.4 and Remark 4.5.5 by symmetry allows us to skip the intervals $(0, a_2(q))$ and $(q^{-1}a_1(q), \infty)$ since they do not lead to a suitable ρ satisfying the q-Pearson equation (3.24) and the boundary conditions. Let us now deal with the last interval $(-\infty, 0)$ which coincide with the one described in Theorem 4.4 i) by symmetry about y-axis. Therefore, here it could be possible to have a suitable ρ. Notice from Figure 4.63E that ρ is decreasing on $(-\infty, 0)$ with the property $\rho \to 0$ as $x \to 0$ since $0 < y_0 < 1$. It seems that $\rho \not\to 0$ as $x \to -\infty$ which leads to $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to -\infty$. Therefore, there is no suitable interval for ρ.

Case 1.i)F: $a_2(q) < 0 < q^{-1}a_1(q), y_0 < 0, 0 < \Lambda_q < 1$. The graph of f is represented in Figure 4.63F. The positivity property enables us to skip the intervals $(a_2(q), 0)$ and $(0, q^{-1}a_1(q))$. The same happens for the interval $(q^{-1}a_1(q), \infty)$ due to Remark 4.5.5. The last interval $(-\infty, a_2(q))$ is the one described in Theorem 4.4 h) by symmetry. That's why, it could be possible to obtain a suitable ρ on this interval. Notice from Figure 4.63F that ρ is decreasing on $(-\infty, a_2(q))$ together with $\rho(qa_2(q), q) = 0$. Therefore, we have $\rho \not\to 0$ as $x \to -\infty$ which leads to $\sigma_1(x,q)\rho(x,q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to -\infty$.

Case 1.ii)G: $0 < q^{-1}a_1(q) < a_2(q)$, $0 < y_0 < 1$, $0 < \Lambda_q < 1$. The graph of f corresponds to this situation is represented in Figure 4.64G. Let us analyse the each interval analog to the before cases. First of all, one can eliminate the interval $(q^{-1}a_1(q), a_2(q))$ due to the positivity. The interval $(-\infty, 0)$ is the one described in Theorem 4.4 i) by symmetry. However, it is seen from Figure 4.64G that ρ is decreasing on $(-\infty, 0)$ with $\rho \to 0$ as $x \to 0$ since $0 < y_0 < 1$ (see Remark 4.46) which leads to $\rho \not\to 0$ as $x \to -\infty \Longrightarrow \sigma_1(x,q)\rho(x,q)x^k \not\to 0$,

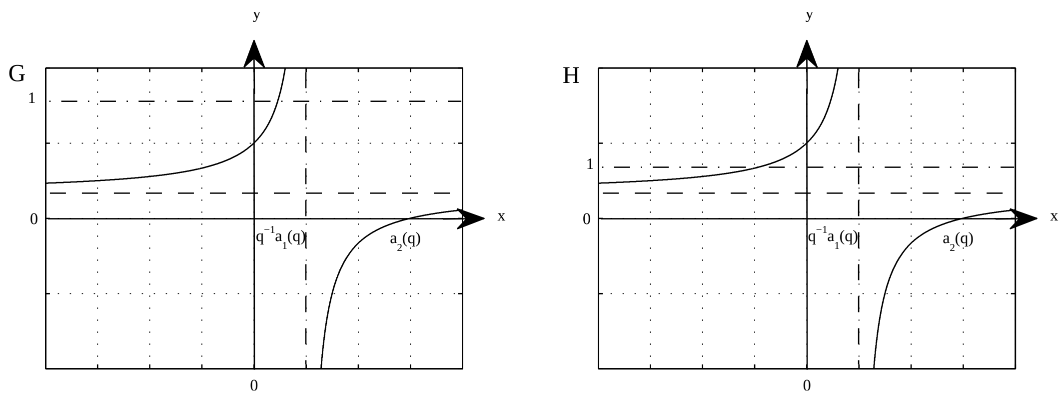


Figure 4.64: Case 1.ii) The function $f(x, q)$ with $0 < \Lambda_q < 1$, $0 < q^{-1}a_1(q) < a_2(q)$, G: $0 < y_0 < 1$, H: $y_0 > 1$.

$k = 0, 1, 2, ...$ as $x \to -\infty$. Let us deal with the interval $(a_2(q), \infty)$ which is the one given in Theorem 4.4 h). Then, here it could be possible to have a suitable ρ. It follows from Figure 4.64G that ρ is increasing on $(a_2(q), \infty)$ with the property that $\rho(qa_2(q), q) = 0$ since $\rho(qa_2(q), q)/\rho(a_2(q), q) = 0$. Then, we have $\rho \not\to 0$ as $x \to \infty$ leading to $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to \infty$.

In order to analyse the last interval $(0, q^{-1}a_1(q))$ which is the one described in Theorem 4.4 b) notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = 0 < x_0 < x = a_2(q)$, then from Figure 4.64G it follows that ρ is increasing on $(0, x_0)$ with $\rho \to 0$ as $x \to 0$ since $0 < y_0 < 1$ and decreasing on $(x_0, q^{-1}a_1(q))$ with $\rho(x, q) \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx, q)/\rho(x, q) \to \infty$. According to the above discussion one can easily sketch the graph of ρ which is represented in Figure 4.65 assuming a positive initial value for the q-weight function in each interval.

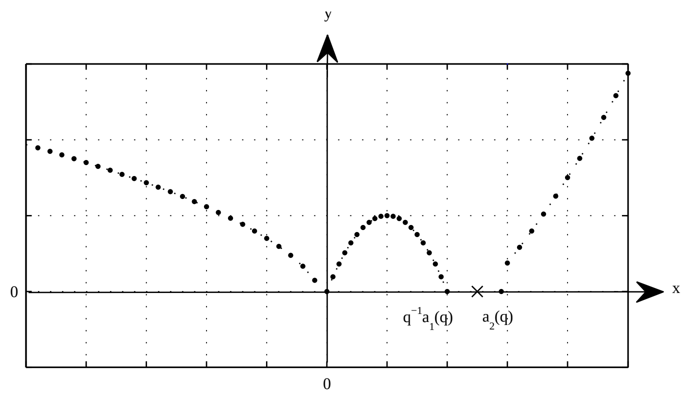


Figure 4.65: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.64G.

We also infer from Figure 4.65 that $(0, a_1(q))$ is suitable interval to have a q-weight function supported at the points $a_1(q)q^k$, $k = 0, 1, \ldots$ (see Theorem 4.4 b)). Thus, according to this result we construct the following theorem.

Theorem 4.47 *Let $a = 0$ and $b = a_1(q)$ be the zeros of $\sigma_1(x, q)$ and assume that $0 < q^{-1}a_1(q) < a_2(q)$, $0 < y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)} \frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] < 1$, and $0 < \Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 1$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b), i.e., they satisfy the orthogonality (4.7) with respect to the q-weight function*

$$\rho(x, q) = x^{\alpha} \frac{(qa_1^{-1}(q)x; q)_\infty}{(a_2^{-1}(q)x; q)_\infty} > 0, \; x \in (a, b) \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0, q)a_2(q)}{\frac{1}{2}\sigma_1''(0, q)a_1(q)} \tag{4.86}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 b)).

This case corresponds to the case IVa3 in Chapter 10 of [35, pages 277 and 311].

An example of such family is the little q-Jacobi polynomials [35] where $a_1(q) = 1$, $a_2(q) = b^{-1}q^{-1}$,

$$\sigma_1(x, q) = q^{-2}x(x - 1), \quad \sigma_2(x, q) = ax(bqx - 1),$$
$$\tau(x, q) = \frac{1 - abq^2}{(1 - q)q}x + \frac{aq - 1}{(1 - q)q}, \quad \lambda_n(q) = -q^{-n}[n]_q \frac{1 - abq^{n+1}}{1 - q}.$$

Little q-Jacobi polynomials are orthogonal on $(0, 1)$ and the conditions $0 < q^2\Lambda_q < 1$, $0 < qy_0 < 1$ and $0 < a_1(q) < a_2(q)$ give us the following restriction for the parameters $0 < a < q^{-1}$, $0 < b < q^{-1}$. By means of Theorem 4.4 b) we can write the orthogonality relation of little q-Jacobi polynomials

$$\int_0^1 x^{\alpha} \frac{(qx; q)_\infty}{(bqx; q)_\infty} P_m(x; a, b|q) P_n(x; a, b|q) d_q x = \frac{(aq)^n(1 - abq)}{(1 - abq^{2n+1})} \frac{(q, bq; q)_n}{(aq, abq; q)_n} \frac{(q, abq^2; q)_\infty}{(aq, bq; q)_\infty} \delta_{mn} \tag{4.87}$$

with $0 < a = q^{\alpha} < q^{-1}$, $0 < b < q^{-1}$.

Case 1.ii)H: $0 < q^{-1}a_1(q) < a_2(q)$, $y_0 > 1$, $0 < \Lambda_q < 1$. The graph of f is represented in Figure 4.64H. It is clear that Figure 4.64G and Figure 4.64H differ only for the y-intercept; y_0. Then, we eliminate the intervals $(q^{-1}a_1(q), a_2(q))$ and $(a_2(q), \infty)$ because of the same reason performed in Case 1.ii)G. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 < x = 0$. Then, from Figure 4.64H it follows that ρ is decreasing on $(-\infty, x_0)$ and increasing on $(x_0, 0)$ with $\rho \to \infty$ as $x \to 0$ since $y_0 > 1$ (see Remark 4.46). Observe that since ρ and $\sigma_1\rho$ have the same graphs then $\sigma_1\rho \to \infty$ as $x \to 0$ which indicates that the boundary condition (3.115) does not satisfied at $x = 0$ when $k = 0$, that's why, $(-\infty, 0)$ is not a suitable interval in order to get a ρ with needed properties. Notice that the same happens for the interval $(0, q^{-1}a_1(q))$ since $\rho \to \infty$ as $x \to 0$.

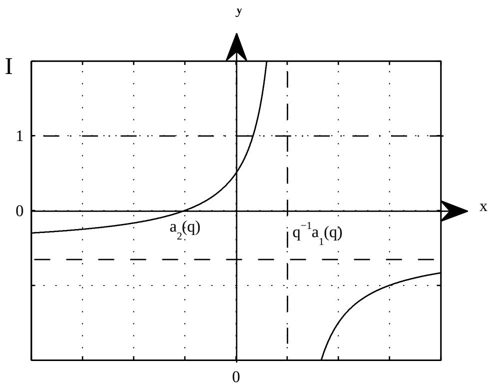


Figure 4.66: Case 2. The function $f(x, q)$ with $\Lambda_q < 0$, $a_2(q) < 0 < q^{-1}a_1(q)$, I: $0 < y_0 < 1$, J: $y_0 > 1$.

Case 2.I: $a_2(q) < 0 < q^{-1}a_1(q)$, $0 < y_0 < 1$, $\Lambda_q < 0$. The graph of f corresponds to this case is represented in Figure 4.66I. The positivity of ρ allows us to skip the intervals $(-\infty, a_2(q))$ and $(q^{-1}a_1(q), \infty)$. On the other hand, $(a_2(q), 0)$ is eliminated with the help of Remark 4.5.4 by symmetry. Notice that the last interval $(0, q^{-1}a_1(q))$ is the one described in Theorem 4.4 b). Then, here it could be possible to have a suitable ρ. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = 0 < x_0 < x = q^{-1}a_1(q)$. Thus, it follows that ρ is increasing on $(0, x_0)$ with $\rho \to 0$ as $x \to 0$ since $0 < y_0 < 1$ (see Remark 4.46) and decreasing on $(x_0, q^{-1}a_1(q))$ with $\rho(x, q) \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx, q)/\rho(x, q) \to +\infty$ as $x \to q^{-1}a_1(q)^-$. Then, according to above discussion it is clear to sketch Figure 4.67 by assuming a positive initial value for the q-weight function in each interval.

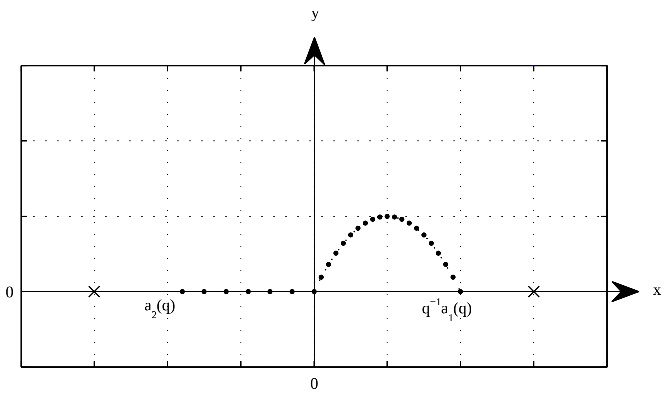


Figure 4.67: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.66I.

It is obvious from Figure 4.67 that there exists a suitable ρ on $(0, a_1(q))$ supported at the points $a_1(q)q^k$, $k = 0, 1, ...$ (see Theorem 4.4 b)) which leads to the following theorem.

Theorem 4.48 *Let $a = 0$ and $b = a_1(q)$ be the zeros of $\sigma_1(x,q)$ and assume that $a_2(q) < 0 < q^{-1}a_1(q)$, $0 < y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] < 1$, and $\Lambda_q = q^{-2}[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.7) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha}\frac{(qa_1^{-1}(q)x;q)_\infty}{(a_2^{-1}(q)x;q)_\infty} > 0,\ x \in (a,b) \quad q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)a_2(q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)} \tag{4.88}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 b)).

This case corresponds to the case IVa4 in Chapter 10 of [35, pages 278 and 312].

