ON THE TIGHT CONTACT STRUCTURES ON SEIFERT FIBRED 3−MANIFOLDS WITH 4 SINGULAR FIBERS

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

ON THE TIGHT CONTACT STRUCTURES ON SEIFERT FIBRED 3−MANIFOLDS WITH 4 SINGULAR FIBERS

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In this thesis, we study the classification problem of Stein fillable tight contact structures on any Seifert fibered 3−manifold M over S^2 with 4 singular fibers. In the case $e_0(M) \leq -4$ we have a complete classification. In the case $e_0(M) \geq 0$ we have obtained upper and lower bounds for the number of Stein fillable contact structures on M.

Keywords: Contact structures, Seifert fibred manifolds

ÖZ

SEİFERT MANİFOLDLARDAKİ TAYT KONTAKT YAPILAR ÜZERİNE

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Bu tezde, M bir 4 tekil lifli Seifert $3-$manifold olmak üzere, M üzerindeki Stein dolabilir tayt yapılar çalışılmıştır. $e_0(M) \leq -4$ olması durumunda Stein dolabilir yapıların sınıflandırması elde edilmiştir. $e_0(M) \geq 0$ olması durumunda ise Stein dolabilir yapıların sayıları için bir üst ve bir alt sınır bulunmuştur.

Anahtar Kelimeler: Kontakt yapılar, Seifert manifoldlar

To my family

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CHAPTER 1

INTRODUCTION

Classification of tight contact structures on 3−manifolds has been an interesting problem since late 1980's. It started with Eliashberg, who showed that there exists a unique tight contact structure on $\mathbb{R}^3$ [9]. Later various people worked on this problem and the classification has been done for several 3−manifolds, including S^3, B^3, $T^2 \times I$, $S^1 \times D^2$, $L(p,q)$, S^1 bundles over surfaces, some surface bundles over S^1 and some Seifert fibred 3−manifolds ([9], [38], [39], [58], [59], [28], [29], [24], [26]).

A Seifert fibred 3−manifold with three or less singular fibers and with base S^2 is called a *small Seifert fibred* 3−*manifold*. The classification of tight contact structures on small Seifert fibred 3−manifolds is as follows: A small Seifert fibred 3−manifold with one or two singular fibers is a Lens space and the exact number of tight contact structures on lens spaces is known [38]. For small Seifert fibred 3−manifolds M with three singular fibers, Wu computed the exact number of tight contact structures on M, whenever $e_0(M) \neq -1$, $e_0(M) \neq -2$ or $e_0(M) \neq 0$ in [59]. Later, Stipsicz, Ghiggini and Lisca gave a complete classification for the case $e_0(M) \geq 0$ in [28]. All the tight contact structures in the above cases are Stein fillable. Note that fillable contact structures are tight (Eliashberg and Gromov, [12]). In the case $e_0(M) = -1$ the classification is more difficult. This is partly because of the existence of non–fillable tight contact structures, which are harder to detect. Another reason is that this case contains some manifolds which do not admit any positive tight contact structure as shown by Honda and Etnyre in [14]. Tight contact structures on small Seifert fibred 3−manifolds with $e_0(M) = -2$ are classified provided that they are L−spaces (Ghiggini, [26]).

For non–small Seifert fibred 3−manifolds, the first result in this direction is the classification of tight contact structures on Seifert fibred 3−manifolds with one singular fiber and with base

T^2, given by Ghiggini in [24]. In this thesis, we study the classification problem of tight contact structures on Seifert fibred 3−manifolds with 4 singular fibers over S^2. We use the methods developed in [58], [59] and [28]. Here is an outline of the thesis. In the Chapter 2, we introduce some preliminary materials such as convex surface theory, Legendrian surgery, classification of tight contact structures on some building blocks of 3−manifolds, Heegaard–Floer theory and open book decompositions.

In the third chapter, using convex surface theory we obtain an upper bound for the number of Stein fillable tight contact structures on some certain Seifert fibred 3−manifolds having 4 singular fibers with base S^2, and then using Legendrian surgery presentations of these Seifert fibred 3−manifolds we get a lower bound for the same number. If a Seifert fibred 3−manifold M has $e_0(M) \leq -4$ then the upper bound and the lower bound we obtain coincide, giving the exact number of Stein fillable contact structures on M. However, in the case $e_0(M) \geq 0$ the two bounds do not match.

The argument we give to get the upper bound depends on whether the contact 3−manifold contains a vertical Legendrian curve with zero twisting or not. We observe that when $e_0(M) \geq 0$ all the tight contact structures contain a vertical Legendrian curve with zero twisting, which is exactly the same situation for the small Seifert fibred manifolds by a result of Wu [58]. It is also proved by Wu [58] that none of the tight contact structures on small Seifert fibred 3−manifolds with $e_0(M) \leq -2$ contains a vertical Legendrian curve with zero twisting. In the case of four singular fibers, a tight contact structure on Seifert fibred 3−manifolds with $e_0(M) \leq -1$ may or may not contain a vertical Legendrian curve with twisting zero. We prove that when $e_0(M) \leq -2$ none of the tight contact structures with zero Giroux torsion admit a Legendrian vertical curve with zero twisting. In the case $e_0(M) \leq -2$, the tight contact structures will be non–fillable if there is a Legendrian vertical curve with twisting zero. At this point we can consider another question: Are the tight but non–fillable contact structures virtually overtwisted or universally tight? The first examples of tight but non–fillable contact structures were found by Etnyre and Honda in [38]. These examples were all virtually overtwisted. Later, several tight but non–fillable examples were exhibited, [45], [46]. These are also all virtually overtwisted. The first of the universally tight but non–fillable examples were constructed by Ghiggini [27] on Seifert fibred 3−manifolds with 4 singular fibers and $e_0(M) = 0$. He used the Ozsváth–Szabó contact invariant to show the non–fillability of his examples. The contact structures he constructed have zero twisting, zero contact invariant and positive Giroux torsion. One can further ask whether there are any tight contact structure with zero twisting, zero contact invariant and zero Giroux torsion or not? Such contact structures were firstly constructed by Massot [50]; he proved that such contact structures exist on any Seifert fibred $3-$manifold with the base genus greater than or equal to 2.

In Chapter 4, we discuss possible generalizations of the above results to the Seifert fibred $3-$manifolds with at least five singular fibers. We give examples of compatible open book decompositions of the Stein fillable contact structures mentioned in Chapter 3.

CHAPTER 2

BACKGROUND

In this chapter we will introduce the classification problem of tight contact structures and give a brief introduction to the methods, which will be used to prove the main theorems. In the last part we will review some important results on the classification of tight contact structures on small Seifert fibred 3−manifolds.

2.1 Classification Problem

On a 3-manifold a plane distribution ξ, which is locally the kernel of a 1-form α with $\alpha \wedge d\alpha \neq 0$ is called a *contact structure*. Martinet in 1971 showed that every oriented closed 3-manifold admits a contact structure [48]. We call a 3-manifold M with a contact structure ξ a *contact* 3-*manifold* and denote it by (M, ξ). As a first example we can look at the standard contact structure on R^3, which is $\xi_{std} = \ker(dz - ydx)$. If we try to draw its picture we will see that the planes are twisting in the y direction as in Figure 2.1. Another example is $\xi_{ot} = \ker (\cos r dz + r \sin r d\theta)$ on R^3 with cylindrical coordinates. In the first example the twisting of planes in the y direction is less than π however in the second example the contact planes rotates along any radial direction infinitely many times.

Definition 2.1.1. A curve L which is everywhere tangent to ξ is called *Legendrian*. We define the *twisting number* $t(L, Fr)$ of a closed Legendrian curve L with respect to a given framing Fr to be the number of counterclockwise (right) 2π twists of ξ along L, relative to Fr. In particular, if L is a connected component of the boundary of a compact oriented surface Σ, $T\Sigma$ gives a natural framing Fr_Σ, then $t(L, Fr_\Sigma)$ is called the *Thurston-Bennequin invariant* $tb(L)$.

Definition 2.1.2. Let L be an oriented Legendrian curve which is the boundary of an embed-

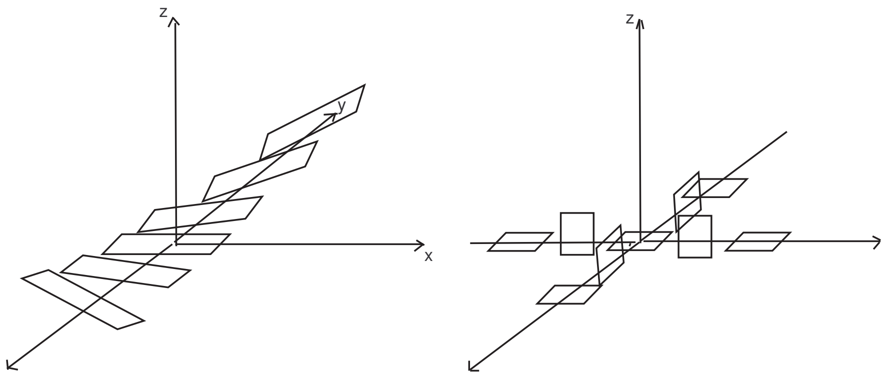

Figure 2.1: Standard and overtwisted contact structures on R^3

ded orientable surface Σ. The winding number of a non–zero tangent vector field along L with respect to any given trivialization of $\xi|_{\partial\Sigma}$ is called the *Rotation number* of a Legendrian curve.

Definition 2.1.3. An embedded disk D in (M, ξ) is called an *overtwisted disk* if the contact planes are tangent to $\partial D = L$ and $tb(L) = 0$. If (M, ξ) contains such a disk then this contact structure is called an *overtwisted contact structure*. If there is no such embedded disk then the contact structure is called a *tight contact structure*.

According to this definition the second example above is an overtwisted contact structure, and we can see the overtwisted disk in Figure 2.1. However, the first example on $\mathbb{R}^3$, which is ξ_{std}, is a tight contact structure. Tightness of this contact structure can be shown using the Bennequin inequality, [2], which says that if L is a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ and $\Sigma \subset \mathbb{R}^3$ is a Seifert surface for L then $tb(L) + |rot(L)| \leq -\chi(\Sigma)$. An overtwisted disk D has $tb(\partial D) = 0$ and $\chi(D) = 1$. Thus by the Bennequin inequality, there can not be such a disk in $(\mathbb{R}^3, \xi_{std})$.

The first example is called the *standard* tight contact structure, since by a theorem of Darboux, locally every tight contact structure look like $(\mathbb{R}^3, \xi_{std})$. Therefore there is no local invariant for contact structures.

Two contact structures ξ_1, ξ_2 on a smooth $3-$manifold M are called *contactomorphic* if there is an orientation preserving diffeomorphism $\varphi : M \to M$ such that $\varphi^*(\xi_1) = \xi_2$. If a contactomorphism is isotopic to the identity then it is called a *contact isotopy*.

The classification of overtwisted contact structures is the same as classification of $2-$plane fields (Eliashberg [8]) .

In this thesis we study the classification problem of (positive) tight contact structures, up to isotopy, on Seifert fibred 3−manifolds with base S^2 and 4 singular fibers.

2.2 Convex Surface Theory

One of the tools used in proof of the main results, is the convex surface theory, which is first used in [33], then developed by Honda in [38].

For a detailed information about convex surface theory one may look at [38], [39],[33].

An embedded surface is called a *convex surface* if there is a *contact vector field* (a vector field whose flow preserves the contact structures) which is transverse to the surface. Let Σ be an embedded convex surface in (M, ξ). By the works of Giroux in [31], and Honda in [38], by an arbitrary small isotopy of the surface Σ any embedded surface can be made convex in the contact 3−manifold (M, ξ).

Definition 2.2.1. Let Σ be a convex embedded surface in (M, ξ) with transverse vector field v. Then $\Gamma_\Sigma = \{x \in \Sigma | \alpha(v_x) = 0\}$ is an embedded curve on Σ, called the *dividing curve* of ξ on Σ.

By a theorem of Giroux [33] we can read the information about the contact structure on a neighborhood of surface from its dividing curves on the surface. The following theorem is called Giroux's Criteria.

Theorem 2.2.2 (Giroux [33]). *Let Σ be an orientable surface (with or without boundary) and $\Sigma \neq S^2$, then a contact structure on $\Sigma \times I$ is tight if and only if Γ_Σ contains no homotopically trivial dividing curve. When $\Sigma = S^2$ then the tight contact structure on $\Sigma \times I$ is tight if and only if Γ_Σ consists of only one component.*

Dividing curves divide the surfaces into + and − regions as $\Sigma - \Gamma_\Sigma = \Sigma_+ \cup \Sigma_-$, where the flow of the transverse vector field v on the convex surface Σ expands (contracts) a volume form on Σ_+ (on $\Sigma-$, resp.) and v points outward from Σ_+ along $\Gamma_\Sigma = \partial\Sigma_+$.

Definition 2.2.3. A bypass is an oriented embedded half overtwisted disk D, whose boundary is the union of two arcs, say α and β so that there are 3 elliptic singular points on α, two of them have the same sign but the third one which is in interior of α has different sign. Along β there are at least 3 elliptic singularity with the same sign but alternating indicies. The sign of a bypass is defined to be the sign of the singularity in the interior of α.

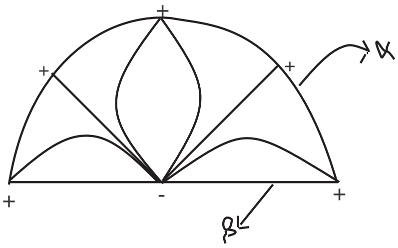

Figure 2.2: Bypass Half Disk

Lemma 2.2.4 (Edge-rounding, [38]). *Let Σ_1 and Σ_2 be convex surfaces with collared Legendrian boundary which intersect transversely inside the ambient contact manifold along a common boundary Legendrian curve. Assume a neighborhood of the common boundary Legendrian curve is locally isomorphic to the neighborhood $N_\epsilon = x^2 + y^2 \leq \epsilon$ of $M = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ with coordinates $((x, y), z)$ and contact $1-$form $\alpha = \sin(2\pi nz)dx + cos(2\pi nz)dy$, for some $n \in \mathbb{Z}^+$, and that Σ_1 and Σ_2, satisfy $\Sigma_1 \cap N_\epsilon = \{x = 0, 0 \leq y \leq \epsilon\}$ and $\Sigma_2 \cap N_\epsilon = \{y = 0, 0 \leq x \leq \epsilon\}$. If we join Σ_1 and Σ_2 along $\{x = y = 0\}$ and round the common edge (take $((\Sigma_1 \cup \Sigma_2) - N_\delta) \cup (\{(x - \delta)^2 + (y - \delta)^2 = \delta^2\} \cap N_\delta)$, where $\delta < \epsilon$), the resulting surface is convex, and the dividing curve $z = \frac{k}{2n}$ on Σ_1 will connect to the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2, where $k = 0, \ldots, 2n - 1$. Here we assume that the orientations of Σ_1 and Σ_2 are compatible and induce the same orientation after rounding.*

Lemma 2.2.5 (Bypass Attachment on T^2, Honda [38]). *If a bypass D is attached to T^2 in standard form, along a Legendrian ruling curve of slope r and if the slope of dividing curve of T^2 is s, then the resulting convex torus T' will have two dividing curves with boundary slope s' which is determined as follows: take the arc $[r, s]$ on the hyperbolic unit disc obtained by starting from r and moving counterclockwise until we hit s. On this arc, s' is the point which is closest to r and has an edge from s' to s.*

In practice this is done as follows: Let $s = \frac{p}{q}$, which we write as $s(T^2) = \frac{p}{q}$. Then the affect of a bypass along a ruling curve with slope $r \neq \frac{p}{q}$ can be found using the Farey tesellation which is the boundary of the hyperbolic unit disk. After bypass attachment as in the Figure 2.3 we obtain a torus T' which is isotopic to T^2 and has slope $s(T') = \frac{p'}{q'}$ such that $\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = 1$, where $p > p'$, $q > q'$ and the slopes, $\frac{p}{q}$ and $\frac{p'}{q'}$ are connected by an edge on the boundary of the hyperbolic unit disk.

