PSEUDOSPIN SYMMETRY AND ITS APPLICATIONS

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ABSTRACT

PSEUDOSPIN SYMMETRY AND ITS APPLICATIONS

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The pseudospin symmetry concept is investigated by solving the Dirac equation for the exactly solvable potentials such as pseudoharmonic potential, Mie-type potential, Woods-Saxon potential and Hulthén plus ring-shaped potential with any spin-orbit coupling term κ. Nikiforov-Uvarov Method, Asymptotic Iteration Method and functional analysis method are used in the calculations. The energy eigenvalue equations of the Dirac particles are found and the corresponding radial wave functions are presented in terms of special functions. We look for the contribution of the ring-shaped potential to the energy spectra of the Dirac particles. Particular cases of the potentials are also discussed. By considering some particular cases, our results are reduced to the well-known ones presented in the literature. In addition, by taking equal mixture of scalar and vector potentials together with tensor potential, solutions of the Dirac equation are found and then the energy splitting between the two states in the pseudospin doublets is investigated. We indicate that degeneracy between members of pseudospin doublet is removed by tensor interactions. Effects of the potential parameters on the pseudospin doublet splitting are also studied. Radial nodes structure of the Dirac spinor are presented.

Keywords: Pseudospin Symmetry, Dirac Equation, Nikiforov-Uvarov Method, Asymptotic Iteration Method, Pseudoharmonic Potential, Mie-Type Potential, Hulthén Potential, Ring-Shaped Potential, Woods-Saxon Potential, Tensor Potential

ÖZ

SANKİSPİN SİMETRİ VE UYGULAMALARI

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Sankispin (psüdospin) simetri kavramı, herhangi bir spin-yörünge çiftlenim terimi κ için sankiharmonik (psüdoharmonik) potansiyel, Mie-tipi potansiyel, Woods-Saxon potansiyel, ve Hulthén artı halka-biçimli (ring-shaped) potensiyel gibi tam olarak çözülebilen potansiyeller ile Dirac denklemi çözülerek, araştırıldı. Hesaplamalarda Nikiforov-Uvarov metodu, asimptotik tekrarlama metodu ve fonksiyon analiz metodu kullanıldı. Dirac parçacıklarının enerji özdeğer denklemleri bulundu ve bunlara karşılık gelen ışınsal dalga fonksiyonları özel fonksiyonlar cinsinden sunuldu. Halka-biçimli (ring-shaped) potansiyelin, Dirac parçacıklarının enerjisine yaptığı katkıyı araştırdık. Potansiyellerin belli durumları tartışıldı. Elde edilen sonuçların, bazı özel durumlarda, literatürdeki bilinen sonuçlara indirgendiği gösterildi. Bununla birlikte, eşit skaler ve vektör potansiyel ile birlikte tensör potansiyel gözönüne alınarak Dirac denkleminin çözümü elde edildi ve sankispin (psüdospin) çiftlerindeki iki düzey arasındaki enerji yarılması incelendi. Sankispin (Psüdospin) çiftindeki enerji yarılmasına potansiyel parametrelerinin etkileri çalışıldı. Dirac spinörünün ışınsal düğüm yapısı sunuldu.

Anahtar Kelimeler: Sankispin (Psüdospin) Simetri, Dirac denklemi, Nikiforov-Uvarov Metodu, Asimptotik Tekrarlama Metodu, Sankiharmonik (Psüdoharmonik) Potansiyel, Mie-Tipi Potansiyel, Hulthén Potasiyel, Halka-biçimli Potansiyel, Woods-Saxon Potansiyel, Tensör Potansiyel

To my wife, Sezen

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CHAPTER 1

INTRODUCTION

Pseudospin concept was introduced for the first time in 1969 [1, 2]. It is based on the experimental observation of quasi-degeneracy in nuclei between single-nucleon states with non-relativistic quantum numbers $(n, l, j = l + \frac{1}{2})$ and $(n-1, l+2, j = l + \frac{3}{2})$, where n, l and j are the radial, the orbital and the total angular momentum quantum numbers, respectively. These two states are considered as a pseudospin doublet with $(\tilde{n} = n - 1, \tilde{l} = l + 1, \tilde{j} = \tilde{l} \pm 1/2)$ because of that they lie very close in energy. Pseudo-orbital angular momentum $\tilde{l}$ and pseudospin $\tilde{s} = 1/2$ quantum numbers have been introduced to express these two states as a doublet structure. The members of the pseudospin doublets have the same pseudo-orbital angular momentum quantum numbers, *i.e.*, $[ns_{1/2}, (n-1)d_{3/2}]$ will have $\tilde{l} = 1$ and can be represented as $[\tilde{n}\tilde{p}_{1/2}, \tilde{n}\tilde{p}_{3/2}]$. In Figure 1.1 which has been taken from Ref. [3], schematic presentation of the pseudospin doublets is presented. In the context of the pseudospin concept, $np_{1/2}$ states are pseudospin singlets represented as $\tilde{n}\tilde{s}_{1/2}$ while $0s_{1/2}, 0p_{3/2}, 0d_{5/2}$ etc. states are intruder orbital states that do not have a partner. This symmetry has been successfully applied to explain many different phenomena in nuclear structure involving deformation [4], superdeformaiton [5], identical bands [6, 7].

Although the history of the pseudospin symmetry has been very well known for a long time, its origin has no accurate explanation in the non-relativistic framework. Thus, lots of studies have been devoted to comprehend the origin of pseudospin symmetry [8, 9, 10, 11, 12, 13]. In Refs. [8, 9, 10], Blokhin *et. al.* have found that there is a connection between the pseudospin symmetry and the relativistic mean field theory. However, there has been no remarkable advance until Ginocchio's work [12]. He has

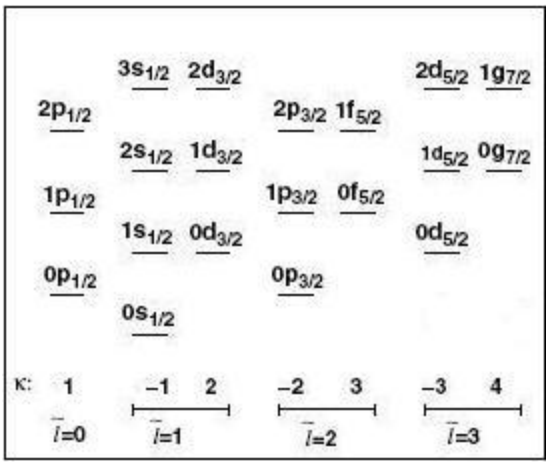

Figure 1.1: Schematic representation of the pseudospin doublets

noticed that pseudo-orbital angular momentum $\tilde{l}$ is nothing but the usual orbital angular momentum of the small (lower) component of the Dirac spinor. One of the inherent characteristic of the relativistic mean field theory is the near equality of an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ in magnitude but different in sign [12]. Ginocchio has indicated that this near equality $S(r) + V(r) \sim 0$ leads to pseudospin symmetry in nuclei (see Ref. [13] and references therein). Meng *et al.* [14, 15] have presented that exact pseudospin symmetry occurs in the Dirac equation when $\frac{d[V(r)+S(r)]}{dr} = 0$ or $V(r) + S(r) = Constant$. After these pioneering studies, numerous works have been made to study the pseudospin symmetry and spin symmetry $\frac{d[V(r)-S(r)]}{dr} = 0$ or $V(r) - S(r) = Constant$ in nuclei and in the Dirac phenomenology [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] (and see Ref. [49] and references therein). Marcos *et al.* [38] have studied the similarities and differences between pseudospin and spin symmetry in the relativistic framework. To do this, they have investigated the effects of the symmetry breaking terms, spin-orbit potential and pseudospin-orbit potential. In their recent paper [42], they have obtained particular aspects of the spin and pseudospin symmetries by considering the different central potentials in the Dirac phenomenology. Under the condition of the spin symmetry, Ginocchio [50] has obtained the energy eigenvalues and corresponding wave functions of the Dirac particles for equal scalar and vector triaxial, axially deformed and spherical oscillator potentials.

Solutions of the Dirac equation for particle dynamics in the relativistic quantum mechanics are very significant in describing the nuclear shell structure under the condition of the pseudospin symmetry [1, 2, 55]. Thus, in recent years, there has been much effort to investigate the pseudospin symmetry and spin symmetry by solving the Dirac equation in terms of different methods for exactly solvable potentials [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79].

In the present thesis, we investigate the pseudospin symmetric solutions of the time-independent Dirac equation for pseudoharmonic potential, Mie-type potential and Hulthén plus ring-shaped potential with any spin-orbit coupling term κ. In the calculations, we use the Nikiforov-Uvarov (NU) method [80] and Asymptotic Iteration (AI) method [81]. Energy eigenvalue equations and corresponding wave functions of the Dirac particles are obtained. Some numerical solutions are given in terms of numerical values of the potential parameters. The contribution of angle-dependent potential to the energy spectra of the Dirac particles is investigated by taking into account the Hulthén plus ring-shaped potential. Particular cases of the potential are also discussed. By considering some special limits, our results are reduced to the well-known ones obtained previously. In addition, by solving the Dirac equation for equal mixture of scalar and vector potentials together with tensor potential, we investigate the energy splitting between the two states in the pseudospin doublets. We show that degeneracy between members of pseudospin doublet is removed by tensor interactions. Effects of the potential parameters on the pseudospin doublet splitting are also studied. Radial node structures of the lower and upper components of the Dirac spinor are presented.

The thesis is organized as follows, pseudospin symmetry in nuclear physics is briefly introduced in the following chapter. The Dirac equation with pseudospin symmetry is presented in chapter 3. In chapter 4, solutions of the Dirac equation with well-known potentials under the condition of the pseudospin symmetry are performed and corresponding results are given. The last chapter is devoted to conclusion.

CHAPTER 2

PSEUDOSPIN SYMMETRY IN NUCLEAR PHYSICS

The pseudospin symmetry has been successfully applied to explain different phenomena in nuclear structure including deformation [4], superdeformation [5] and identical bands [6, 7] since it was introduced in 1969 [1, 2]. Because of these successes, much effort has been put into clarifying the origin of the pseudospin symmetry in atomic nuclei since discovery of this symmetry [8, 9, 10, 11, 12, 13]. Blokhin *et. al.* [8, 9, 10] have shown that there is a connection between the pseudospin symmetry and the relativistic mean field theory. Although the history of the pseudospin symmetry has been very well known for a long time, its origin has no accurate explanation in the non-relativistic framework. In recent years, Ginocchio [13] has noticed that pseudo-orbital angular momentum $\tilde{l}$ is nothing but the usual orbital angular momentum of the small (lower) component of the Dirac spinor. A connection between $V(\vec{r}) + S(\vec{r}) \sim 0$ and pseudospin symmetry has been also established in Refs. [13, 18] and then it has been built that the origin of pseudospin symmetry is relativistic. After these pioneering studies, there has been spent much efforts to investigate the pseudospin symmetry and spin symmetry in nuclei [16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48].

Ginocchio [18, 19] has studied the relativistic magnetic dipole transitions and relativistic Gamov-Teller transitions in nuclei under the condition of pseudospin symmetry. He has shown that although these two transitions between the members of pseudospin doublet are forbidden in the non-relativistic framework, both of them are admissible relativistically. In the other work, he [20] has investigated the pseudospin symmetry in nucleon-nucleus scattering and obtained that it has validity for medium energy nucleon scattering. The violation of the pseudospin symmetry in nucleon-nucleus

scattering has been studied in Ref. [10]. The effects of the pseudospin and spin symmetry breaking on the single-nucleon wave functions in nuclei have been explored in Ref. [30]. By considering Zr and Sn isotopes from the proton drip line to the neutron drip line, Meng *et. al.* [14] have explored the pseudospin symmetry in exotic nuclei. They have found out that the competition between pseudo-centrifugal barrier and the pseudo-orbital potentials determines the quality of the pseudospin symmetry. In order to investigate the pseudospin symmetry in nuclei, solution of the Dirac equation for Woods-Saxon potential has been obtained by Alberto *et. al.* [26]. They have also studied the energy splittings between members of pseudospin doublets with nuclear potential parameters. Based on the relativistic mean field theory, Chen *et. al.* [27] have explored the origin of the pseudospin symmetry and its broken in real nuclei. They have proved that pseudospin symmetry is approximate in real nuclei. The pseudospin symmetry in the resonant states has been studied by solving the Dirac equation for Woods-Saxon potential in Ref. [39].

2.1 The Dirac Hamiltonian

Ginocchio *et. al.* [12, 13] have indicated that origin of the pseudospin symmetry is related with the invariance of the Dirac Hamiltonian for $V(\vec{r}) + S(\vec{r}) = 0$ under an $SU(2)$ algebra. In the presence of an external scalar $S(\vec{r})$ and vector $V(\vec{r})$ potentials, the Dirac Hamiltonian is given as

$$H = [\alpha \cdot \mathbf{p} + \beta\,(M + S(\vec{r})) + V(\vec{r})] \tag{2.1}$$

where $\mathbf{p} = -i\nabla$ is the three-dimensional momentum operator. Here, we take $\hbar = c = 1$ for the simplicity. In this equation, α and β are the 4×4 usual Dirac matrices given as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \text{with } i = 1, 2, 3 \tag{2.2}$$

where I is the 2×2 unitary matrix and the three 2×2 Pauli matrices σ_i are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

For two conditions: $V(\vec{r}) + S(\vec{r}) = C_{ps}$ and $V(\vec{r}) - S(\vec{r}) = C_s$ where C_{ps} and C_s are constants, the Hamiltonian (2.1) is invariant under $SU(2)$ transformation [82, 83]. Energy spectrums of some mesons having small spin-orbit splitting have been explained by using the limit $V(\vec{r}) - S(\vec{r}) = C_s$ [84]. In addition, this limit has been used to investigate the spectrum of an antinucleon in the mean field of nucleons [49, 85]. The second limit leads to pseudospin symmetry in nuclei [12].