An example of such family is the little q-Jacobi polynomials [35] where $a_1(q) = 1$, $a_2(q) = b^{-1}q^{-1}$,

$$\sigma_1(x,q) = q^{-2}x(x-1), \quad \sigma_2(x,q) = ax(bqx-1),$$
$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{aq-1}{(1-q)q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-abq^{n+1}}{1-q}.$$

Little q-Jacobi polynomials are orthogonal on $(0,1)$ and the conditions $q^2\Lambda_q < 0$, $0 < qy_0 < 1$ and $a_2(q) < 0 < a_1(q)$ give us the following restriction for the parameters $0 < a < q^{-1}$, $b < 0$. By means of Theorem 4.4 b) we can write the orthogonality relation of little q-Jacobi polynomials

$$\int_0^1 x^{\alpha}\frac{(qx;q)_\infty}{(bqx;q)_\infty}P_m(x;a,b|q)P_n(x;a,b|q)d_qx = \frac{(aq)^n(1-abq)}{(1-abq^{2n+1})}\frac{(q,bq;q)_n}{(aq,abq;q)_n}\frac{(q,abq^2;q)_\infty}{(aq,bq;q)_\infty}\delta_{mn} \tag{4.89}$$

which coincides with (4.87) but with a different choice of parameters, $0 < a = q^{\alpha} < q^{-1}$, $b < 0$.

Case 2.J: $a_2(q) < 0 < q^{-1}a_1(q)$, $y_0 > 1$, $\Lambda_q < 0$. The graph of f corresponds to this situation is represented in Figure 4.66J. Notice that Figure 4.66J is analog to Figure 4.66I. The difference is the y-intercept; y_0. Then, we all eliminate the intervals except $(0, q^{-1}a_1(q))$ because of the same reason applied in Case 2.I. Notice that since $y_0 > 1$, then $\rho \to \infty$ as $x \to 0$. Thus, the interval $(0, q^{-1}a_1(q))$ does not have the same property with the one in Case 2.I. Observe that the graph of ρ and $\sigma_1\rho$ are same. Hence, we have $\sigma_1\rho \to \infty$ as $x \to 0$ which is the boundary condition when $k = 0$. That's why, this interval can not be used for ρ. Therefore, this case does not lead to any suitable ρ.

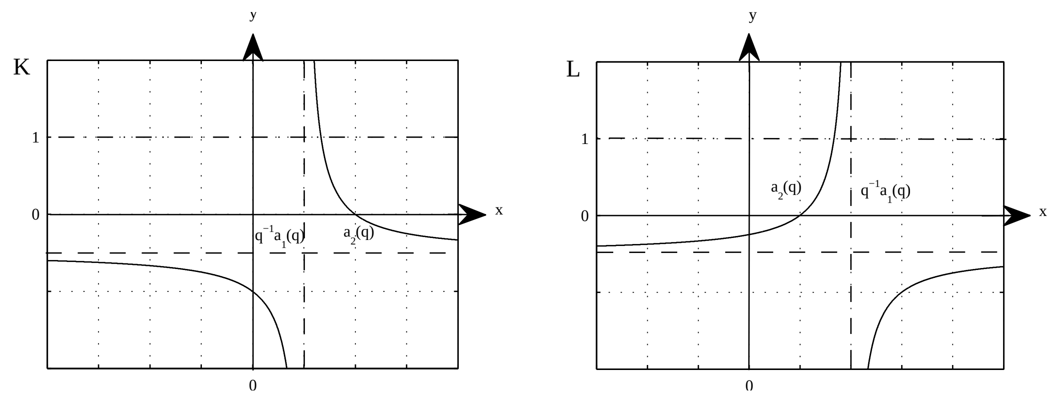


Figure 4.68: Case 2. The function $f(x,q)$ with $\Lambda_q < 0$, $y_0 < 0$, K: $0 < q^{-1}a_1(q) < a_2(q)$, L: $0 < a_2(q) < q^{-1}a_1(q)$.

Case 2.K: $0 < q^{-1}a_1(q) < a_2(q)$, $y_0 < 0$, $\Lambda_q < 0$. The graph of f corresponds to this case is represented in Figure 4.68K. Notice that positivity of ρ enables us to skip the intervals $(-\infty, 0)$, $(0, q^{-1}a_1(q))$ and $(a_2(q), \infty)$. Note that one can also eliminate the rest interval $(q^{-1}a_1(q), a_2(q))$ due to Remark 4.5.4. Therefore, this case does not lead to any suitable ρ.

Case 2.L: $0 < a_2(q) < q^{-1}a_1(q)$, $y_0 < 0$, $\Lambda_q < 0$. The graph of f is represented in Figure 4.68L. $(-\infty, 0)$, $(0, a_2(q))$ and $(q^{-1}a_1(q), \infty)$ are eliminated since ρ should be positive. But, $(a_2(q), q^{-1}a_1(q))$ which coincides with the one described in Theorem 4.4 d) could be possible to construct a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x = a_2(q) < x_0 < x = q^{-1}a_1(q)$. Thus, ρ is increasing on $(a_2(q), x_0)$ with $\rho(qa_2(q), q) = 0$ since $\rho(qa_2(q), q)/\rho(a_2(q), q) = 0$ and decreasing on $(x_0, q^{-1}a_1(q))$ with $\rho \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}a_1(q)^-$. It is obvious that, imposing the above discussion to Figure 4.68L allows us to sketch Figure 4.69.

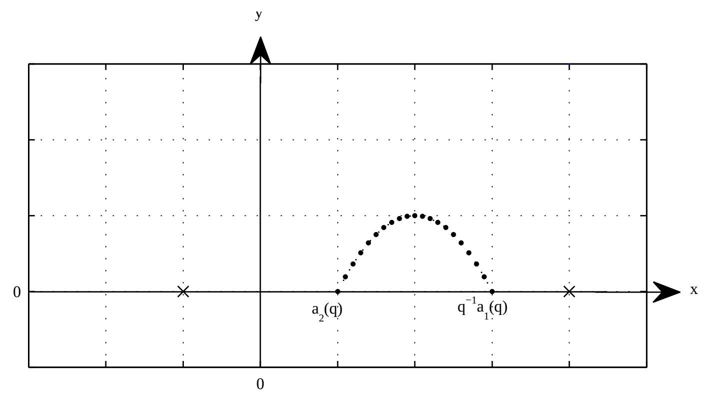


Figure 4.69: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.68L.

We infer from Figure 4.69 that $(a_2(q), q^{-1}a_1(q))$ is the suitable interval in which we can have ρ supported at the points $q^{-k}a_2(q)$, $k = 0, 1, ...$ Therefore, we build the following theorem.

Theorem 4.49 *Let $a = qa_2(q)$ be the zero of $\sigma_2(q^{-1}x, q)$ and $b = a_1(q)$ of $\sigma_1(x, q)$ and assume that $0 < a_2(q) < q^{-1}a_1(q)$, $y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] < 0$, and $\Lambda_q = q^{-2}[1 + (1 - q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b), i.e., they satisfy the orthogonality (4.9) and (4.11) with respect to the q-weight function*

$$\rho(x, q) = x^{\alpha}\sqrt{x^{\log_q x-1}}(qa_2(q)/x, qa_1^{-1}(q)x; q)_\infty > 0,\ x \in (a, b) \tag{4.90}$$

$q^{\alpha} = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)}$ which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 d)).

This case corresponds to the case IIIb with $g = 0$ in Chapter 11 of [35, page 343] which is not given explicitly.

An example of such family is the q-Kravchuk polynomials [35] where $a_1(q) = q^{-N}$, $a_2(q) = 1$,

$$\sigma_1(x, q) = q^{-2}x(x - q^{-N}), \quad \sigma_2(x, q) = -px(x - 1),$$

$$\tau(x, q) = \frac{1 + pq}{(1 - q)q}x - \frac{p + q^{-N-1}}{1 - q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1 + pq^n}{1 - q}.$$

q-Kravchuk polynomials are orthogonal on $(1, q^{-N-1})$ and the conditions $q^2\Lambda_q < 0$, $qy_0 < 0$ and $0 < a_2(q) < a_1(q)$ give us the following restriction for the parameters $p > 0$. By means of Theorem 4.4 d) we can write the orthogonality relation of q-Kravchuk polynomials

$$\int_1^{q^{-N-1}} x^{\alpha+N}\sqrt{x^{\log_q x-1}}(q/x, q^{N+1}x; q)_\infty K_m(x; p, N; q)K_n(x; p, N; q)d_{q^{-1}}x = (q^{-1} - 1)p^{-N}$$
$$\times q^{-\binom{N+1}{2}}(-pq^{-N})^n q^{n^2}\frac{1 + p}{1 + pq^{2n}}(-pq; q)_N(q, q^{N+1}; q)_\infty\frac{(q, -pq^{N+1}; q)_n}{(-p, q^{-N}; q)_n}\delta_{mn} \tag{4.91}$$

associated with $p > 0$. Notice from Theorem 4.4 d) that one can write the orthogonality with finite sum by applying (2.31) to (4.91)

$$\sum_{x=0}^{N}\frac{(q^{-N}; q)_x}{(q; q)_x}(-p)^{-x}K_m(q^{-x}; p, N; q)K_n(q^{-x}; p, N; q) = p^{-N}q^{-\binom{N+1}{2}}(-pq^{-N})^n q^{n^2}$$
$$\times\frac{1 + p}{1 + pq^{2n}}(-pq; q)_N\frac{(q, -pq^{N+1}; q)_n}{(-p, q^{-N}; q)_n}\delta_{mn}. \tag{4.92}$$

4.3.2.5 The q-Classical 0-Jacobi/Bessel Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x \quad [x - a_1(q)]$ and $\tau(x,q) = \tau'(0,q)x + \tau(0,q)$, $\tau'(0,q) \neq 0$ and assume that $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$ and $a_2(q) = 0 \Leftrightarrow \frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = \frac{a_1(q)}{(1-q^{-1})}$. Then, observe from (3.11) that $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right]x^2$. As a result, the q-Pearson equation follows from (4.1)

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x}{q[qx - a_1(q]}. \tag{4.93}$$

Remark 4.50 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.93). Then notice that*

$$\Lambda_q := q^{-2}\left[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0$$

is the horizontal asymptote of the function $f(x,q)$.

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ in (4.93) always intercepts y-axis at $y := y_0 = 0$.

We introduce analogous analysis in order to obtain independent graphs of $\rho(qx,q)/\rho(x,q)$ according to the sign of $a_1(q)$ (zero of σ_1) and of Λ_q by taking Case 1. $\Lambda_q > 0$ and Case 2. $\Lambda_q < 0$. Nevertheless, as before, we need to split 1st case into two separate cases: Case 1.i) when $\Lambda_q > 1$ and Case 1.ii) when $0 < \Lambda_q < 1$.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.93).

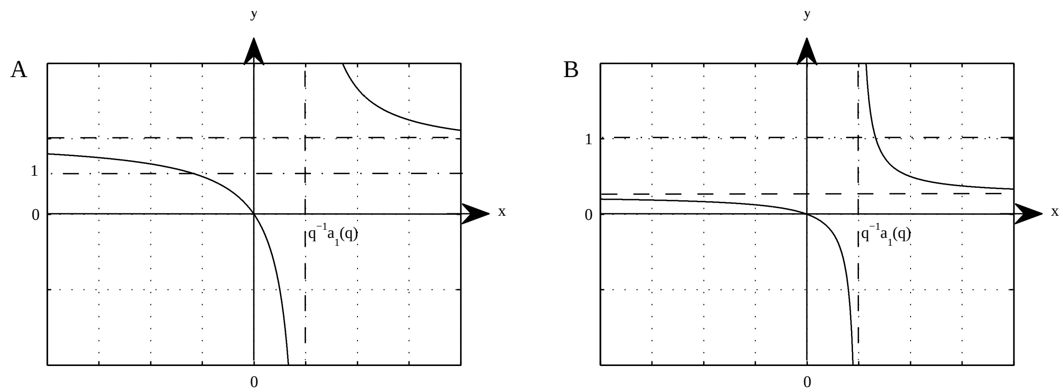


Figure 4.70: The function $f(x,q)$ with $a_1(q) > 0$, Case 1.i)A: $\Lambda_q > 1$, Case 1.ii)B: $0 < \Lambda_q < 1$.

Case 1.i)A: $\Lambda_q > 1$, $a_1(q) > 0$. The graph of f corresponds to this case is represented in Figure 4.70A. First of all the interval $(0, q^{-1}a_1(q))$ is eliminated due to the positivity of ρ.

The same happens for the interval $(q^{-1}a_1(q), \infty)$ because of the reason given in Remark 4.5.5. Notice that the last interval $(-\infty, 0)$ is the one described in Theorem 4.4 i) by symmetry. That's why, this interval could be possible to construct a suitable ρ. Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x_0 < x = 0$. Thus, it follows that ρ is incerasing on $(-\infty, x_0)$ and decreasing on $(x_0, 0)$ with $\rho \to 0$ as $x \to 0^-$ since $\rho(qx, q)/\rho(x, q) \to 0$ as $x \to 0^-$ which leads to $\rho \to 0$ as $x \to -\infty$. But since the interval is infinite we need to check $\sigma_1(x, q)\rho(x, q)x^k \to 0$ as $x \to -\infty$ by using *extended* q-Pearson equation (4.20). We note that the graph of g looks like the one represented Figure 4.70A but with the horizontal asymptote $0 < q^{k+2}\Lambda_q < 1$ which leads to that $\sigma_1(x, q)\rho(x, q)x^k \not\to 0$ as $x \to -\infty$. As a result, this case does not lead to any suitable ρ.