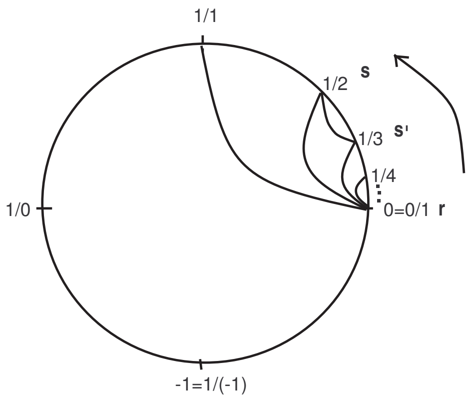

Figure 2.3: An application of Lemma 2.2.5

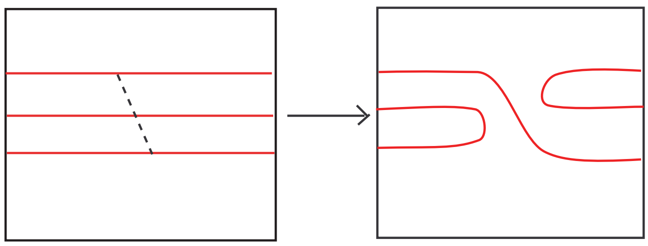

Figure 2.4: Altering of dividing curves after bypass attachment

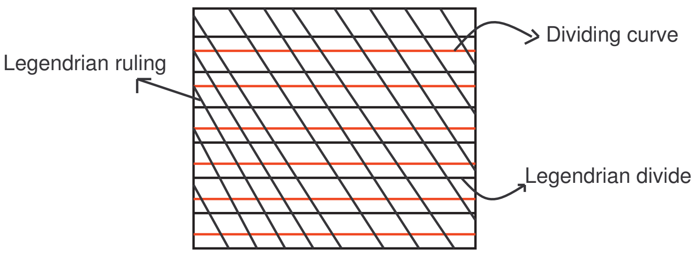

Figure 2.5: Standard torus

Assume $L \subset M$ is a Legendrian curve with a negative twisting number $t(L) = -n$ with respect to some fixed framing. The *standard tubular neighborhood* $N(L)$ of L is defined as the solid torus $S^1 \times D^2$ with coordinates $(z,(x,y))$ and the contact $1-$form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, where $L = \{(z,(x,y)) : x = y = 0\}$. With respect to the fixed framing of L, we may identify $\partial N(L)$ with $\mathbb{R}^2/\mathbb{Z}^2$ such that the meridian is $(1,0)^T$ and the longitude (fixed by the framing) is $(0,1)^T$. Then the boundary slope (which means the slope of the dividing curves of $\partial N(L)$) $s(\partial N(L)) = \dfrac{1}{n}$.

One can observe that the twisting number of a Legendrian curve L, which is the boundary of a convex surface Σ can be calculated using the following $t(L, Fr_\Sigma) = -\#\frac{1}{2}(L \cap \Gamma_\Sigma)$ [38].

Proposition 2.2.6 (Imbalance Principle, Honda [38]). Let $A = S^1 \times [0,1]$ be a convex annulus with Legendrian boundary inside a tight contact manifold. If $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$, then there exists a bypass along $S^1 \times \{0\}$.

Lemma 2.2.7 (Twist Number Lemma, Honda [38]). *Consider a Legendrian curve in a contact manifold (M,ξ) with twisting number n relative to a fixed framing and N a standard tubular neighborhood of L. If there exists a bypass attached to a Legendrian ruling curve of ∂N of slope r with $\dfrac{1}{r} \geq n+1$, then there exists a Legendrian curve with twisting number $n+1$ isotopic to L.*

Definition 2.2.8 (Relative Euler Class). Let (M,ξ) be a contact $3-$manifold, with ∂M. Consider the following exact sequence;

$$H^1(\partial M) \to H^2(M, \partial M) \to H^2(M) \to H^2(\partial M).$$

The Euler class of ξ is denoted by $e(\xi) \in H^2(M)$. If s is a nowhere zero section of the restriction of ξ to the boundary of M, we define the relative Euler class $e(\xi, s) \in H^2(M, \partial M)$ as the obstruction to extend s to M. It follows that $e(\xi, s)$ is a lift of $e(\xi)$.

The relative Euler class can be computed by using $< e(\xi, s), \Sigma >= \chi(\Sigma_+) - \chi(\Sigma_-)$, [38]. The relative Euler class is important since it distinguishes the tight contact structures.

Let $M = T^2 \times [0,1]$ be a contact $3-$manifold with $s(T^2 \times \{0\}) = \frac{p}{q}$ and $s(T^2 \times \{1\}) = \frac{p'}{q'}$, where (q,p) and (q',p') form an integral basis. On M there are two tight contact structures and they can be distinguished by Poincaré duals of the homology classes represented by $\pm((q',p') - (q,p))$. Indeed if a $3-$manifold contains M with the above properties, M is called a *basic slice*.

Using basic slices Honda classified the tight contact structures on $S^1 \times D^2$ and $T^2 \times I$ as follows: Let $-\dfrac{p}{q} = r_0 - \dfrac{1}{r_1 - \frac{1}{r_2 - \cdots - \frac{1}{r_k}}}$ be the continued fraction expansion of the rational number $-\dfrac{p}{q}$ and $p > q \geq 1$.

Theorem 2.2.9 (Honda [38]). *On the solid torus $S^1 \times D^2$ with two dividing curves of slope $-\dfrac{p}{q}$ there are exactly $|(r_o + 1)(r_1 + 1) \cdots (r_{k-1} + 1) r_k|$ non isotopic tight contact structures.*

The ideas in the proof of the following theorem is essential for the proof of our main results.

By minimally twisting in the following theorem, we mean the slope of every torus $T^2 \times \{t\}$ in $T^2 \times [0, 1]$ is between the slope of $T^2 \times \{0\}$ and the slope of $T^2 \times \{1\}$.

Theorem 2.2.10 (Honda [38]). *On $T^2 \times I$ with boundaries $T_i = T \times \{i\}$, for $i = 0, 1$, and boundary slopes $s(T_0) = -1$, $s(T_1) = -\frac{p}{q}$, there are exactly $|(r_o + 1)(r_1 + 1) \cdots (r_{k-1} + 1) r_k|$ minimally twisting tight contact structures, up to isotopy.*

Note that in the above two theorems the slopes are all negative. However, using a suitable diffeomorphism, which is an element of $SL(2, \mathbb{Z})$, we can change a negative slope to a positive one, ([38]).

Definition 2.2.11. Let (M, ξ) be a Seifert fibred $3-$manifold and L be a Legendrian curve isotopic to a regular fiber and $t(L)$ be the twisting of L. Maximal twisting of a contact structure ξ is $t(\xi) = max\{min_{L \in \mathbf{L}} \{t(L), 0\}\}$, where $\mathbf{L}$ is the set of possible Legendrian realizations of L.

To go further we need to understand the tight contact structures on $4-$punctured sphere times S^1. In the following theorem Honda [39] gave a classification of tight contact structures on pair of pants times S^1, with ∞ boundary slopes. Let Σ_0 denote sphere with $3-$punctures. $\partial(\Sigma_0 \times S^1) = T_1 \cup T_2 \cup T_3$ and infinite boundary slope meaning that $s(T_i) = \infty$ for all i.

Theorem 2.2.12 (Honda [39]). *If there exist tight contact structures on $\Sigma_0 \times S^1$ so that all boundary slopes are infinite, then these tight contact structures on $\Sigma_0 \times S^1$ are all S^1-invariant and determined only by the dividing curves Γ_{Σ_0} on Σ_0.*

Theorem 2.2.13 (Giroux [33]). *Let Σ be a surface. On $\Sigma \times S^1$ an S^1 invariant contact structure is (universally) tight if and only if there is no homotopically trivial dividing curve on Σ.*

The theorem below (and its proof) is an analog of a result of Honda on pair of pants times S^1.

Theorem 2.2.14. *Let Σ be a $4-$punctured sphere with $\partial\Sigma \times S^1 = T_1 \cup T_2 \cup T_3 \cup T_4$. Then the tight contact structures on $\Sigma \times S^1$ with minimally convex T_i''s having dividing curve slopes $s_i < \infty \in \mathbb{Z}$, $i = 1, 2, 3, 4$ can be classified as follows:*

1. *A tight contact structure with a vertical Legendrian curve can be factorized as follows,*

$$\Sigma \times S^1 = \Sigma' \times S^1 \cup (T_1 \times I) \cup (T_2 \times I) \cup (T_3 \times I) \cup (T_4 \times I),$$

   where $I = [0, 1]$ and $T_i \times \{1\}$ are the boundaries of $\Sigma' \times S^1$ having dividing curve with ∞ slope and each $T_i \times I$ is minimally twisting.

2. *If we have a universally tight contact structure with a vertical Legendrian curve having zero twisting, then we can extend the tight contact structure uniquely to a tight contact structure on*

$$\Sigma'' \times S^1 = \Sigma \times S^1 \cup (T_1'' \times I) \cup (T_2' \times I) \cup (T_3'' \times I) \cup (T_4'' \times I),$$

   where $I = [-1, 1]$ and all the $T_i'' \times \{-1\}$ which are also boundaries of $\Sigma'' \times S^1$ have ∞ slope.

3. *If $\sum_{i=1}^4 s_i \geq 3$ then*

   (a) *there always exists a vertical Legendrian curve with zero twisting,*

   (b) *universally tight contact structures are as in Part 2,*

   (c) *if $\sum_{i=1}^4 s_i = 4$ then there exists one, if $\sum_{i=1}^4 s_i > 4$ there exist two virtually overtwisted contact structures and if $\sum_{i=1}^4 s_i = 3$ then there exists no tight contact structure.*

4. *If $\sum_{i=1}^4 s_i < 3$ and if there exists a vertical Legendrian curve then tight contact structures are universally tight and are as in Part 2. If there is no vertical Legendrian curve then there exist $3 - \sum_{i=1}^4 s_i$ virtually overtwisted tight contact structures.*

Proof of Theorem Let $\sum_{i=1}^4 s_i \geq 3$. We will first say that there exist a vertical Legendrian curve with twisting zero: Let A_1 and A_2 be two vertical annuli between T_1 and T_2, T_2 and T_3, respectively. Assume that the dividing curves on A_1 and A_2 are parallel connecting one boundary to the other one. (Otherwise, there exist boundary parallel dividing curves which may produce bypasses and after attaching all possible bypasses, we can find an ∞ slope torus and on this torus we can draw a Legendrian vertical curve.) We can cut along $T_1 \cup A_1 \cup T_2 \cup A_2 \cup T_3$. After rounding the eight edges we obtain a $T^2 \times I$, where $T^2 \times \{0\} = T_4$, which has slope s_4, and $T^2 \times \{1\}$ having slope $s_4' = -s_1 - s_2 - s_3 + 2$. By the assumption $s_1 + s_2 + s_3 + s_4 \geq 3$ and this gives $s_4 \geq 3 - s_1 - s_2 - s_3$. Hence, $s_4 \geq 2 - s_1 - s_2 - s_3 = s_4'$ and thus by a Theorem of Honda [38] there exist a T^2 with ∞ slope dividing curve . Once we found a torus with ∞ slope dividing curve connecting a Legendrian divide of this torus with an infinite slope Legendrian ruling on each T_i, $i = 1, 2, 3, 4$ we obtain four vertical annuli. Along each Legendrian ruling on these annuli there are boundary parallel curves which produce bypasses by the imbalance principle. We attach these bypasses and continue this process until we get infinite slope tori. Hence we obtain basic slices $T_i \times I$, for $i = 1, 2, 3, 4$, where $s(T_i \times \{0\}) = s_i$ and $s(T_i \times \{1\}) = \infty$. Now we can write $\Sigma \times S^1 = \Sigma' \times S^1 \cup (T_1 \times I) \cup (T_2 \times I) \cup (T_3 \times I) \cup (T_4 \times I)$ as claimed.

By Proposition 4.4 of Honda in [39], the tight contact structures on $\Sigma' \times S^1$ can be characterized by the dividing curves on Σ'. We have two cases depending on the existence of a boundary parallel dividing curve on Σ'. If there is no boundary parallel dividing curve, which means that there is no bypasses on Σ', then there are three possible dividing curve configurations on Σ as in Figure 3.3. Let each basic slice has a different sign such as; $+, +, -, -$ or $-, +, -, -$ etc., then we can uniquely extend this contact structure to $\Sigma'' \times S^1 = \Sigma \times S^1 \cup (T_1'' \times I) \cup (T_2'' \times I) \cup (T_3'' \times I) \cup (T_4'' \times I)$, where $T_i'' \times [-1, 1] = T_i \times [0, 1] \cup T_i' \times [-1, 0]$. So that on each $T_i'' \times I$ the twisting of tight contact structure is π with each $s(T_i'' \times \{j\}) = \infty$, for $i = \{1, 2, 3, 4\}$ and $j = \{-1, 1\}$. Since the contact structure is tight the sign of the contact structure on $T_i \times [0, 1]$ and the sign on the contact structure on $T_i' \times [-1, 0]$ must be the same by Theorem 1.3 in [39]. Now the classification of tight contact structures on the thickened torus implies that the tight contact structure on $T_i'' \times [-1, 1]$ is universally tight and S^1 invariant. An S^1 invariant contact structure on $\Sigma'' \times S^1$ is universally tight since there is no homotopically trivial dividing curve on Σ'' by Theorem 2.2.13.

If the sign of the basic slices are not mixed, by a suitable diffeomorphism we can arrange the slopes of the dividing curves to be $s_1 = s_2 = s_3 = 1$. First we will show that the contact structures with $s_1 + s_2 + s_3 + s_4 > 3$ are tight. We can construct such tight contact structures as follows; let us start with $S^1 \times D^2$ with boundary slope $s_4 > 0$, then remove standard neighborhoods of three Legendrian curves with twisting -1. To obtain a twisting number -1 curve take stabilizations of Legendrian curves with twisting zero in a way that the contact structure is virtually overtwisted.