The pseudospin generators have the following form [12]

$$S_i = \begin{pmatrix} \tilde{s}_i & 0 \\ 0 & s_i \end{pmatrix}. \tag{2.4}$$

Here, s_i are the usual spin generators and are given as

$$s_i = \frac{\sigma_i}{2} \tag{2.5}$$

$$\tilde{s}_i = U_p s_i U_p \tag{2.6}$$

where $U_p = \frac{\sigma \cdot \mathbf{p}}{p}$ is the momentum-helicity unitary operator [9]. This operator preserves the parity, time-reversal, rotational and translational invariance when it deals with the transformation from the usual shell model space to pseudo shell model space [9]. Commutation relation between the Dirac Hamiltonian (2.1) and the operator S_i is [26]

$$[H, S_i] = \begin{pmatrix} [V(\vec{r}) + S(\vec{r}), \tilde{s}_i] & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.7}$$

When $V(\vec{r}) + S(\vec{r}) = C_{ps}$ or $\frac{d}{dr}(V(\vec{r}) + S(\vec{r})) = 0$, S_i commutes with the Hamiltonian. Therefore, the operator S_i generates an $SU(2)$ symmetry of the Dirac Hamiltonian [33] under the condition of the pseudospin symmetry.

2.2 The Dirac Eigenfunctions

The Dirac wave functions for the two states in the pseudospin doublet structure are given as [13, 14, 15]

$$\Psi_{n,\kappa<0}(\vec{r}) = \begin{pmatrix} \frac{F_{n,\kappa<0}(r)}{r} Y_{jm}^{\tilde{l}-1}(\theta,\phi) \\ i\frac{G_{n,\kappa<0}(r)}{r} Y_{jm}^{\tilde{l}}(\theta,\phi) \end{pmatrix} \tag{2.8}$$

$$\Psi_{n,\kappa>0}(\vec{r}) = \begin{pmatrix} \frac{F_{n,\kappa>0}(r)}{r} Y_{jm}^{\tilde{l}+1}(\theta,\phi) \\ i\frac{G_{n,\kappa>0}(r)}{r} Y_{jm}^{\tilde{l}}(\theta,\phi) \end{pmatrix} \tag{2.9}$$

where $G_{n\kappa}(r)$ and $F_{n\kappa}(r)$ are the lower and upper radial functions, respectively, $Y_{jm}^{\tilde{l}}(\theta,\phi)$ and $Y_{jm}^{l}(\theta,\phi)$ are the pseudospin and spin spherical harmonics, respectively. Here, κ are the eigenvalues of spin-orbit coupling operator $\mathbf{K} = -\beta(\sigma \cdot \mathbf{L} + 1)$ and $\kappa = (j + \frac{1}{2}) > 0$ and $\kappa = -(j + \frac{1}{2}) < 0$ for the unaligned spin $j = l - \frac{1}{2}$ and the aligned spin $j = l + \frac{1}{2}$, respectively. The members of the pseudospin doublet contain unaligned pseudospin for $\kappa < 0$ and aligned pseudospin for $\kappa > 0$. The two states in the doublet have the same energy and connected by the pseudospin generators S_i given by Eq. (2.4). In the pseudospin symmetry, radial wave functions of the lower components which are a doublet according to s_i are equal in shape and magnitude for the members of the doublet [12, 13, 14]. This is the consequence of the relativistic pseudospin symmetry $SU(2)$. Thus, the node structures of the $G_{n,\kappa>0}(r)$ and $G_{n,\kappa<0}(r)$ are also same. On the other hand, upper components $F_{n,\kappa<0}(r)$ and $F_{n,\kappa>0}(r)$ have n and $n-1$ nodes, respectively. One cannot reach these results by considering the pseudospin symmetry in the non-relativistic framework. The structure of the radial nodes occurring in pseudospin doublets has been investigated in Ref. [25] under the condition of the pseudospin symmetry.

2.3 Pseudospin Symmetry Breaking

In the Dirac phenomenology, the following condition is necessary to form bound states in real nuclei

$$\Sigma(\vec{r}) = V(\vec{r}) + S(\vec{r}) < E - M \tag{2.10}$$

where $M - E$ is the binding energy (for more details see Ref. [12, 22]). This means that there has to be a region where binding energy is smaller than the depth of the single-nucleon potential [12]. However, vector potential $V(\vec{r})$ and scalar potential $S(\vec{r})$ have same magnitude but different in sign ($V(\vec{r}) = -S(\vec{r})$) in the exact pseudospin symmetry. Therefore, the bound Dirac valance states cannot be formed in the exact pseudospin symmetry. Thus, pseudospin symmetry must be broken in nuclei [12]. Notwithstanding, it has been shown that the necessary breaking of the pseudospin symmetry in nuclei is small [14, 22, 86, 87]. As a consequence, pseudospin symmetry is an approximate symmetry in real nuclei.

In recent papers, the bound state properties of the nuclei have been explained by considering the pseudospin symmetry breaking (see Ref. [20] and references therein). In addition, Ginocchio has used the pseudospin symmetry breaking to reexamine the nucleon-nucleus scattering. He has obtained that pseudospin symmetry has validity for medium energy nucleon scattering.

2.4 Pseudospin Symmetry and QCD Sum Rules

The near equality in magnitude of a repulsive vector and an attractive scalar potentials $V(\vec{r}) \approx -S(\vec{r})$ in nuclear matter can be proved by using the QCD sum rules (see Ref. [43] and references therein). One can obtain the following scalar and vector potentials for the nucleon by using the detailed QCD sum rules [43]

$$S = -\frac{4\pi^2 \sigma_N \rho_N}{M^2 m_q} \tag{2.11}$$

$$V = \frac{32\pi^2 \rho_N}{M^2} \tag{2.12}$$

where σ_N is scalar density of quarks, ρ_N is taken to be the central matter density of nuclei, q is the quark field operator, m_q is the quark mass and N is the nucleon state. In Eqs. (2.11) and (2.12), σ_N, ρ_N, m_q and M are positive. Thus, scalar potential is attractive and vector potential is repulsive. This result is in good agreement with one obtained in the relativistic mean field theory.

From Eqs. (2.11) and (2.12), the ratio of scalar potential to vector potential is

$$\frac{S}{V} = -\frac{\sigma_N}{8m_q}. \tag{2.13}$$

In QCD, average quark mass m_q and scalar density of quarks σ_N in the proton are approximately $5MeV$ and $45MeV$, respectively. Thus the ratio between S and V becomes

$$\frac{S}{V} \approx -1.1. \tag{2.14}$$

Above result coincides mysteriously with one determined in relativistic mean field models [49].

CHAPTER 3

METHOD

3.1 The Dirac Equation with Pseudospin Symmetry

One of the most significant wave equations is the Dirac equation. It is commonly used to describe the elementary spin-1/2 particles dynamics, such as electron, proton and neutron, in relativistic quantum mechanics. Solutions of the Dirac equation in the presence of the mixed potentials for the pseudospin symmetry are very essential in describing the nuclear shell structure [1, 2, 55]. Thus, in recent years, investigation of the pseudospin symmetry whose detailed discussion will be given in the next section by solving the Dirac equation with exactly solvable potentials in terms of the different methods has attracted much attention [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79]. Solutions of the Dirac and $s(\tilde{s})$-wave Dirac Equations for the Woods-Saxon potential under the conditions of the pseudospin and spin symmetry have been investigated in Refs. [51, 52, 53]. In the pseudospin and spin symmetry limits, Jia *et. al.* [56] have found out the exact solution of the $s(\tilde{s})$-wave Dirac Equation with Eckart potential by using the sypersymmetric quantum mechanics approach and functional analysis method. By considering same potential, Zhang *et. al.* [54] and Soylu *et. al.* [55] have obtained the solutions of the Dirac equation in terms of different methods for any spin-orbit coupling term κ under the conditions of the pseudospin and spin symmetry. Approximate analytical solutions of the Dirac equation with the Pöschl-Teller potential including the spin-orbit coupling term have been studied by Jia *et. al.* [57, 59] and Xu *et. al.* [58]. They have obtained the bound state energy eigenvalues and corresponding wave functions of the Dirac particles in the case of pseudospin symmetry. To get the energy eigenvalues equation and the associated radial wave functions of the Dirac particles

in the spin and pseudospin limits, Taskin [60] and Wei *et. al.* [61] have solved approximately the Dirac equation with the Manning-Rosen potential for the any κ term. Berkdemir [62], Qiang *et. al.* [63] and Bayrak *et. al.* [64] have investigated the pseudospin symmetry by solving the Dirac equation for the Morse potential in terms of different methods. They have also given some numerical solutions for special values of the relevant potential parameters. Pseudospin symmetric solution of the Dirac equation for an angle-dependent potential has been solved by Berkdemir *et. al.* [65]. In their study, they have also discussed the contribution of the angle-dependent potential to the energy eigenvalues of the Dirac particles. In addition, some authors have sought pseudospin symmetric solutions of the Dirac equation for several physical potentials [66, 67, 68, 69, 70, 71, 72].

Recently, exact solutions of the Dirac equation for the quadratic vector and scalar potentials together with a linear tensor potential in the pseudospin and spin symmetry cases have been studied by Lisboa *et al.* [79, 36]. More recently, Akcay [76, 77] has obtained the exact solution of the the Dirac equation for scalar and vector quadratic potentials and Coulomb-like tensor potential under the pseudospin and spin symmetry. Both Lisboa *et al.* and Akcay have argued that the degeneracy between two states in the pseudospin doublet can be removed by tensor interactions.

In the relativistic mean field theory, time-independent Dirac equation for a nucleon with mass M moving in a repulsive vector potential $V(\vec{r})$ and an attractive scalar potential $S(\vec{r})$ reads as follows [88]

$$\left[\alpha \cdot \mathbf{p} + \beta \left(M + S(\vec{r})\right) + V(\vec{r})\right] \Psi_{n\kappa}(\vec{r}) = E_{n\kappa} \Psi_{n\kappa}(\vec{r}) \tag{3.1}$$

where $E_{n\kappa}$ is the relativistic energy of the system. The total angular momentum operator $\mathbf{J}$ and spin-orbit coupling operator $\mathbf{K} = -\beta(\sigma \cdot \mathbf{L} + 1)$, where $\mathbf{L}$ is orbital angular momentum, of the spherical nucleons commute with the Dirac Hamiltonian. The eigenvalues of spin-orbit coupling operator are $\kappa = (j + \frac{1}{2}) > 0$ and $\kappa = -(j + \frac{1}{2}) < 0$ for the unaligned spin $j = l - \frac{1}{2}$ and the aligned spin $j = l + \frac{1}{2}$, respectively. (H, K, J^2, J_z) can be taken as the complete set of the conservative quantities. Thus, the Dirac spinors can be written according to radial quantum number n and spin-orbit coupling quantum number κ as follows

$$\Psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa}(\vec{r}) \\ g_{n\kappa}(\vec{r}) \end{pmatrix} \tag{3.2}$$

where $f_{n\kappa}(\vec{r})$ is the upper (large) and $g_{n\kappa}(\vec{r})$ is the lower (small) components of the Dirac spinors. Inserting Eq. (3.2) into Eq. (3.1), we can get the following coupled differential equations

$$f_{n\kappa}(\vec{r}) = \frac{(\sigma \cdot \mathbf{p})}{[E_{n\kappa} - M - \Sigma(\vec{r})]}\, g_{n\kappa}(\vec{r}) \tag{3.3}$$

$$g_{n\kappa}(\vec{r}) = \frac{(\sigma \cdot \mathbf{p})}{[E_{n\kappa} + M - \Delta(\vec{r})]}\, f_{n\kappa}(\vec{r}) \tag{3.4}$$

where $\Sigma(\vec{r}) = V(\vec{r}) + S(\vec{r})$ and $\Delta(\vec{r}) = V(\vec{r}) - S(\vec{r})$. As we mentioned in the introduction, the pseudospin symmetry occurs in the Dirac equation when $\Sigma(\vec{r}) = C_{ps} = Constant$ and pseudo-orbital angular momentum is normal orbital angular momentum of the lower (small) component of the Dirac spinor [12]. Thus, we deal with the lower component of the Dirac spinor to investigate the pseudospin symmetry in the Dirac phenomenology. By eliminating $f_{n\kappa}(\vec{r})$ from Eq. (3.4), the following uncoupled equation for the lower component can be obtained

$$\left\{ \sigma \cdot \mathbf{p}\, \frac{1}{[E_{n\kappa} - M - \Sigma(\vec{r})]}\, \sigma \cdot \mathbf{p} - [E_{n\kappa} + M - \Delta(\vec{r})] \right\} g_{n\kappa}(\vec{r}) = 0. \tag{3.5}$$

This equation can be separated into the angular and radial wave functions with respect to potentials $\Sigma(\vec{r})$ and $\Delta(\vec{r})$. By considering general form of the second-order differential equation (3.5), we can now give the derivation of the second-order differential equations for the radial and angular wave functions of the Dirac spinor.

