TACTICAL INVENTORY AND BACKORDER DECISIONS FOR SYSTEMS WITH PREDICTABLE PRODUCTION YIELD

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

TURGUT MART

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

MAY 2010

Approval of the thesis:

TACTICAL INVENTORY AND BACKORDER DECISIONS FOR SYSTEMS WITH PREDICTABLE PRODUCTION YIELD

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ABSTRACT

TACTICAL INVENTORY AND BACKORDER DECISIONS FOR SYSTEMS WITH PREDICTABLE PRODUCTION YIELD

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May 2010, 42 pages

We consider a manufacturing system with stochastic demand and predictable production yield. The manufacturer has predetermined prices and limited production capacity in each period. The producer also has the option to save some inventory for future periods even if there is demand in the current period. The demand that is not met is lost or may be backordered for only one period. Our objective is to maximize the expected profit by choosing optimal production, save and backorder quantities in each period. We formulate this problem as a Markov Decision Process where the state of the system is represented by the net inventory and the efficiency parameter. We show that a modified (Y, S, B) policy is optimal in each period. At the end, we have some computational analysis to examine the effects of yield on the optimal decisions.

Keywords: tactical inventory, production yield, dynamic programming

ÖZ

ÖNGÖRÜLEBİLEN VERİM İLE ÇALIŞAN ÜRETİM SİSTEMLERİNDE TAKTİK ENVANTER VE SONRADAN KARŞILAMA KARARLARI

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Mayıs 2010, 42 sayfa

Çalışmamızda öngörülebilir verimle çalışan ve rassal talep ile karşılaşan bir üretim sistemini ele aldık. Bu sistemde üreticinin her periyot için belirli bir üretim kapasitesi ve önceden belirlenmiş ürün fiyatları vardır. Ayrıca, üreticinin ürünlerini mevcut periyotta talep olsa dahi ileriki periyotlarda kullanmak üzere saklama opsiyonu bulunmaktadır. Karşılanmayan talep kaybedilmekte ya da yalnızca bir periyot için sonradan karşılanabilmektedir. Burada bizim amacımız en iyi üretim, saklama ve sonradan karşılama miktarlarını seçerek beklenen karı en yüksek değerine ulaştırmaktır. Bu problemi sistem parametreleri net envanter seviyesi ve verim parametresi olan bir Markov Karar Süreci olarak tasarladık ve modifiye edilmiş (Y, S, B) politikasının her periyot için optimal olduğunu gösterdik. Son olarak verimin modelimizi nasıl etkilediğini göstermek için sayısal analizler yaptık.

Anahtar Kelimeler: taktik envanter, imalat verimi, dinamik programlama

to my family

ACKNOWLEDGMENTS

My sincere thanks belong to my advisors Dr. Serhan Duran and Dr. İsmail Serdar Bakal. Their guidance and support have a great impact on my thesis work. While traveling all around the country as an auditor and trying to complete my work, they were always tolerant and helpful. Without their knowledge and insights this work could not be completed. I would also like to thank Dr. Cem İyigün, Dr. Sinan Gürel and Ertan Yakıcı for their time and effort in serving on my committee.

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CHAPTER 1

INTRODUCTION

In modern production systems, due to highly competitive markets, it is extremely essential to follow and optimize all stages of the production. From raw material to the final customer all processes have to be designed very carefully. Hence, vast numbers of studies were conducted on supply chain systems, many of which are about inventory control systems. Inventory related costs such as holding costs, lost sales etc. have a significant part in total costs explaining the particular interest in inventory problems.

Demand uncertainty is one of the main problems faced in inventory systems. The trade off is between lost (or backordered) sales and inventory holding costs. Almost all inventory strategies are conducted to overcome this problem and minimize costs by controlling this trade off. In traditional inventory control systems the producer meets the demand with all products at hand. However, sometimes it may be more profitable not to sell all items at hand and allow some lost sales. That is especially true for systems with varying prices and production costs. In recent years, an alternative inventory control policy considering this fact has been improved. This policy is mentioned as rationing. With this policy the producer have the option to reserve some of the products for future periods even if there is any demand at current period. Although there are lots of studies in supply chain and inventory control literature, only a few of them are about systems with rationing. Almost none of these studies considers production yield.

In this study we consider a system including a single item producer with production capacity restrictions and one customer class. He has the option to save some inventory for future use and backorder some demand to be met in the following period. The producer determines the prices for each period at the beginning of time horizon. However, these prices are not known by the customers before the beginning of that period. Hence, the customers do not act tactically. This is a general assumption considered in recent studies on rationing such as Federgruen and Henching [7], Chen and Simchi Levi [3], Chan et al [2], Duran et al [5]. Different than the pervious studies, there is a predictable production yield rate in each period in our study. In each period, the producer has to decide how much to produce, save and backorder.

Our model may be adopted successfully for a production system with a new product introduction. In such a system the producer is the only supplier and can decide prices over a time horizon. There are no substitutes and customers may accept backordering. We limit the backorder time as one period to simplify the problem. Also, the firm may reserve products for future use if there are restrictions such as capacity deficiency. Change in the production yield and production costs can be forecasted as they are affected by learning effect and seasonality. In such a system production and inventory policies have to be decided. For a real life example, consider the time Apple introduced i-phone to the market. Before the product launch, the price and probably prices for the following few months were already decided by the firm. Introductory price was set to be a bit higher than the next period prices. That is because people had waited for the product for months and they were much more eager to buy initially. Also, they easily accepted backordering. It is obvious that customers will accept backordering up to a specific time, for example one month. For the costs, we can say that the firm can forecast the labor costs, electronic material costs etc. with past data sets. Lastly, the firm probably had a production yield trend affected by learning effect like we have in our model.

Remainder of this study is organized as follows. In Chapter 2 we review the related literature and discuss recent studies about rationing and yield. Then we present the notation and the assumptions in Chapter 3. We present the profit-to-go function and elaborate this function. In Sections 3.2 and 3.3, we discuss save inventory and backlog demand models respectively. After that we conduct some numerical analysis in Chapter 4 and share some insights. Finally, we present our findings briefly and give some extension ideas in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

In this chapter we present and discuss recent studies related to our study. We review the literature in two main streams; discretionary sales and production yield.

2.1 Studies on Discretionary Sales

Although there are lots of studies conducted on inventory theory, only a few of them considers discretionary sales. We review the literature and present the studies conducted on discretionary sales below.

The earliest study considering discretionary sales is the work of Scarf [16]. Scarf [16] focuses on an inventory planning problem in a multi-period, single-item production system with production capacity limits and fixed production setup costs. He shows that the expected profit functions with discretionary sales are K-concave and the optimal inventory policy is of (s, S) type. He also demonstrates that the optimal discretionary sale amount is dependent on realized demand. Hence, the producer has to wait for the demand realization before deciding discretionary sales.

Chan et al [2] integrate pricing and production decisions in a multi-period, single-item system with discretionary sales. As in Scarf [16], there is a production capacity limit for each period and unsatisfied demand is assumed to be lost. In this study, customers are assumed to act myopically, not strategically. The authors aim to maximize the total profit over a finite horizon by focusing on partial planning strategies; delayed production strategy and delayed pricing strategy. In the former strategy, the pricing decision is made at the beginning of the time horizon and the production decision is made at the beginning of each period. In the second one, the production decision is made at the beginning of the time horizon. The delayed production strategy is in our interest and it is highly related to Scarf [16]. Different than Scarf [16], Chan et al [2] show that the optimal policy is independent of the realized demand and initial inventory.

Duran et al [5] discuss inventory policies in a multi-period, single-item system with two customer classes and rationing among these customer classes. Rationing may be considered as a type of discretionary sales; the difference is that rationing also takes the customer classes into account. In this paper, customer classes are differentiated according to their willingness to pay and wait. There is a production capacity for each period and the unmet demand may be backordered for one period. Prices are determined at the beginning of the horizon and production decision is made in each period. The profit-to-go function over a finite horizon problem is given and discussed in the paper. The authors show that a modified order-up-to policy is optimal and backorder and reserve decisions are independent of the realized demand. Hence, the manufacturer is able to decide the optimal reserve and backorder decisions before demand realizations. A similar study is conducted in Duran et al [6] where customers differ in their patience and one of the classes never accepts backlogging.

Yang et al [20] develop a game theoretic model in an EOQ system with one supplier and two retailers facing demand uncertainty. The pricing, service level and lot sizing decisions are considered together in this paper. The risk sensitivity of the retailers, production capacity and holding costs complicate these decisions. The authors discuss the annual expected profit and certainty equivalent functions and demonstrate the effects of environment change. In this paper, it is shown that the expected profit is increasing in risk aversion and service level investment.

Smith et al [17] consider a joint pricing and inventory planning problem in a multi-period, single-item system with both capacity and inventory constraints. The authors aim to determine the optimal price, production quantity and sales amount for each period. First, the problem is solved exactly with an exponential demand function for a single period using a linear programming model. Then, with this exact solution, a dynamic programming solution is developed to solve the multi-period model. This paper extends Chan et al [2] by solving pricing and production problem simultaneously.

Yan and Liu [19] determine replenishment and discretionary sales jointly in a system with limited capacity, uncertain demand, lost sales and random yield. In this study stochastically proportional yield is used. The objective is to minimize the total cost by choosing optimal replenishment and discretionary sales policy. The authors show that the optimal ordering policy is not a order-up-to, but a threshold type. They also compare the results under random yield with those under certain yield for finite and infinite horizon problem and analyze the effects of random yield on the optimal policies. They present that the reorder and the discretionary sales points are smaller in systems with deterministic yield than those in systems with random yield. This paper is highly related to our study. The main difference is that we consider backordering in addition to lost sales. Also we show that a modified order-up-to policy is optimal.

Since Scarf [16] introduced discretionary sales, not too many studies have been conducted on this area. Chan et al [2] consider a special case of Scarf [16] and show the optimal policy is independent of the realized demand. Duran et al [5] extend Chan et al [2] to a system with two customer classes and backorder. Yan and Liu [19] include random production yield to the problem with lost sales. Our study is highly related to this study. However, we assume that unsatisfied demand can be backlogged and the yield is predictable. Our objective is to characterize the optimal inventory policy and analyze how the optimal production, discretionary sales and backorder decisions are affected by changes in production yield. We also aim to provide managerial insights on our findings.

2.2 Studies on Production Yield

Production yield is an important issue in production planning and inventory control problems, and studied extensively in literature. Yano and Lee [21] present a detailed literature review on this area. They classify and describe the studies up to their time.

Henig and Gerchak [11] provide an extensive analysis of a periodic-review single-item production system with stochastically proportional yield. The authors provide analysis of single period, finite horizon and infinite horizon models, and show that the infinite horizon order point is lower when yield is uncertain. They also show that the order point is independent of yield in single period model. We have similar results indicating that the efficiency in current period does not have impact on the optimal policy of the current period.

Wang and Gerchak [18] consider a production planning problem in a periodic review system with unpredictable capacity, random yield and stochastic demand. The authors discuss simultaneous effects of both uncertain capacity and yield on the lot sizing decision. Their objective is to minimize the expected costs and they show that the objective function is quasi-concave. They prove that the optimal policy is characterized by a single reorder point in each period, but is not of an order-up-to type.

Another study considering yield in a different setting is conducted by Gerchak and Grosfeld-Nir [8]. They study the conformity yield of a product having multiple functionalities with different degrees of conformity to standards when demand is upward-substitutable. The authors model the problem as a complex multiple-lot-sizing production-to-order problem. Expected cost is modeled in dynamic programming and results of numerical examples are provided.

Hsu and Bassok [12] consider a system with n different products and n different customer classes. The products are full downward substitutable and demand is stochastic. The system has a random production yield with a continuous yield coefficient. Probability distributions of the yield coefficients are known. Demand that is not filled is assumed to be lost. Production and delivery lead times are assumed to be zero. The authors present the total profit function and three methods to describe the optimal production decision that maximizes the total profit.

Bollapragada and Morton [1] present three heuristics to decide optimal inventory policy in a single-item periodic review inventory model with stochastically proportional yield and stochastic demand where unsatisfied demand is backlogged. They show that a myopic policy is a good approximation to the optimal policy under fairly general conditions and give some numerical results to show performances of their heuristics.

Duenyas and Tsai [4] consider a continuous review production system with stochastic demand, stochastic production times and random production quality yield. There are two quality classes. The quality of the end product is uncertain and the demand is downward substitutable. However, the manufacturer has the option to refuse satisfying low class customers with high quality goods. The problem is modeled as a Markov Decision Process in the context of a simple $M/M/2$, make-to-stock queue with multiple customer classes, and a heuristic is proposed and numerical analysis is made to test this heuristic.

Grosfeld-Nir et al [9] include inspection costs to a “multiple lot sizing production to order” model with random production yield. The problem is to minimize the total costs by optimizing lot sizing and inspection simultaneously. Three yield patterns are considered; binomial yield, interrupted geometric yield, discrete uniform yield. The authors show that the lot size with costly inspection is smaller than the one with free inspection. They also show that the lot size is independent of inspection costs when yield is binomial, and decreasing in the unit inspection cost for other yield patterns.

Kazaz [13] adopts the production planning problem with random yield to the agriculture industry in the context of olive oil production. A two stage decision making process is considered, growing season and selling season. Demand and yield uncertainty are the main problems considered in the paper.

Gupta and Cooper [10] use a stochastically proportional yield rate model and show that a stochastic improvement on yield rate is not always advantageous. To support this idea, the authors employ stochastic comparison techniques. They provide distribution free bounds on the expected profit in a single period problem and identify properties of yield rate distributions.

Li and Zheng [14] extend combined pricing and inventory control problems to systems with random yield. A single-item, periodic review model is considered in this paper. Demand is independent and price sensitive. Unsatisfied demand is fully backlogged. Production yield is uncertain and a stochastically proportional yield model is used. The aim of the paper is to find optimal dynamic policies that simultaneously determine the production quality and price in each period. The authors explore the operational effects of uncertain yield on the optimal policy and value function. They show that the order threshold is higher in a system with uncertain yield in the multi-period case.