If $s_1 + s_2 + s_3 + s_4 = 4$ then the contact structures with signs $+, +, +, +$ and $-, -, -, -$ are isotopic: Let us connect T_1 and T_2 by a convex vertical annulus A_1 and T_2 to T_3 by a convex vertical annulus A_2. Since we assume the tight contact structures on $T_i \times I's$ are all minimally twisting, on A_1 and A_2 there are no boundary parallel dividing curves. If we cut along $T_1 \cup A_1 \cup T_2 \cup A_2 \cup T_3$ and round the edges we obtain a torus with boundary slope -1 from the T_4 side. Hence, we obtain a thickened torus with two basic slices having slopes, $-1, \infty$ and $\infty, 1$. The tight contact structures in both thickened tori have different signs as $+, -$ or $-, +$ respectively. However, there is only one positive basic slice in both cases, so the contact structures with signs $+, +, +, +$ and $-, -, -, -$ are isotopic.

If $s_1 + s_2 + s_3 + s_4 > 4$ the above proof is not valid anymore. This is because when $s_4 > 1$ there will be more than one positive basic slices in the thickened torus, which we obtain after cutting and rounding edges process. Hence there are two non-isotopic tight contact structures, and they differ by the relative Euler class on Σ.

If $s_1 + s_2 + s_3 + s_4 = 3$ then we will show that the contact structures are overtwisted. As above, after suitable diffeomorphisms we obtain $s_1 = s_2 = s_3 = 1$. Since $s_1 + s_2 + s_3 + s_4 = 3$, we have $s_4 = 0$ and we factor a $T^2 \times I$ layer with slopes $0, 1$. Then we obtain a $\Sigma \times S^1$ having all the boundary slopes as 1. By a similar process, we can construct a tight model using $D^2 \times S^1$. In this tight model, let A_1 and A_2 be two vertical annuli connecting T_1 to T_2 and T_2 to T_3, respectively, on which all the dividing curves are horizontal. Cutting along $T_1 \cup A_1 \cup T_2 \cup A_2 \cup T_3$ and rounding the edges result a T^2 with slope -1. Hence we obtain a thickened torus with two basic slices having slopes $0, \infty$ and $\infty, -1$, respectively. Since the basic slices have different signs, by Theorem 1.3 in [39] the contact structures obtained in this way is not tight. Indeed, also in case $s_4 \leq 0$ we come up with overtwisted contact structure, and hence the same proof covers the case $s_1 + s_2 + s_3 + s_4 \leq 3$.

Now assume that there are boundary parallel arcs on Σ'. If $s_1 + s_2 + s_3 + s_4 \leq 3$ and there is a vertical Legendrian curve then the unmixed case of the signs of basic slices yields an overtwisted disk. In order to get a tight contact structure, signs should be mixed. Hence, we have an S^1 invariant universally tight contact structure and there exists a unique extension to a universally tight contact structure on $\Sigma'' \times S^1$ with ∞ slope dividing curves on each T_i.

If $s_1 + s_2 + s_3 + s_4 < 3$ and there exist a Legendrian vertical curve, then similar to the case $s_1+s_2+s_3+s_4 \geq 3$ there exist universally tight contact structures however, there is no virtually overtwisted contact structures. (We already proved this in the case $s_1 + s_2 + s_3 + s_4 \leq 3$.)

When there is no vertical Legendrian curve then we can connect T_1 to T_2 by A_1 and T_3 to T_4 by A_2, and on A_1 and A_2 there are dividing curves connecting one boundary to the other boundary of the annuli. We can cut along $T_1 \cup A_1 \cup T_2 \cup A_2 \cup T_3$ and round the eight edges to obtain a thickened torus $T^2 \times I$ where $s(T^2 \times \{0\}) = s(T_4)$ and $s(T^2 \times \{1\}) = s(\Sigma \times S^1 \backslash T_1 \cup A_1 \cup T_2 \cup A_2 \cup T_3) = -s_1 - s_2 - s_3 + 2$. Finally by the classification of tight contact structures on thickened torus in [38] there exist $(-s_1 - s_2 - s_3 + 2) - s_4 + 1 = 3 - s_1 - s_2 - s_3 - s_4$ tight contact structures.

2.3 Legendrian Surgery

Given a Legendrian knot L in any contact 3−manifold (M, ξ), a Legendrian surgery on L yields the contact manifold (M', ξ'), where M' is obtained from M by $t(L) - 1$ Dehn surgery on L and ξ' is obtained from ξ as follows: Let N be a standard convex neighborhood of L. Choose a framing on N so that $t(L) = 0$. This choice of framing allows us to make an oriented identification $-\partial(M \backslash N)$ with $\mathbb{R}^2/\mathbb{Z}^2$, where $(1, 0)^T$ is the meridian of N and $(0, 1)^T$ is the longitude of N corresponding to the framing. Now take an identical copy N' of N (with the same framing), and make an oriented identification $\partial N'$ with $(\mathbb{R}^2/\mathbb{Z}^2)$, where $(1, 0)^T$ is the meridian and $(0, 1)^T$ is the longitude. Now $M' = (M \backslash N) \cup_\phi N'$, where $\phi : \partial N' \to -\partial(M \backslash N)$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$.

Definition 2.3.1. A contact 3−manifold (M, ξ) is called *holomorphically fillable* if there is a compact complex surface (X, J) such that the contact structure on ∂X given by the complex tangencies is contactomorphic to (M, ξ).

Definition 2.3.2. A Stein surface is a complex surface X with a real-valued Morse function f on X such that, away from the critical points of f, the field of complex tangencies to the preimage $X_c = f^{-1}(c)$ is a contact structure that induces an orientation on X_c agreeing with the usual orientation as the boundary of $f^{-1}(-\infty, c]$. That is, $f^{-1}(-\infty, c]$ is a Stein filling of $X_c = M$. M is called *Stein fillable* 3−manifold.

The complex structure on a holomorphic filling can be deformed to a blow up of a Stein filling. Hence being holomorphically fillable and Stein fillable are the equivalent.

Definition 2.3.3. A contact $3-$manifold (M, ξ) is called *strongly (symplectically) fillable* if there exist a compact symplectic 4−manifold (X, ω) such that $\partial X = M$ and ω is exact near the boundary with $d\alpha = \omega$.

Definition 2.3.4. A contact $3-$manifold (M, ξ) is called *weakly (symplectically) fillable* or simply *fillable* if there is a compact symplectic $4-$manifold (X, ω) such that $\partial X = M$ and $\omega|_\xi \neq 0$.

We have the following implications: If a contact structure is Stein fillable then it is also strong symplectically fillable. If a contact structure is symplectically fillable then it is also weakly fillable and finally, if a contact structures is weakly fillable then it is tight. The converses of these statements are all false ([25], [11], [14]).

The following theorems relates Legendrian surgery to fillability and are often used to show tightness of many of the contact structures, [38], [59], [30].

Theorem 2.3.5 (Eliashberg [10]). *Suppose $(M'\xi')$ is obtained by Legendrian surgery from (M, ξ). If (M, ξ) is weakly fillable then (M', ξ') is also weakly fillable.*

Theorem 2.3.6 (Eliashberg, Gromov [12]). *If (M, ξ) is fillable then it is tight.*

2.4 Seifert Fibred $3-$Manifolds

Definition 2.4.1. Let M be a fibred $3-$manifold over S^2, with n singular fibers $F_1, F_2, \cdots, F_n$. We can describe M explicitly as follows: Let V_i, $i = 1, 2, \cdots, n$, be a tubular neighborhood of the singular fiber F_i. We identify V_i with $D^2 \times S^1$ and ∂V_i with $\mathbb{R}^2/\mathbb{Z}^2$ by choosing $(1, 0)^T$ as the meridional direction, and $(0, 1)^T$ as the longitudinal direction given by $\{pt\} \times S^1$. We also identify $M \backslash (\bigcup_i V_i)$ with $\Sigma_0 \times S^1$, where Σ_0 is a sphere with n punctures. Finally we choose a diffeomorphism of $-\partial(M \backslash V_i)$ to $\mathbb{R}^2/\mathbb{Z}^2$, which maps $(0, 1)^T$ to the direction of an S^1-fiber, and $(1, 0)^T$ to the direction given by $\partial(M \backslash V_i) \cap (\Sigma_0 \times pt)$. With these identifications we may reconstruct M from $(\Sigma_0 \times S^1) \cup^n_{i=1 A_i} (\bigcup_{i=1} V_i)$, where

$$A_i : \partial V_i \to -\partial(M \backslash V_i),\ A_i = \begin{pmatrix} p_i & p'_i \\ -q_i & q'_i \end{pmatrix} \in SL(2, \mathbb{Z}).$$

This manifold is called a Seifert fibred 3−manifold over S^2 with n singular fiber and we denote it by $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \cdots, \frac{q_n}{p_n})$.

The integer $e_0(M) = \sum_{i=1}^{n} \left\lfloor \frac{q_i}{p_i} \right\rfloor$ is called the e_0 *invariant* of the Seifert fibred manifold.

If $\sum_{i=1}^{n}(1 - \frac{1}{p_i})$, where $\frac{1}{2} \leq (1 - \frac{1}{p_i}) < 1$, is equal to the Euler characteristic of the base , $\chi(S^2)$, then the Seifert fibred 3−manifold is a torus bundle over S^1. These are the Seifert fibred manifolds $M(\pm\frac{1}{4}, \pm\frac{1}{4}, \mp\frac{1}{2})$, $M(\pm\frac{1}{3}, \pm\frac{1}{3}, \mp\frac{2}{3})$, $M(\pm\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2}, \mp\frac{1}{2})$. If $\sum_{i=1}^{n} \frac{q_i}{p_i} = 0$, then the Seifert manifolds contain horizantal incompressible torus. If the genus of base is greater than zero or the number of singular fibers greater than three then there exist vertical incompressible torus. For detailed information one may look at [37]. The classification of the tight contact structures on these T^2 bundles over S^1 is done in [39].

The next section will be a brief introduction to Heegaard–Floer Theory and Ozsváth–Szabó contact invariant ([51], [52].)

2.5 Heegaard Floer Theory

Heegaard Floer homology associates a finitely generated abelian group $\widehat{HF}(M, t)$ to any closed connected oriented Spin^c 3−manifold (M, t). A Spin^c cobordism between two Spin^c manifolds (M_1, t_1) and (M_2, t_2) such that $s_{|M_i} = t_i$ for $i = 1, 2$ yields a homomorphism $F_{W,s}$: $\widehat{HF}(M_1, t_1) \to \widehat{HF}(M_2, t_2)$.

A contact structure ξ on a 3−manifold M determines a Spin^c structure t_ξ on M such that $c_1(t_\xi) = c_1(\xi)$. To any contact manifold (M, ξ) we can associate an element $c(\xi) \in \widehat{HF}(-M, t_\xi)/\pm$, which is an isotopy invariant of ξ (See[51] for details.).

Theorem 2.5.1 (Ozsváth-Szabó [51]). *If* (M, ξ) *is overtwisted then* $c(\xi) = 0$, *if* (M, ξ) *is Stein fillable, then* $c(\xi) \neq 0$.

The contact invariant can also be used to prove the non–fillability of a tight contact structure. Indeed, if a tight contact structure has zero contact invariant then by above theorem this tight contact structure is not Stein fillable, (not even weakly fillable, if one can show that twisted coefficient contact invariant is zero, [52]).

Definition 2.5.2. Let ξ be a contact structure on a 3manifold M. The *Giroux torsion* of (M, ξ)

is the supremum of the integers $n \geq 1$, for which there exists a contact embedding

$$(T^2 \times [0,1],\ \ker(\cos(2\pi nz)dx + \sin(2\pi nz)dy)) \hookrightarrow (M, \xi).$$

We say that Giroux torsion of (M, ξ) is zero if no such embedding exists.

The following theorem relates Giroux torsion to the contact invariant.

Theorem 2.5.3 (Ghiggini, Honda, Van Horn-Morris [22]). *If a closed contact $3-$manifold (M, ξ) has positive $2\pi-$ torsion, then its contact invariant $c(M, \xi)$ in $\widehat{HF}(-M)$ vanishes.*

The above theorem uses $\mathbb{Z}$ coefficient Heegaard Floer homology theory. Using twisted coefficient Heegaard Floer homology theory, Honda and Ghiggini proved the following theorem in [23] (See also [20].).

Theorem 2.5.4 (Ghiggini, Honda, [23]). *If T is a separating pre-Lagrangian torus in a contact $3-$manifold (M, ξ), then for $n > 0$, $c(M, \xi) = 0$ in twisted coefficient Heegaard Floer homology and hence (M, ξ) is not weakly symplectically fillable.*

The theorem below describes how the contact invariant behaves under Legendrian surgery.

Theorem 2.5.5 ([51]). *Suppose (M', ξ') is obtained from (M, ξ) by Legendrian surgery along a Legendrian link. Then there is a cobordism $\widehat{F}_W : \widehat{HF}(-M', \xi') \to \widehat{HF}(-M, \xi)$, which satisfies $\widehat{F}(c(\xi')) = c(\xi)$.*

Definition 2.5.6. A rational homology sphere is an $L-$ space if $\widehat{HF}(M, t) \cong Z$ for all $t \in \mathrm{Spin}^c(M)$.

Characterization of Seifert fibred $L-$spaces, with base S^2 in terms of taut foliations, transverse foliations and transverse contact structures is given in the following theorem.

Theorem 2.5.7 (Lisca, Stipsicz [47]). *Let M be an oriented rational homology $3-$sphere which is Seifert fibred over S^2. Then the following statements are equivalent,*

1. *M is an L-space,*
2. *Either M or $-M$ carries no positive transverse contact structures,*
3. *M carries no transverse foliations,*

4. *M carries no taut foliations.*

The following is an important result of Lisca and Matić in [43] about transverse contact structures on Seifert fibred 3−manifolds over S^2.

Theorem 2.5.8 (Lisca,Matić [43]). *An oriented Seifert fibred rational homology* 3−*sphere* $M(r_1, r_2, \ldots, r_k)$, *with* $r_1 \geq r_2 \geq \cdots \geq r_k$, *admits no positive transverse contact structure if and only if*

- $e_0(M) \geq 0$,
- $e_0(M) = -1$ *and there are no relatively prime integers* $m > a$ *such that* $mr_1 < a < m(1 - r_2)$ *and* $mr_i < 1$ *for all* $i = 3, \cdots, k$.

In the next subsection we will give a brief information about open book decompositions and we will mention some important results.

2.6 Open Book Decompositions

An open book decomposition of a 3−manifold M is a pair (B, h), where B is an oriented link in M, called the *binding* of the open book, and $h : M \backslash B \to S^1$ is a fibration of the complement of B, such that, for each $\theta \in S^1$, $h^{-1}(\theta)$ is the interior of a compact surface $\Sigma \subset M$, whose boundary is B. The surface Σ is called a *page* of the open book and h is called the *monodromy* of the open book.

Alternatively, if we have a compact oriented surface Σ and a homeomorphism $h : \Sigma \to \Sigma$, which is identity near the boundary, we can construct an open book as follows: First form the mapping torus Σ_h. Since h is the identity on $\partial\Sigma$, the boundary of Σ_h is the trivial circle bundle over a union of circles, that is, a union of tori. To complete the construction, solid tori are glued to fill in the boundary tori so that each circle $S^1 \times \{p\} \subset S^1 \times \partial D^2$ is identified with the boundary of a page. In this case, the binding is the collection of n cores $S^1 \times \{q\}$ of the n solid tori glued into the mapping torus, for arbitrarily chosen $q \in D^2$.