3.1.1 The Dirac Equation for Radial Potentials

If the potential considered has only radial dependence $\Delta(\vec{r}) = W(r)$, Eq. (3.5) is reduced into the following form under the condition of the pseudospin symmetry $\Sigma(\vec{r}) = C_{ps} = constant$:

$$\{(\sigma \cdot \mathbf{p})\ \ (\sigma \cdot \mathbf{p}) - [E_{n\kappa} - M - C_{ps}]\,[E_{n\kappa} + M - W(r)]\}\, g_{n\kappa}(\vec{r}) = 0. \tag{3.6}$$

Lower component of the Dirac spinor can be written in terms of radial and angular functions as follows

$$g_{n\kappa}(\vec{r}) = i\frac{G_{n\kappa}(r)}{r}Y^{\tilde{l}}_{jm}(\theta,\phi)\Lambda_{\xi} \tag{3.7}$$

where $G_{n\kappa}(r)$ is the lower radial function, $Y^{\tilde{l}}_{jm}(\theta,\phi)$ is the pseudospin spherical harmonics, $\xi = \pm\frac{1}{2}$ is the projection of pseudo-angular momentum on the z-axis and Λ_{ξ} are two component spinors, i.e $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, pseudo-orbital angular momentum $\tilde{l}$ and pseudospin $\tilde{s}$ can be separately seen in the lower spinor component. Inserting Eq. (3.7) into Eq. (3.6) and using the following relation [88]

$$\sigma\cdot\mathbf{p} = \frac{\sigma\cdot\hat{r}}{r}\left(r\ \hat{r}\cdot\mathbf{p} + i\sigma\cdot\mathbf{L}\right) \tag{3.8}$$

together with the useful properties [89]

$$\sigma\cdot\mathbf{L}\begin{cases} Y^{\tilde{l}}_{jm}(\theta,\phi) \\ Y^{l}_{jm}(\theta,\phi) \end{cases} = \begin{cases} (\kappa-1)Y^{\tilde{l}}_{jm}(\theta,\phi) \\ -(\kappa+1)Y^{l}_{jm}(\theta,\phi) \end{cases} \tag{3.9}$$

$$\frac{\sigma\cdot\vec{r}}{r}\begin{cases} Y^{\tilde{l}}_{jm}(\theta,\phi) \\ Y^{l}_{jm}(\theta,\phi) \end{cases} = \begin{cases} -Y^{l}_{jm}(\theta,\phi) \\ -Y^{\tilde{l}}_{jm}(\theta,\phi) \end{cases} \tag{3.10}$$

Eq. (3.6) turns into a second-order differential equation:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \left(M + E_{n\kappa} - W(r)\right)\left(M - E_{n\kappa} + C_{ps}\right)\right]G_{n\kappa}(r) = 0. \tag{3.11}$$

This is the second-order Schrödinger-like radial equation for the lower component of the Dirac spinors. Here, $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$ and relation between κ and $\tilde{l}$ is given as

$$\kappa \;=\; \begin{cases} (\tilde{l}+1) & j = \tilde{l}+1/2 \qquad \text{aligned} \quad \text{pseudospin} \\ -\tilde{l} & j = \tilde{l}-1/2 \qquad \text{unaligned} \quad \text{pseudospin} \end{cases}. \tag{3.12}$$

3.1.2 The Dirac Equation for Radial Plus Θ-dependent Potentials

Now, we consider radial plus angular-dependent potential $\Delta(\vec{r}) = W(r) + \frac{W(\theta)}{r^2}$. In this case, under the condition of the pseudospin limit $\Sigma(\vec{r}) = C_{ps} = constant$, Eq. (3.5) becomes

$$\left\{\mathbf{p}^2 - [E_{n\kappa} - M - C_{ps}]\left[E_{n\kappa} + M - \left(W(r) + \frac{W(\theta)}{r^2}\right)\right]\right\} g_{n\kappa}(\vec{r}) = 0 \tag{3.13}$$

where $\mathbf{p}^2 = -\vec{\nabla}^2$. In the spherical coordinates, lower component of the Dirac wave functions can be written as follows

$$g_{n\kappa}(\vec{r}) = \frac{G(r)}{r} H(\theta)\Phi(\phi)\tilde{\Lambda}_{\xi}. \tag{3.14}$$

Inserting Eq. (3.14) into Eq. (3.13), we obtain three second-order differential equations for the angular and the radial wave functions as follows

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \tilde{m}^2\Phi(\phi) = 0 \tag{3.15}$$

$$\frac{d^2H(\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dH(\theta)}{d\theta} + \left[\tilde{l}(\tilde{l}+1) - \frac{\tilde{m}^2}{\sin^2\theta} - \gamma W(\theta)\right] H(\theta) = 0 \tag{3.16}$$

$$\frac{d^2G(r)}{dr^2} + \left[\beta^2 - \frac{\tilde{l}(\tilde{l}+1)}{r^2} - \gamma W(r)\right] G(r) = 0 \tag{3.17}$$

with

$$\gamma = E_{n\kappa} - M - C_{ps} \tag{3.18}$$

$$\beta^2 = (E_{n\kappa} + M)(E_{n\kappa} - M - C_{ps}) \tag{3.19}$$

where $\tilde{m}$ and $\tilde{l}$ are separation constants and $\tilde{l}(\tilde{l}+1) = \kappa(\kappa-1)$.

3.1.3 The Dirac Equation for Radial Potential together with Tensor Potential

In the relativistic mean field theories and relativistic Hartree approach model, nuclear properties have been studied by using tensor coupling [90, 91, 92]. In those works,

they have shown that single-particle level splitting in nuclei increases considerably spin-orbit coupling. In this context, Alberto *et. al.* [79, 32, 93] have argued that contribution of the tensor coupling to pseudospin splitting can be significant in nuclei as well. They have found out that effects of the tensor coupling on the two states in the pseudospin doublet are considerable. Thus, investigation of the pseudospin doublet splitting by introducing tensor coupling is very significant.

The time-independent Dirac equation of a nucleon with mass M moving in the presence of a potential (more details can be found in Ref. [79] and references therein) is given as

$$H\Psi(\vec{r}) = E\Psi(\vec{r}) \tag{3.20}$$

where

$$H = \alpha \cdot \mathbf{p} + \beta M + W \tag{3.21}$$

is the Dirac Hamiltonian. Here W is the matrix potential and it can be written as a linear combination of sixteen linearly independent matrices. Under the Lorentz transformation, these matrices can be classified as scalar, pseudoscalar, vector, pseudovector and tensor. In order to investigate the effects of the tensor interaction on the pseudospin doublet splitting, we can consider the following potential

$$W = \beta S(\vec{r}) + V(\vec{r}) - i\beta\alpha \cdot \hat{r}U(r) \tag{3.22}$$

where $V(r)$ is the time component of a vector potential, $S(r)$ is a scalar potential and $U(r)$ is a tensor potential.

As mentioned in the section 2.1 , the Dirac spinor can be written in the following form

$$\Psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa}(\vec{r}) \\ g_{n\kappa}(\vec{r}) \end{pmatrix} = \begin{pmatrix} \frac{F_{n\kappa}(r)}{r} Y^{l}_{jm}(\theta,\phi) \\ i\frac{G_{n\kappa}(r)}{r} Y^{\tilde{l}}_{jm}(\theta,\phi) \end{pmatrix} \tag{3.23}$$

where $Y_{jm}^{l}(\theta,\phi)$ and $Y_{jm}^{\tilde{l}}(\theta,\phi)$ are the spin and pseudospin spherical harmonics, respectively. Inserting Eq. (3.23) into Eq. (3.20), the Dirac equation becomes

$$(\sigma\cdot\mathbf{p}-i\sigma\cdot\hat{r}U(r))\,\frac{iG_{n\kappa}(r)}{r}Y_{jm}^{\tilde{l}}(\theta,\phi)=[E_{n\kappa}-M-\Sigma(\vec{r})]\,\frac{F_{n\kappa}(r)}{r}Y_{jm}^{l}(\theta,\phi) \quad (3.24)$$

$$(\sigma\cdot\mathbf{p}+i\sigma\cdot\hat{r}U(r))\,\frac{F_{n\kappa}(r)}{r}Y_{jm}^{l}(\theta,\phi)=[E_{n\kappa}+M-\Delta(\vec{r})]\,\frac{iG_{n\kappa}(r)}{r}Y_{jm}^{\tilde{l}}(\theta,\phi). \quad (3.25)$$

From now on, we consider that $\Sigma(\vec{r})=C_{ps}=constant$ (pseudospin symmetry) and $\Delta(\vec{r})=W(r)$ has only radial dependence. By using Eqs. (3.8), (3.9) and (3.10), one can obtain the following coupled equations

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r) = [M+E_{n\kappa}-W(r)]\;G_{n\kappa}(r) \quad (3.26)$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r) = [M-E_{n\kappa}+C_{ps}]\;F_{n\kappa}(r). \quad (3.27)$$

Eliminating $F_{n\kappa}(r)$ from Eq. (3.26), second-order differential equation for the lower radial component of the Dirac spinor is computed as

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+(E+M-W(r))\,(E-M-C_{ps})\right.$$
$$\left.+\frac{2\kappa}{r}U(r)-U(r)^2+\frac{dU(r)}{dr}\right]G_{n\kappa}(r) = 0. \quad (3.28)$$

In the literature, several methods such as the NU method, AI method, functional analysis method, supersymmetric WKB approximation, supersymmetric quantum mechanics approach and exact quantization rule have been used to solve the Schrödinger, Klein-Gordon and Dirac equations [54, 55, 56, 62]. In the present thesis, we shall use the NU method, AI method and functional analysis method. In the next subsections, we briefly introduce AI method and NU method.

3.2 The Asymptotic Iteration Method

Ciftci *et al* [81] have proposed the AI method to solve the second-order differential equations which have the following form (see Ref. [64] and references therein)

$$y_n^{''}(r) = \lambda_0(r) y_n^{'}(r) + s_0(r) y_n(r) \tag{3.29}$$

where prime denotes the derivative with respect to r and $\lambda_0(r)$ is different from zero. Here, $\lambda_0(r)$ and $s_0(r)$ are sufficiently differentiable functions. One can differentiate Eq. (3.29) with respect to r and iterate it up to $(k+1)$th and $(k+2)$th derivatives to obtain a general solution to the second-order differential equation. Then, one can find out the following equations

$$y_n^{(k+1)}(r) = \lambda_{k-1}(r) y_n^{'}(r) + s_{k-1}(r) y_n(r) \tag{3.30}$$

$$y_n^{(k+2)}(r) = \lambda_k(r) y_n^{'}(r) + s_k(r) y_n(r) \tag{3.31}$$

where

$$\lambda_k(r) = \lambda_{k-1}^{'}(r) + s_{k-1}(r) + \lambda_0(r)\lambda_{k-1}(r) \tag{3.32}$$

$$s_k(r) = s_{k-1}^{'}(r) + s_0(r)\lambda_{k-1}(r) \tag{3.33}$$

which are called as the recurrence relations. The following equation can be easily obtained from the ratio of the $(k+1)$th and $(k+2)$th derivatives :

$$\frac{d}{dr} \ln \left[y_n^{(k+1)}(r) \right] = \frac{y_n^{(k+2)}(r)}{y_n^{(k+1)}(r)} = \frac{\lambda_k(r) \left[y_n^{'}(r) + \frac{s_k(r)}{\lambda_k(r)} y_n(r) \right]}{\lambda_{k-1}(r) \left[y_n^{'}(r) + \frac{s_{k-1}(r)}{\lambda_{k-1}(r)} y_n(r) \right]}. \tag{3.34}$$

Using asymptotic aspect of the method which is acceptable for adequately large k

$$\frac{s_k(r)}{\lambda_k(r)} = \frac{s_{k-1}(r)}{\lambda_{k-1}(r)} =: \varsigma(r) \tag{3.35}$$

Eq. (3.34) can be turned into

$$\frac{d}{dr} ln \left[y_n^{(k+1)}(r) \right] = \frac{\lambda_k(r)}{\lambda_{k-1}(r)} \tag{3.36}$$

which leads to

$$y_n^{(k+1)}(r) = D_0 \exp\left(\int \frac{\lambda_k(r)}{\lambda_{k-1}(r)} dr\right) = D_0 \lambda_{k-1}(r) \exp\left(\int \left[\varsigma(r) + \lambda_0(r)\right] dr\right) \quad (3.37)$$

where D_0 is the integration constant. Substituting Eq. (3.37) into Eq. (3.30), the first-order differential equation is found as

$$y_n^{'}(r) = -\varsigma(r) y_n(r) + D_0 \exp\left(\int \left[\varsigma(r) + \lambda_0(r)\right] dr\right). \quad (3.38)$$

Physical solution of the Eq. (3.29) can be obtained by using the solution of the first-order differential equation as

$$y_n(r) = D_1 \exp\left(-\int^r \frac{s_n(r_1)}{\lambda_n(r_1)} dr_1\right) \quad (3.39)$$

where D_1 is the integration constant and n represents the radial quantum number. Eq. (3.39) is called as a wave function generator. In the asymptotic iteration method, energy eigenvalues are found out from the root of the following equation

$$\Delta_k(r) = \lambda_k(r) s_{k-1}(r) - \lambda_{k-1}(r) s_k(r) = 0 \quad \text{with} \quad k = 1, 2, 3, ... \quad (3.40)$$

where k is the iteration number. For an exactly solvable potential, the radial second-order differential equation is converted to the form of Eq. (3.29). Then, $s_k(r)$ and $\lambda_k(r)$ parameters are computed by considering the recurrence relations given in Eqs. (3.32) and (3.33) after determining $s_0(r)$ and $\lambda_0(r)$ parameters. Finally, using Eqs. (3.39) and (3.40), energy eigenvalues equations and corresponding radial wave functions can be obtained. As significant point, radial quantum number n is all times smaller than iteration number k in the numerical solutions [81, 64].

3.3 The Nikiforov-Uvarov Method

Nikiforov *et al.* [80] have presented the NU method to obtain the exact solution of the second-order differential equations such as the Schrödinger, Klein-Gordon and Dirac equations. The second-order differential equations can be given as the following form

$$\Psi''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\Psi'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\Psi(s) = 0 \tag{3.41}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are at most second-degree polynomials, and $\widetilde{\tau}(s)$ is a first-degree polynomial. In the Nikiforov-Uvarov method, second-order differential equation is reduced to hypergeometric type equation by using

$$\Psi(s) = \Omega(s)y(s) \tag{3.42}$$

transformation which leads

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{3.43}$$

where

$$\pi(s) = \sigma(s)\frac{\Omega'(s)}{\Omega(s)} \tag{3.44}$$

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s) \quad \text{with} \quad \tau'(s) < 0 \tag{3.45}$$

and

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \quad \text{with} \quad n = 0, 1, 2.... \tag{3.46}$$

Solutions of Eq. (3.43) are given by the Rodrigues relation [94]

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{3.47}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the following condition

$$[\sigma(s)\rho(s)]' = \tau(s)\rho(s) \tag{3.48}$$

where prime denotes the derivative with respect to s. The function $\pi(s)$ is defined as

$$\pi(s) = \frac{\sigma'(s) - \widetilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \widetilde{\tau}(s)}{2}\right)^2 - \widetilde{\sigma}(s) + k\sigma(s)} \tag{3.49}$$

and λ parameter is also given in terms of $\pi'(s)$ and k as

$$\lambda = k + \pi'(s). \tag{3.50}$$

The expression under the square root in Eq. (3.49) must be the square of a polynomial of first degree [80], since $\pi(s)$ is the first-degree polynomial. Then discriminant of the square root has to be zero. Using this relation, an equation for k is found and then, $\pi(s)$ can be easily obtained from relevant k values. Finally, comparing Eq. (3.46) with Eq. (3.50), one can calculate the energy eigenvalues for a given potential.

CHAPTER 4

CALCULATIONS AND RESULTS

4.1 Solutions of the Dirac Equation with Pseudospin Symmetry

Solutions of the Dirac equation in the presence of the mixed potentials for the pseudospin symmetry are very essential in describing the nuclear shell structure [1, 2, 55]. Thus, in recent years, there has been increased attention to investigate the pseudospin symmetry by solving the Dirac equation with exactly solvable potentials such as Woods-Saxon potential [51, 52, 53], Eckart potential [54, 55, 56], Pöschl-Teller potential [57, 58, 59], Manning-Rosen potential [60, 61], Morse potential [62, 63, 64] *etc.*. In this section, we shall investigate the solution of the Dirac equation with some well-known potentials under the condition of the pseudospin symmetry. In the calculations we use different methods such as NU method and AI method.