Li et al [15] discuss on infinite horizon decision problem in a single-stage, single-item, periodic review system. Production yield and demand are uncertain in the system. The authors derive bounds for both the optimal order quantities and order threshold, and then develop a heuristic based on these bounds. In this paper, again stochastically proportional yield is used, and purchasing cost is charged only to the successful units produced and the unsatisfied demand is backlogged.

There are various studies on production yield in production systems. However to the best of our knowledge none of them except Yan and Liu [19] considers discretionary sales which is discussed in Section 2.1. Another important point is that most of the studies use stochastically proportional yield. Hence, we use stochastically proportional production yield in our study.

CHAPTER 3

MODEL ASSUMPTIONS AND DESCRIPTION

We focus on a single product sold at a single manufacturer over a multi-period time horizon, where the manufacturer has limited production capacity in each period. The manufacturer serves a single customer class and makes decisions over a multi-period time horizon, $t = 1, 2, \ldots, T$, with T representing the end of the horizon. Production in each period t is limited by the capacity, q_t, and the manufacturer pays a production cost per each produced unit of k_t. Inventory holding cost is linear, and a charge per unit, h_{t+1}, is assessed to carry inventory from period t to $t + 1$.

The manufacturer has predetermined prices, p_t, that may be different in each period. This is a realistic assumption because separation of pricing and production decisions is very common in current practice. In some companies, pricing decisions are given by the marketing department before the start of a selling season, while production decisions are given by the operations department.

The penalty cost per unit of demand that is rejected and lost is ℓ_t. We define the net revenue of selling to a customer from current inventory as $p_t + h_{t+1} + \ell_t$. Here the producer gains revenue from sold item and saves holding cost of the item and lost sale cost. Manufacturer also has the option of backordering the demand in current period to be (definitely) satisfied in the following period. β_t is the penalty cost per unit of demand that is backlogged at period t. Therefore the net revenue from backlogging a unit demand at period t is $p_t + \ell_t - \beta_t$. Demand function is a general non-stationary stochastic function, d_t, with known probability and cumulative distribution functions ψ_t and Ψ_t, respectively. We assume that the demand function in each period is continuous and differentiable, but no other assumptions are made on the shape of the demand function, so a wide variety of demand models could be used.

Production is a decision made at the beginning of each period and the production leadtime is zero. The net inventory (on hand inventory - backlogs) at the beginning of period t is I_t, and let $\overline{Y}_t$ represent the inventory plus successful production in period t in the presence of an efficiency parameter ε_t.

The sequence of events in every period is as follows. At the beginning of a period, the manufacturer checks the net inventory level I_t and decides the production quantity. Products arrive immediately, but only a fraction (ε_t) of them is successful production. Then the demand in the current period is revealed and the manufacturer decides the amount to reserve from the demand, S_t, and the amount to promise for sale in the following period, B_t. The demand is satisfied according to the $\overline{Y}_t$, S_t, and B_t values. The notations that we defined so far is provided in Table 3.1 for ease of reference. We model the problem as a Markov decision process,

Table 3.1: Notation

q_t	Production capacity at period t
p_t	Unit selling price at period t
k_t	Production cost per produced unit for period t
h_{t+1}	Holding cost per unit for period t
υ	Salvage value of any item left at the end of the horizon
ℓ_t	Lost sale cost for period t
β_t	Penalty cost per unit of demand that is backlogged at period t
d_t	Random demand for period t
ε_t	Efficiency parameter at period t
I_t	Net inventory at the beginning of period t
$\overline{Y}_t$	Inventory after backorders are satisfied and production is realized at period t
S_t	Max. amount of inventory to reserve in period t
B_t	Max. amount of demand to be backlogged in period t
$J_t(I_t, \varepsilon_t)$	Expected profit from period t to the end of the horizon
$G_t(\overline{Y}_t)$	Expected profit-to-go with $\overline{Y}_t$ units of product available after production
A_t	Realized backorder amount at period t

where the state of the system is represented by the net inventory and the efficiency parameter. For clarity of exposition, we present the model with the S_t and B_t decisions given *ex ante*. However, in our analysis we show that the optimal (S_t, B_t) decision is the same whether they are made before or after demand revelation.

3.1 Base Model

Let $J_t(I_t, \varepsilon_t)$ be the total expected profit from period t to the end of the horizon, or the *profit-to-go* with the efficiency parameter ε_t. Let $G_t(\overline{Y}_t)$ be the expected profit-to-go with $\overline{Y}_t$ units of product available after production. We can now write the optimal expected profit in period t and onward for the problem as the following recursive equation.

$$J_t(I_t, \varepsilon_t) = \max_{\overline{Y}_t : \max\{0, I_t\} \leq \overline{Y}_t \leq I_t + \varepsilon_t q_t} [-\frac{k_t}{\varepsilon_t}(\overline{Y}_t - I_t) + G_t(\overline{Y}_t)] \tag{3.1}$$

$$\begin{aligned} G_t(\overline{Y}_t) &= \max_{S_t : 0 \leq S_t \leq \overline{Y}_t,\, B_t : 0 \leq B_t \leq \varepsilon_{t+1} q_{t+1}} \Big\{ \int [p_t(\min(d_t, \overline{Y}_t - S_t + B_t)) \\ &- h_{t+1}(S_t + \max(0, \overline{Y}_t - S_t - d_t)) \\ &- \ell_t \max(0, [d_t - \overline{Y}_t - S_t - B_t]) - \beta_t \min([d_t - \overline{Y}_t + S_t]^+, B_t) \\ &+ J_{t+1}(S_t - B_t + \max(0, \overline{Y}_t - S_t - d_t + B_t), \varepsilon_{t+1})] d\Psi_t(d_t) \Big\} \end{aligned} \tag{3.2}$$

In Equation (3.1), the objective is to maximize the profit by the production decision. The produce-up-to level $\overline{Y}_t$ is greater than or equal to $\max\{0, I_t\}$ since backorders need to be fully satisfied at the current period and less than or equal to $I_t + \varepsilon_t q_t$ since the production is limited by the capacity and the efficiency parameter. The first term is the production cost, that is charged to all produced products. Next term is the profit-to-go function of the remainder from the horizon with the inventory after the backorders are satisfied and the production is realized.

In Equation (3.2), the profit-to-go function is maximized over the reserve and backorder decisions. The constraints ensure that the reserve decision can not be greater than the inventory at hand and no more than the capacity times the efficiency factor of the following period can be backlogged. The first term is the revenue from both sold and backlogged items. The second item is the inventory holding cost of the items reserved and the items that are not sold. Next term is the lost sale cost incurred from the demand that is neither satisfied nor backlogged. The fourth term is the penalty cost to be paid for the backlogged demand. Finally, the last term is the maximum expected profit in future periods with the available inventory. In the last period, when $t = T$, J_{t+1} is replaced by $\upsilon(\overline{Y}_t - d_t)^+$ which is the salvage revenue.

According to this model at each period the manufacturer have to decide the amount of items to reserve and the amount of demand to backorder. This is not an easy problem to solve, however we show that in an optimal policy for a period at least one of these decisions must be zero.

Lemma 3.1.1 *In an optimal policy,* $S_t \,.\, B_t = 0$ *for any* $t = 1, 2, \ldots, T$.

That means if it is good to reserve some items in a period, it is not reasonable to backorder items at the same period or the opposite. We will use contradiction to prove this.

Proof. Assume that there is an optimal solution, $(S_t,\ B_t)$, where both $S_t > 0$ and $B_t > 0$. We will show that there exists an alternate policy that is at least as good as and sometimes better than the "optimal" solution. Let V_t and $V_t^{'}$ be the expected profit starting from period t under the optimal policy and the alternate policy, respectively. We will consider two main market environments;

1) current net revenue from selling out of inventory > future marginal expected profit,

2) current net revenue from backlogging < future marginal expected profit.

Case 1: Since current net revenue from selling out of inventory is bigger than future marginal expected profit, we do not have a motivation to reserve items to sell in the following periods, thus the alternate policy is chosen to be reserving one less item. We can represent the alternate policy as $(S_t - 1,\ B_t)$.

- When $d_t < \overline{Y}_t - S_t$

$$\begin{aligned} V_t &= p_t d_t - h_{t+1}(Y_t - d_t) + J_{t+1}(\overline{Y}_t - d_t, \varepsilon_{t+1}) \\ V_t^{'} &= p_t d_t - h_{t+1}(Y_t - d_t) + J_{t+1}(\overline{Y}_t - d_t, \varepsilon_{t+1}) = V_t, \end{aligned}$$

- When $\overline{Y}_t - S_t \leq d_t < \overline{Y}_t - S_t + B_t$

$$\begin{aligned} V_t &= p_t d_t - h_{t+1} S_t - \beta_t (d_t - (\overline{Y}_t - S_t)) + J_{t+1}(\overline{Y}_t - d_t, \varepsilon_{t+1}) \\ V_t^{'} &= p_t d_t - h_{t+1}(S_t - 1) - \beta_t (d_t - \overline{Y}_t + (S_t - 1)) + J_{t+1}(\overline{Y}_t - d_t, \varepsilon_{t+1}) \\ &= V_t + h_{t+1} + \beta_t > V_t, \end{aligned}$$

- When $d_t \geq \overline{Y}_t - S_t + B_t$

$$\begin{aligned} V_t &= p_t(\overline{Y}_t - S_t + B_t) - h_{t+1} S_t - \ell_t (d_t - \overline{Y}_t + S_t - B_t) - \beta_t B_t + J_{t+1}(S_t - B_t, \varepsilon_{t+1}) \\ V_t^{'} &= p_t(\overline{Y}_t - (S_t - 1) + B_t) - h_{t+1}(S_t - 1) - \ell_t (d_t - \overline{Y}_t + (S_t - 1) - B_t) - \beta_t B_t \\ &+ J_{t+1}(S_t - 1 - B_t, \varepsilon_{t+1}) \\ &= V_t + (p_t + h_{t+1} + \ell_t) - \Big[J_{t+1}(S_t - B_t, \varepsilon_{t+1}) - J_{t+1}(S_t - B_t - 1, \varepsilon_{t+1}) \Big] > V_t. \end{aligned}$$

Case 2: Since current net revenue from backlogging is smaller than future marginal expected profit, we do not have a motivation to backorder items to sell the future capacity in period t, thus the alternate policy is chosen to be backordering one less item. We can represent the alternate policy as $(S_t,\ B_t - 1)$.

- When $d_t < \overline{Y}_t - S_t$

$$\begin{aligned} V_t &= p_t d_t - h_{t+1}(Y_t - d_t) + J_{t+1}(Y_t - d_t, \varepsilon_{t+1}) \\ V_t^{'} &= p_t d_t - h_{t+1}(Y_t - d_t) + J_{t+1}(Y_t - d_t, \varepsilon_{t+1}) = V_t, \end{aligned}$$

- When $\overline{Y}_t - S_t \leq d_t < \overline{Y}_t - S_t + B_t$

$$\begin{aligned} V_t &= p_t d_t - h_{t+1} S_t - \beta_t (d_t - (\overline{Y}_t - S_t)) + J_{t+1}(\overline{Y}_t - d_t, \varepsilon_{t+1}) \\ V_t^{'} &= p_t d_t - h_{t+1} S_t - \beta_t (d_t - (\overline{Y}_t - S_t)) + J_{t+1}(\overline{Y}_t - d_t, \varepsilon_{t+1}) = V_t, \end{aligned}$$

- When $d_t \geq \overline{Y}_t - S_t + B_t$

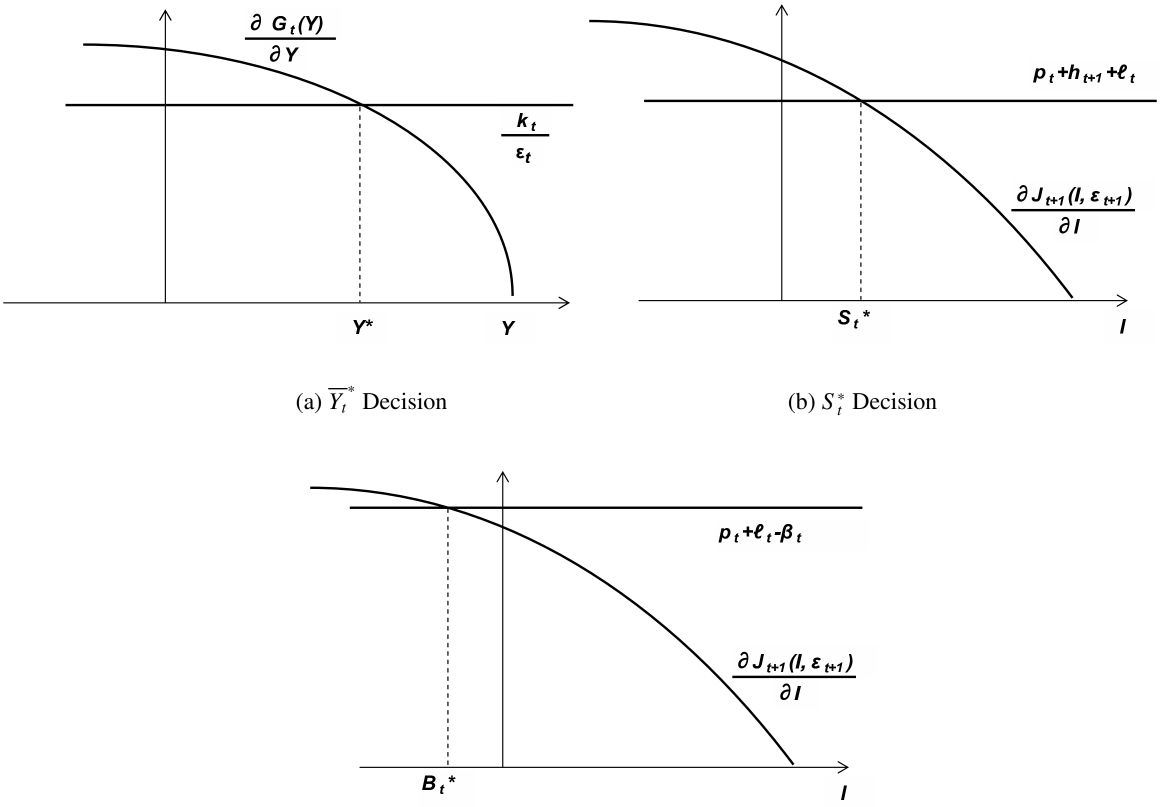
$$\begin{aligned} V_t &= p_t(\overline{Y}_t - S_t + B_t) - h_{t+1} S_t - \ell_t(d_t - \overline{Y}_t + S_t - B_t) - \beta_t B_t + J_{t+1}(S_t - B_t, \varepsilon_{t+1}) \\ V_t^{'} &= p_t(\overline{Y}_t - S_t + (B_t - 1)) - h_{t+1} S_t - \ell_t(d_t - \overline{Y}_t + S_t - (B_t - 1)) - \beta_t(B_t - 1) \\ &+ J_{t+1}(S_t - (B_t - 1), \varepsilon_{t+1}) \\ &= V_t + \Big[J_{t+1}(S_t - B_t + 1, \varepsilon_{t+1}) - J_{t+1}(S_t - B_t, \varepsilon_{t+1})\Big] - (p_t + \ell_t - \beta_t) > V_t. \end{aligned}$$