Alexander proved that every closed oriented 3−manifold admits an open book decomposition [1]. A contact structure is compatible with an open book decomposition if away from the binding, the contact distribution is isotopic to the tangent spaces of the pages through confoliations. Every open book decomposition supports a contact structure (Thurston–Winkelnkemper, [55]). By a theorem of Giroux in [34] one can relate the contact structures on $3-$manifolds up to isotopy with their open book decompositions up to positive stabilizations. Here by a positive stabilization we mean modifying the page by adding a $2-$dimensional $1-$handle and composing the monodromy by the positive Dehn twist along a curve that runs over that handle exactly once.

Seifert fibred $3-$manifolds can be seen as the boundaries of plumbed $4-$manifolds. There are several constructions for open book decompositions of some certain Seifert fibred $3-$manifolds using the plumbing diagrams. By adding a $2-$dimensional $1-$handle it is always possible to increase the page genus of an open book decomposition, however finding the minimal page genus of an open book decomposition that supports a given contact structure is still an open problem. For the plumbings with no bad vertex, which means framing coefficient of each vertex is less than or equal to minus the number of edges going out form that vertex, in [53] Schönenberger constructed open book decompositions with page genus zero (planar pages). In [19] Etnyre and Özbağcı constructed open book decompositions for plumbings with some bad vertices also. However, the open books they constructed have positive page genus.

In [19] it is also proved that the small Seifert fibred $3-$manifolds with $e_0(M) \geq 0$ and certain small Sefiert manifolds with $e_0(M) = -1$ admit planar open book decompositions. Moreover, in [47] it is shown that a small Seifert fibred $3-$manifold, which has zero twisting Legendrian vertical curve, has compatible planar open book decompositions.

Definition 2.6.1. Consider an embedding of a normal complex singularity (X, x) in $(\mathbb{C}^{2N}, 0)$. The $3-$manifold $Y = S_\epsilon^{2N-1} \cap X$, called the link of the singularity, has the canonical contact structure ξ_{can}, induced by the complex structure. If (M, ξ) is isomorphic to such (Y, ξ_{can}) then it is called a *Milnor fillable* contact $3-$manifold.

Grauert [36] showed that a small Seifert fibred $3-$manifold is Milnor fillable if and only if the plumbing is negative definite. It is also shown in [5] that every closed oriented $3-$manifold admits at most one Milnor fillable contact structure.

One can ask what is the minimal page genus (Milnor genus) of an open book (Milnor open book), which is compatible with the unique Milnor fillable contact structure. It is shown in

[4] that if a contact structure is Milnor fillable then it is Stein fillable. Therefore the Milnor genus is greater than or equal to the support genus of an open book.

The results we mention so far are about small Seifert fibred 3−manifolds with $e_0(M) \neq -2$. For $e_0(M) = -2$, there are also some results. In [3], Bhupal and Özbağcı constructed some families of small Seifert fibred 3−manifolds, where the Milnor genus equals to the support genus, but these examples are all non-planar. All of these examples for the case $e_0(M) = -2$ are L−spaces.

There is also another recent interesting result of Lekili and Özbağcı [42], which says that Milnor fillable contact structures are universally tight. They also say that these contact structures do not come from taut foliations. Combining this result with a result of Ghiggini, Lisca and Stipsicz, it can be deduced that Milnor fillable contact structures are all L−spaces.

As we mentioned before there are six Seifert fibred 3−manifolds, which are torus bundles over S^1. Open book decompositions for these manifolds are given in [56], [16]. All these open books have page genus one.

Plamenevskaya and Van Horn-Morris in [54] constructed open book decompositions for non–fillable contact structures on Seifert fibred 3−manifold $M(-1; r_1, r_2, r_3)$, with $r_1, r_2 \geq \frac{1}{2}$, $r_3 = \frac{1}{p}$. They also found a more general non–fillable family of Seifert fibred 3−manifold with $e_0(M) = -1$.

In Section 4, using the above results we will describe compatible open book decompositions of some tight contact structures on Seifert fibred 3−manifolds with 4 singular fibers.

In the last part of this chapter, we will review some results about the classification of tight contact structures on small Seifert fibred 3−manifolds.

2.7 Results on Small Seifert Fibred 3− Manifolds

Let $-\frac{p_i}{q_i} = r_0^i - \cfrac{1}{r_1^i - \frac{1}{r_2^i \cdots - \frac{1}{r_{m_i}^i}}}$ be the continued fraction expansion of $-\frac{p_i}{q_i}$, with all $r_j^i \leq -2$, where $p_i > q_i > 0$ and $(p_i, q_i) = 1$.

In the following theorems when $e_0(M) > 0$, we can assume that $\frac{q_1}{p_1} > 1$, $0 < \frac{q_i}{p_i} < 1$, so that in the continued fraction expansion we may have $r_0^i \leq -1$. In the following theorem, Wu classified the tight contact structures on small Seifert manifolds with $e_0(M) \neq -2, -1, 0$.

Theorem 2.7.1 (Wu [58], [59]). *1. Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. If $e_0(M) > 0$, then there are exactly $|\prod_{i=1}^{3} r_0^{(i)} \prod_{j=1}^{m_i}(r_j^{(i)} + 1)|$ tight contact structures and all of these contact structures can be obtained by Legendrian surgery on a link in S^3, therefore they are all Stein fillable.*

2. Let $M = M(e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. If $e_0(M) \leq -3$, then there are exactly $|(e_0(M) + 1) \prod_{i=1}^{3} \prod_{j=1}^{m_i}(r_j{}^i + 1)|$ Stein fillable tight contact structures.

Theorem 2.7.2 (Ghiggini, Lisca, Stipsicz [28]). *On a small Seifert fibred $3-$manifold $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ with $e_0(M) \geq 0$, there are exactly $|(\prod_{i=1}^{3}(r_0{}^i + 1) - \prod_{i=1}^{3} r_0{}^i) \prod_{i=1}^{3} \prod_{k=1}^{m_i}(r_k{}^i + 1)|$ Stein fillable tight contact structures.*

The above theorems cover the cases when $e_0(M) \neq -1, -2$.

The classification of tight contact structures on small Seifert fibred $3-$manifolds with $e_0(M) = -1$ is harder, since there are non-fillable tight contact structures. Indeed on some Seifert fibred $3-$manifolds with $e_0(M) = -1$, there is no positive tight contact structures. The following theorems are proved by Honda and Etnyre using convex surface theory;

Theorem 2.7.3 (Etnyre, Honda [14]). *There exist no positive tight contact structure on the Poincaré homology sphere $M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ with reverse orientation.*

Theorem 2.7.4 (Etnyre, Honda [18]). *On $M(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ there exist one tight contact structure and $M(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ there are two non–isotopic tight contact structures which are not weakly symplectically fillable.*

These were the first examples of tight but not fillable contact structures. The main problem for constructing non–fillable tight examples is showing the tightness of the contact structure, since Theorem 2.3.6 is not applicable anymore. The tightness of the non–fillable contact structures in the previous theorem is proved by the convex surface theory. Later, Stipsicz and Lisca produced infinitely many non–fillable tight examples, using Legendrian surgery diagrams to compute the Ozváth-Szabó contact invariants, whose non–triviality implies the tightness of the contact structure.

The above tight but not fillable contact structures are all virtually overtwisted and all have non-zero Osváth-Szabó contact invariant. We may ask that, do all tight but not fillable contact structures have non-zero contact invariant, or are there any tight but not-fillable contact structures which are not virtually overtwisted?

In 2006 Ghiggini answered both questions. He showed that there exist infinitely many tight contact structures with trivial contact invariant which are all universally tight. His examples are Seifert fibred $3-$manifolds over S^2 with four singular fibers and $e_0(M) = 0$. Later in 2007 Honda, Ghiggini and Van Horn-Morris showed that if a tight contact structure has positive Giroux torsion then its contact invariant vanishes. This result provided many other examples of tight contact structures with trivial contact invariant [49].

In the case $e_0(M) = -1$ there is no complete classification, however there are some partial results as in the following theorems;

Theorem 2.7.5 (Ghiggini, Schönenberger [30]). *On the small Seifert fibred $3-$manifold $M(\frac{1}{2}, -\frac{1}{3}, -\frac{2}{11})$ there exist two Stein fillable contact structures and on the small Seifert fibred $3-$manifold $M(-\frac{1}{2}, \frac{2}{3}, \frac{2}{11})$ there exist a unique Stein fillable tight contact structure.*

In the proof of the above theorem, convex surface theory is used to find an upper bound and Legendrian surgery is used to find a lower bound. The manifolds in the above theorem are Brieskorn homology spheres $\pm\Sigma(2, 3, 11)$. Recently Ghiggini and Horn-Morris gave a complete classification of the tight contact structures on $-\Sigma(2, 3, 6n - 1)$, for $n \geq 2$.

Later using Heegaard Floer homology and Legendrian surgery Ghiggini, Lisca and Stipsicz in [29] gave a classification for some small Seifert fibred $3-$manifold with $\frac{q_1}{p_1}, \frac{q_2}{p_2} \geq \frac{1}{2}$ and $e_0(M) = -1$. They also showed that on $M(\frac{1}{2}, \frac{1}{2}, \frac{1}{p})$ there are strong but not Stein fillable contact structures on M, for each $p \geq 2$, and that there are no strongly fillable contact structures for some $p \geq 2$.

Theorem 2.7.6 (Ghiggini [26]). *Let $M(e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ with $e_0(M) = -2$ be an $L-$space. Then there are exactly $| \prod_{i=1}^{m_1}(r_i^1) \prod_{i=1}^{m_2}(r_i^2) \prod_{i=1}^{m_3}(r_i^3)|$ Stein fillable tight contact structures.*

In next chapter we will try to extend of some of the results to Seifert $3-$manifolds with 4 singular fibers.

CHAPTER 3

STATEMENTS AND PROOFS OF THE MAIN THEOREMS

The following theorems are what we were able to prove regarding the classification of tight contact structures on Seifert fibred $3-$manifolds with four singular fibers and base S^2.

3.1 Main Results

In the following theorems let M be a Seifert fibred $3-$manifold which admits a Seifert fibration over S^2 with 4 singular fibers.

Theorem 3.1.1. *If $e_0(M) \leq -2$ then no tight contact structures on M with zero Giroux torsion contains a Legendrian curve with zero twisting.*

Let $M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}, -\frac{q_4}{p_4})$ be a Seifert manifold, where $-\dfrac{q_i}{p_i} = [a^i_0, a^i_1, \cdots, a_{m_i}^i]$ and $a_i \leq -2,\ i = 1, 2, 3, 4$ is the continued fraction expansion of $-\dfrac{q_i}{p_i}$ with $(p_i, q_i) = 1,\ p_i > 1,$ $q_i > 0$.

Theorem 3.1.2. *If $e_0(M) \leq -4$ then there are exactly $|(e_0(M) + 1) \prod_{i=1}^{4} \prod_{j=1}^{m_i} (a_j^{(i)} + 1)|$ Stein fillable contact structures (up to isotopy) on M. Moreover M has infinitely many non–fillable tight contact structures with positive Giroux torsion and a Legendrian vertical curve with zero twisting.*

Let now $M = M(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4})$ be a Seifert manifold, where $-\dfrac{p_i}{q_i} = [a^i_0, a^i_1, \cdots, a_{m_i}^i]$ and $a_1 \leq -1$, $a_i \leq -2$ is the continued fraction expansion of $-\dfrac{p_i}{q_i}$ with $(p_i, q_i) = 1,\ p_i \geq 1, q_i > 1.$

Theorem 3.1.3. *If $e_0(M) \geq 0$ then, every tight contact structure on M contains a Legendrian* *vertical curve with zero twisting. There are at most*

$$2|(\prod_{i=1}^{4}({a_0}^i+1)-\prod_{i=1}^{4}{a_0}^i)\prod_{i=1}^{4}\prod_{k=1}^{m_i}({a_k}^i+1)|$$

and at least

$$|(\prod_{i=1}^{4}({a_0}^i+1)-\prod_{i=1}^{4}{a_0}^i)\prod_{i=1}^{4}\prod_{k=1}^{m_i}({a_k}^i+1)|$$

many Stein fillable contact structures (up to isotopy) on M. Moreover, there are infinitely many non–fillable tight contact structures on M with positive Giroux torsion.

Proof of Theorem 3.1.1 On contrary to the theorem, assume that there exists a Legendrian vertical curve L with twisting zero. By adding cusps we can assume the twisting of any singular fiber is some number $t_i \leq -1$, for each $i = 1, 2, 3, 4$. Let V_i denote a standard neighborhood of each singular fiber. Then slope of this dividing curves on ∂V_i is $s(\partial V_i) = \dfrac{1}{t_i}$. Since $e_0(M) \leq -2$, M can be written in the form

$$M(\frac{q_1}{p_1}, e_0+2+\frac{q_2}{p_2}, \frac{q_3}{p_3}, \frac{q_4}{p_4}),$$

where $p_i > q_i > 0$, for $i = 3, 4$, $q_1, q_2 < 0$ and $p_i > 1$. Let the orientation preserving diffeomorphism given by $\varphi_i : \partial V_i \longrightarrow T_i = \partial(M \backslash V_i)$, $\varphi_i = \begin{pmatrix} p_i & u_i \\ -(e_0+2)p_i - q_i & v_i \end{pmatrix}$, for $i = 2$ and $\varphi_i : \partial V_i \longrightarrow T_i$, by $\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}$, for $i \neq 2$. Using φ_i we can compute the slope of dividing curves on T_i as $s_2 = s(T_2) = \dfrac{-((e_0+2)p_2+q_2)t_2+v_2}{p_2t_2+u_2}$ and $s_i = s(T_i) = \dfrac{-q_it_i+v_i}{p_it_i+u_i}$, for $i = 1, 3, 4$. One can observe that for $i = 2$, $-(e_0+2) < s_i < -(e_0+2) - \frac{q_i}{p_i}$, for $i = 1$, $0 < s_i < -\frac{q_i}{p_i}$, and for $i = 3, 4$, $-1 < s_i < -\frac{q_i}{p_i}$.

Connect L with an $\infty-$slope Legendrian ruling curve on each T_i. This produces four vertical annuli. Since L has twisting zero, there will be no dividing curves starting or ending on L. By the imbalance principle there will be bypasses along each ruling curve on each annuli. After attaching all possible bypasses, we obtain tori T_i' with slope ∞ isotopic to V_i. Using Farey tesellation, when we start from $\infty-$slope and hit s_i going counterclockwise on the boundary of hyperbolic disk, we know that all the intermediate slopes can be realized, [38]. Therefore around each singular fiber we obtain a thickened tori containing basic slices with slopes as in Figure 3.1. All thickened tori contain the basic slice with slopes $-1, \infty$. According to the sign of this basic slice there are several cases. In each case we either end up with an inappropriate

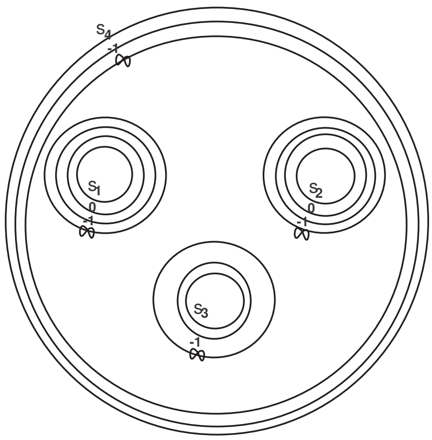

Figure 3.1: Disk with 3 holes and boundary slopes of thickened tori

therefore overtwisted contact structure or a contact structure with positive Giroux torsion, both of which contradict with the assumptions of the theorem. In the following, we will give one case which ends up with positive Giroux torsion and another case which ends up with an inappropriate contact structure. All the other cases are similar.