4.1.1 Pseudoharmonic Potential

The pseudoharmonic potential plays a fundamental role in chemical and molecular physics, since it can be used to describe the molecular vibrations and to obtain the energy spectrum of linear and non-linear systems. One can consider the pseudoharmonic potential as an intermediate potential between the harmonic oscillator potential and the Morse-type potentials which are more realistic anharmonic potentials in good agreement with experimental data [94, 95]. The pseudoharmonic potential and anharmonic potentials have wide applications in molecular and chemical physics [96, 97, 98, 99, 100]. The pseudoharmonic potential has some advantages according to the other anharmonic potentials such as Morse potential. For instance, wave function of the pseudoharmonic potential vanishes at the origin, but the Morse po-

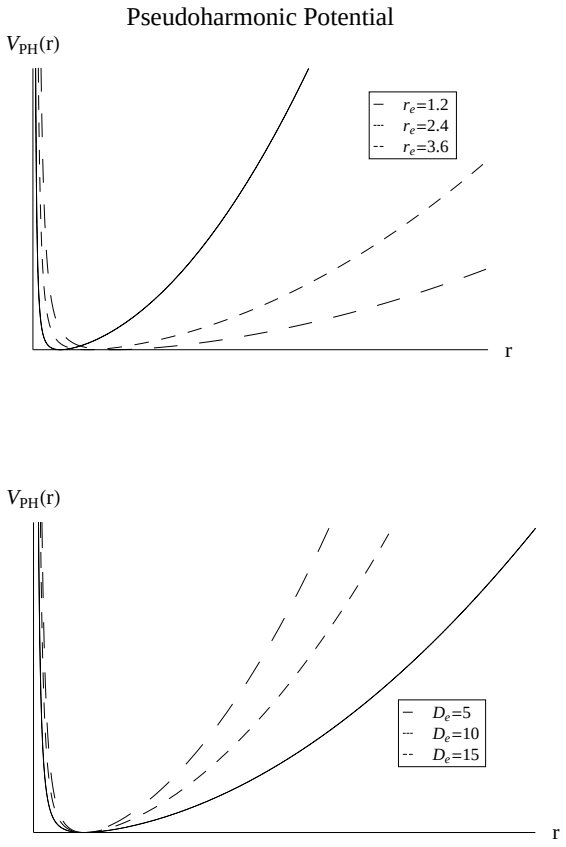

Figure 4.1: Pseudoharmonic potential as functions of the equilibrium intermolecular separation r_e and the dissociation energy D_e

tential does not. Besides, the pseudoharmonic potential is one of the exactly solvable potentials like Coulombic potential. The pseudoharmonic potential is also important to define the real physical systems which have generally anharmonical properties. Recently, solutions of the Schrödinger and the Klein-Gordon equations for the pseudoharmonic potential have been investigated [101, 102, 103, 104, 105, 106]. In those works, the authors have obtained the energy spectra of all the bound states. In other work, Gang *et al.* [107] have considered the special case of the pseudoharmonic potential evaluated the energy spectra and corresponding wave functions of the Klein-Gordon and Dirac equations by using supersymmetric quantum mechanics, shape invariance and alternative method. Recently, pseudoharmonic potential has also been studied by Dong to investigate the realization of the creation and annihilation operators [108].

The pseudoharmonic potential has the following form [109]

$$V(r) = D_e \left(\frac{r}{r_e} - \frac{r_e}{r} \right)^2 \tag{4.1}$$

where D_e is the dissociation energy and r_e is the equilibrium intermolecular separation. This potential can be simply written as [102, 110, 111]

$$V(r) = Ar^2 + \frac{B}{r^2} + C \tag{4.2}$$

where

$$A = D_e r_e^{-2}, \quad B = D_e r_e^2 \quad \text{and} \quad C = -2D_e. \tag{4.3}$$

We now investigate the solution of the Dirac equation with pseudoharmonic potential given in Eq. (4.2) under the condition of the pseudospin symmetry. By considering Eq. (3.11), we obtain the following second order differential equation for the relevant potential

$$\left[\frac{d^2}{dr^2} - \frac{\alpha}{r^2} - \nu^2 r^2 - \mu \right] G_{n\kappa}(r) = 0 \tag{4.4}$$

where α, ν and μ are the dimensionless parameters given as

$$\alpha = \kappa(\kappa - 1) + \gamma B \tag{4.5}$$

$$\nu = \sqrt{A\gamma} \tag{4.6}$$

$$\mu = \beta + C\gamma \tag{4.7}$$

with

$$\gamma = E_{n\kappa} - M - C_{ps} \tag{4.8}$$

$$\beta = (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}). \tag{4.9}$$

By using the following coordinate transformation

$$s = r^2 \tag{4.10}$$

we obtain

$$\frac{d^2G_{n\kappa}(s)}{ds^2} + \frac{1}{2s}\frac{dG_{n\kappa}(s)}{ds} - \frac{1}{4s^2}\left[\nu^2 s^2 + \mu s + \alpha\right] G_{n\kappa}(s) = 0. \tag{4.11}$$

In this form of the above equation, we can use the Nikiforov-Uvarov method to evaluate the solution of the relevant second-order differential equation. Thus, the following polynomials are obtained by comparing Eq. (4.11) with Eq. (3.41):

$$\widetilde{\tau}(s) = 1, \quad \sigma(s) = 2s, \quad \text{and} \quad \widetilde{\sigma}(s) = -(\nu^2 s^2 + \mu s + \alpha). \tag{4.12}$$

From Eqs. (3.49) and (4.12), we get the $\pi(s)$ as follow

$$\pi(s) = \frac{1}{2} \pm \frac{1}{2}\sqrt{4\nu^2 s^2 + 4(\mu + 2k)s + 1 + 4\alpha}. \tag{4.13}$$

In this equation, discriminant of the square root has to be zero due to $\pi(s)$ is at most first-degree polynomial. Then, we can determine the k values as

$$k_{\pm} = -\frac{\mu}{2} \pm \frac{\nu}{2}\sqrt{1 + 4\alpha} \tag{4.14}$$

which yields

$$\pi(s) = \left\{ \begin{array}{ll} \frac{1}{2} + \frac{1}{2}(2\nu s - \sqrt{1+4\alpha}) & \text{for} \quad k_- = -\frac{\mu}{2} - \frac{\nu}{2}\sqrt{1+4\alpha} \\ \frac{1}{2} - \frac{1}{2}(2\nu s - \sqrt{1+4\alpha}) & \text{for} \quad k_- = -\frac{\mu}{2} - \frac{\nu}{2}\sqrt{1+4\alpha} \\ \frac{1}{2} + \frac{1}{2}(2\nu s + \sqrt{1+4\alpha}) & \text{for} \quad k_+ = -\frac{\mu}{2} + \frac{\nu}{2}\sqrt{1+4\alpha} \\ \frac{1}{2} - \frac{1}{2}(2\nu s + \sqrt{1+4\alpha}) & \text{for} \quad k_+ = -\frac{\mu}{2} + \frac{\nu}{2}\sqrt{1+4\alpha} \end{array} \right\}. \tag{4.15}$$

In the Nikiforov-Uvarov method, $\tau'(s) < 0$ must be satisfied in order to obtain a physical eigenfunction. To do this, we can take the $k_- = -\frac{\mu}{2} - \frac{\nu}{2}\sqrt{1+4\alpha}$ and then, we have

$$\pi(s) = \frac{1}{2} - \frac{1}{2}(2\nu s - \sqrt{1+4\alpha}). \tag{4.16}$$

$\tau(s)$ is obtained from Eq. (3.45) as

$$\tau(s) = -2\nu s + 2 + \sqrt{1+4\alpha}. \tag{4.17}$$

Using Eqs. (3.46) and (3.50) and combining Eqs. (4.16) and (4.17), we find the λ_n and λ, respectively, as

$$\lambda_n = \lambda = 2n\nu \tag{4.18}$$

$$\lambda = -\frac{\mu}{2} - \frac{\nu}{2}\left(2 + \sqrt{1+4\alpha}\right). \tag{4.19}$$

Comparing Eq (4.18) with Eq. (4.19) and recalling the values of α, ν, μ, γ and β given in Eqs. (4.5), (4.6), (4.7), (4.8) and (4.9), respectively, we can consequently obtain the following energy eigenvalue equation

$$A\left(2 + 4n + \sqrt{(2\kappa-1)^2 + 4(E_{n\kappa} - M - C_{ps})B}\right)^2 - (E_{n\kappa} - M - C_{ps})(E_{n\kappa} + M - C)^2 = 0. \tag{4.20}$$

In the pseudospin symmetry limit, the Dirac wave function $G_{n\kappa}(r)$ can be found out in terms of special orthogonal functions after obtaining $\Omega(s)$ and $y_{n\kappa}(s)$ given in Eqs. (3.44) and (3.47), respectively. Using Eq. (3.44) with Eqs. (4.12) and (4.16), we have

$$\Omega(s) = s^{\frac{1}{4}+\frac{1}{4}\sqrt{1+4\alpha}} e^{-\frac{\nu s}{2}}. \tag{4.21}$$

Considering Eq. (3.48) with Eqs. (4.12) and (4.17), $\rho(s)$ is obtained as

$$\rho(s) = s^{\frac{1}{2}\sqrt{1+4\alpha}} e^{-\nu s} \tag{4.22}$$

and then, we get the $y_{n\kappa}(s)$ given by equation (3.47) as

$$y_{n\kappa}(s) = B_{n\kappa} e^{\nu s} s^{-\frac{1}{2}\sqrt{1+4\alpha}} \frac{d^n}{ds^n} \left((2s)^n s^{\frac{1}{2}\sqrt{1+4\alpha}} e^{-\nu s} \right) \tag{4.23}$$

where $B_{n\kappa}$ is the normalization constant. $y_{n\kappa}(s)$ can be expressed in terms of the generalized Laguerre Polynomials as follows [106]

$$y_{n\kappa}(s) = B_{n\kappa} L_n^{\epsilon}(\nu s) \tag{4.24}$$

where $\epsilon = \frac{1}{2}\sqrt{1+4\alpha}$. Finally, the lower radial wave function $G_{n\kappa}(r)$ is obtained as

$$G_{n\kappa}(r) = C_{n\kappa} r^{\frac{1}{2}+\epsilon} e^{-\frac{\nu r^2}{2}} L_n^{\epsilon}(\nu r^2). \tag{4.25}$$

Here, normalization constant $C_{n\kappa}$ is given in Refs. [94, 135] as

$$C_{n\kappa} = \sqrt{\frac{2\nu^{(1+\frac{\epsilon}{2})} n!}{(n+\frac{\epsilon}{2})!}}. \tag{4.26}$$

Consequently, upper radial wave function can be found from the following equation

$$F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} \right) G_{n\kappa}(r)$$

4.1.2 Mie-Type Potential

Mie-type potentials [112] can be taken by the following general form

$$V_{Mt} = \frac{A}{r^2} - \frac{B}{r} + C. \tag{4.27}$$

Kratzer-Fues potential can be given as an example on the Mie-type potentials by setting $A = D_e r_e^2$, $B = 2D_e r_e$ and $C = 0$ [113, 114]

$$V_{KF} = -D_e \left(\frac{2r_e}{r} - \frac{r_e^2}{r^2} \right) \tag{4.28}$$

where D_e is the dissociation energy and r_e is the equilibrium intermolecular length.

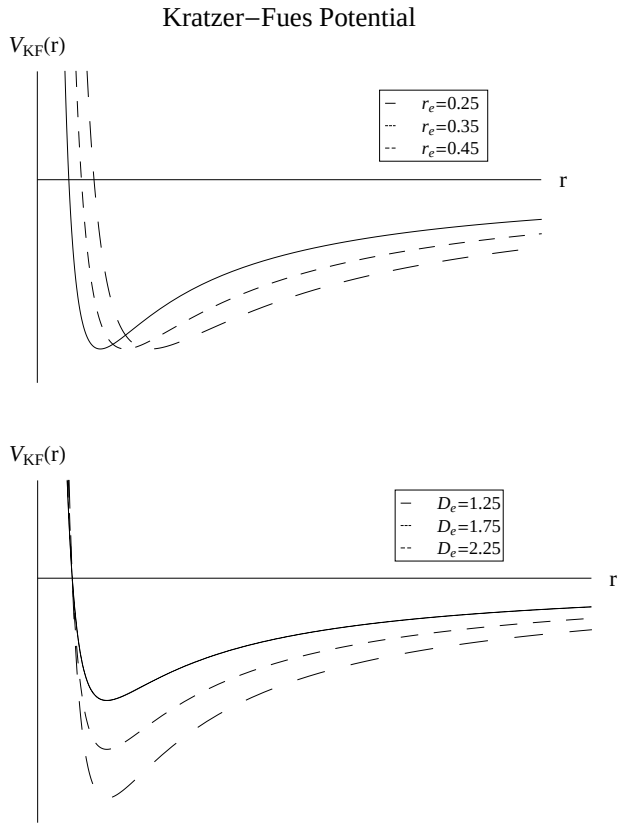

Figure 4.2: Kratzer-Fues potential as functions of the equilibrium intermolecular separation r_e and the dissociation energy D_e

The other example is the modified Kratzer potential obtained by setting $A = D_e r_e^2$, $B = 2D_e r_e$ and $C = D_e$ (see Ref. [115] and references therein) as

$$V_{MK} = D_e \left(\frac{r - r_e}{r} \right)^2 . \tag{4.29}$$

One can easily see that this potential is shifted in amount of D_e from the standard Kratzer potential. Kratzer-Fues and modified Kratzer potentials have been considered as a model to describe internuclear vibration of a diatomic molecule [116, 117, 118, 119, 120]. To find out the exact energy levels of all bound states and corresponding wave functions, some authors have solved the Schrödinger equation for these Mie-type potentials [130, 122]. In the pseudospin symmetry notion, one can include the diatomic potential models with the reduced mass ($\mu = \frac{m_1 m_2}{m_1 + m_2}$) if the nuclei have masses m_1 and m_2 [54]. Recently, considering some diatomic molecular potential models such

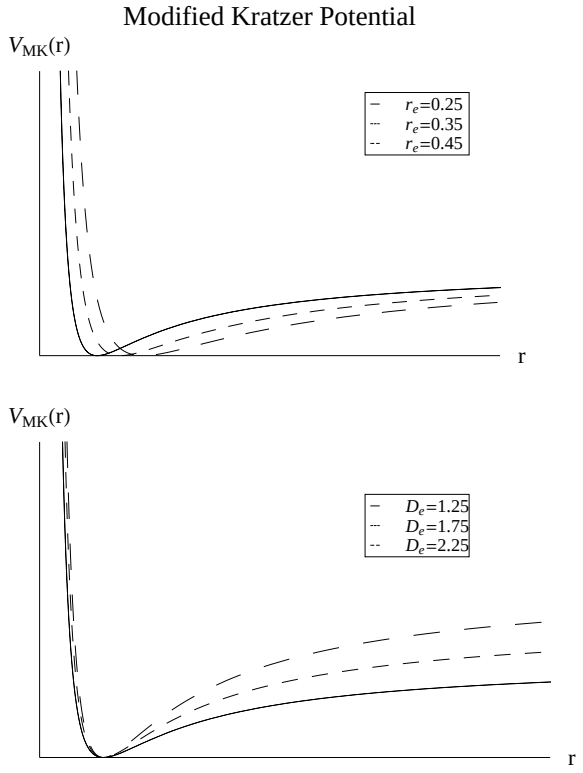

Figure 4.3: The modified Kratzer potential as functions of the equilibrium intermolecular separation r_e and the dissociation energy D_e, respectively

as Eckart potential [54] and Pöschl-Teller potential [58], the pseudospin symmetry and spin symmetry have been investigated by solving the Dirac equation in terms of different methods. We now investigate the bound state solution of the Dirac equation for the nuclei by considering Mie-type potential under the pseudospin symmetry.