Case 1.ii)B: $0 < \Lambda_q < 1$, $a_1(q) > 0$. The graph of f corresponds to this case is represented in Figure 4.70B. Notice that an analogous analysis as the one that has been done in Case 1.i)A leads to eliminate the intervals $(0, q^{-1}a_1(q))$ and $(q^{-1}a_1(q), \infty)$. Thus, we only analyse the interval $(-\infty, 0)$ which coincides with the one described in Theorem 4.4 i) by symmetry. Observe from Figure 4.70B that ρ is decreasing on $(-\infty, 0)$ with $\rho \to 0$ as $x \to 0^-$ since $\rho(qx, q)/\rho(x, q) \to 0$ as $x \to 0^-$ which leads to $\rho \not\to 0$ as $x \to -\infty \Longrightarrow \sigma_1(x, q)\rho(x, q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to -\infty$. Hence, we can also not use the interval $(-\infty, 0)$ for constructing ρ.

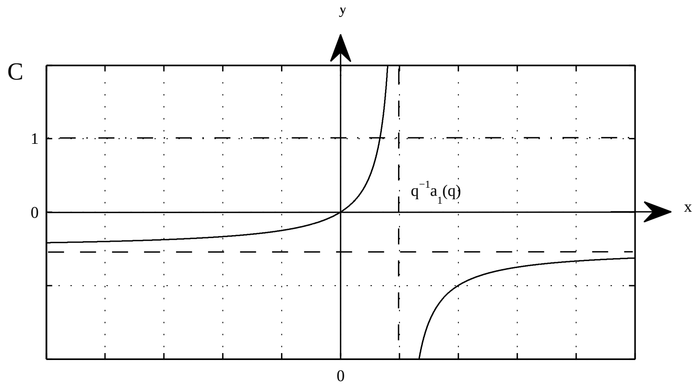


Figure 4.71: The function $f(x, q)$ with Case 2.C: $\Lambda_q < 0$, $a_1(q) > 0$.

Case 2.C: $\Lambda_q < 0$, $a_1(q) > 0$. The graph of f corresponds to this case is represented in Figure 4.71C. Positivity of ρ allows us to skip the intervals $(-\infty, 0)$ and $(q^{-1}a_1(q), \infty)$. Thus, we only need to analyse the interval $(0, q^{-1}a_1(q))$ which is the one defined in Theorem 4.4 b). Thus, this interval could be possible for constructing ρ. Notice that $\rho(qx, q)/(x, q) = 1$ at $x_0 = -\tau(0, q)/'(0, q)$, $x = 0 < x_0 < x = q^{-1}a_1(q)$. Then, it follows that ρ is increasing on $(0, x_0)$ with $\rho \to 0$ as $x \to 0^+$ since $\rho(qx,q)/(x,q) = 0$ at $x = 0$ and decreasing on $(x_0, q^{-1}a_1(q))$ with $\rho \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx,q)/(x,q) \to \infty$ which leads to Figure 4.72 for corresponding ρ.

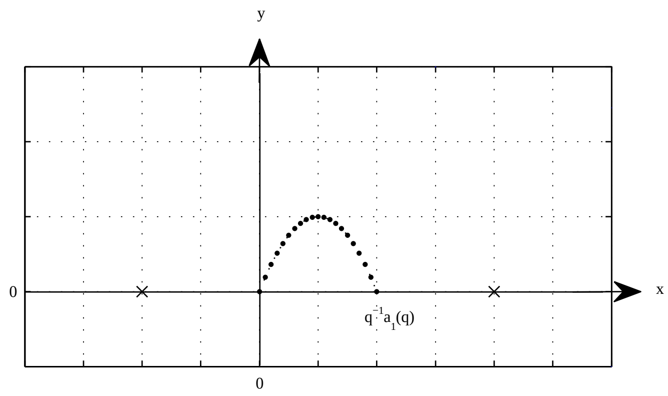


Figure 4.72: Possible positive graph of corresponding $\rho(x,q)$ for Figure 4.71.

It is obvious from Figure 4.72 that there exists a suitable ρ defined on $(0, a_1(q))$ supported at the points $a_1(q)q^k$, $k = 0, 1, ...$ (see Theorem 4.4 b)). Thus, we have the following theorem.

Theorem 4.51 *Let $a = 0$ and $b = a_1(q)$ be the zeros of $\sigma_1(x,q)$ and assume that $a_1(q) > 0$ and $\Lambda_q = q^{-2}[1 + (1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 0$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a,b), i.e., they satisfy the orthogonality (4.7) with respect to the q-weight function*

$$\rho(x,q) = x^{\alpha}\sqrt{x^{\log_q x - 1}}(qx/a_1(q);q)_\infty > 0,\ x \in (a,b) \quad q^{\alpha} = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)} \tag{4.94}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 b)).

This case corresponds to the case IVa5 in Chapter 10 of [35, pages 278 and 313].

An example of such family is the alternative q-Charlier polynomials [35] where $a_1(q) = 1$,

$$\sigma_1(x,q) = q^{-2}x(1-x), \quad \sigma_2(x,q) = ax^2,$$

$$\tau(x,q) = -\frac{1+aq}{(1-q)q}x + \frac{1}{(1-q)q}, \quad \lambda_n(q) = q^{-n}[n]_q\frac{1+aq^n}{1-q}.$$

Alternative q-Charlier polynomials are orthogonal on $(0,1)$ and the conditions $q^2\Lambda_q < 0$ and $a_1(q) > 1$ give us the following restriction for the parameters $a > 0$. By means of Theorem

4.4 b) we can write the orthogonality relation of alternative q-Charlier polynomials

$$\int_0^1 x^{\alpha}\sqrt{x^{\log_q x-1}}(qx;q)_\infty K_m(x;a;q)K_n(x;a;q)d_qx = \frac{a^n q^{\binom{n+1}{2}}}{1+aq^{2n}}(q;q)_n(-aq^n,q;q)_\infty\delta_{mn} \quad (4.95)$$

associated with $a = q^{\alpha} > 0$.

4.3.2.6 The q-Classical 0-Jacobi/Laguerre Polynomials

Let $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$ and $\tau(x,q) = \tau'(0,q)x+\tau(0,q)$, $\tau'(0,q) \neq 0$ and assume that $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = -\frac{1}{(1-q^{-1})}$. Then, observe from (3.11) that $\sigma_2(x,q) = \sigma_2'(0,q)x = q\left[(1-q^{-1})\tau(0,q)-\frac{1}{2}\sigma_1''(0,q)a_1(q)\right]x$ and therefore the q-Pearson equation follows from (4.1) as

$$\begin{aligned}\frac{\rho(qx,q)}{\rho(x,q)} &= \frac{\sigma_1(x,q)+(1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} \\ &= \frac{(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}-a_1(q)}{q[qx-a_1(q]}. \end{aligned} \quad (4.96)$$

Remark 4.52 *Let $f(x,q) = \rho(qx,q)/\rho(x,q)$ be the function defined in (4.96). Then notice that $y = 0$ is the horizontal asymptote of the function $f(x,q)$.*

Before starting the analysis let us point out that $\rho(qx,q)/\rho(x,q)$ in (4.96) always intercepts y-axis at the point

$$y := y_0 = q^{-1}\left[1-\frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

Notice for the zero cases that a or b could be zero. That's why, we should know the behaviour of ρ at $x = 0$. To learn this we perform the following remark obtained from Remark 4.35.

Remark 4.53 *Behaviour of the q-weight function at $x = 0$ depends on the succesive solution of the q-Pearson equation*

$$\begin{aligned}\rho(qx,q) &= q^{-1}\left[1-\frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{1}{(1-qx/a_1(q))}\rho(x,q) \\ \Leftrightarrow \rho(q^kx,q) &= q^{-k}\left[1-\frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]^k\frac{1}{(qx/a_1(q);q)_k}\rho(x,q). \end{aligned} \quad (4.97)$$

It is apparent that as $k \to \infty$ the behaviour of ρ at $x = 0$ is accomplished. Notice that if $0 < y_0 = q^{-1}[1-\frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}] < 1$, $\rho(x,q) \to 0$ as $x \to 0$ otherwise it tends to $\mp\infty$.

We introduce analogous analysis in order to obtain independent graphs of $\rho(qx,q)/\rho(x,q)$ according to the sign of $a_1(q)$ (zero of σ_1) and of y_0 by taking Case 1. $y_0 > 0$ and Case 2. $y_0 < 0$. Nevertheless, as before, we need to split 1st case into two separate cases: Case 1.i) when $y_0 > 1$ and Case 1.ii) when $0 < y_0 < 1$.

Let $f(x,q) := \rho(qx,q)/\rho(x,q)$ be the function defined in (4.96).

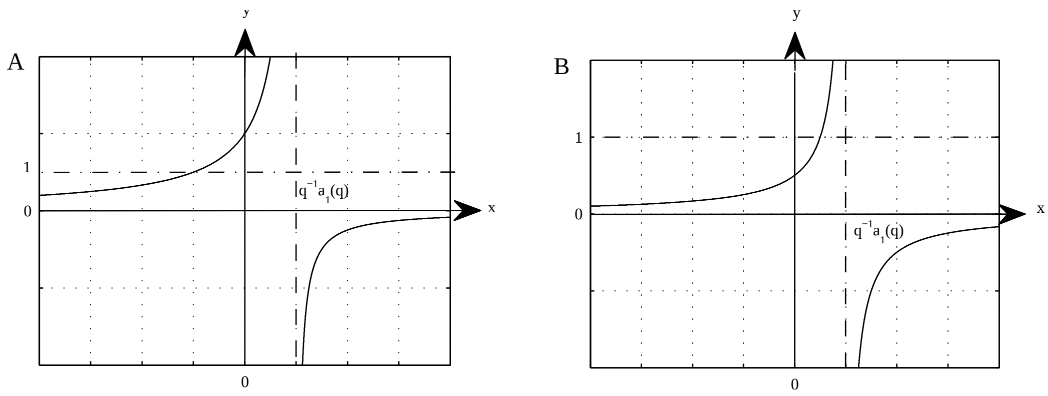


Figure 4.73: The function $f(x,q)$ with $a_1(q) > 0$, Case 1.i)A: $y_0 > 1$, Case 1.ii)B: $0 < y_0 < 1$.

Case 1.i)A: $y_0 > 1$, $a_1(q) > 0$. The graph of f corresponds to this case is represented in Figure 4.73A. We first begin with considering the positivity of ρ which allows us to eliminate the interval $(q^{-1}a_1(q), \infty)$. Let us consider the next interval $(-\infty, 0)$ which coincides with the one described in Theorem 4.4 i) by symmetry. Thus, this interval could be possible to have a suitable ρ. Notice that $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q)$, $x_0 < x = 0$. Therefore, it follows that ρ is decreasing on $(-\infty, x_0)$ and increasing on $(x_0, 0)$ with $\rho \to \infty$ as $x \to 0^-$ since $y_0 > 1$ (see Remark 4.53). Notice from (4.20) and (4.21) that graphs of ρ and $\sigma_1\rho$ have the same properties which lead to that $\sigma_1\rho \to \infty$ as $x \to 0^-$. Observe that this is the boundary condition when $k = 0$. That's why, this interval does not lead to any suitable ρ with the needed properties. Let us deal with the last interval $(0, q^{-1}a_1(q))$ which is the one given in Theorem 4.4 b). Thus, we have a possibility to construct a suitable ρ on this interval. Notice from Figure 4.73A that ρ is decreasing on $(0, q^{-1}a_1(q))$ with $\rho \to \infty$ as $x \to 0^+$ since $y_0 > 1$ (see Remark 4.53) and $\rho \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}a_1(q)^-$. Since $\rho \to \infty$ as $x \to 0^+$, then $\sigma_1\rho \to \infty$ as $x \to 0^+$ because of the same reason that we used for the interval $(-\infty, 0)$. As a result, this case does not lead to any suitable ρ with the needed properties.

Case 1.ii)B: $0 < y_0 < 1$, $a_1(q) > 0$. The graph of f corresponds to this case is represented in Figure 4.73B. Notice that Figure 4.73A and Figure 4.73B are similar except y-intercept; y_0. Thus, we exclude the interval $(q^{-1}a_1(q), \infty)$ due to the positivity of ρ. The next interval $(-\infty, 0)$ is the one described in Theorem 4.4 i) by symmetry. Notice that ρ is decreasing on this interval with $\rho \to 0$ as $x \to 0^-$ which leads to that $\rho \to \infty$ as $x \to -\infty$ (Observe that $\rho(qx, q)/\rho(x, q) \to 0$ as $x \to -\infty$) $\Longrightarrow \sigma_1(x, q)\rho(x, q)x^k \not\to 0$, $k = 0, 1, 2, ...$ as $x \to -\infty$. Hence, it is not possible to have a suitable ρ on this interval. On the other hand the last interval $(0, q^{-1}a_1(q))$ could also be possible for constructing ρ since it coincides with the one given Theorem 4.4 b). Notice that $\rho(qx, q)/\rho(x, q) = 1$ at $x_0 = -\tau(0, q)/\tau'(0, q)$, $x = 0 < x_0 < x = q^{-1}a_1(q)$. Therefore, it follows that ρ is increasing on $(0, x_0)$ with $\rho \to 0$ as $x \to 0^+$ since $0 < y_0 < 1$ (see Remark 4.53) and decreasing on $(x_0, q^{-1}a_1(q))$ with $\rho(qx, q)/\rho(x, q) \to 0$ as $x \to q^{-1}a_1(q)^-$ since $\rho(qx, q)/\rho(x, q) \to \infty$ as $x \to q^{-1}a_1(q)^-$ which allows us to build Figure 4.74 for corresponding ρ.