Let $T_1 \times [1/2, 1]$ and $T_2 \times [1/2, 1]$, where $s(T_i \times \{1/2\}) = -1$ and $s(T_i \times \{1\}) = \infty$, for $i = 1, 2$ have the same sign, say $+$. Then for the remaining two basic slices $T_3 \times [1/2, 1]$ and $T_4 \times [1/2, 1]$ where, $s(T_i \times \{1/2\}) = -1$ and $s(T_i \times \{1\}) = \infty$, for $i = 3, 4$, and there are four cases according to their signs: $\{-, -\}, \{+, -\}, \{-, +\}, \{+, +\}$.

Connecting an infinite slope Legendrian ruling curve on $T_1 \times \{1/2\}$ with an infinite slope Legendrian curve on $T_2 \times \{1/2\}$ yields a vertical annulus A with no boundary parallel dividing curve. Therefore, cutting along $T_1 \times \{1/2\} \cup A \cup T_2 \times \{1/2\}$ and rounding the edges gives a torus with slope one. Now we have a pair of pants times S^1 with boundary slopes $1, -1, -1$ and there are ∞ slope tori in the neighborhood of T_3 and T_4, which implies that we can also find one in the neighborhood of the torus with slope 1.

Consider the case $\{+, +\}$. We can cut along $T_3 \times \{1/2\} \cup A \cup T_4 \times \{1/2\}$, where $s(T_i \times \{1/2\}) = -1$, for $i = 1, 2$ and $s(T_i \times \{1\}) = \infty$ for $i = 3, 4$. After rounding the edges we obtain a torus with slope one. Therefore we obtain two tori with slope one, which are not boundaries of singular fibers. Since we assume the existence of a Legendrian vertical curve with twisting zero, we can find a torus with $\infty-$slope between these slope one tori. This means there exist an embedded $T^2 \times I$, which have at least π twisting. So the Giroux torsion is positive The case $\{-, -\}$ will be the same.

Now consider the case $\{+, -\}$. We have three of the four basic slices with slopes -1 and ∞ of the same sign. Connecting an infinite slope Legendrian ruling curve on $T_1 \times \{1/2\}$ with an infinite slope Legendrian ruling curve on $T_2 \times \{1/2\}$ yields a vertical annulus A_1 and connecting an infinite slope Legendrian ruling curve on $T_2 \times \{1/2\}$ with an infinite slope Legendrian curve on $T_3 \times \{1/2\}$ yields a vertical annulus A_2. On A_1 and A_2 there are no boundary parallel dividing curves. We can extend the annulus A_1 to an annulus A_1', which is in between the infinite slope Legendrian ruling curves on $T_1 \times \{1/4\}$ and $T_2 \times \{1/4\}$, where $s(T_i \times \{1/4\}) = 0$ for $i = 1, 2$. Similarly, we can extend the annulus to A_3', which is in between infinite the slope Legendrian ruling curves on $T_2 \times \{1/4\}$ and $T_3 \times \{1/2\}$. If we cut along $T_1 \times \{1/4\} \cup A_1' \cup T_2 \times \{1/4\} \cup A_3' \cup T_3 \times \{1/2\}$, and round the edges, we obtain a torus with slope -1, which is in the neighborhood of the remaining fourth singular fiber. Thus, we obtain a π twisting thickened torus in the neighborhood of fourth singular fiber. However using Proposition 4.16 of Honda ([38]) we can realize any rational slope in this thickened torus. Therefore, we can realize the meridian of the neighborhood of singular fiber as a dividing curve. This gives an embedded disk with twisting zero, which is nothing but an overtwisted disk. Hence, we are done. The case $\{-, +\}$ will be similar. $\square$

Proof of Theorem 3.1.2 Let $M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}, -\frac{q_4}{p_4})$, with $-\frac{q_i}{p_i} = [a_0^{(i)}, a_1^{(i)}, \ldots, a_{m_i}^{(i)}]$, where all $a_j^{(i)}$ are integers $a_0^{(i)} = -(\left\lfloor \frac{q_i}{p_i} \right\rfloor + 1) \leq -1$, $a_j^{(i)} \leq -2$ for $j \geq 1$. We claim that up to isotopy there are $|(e_0(M) + 1) \prod_{i=1}^{4} \prod_{j=1}^{m_i} (a_j^{(i)} + 1)|$ tight contact structures on M which do not contain a Legendrian vertical curve with twisting zero, i.e having negative maximal twisting.

Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \longrightarrow T_i$ by $\varphi_i = \begin{pmatrix} p_i & u_i \\ q_i & v_i \end{pmatrix}$. Then

$$M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}, -\frac{q_4}{p_4}) = (\Sigma \times S^1) \bigcup_{\varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4} (V_1 \cup V_2 \cup V_3 \cup V_4).$$

Let ξ be a tight contact structure on M. Isotope ξ to make each V_i a standard neighborhood of a Legendrian circle F_i, isotopic to the i^{th} singular fiber with twisting $t_i < -2$. Then ∂V_i is convex and has two dividing curves with slope $\dfrac{1}{t_i}$. Then, since $T_i = \partial V_i$ we can compute the slope of dividing curves on T_i, $s_i = s(T_i)$, using the map φ_i. We see that $s_i = \dfrac{t_i q_i + v_i}{t_i p_i + u_i} = \dfrac{q_i}{p_i} + \dfrac{1}{p_i(t_i p_i + u_i)}$. Since $t_i < -2$ we have $\left\lfloor \dfrac{q_i}{p_i} \right\rfloor < s_i < \dfrac{q_i}{p_i}$. Using Giroux Flexibility theorem we can assume that each T_i has Legendrian rulings L_i of slope ∞ when measured in the coordinates of

T_i. Let $A_1, A_2 \subset \Sigma \times S^1$ be convex vertical annuli such that $\partial A_1 = L_1 \cup L_2$ and $\partial A_2 = L_2 \cup L_3$. By the assumption ξ has negative maximal twisting so there must be dividing curves of A_1 and A_2 that connect the two boundary components of A_1 and A_2. If there are ∂–parallel dividing curves on A_1 and A_2, by imbalance principle there are bypasses. To get rid of all $\partial-$parallel dividing curves on A_1 and A_2 we attach all bypasses. After the isotpoy the slopes of dividing curves of T_1, T_2 and T_3 become $s_1' = \dfrac{q_1}{p}$, $s_2' = \dfrac{q_2}{p}$, $s_3' = \dfrac{q_3}{p}$, $p \geq 1, (p, q_i) = 1$, for $i = 1, 2, 3$. Since $s_i > \left\lfloor \frac{q_i}{p_i} \right\rfloor$ we have $s_i' \geq \left\lfloor \frac{q_i}{p_i} \right\rfloor \geq 0$, because otherwise if $s_i' < \left\lfloor \frac{q_i}{p_i} \right\rfloor$ there exist an $\infty-$slope torus, on which we can find a twisting zero vertical Legendrian curve. This contradicts to the assumption that ξ has negative maximal twisting. If we cut M along $T_1 \cup A_1 \cup T_2 \cup A_2 \cup T_3$ and round the edges, we obtain a torus with slope $s_4' = -\dfrac{q_1 + q_2 + q_3 + 2}{p}$. The slope s_4' corresponds to the slope $s_4'' = -\dfrac{pq_4 + (q_1 + q_2 + q_3 + 2)p_4}{qv_4 + (q_1 + q_2 + q_3 + 2)u_4}$ in the coordinates of ∂V_4. One can observe that $s_4'' < -\dfrac{q_4}{v_4}$. By a theorem in [38] we can find a torus with slope $-\dfrac{q_4}{v_4}$. In the coordinates of T_4 this slope becomes zero. This implies that the maximal twisting number is -1. When we have a twisting number -1 Legendrian curve γ then we can find an annuli between the curve γ curve and other four $\infty-$slope Legendrian ruling curves on boundary of V_i. On this annuli we apply the imbalance principle. Then adding bypass operations we can assume that all the boundary slopes of T_i become $\left\lfloor \frac{q_i}{p_i} \right\rfloor$. When measured in the coordinates of ∂V_i, the slopes of the dividing curves become $-\dfrac{q_i - \left\lfloor \frac{q_i}{p_i} \right\rfloor p_i}{v_i - \left\lfloor \frac{q_i}{p_i} \right\rfloor u_i} = -\dfrac{q_i + (a_0^{(i)} + 1)p_i}{v_i + (a_0^{(i)} + 1)u_i}$. As in [58], one can observe that this quotient has $[a_{m_i}^{(i)}, a_{m_i-1}^{(i)}, \cdots, a_1 + 1]$ as its continued fraction expansion. Therefore using the classification of the tight contact structures on solid torus, we conclude that on the neighborhood of each singular fiber there are $|\prod_{j=1}^{m}{}_i(a_j^{(i)} + 1)|$ tight contact structures up to isotopy.

By extending a theorem of Ko Honda in [39], to Seifert fibred $3-$manifold with four singular fibers as in the last part of Theorem 2.2.14, we can say there are exactly $3 + \left\lfloor \frac{q_1}{p_1} \right\rfloor + \left\lfloor \frac{q_2}{p_2} \right\rfloor + \left\lfloor \frac{q_3}{p_3} \right\rfloor + \left\lfloor \frac{q_4}{p_4} \right\rfloor = |e_0(M) + 1|$ tight contact structures on $\Sigma \times S^1$, where Σ is a sphere with 4 punctures. As it is written in Theorem 2.2.14, none of these tight contact structures contain a Legendrian vertical curve with twisting zero. So, there are at most $|(e_0(M) + 1) \prod_{j=1}^{m_i}(a_j^{(i)} + 1)|$ tight contact structures.

For the lower bound we use Legendrian surgery diagram. Indeed in Figure 3.2 we draw the smooth surgery diagram of M. In the Figure each $q_i' = q_i + \left\lfloor -\frac{q_i}{p_i} \right\rfloor p_i$. We also draw the possible Legendrian realizations of a smooth surgery an unknot with framing e_0, which is less than or equal to minus two. There are $e_0 + 1$ ways to obtain different Legedrian diagram, which gives different rotation numbers. All the contact structures obtained by Legendrian surgery will be Stein fillable by Eliashberg [10]. Moreover, by Lisca and Matić [44], they can be distinguished by their first Chern classes which is the rotation number of the knot. Since all the framing coefficients are less than or equal to -2 we can do the same for all the unknots in Figure 3.2. Hence there are at least

$$|(e_0(M) + 1)\prod_{j=1}^{m_i}(a_j^{(i)} + 1)|$$

Stein fillable contact structures on M with $e_0 \leq -2$.

We now obtained an upper bound using convex surface theory for the negative maximally twisting tight contact structures on M. This number matches with the lower bound coming from Legendrian surgery, which is the number of Stein fillable (therefore Giroux torsion zero) contact structures. By Theorem 3.1.1 all Giroux torsion zero contact structures are negative maximally twisting. Therefore the number we obtain gives the exact number of Stein fillable contact structure on M with $e_0(M) \leq -4$. This finalizes the proof of the first part of the theorem.

For the second statement note that there are infinitely many universally tight contact structures when there is an incompressible torus [6]. When the number of singular fiber of a Seifert fibred 3−manifold is greater than three there exists a vertical incompressible torus. Along this incompressible torus one can embed $T^2 \times I$ with the special contact structure on it. This does not change the manifold, but the contact structure changes. By a result of Honda, Van-Horn Morris, Ghiggini [23], [22] all of such contact structures will be non–fillable. □

Proof of Theorem 3.1.3 We will first show that on a Seifert fibred manifold M with 4 singular fibers and $e_0(M) \geq 0$, no tight contact structure has negative maximal twisting number. (See also Theorem1.3 in [58]). We assume the set up in the above proof so that, $-\partial\Sigma \times S^1 = T_1 \cup T_2 \cup T_3 \cup T_4$, $\varphi_i : \partial V_i \to \partial(M \backslash V_i)$, $\begin{pmatrix} p_i & q_i \\ -q_i & u_i \end{pmatrix}$, where $p_i u_i + v_i q_i = 1$, $-\partial\Sigma \times \{pt\} \to (0,1)^T$. First we will prove the following claim.

Claim: Let ξ be a tight contact structure on $\Sigma \times S^1$ with $s_i = s(T_i)$, such that all T_i's are convex. Then there exist collar neighborhoods $T_1 \times I$, $T_2 \times I$ and $T_3 \times I$ of T_1, T_2 and T_3, properly embedded vertical convex annuli A_1, A_2 in $(\Sigma \times S^1) \backslash (T_1 \times I \cup T_2 \times I \cup T_3 \times I)$, where A_1 is connecting $T_1 \times \{1\}$ to $T_2 \times \{1\}$ and A_2 is connecting $T_2 \times \{1\}$ to $T_3 \times \{1\}$ with Legendrian

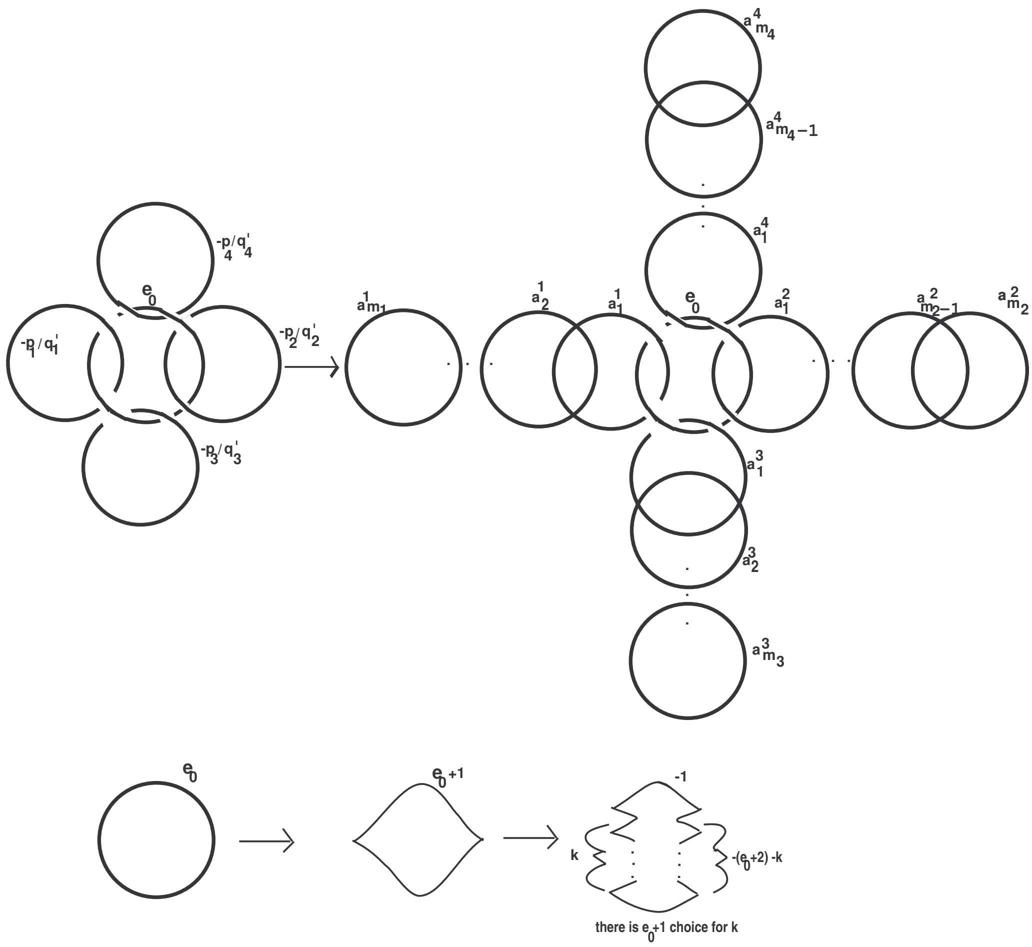

Figure 3.2: Smooth surgery diagram of M with $e_0(M) \leq -2$ and Legendrian realization of a smooth surgery on the unknot with coefficient $e_0 \leq -2$.

boundaries such that

1) $T_1 \times I$, $T_2 \times I$ and $T_3 \times I$ are mutually disjoint form T_4

2) for $i = 1, 2, 3, 4$, $T_i \times \{0\} = T_i$, and each $T_i \times \{1\}$ is convex with dividing curve slope $s'_i \leq s_i$

3) A_1 and A_2 has no ∂ parallel dividing curves.