In Eq. (3.11), if we take the $W(r)$ as the Mie-type potential, then we have the following second-order differential equation for the lower (small) component of the Dirac spinor

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \gamma\left(\frac{A}{r^2} - \frac{B}{r} + C\right) - \beta^2\right] G_{n\kappa}(r) = 0 \tag{4.30}$$

where

$$\gamma = E_{n\kappa} - M - C_{ps}, \tag{4.31}$$

$$\beta^2 = (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}). \tag{4.32}$$

In order to obtain the solution of Eq. (4.30), we should convert this equation into the form of Eq. (3.29). If we take $G_{n\kappa}(r) = r^{\frac{1}{2}} g_{n\kappa}(r)$, then Eq. (4.30) reduces to the following form

$$\frac{d^2 g_{n\kappa}(r)}{dr^2} + \frac{1}{r}\frac{dg_{n\kappa}(r)}{dr} + \left[-\frac{\varepsilon^2}{r^2} + \frac{\nu^2}{r} - \mu^2\right] g_{n\kappa}(r) = 0 \tag{4.33}$$

where $\varepsilon^2 = \kappa(\kappa-1)+\gamma A+\frac{1}{4}$, $\nu^2 = \gamma B$ and $\mu^2 = \gamma C+\beta^2$. Considering the asymptotic behavior of the radial wave function as $g_{n\kappa}(\infty) \sim e^{-\mu r}$ for $r \to \infty$ and $g_{n\kappa}(0) \sim r^{\varepsilon}$ for $r \to 0$, the reasonable physical solution can be expressed as

$$g_{n\kappa}(r) = r^{\varepsilon} e^{-\mu r} \Lambda_{n\kappa}(r). \tag{4.34}$$

Substituting Eq. (4.34) into Eq. (4.33), we get the following second-order homogeneous linear differential equation

$$\frac{d^2\Lambda_{n\kappa}(r)}{dr^2} = \left(\frac{2\mu r - 2\varepsilon - 1}{r}\right)\frac{d\Lambda_{n\kappa}(r)}{dr} + \left(\frac{2\varepsilon\mu + \mu - \nu^2}{r}\right)\Lambda_{n\kappa}(r) \tag{4.35}$$

We can determine the $s_0(r)$ and $\lambda_0(r)$ parameters by comparing Eq. (4.35) with Eq. (3.29) as

$$s_0(r) = \frac{2\varepsilon\mu + \mu - \nu^2}{r} \tag{4.36}$$

$$\lambda_0(r) = \frac{2\mu r - 2\varepsilon - 1}{r} \tag{4.37}$$

From Eqs. (4.36) and (4.37), one can easily see that $s_0(r)$ and $\lambda_0(r)$ parameters are sufficiently differentiable functions and $\lambda_0(r)$ is different from zero. Thus, Eq. (4.35) is suitable to use the asymptotic iteration method. Considering the recurrence relations given in Eqs. (3.32) and (3.33), we can calculate the other parameters as follows

$$s_1(r) = \frac{4\varepsilon\mu^2 + 2\mu^2 - 2\mu\nu^2}{r} + \frac{2\nu^2 + 2\varepsilon\nu^2 - 6\varepsilon\mu - 2\mu - 4\varepsilon^2\mu}{r^2} \tag{4.38}$$

$$\lambda_1(r) = 4\mu^2 - \frac{6\mu\varepsilon + 3\mu + \nu^2}{r} + \frac{2(\varepsilon(2\varepsilon+3)+1)}{r^2} \tag{4.39}$$

$$s_2(r) = \frac{(\mu + 2\varepsilon\mu - \nu^2)(6 + 4\varepsilon^2 + \varepsilon(10 - 6\mu r) - 5\mu r + 4\mu^2 r^2 - r\nu^2)}{r^3} \tag{4.40}$$

$$\lambda_2(r) = \frac{2(1+\varepsilon)(1+2\varepsilon)(4\mu r - 2\varepsilon - 3) + 8\mu^2(\mu r - 2\varepsilon - 1)r^2}{r^3} + \frac{4(\varepsilon - \mu r + 1)\nu^2 r}{r^3} \tag{4.41}$$

... etc.

Using Eq. (3.40), we get the following eigenvalues equations

$$\varepsilon_0 = \frac{\nu^2 - \mu}{2\mu} \quad \text{from} \quad s_0(r)\lambda_1(r) - s_1(r)\lambda_0(r) = 0 \tag{4.42}$$

$$\varepsilon_1 = \frac{\nu^2 - 3\mu}{2\mu} \quad \text{from} \quad s_1(r)\lambda_2(r) - s_2(r)\lambda_1(r) = 0 \tag{4.43}$$

$$\varepsilon_2 = \frac{\nu^2 - 5\mu}{2\mu} \quad \text{from} \quad s_2(r)\lambda_3(r) - s_3(r)\lambda_2(r) = 0 \tag{4.44}$$

... etc.

We can easily generalize the above equations as

$$\varepsilon_n = \frac{\nu^2 - (2n+1)\mu}{2\mu}, \qquad n = 0, 1, 2, ... \tag{4.45}$$

Thus, the energy eigenvalues equation for the nuclei in the relativistic Mie-type potentials is obtained by using the Eq. (4.45) and recalling the values of $\varepsilon, \mu, \nu, \gamma$ and β as follow

$$(M + E_{n\kappa} - C)\left[1 + 2n + \sqrt{(2\kappa-1)^2 + 4(E_{n\kappa} - M - C_{ps})A}\right]^2 + (E_{n\kappa} - M - C_{ps})B^2 = 0 \tag{4.46}$$

Eq. (4.46) is a rather complicated transcendental equation. However, energy eigenvalues can be found by setting the parameters in the Mie-type potentials to suitable

values.

Now, let's evaluate the eigenfunction for the Mie-type potentials in the pseudospin symmetry. The eigenfunctions can be calculated by using Eq. (3.31) as follows

$$\begin{aligned}
\Lambda_0(r) &= D_1 && (4.47)\\
\Lambda_1(r) &= -D_1(\nu^2-2\mu)\left(1-\frac{2\mu r}{\frac{\nu^2-3\mu}{\mu}+1}\right) && (4.48)\\
\Lambda_2(r) &= D_1(\nu^2-4\mu)(\nu^2-3\mu) \\
&\times \left(1-\frac{4\mu r}{\frac{\nu^2-5\mu}{\mu}+1}+\frac{4\mu^2 r^2}{\left(\frac{\nu^2-5\mu}{\mu}+1\right)\left(\frac{\nu^2-5\mu}{\mu}+2\right)}\right) && (4.49)
\end{aligned}$$

... etc.

General formula for $\Lambda_n(r)$ can be obtained from above equations as [64]

$$\Lambda_n(r) = D_1(-1)^n\left[\prod_{k=n}^{2n-1}(\nu^2-(k+1)\mu)\right]\ {}_1F_1(-n,2\varepsilon_n+1;2\mu r) \tag{4.50}$$

which leads

$$G_{n\kappa}(r) = r^{\varepsilon_n+\frac{1}{2}}e^{-\mu r}C_2(-1)^n\left[\prod_{k=n}^{2n-1}(\nu^2-(k+1)\mu)\right]\ {}_1F_1(-n,2\varepsilon_n+1;2\mu r). \tag{4.51}$$

Radial wave function of lower component of the Dirac spinor can be converted into the following form when hypergeometric function ${}_1F_1$ is written in terms of the Laguerre polynomials

$$G_{n\kappa}(r) = N_{n\kappa}r^{\varepsilon_n+\frac{1}{2}}e^{-\mu r}L_n^{2\varepsilon_n}(2\mu r) \tag{4.52}$$

where N is the normalization constant given as [64]

$$N_{n\kappa} = \frac{1}{n!}(2\mu)^{\varepsilon_n+\frac{1}{2}}\sqrt{\frac{(n-2\varepsilon_n)!}{n!}}. \tag{4.53}$$

And upper component of the Dirac spinor can be calculated from the following relation for the pseudospin symmetry case

$$F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r} \right) G_{n\kappa}(r) \tag{4.54}$$

Here, $F_{n\kappa}(r)$ is admissible for $E \neq M + C_{ps}$ which is valid only for negative energy solutions. Thus, only negative energy spectrum is obtained in the pseudospin symmetry limit.

4.1.3 Hulthén plus Ring-Shaped Potential

One of the significant short-range potentials is the Hulthén potential [123] which acts like a Coulomb potential under $r \to 0$ limitation. It has been widely used in the solid-state physics [124], atomic physics [125], nuclear and particle physics [126] and chemical physics [127, 128] (see Ref. [128] and more references therein). Recently, some authors have also studied the Hulthén potential in the non-relativistic and the relativistic quantum mechanics [129, 130, 131, 132]. Ring-shaped potentials which have applications to ring-shaped cyclic polyene and benzene type organic molecules have been considered to solve the Schrödinger equation and Dirac equation [65, 133, 134]. Thus, it is worth to investigate the solution of the Dirac equation for the generalized Hulthén potential plus a ring-shaped potential under the condition of the pseudospin symmetry.

The Hulthén plus ring-shaped potential can be given as

$$V(r,\theta) = -Z\alpha \frac{e^{-\alpha r}}{1 - qe^{-\alpha r}} + \beta \frac{\cos^2\theta}{r^2 \sin^2\theta} \tag{4.55}$$

where α and β are the screening parameter and a positive real constant, respectively. Z is related with atomic number when the potential is used for atomic structure and q is the deformation parameter. The Hulthén plus ring-shaped potential has both radial and angular dependence. Thus, we consider Eqs. (3.15), (3.16) and (3.17) to investigate the pseudospin symmetric solution of the Dirac equation for the relevant

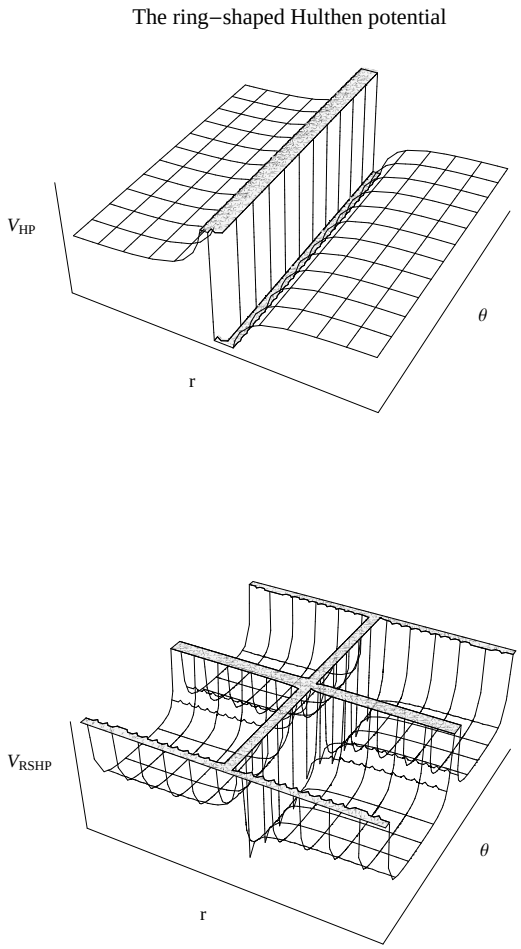

Figure 4.4: The Hulthén and the Hulthén plus ring-shaped potentials as functions of potential parameters and θ.

potential. Then, we obtain the following three second-order differential equations by using Eqs. (3.15), (3.16) and (3.17)

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \widetilde{m}^2\Phi(\phi) = 0 \qquad (4.56)$$

$$\frac{d^2H(\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dH(\theta)}{d\theta} + \left(\widetilde{l}(\widetilde{l}+1) - \frac{\widetilde{m}^2 + \gamma\beta\cos^2\theta}{\sin^2\theta}\right)H(\theta) = 0 \qquad (4.57)$$

$$\frac{d^2R(r)}{dr^2} + \left(-\frac{\widetilde{l}(\widetilde{l}+1)}{r^2} + \varsigma^2 + \frac{\gamma Z\alpha e^{-\alpha r}}{1-qe^{-\alpha r}}\right)R(r) = 0 \qquad (4.58)$$

For bound states, we have the boundary conditions that $\Phi(\phi + 2\pi) = \Phi(\phi)$, $H(0)$

and $H(\pi)$ have a finite value and $R(0) = R(\infty) = 0$ in Eqs. (4.56), (4.57) and (4.58), respectively.