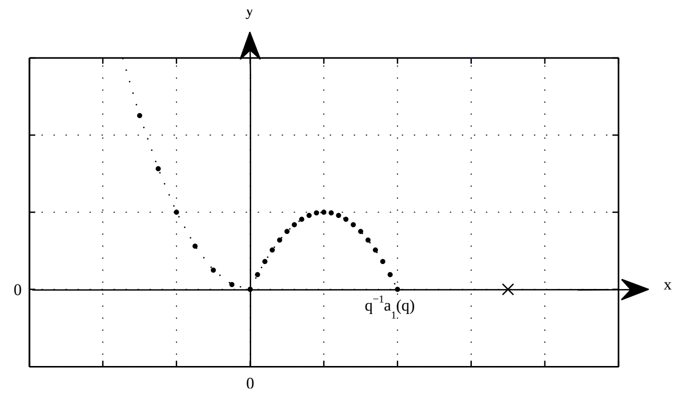


Figure 4.74: Possible positive graph of corresponding $\rho(x, q)$ for Figure 4.73B.

It is also clear from Figure 4.74 that the boundary condition (3.119) holds at $x = 0$ and $x = a_1(q)$, hence there exists a suitable ρ satisfying the needed properties on $(0, a_1(q))$ supported at the points $q^k a_1(q)$, $k = 0, 1, ...$ (see Theorem 4.4 b)). Thus we construct the following theorem according as the result of this case.

Theorem 4.54 *Let $a = 0$ and $b = a_1(q)$ be the zeros of $\sigma_1(x, q)$ and assume that $a_1(q) > 0$ and $0 < y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)} \frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] < 1$. Then, there exists a sequence of polynomials $(P_n)_n$ orthogonal on (a, b), i.e., they satisfy the orthogonality (4.7) with respect to the q-weight* *function*

$$\rho(x,q) = x^{\alpha}(qx/a_1(q);q)_\infty > 0,\ x \in (a,b) \quad q^{\alpha} = -\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\frac{1}{2}\sigma_1''(0,q)a_1(q)} \tag{4.98}$$

which satisfies the q-Pearson equation and the boundary condition (see Theorem 4.4 b)).

This case corresponds to the case IVa4 in Chapter 10 of [35, pages 278 and 312].

An example of such family is the little q-Laguerre (Wall) polynomials [35] where $a_1(q) = 1$,

$$\sigma_1(x,q) = q^{-2}x(1-x), \quad \sigma_2(x,q) = ax,$$

$$\tau(x,q) = -\frac{1}{(1-q)q}x + \frac{1-aq}{(1-q)q}, \quad \lambda_n(q) = \frac{q^{-n}}{1-q}[n]_q.$$

Little q-Laguerre (Wall) polynomials are orthogonal on $(0,1)$ and the conditions $0 < qy_0 < 1$ and $a_1(q) > 1$ give us the following restriction for the parameters $0 < a < q^{-1}$. By means of Theorem 4.4 b) we can write the orthogonality relation of little q-Laguerre (Wall) polynomials

$$\int_0^1 x^{\alpha}(qx;q)_\infty P_m(x;\alpha|q)P_n(x;\alpha|q)d_qx = q^{(\alpha+1)n}\frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}\frac{(q;q)_n}{(q^{\alpha+1};q)_n}\delta_{mn} \tag{4.99}$$

together with $0 < a = q^{\alpha} < q^{-1}$.

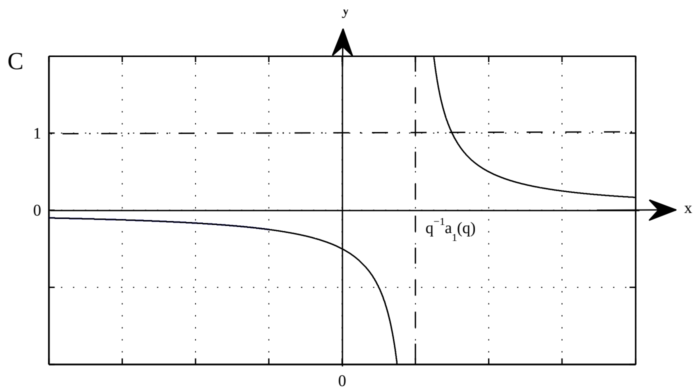


Figure 4.75: The function $f(x,q)$ with Case 2.C: $y_0 < 0$, $a_1(q) > 0$.

Case 2.C: $y_0 < 0$, $a_1(q) > 0$. The graph of f corresponds to this case is represented in Figure 4.75C. Notice that the intervals $(-\infty,0)$ and $(0,q^{-1}a_1(q))$ are both eliminated due to the positivity of ρ. The same happens for the interval $(q^{-1}a_1(q),\infty)$ due to Remark 4.5.5.

As a result of the qualitative analysis, in the following tables we show the main intervals of orthogonality depending on the range of the parameters of each family. We note that the relations given with * lead to the new relations obtained with our approach. Actually, they are the ones which have not been reported in the q-Askey scheme in [36]. However, the relations in Table 4.6 and Table 4.14 have been mentioned in the very recent book [35].

Table 4.2: $\emptyset$-Jacobi/Jacobi $\Leftrightarrow$ Big q-Jacobi Polynomials

$\sigma_1(x,q) = q^{-2}(x-aq)(x-cq)$,	$\sigma_2(x,q) = aq(x-1)(bx-c)$
(cq, aq)	$c<0,\ 0<b<q^{-1},\ 0<a<q^{-1}$
	* $c<0,\ b<0,\ abc^{-1}q \le 1,\ 0<a<q^{-1}$

Table 4.3: $\emptyset$-Laguerre/Jacobi $\Leftrightarrow$ Alternative Big q-Jacobi Polynomials

$\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$,	$\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]$
$(a_1(q), b_1(q))$	* $1+(1-q^{-1})\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} < 0,\ a_2(q), b_2(q) \in \mathbb{C}$

Table 4.4: $\emptyset$-Jacobi/Jacobi $\Leftrightarrow$ q-Hahn Polynomials

$\sigma_1(x,q) = q^{-2}(x-q^{-N})(x-\alpha q)$,	$\sigma_2(x,q) = \alpha q(x-1)(\beta x - q^{-N-1})$
$(1, q^{-N-1})$	$0<\alpha<q^{-1},\ 0<\beta<q^{-1}$
	* $0<\alpha<q^{-1}, \beta<0$
	$\alpha \ge q^{-N-1}, \beta \ge q^{-N-1}$
	* $\alpha<0, \beta \ge q^{-N-1}$

Table 4.5: $\emptyset$-Laguerre/Jacobi $\Leftrightarrow$ q-Meixner Polynomials

$\sigma_1(x,q) = cq^{-2}(x-bq)$,	$\sigma_2(x,q) = (x-1)(x+bc)$
$(1, \infty)$	$c>0,\ 0<b<q^{-1}$
	* $c>0,\ b<0,\ 0<-bc \le 1$

Table 4.6: $\emptyset$-Laguerre/Jacobi $\Leftrightarrow$ Alternative q-Meixner Polynomials

$\sigma_1(x,q) = \sigma_1'(0,q)[x - a_1(q)]$,	$\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)[x - a_2(q)][x - b_2(q)]$
$(a_1(q), \infty)$	* $\frac{\tau'(0,q)}{\sigma_1'(0,q)} < 0,\ a_2(q) \le b_2(q) < a_1(q) < 0$ * $\frac{\tau'(0,q)}{\sigma_1(0,q)} < 0,\ a_2(q), b_2(q) \in \mathbb{C}$ and $a_1(q) < 0$

Table 4.7: $\emptyset$-Laguerre/Jacobi $\Leftrightarrow$ Quantum q-Kravchuk Polynomials

$\sigma_1(x,q) = -q^{-2}(x - q^{-N})$,	$\sigma_2(x,q) = (x-1)(px - q^{-N-1})$
$(1, q^{-N-1})$	$p \ge q^{-N-1}$

Table 4.8: $\emptyset$-Hermite/Jacobi $\Leftrightarrow$ Al-Salam Carlitz II Polynomials

$\sigma_1(x,q) = aq^{-1}$,	$\sigma_2(x,q) = (1-x)(a-x)$
$(1, \infty)$	$0 < a \le 1$

Table 4.9: $\emptyset$-Hermite/Jacobi $\Leftrightarrow$ Discrete q^{-1}-Hermite II Polynomials

$\sigma_1(x,q) = q^{-1}$,	$\sigma_2(x,q) = 1 + x^2$
$(-\infty, \infty)$	

Table 4.10: $\emptyset$-Jacobi/Laguerre $\Leftrightarrow$ Big q-Laguerre Polynomials

$\sigma_1(x,q) = q^{-2}(x - aq)(x - bq)$,	$\sigma_2(x,q) = abq(1-x)$
(bq, aq)	$b < 0,\ 0 < a < q^{-1}$

Table 4.11: $\emptyset$-Jacobi/Laguerre $\Leftrightarrow$ Affine q-Kravchuk Polynomials

$\sigma_1(x,q) = q^{-1}(x - q^{-N})(x - pq)$,	$\sigma_2(x,q) = -pq^{1-N}(x-1)$
$(1, q^{-N-1})$	$0 < p < q^{-1}$

Table 4.12: ∅-Jacobi/Hermite ⇔ Al-Salam Carlitz I Polynomials

$\sigma_1(x,q) = q^{-1}(1-x)(a-x)$,	$\sigma_2(x,q) = a$
$(a,1)$	$a < 0$

Table 4.13: ∅-Jacobi/Hermite ⇔ Discrete q-Hermite I Polynomials

$\sigma_1(x,q) = -q^{-1}(1-x^2)$,	$\sigma_2(x,q) = -1$
$(-1,1)$	

Table 4.14: 0-Jacobi/Jacobi ⇔ Little q-Jacobi Polynomials

$\sigma_1(x,q) = q^{-2}x(x-1)$,	$\sigma_2(x,q) = ax(bqx-1)$
$(0,1)$	$0 < a < q^{-1}$, $0 < b < q^{-1}$ * $0 < a < q^{-1}$, $b < 0$

Table 4.15: 0-Jacobi/Jacobi ⇔ q-Kravchuk Polynomials

$\sigma_1(x,q) = q^{-2}x(x-q^{-N})$,	$\sigma_2(x,q) = -px(x-1)$
$(1,q^{-N-1})$	$p > 0$

Table 4.16: 0-Laguerre/Jacobi ⇔ q-Laguerre Polynomials

$\sigma_1(x,q) = q^{-2}x$,	$\sigma_2(x,q) = q^{\alpha}x(x+1)$
$(0,\infty)$	$\alpha > -1$

Table 4.17: 0-Laguerre/Jacobi ⇔ q-Charlier Polynomials

$\sigma_1(x,q) = aq^{-2}x$,	$\sigma_2(x,q) = x(x-1)$
$(1,\infty)$	$a > 0$

Table 4.18: 0-Jacobi/Bessel ⇔ Alternative q-Charlier Polynomials

$\sigma_1(x,q) = q^{-2}x(1-x)$,	$\sigma_2(x,q) = ax^2$
$(0,1)$	$a > 0$

Table 4.19: 0-Laguerre/Bessel ⇔ Stieltjes-Wigert Polynomials

$\sigma_1(x,q) = q^{-2}x$,	$\sigma_2(x,q) = x^2$
$(0,\infty)$	

Table 4.20: 0-Jacobi/Laguerre ⇔ Little q-Laguerre (Wall) Polynomials

$\sigma_1(x,q) = q^{-2}x(1-x)$,	$\sigma_2(x,q) = ax$
$(0,1)$	$0 < a < q^{-1}$

CHAPTER 5

RELATIONS BETWEEN THE Q-CLASSICAL POLYNOMIALS

In this chapter, we introduce the relations between the q-classical polynomials and the classical continues (those of Jacobi, Laguerre, Hermite) and discrete ones (those of Hahn, Meixner, Kravchuk, Charlier). First of all let us construct the following Table 5.1 [3, 35, 36, 6, 42] according to the identification of the q-polynomials that we found in chapter 4.

Table 5.1: Relation between the q-Classical and the q-Askey polynomials

Cases in KLS's book	⇔q-Classical family	⇔q-Askey scheme
CaseVIIa1.Chp10/ IIIb5/9.Chp11	⇔∅-Jacobi /Jacobi	⇔The big q-Jacobi, q-Hahn
CaseIIa2.Chp11/ IIb1.Chp11	⇔∅-Laguerre/Jacobi	⇔q-Meixner, quantum q-Kravchuk
CaseIa1.Chp11/ Va2.Chp10	⇔∅-Hermite/Jacobi	⇔Al-Salam-Carlitz II, discrete q^{-1}-Hermite II
CaseVIIa1.Chp10/ IIIb3.Chp11	⇔∅-Jacobi/Laguerre	⇔Big q-Laguerre, affine q-Kravchuk
CaseVIIa1.Chp.10	⇔∅-Jacobi/Hermite	⇔Al-Salam-Carlitz I, discrete q-Hermite
CaseIVa3/4.Chp10/ IIIb.Chp11	⇔0-Jacobi/Jacobi	⇔The little q-Jacobi, q-Kravchuk
CaseIIIa2.Chp10/ IIa2.Chp11	⇔0-Laguerre/Jacobi	⇔q-Laguerre, q-Charlier
CaseIVa5.Chp10	⇔0-Jacobi/Bessel	⇔Alternative q-Charlier
CaseIIIa2.Chp10	⇔0-Laguerre/Bessel	⇔Stieltjes-Wigert
CaseIVa4.Chp10	⇔0-Jacobi/Laguerre	⇔Little q-Laguerre (Wall)

We remark that the cases given in first column belong to [35] where

$$\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q) = ex^2 + 2fqx + gq^2, \tag{5.1}$$

$$q^2\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \alpha x^2 + \beta qx + gq^2 \tag{5.2}$$

in Chapter 10 (with the lattice $x(s) = q^s$) and

$$q^2\sigma_1(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q) = eq^2 + 2fq^{x+1} + gq^{2x}, \tag{5.3}$$

$$\sigma_2(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \alpha^* + \beta^* q^{x-1} + gq^{2x-1} \tag{5.4}$$

in Chapter 11 (with the lattice $x(s) = q^{-s}$).