For all cases we have the same or better performance from the alternate policy. Then the policy with both decisions positive can not be optimal, at least one of them must be zero. ■

By Lemma 3.1.1, the problem can be simplified into two candidate policies, $G_t^S(\overline{Y_t})$ and $G_t^B(\overline{Y_t})$, where, $G_t^S(\overline{Y_t})$ is the "save-inventory" policy with positive S_t value and $G_t^B(\overline{Y_t})$ is the "backlog-demand" policy with positive B_t value. Then the base problem can be rewritten as

$$G_t(\overline{Y_t}) = \max\{G_t^S(\overline{Y_t}), G_t^B(\overline{Y_t})\}.$$

This simplification means that the better policy among those two policies will be chosen in each period. The result that is proven formally is also true intuitively. It is better to choose the save-inventory policy in an environment where the marginal expected profit from selling each reserved item in the following period is greater than the net revenue from selling each item in the current period. For example, in a system with prices increasing by periods save inventory policy is better. Because selling an item in future is more profitable. On the other hand, when the net revenue of backlogging an item is greater than the marginal expected profit from selling an item in the following period, the backlog-demand policy is the best one to choose. A system with prices decreasing by periods would be a good candidate for this situation.



(a) $\overline{Y_t}^*$ Decision (b) S_t^* Decision

(c) B_t^* Decision

Figure 3.1: Optimal Unconstrained Decisions

In Sections 3.2 and 3.3, we will prove some structural (e.g. concave, non-increasing, etc.) properties of the profit-to-go functions. The decision variables $\overline{Y_t}$, S_t and B_t are all constrained variables and they are limited by parameters and state variables of period t as seen in Equations (3.1) and (3.2). But if we let $\overline{Y_t}^*$, S_t^* and B_t^* to be the unconstrained optimal decisions and define them as;

$$
\begin{array}{lll}
\overline{Y_t}^* & = \max\{\overline{Y} : \frac{k_t}{\varepsilon_t} \leq \frac{\partial G_t(\overline{Y})}{\partial \overline{Y}}\} & if\, \frac{k_t}{\varepsilon_t} \leq \frac{\partial G_t(0)}{\partial \overline{Y}}, \\
S_t^* & = \max\{I : p_t + h_{t+1} + \ell_t \leq \frac{\partial J_{t+1}(I,\varepsilon_{t+1})}{\partial I}\} & if\; p_t + \ell_t + h_{t+1} < \frac{\partial J_{t+1}(0,\varepsilon_{t+1})}{\partial I}, \\
B_t^* & = \min\{I : p_t + \ell_t - \beta_t \leq \frac{\partial J_{t+1}(-I,\varepsilon_{t+1})}{\partial I}\} & if\; p_t + \ell_t - \beta_t > \frac{\partial J_{t+1}(0,\varepsilon_{t+1})}{\partial I}
\end{array}
$$

where the decisions are zero if the corresponding conditions are not satisfied, we will be able to prove the desired structural properties of the policies. Unconstrained optimal decisions and how they are determined are illustrated in Figure 3.1.

Note that the choice of $\overline{Y_t}^*$ decision can easily be verified by taking derivative of unconstrained Equation (3.1) with respect to $\overline{Y_t}$. However the verification for S_t^* and B_t^* decisions are not trivial. But we can observe that when $p_t + \ell_t + h_{t+1}$ (net revenue from selling out of inventory) crosses the marginal expected profit curve in period $t + 1$ ($\frac{\partial J_{t+1}(I,\varepsilon_{t+1})}{\partial I}$) at a positive I value, we have a positive S_t^* value but since $p_t + \ell_t + h_{t+1} > p_t + \ell_t - \beta_t$, we have $B_t^* = 0$. Similarly, when $p_t + \ell_t - \beta_t$ crosses the the marginal expected profit curve in period $t + 1$ at a negative I value, we have a positive B_t^* decision but a zero S_t^* value. These relations comply with Lemma 3.1.1 and verify the choice of S_t^* and B^* decisions. Now we can discuss the two models representing these two candidate policies.

3.2 Save-Inventory Policy

For the save-inventory policy, the optimal expected profit can be written as;

$$
\begin{aligned}
G_t^S(\overline{Y_t}) \;=\;& \max_{S_t : 0 \leq S_t \leq \overline{Y_t}} \Big\{ \int [p_t(\min(d_t, \overline{Y_t} - S_t)) - h_{t+1}(S_t + \max(0, \overline{Y_t} - S_t - d_t)) \\
-\;& \ell_t \max(0, [d_t - \overline{Y_t} - S_t]) + J_{t+1}(S_t + \max(0, \overline{Y_t} - S_t - d_t), \varepsilon_{t+1})] d\Psi_t(d_t) \Big\} \qquad (3.3) \\
=\;& \int_0^{\overline{Y_t} - S_t} p_t x d\Psi_t(x) + \int_{\overline{Y_t} - S_t}^{\infty} p_t(\overline{Y_t} - S_t) d\Psi_t(x) - \ell_t \int_{\overline{Y_t} - S_t}^{\infty} (x - (\overline{Y_t} - S_t)) d\Psi_t(x) \\
+\;& \int_0^{\overline{Y_t} - S_t} [J_{t+1}(\overline{Y_t} - x, \varepsilon_{t+1}) - (\overline{Y_t} - x) h_{t+1}] d\Psi_t(x) + \int_{\overline{Y_t} - S_t}^{\infty} [J_{t+1}(S_t, \varepsilon_{t+1}) - S_t h_{t+1}] d\Psi_t(x).
\end{aligned}
$$

In Equation (3.3) the profit-to-go function is maximized over the reserve decision. It is a simpler version of Equation (3.2) since there is only one decision to be considered in this equation. The constraint ensures that reserve decision can not be more than the inventory level after the production realization and can not take a negative value. The first term in the equation is the revenue gained from sold items. The producer reserves S_t items for future periods, and makes the remaining items on hand $(\overline{Y_t} - S_t)$ available to the customers in period t. Obviously the producer can not sell more than the current demand. The second term is the inventory holding cost to be paid for reserved items and items that are not sold. Next term is the lost sale cost from unsatisfied demand. Last term is the maximum expected profit in future periods with the available inventory. This term is replaced with salvage revenue in the last period.

It is important to note that we start period t with I_t units and then decide $\overline{Y_t}$, which is limited by $I_t + \varepsilon_t q_t$. Therefore G_t^S is independent of the ε_t. Next we will show that the profit-to-go functions are concave and the optimal decisions are not dependent on the current period's efficiency parameter.

Theorem 3.2.1 *For the profit-to-go functions,*

- $\frac{\partial G_t^S(\overline{Y_t})}{\partial \overline{Y_t}}$ *is non-increasing in* $\overline{Y}_t$ *and* $G_t^S(\overline{Y_t})$ *is concave in* $\overline{Y}_t$*,*
- $\frac{\partial J_t(I_t,\varepsilon_t)}{\partial I_t}$ *is non-increasing in* I_t *and* $J_t(I_t,\varepsilon_t)$ *is concave in* I_t*,*
- *for any* $(\varepsilon_t^1, \varepsilon_t^2)$ *pair;* $\overline{Y}_t^{*\varepsilon_t^1} = \overline{Y}_t^{*\varepsilon_t^2}$ $(\frac{\partial^2 G_t^S(I_t)}{\partial I_t \partial \varepsilon_t} = 0)$,
- *for any* $(\varepsilon_t^1, \varepsilon_t^2)$ *pair;* $S_t^{*\varepsilon_t^1} = S_t^{*\varepsilon_t^2}$ $(\frac{\partial^2 J_{t+1}(I_{t+1},\varepsilon_{t+1})}{\partial I_{t+1}\partial \varepsilon_t} = 0)$.

Proof. We use induction for this proof. We first focus on $G_t^S(\overline{Y_t})$ and start with period T. Obviously $S_T = 0$, and $J_{T+1}(\cdot,\cdot) = 0$ since there is not any meaningful decision in last period. We see from Equation 3.3 that,

$$G_T^S(\overline{Y_T}) = \int_0^{\overline{Y_T}} p_T x d\Psi_T(x) + \int_{\overline{Y_T}}^{\infty} p_T(\overline{Y_T}) d\Psi_T(x) + \upsilon \int_0^{\overline{Y_T}} (\overline{Y_T} - x) d\Psi_T(x) - \ell_T \int_{\overline{Y_T}}^{\infty} (x - \overline{Y_T}) d\Psi_T(x)$$

where υ is the salvage value per item at the end of the horizon. Using Leibnitz's rule we obtain,

$$\begin{aligned} \frac{\partial G_T^S(\overline{Y_T})}{\partial \overline{Y_T}} &= p_T(1 - \Psi_T(\overline{Y_T})) + \ell_T(1 - \Psi_T(\overline{Y_T})) + \upsilon \Psi_T(\overline{Y_T}), \\ \frac{\partial^2 G_T^S(\overline{Y_T})}{\partial \overline{Y}_T^2} &= (\upsilon - \ell_T - p_T)\psi_T(\overline{Y_T}) \end{aligned}$$

Since by the assumption $p_T + \ell_T > \upsilon$, $\frac{\partial^2 G_T^S(\overline{Y_T})}{\partial \overline{Y}_T^2} \leq 0$ Therefore $G_T^S(\overline{Y_T})$ is concave in $\overline{Y}_T$ and $\frac{\partial G_T^S(\overline{Y_T})}{\partial \overline{Y_T}}$ is non-increasing in $\overline{Y}_T$.

Now we show that $J_T(I_T, \varepsilon_T)$ is concave. Define $\overline{Y}_T^*$ as 0 if $\frac{k_T}{\varepsilon_T} > \frac{\partial G_T^S(0)}{\partial \overline{Y_T}}$, otherwise as $\max\Big\{\overline{Y} : \frac{k_T}{\varepsilon_T} \leq \frac{\partial G_T^S(\overline{Y})}{\partial \overline{Y}}\Big\}$. Because $G_T^S(\overline{Y_T})$ is concave in $\overline{Y}_T$ and by the choice of $\overline{Y}_T^*$, $\overline{Y}_T^*$ maximizes $J_T(I_T, \varepsilon_T)$ with production realization of $\max\{\varepsilon_T q_T, (\overline{Y}_T^* - I_T)\}$. Hence,

$$J_T(I_T, \varepsilon_T) = \begin{cases} -k_T q_T + G_T^S(I_T + \varepsilon_T q_T) & if\ I_T \leq \overline{Y}_T^* - \varepsilon_T q_T \\ -k_T(\overline{Y}_T^* - I_T) + G_T^S(\overline{Y}_T^*) & if\ \overline{Y}_T^* - \varepsilon_T q_T < I_T \leq \overline{Y}_T^* \\ G_T^S(I_T) & if\ \overline{Y}_T^* < I_T. \end{cases}$$

Therefore,

$$\frac{\partial J_T(I_T, \varepsilon_T)}{\partial I_T} = \begin{cases} \frac{\partial G_T^S(I_T + \varepsilon_T q_T)}{\partial I_T} \geq \frac{k_T}{\varepsilon_T} & if\ I_T \leq \overline{Y}_T^* - \varepsilon_T q_T \\ \frac{k_T}{\varepsilon_T} & if\ \overline{Y}_T^* - \varepsilon_T q_T < I_T \leq \overline{Y}_T^* \\ \frac{\partial G_T^S(I_T)}{\partial I_T} \leq \frac{k_T}{\varepsilon_T} & if\ \overline{Y}_T^* < I_T. \end{cases} \tag{3.4}$$

The two inequalities in Equation 3.4 hold due to the choice of $\overline{Y}_T^*$ decision. Since $\frac{\partial G_T^S(\overline{Y_T})}{\partial \overline{Y_T}}$ is non-increasing in $\overline{Y}_T$, $\frac{\partial J_T(I_T, \varepsilon_T)}{\partial I_T}$ is also non-increasing in I_T and $J_T(I_T, \varepsilon_T)$ is concave in I_T. It is seen that $\frac{\partial J_T(I_T, \varepsilon_T)}{\partial I_T}$ is biggest when I_T is small ($\geq \frac{k_T}{\varepsilon_T}$) then decreases as I_T increases. Thus, $\frac{\partial J_T(I_T, \varepsilon_T)}{\partial I_T}$ is non-increasing and $J_T(I_T, \varepsilon_T)$ is concave for all I_T values.

Also noting that $\frac{\partial G_T^S(\overline{Y_T})}{\partial \overline{Y_T}}$ is independent of ε_T, for any $(\varepsilon_T^1, \varepsilon_T^2)$ pair, we have $\overline{Y}_T^{*\varepsilon_T^1} = \overline{Y}_T^{*\varepsilon_T^2}$, and $S_T^{*\varepsilon_T^1} = S_T^{*\varepsilon_T^2} = 0$.