4.1.3.1 The ϕ-dependent equation

Taking into account the boundary condition $\Phi(\phi + 2\pi) = \Phi(\phi)$, solution of Eq. (4.56) can be obtained immediately as

$$\Phi_{\widetilde{m}}(\phi) = \frac{1}{\sqrt{2\pi}} e^{i\widetilde{m}\phi} \tag{4.59}$$

where $\widetilde{m} = 0, \pm 1, \pm 2....$

4.1.3.2 The θ-dependent equation

In order to obtain the solution of Eq. (4.57), we introduce a new variable $s = \cos\theta$ and set

$$\widetilde{\Re} = \widetilde{l}(\widetilde{l}+1) - \widetilde{m}^2, \quad \widetilde{\Im} = \widetilde{l}(\widetilde{l}+1) + \gamma\beta. \tag{4.60}$$

Then, Eq. (4.57) becomes

$$\frac{d^2H(s)}{ds^2} - \frac{2s}{1-s^2}\frac{dH(s)}{ds} + \frac{1}{(1-s^2)^2}\left(\widetilde{\Re} - \widetilde{\Im}s^2\right) H(s) = 0. \tag{4.61}$$

Comparing Eq. (4.61) with Eq. (3.41), we get the following polynomials

$$\widetilde{\tau}(s) = -2s, \quad \sigma(s) = 1 - s^2, \quad \widetilde{\sigma}(s) = \widetilde{\Re} - \widetilde{\Im}s^2. \tag{4.62}$$

Using Eq. (3.49), $\pi(s)$ is obtained as

$$\pi(s) = \pm\sqrt{(\widetilde{\Im} - k)s^2 + (k - \widetilde{\Re})}. \tag{4.63}$$

As mentioned, discriminant of the square root has to be zero. Then, the function $\pi(s)$ is obtained in the following four possible values

$$\pi(s) = \pm \left\{ \begin{array}{ll} \sqrt{\widetilde{\Im} - \widetilde{\Re}} & \text{for} \quad k_+ = \widetilde{\Im} \\ \left(\sqrt{\widetilde{\Im} - \widetilde{\Re}}\right) s & \text{for} \quad k_- = \widetilde{\Re} \end{array} \right\} \tag{4.64}$$

In the Nikiforov-Uvarov method, $\tau'(s) < 0$ must be satisfied in order to obtain a physical solution. Thus, one can choose the following k values to satisfy this condition

$$k_- = \widetilde{\Re} \tag{4.65}$$

which leads to

$$\pi(s) = -\left(\sqrt{\widetilde{\Im} - \widetilde{\Re}}\right) s. \tag{4.66}$$

From Eq. (3.45), $\tau(s)$ is calculated as

$$\tau(s) = -2s\left(1 + \sqrt{\widetilde{\Im} - \widetilde{\Re}}\right). \tag{4.67}$$

Using Eqs. (3.46) and (3.50), we obtain

$$\lambda_{n'} = n'^2 + n'\left(1 + 2\sqrt{\widetilde{\Im} - \widetilde{\Re}}\right) \tag{4.68}$$

$$\lambda = \widetilde{\Re} - \left(\sqrt{\widetilde{\Im} - \widetilde{\Re}}\right) \tag{4.69}$$

where we put prime on n coming from θ-dependent part to distinguish it from n coming from radial part. Recalling the values of $\widetilde{\Im}$, $\widetilde{\Re}$ and γ and comparing Eq. (4.68) with Eq. (4.69), we have

$$\widetilde{l}(\widetilde{l}+1) - \widetilde{m}^2 - \sqrt{(E - M - C_{ps})\beta + \widetilde{m}^2} = n'^2 + n' + 2n'\sqrt{(E - M - C_{ps})\beta + \widetilde{m}^2}. \tag{4.70}$$

Making some arrangement on Eq. (4.70), one can obtain

$$\widetilde{l}(\widetilde{l}+1)+(E-M-C_{ps})\beta = \left(n'+\sqrt{(E-M-C_{ps})\beta+\widetilde{m}^2}\right) \times \left(n'+\sqrt{(E-M-C_{ps})\beta+\widetilde{m}^2}+1\right). \quad (4.71)$$

By setting $l'(l'+1)=\widetilde{l}(\widetilde{l}+1)+(E-M-C_{ps})\beta$, Eq. (4.71) becomes

$$l'(l'+1)=\left(n'+\sqrt{(E-M-C_{ps})\beta+\widetilde{m}^2}\right)\left(n'+\sqrt{(E-M-C_{ps})\beta+\widetilde{m}^2}+1\right). \quad (4.72)$$

From Eq. (4.72), one can easily see that there arises the following relation between l' and n'

$$n'=l'-\mu \qquad\qquad n'=0,1,2... \quad (4.73)$$

where $\mu=\sqrt{(E-M-C_{ps})\beta+\widetilde{m}^2}$. Here, although the parameter l' does not need to be an integer, n' has to be an integer. Therefore, $l'-\mu$ term has to be an integer. In Eq. (4.73), l' term is called as the 'modified' pseudo-orbital angular quantum number because of that the usual pseudo-orbital angular quantum number $\widetilde{l}$ is destroyed by the contribution coming from the angle-dependent potential. This result coincides with previous one obtained in Ref. [65].

To obtain the θ-dependent angular wave function, we should first calculate $\Omega(s)$ and $y(s)$. $\Omega(s)$ can be found out by using Eq. (3.44) as

$$\Omega(s)=\left(1-s^2\right)^{\frac{\mu}{2}} \quad (4.74)$$

After calculating the weight function $\rho(s)$ given in Eq. (3.48), we can obtain the solution of hypergeometric type equation (3.43). By using Eqs. (4.62) and (4.67), we first get the following weight function

$$\rho(s)=\left(1-s^2\right)^{\mu}. \quad (4.75)$$

Then, $y_{n'}(s)$ is found from Eq. (3.47) as

$$y_{n'}(s) = B_{n'}\left(1-s^2\right)^{-\mu}\frac{d^{l'-\mu}}{ds^{l'-\mu}}\left[(1-s^2)^{l'}\right] \tag{4.76}$$

$$y_{n'}(s) = B_{n'}P_{n'}^{\mu,\,\mu}(s) \tag{4.77}$$

where $B_{n'}$ is the normalization constant. Finally, we obtain the θ-dependent wave function in terms of associated-Legendre functions by setting $s=\cos\theta$ as follows [94]

$$H_{n'}(\theta) = B_{n'}\left(\sin\theta\right)^{\mu}P_{n'}^{\mu,\mu}(\cos\theta) \tag{4.78}$$

The normalization constant can be obtained from the orthogonality relation of the associated-Legendre functions as [136]

$$B_{n'} = \sqrt{\frac{(2n'+2\mu+1)\,n'!}{2\Gamma\left(n'+2\mu\right)}}. \tag{4.79}$$

4.1.3.3 The radial equation

Now, we study the radial equation given in Eq. (4.58). Due to the centrifugal-like term, radial equation cannot be solved exactly. Therefore, the following approximation which is valid only small α value can be used to get an approximate analytical solution of the radial Eq. (4.58) [63]

$$\frac{1}{r^2} \approx \frac{\alpha^2 e^{-\alpha r}}{\left(1-qe^{-\alpha r}\right)^2}. \tag{4.80}$$

Considering above approximation and introducing a new variable $s=e^{-\alpha r}$, Eq. (4.58) becomes

$$\frac{d^2R(s)}{ds^2}+\frac{(1-qs)}{s(1-qs)}\frac{dR(s)}{ds}+\frac{1}{s^2(1-qs)^2}\left(\nu s^2+\eta s-\varepsilon^2\right)R(s)=0 \tag{4.81}$$

with

$$\varepsilon^2 = -\frac{\varsigma^2}{\alpha^2} \tag{4.82}$$

$$\nu = q^2\frac{\varsigma^2}{\alpha^2} - q\frac{\gamma Z}{\alpha} \tag{4.83}$$

$$\eta = \frac{\gamma Z}{\alpha} - 2q\frac{\varsigma^2}{\alpha^2} - \widetilde{L}(\widetilde{L}+1) \tag{4.84}$$

$$\widetilde{L}(\widetilde{L}+1) = (n'+\mu)(n'+\mu+1) + \mu^2 - \widetilde{m}^2. \tag{4.85}$$

Comparing Eq. (4.81) with Eq. (3.41) and tracking similar calculations used to find out the solution of the angle-dependent second-order differential equation, we get the following polynomials

$$\widetilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s(1-qs), \quad \widetilde{\sigma}(s) = \nu s^2 + \eta s - \varepsilon^2 \tag{4.86}$$

$$\pi(s) = -q\frac{s}{2} - \frac{1}{2}\left[\left(2q\varepsilon + q\sqrt{1 + \frac{4\widetilde{L}(\widetilde{L}+1)}{q}}\right)s - 2\varepsilon\right] \tag{4.87}$$

$$\tau(s) = 1 + 2\varepsilon - q\left(2 + 2\varepsilon + \sqrt{1 + \frac{4\widetilde{L}(\widetilde{L}+1)}{q}}\right)s \tag{4.88}$$

$$\lambda_n = qn^2 + qn\left(1 + 2\varepsilon + \sqrt{1 + \frac{4\widetilde{L}(\widetilde{L}+1)}{q}}\right) \tag{4.89}$$

$$\lambda = \frac{\gamma Z}{\alpha} - \widetilde{L}(\widetilde{L}+1) - q\left(\varepsilon + \frac{1}{2}\right)\left(1 + \sqrt{1 + \frac{4\widetilde{L}(\widetilde{L}+1)}{q}}\right). \tag{4.90}$$

Recalling γ and ε and comparing Eq. (4.89) with Eq. (4.90), relativistic energy eigenvalues equation is obtained as

$$[M - E + C_{ps}][M + E] = \alpha^2\left[\frac{-\frac{(M-E+C_{ps})Z}{q\alpha} - \frac{\widetilde{L}(\widetilde{L}+1)}{q} - n^2 - n\sqrt{1 + \frac{4\widetilde{L}(\widetilde{L}+1)}{q}}}{2n + 1 + \sqrt{1 + \frac{4\widetilde{L}(\widetilde{L}+1)}{q}}} - \frac{1}{2}\right]^2 \tag{4.91}$$

Radial wave function can be obtained by using the same procedure used to find the θ-dependent angular wave function as

$$R(r) = Be^{-\alpha\varepsilon r}\left(1 - qe^{-\alpha r}\right)^{\frac{1}{2}+\varrho} P_n^{2\varepsilon, 2\varrho}(1 - 2qe^{-\alpha r}) \tag{4.92}$$

where $\varrho = \frac{1}{2}\sqrt{1+\frac{4\widetilde{L}(\widetilde{L}+1)}{q}}$.

4.1.4 Woods-Saxon plus Coulomb-like Tensor Potential

The Woods-Saxon potential which is a spherically symmetric potential plays an essential role in nuclear physics and microscopic physics, since it can be used to describe the nucleon-heavy nucleus interactions. The Woods-Saxon potential is given by [137]

$$V(r) = -\frac{V_0}{1+e^{\frac{r-R}{a}}} \tag{4.93}$$

where V_0 is the potential depth, a is the diffusivity related with the surface thickness and R is the width of the potential.

Under the condition of the pseudospin symmetry, the radial dependent potential $W(r)$ and the tensor potential $U(r)$ can be taken as

$$W(r) = -\frac{V_0}{1+e^{\frac{(r-R)}{a}}}, \quad U(r) = -\frac{D}{r} \tag{4.94}$$

where D is a constant. By inserting above potentials into Eq. (3.28) and defining the following new parameters together with $x = r - R$

$$\widetilde{\gamma} = \frac{(\kappa+D)(\kappa+D-1)}{R^2} \tag{4.95}$$

$$\widetilde{\mu} = (E-M-C_{ps})V_0 \tag{4.96}$$

$$\widetilde{\beta} = (E+M)(E-M-C_{ps}) \tag{4.97}$$

$$\nu = \frac{1}{a} \tag{4.98}$$

the second-order differential equation obtained for the lower radial wave function $G_{n\kappa}(r)$ transforms into

$$\frac{d^2G_{n\kappa}(x)}{dx^2} - \left(\frac{\widetilde{\gamma}}{\left(1+\frac{x}{R}\right)^2} - \frac{\widetilde{\mu}}{1+e^{\nu x}} - \widetilde{\beta}\right) G_{n\kappa}(x) = 0. \tag{4.99}$$

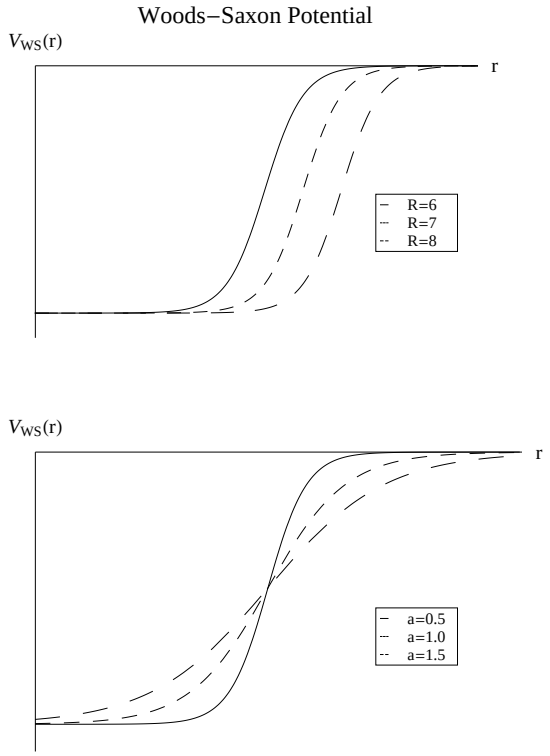

Figure 4.5: The Woods-Saxon potential as functions of the width R and the diffusivity a of the potential.

This equation can not be solved analytically for $\widetilde{\gamma} \neq 0$ because of the pseudospin-orbit coupling term $\frac{\widetilde{\gamma}}{\left(1+\frac{x}{R}\right)^2}$. Therefore, we shall use the Pekeris approximation [138] in order to deal with the pseudospin-orbit coupling term.

In the Pekeris approximation, the centrifugal potential is expanded in a series around $x = 0$ as

$$\begin{aligned} V_{so}(r) &= \frac{(\kappa+D)(\kappa+D-1)}{r^2} = \frac{\widetilde{\gamma}}{\left(1+\frac{x}{R}\right)^2} \\ &= \widetilde{\gamma}\left(1-2\frac{x}{R}+3\left(\frac{x}{R}\right)^2-4\left(\frac{x}{R}\right)^3+...\right). \end{aligned} \tag{4.100}$$

This series expansion is valid only for low vibrational energy states. Thus, we can use this expansion near the minimum point $r \approx R$ where the maximum force is experienced (see Ref. [139] and references therein). Thus, it is sufficient to keep expansion terms only up to the second order. The following form of the potential can be used instead

of the centrifugal potential in the Pekeris approximation [62]

$$\widetilde{V}_{so}(x) = \widetilde{\gamma}\left(D_0 + \frac{D_1}{1+e^{\nu x}} + \frac{D_2}{(1+e^{\nu x})^2}\right) \tag{4.101}$$

where D_0, D_1 and D_2 are new arbitrary parameters. The expression of Eq. (4.101) can be expanded around $x = 0$ up to the second order term as

$$\widetilde{V}_{so}(x) = \widetilde{\gamma}\left(\left(D_0 + \frac{D_1}{2} + \frac{D_2}{4}\right) - \frac{\nu}{4}(D_1 + D_2)x + \frac{\nu^2}{16}D_2x^2...\right). \tag{4.102}$$

Comparing the equal powers of Eqs. (4.102) and (4.100), arbitrary parameters D_0, D_1 and D_2 are found in terms of $a(= 1/\nu)$ and R as follows

$$D_0 = 1 - \frac{4}{\nu R} + \frac{12}{\nu^2 R^2} \tag{4.103}$$

$$D_1 = \frac{8}{\nu R} - \frac{48}{\nu^2 R^2} \tag{4.104}$$

$$D_2 = \frac{48}{\nu^2 R^2}. \tag{4.105}$$

Now, we can take the $\widetilde{V}_{so}(x)$ potential instead of the pseudospin-orbit coupling potential.