Proof of the Claim First we consider the case $s_1 = s_2 = s_3 = \infty$. Connect a Legendrian divide of T_1 to a Legendrian divide of T_2 and a Legendrian divide of T_2 to a Legendrian divide of T_3 so that we obtain two annuli on which there are no boundary parallel curves. This finishes the proof for this case.

Now, If $s_1 = \infty$ and all the other s_i are finite then connect a Legendrian divide of T_1 to a Legendrian ruling with ∞ slope on T_2 and to a Legendrian vertical ruling with ∞ slope on T_3

by annuli B_1 and B_2. T_1 side of the annulus does not intersect any dividing curves. If there are $\partial-$parallel arcs on T_2 and T_3 sides of the annuli B_1 and B_2 adding bypasses we obtain $\infty-$slope tori isotopic to T_2 and T_3.

The case $s_1 = s_2 = \infty$ and $s_3$and s_4 are finite can be handled similarly.
Finally assume all $s_i's$ are finite. After renaming the tori T_i's we may assume that the torus T_2 has slope with smallest denominator among those of all the T_i's. Let $s_i = \dfrac{q_i}{p_i}$, for $i = 1, 2$, where $p_i > 0$. Connect vertical Legendrian ruling curves on each T_1, T_2 and T_3, to obtain vertical convex annuli A_1 and A_2. If A_1 and A_2 has no ∂ parallel curve then we are done. Since we assume that slope of T_2 has the smallest denominator, by the imbalance principle there may be ∂–parallel curve on T_1 and T_3 side of the annuli A_1 and A_2. If this is the case then we can attach these bypasses to T_1 and T_2 along the $\infty-$slope Legendrian ruling curve. Repeat the procedure until there are no more $\partial-$parallel dividing curves on A_1 and A_2, or until the slopes s_1, s_2 and s_3 become all ∞, which is the case we already handled. $\square$

Since $e_0(M) \geq 0$, we may assume that $\dfrac{q_1}{p_1}, \dfrac{q_2}{p_2}, \dfrac{q_3}{p_3} > 0$, and $\dfrac{q_1}{p_1} + \dfrac{q_4}{p_4} \geq 0$. Define an orientation preserving diffeomorphism by $\varphi_i : \partial V_i \to T_i$, $\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}$, where $p_i u_i + v_i q_i = 1$.

Let ξ be a tight contact structure on M. First isotope ξ to make each V_i a standard neighborhood of a Legendrian circle L_i, isotopic to the i^{th} singular fiber with twisting number $t_i < 0$. Then each V_i is a convex tori with two dividing curves having slopes $\dfrac{1}{t_i}$. We can calculate the slope of T_i as $s(T_i) = s_i$, where

$$s_i = \frac{-t_i q_i + v_i}{t_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)} < -\frac{q_i}{p_i}.$$

By the previous claim we can thicken V_1, V_2 and V_3 to V_1',V_2' and V_3' such that

1) V_1', V_2', V_3' and V_4 are pairwise disjoint,

2) $T_i' = \varphi_i(\partial V_i')$, $s_i' = -\frac{q_i'}{p} \leq s_i$, p, $q_i > 0$, $i = 1, 2, 3$, and

3) there exist properly embedded vertical annuli A_1 connecting T_1' to T_2' and A_2 connecting T_2' and T_3' that have no ∂–parallel dividing curves.

If all the dividing curves of A_1 (or A_2) are ∂–parallel then there is a Legendrian vertical curve on this annulus, which has zero intersection with the dividing curves of the annulus so that it has twisting zero, and hence we are done.

If there are dividing curves connecting the two components of ∂A_1 and ∂A_2 then we cut M

along $V_1' \cup A_1 \cup V_2' \cup A_2 \cup V_3'$. This gives an embedded thickened torus $T_4 \times I$, whose boundary is convex with two dividing curves of slopes s_4 and s_4', where

$$s_4' = \frac{q_1' + q_2' + q_3' - 2}{p} \geq \frac{q_1'}{p} \geq -s_1 > \frac{q_1}{p_1} \geq -\frac{q_4}{p_4} > s_4 \, .$$

Since $s_4' > 0 > s_4$, when we go counterclockwise from s_4' to s_4 along the hyperbolic unit disk we pass through the infinite slope. This guarantees the existence of a torus with slope ∞, so that we have a twisting number zero curve on this torus. This concludes the proof of the claim.

In the second part of the proof, we will find an upper bound for the number of tight contact structures with zero Giroux torsion using the convex surface theory. Since there exists a Legendrian vertical curve L with zero twisting, we can connect L with all the vertical Legendrian ruling curves on the boundaries of neighborhoods of singular fibers by vertical annuli A_i. Since L has twisting zero there is no intersection on the L side of the annuli $A_i's$ and hence there will be boundary parallel dividing curves on the T_i sides of the annuli. After attaching all possible bypasses we obtain an $\infty-$slope tori, say T_i', parallel to T_i, $i = 1, 2, 3, 4$. Using the boundary map we see this $\infty-$slope dividing curve corresponds to $-\frac{p_i}{v_i}$ in the coordinates of the U_i, where $T_i' \cong \partial(M \backslash U_i)$. One can observe that $-\frac{p_i}{v_i} = [a^i_{m_i}, \ldots, a^i_2, a^i_1, a^i_0]$. Using the classification of tight contact structures on the solid torus with slope $-\frac{p_i}{v_i}$ there are $| \prod_{i=1}^4 {a_0}^i \prod_{i=1}^4 \prod_{k=1}^{m_i} ({a_k}^i + 1)|$ many tight contact structures.

However, in the case $e_0(M) = 0$, some of these contact structures are isotopic to each other. Indeed, the tight structures, where all basic slices with slopes $-1, \infty$ have positive sign is isotopic to the structure when all basic slices with slopes $-1, \infty$ have negative sign. To see this first note that since $e_0(M) = 0$, we have $-1 < s_i < -\frac{q_i}{p_i}$. We showed that all the tight contact structures on M with $e_0(M) \geq 0$ contain a Legendrian vertical curve L with twisting zero. Connect L with vertical Legendrian ruling curves on the neighborhoods of each singular fiber and obtain vertical annuli A_i. The imbalance principle says that there are boundary parallel curves on T_i side. Attaching all possible bypasses gives a thickened tori around each singular fiber, which contains basic slices with slope $-1, \infty$. If the sign of these basic slices are all the same then we can connect Legendrian vertical ruling curves on each -1 slope tori, and obtain two annuli, B_1 between $T_1 \times \{1/2\}$ and $T_2 \times \{1/2\}$, and B_2 between $T_2 \times \{1/2\}$ and $T_3 \times \{1/2\}$, where the slope of each torus is $s(T_i \times \{1/2\}) = -1$, for $i = 1, 2, 3$. According to the previous claim there is no boundary parallel dividing curves on these annuli. Cutting along $T_1 \times \{1/2\} \cup B_1 \cup T_2 \times \{1/2\} \cup B_2 \cup T_3 \times \{1/2\}$ and rounding the edges yields a torus with slope -1. Then, we obtain a thickened torus with two basic slices in the neighborhood of the fourth singular fiber which have slopes $-1, \infty, 1$. The sign of the first basic slice is positive as we assumed at the beginning, and the sign of the new basic slice with slopes $\infty, 1$ is negative.

Similarly if we assume that all the signs of the basic slices with slope $-1, \infty$ are negative we obtain a thickened torus containing basic slices with same slopes but opposite signs in this case. However, in both cases, there is only one positive basic slice in the thickened tori, so that these contact structures must be isotopic [38].

The number of tight contact structures, which contains the basic slices having slopes $-1, \infty$ with positive (or negative) sign is $|\prod_{i=1}^{4}({a_0}^i + 1) \prod_{i=1}^{4} \prod_{k=1}^{m_i}({a_k}^i + 1)|$. Then $|\prod_{i=1}^{4} {a_0}^i - \prod_{i=1}^{4}({a_0}^i + 1) \prod_{i=1}^{4} \prod_{k=1}^{m_i}({a_k}^i + 1)|$ is the number of tight contact structures on the neighborhoods of singular fibers which is $M \backslash (\Sigma \times S^1)$.

Next, we need an upper bound on the number of tight contact structures with zero Giroux torsion on $\Sigma \times S^1$ with ∞ boundary slopes.

Let $M \backslash (\bigcup_{i=1}^{4} U_i)$ be the background of $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}, \frac{q_4}{p_4})$, which is diffeomorphic to $\Sigma \times S^1$, where Σ is a 4−punctured sphere. So $\partial\Sigma \times S^1 = T_1 \cup T_2 \cup_3 \cup T_4$. The S^1−invariant tight contact structures on $\Sigma \times S^1$ by the ∞ boundary slopes are determined with dividing curves on any section. Figure 3.3 shows all possible dividing curve configurations on a section. The relative Euler class calculations show that the structures corresponding to the configurations are mutually non–isotopic except the ones corresponding to (3) and (4), which are isotopic. We call these structures ξ_+ ξ_-, and ξ_0 as in Figure 3.3.

In this part of the proof, we will show that the configurations in Figure 3.4(A) correspond to a Stein fillable contact structure and those in Figure 3.4(B) correspond to non-fillable tight contact structures.

We construct a tight contact structure ζ on $M(r_1, r_2, r_3)$, such that after some negative Legendrian surgery on $(M(r_1, r_2, r_3), \zeta)$ we obtain $(M(r_1, r_2, r_3, r_4), \xi)$ as in the following: Let $(M \backslash V_4, \xi|_{M \backslash V_4})$ have infinite boundary slope and on $(D^2 \times S^1)$, let λ be the unique tight contact structure with slope ∞. Then set

$$(M(r_1, r_2, r_3), \zeta) = (M \backslash V_4, \xi|_{M_4}) \cup_{id} (D^2 \times S^1, \lambda)$$

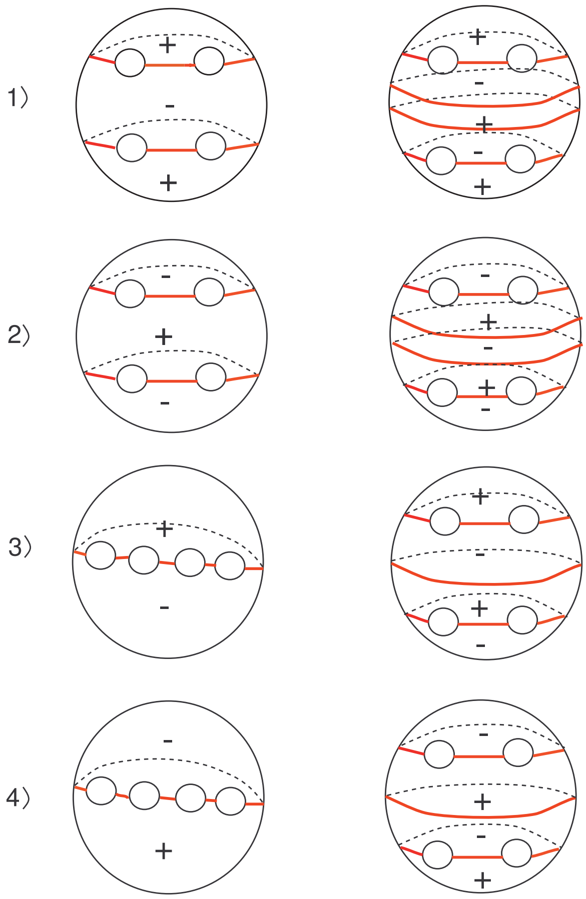

Figure 3.3: Possible dividing curves on the background of M.

by id : $\partial D^2 \times S^1 \longrightarrow -\partial(M \backslash V_4)$.

In $(M(r_1, r_2, r_3), \zeta)$ constructed as above, there is a $D^2 \times S^1$ with ∞ slope, which is the neighborhood of a Legendrian curve with twisting zero. So, $(M \backslash V_4, \xi|_{M \backslash V_4})$ is the complement of the standard neighborhood of a regular fiber with twisting number zero in $M(r_1, r_2, r_3)$. So when we perform smooth $-\frac{1}{r_4}$ surgery on this fiber we get $M(r_1, r_2, r_3, r_4)$, and since the twisting number of the curve on which we apply the Legendrian surgery is zero, the smooth surgery coefficient is equal to the contact surgery coefficient. Thus, we have $-\frac{1}{r_4}$ negative Legendrian surgery and this can be written as a sequence of -1 Legendrian surgeries.