Now assume that $\frac{\partial G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is non-increasing and $J_T(I_T, \varepsilon_T)$ is concave for period $t = t...T$, we show that $G_{t-1}^S(\overline{Y}_{t-1})$ is non-increasing. We have two $G_{t-1}^S(\overline{Y}_{t-1})$ equations for the following cases,

- Case 1: When $\overline{Y}_{t-1} \leq S_{t-1}^{*\varepsilon_{t-1}}$:

$$\begin{aligned} G_{t-1}^S(\overline{Y}_{t-1}) &= [J_t(\overline{Y}_{t-1}, \varepsilon_t) - \overline{Y}_{t-1} h_t] - \ell_{t-1} \int_0^\infty x d\Psi_{t-1}(x) \\ \frac{\partial G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}} &= \frac{\partial J_t(\overline{Y}_{t-1}, \varepsilon_t)}{\partial \overline{Y}_{t-1}} - h_t \\ \frac{\partial^2 G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}^2} &= \frac{\partial^2 J_t(\overline{Y}_{t-1}, \varepsilon_t)}{\partial \overline{Y}_{t-1}^2} \leq 0 \qquad (\textit{due to induction hypothesis}) \end{aligned}$$

- Case 2: When $\overline{Y}_{t-1} > S_{t-1}^{*\varepsilon_{t-1}}$:

$$\frac{\partial G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}} = \int_{\overline{Y}_{t-1}-S_{t-1}^*}^{\infty} (p_{t-1} + \ell_{t-1}) d\Psi_{t-1}(x) + \int_0^{\overline{Y}_{t-1}-S_{t-1}^*} [\frac{\partial J_t(\overline{Y}_{t-1} - x, \varepsilon_t)}{\partial \overline{Y}_{t-1}} - h_t] d\Psi_{t-1}(x)$$

$$\begin{aligned}\frac{\partial^2 G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}^2} &= [\frac{\partial J_t(S_{t-1}^*, \varepsilon_t)}{\partial \overline{Y}_{t-1}} - (p_{t-1} + h_t + \ell_{t-1})]\psi_{t-1}(\overline{Y}_{t-1} - S_{t-1}^*) \\ &+ \int_0^{\overline{Y}_{t-1}-S_{t-1}^*} \frac{\partial^2 J_t(\overline{Y}_{t-1} - x, \varepsilon_t)}{\partial \overline{Y}_{t-1}^2} d\Psi_t(x) \leq 0\end{aligned}$$

Inequality is due to the induction hypothesis and the choice of $S_{t-1}^{*\varepsilon_{t-1}}$ decision; which is 0 if $p_{t-1} + h_t + \ell_{t-1} > \frac{\partial J_t(0,\varepsilon_t)}{\partial I}$, otherwise is equal to $\max\{I : p_{t-1} + h_t + \ell_{t-1} \leq \frac{\partial J_t(I,\varepsilon_t)}{\partial I}\}$. Thus, $\frac{\partial G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is non-increasing in $\overline{Y}_{t-1}$, and $G_{t-1}^S(\overline{Y}_{t-1})$ is concave in $\overline{Y}_{t-1}$.

Now define $\overline{Y}_{t-1}^*$ as 0 if $\frac{k_{t-1}}{\varepsilon_{t-1}} > \frac{\partial G_{t-1}^S(0)}{\partial \overline{Y}_{t-1}}$, otherwise as $\max\left\{\overline{Y} : \frac{k_{t-1}}{\varepsilon_{t-1}} \leq \frac{\partial G_{t-1}^S(\overline{Y})}{\partial \overline{Y}}\right\}$. Because $G_{t-1}^S(\overline{Y}_{t-1})$ is concave in $\overline{Y}_{t-1}$ and by the choice of $\overline{Y}_{t-1}^*$, $\overline{Y}_{t-1}^*$ maximizes $J_{t-1}(I_{t-1}, \varepsilon_{t-1})$ with production realization of $\max\{\varepsilon_{t-1}q_{t-1}, (\overline{Y}_{t-1}^* - I_{t-1})\}$. Hence,

$$\frac{\partial J_{t-1}(I_{t-1}, \varepsilon_{t-1})}{\partial I_{t-1}} = \begin{cases} \frac{\partial G_{t-1}^S(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}} \geq \frac{k_{t-1}}{\varepsilon_{t-1}} & if\ I_{t-1} \leq \overline{Y}_{t-1}^* - \varepsilon_{t-1}q_{t-1} \\ \frac{k_{t-1}}{\varepsilon_{t-1}} & if\ \overline{Y}_{t-1}^* - \varepsilon_{t-1}q_{t-1} < I_{t-1} \leq \overline{Y}_{t-1}^* \\ \frac{\partial G_{t-1}^S(I_{t-1})}{\partial I_{t-1}} \leq \frac{k_{t-1}}{\varepsilon_{t-1}} & if\ \overline{Y}_{t-1}^* < I_{t-1}. \end{cases} \tag{3.5}$$

The two inequalities in Equation 3.5 hold due to the choice of $\overline{Y}_{t-1}^*$ decision. Since $\frac{\partial G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is non-increasing in $\overline{Y}_{t-1}$, $\frac{\partial J_{t-1}(I_{t-1},\varepsilon_{t-1})}{\partial I_{t-1}}$ is also non-increasing in I_{t-1} and $J_{t-1}(I_{t-1}, \varepsilon_{t-1})$ is concave in I_{t-1}. This is true for all I_{t-1} values as mentioned before.

Again, we see that $\frac{\partial G_{t-1}^S(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is independent of ε_{t-1}, for any $(\varepsilon_{t-1}^1, \varepsilon_{t-1}^2)$ pair, we have $\overline{Y}_{t-1}^{*\varepsilon_{t-1}^1} = \overline{Y}_{t-1}^{*\varepsilon_{t-1}^2}$.

Similarly, to see whether $S_{t-1}^{*\varepsilon_{t-1}^1} = S_{t-1}^{*\varepsilon_{t-1}^2}$ or not, we need to check if $\frac{\partial J_t(I_t,\varepsilon_t)}{\partial I_t}$ depends on ε_{t-1} for any I_t. We have

$$\frac{\partial^2 J_t(I_t, \varepsilon_t)}{\partial I_t \partial \varepsilon_{t-1}} = \begin{cases} \frac{\partial^2 G_t^S(I_t+\varepsilon_t q_t)}{\partial I_t \partial \varepsilon_{t-1}} = 0 & if\ I_t + \varepsilon_t q_t \leq \overline{Y}_t^* \\ 0 & if\ \overline{Y}_t^* < I_t + \varepsilon_t q_t, \end{cases}$$

since for period t, we have,

- Case 1: When $I_t + \varepsilon_t q_t \leq S_t^{*\varepsilon_t}$:

$$
\begin{aligned}
G_t^S(I_t + \varepsilon_t q_t) &= [J_{t+1}(I_t + \varepsilon_t q_t, \varepsilon_{t+1}) - (I_t + \varepsilon_t q_t)h_{t+1}] - \ell_t \int_0^\infty x d\Psi_t(x) \\
\frac{\partial G_t^S(I_t + \varepsilon_t q_t)}{\partial I_t} &= \frac{\partial J_{t+1}(I_t + \varepsilon_t q_t, \varepsilon_{t+1})}{\partial I_t} - h_{t+1} \\
\frac{\partial^2 G_t^S(I_t + \varepsilon_t q_t)}{\partial I_t \partial \varepsilon_{t-1}} &= 0
\end{aligned}
$$

- Case 2: When $I_t + \varepsilon_t q_t > S_t^{*\varepsilon_t}$:

$$
\begin{aligned}
\frac{\partial G_t^S(I_t + \varepsilon_t q_t)}{\partial I_t} &= \int_{I_t+\varepsilon_t q_t - S_t^*}^{\infty} (p_t + \ell_t) d\Psi_t(x) \\
&+ \int_0^{I_t+\varepsilon_t q_t - S_t^*} [\frac{\partial J_{t+1}(I_t + \varepsilon_t q_t - x, \varepsilon_{t+1})}{\partial I_t} - h_{t+1}] d\Psi_t(x) \\
\frac{\partial^2 G_t^S(I_t + \varepsilon_t q_t)}{\partial I_t \partial \varepsilon_{t-1}} &= 0
\end{aligned}
$$

Thus $\frac{\partial J_t(I_t,\varepsilon_t)}{\partial I_t}$ does not depend on ε_{t-1} and $S_{t-1}^{*\varepsilon_{t-1}^1} = S_{t-1}^{*\varepsilon_{t-1}^2}$ ■

We show in Theorem 3.2.1 that the optimal unconstrained reserve and production decisions in Save-Inventory policy are independent of the current period's efficiency parameter and current demand. It is intuitively true, because the efficiency parameter is known at the beginning of the period and the producer takes this value into consideration. The producer would change the production amount to compensate the change in the efficiency parameter. Hence, as we assumed, the producer can decide the optimal policy at the beginning of each period before demand and production realization. Theorem 3.2.1 implies the optimal policy for the Save-Inventory policy; thus we have the following corollary.

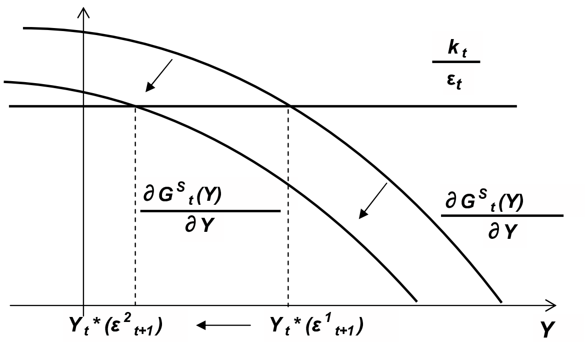
Corollary 3.2.2 *Given a vector of prices, there exists an optimal modified base stock policy for the Save-Inventory policy with an optimal reserve-up-to level (S_t^*) and optimal produce-up-to level ($\overline{Y}_t^*$).*

In an optimal policy, the realized production level ($\overline{Y}_t$) and reserve (S_t) decisions may not be equal to the optimal produce-up-to ($\overline{Y}_t^*$) and the optimal reserve-up-to (S_t^*) levels. Capacity and efficiency parameter may limit the production and if it is not possible to produce up to ($\overline{Y}_t^*$) then the producer should produce as much as possible. Similarly the (S_t) decision is limited by ($\overline{Y}_t$). If there is not enough inventory on hand to reserve (S_t^*) units then as much units as possible should be reserved. Next, we discuss how decisions in period t changes with ε_{t+1}.

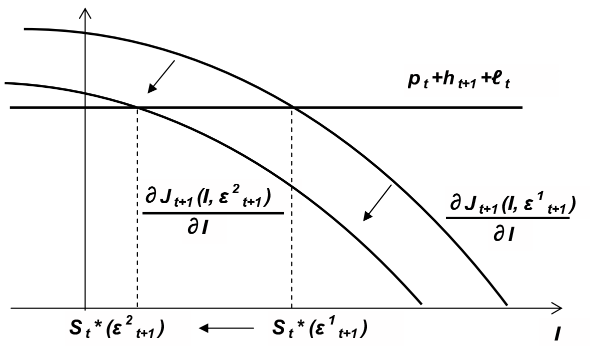
Theorem 3.2.3 *Under the save-inventory policy*

- $\frac{\partial^2 G_t^S(\overline{Y}_t)}{\partial \overline{Y}_t \partial \varepsilon_{t+1}} \leq 0\ \forall \overline{Y}_t$ *and* $\overline{Y}_t^*$ *decision is non-increasing in* ε_{t+1}.
- $\frac{\partial^2 J_{t+1}(I_{t+1}, \varepsilon_{t+1})}{\partial I_{t+1} \partial \varepsilon_{t+1}} \leq 0\ \forall I_{t+1}$ *and* $\overline{S}_t^*$ *decision is non-increasing in* ε_{t+1}.

Proof. We will show that $\frac{\partial G_t^S(\overline{Y}_t)}{\partial \overline{Y}_t}$ and $\frac{\partial J_{t+1}(I_{t+1}, \varepsilon_{t+1})}{\partial I_{t+1}}$ curves are never shifting upwards (for any I_{t+1} or $\overline{Y}_t$ value, respectively) as ε_{t+1} increases. Figure 3.2 depicts that when this is the case, both $\overline{Y}_t^*$ and $\overline{S}_t^*$ decisions are non-increasing in ε_{t+1}.