Defining a new variable $y = \frac{1}{1+e^{\nu x}}$ and inserting it into Eq. (4.99) yields

$$\begin{aligned} &y(1-y)\frac{d^2}{dy^2}G_{n\kappa}(y) + (1-2y)\frac{d}{dy}G_{n\kappa}(y) \\ &\quad -\frac{1}{\nu^2 y(1-y)}\left((\widetilde{\gamma}D_0 - \widetilde{\beta}) + (\widetilde{\gamma}D_1 - \widetilde{\mu})y + \widetilde{\gamma}D_2y^2\right)G_{n\kappa}(y) = 0. \end{aligned} \tag{4.106}$$

By setting the following factorization

$$G(y) = y^{\widetilde{\sigma}}(1-y)^{\widetilde{\tau}}g(y) \tag{4.107}$$

with

$$\widetilde{\sigma} = \sqrt{\frac{\widetilde{\gamma}D_0}{\nu^2} - \frac{\widetilde{\beta}}{\nu^2}} \tag{4.108}$$

$$\widetilde{\tau} = \sqrt{\frac{\widetilde{\gamma}(D_0+D_1+D_2)}{\nu^2} - \frac{\widetilde{\beta}+\widetilde{\mu}}{\nu^2}} \tag{4.109}$$

and substituting it into Eq. (4.106), we obtain

$$y(1-y)\frac{d^2g(y)}{dy^2} + \left(\widetilde{c} - (\widetilde{a}+\widetilde{b}+1)y\right)\frac{dg(y)}{dy} - \widetilde{a}\,\widetilde{b}\;g(y) = 0 \tag{4.110}$$

where

$$\widetilde{a} = \widetilde{\sigma} + \widetilde{\tau} + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\frac{\widetilde{\gamma}D_2}{\nu^2}} \tag{4.111}$$

$$\widetilde{b} = \widetilde{\sigma} + \widetilde{\tau} + \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{\widetilde{\gamma}D_2}{\nu^2}} \tag{4.112}$$

$$\widetilde{c} = 2\widetilde{\sigma} + 1. \tag{4.113}$$

Eq. (4.110) is the well-known hypergeometric equation and its solution can be written in terms of hypergeometric functions. Thus, recalling the variables y and x and the value of ν, final solution of the lower radial wave function of the Dirac spinor can be written as follows

$$G_{n\kappa}(r) = C_1\left(1+e^{\frac{r-R}{a}}\right)^{\widetilde{q}} e^{\frac{\widetilde{\tau}(r-R)}{a}}\;{}_2F_1\left(\widetilde{a},\widetilde{b},\widetilde{c};\frac{1}{1+e^{\frac{r-R}{a}}}\right) \tag{4.114}$$

where

$$\widetilde{q} = -(\widetilde{\sigma}+\widetilde{\tau}). \tag{4.115}$$

We can also obtain the other component of the Dirac spinor under the pseudospin symmetry by using Eq. (3.27) as follows

$$F_{n\kappa}(r) = \frac{C_1}{M-E+C_{ps}}\left(1+e^{\frac{r-R}{a}}\right)^{\widetilde{q}} e^{\frac{\widetilde{\tau}(r-R)}{a}}\left\{\left[-\frac{\kappa+D}{r} + \frac{1}{a}\left(\frac{\widetilde{q}}{1+e^{\frac{-(r-R)}{a}}} + \widetilde{\tau}\right)\right]\right.$$
$$\left.\times\;{}_2F_1\left(\widetilde{a},\widetilde{b},\widetilde{c};\frac{1}{1+e^{\frac{r-R}{a}}}\right) + \frac{\widetilde{a}\widetilde{b}}{\widetilde{c}}\;{}_2F_1\left(\widetilde{a}+1,\widetilde{b}+1,\widetilde{c}+1;\frac{1}{1+e^{\frac{r-R}{a}}}\right)\right\} \tag{4.116}$$

where we have used the following identity of the hypergeometric function [140]

$$\frac{d}{dz}\ {}_2F_1(\xi_1,\xi_2,\xi_3;z)=\frac{\xi_1\xi_2}{\xi_3}\ {}_2F_1(\xi_1+1,\xi_2+1,\xi_3+1;z). \tag{4.117}$$

Here, $F_{n\kappa}(r)$ is admissible for $E\neq M+C_{ps}$ which is valid only for negative energy solutions. Thus, the energy spectrum obtained in the pseudospin symmetry limit is negative.

The hypergeometric function ${}_2F_1$ can be reduced to polynomial degree n if $\widetilde{a}=-n$ or $\widetilde{b}=-n$, where n is an integer. Therefore, one can determine that lower radial wave function $G_{n\kappa}(r)$ can be finite under the following quantum condition

$$\widetilde{a}=-n,\quad n=0,1,2,... \tag{4.118}$$

By substituting Eqs. (4.108) and (4.109) into Eq. (4.111) together with quantum condition $\widetilde{a}=-n$, we can get the energy eigenvalues equation for the nuclei under the pseudospin symmetry limit as

$$E^2-M^2-C_{ps}(E+M)=$$
$$\widetilde{\gamma}D_0-\frac{1}{a^2}\left(n+\frac{1}{2}-\frac{1}{2}\sqrt{1+4\widetilde{\gamma}D_2a^2}+\sqrt{\widetilde{\gamma}D_3a^2-(E-M-C_{ps})(E+M+V_0)a^2}\right)^2 \tag{4.119}$$

4.1.5 Pseudoharmonic Potential together with Tensor Potential

We shall consider the pseudoharmonic potential given in Eq. (4.1) as a radial potential $W(r)$ whereas the linear tensor potential is taken as

$$U(r)=\sigma r \tag{4.120}$$

where σ is a constant. By inserting these potentials into Eq. (3.28), the second-order differential equation can be transformed into

$$\frac{d^2}{dr^2}G_{n\kappa}(r)+\left[\beta-\frac{\mu(\mu-1)}{r^2}-\nu^2r^2\right]G_{n\kappa}(r)=0 \tag{4.121}$$

where

$$\gamma = E_{n\kappa} - M - C_{ps} \tag{4.122}$$

$$\nu = \sqrt{\sigma^2 + A(E_{n\kappa} - M - C_{ps})} \tag{4.123}$$

$$\mu = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\widetilde{l}(\widetilde{l}+1) + 4(E_{n\kappa} - M - C_{ps})B} \tag{4.124}$$

$$\beta = (E_{n\kappa} - M - C_{ps})(E_{n\kappa} + M - C) + 2\kappa\sigma + \sigma \tag{4.125}$$

From Eq. (4.121) one can see that the behavior of the solution of the second-order differential equation is determined by the centrifugal term at $r = 0$ and asymptotic behavior of this equation is determined by the oscillator term [141]. In this context, we can propose the following factorization

$$G_{n\kappa}(r) = r^{\mu} e^{-\frac{1}{2}\nu r^2} h_{n\kappa}(r) \tag{4.126}$$

Inserting above expression into Eq. (4.121), the equation for $h_{n\kappa}(r)$ reads

$$\frac{d^2}{dr^2}h_{n\kappa}(r) + \left[\frac{2\mu}{r} - 2\nu r\right]\frac{d}{dr}h_{n\kappa}(r) + \left[\beta - \nu(1+2\mu)\right]h_{n\kappa}(r) = 0 \tag{4.127}$$

It is suitable for introducing the following variable

$$x = \nu r^2 \tag{4.128}$$

which leads to

$$x\frac{d^2}{dx^2}h_{n\kappa}(x) + [b - x]\frac{d}{dx}h_{n\kappa}(x) - a h_{n\kappa}(x) = 0 \tag{4.129}$$

where

$$a = \frac{1}{2}\left(\mu + \frac{1}{2} - \frac{\beta}{2\nu}\right) \tag{4.130}$$

$$b = \mu + \frac{1}{2} \tag{4.131}$$

Eq. (4.129) is a confluent hypergeometric equation with the following general solution [140]

$$h_{n\kappa}(r) = \Lambda_1 \ {}_1F_1(a, b; \nu r^2) + \Lambda_2 \nu^{-(\mu+\frac{1}{2})} r^{-(2\mu+1)} \ {}_1F_1(a - c + 1, 2 - c; \nu r^2) \quad (4.132)$$

where Λ_1 and Λ_2 are normalization constant and ${}_1F_1$ is the confluent hypergeometric function. In this solution, second part is in contradiction with normalization at $r = 0$ [141]. Thus, Λ_2 should be zero. Consequently, lower component of the Dirac spinor can be written by considering Eqs. (4.126), (4.130), (4.131) and (4.132) as follows

$$G_{n\kappa}(r) = \Lambda_1 r^{\mu} e^{-\frac{1}{2}\nu r^2} \ {}_1F_1\left(\frac{1}{2}\left(\mu + \frac{1}{2} - \frac{\beta}{2\nu}\right), \mu + \frac{1}{2}; \nu r^2\right) \quad (4.133)$$

By using special interest among confluent hypergeometric function and generalized Laguerre polynomials [141], we can write the lower component in terms of generalized Laguerre polynomials as follows

$$G_{n\kappa}(r) = \Lambda_1 \frac{n!(\mu - \frac{1}{2})!}{(n + \mu - \frac{1}{2})!} r^{\mu} e^{-\frac{1}{2}\nu r^2} \ L_n^{(\mu-\frac{1}{2})}(\nu r^2). \quad (4.134)$$

Upper component of the Dirac spinor can be calculated by using the following relation obtained from Eq. (3.27)

$$F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C_{ps}} \left[\frac{d}{dr} - \frac{\kappa}{r} + \sigma r\right] G_{n\kappa}(r). \quad (4.135)$$

In Eq. (4.135), one can see that $F_{n\kappa}(r)$ is admissible for $E \neq M + C_{ps}$ which is valid only for negative energy solutions. Thus, the energy spectrum obtained in the pseudospin symmetry limit is negative. From Eq. (4.133), we must have $a = -n$, with $n = 0, 1, 2...$, for bound states [141]. Moreover, the relativistic energy spectrum is derived from $a = -n$ condition as

$$\sqrt{\sigma^2 + A(E_{n\kappa} - M - C_{ps})}\left(2 + 4n + \sqrt{1 + 4\widetilde{l}(\widetilde{l} + 1) + 4(E_{n\kappa} - M - C_{ps})B}\right) - (E_{n\kappa} - M - C_{ps})(E_{n\kappa} + M - C) - 2\kappa\sigma - \sigma = 0 \quad (4.136)$$

Eq. (4.136) is the energy eigenvalue equation of the Dirac particle for the pseudoharmonic potential in the presence of the tensor potential under the condition of the pseudospin symmetry.

4.2 Results

4.2.1 Pseudoharmonic potential

The energy eigenvalues equation (4.20) and corresponding lower radial wave function (4.25) have been obtained for the Dirac particles with pseudoharmonic potential under the condition of the pseudospin symmetry. It is found out that the energy eigenvalues equation (4.20) is a rather complicated intangible equation. However, one can carry out the energy eigenvalues by choosing the suitable parameters in the pseudoharmonic potential. Equation (4.20) shows that energy eigenvalues $E_{n\kappa}$ depend on n, $\kappa(\widetilde{l})$, C_{ps} as well as parameters A, B, C and M.

Using energy eigenvalues equation (4.20), we have computed the some energy levels with the help of the computer for several values of n and κ. Energy eigenvalues equation yields three values for the each states. However, we have seen that only one of them is real and physical. In Table (4.1), we give some numerical solutions of the equation (4.20) with parameters $M = 10.0fm^{-1}$, $r_e = 2.4fm$, $D_e = 5.0fm^{-1}$, $C_{ps} = -10.3834fm^{-1}$ and $C_{ps} = -11.5fm^{-1}$. Ginocchio [49] showed that there are only negative energy bound states in the pseudospin symmetry limit. From Table (4.1), one can observe that there are only negative energy bound state solutions for $C_{ps} \leq -10.3834fm^{-1}$. However, there are no negative energy bound state solutions if we take $C_{ps} > -10.3834fm^{-1}$. One can also see from Table (4.1) that there are degeneracies between the eigenstates $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$ *etc.*. In fact, each of these eigenstates forms a pseudospin doublet. For instance, $1p_{3/2}$ with $n = 1$ and $\kappa = -2$ is the partner of $0f_{5/2}$ with $n - 1 = 0$ and $\kappa = 3$.

On the other hand, by using Eq. (4.20), energy eigenvalues equation of harmonic oscillator potential can be obtained by setting $A = \frac{1}{2}Mw_1^2$, $B = C = C_{ps} = 0$ and recalling $\kappa(\kappa - 1) = \widetilde{l}(\widetilde{l} + 1)$ in the pseudospin symmetry case as

Table 4.1: The bound state energy eigenvalues in units of fm^{-1} for several values of n and κ with $C_{ps} = -10.3834 fm^{-1}$ and $C_{ps} = -11.5 fm^{-1}$ in the pseudospin symmetry.