One can also find the relation with the Nikiforov-Uvarov scheme [5, 47] by considering second order linear difference equation of hypergeometric type on non-uniform lattices $x(s)$ given with (1.23)

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n[x(s)]}{\nabla x(s)} + \tau(s)\frac{\Delta P_n[x(s)]}{\Delta x(s)} + \hat{\lambda}_n P_n[x(s)] = 0. \tag{5.5}$$

Notice that if the lattice is q-linear of the form, $x(s) = c_1 q^s := x$, then

$$\frac{\Delta P_n[x(s)]}{\Delta x(s)} = D_q P_n(x) \quad \text{and} \quad \frac{\nabla \mathrm{P_n[x(s)]}}{\nabla \mathrm{x(s)}} = \mathrm{D_{q^{-1}} P_n(x)}, \quad \mathrm{P_n(x) := P_n[x(s)]}$$

and $\Delta x(s-\frac{1}{2}) = q^{-1/2}\Delta x(s)$. Therefore, (5.5) becomes

$$\sigma(s) D_q D_{q^{-1}} P_n(x) + q^{-1/2}\tau(s) D_q P_n(x) + q^{-1/2}\hat{\lambda}_n P_n(x) = 0. \tag{5.6}$$

Furthermore, using the operational equivalences defined by (3.3) and (3.4) provide us to obtain the alternative equation as the following form

$$\left[\sigma(s) + (q-1)xq^{-1/2}\tau(s)\right] D_q D_{q^{-1}} P_n(x) + q^{-1/2}\tau(s) D_{q^{-1}} P_n(x) + q^{-1/2}\hat{\lambda}_n P_n(x) = 0. \tag{5.7}$$

Notice that (5.6) and (5.7) are the q-EHT of the 1st and 2nd kinds of the form (3.5) and (3.10), respectively, where

$$q^{-1}\sigma(s) = \sigma_1(x,q), \quad \sigma(s) + (q-1)xq^{-1/2}\tau(s) = \sigma_2(x,q),$$

$$q^{-1/2}\tau(s) = \tau(x,q), \quad q^{-1/2}\hat{\lambda}_n = \lambda_n(q).$$

On the other hand, setting the lattice as q-linear of the form $x(s) = c_1 q^{-s} := x$ provides

$$\frac{\Delta P_n[x(s)]}{\Delta x(s)} = D_{q^{-1}} P_n(x) \quad \text{and} \quad \frac{\nabla \mathrm{P_n[x(s)]}}{\nabla \mathrm{x(s)}} = \mathrm{D_q P_n(x)}, \quad \mathrm{P_n(x) := P_n[x(s)]}$$

and $\Delta x(s-\frac{1}{2}) = q^{1/2}\Delta x(s)$. Inserting these values in (5.5) leads to

$$q\sigma(s) D_q D_{q^{-1}} P_n(x) + q^{1/2}\tau(s) D_{q^{-1}} P_n(x) + q^{1/2}\hat{\lambda}_n P_n(x) = 0 \tag{5.8}$$

and with the help of operational equivalences defined by (3.3) and (3.4) we get

$$\left[q\sigma(s) + (1-q)xq^{1/2}\tau(s)\right] D_q D_{q^{-1}} P_n(x) + q^{1/2}\tau(s) D_q P_n(x) + q^{1/2}\hat{\lambda}_n P_n(x) = 0. \tag{5.9}$$

Notice that (5.8) and (5.9) are the q-EHT of the 2nd and 1st kinds of the form (3.10) and (3.5), respectively, where

$$\sigma(s) + (q^{-1}-1)xq^{1/2}\tau(s) = \sigma_1(x,q), \quad q\sigma(s) = \sigma_2(x,q),$$

$$q^{1/2}\tau(s) = \tau(x,q), \quad q^{1/2}\hat{\lambda}_n = \lambda_n(q).$$

Before introducing the limit relations we construct the hypergeometric representations of q-classical polynomials in the Hahn sense by use of the formulas obtained in Chapter 3. Moreover, we perform the relations between them.

5.1 ∅-Jacobi/Jacobi ⇔ Big q-Jacobi polynomials

By choosing the coefficients as

$$\sigma_1(x,q) = q^{-2}(x-aq)(x-cq), \quad \sigma_2(x,q) = aq(x-1)(bx-c),$$

$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{a(bq-1)+c(aq-1)}{1-q}, \quad \lambda_n(q) = q^{-n}[n]_q\frac{1-abq^{n+1}}{q-1},$$

we get the big q-Jacobi polynomials $P_n(x;a,b,c;q)$. Setting $a_1(q) = cq$, $b_1(q) = aq$, $a_2(q) = b^{-1}c$, $b_2(q) = 1$ in the representation formula identified by (3.86) leads to the following hypergeometric representation of the monic big q-Jacobi polynomials

$$P_n(x;q) := P_n(x;a,b,c;q) = \frac{(aq,cq;q)_n}{(abq^{n+1};q)_n}{}_3\varphi_2\left(\begin{matrix} q^{-n}, & abq^{n+1}, & x \\ & aq, & cq \end{matrix}\middle| q;q\right). \tag{5.10}$$

5.2 ∅-Jacobi/Jacobi ⇔ q-Hahn polynomials

Coefficients of the q-difference equation of hypergeometric type for the q-Hahn polynomials $Q_n(x;a,b,N;q)$ look like

$$\sigma_1(x,q) = q^{-2}(x-q^{-N})(x-\alpha q), \quad \sigma_2(x,q) = \alpha q(x-1)(\beta x - q^{-N-1}),$$

$$\tau(x,q) = \frac{1-\alpha\beta q^2}{(1-q)q}x + \frac{\alpha q^{-N}+\alpha\beta q-\alpha-q^{-N-1}}{1-q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-\alpha\beta q^{n+1}}{1-q}.$$

Hypergeometric representation of the monic q-Hahn polynomials $Q_n(x;\alpha,\beta,N;q)$

$$P_n(x;q) := Q_n(x;\alpha,\beta,N;q) = \frac{(\alpha q,q^{-N};q)_n}{(\alpha\beta q^{n+1};q)_n}{}_3\varphi_2\left(\begin{matrix} q^{-n}, & \alpha\beta q^{n+1}, & x \\ & \alpha q, & q^{-N} \end{matrix}\middle| q;q\right) \tag{5.11}$$

is obtained by use of the formula identified by (3.86) or (3.87) by setting $a_1(q) = q^{-N}$, $b_1(q) = \alpha q$, $a_2(q) = 1$ and $b_2(q) = \beta^{-1}q^{-N-1}$.

Notice that setting $a = \alpha,\ b = \beta,\ c = q^{-N-1}$ in the big q-Jacobi polynomials (5.10) allows us to get the representation of the q-Hahn polynomials (5.11) (see [6])

$$P_n(x,\alpha,\beta,q^{-N-1};q) = Q_n(x,\alpha,\beta,N;q). \tag{5.12}$$

5.3 $\emptyset$-Laguerre/Jacobi ⇔ q-Meixner polynomials

The q-Meixner polynomials $M_n(x;b,c;q)$ have the following coefficients

$$\sigma_1(x,q) = cq^{-2}(x-bq), \quad \sigma_2(x,q) = (x-1)(x+bc),$$
$$\tau(x,q) = -\frac{1}{1-q}x + \frac{cq^{-1}-bc+1}{1-q}, \quad \lambda_n(q) = \frac{[n]_q}{1-q}.$$

Representation formula defined by (3.97) associated with $a_1(q) = bq$, $a_2(q) = -bc$, $b_2(q) = 1$ leads to the hypergeometric representation of the monic q-Meixner polynomials $M_n(x;b,c;q)$ as the following form

$$P_n(x;q) := M_n(x;b,c;q) = (-c)^n q^{-n^2}(bq;q)_n\,{}_2\varphi_1\left(\begin{matrix} q^{-n}, & x \\ & bq \end{matrix}\middle| q; -\frac{q^{n+1}}{c}\right). \tag{5.13}$$

Observe that the q-Meixner polynomials $M_n(x;b,c;q)$ defined by (5.13) are also obtained from the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ (5.10) and the q-Hahn polynomials $Q_n(x;\alpha,\beta,N;q)$ by using the limit $a \to \infty$ with $b := -c^{-1}$, $c := b$ and $N \to \infty$ with $\alpha := b, \beta := -b^{-1}c^{-1}q^{-N-1}$, respectively (see [36]),

$$\lim_{a\to\infty} P_n(x;a,-c^{-1},b;q) = M_n(x;b,c;q), \tag{5.14}$$

$$\lim_{N\to\infty} Q_n(x;b,-b^{-1}c^{-1}q^{-N-1},N;q) = M_n(x;b,c;q). \tag{5.15}$$

5.4 $\emptyset$-Laguerre/Jacobi ⇔ Quantum q-Kravchuk polynomials

Coefficients of the q-difference equation for the quantum q-Kravchuk polynomials $K_n^{qtm}(x;p,N;q)$ are as follows:

$$\sigma_1(x,q) = -q^{-2}(x-q^{-N}), \quad \sigma_2(x,q) = (x-1)(px-q^{-N-1}),$$

$$\tau(x,q) = -\frac{p}{1-q}x + \frac{p-q^{-1}+q^{-N-1}}{1-q}, \quad \lambda_n(q) = \frac{p}{1-q}[n]_q.$$

Then, representation of the monic quantum q-Kravchuk polynomials $K_n^{qtm}(x;p,N;q)$ follows from the formula defined by (3.97) as

$$P_n(x;q) := K_n^{qtm}(x;p,N;q) = p^{-n}q^{-n^2}(q^{-N};q)_n {}_2\varphi_1\left(\begin{array}{cc} q^{-n}, & x \\ & q^{-N} \end{array}\middle| q; pq^{n+1}\right). \tag{5.16}$$

Notice that the quantum q-Kravchuk polynomials $K_n^{qtm}(x;p,N;q)$ (5.16) can also be obtained by inserting $b = q^{-N-1}, c = -p^{-1}$ into the q-Meixner polynomials $M_n(x;b,c;q)$ (5.13) (see [6]) and by setting $\beta := p$, $\alpha \to \infty$ in the q-Hahn polynomials $Q_n(x;\alpha,\beta,N;q)$ identified by (5.16) (see [36]);

$$M_n(x;q^{-N-1},-p^{-1},N;q) = K_n^{qtm}(x;p,N;q), \tag{5.17}$$

$$\lim_{\alpha\to\infty} Q_n(x;\alpha,p,N;q) = K_n^{qtm}(x;p,N;q). \tag{5.18}$$

5.5 $\emptyset$-Hermite/Jacobi $\Leftrightarrow$ Al-Salam-Carlitz II polynomials

The Al-Salam-Carlitz II polynomials $V_n^{(a)}(x;q)$ have

$$\sigma_1(x,q) = aq^{-1}, \quad \sigma_2(x,q) = (1-x)(a-x),$$

$$\tau(x,q) = -\frac{1}{1-q}x - \frac{1+a}{q-1}, \quad \lambda_n(q) = \frac{1}{1-q}[n]_q,$$

and therefore the representation of monic $V_n^{(a)}(x;q)$ becomes

$$P_n(x;q) := V_n^{(a)}(x;q) = (-a)^n q^{-\binom{n}{2}} {}_2\varphi_0\left(\begin{array}{cc} q^{-n}, & x \\ & - \end{array}\middle| q; \frac{q^n}{a}\right) \tag{5.19}$$

by use of the formula (3.100).

Observe that the Al-Salam Carlitz II polynomials can also be obtained by use of the limit relations $c \to 0$ with $b := -a/c$ and $N \to \infty$ with $p := a^{-1}q^{-N-1}$ in the definitions (5.13) and (5.16) of the q-Meixner and the quantum q-Kravchuk polynomials, respectively (see [36]),

$$\lim_{c\to 0} M_n(x;-\frac{a}{c},c;q) = q^{-n}V_n^{(a)}(x;q), \tag{5.20}$$

$$\lim_{N\to\infty} K_n^{qtm}(x;a^{-1}q^{-N-1},N;q) = V_n^{(a)}(x;q). \tag{5.21}$$

5.6 $\emptyset$-Hermite/Jacobi ⇔ Discrete q-Hermite II polynomials

The discrete q-Hermite II polynomials $\widetilde{h}_n(x;q)$ have the following specific values

$$\sigma_1(x,q) = q^{-1}, \quad \sigma_2(x,q) = 1 + x^2,$$

$$\tau(x,q) = -\frac{1}{1-q}x, \quad \lambda_n(q) = \frac{1}{1-q}[n]_q,$$

which lead to the hypergeometric representation of the monic $\widetilde{h}_n(x;q)$ with the help of (3.100)

$$P_n(x;q) := \widetilde{h}_n(x;q) = (i)^{-n} q^{-\binom{n}{2}} {}_2\varphi_0\left(\begin{array}{cc} q^{-n}, & ix \\ & - \end{array} \middle| q; -q^n\right). \tag{5.22}$$

Notice that the Al-Salam Carlitz II polynomials defined by (5.19) together with the substitution $a = -1$, $x \to ix$ lead to the discrete q-Hermite II polynomials identified by (5.22),

$$i^{-n} V_n^{(-1)}(ix;q) = \widetilde{h}_n(x;q). \tag{5.23}$$

5.7 $\emptyset$-Jacobi/Laguerre ⇔ Big q-Laguerre polynomials

Letting

$$\sigma_1(x,q) = q^{-2}(x-aq)(x-bq), \quad \sigma_2(x,q) = abq(1-x),$$

$$\tau(x,q) = -\frac{q^{-1}}{q-1}x + \frac{a+b-abq}{q-1}, \quad \lambda_n(q) = \frac{q^{-n}}{q-1}[n]_q,$$

give the big q-Laguerre polynomials $P_n(x;a,b;q)$.