(a) Change in $\overline{Y}_t^*$ Decision



(b) Change in S_t^* Decision

Figure 3.2: ε_{t+1} effect on decisions in period t under Save-Inventory Policy

We use induction for this proof. First we focus on $G_T^S(I_T + \varepsilon_T q_T)$ and period $T-1$

$$
\begin{aligned}
G_T^S(I_T + \varepsilon_T q_T) \quad &= \int_0^{I_T+\varepsilon_T q_T} p_T x d\Psi_T(x) + \int_{I_T+\varepsilon_T q_T}^{\infty} p_T(I_T + \varepsilon_T q_T) d\Psi_T(x) \\
&+ \quad \upsilon \int_0^{I_T+\varepsilon_T q_T} (I_T + \varepsilon_T q_T - x) d\Psi_T(x) - \ell_T \int_{I_T+\varepsilon_T q_T}^{\infty} (x - I_T - \varepsilon_T q_T) d\Psi_T(x)
\end{aligned}
$$

then we obtain,

$$
\begin{aligned}
\frac{\partial G_T^S(I_T + \varepsilon_T q_T)}{\partial I_T} \quad &= \quad (p_T + \ell_T)(1 - \Psi_T(I_T + \varepsilon_T q_T)) + \upsilon \Psi_T(I_T + \varepsilon_T q_T) \\
\frac{\partial^2 G_T^S(I_T + \varepsilon_T q_T)}{\partial I_T \partial \varepsilon_T} \quad &= \quad q_T(\upsilon - p_T - \ell_T)\psi_T(I_T + q_T \varepsilon_T) < 0 \quad \forall I_T. \ (*)
\end{aligned}
$$

The inequality is true since $p_T + \ell_T > \upsilon$ by assumption. Then,

$$\frac{\partial^2 J_T(I_T,\varepsilon_T)}{\partial I_T \partial \varepsilon_T} = \begin{cases} \frac{\partial^2 G_T^S(I_T+\varepsilon_T q_T)}{\partial I_T \partial \varepsilon_T} < 0 \quad (from\ *) & if\ I_T \leq \overline{Y}_T^* - \varepsilon_T q_T \\ -\frac{k_T}{\varepsilon_T^2} < 0 & if\ \overline{Y}_T^* - \varepsilon_T q_T < I_T \leq \overline{Y}_T^* \quad (**) \\ \frac{\partial^2 G_T^S(I_T)}{\partial I_T \partial \varepsilon_T} = 0 & if\ \overline{Y}_T^* < I_T. \end{cases}$$

Therefore, as I_T increases, the reserve-up-to decision $S_{T-1}^{*\varepsilon_{T-1}}$ first decreases (where $0 \leq I_T \leq \overline{Y}_T^*$) and then remains unchanged. Moreover, there are two cases considered for the $G_{T-1}^S(\overline{Y}_{T-1})$ function. We present derivatives of the function below;

- Case 1: When $\overline{Y}_{T-1} \leq S_{T-1}^{*\varepsilon_{T-1}}$:

$$\begin{aligned} G_{T-1}^S(\overline{Y}_{T-1}) &= [J_T(\overline{Y}_{T-1},\varepsilon_T) - \overline{Y}_{T-1}h_T] - \ell_{T-1}\int_0^\infty x d\Psi_{T-1}(x) \\ \frac{\partial G_{T-1}^S(\overline{Y}_{T-1})}{\partial \overline{Y}_{t-1}} &= \frac{\partial J_T(\overline{Y}_{T-1},\varepsilon_T)}{\partial \overline{Y}_{T-1}} - h_T \\ \frac{\partial^2 G_{T-1}^S(\overline{Y}_{T-1})}{\partial \overline{Y}_{T-1}\partial \varepsilon_T} &= \frac{\partial^2 J_T(\overline{Y}_{T-1},\varepsilon_T)}{\partial \overline{Y}_{T-1}\partial \varepsilon_T} \leq 0 \ \ due\ to\ (**). \end{aligned}$$

- Case 2: When $\overline{Y}_{T-1} > S_{T-1}^{*\varepsilon_{T-1}}$:

$$\begin{aligned} \frac{\partial G_{T-1}^S(\overline{Y}_{T-1})}{\partial \overline{Y}_{T-1}} &= \int_{\overline{Y}_{T-1}-S_{T-1}^*}^{\infty} (p_{T-1}+\ell_{T-1}) d\Psi_{T-1}(x) \\ &+ \int_0^{\overline{Y}_{T-1}-S_{T-1}^*} [\frac{\partial J_T(\overline{Y}_{T-1}-x,\varepsilon_T)}{\partial \overline{Y}_{T-1}} - h_T] d\Psi_{T-1}(x) \\ \frac{\partial^2 G_{T-1}^S(\overline{Y}_{T-1})}{\partial \overline{Y}_{T-1}\partial \varepsilon_T} &= [(p_{T-1}+h_T+\ell_{T-1}) - \frac{\partial J_T(S_{T-1}^*,\varepsilon_T)}{\partial \overline{Y}_{T-1}}]\psi_{T-1}(\overline{Y}_{T-1}-S_{T-1}^*)\frac{\partial S_{T-1}^*}{\partial \varepsilon_T} \\ &+ \int_0^{\overline{Y}_{T-1}-S_{T-1}^*} \frac{\partial^2 J_t(\overline{Y}_{T-1}-x,\varepsilon_T)}{\partial \overline{Y}_{T-1}\partial \varepsilon_T} d\Psi_T(x) \leq 0 \ \ due\ to\ (**)\ and\ S_{T-1}^*\ decision. \end{aligned}$$

Now for periods up to $t-1$, assume,

$$\frac{\partial^2 G_t^S(I_t+\varepsilon_t q_t)}{\partial I_t \partial \varepsilon_t} < 0 \quad \forall I_t.\ (***)$$

Then we have,

$$\frac{\partial^2 J_t(I_t,\varepsilon_t)}{\partial I_t \partial \varepsilon_t} = \begin{cases} \frac{\partial^2 G_t^S(I_t+\varepsilon_t q_t)}{\partial I_t \partial \varepsilon_t} < 0 \quad (from\ \ ***) & if\ I_t \leq \overline{Y}_t^* - \varepsilon_t q_t \\ -\frac{k_t}{\varepsilon_t} < 0 & if\ \overline{Y}_t^* - \varepsilon_t q_t < I_t \leq \overline{Y}_t^* \quad (****) \\ \frac{\partial^2 G_t^S(I_t)}{\partial I_t \partial \varepsilon_t} = 0 & if\ \overline{Y}_t^* < I_t. \end{cases}$$

Moreover,

$$\frac{\partial^2 G^S_{t-1}(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}\partial\varepsilon_t} = \begin{cases} \frac{\partial^2 J_t(\overline{Y}_{t-1},\varepsilon_t)}{\partial \overline{Y}_{t-1}\partial\varepsilon_t} \leq 0 & if\ \overline{Y}_{t-1} \leq S^*_{t-1} \\ \int\limits_0^{\overline{Y}_{t-1}-S^*_{t-1}} \frac{\partial^2 J_t(\overline{Y}_{t-1}-x,\varepsilon_t)}{\partial \overline{Y}_{t-1}\partial\varepsilon_t} \leq 0 & otherwise \end{cases} \qquad due\ to\ (****).$$

To complete the proof it is enough to check whether

$$\frac{\partial^2 G^S_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial\varepsilon_{t-1}} < 0 \quad \forall I_{t-1}$$

holds for period $t-2$ or not. We have two cases;

- Case 1: When $I_{t-1}+\varepsilon_{t-1}q_{t-1} \leq S^{*\varepsilon_{t-1}}_{t-1}$:

$$G^S_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1}) = [J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t) - (I_{t-1}+\varepsilon_{t-1}q_{t-1})h_t] - \ell_{t-1}\int\limits_0^\infty x d\Psi_{t-1}(x)$$

then we obtain,

$$\begin{aligned} \frac{\partial G^S_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}} &= \frac{\partial J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t)}{\partial I_{t-1}} - h_t \\ \frac{\partial^2 G^S_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial\varepsilon_{t-1}} &= \frac{\partial^2 J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t)}{\partial I_{t-1}\partial\varepsilon_{t-1}}, \end{aligned}$$

where

$$J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t) = \begin{cases} k_t q_t + G^S_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}+\varepsilon_t q_t) & if\ I_{t-1}+\varepsilon_{t-1}q_{t-1} \leq \overline{Y}^*_t - \varepsilon_t q_t \\ -\frac{k_t}{\varepsilon_t}(\overline{Y}^*_t - I_{t-1} - \varepsilon_{t-1}q_{t-1}) + G^S_t(\overline{Y}^*_t) & otherwise \\ G^S_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}) & if\ \overline{Y}^*_t < I_{t-1}+\varepsilon_{t-1}q_{t-1} \end{cases}$$

and,

$$\frac{\partial^2 J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t)}{\partial I_{t-1}\partial\varepsilon_{t-1}} = \begin{cases} \frac{\partial^2 G^S_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}+\varepsilon_t q_t)}{\partial I_{t-1}\partial\varepsilon_{t-1}} < 0 & if\ I_{t-1}+\varepsilon_{t-1}q_{t-1} \leq \overline{Y}^*_t - \varepsilon_t q_t \\ 0 & otherwise \\ \frac{\partial^2 G^S_t(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial\varepsilon_{t-1}} < 0 & if\ \overline{Y}^*_t < I_{t-1}+\varepsilon_{t-1}q_{t-1} \end{cases} \qquad from\ (***).$$

- Case 2: When $I_{t-1}+\varepsilon_{t-1}q_{t-1} > S^{*\varepsilon_{t-1}}_{t-1}$:

$$\begin{aligned} \frac{\partial G^S_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}} &= \int\limits_{I_{t-1}+\varepsilon_{t-1}q_{t-1}-S^*_{t-1}}^{\infty} (p_{t-1}+\ell_{t-1})d\Psi_{t-1}(x) \\ &+ \int\limits_0^{I_{t-1}+\varepsilon_{t-1}q_{t-1}-S^*_{t-1}} [\frac{\partial J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}-x,\varepsilon_t)}{\partial I_{t-1}} - h_t]d\Psi_{t-1}(x) \end{aligned}$$

$$
\begin{aligned}
&\frac{\partial^2 G^S_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial\varepsilon_{t-1}}\\
= \;& (q_{t-1}-\frac{\partial S^*_{t-1}}{\partial\varepsilon_t})\Big(\frac{\partial J_t(S^*_{t-1},\varepsilon_t)}{\partial I_{t-1}} - (p_{t-1}+\ell_{t-1}+h_t)\Big)\psi_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1}-S^*_{t-1})\\
+ \;& \int_0^{I_{t-1}+\varepsilon_{t-1}q_{t-1}-S^*_{t-1}} \frac{\partial^2 J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}-x,\varepsilon_t)}{\partial I_{t-1}\partial\varepsilon_{t-1}} d\Psi_{t-1}(x) \leq 0.
\end{aligned}
$$

Therefore it holds. ■

In this theorem we present that in Save-Inventory policy the current decisions are effected by the efficiency parameter of the future periods. Initial inventory of the following period is determined by current decisions and change in future periods effects the current decisions because of the demand uncertainty. As efficiency parameter of the future periods increases the producer can produce more in these periods and reserves less inventory to avoid holding cost and lost sale cost. Therefore, there would be a lower S^*_t value.

3.3 Backlog-Demand Policy

Next model to consider is the model with positive B_t value. In this policy reserve-up-to decision S_t is zero and there will be a positive backorder-up-to decision B_t. The profit-to-go function will be as the following;

$$
\begin{aligned}
G^B_t(\overline{Y_t}) = \;& \max_{B_t:0\leq B_t\leq\ \varepsilon_{t+1}q_{t+1}} \Big\{\int [p_t(\min(d_t,\overline{Y_t}+B_t)) - h_{t+1}(\overline{Y_t}-d_t)^+\\
- \;& \ell_t[d_t-(\overline{Y_t}+B_t)]^+ - \beta_t A_t + J_{t+1}((\overline{Y_t}-d_t)^+ - A_t,\varepsilon_{t+1})]d\Psi_t(d_t)\Big\} \qquad (3.6)\\
= \;& \int_0^{\overline{Y_t}+B^*_t} p_t x d\Psi_t(x) + \int_{\overline{Y_t}+B^*_t}^{\infty} p_t(\overline{Y_t}+B^*_t)d\Psi_t(x) - \ell_t \int_{\overline{Y_t}+B^*_t}^{\infty} (x-(\overline{Y_t}+B^*_t))d\Psi_t(x)\\
- \;& \beta_t \int_{\overline{Y_t}}^{\overline{Y_t}+B^*_t} (x-\overline{Y_t})d\Psi_t(x) - \beta_t \int_{\overline{Y_t}+B^*_t}^{\infty} B^*_t d\Psi_t(x) - h_t \int_0^{\overline{Y_t}} (\overline{Y_t}-x)d\Psi_t(x)\\
+ \;& \int_0^{\overline{Y_t}+B^*_t} J_{t+1}(\overline{Y_t}-x,\varepsilon_{t+1})d\Psi_t(x) + \int_{\overline{Y_t}+B^*_t}^{\infty} J_{t+1}(-B^*_t,\varepsilon_{t+1})d\Psi_t(x)
\end{aligned}
$$

The profit-to-go function is maximized over the backorder decision in Equation (3.6). The constraint means that the back-order amount is limited by the efficiency parameter and the capacity of the following period. The first term in the equation is the revenue from the sold products and backlogged demand. The second term is again the inventory holding cost and the third one is the penalty cost to be paid for unsatisfied demand due to capacity and efficiency restrictions. Next term where $A_t = \min([d_t - \overline{Y_t}]^+, B_t)$ is the realized backlogged demand. The last term is same as in the previous equations, the maximum expected profit in future periods with the available inventory which is replaced with salvage revenue in period T.

In backlog demand policy we have a similar theorem with the one in Save-Inventory policy.

Theorem 3.3.1 *For the profit-to-go functions,*

- $\frac{\partial G_t^B(\overline{Y_t})}{\partial \overline{Y_t}}$ *is non-increasing in* $\overline{Y}_t$ *and* $G_t^B(\overline{Y_t})$ *is concave in* $\overline{Y}_t$*,*
- $\frac{\partial J_t(I_t,\varepsilon_t)}{\partial I_t}$ *is non-increasing in* I_t *and* $J_t(I_t,\varepsilon_t)$ *is concave in* I_t*,*
- *for any* $(\varepsilon_t^1, \varepsilon_t^2)$ *pair;* $\overline{Y}_t^{*\varepsilon_t^1} = \overline{Y}_t^{*\varepsilon_t^2}$ $(\frac{\partial^2 G_t^B(I_t)}{\partial I_t \partial \varepsilon_t} = 0)$*,*
- *for any* $(\varepsilon_t^1, \varepsilon_t^2)$ *pair;* $B_t^{*\varepsilon_t^1} = B_t^{*\varepsilon_t^2}$ $(\frac{\partial^2 J_{t+1}(I_{t+1},\varepsilon_{t+1})}{\partial I_{t+1}\partial \varepsilon_t} = 0)$.