In [28] it has been proved that if a contact Seifert fibred 3−manifold with three singular fibers over S^2 has a background diffeomorphic to the one in Figure 3.5 then the contact structure on M is Stein fillable. Using the above construction and the theorem of Eliashberg, which says

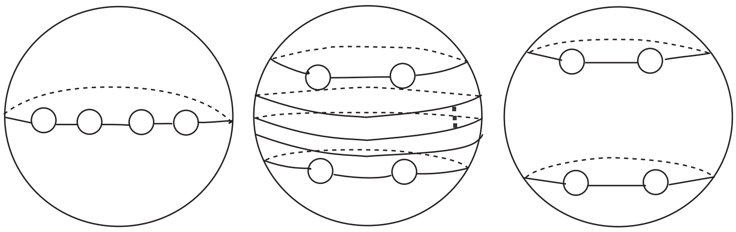

Figure 3.4: Possible dividing curves on a section of 4 punctured sphere times S^1.

that Legendrian surgery preserves Stein fillability, we deduce that a tight contact structure on a Seifert fibred $3-$manifold with 4 singular fiber which has a background diffeomorphic to the one in Figure 3.4(A) is also Stein fillable. For the background given in Figure 3.4(B),

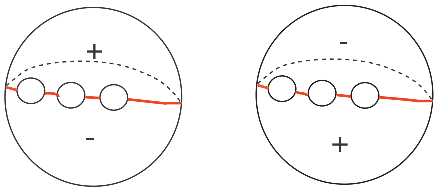

Figure 3.5: There is a unique dividing curve configuration for pair of pants times S^1 with ∞ boundary slopes.

we refer to the Proposition 3.6 in [27]. It is proved that such a background corresponds to an embedded $T^2 \times I$ with $\xi_n = \ker(\cos 2\pi nzdx + \sin 2\pi nzdy)$ along incompressible tori which separates the manifolds into pieces. This embedding yields positive Giroux torsion, so that the corresponding tight contact structure on M is non–fillable (See also [57], and [49].).

However, the above argument does not work for the background in Figure 3.4(C). Besides, by a Theorem of Giroux, since there is no homotopically trivial curve on a section of the background, any S^1-invariant contact structure is universally tight.

Existence of these contact structures can be guaranteed by Legendrian surgery. Figure 3.6 shows the handlebody decomposition of a $4-$manifold whose boundary is diffeomorphic to a Seifert fibred $3-$manifold with $e_0(M) \geq 0$. Legendrian realizations of these unknots give possible Stein fillable contact structures. We refer to Proposition 3.1 in [28] to deduce that there are $|\prod_{i=1}^{4} a_0{}^i - \prod_{i=1}^{4}(a_0{}^i + 1)\prod_{i=1}^{4}\prod_{k=1}^{m_i}(a_k{}^i + 1)|$ Stein fillable contact structures on M.

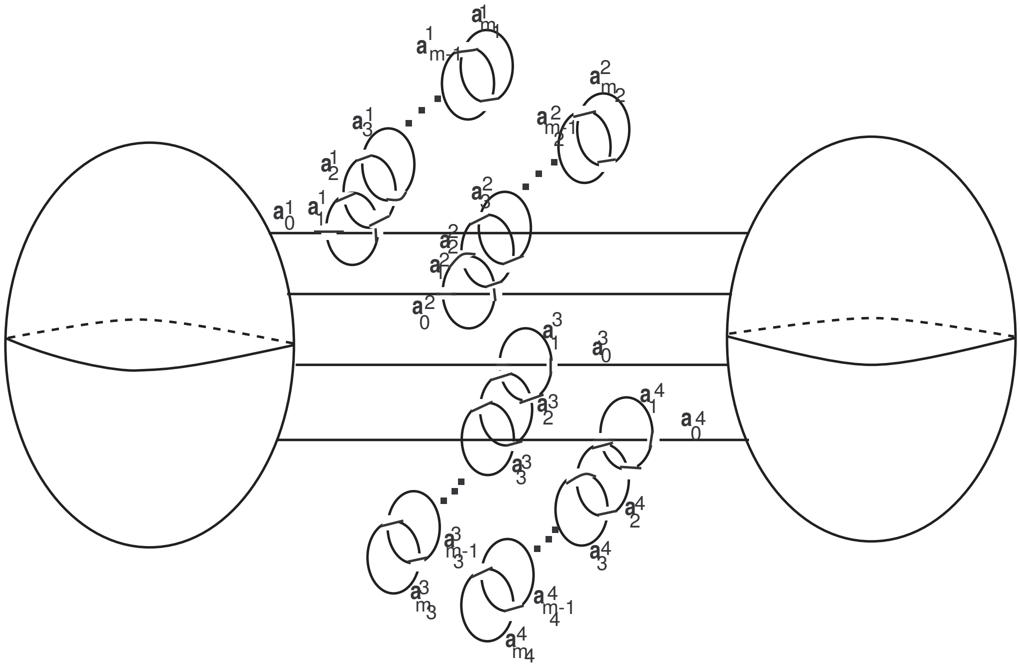

Figure 3.6: Handlebody decomposition of a 4 manifold whose boundary is M with $e_0(M) \geq 0$.

Therefore as an upper bound we obtain $2|(\prod_{i=1}^4 a_0{}^i - \prod_{i=1}^4 (a_0{}^i + 1)) \prod_{i=1}^4 \prod_{k=1}^{m_i} (a_k{}^i + 1)|$, and by above explanations at least $|(\prod_{i=1}^4 a_0{}^i - \prod_{i=1}^4 (a_0{}^i + 1)) \prod_{i=1}^4 \prod_{k=1}^{m_i} (a_k{}^i + 1)|$ many of them are Stein fillable.□

CHAPTER 4

GENERALIZATIONS AND OPEN BOOK DECOMPOSITIONS

In this chapter, we discuss the possible generalizations of the main results to Seifert fibered 3−manifold with more than 4 singular fibers. We also study the open book decompositions of Seifert fibered 3−manifolds with 4 singular fibers.

4.1 Possible Generalizations

Let ξ be a tight contact structure on the Seifert fibered 3−manifold $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \cdots, \frac{q_n}{p_n})$ with base S^2 and n singular fibers. Let $M \setminus \bigcup_{i=1}^{n} U_i$ be the background of $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \cdots, \frac{q_n}{p_n})$, and choose a diffeomorphism to $\Sigma \times S^1$ where Σ is an n punctured sphere, so that $\partial\Sigma \times S^1 = T_1 \cup T_2 \cup \cdots T_n$. If the slopes of the dividing curves on T_i are all $s(T_i) = \infty$, then there are three homotopy class of contact structures on $\Sigma \times S^1$ (Recall the background for 4 fibered case and Figure 3.3).

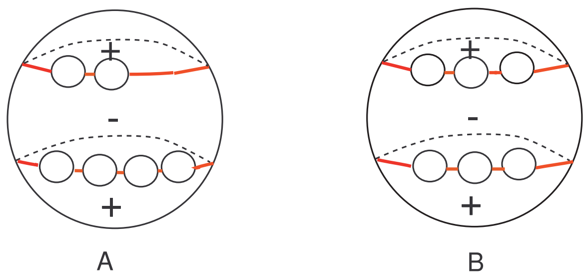

Figure 4.1: Examples of dividing curves on the background of M with six singular fibers.

However, in each homotopy class there may be non-isotopic contact structures. For example in Figure 4.1, although they have the same relative Euler class, the structure corresponding to Figure 4.1 (A) may not be isotopic to the structure in Figure 4.1(B). Once we have a tool to distinguish these tight contact structures we can find an upper bound using exactly the same techniques employed in the four fiber case. Finding a lower bound can be done using Legendrian surgery diagrams similar to previous results.

4.2 Open Book Decompositions

For small Seifert fibered $3-$manifolds with $e_0(M) \leq -3$ it is shown in [53] that all tight contact structures are compatible with planar open books and all are Stein fillable. Schönenberger [53] generalizes this result to Seifert fibered $3-$manifolds with n singular fibers and says that on a plumbing, when framing coefficient of central vertex is less than or equal to minus the number of edges going out of that vertex (which means that there is no bad vertex or a non-positive plumbing diagram), any Legendrian realization give a Stein fillable contact structure and the compatible open book is planar.

As a result of Theorem 3.1.2 and Schönenberger's theorem we can say that on a Seifert fibered $3-$manifold M with 4 singular fibers and $e_0(M) \leq -4$ all Stein fillable contact structures are compatible with planar open books.

In Figure 4.2, we present a possible planar open book decomposition. There are many other ones depending on the position of cusps which are used to stabilize the Legendrian knot. However, all the open books corresponding to Stein fillable contact structures in the case $e_0(M) \leq -4$, will be planar. In Figure 4.2, all the curves on the page represent one positive Dehn twist. In Figure 4.2 and Figure 4.6 we assume all r_i's are greater than 1 and $-r_1 = [-a_1, -a_2, \cdots, -a_k]$, $-r_2 = [-d_1, -d_2, \cdots, -d_l]$, $-r_3 = [-b_1, -b_2, \cdots, -b_m]$, $-r_4 = [-c_1, -c_2, \cdots, -c_n]$ where all $-a_i$'s, $-b_i$'s, $-c_i$'s and $-d_i$'s are less than or equal to -2. In both cases we start with an open book of S^3, whose page is annulus and monodromy is a right handed Dehn twist around the core curve of the annulus.

Similarly, it is shown in [19] that all Stein fillable tight contact structures on small Seifert fibred $3-$manifolds with $e_0(M) \geq 0$ are compatible with planar open books. An example can be seen in Figure 4.3. We can construct planar open books for the Stein fillable contact structures on a Seifert fibred $3-$manifold with 4 singular fibers and $e_0(M) \geq 0$. Similarly, when $e_0(M) = k \geq 0$, an example of an open book decomposition is as in Figure 4.4. In Figure 4.3 we assume all $r_i > 1$ and $-r_i = [a^i_0, a^i_1, \cdots, a^i_{m_i}]$, for all $a^i_j \leq -2$. In Figure 4.4 we assume $0 < r_1 < 1$ and $r_i > 1$ for $i = 2, 34$, and $-r_i = [a^i_0, a^i_1, \cdots, a^i_{m_i}]$, $a^1_0 \leq -1$ and $a^i_j \leq -2$. In both cases we start with an open book of $S^1 \times S^2$, whose page is annulus and monodromy is trivial. Therefore Figure 4.3 and Figure 4.4 can be obtained in a similar way. When $e_0(M) = -2$, for a Seifert 3−manifold with 4 singular fibers there is a bad vertex in its plumbing diagram as explained in [19]. We can describe its open book decomposition and it turns out to be non-planar. These contact structures are Stein fillable and have zero Giroux torsion. For the special case $M(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ the manifold is actually a torus bundle over the circle and a compatible open book decomposition described separately in [16], [56],[19].

Dalyan in [60] also says that if the plumbing diagram contains Figure 4.5 as a subgraph then its compatible open books will be non-planar. Also using rolling up as in [53] and [19] a non–planar open book decomposition for a Seifert fibred 3−manifold with 4 singular fibers and $e_0(M) = -2$ is given as in Figure 4.6. (The case $e_0(M) = -3$ is similar.)

As in Figure 4.7 Seifert fibred 3−manifolds with 4 or more singular fibers which are also L−spaces, and are compatible with planar open books. Since there is one negative Dehn twist around the blue curve, there may be non–fillable contact structures, and non–fillability of these contact structures can be shown using the technique in [54] improved by Jeremy Van–Horn Morris and Olga Plamenevskaya, which shows that this negative Dehn twist can not be removed from the monodromy of the open book. As it is written in the same paper open book decompositions of non–fillable contact structures on Seifert fibred 3−manifolds with 4 or more singular fibers can be obtained using similar techniques. However these may not be the whole list of tight contact structures on M. For the case $e_0(M) = -1$, we can draw the following Legendrian surgery diagram, and it is easy to draw a compatible planar open book for this surgery diagram if the surgeries have integer coefficients (Figure 4.7). Note that for each open book decomposition figure given in this chapter, on each page, addition to the right handed Dehn twists along curves drawn in the figures, there are right handed Dehn twists along the curves which are parallel to the holes on the pages.

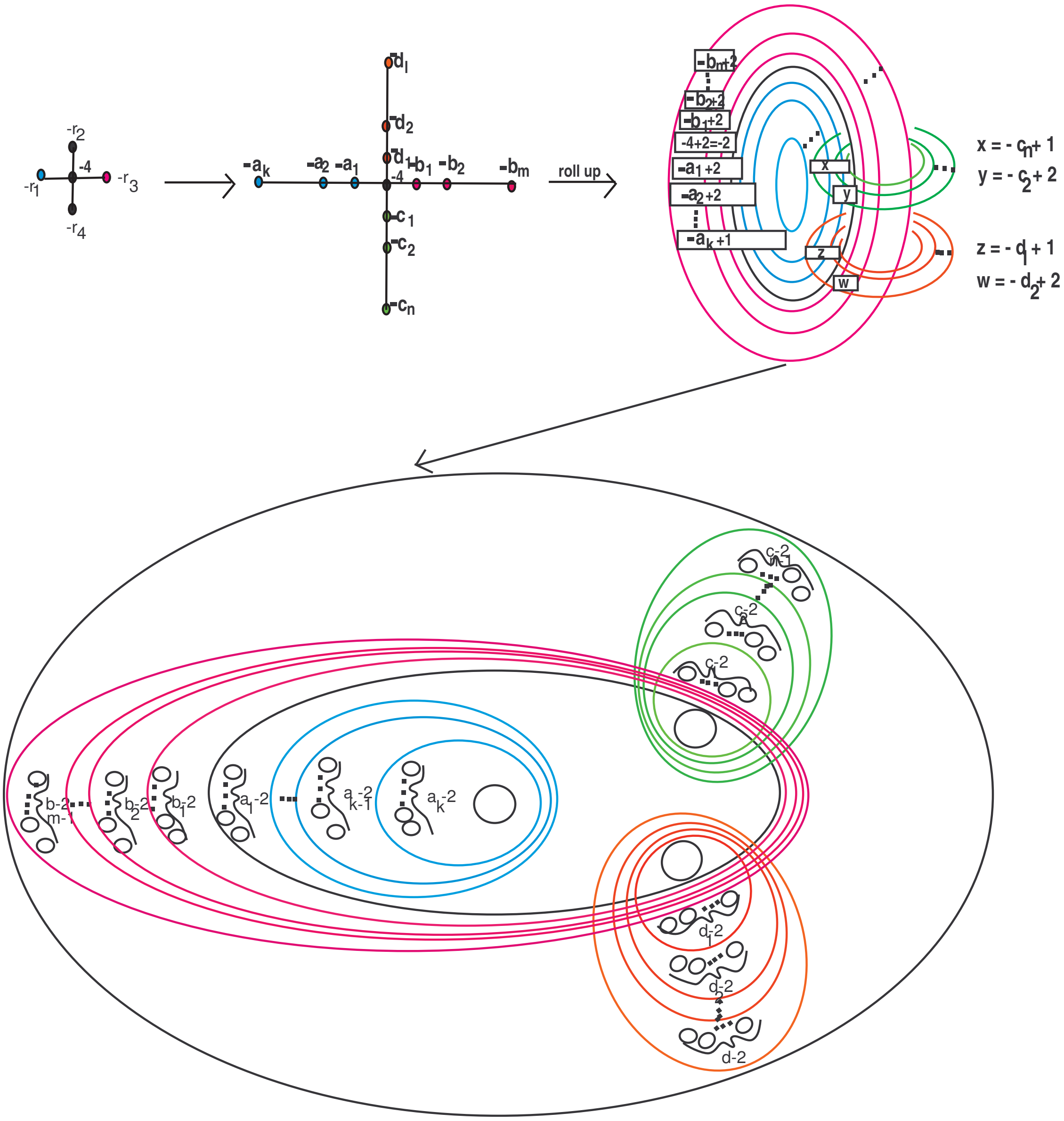

Figure 4.2: Surgery diagram to plumbing, rolling up plumbing to a page of the compatible open book