In addition to these values, one can get the hypergeometric representation of the big q-Laguerre polynomials $P_n(x;a,b;q)$ by use of (3.91) as the following form

$$P_n(x;q) := P_n(x;a,b;q) = (aq,bq;q)_n {}_3\varphi_2\left(\begin{array}{ccc} q^{-n}, & x, & 0 \\ aq, & bq & \end{array} \middle| q;q\right). \tag{5.24}$$

We remark that the big q-Laguerre polynomials defined by (5.24) can be derived from the big q-Jacobi polynomials defined by (5.10) with substitution $b := 0, c := b$ [36],

$$P_n(x;a,0,b;q) = P_n(x;a,b;q). \tag{5.25}$$

5.8 $\emptyset$-Jacobi/Laguerre $\Leftrightarrow$ Affine q-Kravchuk polynomials

By choosing

$$\sigma_1(x,q) = q^{-1}(x-q^{-N})(x-pq), \quad \sigma_2(x,q) = -pq^{1-N}(x-1),$$
$$\tau(x,q) = \frac{1}{1-q}x - \frac{pq+q^{-N}-pq^{1-N}}{1-q}, \quad \lambda_n(q) = \frac{1}{q-1}[n]_{q^{-1}},$$

we obtain the affine q-Kravchuk polynomials $K_n^{aff}(x;p,N;q)$

$$P_n(x;q) := K_n^{aff}(x;p,N;q) = (q^{-N},pq;q)_n {}_3\varphi_2\left(\begin{array}{ccc} q^{-n}, & x, & 0 \\ q^{-N}, & pq & \end{array}\middle| q;q\right) \tag{5.26}$$

with the help of (3.91). Notice that the monic affine q-Kravchuk polynomials $K_n^{aff}(x;p,N;q)$ (5.26) can also be obtained from the monic big q-Laguerre (5.24) and the q-Hahn (5.11) polynomials by taking $a = q^{-N-1}$, $b = p$ (see [6]) and $\alpha := p, \beta := 0$ (see [36]), respectively,

$$P_n(x;q^{-N-1},p;q) = K_n^{aff}(x;p,N;q), \tag{5.27}$$

$$Q_n(x;p,0,N;q) = K_n^{aff}(x;p,N;q). \tag{5.28}$$

5.9 $\emptyset$-Jacobi/Hermite $\Leftrightarrow$ Al-Salam-Carlitz I polynomials

We enter the Al-Salam-Carlitz I polynomials $U_n^{(a)}(x;q)$ with the following coefficients

$$\sigma_1(x,q) = q^{-1}(1-x)(a-x), \quad \sigma_2(x,q) = a,$$
$$\tau(x,q) = \frac{1}{1-q}x - \frac{1+a}{1-q}, \quad \lambda_n(q) = \frac{q^{1-n}}{q-1}[n]_q,$$

having the following representation

$$P_n(x;q) := U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\varphi_1\left(\begin{array}{cc} q^{-n}, & x^{-1} \\ 0 & \end{array}\middle| q;\frac{qx}{a}\right) \tag{5.29}$$

obtained from (3.94). We remark that the monic Al-Salam Carlitz I polynomials are derived by use of the transformation $x \to aqx$, $b \to ab$ and $a \to 0$ in the representation formula obtained by applying the transformation formula (2.47) to the monic big q-Laguerre polynomials identified by (5.24) [36],

$$\lim_{a\to 0} \frac{P_n(aqx;a,ab;q)}{a^n} = U_n^{(b)}(x;q). \tag{5.30}$$

Remark 5.1 *We remark that the discrete q-Hermite I polynomials are obtained from the Al-Salam-Carlitz I polynomials by setting* $a = -1$ *(see [36] for further details).*

5.10 0-Jacobi/Jacobi ⇔ Little q-Jacobi polynomials

Setting

$$\sigma_1(x,q) = q^{-2}x(x-1), \quad \sigma_2(x,q) = ax(bqx-1),$$

$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{aq-1}{(1-q)q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1-abq^{n+1}}{1-q},$$

leads to the little q-Jacobi polynomials $P_n(x;a,b;q)$

$$P_n(x;q) := P_n(x;a,b;q) = \frac{(-1)^n q^{\binom{n}{2}}(aq;q)_n}{(abq^{n+1};q)_n}\,{}_2\varphi_1\left(\begin{matrix} q^{-n}, & abq^{n+1} \\ & aq \end{matrix}\middle| q; qx\right) \tag{5.31}$$

with the help of (3.103). Note that one can also get the monic little q-Jacobi polynomials by setting $x \to cqx$ with $c \to \infty$ and $x \to q^{-N}x$, $\alpha := a, \beta := b$ with $N \to \infty$ in the definitions of the monic big q-Jacobi polynomials defined by (5.10) and the q-Hahn polynomials given by (5.11), respectively, [36]

$$\lim_{c\to\infty} \frac{P_n(cqx;a,b,c;q)}{c^n} = P_n(x;a,b;q), \tag{5.32}$$

$$\lim_{N\to\infty} q^N Q_n(q^{-N}x;a,b,N;q) = P_n(x;a,b;q). \tag{5.33}$$

5.11 0-Jacobi/Jacobi ⇔ q-Kravchuk polynomials

The q-Kravchuk polynomials $K_n(x;p,N;q)$ have the following coefficients

$$\sigma_1(x,q) = q^{-2}x(x-q^{-N}), \quad \sigma_2(x,q) = -px(x-1),$$

$$\tau(x,q) = \frac{1+pq}{(1-q)q}x - \frac{p+q^{-N-1}}{1-q}, \quad \lambda_n(q) = -q^{-n}[n]_q\frac{1+pq^n}{1-q}.$$

Moreover, from (3.103) we derive $K_n(x;p,N;q)$ hereinbelow

$$P_n(x;q) := K_n(x;p,N;q) = \frac{(-1)^n q^{-Nn+\binom{n}{2}}(-pq^{N+1};q)_n}{(-pq^n;q)_n}\,{}_2\varphi_1\left(\begin{matrix} q^{-n}, & -pq^n \\ & -pq^{N+1} \end{matrix}\middle| q; q^{N+1}x\right). \tag{5.34}$$

Observe that the monic q-Kravchuk polynomials with the formula obtained by applying the transformation formula (2.48) to (5.34) are derived by use of the transformation $\beta = -\alpha^{-1}q^{-1}p$ and $\alpha \to 0$ in the monic q-Hahn polynomials [36],

$$\lim_{\alpha\to 0} Q_n(x;\alpha,-\alpha^{-1}q^{-1}p;q) = K_n(x;p,N;q). \tag{5.35}$$

5.12 0-Laguerre/Jacobi ⇔ q-Laguerre polynomials

By setting

$$\sigma_1(x,q) = q^{-2}x, \quad \sigma_2(x,q) = q^{\alpha}x(x+1),$$
$$\tau(x,q) = -\frac{q^{\alpha}}{1-q}x + \frac{q^{-1}-q^{\alpha}}{1-q}, \quad \lambda_n(q) = [n]_q\frac{q^{\alpha}}{1-q},$$

the hypergeometric representation of the monic q-Laguerre polynomials $L_n^{\alpha}(x;q)$ follows from (3.109)

$$P_n(x;q) := L_n^{\alpha}(x;q) = (-1)^n q^{-n^2} q^{-\alpha n}{}_2\varphi_1\left(\begin{matrix} q^{-n}, & -x \\ & 0 \end{matrix}\middle| q; q^{\alpha+n+1}\right). \tag{5.36}$$

Note that one can obtain the monic q-Laguerre polynomials with the formula obtained by applying successively the transformation formulas (2.49) with $c \to 0$ and (2.50) to (5.36) by setting $a = q^{\alpha}$, $x \to -b^{-1}q^{-1}x$ with $b \to \infty$ and $b = q^{\alpha}$, $x \to cq^{\alpha}x$ with $c \to \infty$ in the monic little q-Jacobi (5.31) and the q-Meixner (5.13) polynomials, respectively,

$$\lim_{b\to\infty} q^{-\binom{n}{2}} P_n(-\frac{x}{bq}; q^{\alpha}, b; q) = q^{n^2+\alpha n} L_n^{\alpha}(x;q), \tag{5.37}$$

$$\lim_{c\to\infty} \frac{M_n(cq^{\alpha}x; q^{\alpha}, c; q)}{c^n} = q^{\alpha n} L_n^{\alpha}(x;q). \tag{5.38}$$

5.13 0-Laguerre/Jacobi ⇔ q-Charlier polynomials

By choosing

$$\sigma_1(x,q) = aq^{-2}x, \quad \sigma_2(x,q) = x(x-1),$$
$$\tau(x,q) = -\frac{1}{1-q}x + \frac{a+q}{(1-q)q}, \quad \lambda_n(q) = [n]_q\frac{1}{1-q},$$

we get the q-Charlier polynomials $C_n(x;a;q)$. Note that the representation formula is obtained from the formula (3.109) as the following form

$$P_n(x;q) := C_n(x;a;q) = (-a)^n q^{-n^2}{}_2\varphi_1\left(\begin{matrix} q^{-n}, & x \\ & 0 \end{matrix}\middle| q; -\frac{q^{n+1}}{a}\right). \tag{5.39}$$

Observe that the monic q-Laguerre polynomials (5.36) together with substitution $x \to -x$, $q^{\alpha} = -a^{-1}$, the monic q-Meixner polynomials (5.13) associated with $b \to 0$ and the monic q-Kravchuk polynomials with the formula obtained by applying the transformation formula

(2.48) to (5.34) together with the substitution $p = a^{-1}q^{-N}$, $N \to \infty$ generate the monic q-Charlier polynomials in the following way (see [36]),

$$(-1)^n L_n(-x; -a^{-1}; q) = C_n(x; a; q), \tag{5.40}$$

$$M_n(x; 0, a; q) = C_n(x; a; q), \tag{5.41}$$

$$\lim_{N\to\infty} K_n(x; a^{-1}q^{-N}, N; q) = C_n(x; a; q). \tag{5.42}$$

5.14 0-Jacobi/Bessel ⇔ Alternative q-Charlier polynomials

Choosing the values for the alternative q-Charlier polynomials $K_n(x; a; q)$ hereinbelow

$$\sigma_1(x, q) = q^{-2}x(1 - x), \quad \sigma_2(x, q) = ax^2,$$

$$\tau(x, q) = -\frac{1 + aq}{(1 - q)q}x + \frac{1}{(1 - q)q}, \quad \lambda_n(q) = q^{-n}[n]_q \frac{1 + aq^n}{1 - q},$$

leads to the following representation

$$P_n(x; q) := K_n(x; a; q) = \frac{(-1)^n q^{\binom{n}{2}}}{(-aq^n; q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, & -aq^n \\ & 0 \end{matrix}\middle| q; qx\right) \tag{5.43}$$

with the help of (3.106). Observe that one can get the alternative q-Charlier polynomials (5.43) by setting $b \to -a^{-1}q^{-1}b$ with $a \to 0$ and $x \to q^{-N}x$ with $N \to \infty$ in the definitions of the little q-Jacobi (5.31) and the q-Kravchuk (5.34) polynomials, respectively, in the following way [36],

$$\lim_{a\to 0} P_n(x; a, -\frac{b}{aq}; q) = K_n(x; b; q), \tag{5.44}$$

$$\lim_{a\to 0} q^{Nn} K_n(q^{-N}x; p, N; q) = K_n(x; p; q). \tag{5.45}$$

5.15 0-Laguerre/Bessel ⇔ Stieltjes-Wigert polynomials

In this case, (3.17), (3.18) and (3.19) become

$$\sigma_1(x, q) = q^{-2}x, \quad \sigma_2(x, q) = x^2,$$

$$\tau(x, q) = -\frac{1}{1 - q}x + \frac{1}{(1 - q)q},$$

and therefore we attain $\lambda_n(q) = [n]_q \frac{1}{1-q}$ from (3.52). Notice that with the help of the formula (3.113) we get the representation of the Stieltjes-Wigert polynomials $S_n(x; q)$

$$P_n(x; q) := S_n(x; q) = (-1)^n q^{-n^2} {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \middle| q; -q^{n+1}x\right). \tag{5.46}$$

We remark that the Stieltjes-Wigert polynomials can be derived from the q-Laguerre (5.36), the alternative q-Charlier (5.43) and the q-Charlier (5.39) polynomials by setting $x \to q^{-\alpha}x$ with $\alpha \to \infty$, $x \to a^{-1}x$ with $a \to \infty$ and $x \to ax$ with $a \to \infty$, respectively, [36]

$$\lim_{\alpha\to\infty} q^{\alpha n} L_n^{\alpha}(q^{-\alpha}x; q) = S_n(x; q), \tag{5.47}$$

$$\lim_{a\to\infty} a^n K_n(\frac{x}{a}; a; q) = S_n(x; q), \tag{5.48}$$

$$\lim_{a\to\infty} \frac{C_n(ax; a; q)}{a^n} = S_n(x; q). \tag{5.49}$$

5.16 0-Jacobi/Laguerre ⇔ Little q-Laguerre (Wall) polynomials

Polynomial coefficients of the little q-Laguerre (wall) $P_n(x; a; q)$ polynomials look like

$$\sigma_1(x, q) = q^{-2}x(1 - x), \quad \sigma_2(x, q) = ax,$$

$$\tau(x, q) = -\frac{1}{(1-q)q}x + \frac{1-aq}{(1-q)q}, \quad \lambda_n(q) = \frac{q^{-n}}{1-q}[n]_q,$$

which lead to the representation of $P_n(x; a; q)$ as

$$P_n(x; q) := P_n(x; a; q) = (-1)^n q^{\binom{n}{2}} (aq; q)_n {}_2\varphi_1\left(\begin{matrix} q^{-n}, \quad 0 \\ aq \end{matrix} \middle| q; qx\right) \tag{5.50}$$

by inserting the needed values in (3.112). Note that the little q-Laguerre polynomials can be obtained from the big q-Laguerre, the little q-Jacobi and the affine q-Kravchuk polynomials by setting $x \to bqx$ with $b \to \infty$, $b = 0$ and $x \to q^{-N}x$ with $N \to \infty$, respectively, in the following way,

$$\lim_{b\to\infty} \frac{P_n(bqx, a, b; q)}{b^n} = P_n(x; a; q), \tag{5.51}$$

$$P_n(x, a, 0; q) = P_n(x; a; q), \tag{5.52}$$

$$\lim_{N\to\infty} q^N K_n^{aff}(q^{-N}x, p, N; q) = P_n(x; p; q). \tag{5.53}$$

5.17 Limit Relations

Limit relations between the q-Hahn polynomials have been performed in each case above. In the present section, we deal with the relations between the q-polynomials on linear lattice (the q-Hahn class) and the classical continuous and discrete ones identified by Table 1.1 and Table 1.2, respectively. By using the properties of the limit relation between the hypergeometric series ${}_r\varphi_s$ and ${}_rF_s$ and q-shifted factorial and Pochhammer symbol defined by (2.41) and (2.40), respectively, all these limit relations are extracted from [36].