Proof. We follow a similar way in this proof. First, we consider $G_T^B(\overline{Y_T})$. Obviously $B_T = 0$, and $J_{T+1}(\cdot,\cdot) = 0$ then:

$$G_T^B(\overline{Y_T}) = \int_0^{\overline{Y_T}} p_T x d\Psi_T(x) + \int_{\overline{Y_T}}^{\infty} p_T(\overline{Y_T}) d\Psi_T(x) + \upsilon \int_0^{\overline{Y_T}} (\overline{Y_T} - x) d\Psi_T(x) - \ell_T \int_{\overline{Y_T}}^{\infty} (x - \overline{Y_T}) d\Psi_T(x)$$

using Leibnitz's rule we obtain

$$\frac{\partial G_T^B(\overline{Y_T})}{\partial \overline{Y_T}} = p_T(1 - \Psi_T(\overline{Y_T})) + \ell_T(1 - \Psi_T(\overline{Y_T})) + \upsilon \Psi_T(\overline{Y_T})$$
$$\frac{\partial^2 G_T^B(\overline{Y_T})}{\partial \overline{Y}_T^2} = (\upsilon - \ell_T - p_T)\psi_T(\overline{Y_T}) < 0 \qquad (since\ p_T + \ell_T > \upsilon)$$

Hence, $G_T^B(\overline{Y_T})$ is also concave in $\overline{Y}_T$ and $\frac{\partial G_T^B(\overline{Y_T})}{\partial \overline{Y_T}}$ is non-increasing in $\overline{Y}_T$. Now we focus on $J_T(I_T, \varepsilon_T)$. Define $\overline{Y}_{t-1}^*$ as 0 if $\frac{k_{t-1}}{\varepsilon_{t-1}} > \frac{\partial G_{t-1}^S(0)}{\partial \overline{Y}_{t-1}}$, otherwise as $\max\left\{\overline{Y} : \frac{k_{t-1}}{\varepsilon_{t-1}} \le \frac{\partial G_{t-1}^S(\overline{Y})}{\partial \overline{Y}}\right\}$ then,

$$J_T(I_T, \varepsilon_T) = \begin{cases} -k_T q_T + G_T^B(I_T + \varepsilon_T q_T) & if\ I_T \le \overline{Y}_T^* - \varepsilon_T q_T \\ -k_T(\overline{Y}_T^* - I_T) + G_T^B(\overline{Y}_T^*) & if\ \overline{Y}_T^* - \varepsilon_T q_T < I_T \le \overline{Y}_T^* \\ G_T^B(I_T) & if\ \overline{Y}_T^* < I_T. \end{cases}$$

Therefore,

$$\frac{\partial J_T(I_T,\varepsilon_T)}{\partial I_T} = \begin{cases} \frac{\partial G_T^B(I_T+\varepsilon_T q_T)}{\partial I_T} \geq \frac{k_T}{\varepsilon_T} & if\ I_T \leq \overline{Y}_T^* - \varepsilon_T q_T \\ \frac{k_T}{\varepsilon_T} & if\ \overline{Y}_T^* - \varepsilon_T q_T < I_T \leq \overline{Y}_T^* \\ \frac{\partial G_T^B(I_T)}{\partial I_T} \leq \frac{k_T}{\varepsilon_T} & if\ \overline{Y}_T^* < I_T. \end{cases} \tag{3.7}$$

The inequalities in Equation 3.7 hold due to the choice of $\overline{Y}_T^*$. Since $\frac{\partial G_T^B(\overline{Y_T})}{\partial \overline{Y_T}}$ is non-increasing in $\overline{Y}_T$, $\frac{\partial J_T(I_T,\varepsilon_T)}{\partial I_T}$ is also non-increasing in I_T and $J_T(I_T,\varepsilon_T)$ is concave in I_T. It is seen that $\frac{\partial J_T(I_T,\varepsilon_T)}{\partial I_T}$ is biggest when I_T is small ($\geq \frac{k_T}{\varepsilon_T}$) then decreases as I_T increases. Thus, $\frac{\partial J_T(I_T,\varepsilon_T)}{\partial I_T}$ is non-increasing and $J_T(I_T,\varepsilon_T)$ is concave for all I_T values.

Also noting that $\frac{\partial G_T^B(\overline{Y_T})}{\partial \overline{Y_T}}$ is independent of ε_T, for any $(\varepsilon_T^1, \varepsilon_T^2)$ pair, we have $\overline{Y}_T^{*\varepsilon_T^1} = \overline{Y}_T^{*\varepsilon_T^2}$, and $B_T^{*\varepsilon_T^1} = B_T^{*\varepsilon_T^2} = 0$.

Now assume that $\frac{\partial G_{t-1}^B(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is non-increasing and $J_T(I_T,\varepsilon_T)$ is concave for period $t = t...T$, we show that $G_{t-1}^B(\overline{Y}_{t-1})$ is non-increasing.

$$\begin{aligned} G_{t-1}^B(\overline{Y}_{t-1}) &= \int_0^{\overline{Y}_{t-1}+B_{t-1}^*} p_{t-1} x\, d\Psi_{t-1}(x) + \int_{\overline{Y}_{t-1}+B_{t-1}^*}^{\infty} p_{t-1}(\overline{Y}_{t-1} + B_{t-1}^*) d\Psi_{t-1}(x) \\ &- \ell_{t-1} \int_{\overline{Y}_{t-1}+B_{t-1}^*}^{\infty} (x - (\overline{Y}_{t-1} + B_{t-1}^*)) d\Psi_{t-1}(x) + \int_0^{\overline{Y}_{t-1}+B_{t-1}^*} J_t(\overline{Y}_{t-1} - x, \varepsilon_t) d\Psi_{t-1}(x) \\ &- \beta_{t-1} \int_{\overline{Y}_{t-1}}^{\overline{Y}_{t-1}+B_{t-1}^*} (x - \overline{Y}_{t-1}) d\Psi_{t-1}(x) - \beta_{t-1} \int_{\overline{Y}_{t-1}+B_{t-1}^*}^{\infty} B_{t-1}^* d\Psi_{t-1}(x) \\ &+ \int_{\overline{Y}_{t-1}+B_{t-1}^*}^{\infty} J_t(-B_{t-1}^*, \varepsilon_t) d\Psi_{t-1}(x) - h_{t-1} \int_0^{\overline{Y}_{t-1}} (\overline{Y}_{t-1} - x) d\Psi_{t-1}(x) \end{aligned}$$

$$\begin{aligned} \frac{\partial G_{t-1}^B(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}} &= \int_{\overline{Y}_{t-1}+B_{t-1}^*}^{\infty} (p_{t-1} + \ell_{t-1}) d\Psi_{t-1}(x) + \int_0^{\overline{Y}_{t-1}+B_{t-1}^*} \frac{\partial J_t(\overline{Y}_{t-1} - x, \varepsilon_t)}{\partial \overline{Y}_{t-1}} d\Psi_{t-1}(x) \\ &- h_{t-1} \int_0^{\overline{Y}_{t-1}} d\Psi_{t-1}(x) + \beta_{t-1} \int_{\overline{Y}_{t-1}}^{\overline{Y}_{t-1}+B_{t-1}^*} d\Psi_{t-1}(x) \end{aligned}$$

$$\frac{\partial^2 G^B_{t-1}(\overline{Y}_{t-1})}{\partial \overline{Y}^2_{t-1}} = [\frac{\partial J_t(-B^*_{t-1},\varepsilon_t)}{\partial \overline{Y}_{t-1}} - (p_{t-1}+\ell_{t-1}-\beta_{t-1})]\psi_{t-1}(\overline{Y}_{t-1}+B^*_{t-1}) - (h_t+\beta_{t-1})\psi_{t-1}(\overline{Y}_{t-1}) + \int_0^{\overline{Y}_{t-1}+B^*_{t-1}} \frac{\partial^2 J_t(\overline{Y}_{t-1}-x,\varepsilon_t)}{\partial \overline{Y}^2_{t-1}} d\Psi_t(x) \le 0$$

Inequality is due to the induction hypothesis and the choice of $B^{*\varepsilon_{t-1}}_{t-1}$ decision; which is 0 if $p_{t-1}+\ell_{t-1}-\beta_{t-1} < \frac{\partial J_t(0,\varepsilon_t)}{\partial I}$, otherwise equal to $\min\{I : p_{t-1}+\ell_{t-1}-\beta_{t-1} \le \frac{\partial J_t(-I,\varepsilon_t)}{\partial I}\}$. Thus, $\frac{\partial G^B_{t-1}(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is non-increasing in $\overline{Y}_{t-1}$, and $G^B_{t-1}(\overline{Y}_{t-1})$ is concave in $\overline{Y}_{t-1}$.

Now define $\overline{Y}^*_{t-1}$ as 0 if $\frac{k_{t-1}}{\varepsilon_{t-1}} > \frac{\partial G^B_{t-1}(0)}{\partial \overline{Y}_{t-1}}$, otherwise as $\max\left\{\overline{Y} : \frac{k_{t-1}}{\varepsilon_{t-1}} \le \frac{\partial G^B_{t-1}(\overline{Y})}{\partial \overline{Y}}\right\}$. Because $G^B_{t-1}(\overline{Y}_{t-1})$ is concave in $\overline{Y}_{t-1}$ and by the choice of $\overline{Y}^*_{t-1}$, $\overline{Y}^*_{t-1}$ maximizes $J_{t-1}(I_{t-1},\varepsilon_{t-1})$ with production realization of $\max\{\varepsilon_{t-1}q_{t-1}, (\overline{Y}^*_{t-1}-I_{t-1})\}$. Hence,

$$\frac{\partial J_{t-1}(I_{t-1},\varepsilon_{t-1})}{\partial I_{t-1}} = \begin{cases} \frac{\partial G^B_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}} \ge \frac{k_{t-1}}{\varepsilon_{t-1}} & if\ I_{t-1} \le \overline{Y}^*_{t-1} - \varepsilon_{t-1}q_{t-1} \\ \frac{k_{t-1}}{\varepsilon_{t-1}} & if\ \overline{Y}^*_{t-1} - \varepsilon_{t-1}q_{t-1} < I_{t-1} \le \overline{Y}^*_{t-1} \\ \frac{\partial G^B_{t-1}(I_{t-1})}{\partial I_{t-1}} \le \frac{k_{t-1}}{\varepsilon_{t-1}} & if\ \overline{Y}^*_{t-1} < I_{t-1}. \end{cases}$$

$\frac{\partial G^B_{t-1}(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is non-increasing in $\overline{Y}_{t-1}$, then $\frac{\partial J_{t-1}(I_{t-1},\varepsilon_{t-1})}{\partial I_{t-1}}$ is also non-increasing in I_{t-1} and $J_{t-1}(I_{t-1},\varepsilon_{t-1})$ is concave in I_{t-1}.

As seen before $\frac{\partial G^B_{t-1}(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}}$ is independent of ε_{t-1}. Similarly, to see whether $B^{*\varepsilon^1_{t-1}}_{t-1} = B^{*\varepsilon^2_{t-1}}_{t-1}$ or not, we need to check if $\frac{\partial J_t(I_t,\varepsilon_t)}{\partial I_t}$ depends on ε_{t-1} for any I_t. We have

$$\frac{\partial^2 J_t(I_t,\varepsilon_t)}{\partial I_t \partial \varepsilon_{t-1}} = \begin{cases} \frac{\partial^2 G^B_t(I_t+\varepsilon_t q_t)}{\partial I_t \partial \varepsilon_{t-1}} = 0 & if\ I_t + \varepsilon_t q_t \le \overline{Y}^*_t \\ 0 & if\ \overline{Y}^*_t < I_t + \varepsilon_t q_t, \end{cases}$$

since for period t, we have,

$$\frac{\partial G^B_t(I_t+\varepsilon_t q_t)}{\partial I_t} = \int_{I_t+\varepsilon_t q_t + B^*_t}^{\infty} (p_t+\ell_t) d\Psi_t(x) + \int_0^{I_t+\varepsilon_t q_t+B^*_t} \frac{\partial J_{t+1}(I_t+\varepsilon_t q_t - x, \varepsilon_{t+1})}{\partial I_t} d\Psi_t(x) - \int_0^{I_t+\varepsilon_t q_t} h_t d\Psi_t(x) + \beta_t \int_{I_t+\varepsilon_t q_t}^{I_t+\varepsilon_t q_t+B^*_t} d\Psi_t(x)$$

$$\frac{\partial^2 G^B_t(I_t+\varepsilon_t q_t)}{\partial I_t \partial \varepsilon_{t-1}} = 0.$$

Thus $B^{*\varepsilon^1_T}_T = B^{*\varepsilon^2_T}_T = 0$. ■

We show in Theorem 3.3.1 that the optimal production and backlog decisions under backlog-demand policy are also independent of the current demand and the efficiency parameter of the current period. It is intuitively true, because as in the save inventory policy the efficiency parameter is known and the producer takes this value into consideration. The producer would change the production amount to compensate the change in the efficiency parameter. Therefore the producer can decide the optimal policy at the beginning of each period. Theorem 3.3.1 implies the optimal policy for the Backlog-Demand policy; thus we have the following corollary.

Corollary 3.3.2 *Given a vector of prices, there exists an optimal modified base stock policy for the Backlog-Demand policy with an optimal backlog-up-to level (B_t^*) and optimal produce-up-to level ($\overline{Y}_t^*$).*

We have mentioned that the realized production level ($\overline{Y}_t$) decision may be different than the optimal produce-up-to level ($\overline{Y}_t^*$) for the Save-Inventory policy. It is same for the Backlog-Demand policy. In addition backlog (B_t) decision may not be equal to the optimal backlog-up-to level (B_t^*). It is limited by the following period's capacity and efficiency parameter. Next, we discuss how decisions in period t changes with ε_{t+1}.