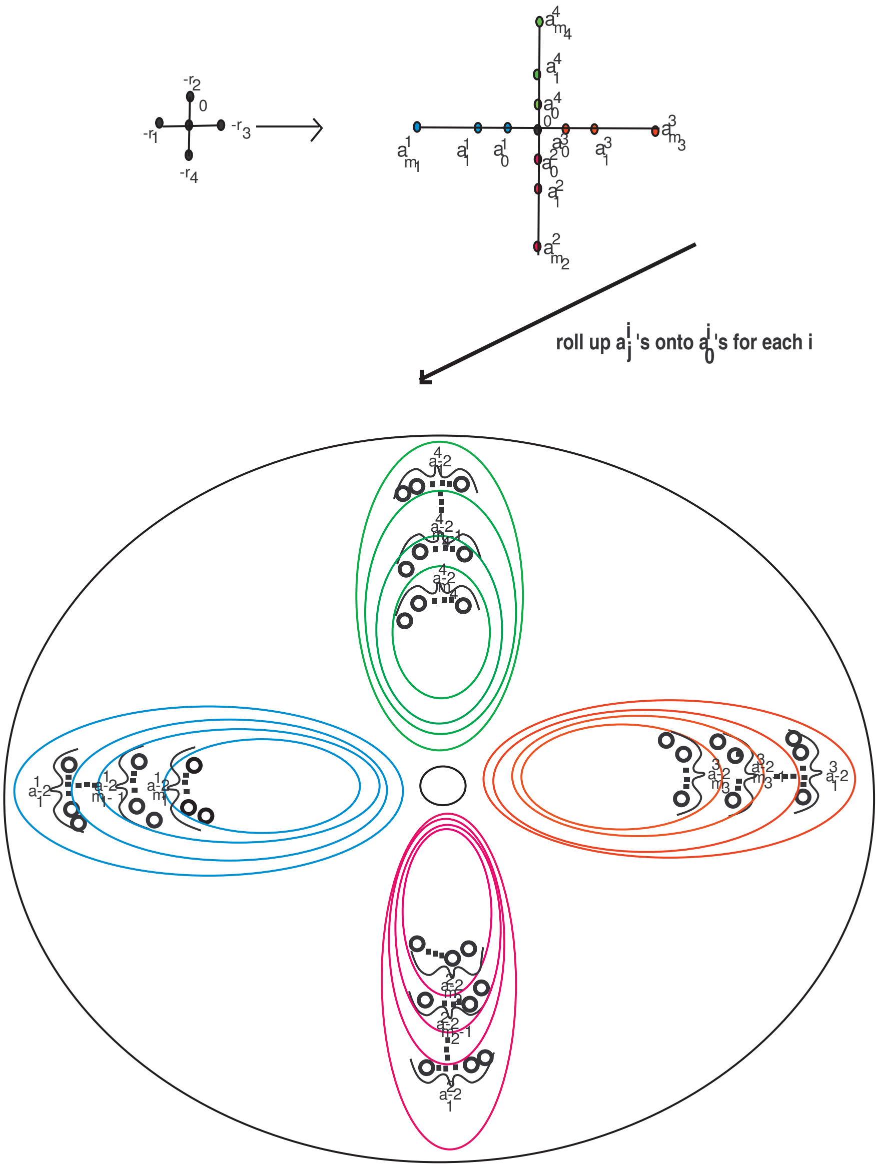

Figure 4.3: Plumbing to an open book for Seifert fibred 3−manifold with $e_0(M) = 0$

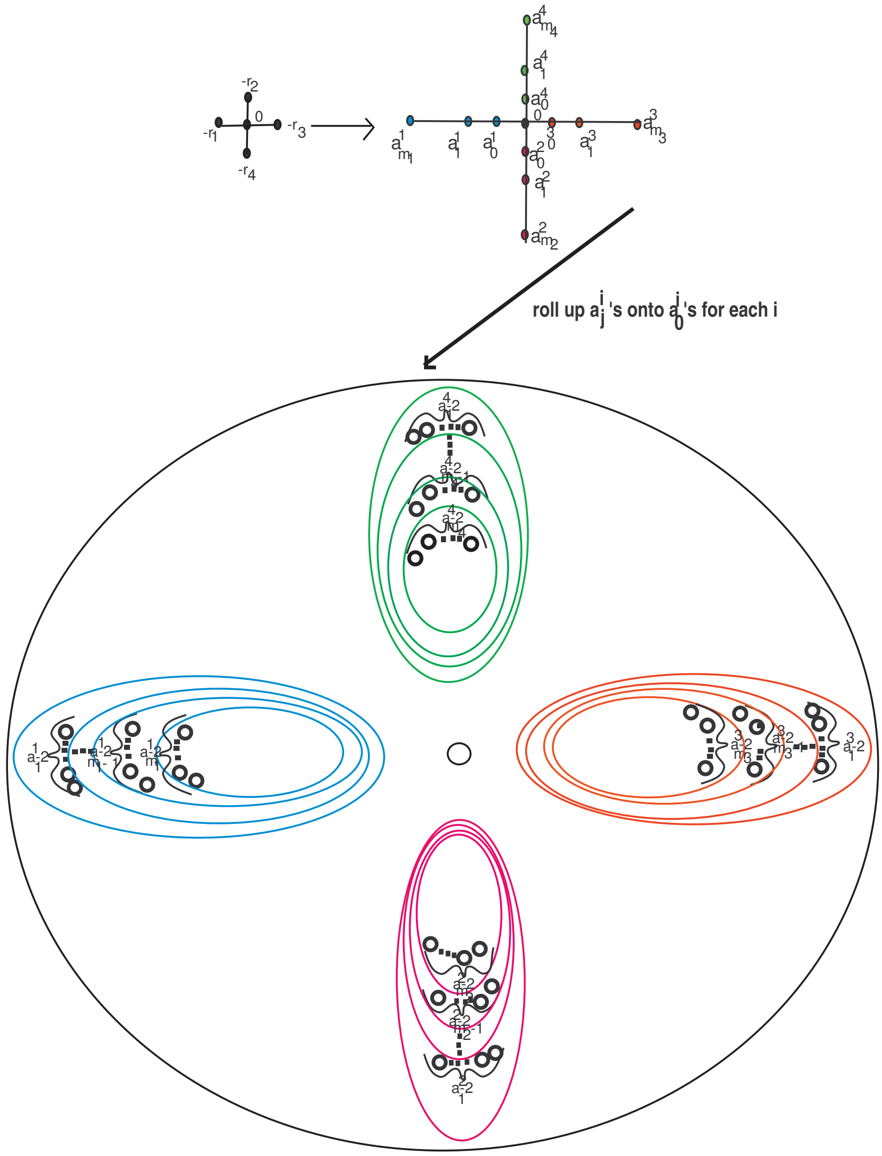

Figure 4.4: Plumbing to an open book for a Seifert fibred 3−manifold with $e_0(M) = k \geq 0$.

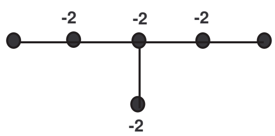

Figure 4.5: Plumbing diagram of a small Seifert fibred 3−manifold with $e_0(M) = -2$.

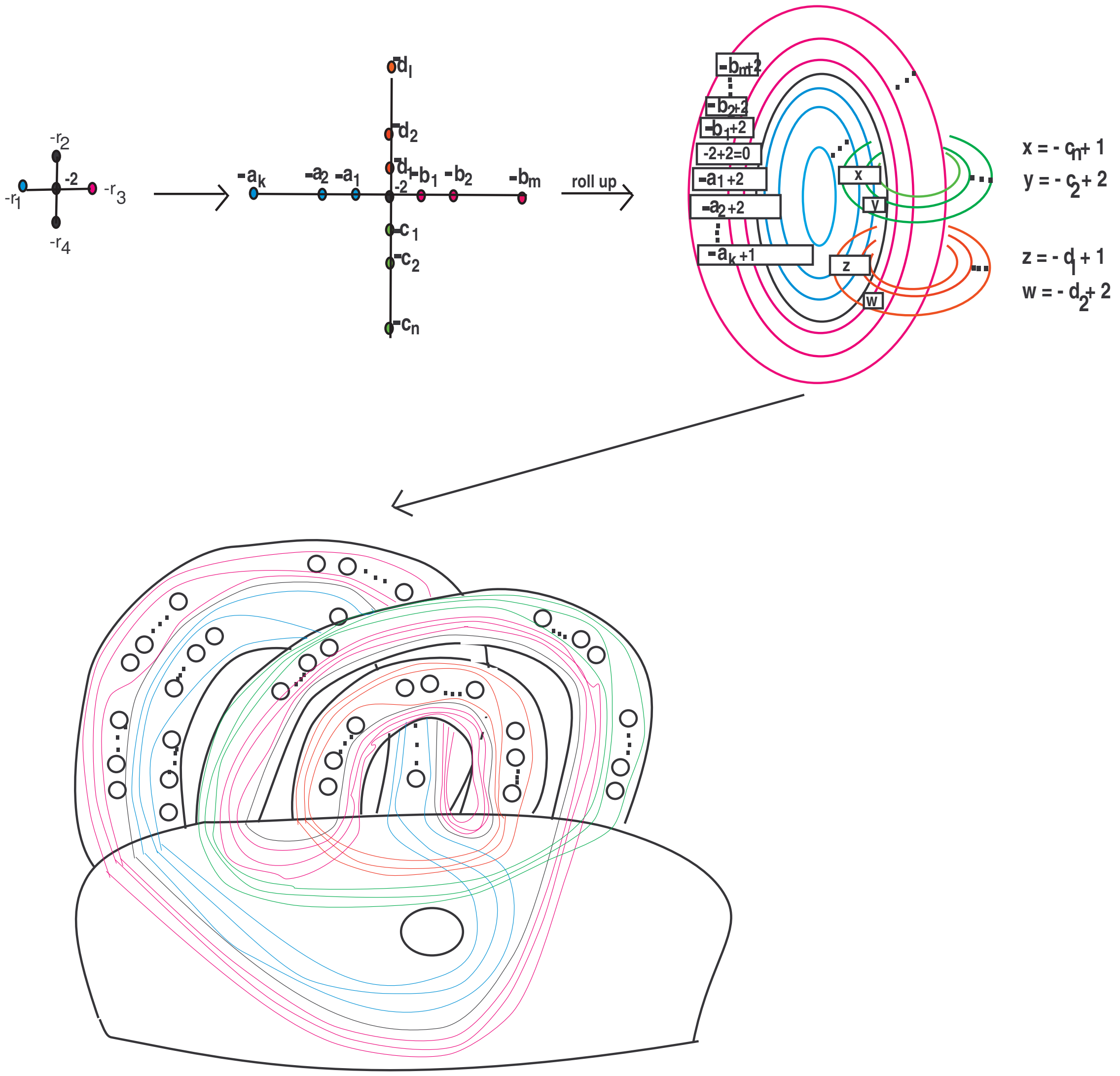

Figure 4.6: Plumbing to a compatible open book for a Seifert fibred 3−manifold with $e_0(M) = -2$.

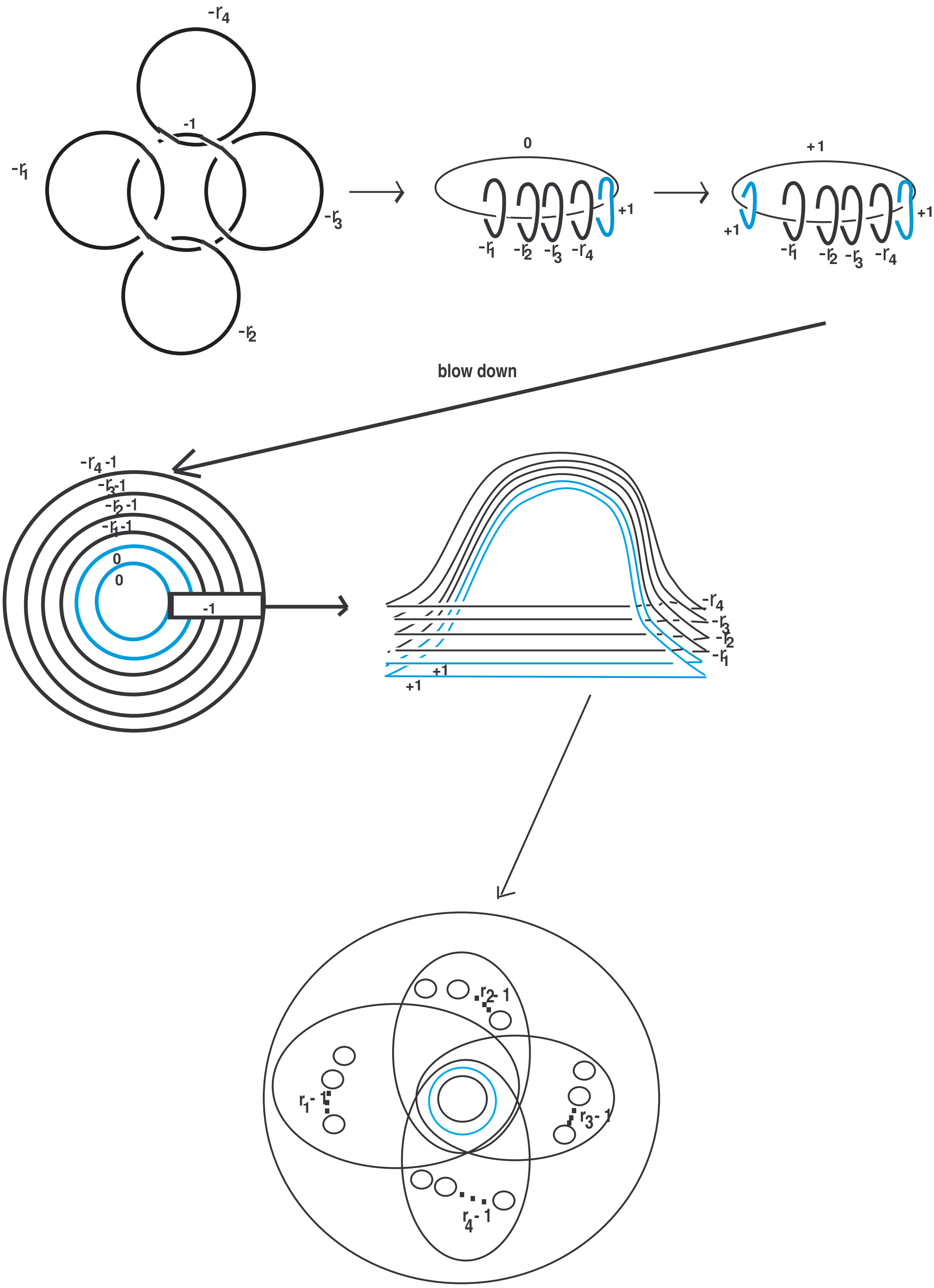

Figure 4.7: Legendrian realization of a surgery diagram and a compatible open book decomposition of a Seifert fibred 3−manifold with $e_0(M) = -1$ and $r_i \in \mathbb{Z}^{\geq 2}$ for each i.

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