Notice that the q-EHT of the 1st kind

$$\sigma_1(x;q)D_{q^{-1}}D_q y(x,q) + \tau(x,q)D_q y(x,q) + \lambda(q)y(x,q) = 0$$

where

$$\sigma_1(x;q) := \frac{2}{1+q}\left[\sigma(x) - \frac{1}{2}(q-1)x\tau(x)\right],\ \tau(x,q) := \tau(x),\ \lambda(q) := \lambda$$

and the 2nd kind

$$\sigma_2(x;q)D_q D_{q^{-1}} y(x,q) + \tau(x,q)D_{q^{-1}} y(x,q) + \lambda(q)y(x,q) = 0$$

where

$$\sigma_2(x,q) := q\left[\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)\right]$$

approach to the classical EHT

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

as $q \to 1$. Observe that, both $\sigma_1(x,q)$ and $\sigma_2(x,q)$ tend to $\sigma(x)$ as $q \to 1$. As a result, by using this property we obtain the well-known classical orthogonal polynomials (those of Jacobi, Laguerre, Hermite) in the following.

We first start to get the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which have quadratic $\sigma(x)$ with distinct roots. That's why, $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi polynomials converge to the Jacobi Polynomials as $q \to 1$. Some examples of such relations are as the following.

Big q-Jacobi → Jacobi

By setting $c = 0$, $a = q^\alpha$, $b = q^\beta$, $q \to 1$ in the q-EHT for the big q-Jacobi polynomials we get

$$\sigma_1(x,q) = q^{-2}(x-aq)(x-cq) \to x(x-1), \quad \sigma_2(x,q) = aq(x-1)(bx-c) \to x(x-1),$$

$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{a(bq-1)+c(aq-1)}{1-q} \to (\alpha+\beta+2)x - (\alpha+1),$$

$$\lambda_n(q) = q^{-n}[n]_q \frac{1-abq^{n+1}}{q-1} \to -n(n+\alpha+\beta+1)$$

with the help of the limit relation of the q-numbers defined by (2.40) which leads to

$$x(1-x)y''(x) + [(\alpha+1) - (\alpha+\beta+2)x]\,y'(x) + n(n+\alpha+\beta+1)y(x) = 0.$$

Notice that the transformation $x = \frac{1-t}{2}$ leads to

$$(1-t^2)y''(t) + [(\beta-\alpha) - (\alpha+\beta+2)t]\,y'(t) + n(n+\alpha+\beta+1)y(t) = 0$$

which is the differential equation for the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$.

We remark that one can also obtain the hypergeometric representation of the Jacobi polynomials from the one defined by (5.10) of the monic big q-Jacobi polynomials with the same relation on the parameters together with the definition of ${}_r\phi_s$ in (2.38) and the limit relation between ${}_r\phi_s$ and ${}_rF_s$, q-numbers and Pochhammer symbol defined by (2.41) and (2.40), respectively, in the following way

$$\lim_{q\to 1} P_n(x; q^\alpha, q^\beta, 0; q) = \frac{P_n^{(\alpha,\beta)}(2x-1)}{2^n} \tag{5.54}$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\, {}_2F_1\left(\begin{matrix} -n, & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix}\;\middle|\; \frac{1-x}{2}\right) \tag{5.55}$$

here, ${}_2F_1$ is defined by (2.36) with $r = 2$, $s = 1$.

Little q-Jacobi → Jacobi

Let $a = q^\alpha$, $b = q^\beta$ in the definition of the little q-Jacobi polynomials (5.31) and take the limit as $q \to 1$ by using definition of ${}_r\phi_s$ in (2.38) and the limit relation between ${}_r\phi_s$ and ${}_rF_s$ defined by (2.41), we obtain the Jacobi polynomials identified by (5.55) in the following way

$$\lim_{q\to 1} P_n(x; q^\alpha, q^\beta; q) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{(-2)^n}. \tag{5.56}$$

One can also obtain the differential equation for the Jacobi polynomials from the q-EHT for the little q-Jacobi polynomials by use of the analogous transformation.

Notice that the Laguerre polynomials $L_n^\alpha(x)$ can be obtained by use of the convenient limit relations when σ_1 and σ_2 are quadratic or linear since $L_n^\alpha(x)$ has linear $\sigma(x)$. Some examples of such relations are as the following.

Big q-Laguerre → Laguerre

If we set $a = q^\alpha$, $b = (1-q)^{-1}q^\beta$ and $q \to 1$ in the q-EHT for the big q-Laguerre polynomials, we get

$$(q-1)\sigma_1(x,q) = q^{-2}(q-1)(x-aq)(x-bq) \to (x-1),\ (q-1)\sigma_2(x,q) = (q-1)abq(1-x) \to (x-1),$$

$$(q-1)\tau(x,q) = -(q-1)[\frac{q^{-1}}{q-1}x + \frac{a+b-abq}{q-1}] \to (\alpha+1-x+1),\ (q-1)\lambda_n(q) = q^{-n}[n]_q \to n,$$

with the help of the limit relation of the q-numbers defined by (2.40) which leads to

$$(x-1)y''(x) + [\alpha + 1 - (x-1)]\, y'(x) + ny(x) = 0.$$

Notice that solution of this equation is $L_n^\alpha(x-1)$.

By letting same transformation $a = q^\alpha$, $b = (1-q)^{-1}q^\beta$ in the definition

$$P_n(x;a,b;q) = (-b)^n q^{\binom{n}{2}} (aq;q)_n {}_2\varphi_1 \left(\begin{matrix} q^{-n}, & aqx^{-1} \\ & aq \end{matrix} \middle| q; \frac{x}{b} \right) \tag{5.57}$$

of the monic big q-Laguerre polynomials obtained by using the transformation formula (2.47) to (5.24) and then concerning limit as $q \to 1$ with the help of the definition of ${}_r\phi_s$ in (2.38) and the limit relation between ${}_r\phi_s$ and ${}_rF_s$ defined by (2.41), we can also arrive at the monic Laguerre polynomials in the following way

$$\lim_{q\to 1}(1-q)^n P_n(x, q^\alpha, (1-q)^{-1}q^\beta; q) = L_n^\alpha(x-1) \tag{5.58}$$

where the monic Laguerre polynomials are identified by

$$L_n^\alpha(x) = (-1)^n(\alpha+1)_n {}_1F_1 \left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| x \right). \tag{5.59}$$

Little q-Jacobi → Laguerre

Assuming that $a = q^\alpha$, $b = -q^\beta$ and $x \to \frac{1}{2}(1-q)x$ and $q \to 1$ in the definition (5.31) of the little q-Jacobi polynomials together with the idea in (2.40) and (2.41) lead to the Laguerre polynomials (5.59) hereinbelow

$$\lim_{q\to1} 2^n q^{-\binom{n}{2}} P_n(\frac{1}{2}(1-q)x; q^\alpha, -q^\beta; q) = L_n^\alpha(x). \tag{5.60}$$

One can also obtain the differential equation for the Laguerre polynomials from the q-EHT for the little q-Jacobi polynomials by use of the analogous transformation.

Little q-Laguerre/Wall → Laguerre

Let $a = q^\alpha$ and $x \to (1-q)x$ with the limit $q \to 1$ in the little q-Laguerre polynomials identified by (5.50), then we get the Laguerre polynomials (5.59) by use of (2.40) and (2.41) as follows:

$$\lim_{q\to1} q^{-\binom{n}{2}} P_n((1-q)x; q^\alpha; q) = L_n^\alpha(x). \tag{5.61}$$

One can also obtain the differential equation for the Laguerre polynomials from the q-EHT for the little q-Laguerre/Wall polynomials by use of the analogous transformation.

q-Laguerre → Laguerre

If we set $x \to (1-q)x$ with $q \to 1$ in the definition

$$L_n^\alpha(x;q) = (-1)^n q^{-n^2-\alpha n}(q^{\alpha+1};q)_n {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}\middle| q; -xq^{n+\alpha+1}\right) \tag{5.62}$$

of the monic q-Laguerre polynomials obtained by applying the transformation formulas (2.49) with $c \to 0$ and (2.50) to (5.36), we arrive at the Laguerre polynomials (5.59) with the help of (2.40) and (2.41) hereinbelow

$$\lim_{q\to1} q^{n^2+\alpha n} L_n^\alpha((1-q)x; q) = L_n^\alpha(x). \tag{5.63}$$

One can also obtain the differential equation for the Laguerre polynomials from the q-EHT for the q-Laguerre polynomials by use of the analogous transformation.

In order to get the Hermite polynomials $H_n(x)$ we use the convenient limit relations when σ_1 and σ_2 are quadratic, linear or constant since $H_n(x)$ has constant $\sigma(x)$. Some examples of such relations are as the following.

Discrete q-Hermite I → Hermite

By setting $x \to x\sqrt{1-q^2}$ and $q \to 1$ in the q-EHT for the discrete q-Hermite I polynomials, we get

$$\sigma_1(x,q) = -q^{-1}(1-x)(x+1) \to -1, \quad \sigma_2(x,q) = -1,$$
$$\frac{1-q^2}{\sqrt{1-q^2}}\tau(x,q) = \frac{1-q^2}{\sqrt{1-q^2}}\frac{1}{1-q}x \to 2x, \ (1-q^2)\lambda_n(q) = -(1+q)q^{1-n}[n]_q \to -2n$$

with the help of the limit relation of the q-numbers defined by (2.40) which leads to

$$y''(x) - 2xy'(x) + 2ny(x) = 0.$$

Notice that solution of this equation is $H_n(x)$.

By letting identical transformation $x \to x\sqrt{1-q^2}$ in the definition

$$h_n(x;q) = q^{-n}x^n{}_2\varphi_1\left(\begin{matrix} q^{-n}, & q^{-n+1} \\ & - \end{matrix}\,\middle|\, q^2; \frac{q^{2n-1}}{x^2}\right) \tag{5.64}$$

of the monic discrete q-Hermite I polynomials obtained by using the transformation (2.54) to (5.29) with $a = -1$ and then concerning limit as $q \to 1$ with the help of the definition of ${}_r\phi_s$ in (2.38) and the limit relation between ${}_r\phi_s$ and ${}_rF_s$ defined by (2.41) together with the q-shifted factorial identified by (2.20) and (2.40), we can also arrive at the monic Hermite polynomials in the following way

$$\lim_{q\to 1}\frac{h_n(\sqrt{1-q^2}x, q)}{(1-q)^{n/2}} = H_n(x) \tag{5.65}$$

where the monic Hermite polynomials are identified by

$$H_n(x) = x^n{}_2F_0\left(\begin{matrix} -n/2, & -(n-1)/2 \\ & - \end{matrix}\,\middle|\, -\frac{1}{x^2}\right). \tag{5.66}$$

Discrete q-Hermite II → Hermite

By setting $x \to x\sqrt{1-q^2}$ and $q \to 1$ in the in the definition

$$\widetilde{h}_n(x;q) = x^n{}_2\varphi_0\left(\begin{matrix} q^{-n}, & q^{-n+1} \\ & 0 \end{matrix}\,\middle|\, q^2; \frac{-q^2}{x^2}\right) \tag{5.67}$$

of the monic discrete q-Hermite II polynomials obtained by using the transformation (2.55) to (5.22) and then concerning limit as $q \to 1$ with the help of the definition of ${}_r\phi_s$ in (2.38) and the limit relation between ${}_r\phi_s$ and ${}_rF_s$ defined by (2.41) together with the q-shifted factorial identified by (2.20) and (2.40), we can also arrive at the monic Hermite polynomials in the following way

$$\lim_{q\to 1}\frac{\widetilde{h}_n(\sqrt{1-q^2}x,q)}{(1-q)^{n/2}} = H_n(x) \tag{5.68}$$

where the monic Hermite polynomials are identified by (5.66).

One can also obtain the differential equation for the Hermite polynomials from the q-EHT for the discrete q-Hermite II polynomials by use of the analogous transformation.

In the further study, we obtain the well-known classical discrete orthogonal polynomials (those of Hahn, Meixner, Kravchuk, Charlier). To this end, we first obtain the relation between q-EHT and difference equation on uniform lattice. Notice that the q-EHT of the 1st kind leads to

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0$$

together with the transformation $x \to q^x$, $q \to 1$ and the relation

$$\lim_{q\to 1}\frac{1-q^x}{1-q} = x.$$

We note that $(1-q)^{-2}\sigma_1(x)$ tends to $\sigma(x)$ and $(1-q)^{-2}\sigma_2(x,q)$ tends to $\sigma(x)+\tau(x)$.