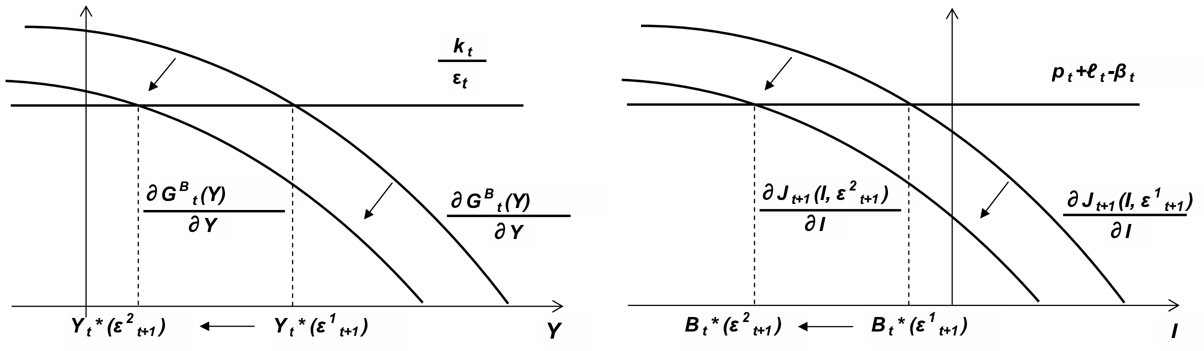
Theorem 3.3.3 *Under backlog-demand policy*

- $\frac{\partial^2 G_t^B(\overline{Y}_t)}{\partial \overline{Y}_t \partial \varepsilon_{t+1}} \leq 0 \ \forall \overline{Y}_t$ *and* $\overline{Y}_t^*$ *decision is non-increasing in* ε_{t+1}.
- $\frac{\partial^2 J_{t+1}(I_{t+1}, \varepsilon_{t+1})}{\partial I_{t+1} \partial \varepsilon_{t+1}} \leq 0 \ \forall I_{t+1}$ *and* $\overline{B}_t^*$ *decision is non-decreasing in* ε_{t+1}.

Proof. We will show that $\frac{\partial G_t^B(\overline{Y}_t)}{\partial \overline{Y}_t}$ and $\frac{\partial J_{t+1}(I_{t+1}, \varepsilon_{t+1})}{\partial I_{t+1}}$ curves are never shifting upwards (for any I_{t+1} or $\overline{Y}_t$ value, respectively) as ε_{t+1} increases. Figure 3.3 depicts that when this is the case, $\overline{Y}_t^*$ decision is non-increasing and $\overline{B}_t^*$ decision is non-decreasing in ε_{t+1}.

Induction is used for this proof. First we focus on $G_T^B(I_T + \varepsilon_T q_T)$

$$\begin{aligned} G_T^B(I_T + \varepsilon_T q_T) &= \int_0^{I_T+\varepsilon_T q_T} p_T x d\Psi_T(x) + \int_{I_T+\varepsilon_T q_T}^{\infty} p_T(I_T + \varepsilon_T q_T) d\Psi_T(x) \\ &+ \upsilon \int_0^{I_T+\varepsilon_T q_T} (I_T + \varepsilon_T q_T - x) d\Psi_T(x) - \ell_T \int_{I_T+\varepsilon_T q_T}^{\infty} (x - I_T - \varepsilon_T q_T) d\Psi_T(x) \\ \frac{\partial G_T^B(I_T + \varepsilon_T q_T)}{\partial I_T} &= (p_T + \ell_T)(1 - \Psi_T(I_T + \varepsilon_T q_T)) + \upsilon \Psi_T(I_T + \varepsilon_T q_T) \end{aligned}$$



(a) Change in $\overline{Y}_t^*$ Decision (b) Change in B_t^* Decision

Figure 3.3: ε_{t+1} effect on decisions in period t under Backlog-Demand Policy

$$\frac{\partial^2 G_T^B(I_T + \varepsilon_T q_T)}{\partial I_T \partial \varepsilon_T} = q_T(\upsilon - p_T - \ell_T)\psi_T(I_T + q_T\varepsilon_T) < 0 \quad \forall I_T.\ (*)$$

The inequality hold due to the assumption $p_T + \ell_T > \upsilon$

$$\frac{\partial^2 J_T(I_T, \varepsilon_T)}{\partial I_T \partial \varepsilon_T} = \begin{cases} \frac{\partial^2 G_T^B(I_T+\varepsilon_T q_T)}{\partial I_T \partial \varepsilon_T} < 0 \ \ (from\ *) & if\ I_T \le \overline{Y}_T^* - \varepsilon_T q_T \\ \frac{-k_T}{\varepsilon_T^2} < 0 & if\ \overline{Y}_T^* - \varepsilon_T q_T < I_T \le \overline{Y}_T^* \quad (**) \\ \frac{\partial^2 G_T^B(I_T)}{\partial I_T \partial \varepsilon_T} = 0 & if\ \overline{Y}_T^* < I_T. \end{cases}$$

Therefore, as I_T increases, the backorder-up-to decision $B_{T-1}^{*\varepsilon_{T-1}}$ first decreases and then remains unchanged since B_{T-1}^* is defined as $\min\{I : p_{T-1} + \ell_{T-1} - \beta_{T-1} \le \frac{\partial J_T(-I, \varepsilon_T)}{\partial I}\}$. Moreover,

$$\begin{aligned} \frac{\partial G_{T-1}^B(\overline{Y}_{T-1})}{\partial \overline{Y}_{T-1}} &= \int_{\overline{Y}_{T-1}+B_{T-1}^*}^{\infty} (p_{T-1} + \ell_{T-1}) d\Psi_{T-1}(x) + \beta_{T-1} \int_{\overline{Y}_{T-1}}^{\overline{Y}_{T-1}+B_{T-1}^*} d\Psi_{T-1}(x) \\ &+ \int_0^{\overline{Y}_{T-1}+B_{T-1}^*} \frac{\partial J_T(\overline{Y}_{T-1} - x, \varepsilon_T)}{\partial \overline{Y}_{T-1}} d\Psi_{T-1}(x) - h_T \int_0^{\overline{Y}_{T-1}} d\Psi_{T-1} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 G_{T-1}^B(\overline{Y}_{T-1})}{\partial \overline{Y}_{T-1} \partial \varepsilon_T} &= [-(p_{T-1} + \ell_{T-1} - \beta_{T-1}) + \frac{\partial J_T(-B_{T-1}^*, \varepsilon_T)}{\partial \overline{Y}_{T-1}}]\psi_{T-1}(\overline{Y}_{T-1} + B_{T-1}^*)\frac{\partial B_{T-1}^*}{\partial \varepsilon_T} \\ &+ \int_0^{\overline{Y}_{T-1}+B_{T-1}^*} \frac{\partial^2 J_t(\overline{Y}_{T-1} - x, \varepsilon_T)}{\partial \overline{Y}_{T-1} \partial \varepsilon_T} d\Psi_T(x) \ \le\ 0 \ due\ to\ (**)\ and\ B_{T-1}^*\ decision. \end{aligned}$$

Up to period $t-1$,assume,

$$\frac{\partial^2 G_t^B(I_t + \varepsilon_t q_t)}{\partial I_t \partial \varepsilon_t} < 0 \quad \forall I_t.\ (***)$$

Then we have,

$$\frac{\partial^2 J_t(I_t,\varepsilon_t)}{\partial I_t \partial \varepsilon_t} = \begin{cases} \frac{\partial^2 G_t^B(I_t+\varepsilon_t q_t)}{\partial I_t \partial \varepsilon_t} < 0 \quad (from\ \ ***) & if\ I_t \leq \overline{Y}_t^* - \varepsilon_t q_t \\ \frac{-k_t}{\varepsilon_t} < 0 & if\ \overline{Y}_t^* - \varepsilon_t q_t < I_t \leq \overline{Y}_t^* \quad (****) \\ \frac{\partial^2 G_t^B(I_t)}{\partial I_t \partial \varepsilon_t} = 0 & if\ \overline{Y}_t^* < I_t. \end{cases}$$

$\overline{B}_{t-1}^*$ decision is non-decreasing in ε_t since it is defined as $\min\{I : p_{t-1}+\ell_{t-1}-\beta_{t-1} \leq \frac{\partial J_t(-I,\varepsilon_t)}{\partial I}\}$. Moreover,

$$\frac{\partial^2 G_{t-1}^B(\overline{Y}_{t-1})}{\partial \overline{Y}_{t-1}\partial \varepsilon_t} = \int_0^{\overline{Y}_{t-1}+B_{t-1}^*} \frac{\partial^2 J_t(\overline{Y}_{t-1}-x,\varepsilon_t)}{\partial \overline{Y}_{t-1}\partial \varepsilon_t} \leq 0 \ \ due\ to\ (****).$$

Finally for period $t-2$: it is enough to check whether

$$\frac{\partial^2 G_{t-1}^B(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial \varepsilon_{t-1}} \leq 0 \quad \forall I_{t-1}$$

holds or not.

$$\begin{aligned} \frac{\partial G_{t-1}^B(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}} &= \int_{I_{t-1}+\varepsilon_{t-1}q_{t-1}+B_{t-1}^*}^{\infty} (p_{t-1}+\ell_{t-1})d\Psi_{t-1}(x) \\ &+ \int_0^{I_{t-1}+\varepsilon_{t-1}q_{t-1}+B_{t-1}^*} \frac{\partial J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}-x,\varepsilon_t)}{\partial I_{t-1}} d\Psi_{t-1}(x) \\ &- h_{t-1}\int_0^{I_{t-1}+\varepsilon_{t-1}q_{t-1}} d\Psi_{t-1}(x) + \beta_{t-1}\int_{I_{t-1}+\varepsilon_{t-1}q_{t-1}}^{I_{t-1}+\varepsilon_{t-1}q_{t-1}+B_{t-1}^*} d\Psi_{t-1}(x) \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 G_{t-1}^B(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial \varepsilon_{t-1}} \\ =\ & (q_{t-1}+\frac{\partial B_{t-1}^*}{\partial \varepsilon_t})\Big(\frac{\partial J_t(-B_{t-1}^*,\varepsilon_t)}{\partial I_{t-1}} - (p_{t-1}+\ell_{t-1}-\beta_{t-1})\Big)\psi_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1}+B_{t-1}^*) \\ -\ & (h_{t-1}+\beta_{t-1})q_{t-1}\psi_{t-1}(I_{t-1}+\varepsilon_{t-1}q_{t-1}) \\ +\ & \int_0^{I_{t-1}+\varepsilon_{t-1}q_{t-1}+B_{t-1}^*} \frac{\partial^2 J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1}-x,\varepsilon_t)}{\partial I_{t-1}\partial \varepsilon_{t-1}} d\Psi_{t-1}(x) \ \leq \ 0, \end{aligned}$$

due to the B_{t-1}^* and since

$$J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t) = \begin{cases} k_t q_t + G_t^B(I_{t-1}+\varepsilon_{t-1}q_{t-1}+\varepsilon_t q_t) & if\ I_{t-1}+\varepsilon_{t-1}q_{t-1} \leq \overline{Y}_t^* - \varepsilon_t q_t \\ -\frac{k_t}{\varepsilon_t}(\overline{Y}_t^* - I_{t-1} - \varepsilon_{t-1}q_{t-1}) + G_t^B(\overline{Y}_t^*) & otherwise \\ G_t^B(I_{t-1}+\varepsilon_{t-1}q_{t-1}) & if\ \overline{Y}_t^* < I_{t-1}+\varepsilon_{t-1}q_{t-1} \end{cases}$$

and,

$$\frac{\partial^2 J_t(I_{t-1}+\varepsilon_{t-1}q_{t-1},\varepsilon_t)}{\partial I_{t-1}\partial\varepsilon_{t-1}} = \begin{cases} \frac{\partial^2 G_t^B(I_{t-1}+\varepsilon_{t-1}q_{t-1}+\varepsilon_t q_t)}{\partial I_{t-1}\partial\varepsilon_{t-1}} < 0 & if\ I_{t-1}+\varepsilon_{t-1}q_{t-1} \leq \overline{Y}_t^* - \varepsilon_t q_t \\ 0 & otherwise \qquad from\ (***). \\ \frac{\partial^2 G_t^B(I_{t-1}+\varepsilon_{t-1}q_{t-1})}{\partial I_{t-1}\partial\varepsilon_{t-1}} < 0 & if\ \overline{Y}_t^* < I_{t-1}+\varepsilon_{t-1}q_{t-1} \end{cases}$$

Therefore, it holds. ■

In this theorem we present that the current decisions are effected by the efficiency parameter of the future periods. Change in future periods effects the current decisions because of the demand uncertainty. Also initial inventory of the following period is determined by the current decisions. As efficiency parameter of the future periods increases available amount for the backlogged demand increases and there would be a higher B_t^* value.

CHAPTER 4

NUMERICAL STUDY

We conduct a numerical analysis and present the results in this chapter. Our aim is to obtain insights about the model and how it is affected by parameter changes. We have built a program (available upon request) in C++ and enumerate all possible solutions to find the optimal policies in different problem environments.

We consider a finite time horizon with six periods and assume that demand in each period is normally distributed with an average of 50. The coefficient of variation of demand in a given period is assumed to be $CV_u = std(d_t)/E(d_t)$, where $std(d_t)$ denotes the standard deviation and $E(d_t)$ denotes the expected value of demand. In all cases we set the coefficient of variation of demand as 0.2 and truncate the demand distribution to be $\leq$ 3 standard deviations from the mean. Hence, demand in each period is distributed between 20 and 80.

For the initial case we set the production capacity to 80% of the expected demand. Production capacity is assumed to be same in each period. The production cost and the selling price are considered to be 100 and 500 respectively. Both production cost and price are also constant in the initial case. We assume the producer gives a 10% discount for back-ordered items and set the penalty cost of back-ordered demand to 50. Holding cost is assumed to be 20% of the initial production cost. For simplicity, lost sales cost per unit is assumed to be zero in the initial case. These three parameters are also assumed to be constant over periods in the initial case.

We also modify some parameters of this initial case and ran experiments with different problem settings. Our findings are presented in the following sections.

4.1 Marginal Expected Profit Curves

In this section we want to illustrate Theorem 3.2.3 and Theorem 3.3.3 numerically. We focus on a production system with prices decreasing over periods and present the effect of efficiency parameter on marginal expected profit curves. We start with the initial case setting and allow prices to decrease with a rate of 10% of the initial price. We solve the problem for two cases; in the first one we set the efficiency parameter of Period 3 to 0.4, others to 0.8 and in the second one all efficiency parameters are considered to be 0.8 as in the initial case. Marginal expected profit curves for Period 2 are given in Figure 4.1 and Figure 4.2 respectively.

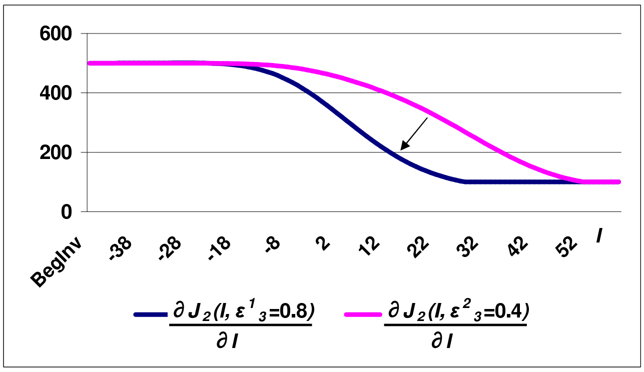


Figure 4.1: ε_3 effect on Marginal J curve in period 2

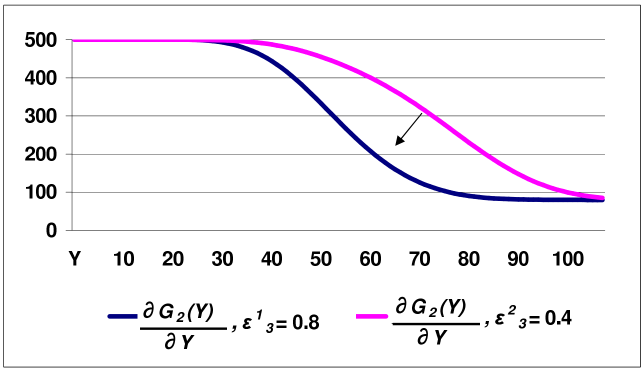


Figure 4.2: ε_3 effect on Marginal G curve in period 2

In this experiment it is observed that both curves are shifted to the left as efficiency parameter of the following period increases. On the other hand the efficiency parameter does not have any impact on the current or next periods' curves. These findings are consistent with our theorems.

4.2 Decreasing Price

In this section, we assume that the selling prices have a decreasing trend with a rate of 10% of the initial price as in the previous section. We ran our program for four cases with different efficiency parameter values; 0.9, 0.8, 0.7, 0.6. These values are not allowed to differ by periods. Optimal decisions are given in Table 4.1.

Table 4.1: Optimal Decisions for a System with Decreasing Prices

	$\varepsilon_t = 0.9$		$\varepsilon_t = 0.8$		$\varepsilon_t = 0.7$		$\varepsilon_t = 0.6$	
Period	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*
1	36	-36	32	-32	28	-28	24	-24
2	76	-36	72	-32	68	-28	64	-24
3	98	-36	108	-32	108	-28	104	-24
4	85	-36	92	-32	100	-28	107	-24
5	71	-36	75	-32	79	-28	83	-24
6	53	0	53	0	53	0	53	0

While prices are decreasing the producer is expected to have a tendency to sell future capacity in the current period and as we expect back-ordering strategy is used in this production setting.

At the beginning of the time horizon the optimal $\overline{Y}_t^*$ amount is smaller because of the demand uncertainty and holding cost. Hence, the producer decides to produce less and backlog demand exceeding inventory on hand. By doing this the producer produces items after demand realization and reduces demand uncertainty. In the later periods optimal production amount is increased to satisfy the expected back-orders. Coming up to the final period production decreases again to decrease inventory on hand. It is also seen in Table 4.1 that B_t^* is non-decreasing in efficiency parameter.

4.3 Increasing Price

In this section the selling prices are assumed to have an increasing trend with a rate of 10% of the initial price value. As in the previous section we consider four efficiency parameter values; 0.9, 0.8, 0.7, 0.6. Optimal decisions are given in Table 4.2.

In this setting reserve decision is more advantageous than back-order decision and it is seen

Table 4.2: Optimal Decisions for a System with Increasing Prices

	$\varepsilon_t = 0.9$		$\varepsilon_t = 0.8$		$\varepsilon_t = 0.7$		$\varepsilon_t = 0.6$	
Period	$\overline{Y}_t^*$	S_t^*	$\overline{Y}_t^*$	S_t^*	$\overline{Y}_t^*$	S_t^*	$\overline{Y}_t^*$	S_t^*
1	36	40	32	40	28	40	24	40
2	76	38	72	54	68	70	64	80
3	111	23	112	35	108	47	104	59
4	97	9	105	17	113	25	121	33
5	81	0	85	3	88	7	92	11
6	59	0	59	0	59	0	59	0

that S_t^* is non-decreasing in efficiency parameter. As in Section 4.2 the optimal $\overline{Y_t}^*$ amount is higher in middle periods because of the same reason.

4.4 Increasing and Decreasing Price

In some production environments there may be both back-order and reserve decisions in the optimal solution. In Section 4.4 we analyze a production system with both decreasing and increasing prices. The optimal decision variables are given in Table 4.3. In the optimal solution there are both reserve and back-order decisions. It seems that in periods with decreasing prices reserve decision is better and in others back-order decision is better which is intuitively true.

Table 4.3: Optimal Decisions for a System with Increasing and Decreasing Prices

Period	$\overline{Y}_t^*$	S_t^*	B_t^*
1	32	40	0
2	72	36	0
3	112	14	0
4	96	0	-32
5	78	0	-32
6	54	0	0

4.5 Decreasing Cost

In Section 4.5 we let the production cost decrease with a rate of 10% of the initial cost. In this case as the prices are constant it is not important when to sell an item and change in the production costs effects the decision when to produce. In such a production system if the capacity is tight, the producer would choose to produce as much as possible and sells all inventory on hand at current period since demand is higher. There is not any other options. However, if the capacity is higher, then the producer may choose to use previous (reserve) or next (backorder) periods' excess capacities. Hence, in this section we consider the production capacity in each period to be 120% of the expected demand.

Optimal decisions are given in Table 4.4. It is observed that backlogging is better than reserving. This is also intuitively true since the costs are decreasing and it is more profitable to produce later.

Table 4.4: Optimal Decisions for a System with Decreasing Costs

Period	$\overline{Y}_t^*$	S_t^*	B_t^*
1	48	0	-16
2	78	0	-14
3	76	0	-20
4	73	0	-16
5	70	0	-28
6	63	0	0

4.6 Increasing Lost Sale Cost

In the initial case we assume lost sale costs to be zero and in Section 4.6 we analyze the lost sale cost effect. In this case since the prices and the costs are considered to be constant we assume the capacity to be 120% of the expected demand. Otherwise, as mentioned before in Section 4.5 there would be no tactical decisions. In this section we ran experiments with four different lost sale cost values; 25, 50, 75 and 100. Lost sale costs are assumed to be constant over periods. Optimal decision variables are given in Table 4.5.

It is seen that as the lost sale increases optimal back-order amount decreases. This is intu-

Table 4.5: Optimal Decisions for a System with Increasing Lost Sale Costs

	$\ell_t = 25$		$\ell_t = 50$		$\ell_t = 75$		$\ell_t = 100$	
Period	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*
1	48	-10	48	-7	48	-4	48	-1
2	79	-12	79	-8	80	-5	81	-3
3	77	-13	77	-10	78	-7	78	-5
4	74	-15	74	-11	75	-9	75	-7
5	70	-15	70	-11	71	-9	71	-8
6	63	0	63	0	64	0	64	0

itively true since as the lost sale cost increases the producer will be more likely to avoid lost sales. Because of demand uncertainty the optimal back-order amount decreases, next period's capacity is used for next period's demand. Changes in lost sale costs affect the optimal decision amount; however similar results are obtained with the existence of lost sale costs. Hence, our assumption of zero lost sale costs does not invalidate our study.

4.7 Increasing Back-Order Penalty

We present the effects of changes in back-order penalty costs in this section. Capacity is considered to be 120% of the expected demand in this section as in Section 4.5 and Section 4.6, because the prices and the costs are constant. We consider four production settings with different back-order penalty costs; 25, 50, 75 and 100. Back-order penalty costs are considered to be constant over periods. Optimal decision variables are given in Table 4.6.

Table 4.6: Optimal Decisions for a System with Increasing Backorder Penalty Costs

	$\beta_t = 25$		$\beta_t = 50$		$\beta_t = 75$		$\beta_t = 100$	
Period	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*	$\overline{Y}_t^*$	B_t^*
1	48	-10	48	-7	48	-4	48	-2
2	78	-11	79	-8	80	-6	81	-4
3	77	-12	77	-10	78	-8	78	-6
4	74	-12	74	-11	75	-10	75	-9
5	70	-11	70	-11	71	-11	71	-11
6	63	0	63	0	63	0	63	0

It is observed that as the back-order penalty cost increases the optimal back-order quantity decreases. This is also intuitively true because as penalty cost increases cost of backlogging any item increases and the optimal amount decreases.

4.8 Tactical vs. Traditional Inventory

As mentioned before there is no reserve option in traditional inventory control systems, the producer meets the demand as much as there is sufficient inventory on hand. Also, in some traditional systems backlogging is not allowed and unsatisfied demand is fully lost. In this section, we compare tactical inventory control systems with such a traditional inventory control system.

Our objective is to present the efficiency of our tactical inventory policy over traditional inventory control policy in different problem settings and determine the situations where tactical inventory policies can provide significant improvements.

Four parameters (production capacity, efficiency parameter, cost and price) are allowed to have three cases, so we have 81 different production settings. Production capacity is assumed to be 60%, 80% and 100% of the expected demand. Efficiency parameter is assumed to be 0.6, 0.8 and 1. Both of these parameters are assumed to be constant over periods. Prices and costs are assumed to have trends such as increasing, constant and decreasing.

Optimal total expected profit of six periods are computed and compared to those with traditional inventory control policy. Improvement percentages are given in the Table 4.7.

Table 4.7: Efficiency of Tactical Inventory Policy over Traditional Inventory Policy

		q_t = 60%			80%			100%		
cost	price	ε_t = 0.6	0.8	1	0.6	0.8	1	0.6	0.8	1
dec	dec	16.67	15.11	15.01	16.96	16.43	18.64	17.78	19.54	20.12
	cons	0.00	0.01	0.20	0.02	0.27	1.57	0.23	1.64	6.24
	inc	17.85	11.84	7.67	13.32	6.72	2.28	9.20	2.44	2.90
cons	dec	20.00	16.94	16.19	20.26	18.12	19.50	20.94	20.84	20.29
	cons	0.00	0.02	0.22	0.02	0.29	1.66	0.26	1.77	6.46
	inc	21.43	13.33	8.39	16.00	7.59	2.57	11.08	2.83	2.84
inc	dec	25.00	19.29	17.61	25.20	20.29	20.52	25.68	22.50	20.32
	cons	0.00	0.02	0.23	0.02	0.32	1.75	0.30	1.91	6.61
	inc	26.78	15.26	9.29	20.03	8.76	2.97	13.93	3.40	2.98

As expected tactical inventory policy is always better than or at least equal to traditional inventory policy. It is mostly efficient in a system with tight capacity, small efficiency parameter, and increasing costs and prices. In cases with non-constant prices the policy is more efficient. This is because reserve and backorder decisions are more effective in such production environments. When the prices are constant over periods the backorder and reserve decisions become less important. Efficiency in systems with decreasing prices is always better because in such systems demand uncertainty is reduced by backlogging demand.

It is also observed that when there is an increasing trend in prices, tactical inventory policy is more efficient in systems with lower capacity. When there is a decreasing trend in prices tactical inventory policy is more efficient in systems with higher capacity. In a system with increasing prices save inventory policy is better. In this policy producer produces an item at current period and sells later. If the capacity is high the producer may produce and sell later, and there would be no holding costs. Hence, it is not profitable to save much inventory and the efficiency decreases. On the other hand, in a system with decreasing prices backorder policy is better. In this policy backorder amount is limited by the production capacity and the efficiency parameter of the following period. Then high capacity means more backlogging option and more efficiency.

CHAPTER 5

CONCLUSIONS

In this study we focused on a single-item multi-period production system with limited capacity and predictable yield. The producer has the option to refuse to satisfy part of the demand even when there is inventory on hand. This strategy is referred as discretionary sales and is beneficial when costs and prices are variable. The producer also has the option to backlog demand for one period. Demand across periods is assumed to be independent and unsatisfied demand that is not backlogged is lost. Prices are assumed to be predetermined and not known by the customers before the period. Hence, the customers do not act tactically. The manufacturer has to decide optimal lot sizing, reserve and backorder amounts in each period.

We analyze the profit-to-go functions and show that the optimal policy is of a modified produce-up-to type. This optimal policy is characterized by three parameters (Y, S, B) where Y is the produce-up-to level, S is the reserve and B is the backlog-up-to levels. We prove that the reserve and backlog decisions cannot be positive at the same period which is intuitively true too. Hence, we have two candidate policies to be optimal; save-inventory policy and backlog-demand policy. In each period the better one is chosen to be the optimal policy.

We analyze and present how the optimal policy is affected by changes in the production yield. The production yield of the current period is shown to have no impact on the optimal policy for the remaining periods. On the other hand, in the save-inventory strategy the optimal reserve decision is non-increasing while in the backlog-demand strategy the optimal backlog decision is non-decreasing in the production yield of the following period. The production decision is non-increasing for both strategies.

In order to support our findings and test the performance of the tactical inventory control policy we conduct some computational studies. We consider an initial case and analyze how the optimal decision variables change when some of the parameters change. We also compute the improvement of the optimal expected total profit in tactical inventory control policy over the one in traditional inventory control policy. We present the results in different problem settings and show when it is best to use tactical inventory policy. Moreover the numerical analysis was very supportive to provide managerial insights on the model.

There are some potential extensions we suggest for future research. One may consider random production yield and analyze how the policies are affected. Random capacity limits may be included to the model as some industries have unpredicted capacities due to production line problems. Another extension of our study may allow backlogged items to be satisfied till the end of the time horizon.

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