On the other hand, taking account of the transformation $x \to q^{-x}$ with $q \to 1$ leads to

$$[\sigma(x)+\tau(x)]\Delta\nabla y(x) + \tau(x)\nabla y(x) + \lambda y(x) = 0$$

where $(1-q)^{-2}\sigma_1(x)$ tends to $\sigma(x)+\tau(x)$ and $(1-q)^{-2}\sigma_2(x,q)$ tends to $\sigma(x)$. As a result of this property, we perform the following limits.

We first start to get the Hahn polynomials where $\sigma(x)$ and $\sigma(x)+\tau(x)$ are quadratic. That's why, we deal with the ones when σ_1 and σ_2 are both quadratic. An example of such relation is as the following.

q-Hahn → Hahn

By setting $\alpha = q^\alpha, \beta = q^\beta$, $x \to q^{-x}$ and $q \to 1$ in the q-Hahn polynomials, we get

$$(q-1)^{-2}\sigma_1(x,q) = q^{-2}\frac{(x-q^{-N})}{1-q}\frac{(x-\alpha q)}{1-q} \to (x-N)(x+\alpha+1) = \sigma(x)+\tau(x),$$

$$(q-1)^{-2}\sigma_2(x,q) = \alpha q \frac{(x-1)}{1-q}\frac{(\beta x - q^{-N-1})}{1-q} \to x(x-\beta-N-1) = \sigma(x),$$
$$\frac{\tau(x,q)}{1-q} = \frac{1-\alpha\beta q^2}{(1-q)^2 q}x + \frac{\alpha q^{-N}+\alpha\beta q - \alpha - q^{-N-1}}{(1-q)^2} \to (\alpha+\beta+2)x - (\alpha+1)N = \tau(x),$$
$$\lambda_n(q) = -q^{-n}[n]_q \frac{1-\alpha\beta q^{n+1}}{1-q} \to n(n+\alpha+\beta+1) = \lambda_n,$$

with the help of the limit relation of the q-numbers defined by (2.40) which leads to

$$x(x-\beta-N-1)\Delta\nabla y(x) + \Big[(\alpha+\beta+2)x - (\alpha+1)N\Big]\Delta y(x) + n(n+\alpha+\beta+1)y(x) = 0.$$

Notice that solution of this equation is $h_n^{\alpha,\beta}(x)$.

Letting same transformation $\alpha = q^\alpha, \beta = q^\beta$, $x \to q^{-x}$ and $q \to 1$ in the hypergeometric representation of the q-Hahn polynomials defined by (5.11) we also arrive at the Hahn polynomials with the help of (2.40) and (2.41) as the following form

$$\lim_{q\to 1} Q_n(q^{-x}; q^\alpha, q^\beta, N; q) = Q_n(x;\alpha,\beta,N) \tag{5.69}$$

where $Q_n(x;\alpha,\beta,N)$ is the Hahn polynomials identified as the following

$$Q_n(x;\alpha,\beta,N) = \frac{(\alpha+1)_n(-N)_n}{(\alpha+\beta+n+1)_n}\,{}_3F_2\left(\begin{array}{ccc} -n, & n+\alpha+\beta+1, & -x \\ & \alpha+1, \quad -N & \end{array}\Big| 1\right). \tag{5.70}$$

We remark that the q-Meixner and the q-Kravchuk polynomials have linear $\sigma(x)$ and $\sigma(x)+\tau(x)$. That's why, q-polynomials which have quadratic and linear σ_1 and σ_2 lead to the q-Meixner and the q-Kravchuk polynomials. Some examples of such relations are as the following.

q-Meixner → Meixner

By setting $b = q^{\beta-1}$, $c \to (1-c)^{-1}c$, $x \to q^{-x}$ in the definition (5.13) of the q-Meixner polynomials and then letting $q \to 1$ concerning with (2.40) and (2.41) bring about the Meixner polynomials as

$$\lim_{q\to 1} M_n(q^{-x}; q^{\beta-1}, \frac{c}{1-c}; q) = M_n(x;\beta,c) \tag{5.71}$$

where $M_n(x;\beta,c)$ is the Meixner polynomials identified by

$$M_n(x;\beta,c) = (\frac{c}{c-1})^n(\beta)_n\,{}_2F_1\left(\begin{array}{cc} -n, & -x \\ \beta & \end{array}\Big| 1-\frac{1}{c}\right). \tag{5.72}$$

One can also obtain the difference equation for the Meixner polynomials from the q-EHT for the q-Meixner polynomials by use of the analogous transformation.

Quantum q-Kravchuk → Kravchuk

If we set $p \to p^{-1}$, $x \to q^{-x}$ in the definition (5.16) of the quantum q-Kravchuk polynomials and then letting $q \to 1$ together with applying the limit in (2.40) and (2.41), we get the Kravchuk polynomials in the following way

$$\lim_{q\to 1} K_n^{qtm}(q^{-x}; p^{-1}, N; q) = K_n(x; p, N) \tag{5.73}$$

where $K_n(x; p, N)$ is the Kravchuk polynomials defined by

$$K_n(x; p, N) = p^n(-N)_n {}_2F_1\left(\begin{matrix} -n, & -x \\ & -N \end{matrix} \middle| \frac{1}{p}\right). \tag{5.74}$$

One can also obtain the difference equation for the Kravchuk polynomials from the q-EHT for the quantum q-Kravchuk polynomials by use of the analogous transformation.

q-Kravchuk → Kravchuk

Setting $x \to q^{-x}$ in the definition

$$K_n(x; p, N; q) = \frac{(q^{-N}; q)_n}{(-pq^n; q)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, & -pq^n, & x \\ & q^{-N}, & 0 \end{matrix} \middle| q; q\right) \tag{5.75}$$

of the q-Kravchuk polynomials obtained by applying the transformation formula (2.48) to (5.34), then letting $q \to 1$ with the help of (2.40) and (2.41) lead to the Kravchuk polynomials (5.74) as the following form

$$\lim_{q\to 1} K_n(q^{-x}; p, N; q) = K_n(x; \frac{1}{1+p}, N). \tag{5.76}$$

One can also obtain the difference equation for the Kravchuk polynomials from the q-EHT for the q-Kravchuk polynomials by use of the analogous transformation.

Affine q-Kravchuk → Kravchuk

Let $p \to 1 - p$, $x \to q^{-x}$ and $q \to 1$ in the definition (5.26) of the affine q-Kravchuk polynomials, then we obtain the Kravchuk polynomials (5.74) in the following way

$$\lim_{q\to 1} K_n^{aff}(q^{-x}; 1 - p, N; q) = K_n(x; p, N) \tag{5.77}$$

by use of the definition of q-shifted factorial defined by (2.15) and the limit relations (2.40), (2.41).

One can also obtain the difference equation for the Kravchuk polynomials from the q-EHT for the affine q-Kravchuk polynomials by use of the analogous transformation.

We note that the q-Charlier polynomials have linear $\sigma(x)$ and constant $\sigma(x)+\tau(x)$. That's why, q-polynomials which have quadratic, linear and constant σ_1 and σ_2 lead to the q-Charlier polynomials. Some examples of such relations are as the following.

Little q-Laguerre/Wall → Charlier

Letting $a \to (1-q)a$ and $x \to q^x$ in the q-EHT for the little q-Laguerre polynomials, we get

$$\frac{\sigma_1(x,q)}{(q-1)} = q^{-2}x\frac{(1-x)}{q-1} \to -x = \sigma(x), \quad \frac{\sigma_2(x,q)}{(q-1)} = a\frac{x}{q-1} \to -a = \sigma(x)+\tau(x),$$

$$\tau(x,q) = -\frac{1}{(1-q)q}x + \frac{1-aq}{(1-q)q} \to x-a = \tau(x), \quad (q-1)\lambda_n(q) = q^{-n}[n]_q \to -n = \lambda_n,$$

with the help of the limit relation of the q-numbers defined by (2.40) which lead to

$$x\Delta\nabla y(x) + (a-x)\Delta y(x) + ny(x) = 0.$$

Notice that solution of this equation is $C_n(x,a)$.

Setting identical transformation $a \to (1-q)a$ and $x \to q^x$ in the definition

$$P_n(x;a;q) = a^n q^{n(n-1)} {}_2\varphi_0\left(\begin{matrix} q^{-n}, & x^{-1} \\ & - \end{matrix}\middle| q; \frac{x}{a}\right) \tag{5.78}$$

of the little q-Laguerre (Wall) polynomials obtained from (5.50) by applying the transformation formulas (2.49) with $b \to 0$ and (2.51) successively, we also arrive at the Charlier polynomials as

$$\lim_{q\to 1}\frac{P_n(q^x;(1-q)a;q)}{(1-q)^n} = C_n(x,a) \tag{5.79}$$

by using (2.40) and (2.41). Here, $C_n(x,a)$ is the Charlier polynomials given by

$$C_n(x;a) = (-a)^n {}_2F_0\left(\begin{matrix} -n, & -x \\ & - \end{matrix}\middle| -\frac{1}{a}\right). \tag{5.80}$$

q-Laguerre → Charlier

Using the transformation $x \to -q^{-x}$, $q^{\alpha} = a^{-1}(q-1)^{-1} \Leftrightarrow \alpha = -\frac{\ln(q-1)a}{\ln q}$ in the definition (5.36) of the q-Laguerre polynomials and letting $q \to 1$ associated with the properties in (2.40) and (2.41) lead to the Charlier polynomials (5.80) in the following way,

$$\lim_{q\to 1} \frac{L_n^{-\frac{\ln(q-1)a}{\ln q}}(-q^{-x};q)}{(q-1)^n} = C_n(x;a). \tag{5.81}$$

One can also obtain the difference equation for the Charlier polynomials from the q-EHT for the q-Laguerre polynomials by use of the analogous transformation.

q-Charlier → Charlier

Assuming $x \to q^{-x}$, $a \to (1-q)a$ in the definition (5.39) of the q-Charlier polynomials and then taking the limit as $q \to 1$ together with the properties in (2.40) and (2.41), we arrive at the Charlier polynomials (5.80) as

$$\lim_{q\to 1} \frac{C_n(q^{-x};a(1-q);q)}{(1-q)^n} = C_n(x;a). \tag{5.82}$$

One can also obtain the difference equation for the Charlier polynomials from the q-EHT for the q-Charlier polynomials by use of the analogous transformation.

Alternative q-Charlier → Charlier

Inserting $x \to q^x$ and $a \to (1-q)a$ into the definition

$$K_n(x;a;q) = \frac{a^n q^{n^2+\binom{n}{2}} x^n}{(-aq^n;q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, & x^{-1} \\ & 0 \end{matrix} \,\middle|\, -\frac{q^{-n+1}}{a}\right) \tag{5.83}$$

of the alternative q-Charlier polynomials obtained from (5.43) by applying the transformation formula (2.49) with $c \to 0$ and then letting $q \to 1$ associated with performing the properties in (2.40) and (2.41) bring about the Charlier polynomials (5.80) as the following form

$$\lim_{q\to 1} \frac{K_n(q^x;a(1-q);q)}{(q-1)^n} = C_n(x;a). \tag{5.84}$$

One can also obtain the difference equation for the Charlier polynomials from the q-EHT for the alternative q-Charlier polynomials by use of the analogous transformation.

Al-Salam Carlitz I → Charlier

Replacing $x \to q^x$ and $a \to a(q-1)$ in the definition (5.29) of the Al-Salam Carlitz I polynomials and then taking the limit as $q \to 1$ together with the properties in (2.40) and (2.41) produce the Charlier polynomials as

$$\lim_{q\to 1} \frac{U_n^{(a(q-1))}(q^x;q)}{(q-1)^n} = C_n(x;a). \tag{5.85}$$

One can also obtain the difference equation for the Charlier polynomials from the q-EHT for the Al-Salam Carlitz I polynomials by use of the analogous transformation.

Al-Salam Carlitz II → Charlier

Inserting $x \to q^{-x}$ and $a \to a(q-1)$ into the definition (5.19) of the Al-Salam Carlitz II polynomials and taking the limit as $q \to 1$ together with applying the properties (2.40) and (2.41) lead to the Charlier polynomials as the following form

$$\lim_{q\to 1} \frac{V_n^{(a(q-1))}(q^{-x};q)}{(q-1)^n} = C_n(x;a). \tag{5.86}$$

One can also obtain the difference equation for the Charlier polynomials from the q-EHT for the Al-Salam Carlitz II polynomials by use of the analogous transformation.

CHAPTER 6

CONCLUSION

The investigation of q-difference equations is an old problem which has been studied by several authors for a long time, especially, from the view point of the Favard theorem. In this thesis, on the other hand, we present a more direct and a simpler geometrical approach based on the qualitative analysis of solutions of the q-Pearson equation. In this way, we show that it is possible to introduce in a unified manner all polynomial solutions of the q-difference equation of the hypergeometric type, which are orthogonal on certain intervals.

Besides its simplicity and clarity, our approach enables to introduce some new orthogonality relations which have not been reported previously. The appearance of such new relations, see, for example, Theorem 4.13, is due to the fact that we have considered the polynomial coefficients of the q-hypergeometric equation in their full generality dealing with all suitable structures.

Recall once more that q-polynomials of the Hahn class, defined on the q-linear lattices of forms $x(s) = q^s$ and $x(s) = q^{-s}$, have been examined in this thesis. Actually, satisfactory results are obtained and research articles on the subject are in progress [8, 9]. Furthermore, we have just started to study along the same lines the q-polynomials on a q-quadratic lattice of the form $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, where c_1, c_2 and c_3 are definite constants. As another extension of our thesis, we consider the q-Krall type polynomials on the non-uniform q-quadratic lattices. Some partial results for q-Racah and for q-dual Hahn polynomials have been obtained, which will be reported in due course [7].

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