$\widetilde{l}$	$n, \kappa_{<0}$	(l, j)	$C_{ps} = -10.3834$ $E_{n,\kappa_{<0}}$	$C_{ps} = -11.5$ $E_{n,\kappa_{<0}}$	$n-1, \kappa_{>0}$	$(l+2, j+1)$	$C_{ps} = -10.3834$ $E_{n-1,\kappa_{>0}}$	$C_{ps} = -11.5$ $E_{n-1,\kappa_{>0}}$
1	1,-1	$1s_{1/2}$	-0.000014	-0.985845	0,2	$0d_{3/2}$	-0.000014	-0.985845
2	1,-2	$1p_{3/2}$	0.098293	-0.876620	0,3	$0f_{5/2}$	0.098293	-0.876620
3	1,-3	$1d_{5/2}$	0.222963	-0.737799	0,4	$0g_{7/2}$	0.222963	-0.737799
4	1,-4	$1f_{7/2}$	0.367395	-0.577439	0,5	$0h_{9/2}$	0.367395	-0.577439
1	2,-1	$2s_{1/2}$	0.478572	-0.424705	1,2	$1d_{3/2}$	0.478572	-0.424705
2	2,-2	$2p_{3/2}$	0.577344	-0.319515	1,3	$1f_{5/2}$	0.577344	-0.319515
3	2,-3	$2d_{5/2}$	0.707842	-0.179754	1,4	$1g_{7/2}$	0.707842	-0.179754
4	2,-4	$2f_{7/2}$	0.861192	-0.015172	1,5	$1h_{9/2}$	0.861192	-0.015172

$$(E_{n\widetilde{l}} + M)\sqrt{\frac{(E_{n\widetilde{l}} - M)}{2M}} = w_1 \left[2n + \widetilde{l} + \frac{3}{2}\right]. \tag{4.137}$$

Equation (4.137) is same as that Eq. (62) obtained for the harmonic oscillator potential in Ref. [36]. One can easily see that there is no pseudospin-orbit coupling term in the equation. Thus, the states which have same radial n and pseudo-orbital angular momentum $\widetilde{l}$ quantum numbers with $j = \widetilde{l} + \frac{1}{2}$ and $j = \widetilde{l} - \frac{1}{2}$ are degenerate. In the non-relativistic limit $E_{n\widetilde{l}} - M \to E_{n\widetilde{l}}$ and $M + E_{n\widetilde{l}} \to 2M$ [36], equation (4.137) yields

$$E_{n\widetilde{l}} = \frac{w_1^2}{2M}\left[2n + \widetilde{l} + \frac{3}{2}\right]^2 \tag{4.138}$$

where $w_1 = \sqrt{\frac{2D_e}{Mr_e^2}}$ is the classical frequency for small harmonic vibrations [141]. Right-hand side of the Equation (4.138) is always positive; therefore there are only positive energy eigenvalues in the non-relativistic limit for harmonic oscillator potential [36].

4.2.2 Mie-type potential

In the section 4.1.2, we have obtained the energy eigenvalues equation (4.46) and corresponding wave function (4.52) of the Dirac particles for the general form of the Mie-type potential under the pseudospin symmetry limit. Kratzer-Fues and Modified Kratzer potentials can be given as examples on the Mie-type potentials. Thus, we discuss the bound state energy equations of these two Mie-type potentials. In addition, we give some particular cases of the energy eigenvalues equations.

4.2.2.1 Kratzer-Fues potential

Using Eq. (4.46), we can obtain the energy eigenvalues equation for the nuclei in the relativistic Kratzer-Fues potential (4.28) by setting $A = D_e r_e^2, B = 2D_e r_e$ and $C = 0$ under the condition of pseudospin symmetry limit as

$$(M + E_{n\kappa}) \left[1 + 2n + \sqrt{(2\kappa - 1)^2 + 4(E_{n\kappa} - M - C_{ps})D_e r_e^2}\right]^2 + (E_{n\kappa} - M - C_{ps})(2D_e r_e)^2 = 0 \qquad (4.139)$$

If we take C_{ps} equal to zero, our result agrees with one obtained in Ref. [65] for the pseudospin symmetry case given in Eq. (42) computed by using the Nikiforov-Uvarov method.

We have calculated some energy levels of the pseudospin symmetry Kratzer-Fues potential for several values of n and κ with the help of the computer by using Eq. (4.139). In the calculation, we use the parameters $M = 5fm^{-1}$, $D_e = 1.25fm^{-1}$, $r_e = 0.35fm$ and $C_{ps} = 0$. The numerical results are presented in Table (4.2) for the pseudospin symmetry case. From Table (4.2), we observe that energies of bound states given in same line are degenerate such as $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$ *etc.*. Thus, each pair is considered as the pseudospin doublet. Each of the two states in the pseudospin doublet has the same pseudo-orbital angular quantum number and pseudospin quantum number. In addition, in Table (4.2), one can observe that levels in the pseudospin doublets have the negative energy as mentioned.

4.2.2.2 Modified Kratzer potential

For the modified Kratzer potential (4.29), we can obtain the following energy eigenvalues equation by using Eq. (4.46) in the pseudospin symmetry limit,

$$(M + E_{n\kappa} - D_e) \left[1 + 2n + \sqrt{(2\kappa - 1)^2 + 4(E_{n\kappa} - M - C_{ps})D_e r_e^2}\right]^2 + (E_{n\kappa} - M - C_{ps})(2D_e r_e)^2 = 0 \qquad (4.140)$$

Table 4.2: The bound state energy eigenvalues in units of fm^{-1} of the pseudospin symmetry Kratzer-Fues potential for several values of n and κ with $C_{ps} = 0$.

$\widetilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$
1	1,-1	$1s_{1/2}$	-4.672305	0,2	$0d_{3/2}$	-4.672305
2	1,-2	$1p_{3/2}$	-4.860421	0,3	$0f_{5/2}$	-4.860421
3	1,-3	$1d_{5/2}$	-4.916782	0,4	$0g_{7/2}$	-4.916782
4	1,-4	$1f_{7/2}$	-4.943953	0,5	$0h_{9/2}$	-4.943953
1	2,-1	$2s_{1/2}$	-4.833547	1,2	$1d_{3/2}$	-4.833547
2	2,-2	$2p_{3/2}$	-4.913195	1,3	$1f_{5/2}$	-4.913195
3	2,-3	$2d_{5/2}$	-4.942941	1,4	$1g_{7/2}$	-4.942941
4	2,-4	$2f_{7/2}$	-4.959104	1,5	$1h_{9/2}$	-4.959104

Eq. (4.140) is also complicated transcendental equation. However, energy eigenvalues can be obtained by setting the parameters to suitable values. To compute some energy levels of the pseudospin symmetry modified Kratzer potential for several values of n and κ with the help of the computer by using Eq. (4.140), we consider the parameters $M = 5fm^{-1}$, $D_e = 1.25fm^{-1}$, $r_e = 0.35fm$ and $C_{ps} = 0fm^{-1}$. In Table (4.3), we can observe that levels given in same line have same energy such as $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$ *etc.*. These levels with same energy form pseudospin doublets. We can also see in Table (4.3), all levels have negative energy in the pseudospin symmetry as expected.

In addition, in the introduction section, we mentioned that modified Kratzer potential is shifted in amount of D_e from Kratzer-Fues potential. This shift also arises in the energy eigenvalues equations and energy eigenvalues obtained in the pseudospin limit. Comparing Eq. (4.139) with Eq. (4.140), we can observe that energy eigenvalues equations obtained in the pseudospin case are different from each other only D_e term. From Table (4.2) and Table (4.3), we can see that difference between energies of the same states is nearly in amount of D_e.

4.2.2.3 Particular case I : The $A = 0$ and the $C = 0$

In this case, general form of Mie-type potentials is reduced to well-known Coulomb-like potential. Energy eigenvalues equation for the Coulomb-like potential can be found out in the following forms under the condition of pseudospin symmetry by using Eq. (4.46)

Table 4.3: The bound state energy eigenvalues in units of fm^{-1} of the pseudospin symmetry Modified-Kratzer potential for several values of n and κ with $C_{ps} = 0$.

$\widetilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$
1	1,-1	$1s_{1/2}$	-3.484888	0,2	$0d_{3/2}$	-3.484888
2	1,-2	$1p_{3/2}$	-3.630626	0,3	$0f_{5/2}$	-3.630626
3	1,-3	$1d_{5/2}$	-3.678048	0,4	$0g_{7/2}$	-3.678048
4	1,-4	$1f_{7/2}$	-3.701324	0,5	$0h_{9/2}$	-3.701324
1	2,-1	$2s_{1/2}$	-3.612693	1,2	$1d_{3/2}$	-3.612693
2	2,-2	$2p_{3/2}$	-3.675416	1,3	$1f_{5/2}$	-3.675416
3	2,-3	$2d_{5/2}$	-3.700567	1,4	$1g_{7/2}$	-3.700567
4	2,-4	$2f_{7/2}$	-3.714444	1,5	$1h_{9/2}$	-3.714444

$$E_{n\kappa}^{ps} = \frac{(M + C_{ps})B^2 - 4(n+\kappa)^2 M}{B^2 + 4(n+\kappa)^2}. \tag{4.141}$$

Eq. (4.141) agrees with Eq. (37) in Ref. [65] when $C_{ps} = 0$.

4.2.2.4 Particular case II : The $B = 0$ and the $C = 0$

In this limit, Mie-type potential is reduced to pseudo-centrifugal potential. We can obtain the bound state energy equation as follows

$$E_{n\kappa}^{ps} = M + C_{ps} + \frac{(n-\kappa+1)(n+\kappa)}{A} \tag{4.142}$$

For $n = 0$ and $\kappa = 1$ (ground state), energy equation is reduced into $E_{n\kappa}^{ps} = M + C_{ps}$ in the pseudospin symmetry limit.

4.2.3 Hulthén plus Ring-Shaped Potential

From Eq. (4.73) it is shown that although the parameter l' does not need to be an integer, n' has to be an integer. Therefore, $l' - \mu$ term has to be an integer. Here, l' term is called as the 'modified' pseudo-orbital angular quantum number because of that the usual pseudo-orbital angular quantum number $\widetilde{l}$ is damaged by the contribution coming from the angle-dependent potential. This result is coincidence with previous one obtained in Ref. [65].

From Eq. (4.91), we can say that energy eigenvalues equation depends on the potential parameters Z, α, q, nucleon mass M and constant C_{ps} as well as $\widetilde{L}$ including n'. On the other hand, the total wave function $G(\vec{r})$ in the pseudospin symmetry can be written by combining Eq. (3.14) with Eqs. (4.59), (4.78) and (4.92) as

$$G(\vec{r}) = \frac{B}{\sqrt{2\pi}} \sqrt{\frac{(2n'+2\mu+1)\,n'!}{2\Gamma\left(n'+2\mu\right)}}\ \frac{e^{-\alpha\varepsilon r}}{r}\ \left(1-qe^{-\alpha r}\right)^{\frac{1}{2}+\varrho} \times P_n^{2\varepsilon,2\varrho}(1-2qe^{-\alpha r})\ (\sin\theta)^{\mu}\, P_{n'}^{\mu,\mu}(\cos\theta)\ e^{i\widetilde{m}\phi}\ \widetilde{\Lambda}_{\xi}. \tag{4.143}$$

4.2.3.1 Particular case : The $\beta = 0$ and $q = 1$

In this case, generalized Hulthén potential plus a ring-shaped potential is reduced to the well-known Hulthén potential. In the limits $\beta = 0$ and $q = 1$, energy eigenvalues equation (4.91) becomes

$$[M-E+C_{ps}][M+E] = \alpha^2 \left[\frac{-\frac{(M-E+C_{ps})Z}{\alpha} - \widetilde{l}(\widetilde{l}+1) - n^2 - n\sqrt{1+4\widetilde{l}(\widetilde{l}+1)}}{2n+1+\sqrt{1+4\widetilde{l}(\widetilde{l}+1)}} - \frac{1}{2}\right]^2 \tag{4.144}$$

Remaining $\widetilde{l}(\widetilde{l}+1) = \kappa(\kappa-1)$ and making some arrangements on Eq. (4.144), one can obtain the following energy eigenvalues equation

$$[M-E+C_{ps}][M+E] = \alpha^2 \left[-\frac{(M-E+C_{ps})Z}{2\alpha(n+\kappa)} - \frac{(n+\kappa)}{2}\right]^2. \tag{4.145}$$

Eq. (4.145) is just expression Eq. (47) in Ref. [66] obtained by using asymptotic iteration method under the pseudospin symmetry limit.

4.2.4 Woods-Saxon plus Coulomb-like Tensor Potential

We have obtained the energy eigenvalues equation (4.119) that is an algebraic equation in E. The solution of this algebraic equation with respect to E can be obtained in terms of particular values of radial n and pseudo-orbital angular momentum $\widetilde{l}$ quantum numbers. From Eq. (4.119), we can obtain the energy eigenvalues equation of the

Woods-Saxon potential by setting $D = 0$ for any spin-orbit quantum number κ as follows

$$E^2 - M^2 - C_{ps}(E+M) =$$
$$\widetilde{\gamma}D_0 - \frac{1}{a^2}\left(n + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\widetilde{\gamma}D_2a^2} + \sqrt{\widetilde{\gamma}D_3a^2 - (E - M - C_{ps})(E + M + V_0)a^2}\right)^2 \tag{4.146}$$

where $\widetilde{\gamma} = \frac{\kappa(\kappa-1)}{R^2}$. If we take $\kappa = 1$ in Eq. (4.146), it becomes

$$a\left(\sqrt{(M - E + C_{ps})(E + M + V_0)} + \sqrt{(M - E + C_{ps})(E + M)}\right) = -n \tag{4.147}$$

Eq. (4.147) is just expression (38) of Ref. [52], which is the energy eigenvalue equation of the $\tilde{s}$-wave Dirac equation for the Woods-Saxon potential with pseudospin symmetry.

We can calculate the energy eigenvalues from Eq. (4.119) for different values of the quantum numbers (n, κ) in the pseudospin symmetry limit. They are presented in Table (4.4). We take the following parameters $M = 10fm^{-1}$, $a = 0.5fm$, $R = 7fm$, $V_0 = 10fm^{-1}$ and $C_{ps} = -10.1fm^{-1}$ in the calculations. From Table (4.4), one can observe that every pair of orbitals $(ns_{1/2}, (n-1)d_{3/2})$ with $\widetilde{l} = 1$, $(np_{3/2}, (n-1)f_{5/2})$ with $\widetilde{l} = 2$ and $(nd_{5/2}, (n-1)g_{7/2})$ with $\widetilde{l} = 3$ has the same energy in the absence of the tensor potential $(D = 0)$. Thus, each of the pairs can be viewed as the pseudospin doublets, i.e, the state $1s_{1/2}$ with $n = 1$, $\kappa = -1$ forms a pseudospin doublet with $0d_{3/2}$ state with $n - 1 = 0$, $\kappa = 2$. In addition, one can see in Table (4.4) that the degeneracy between two states in the pseudospin doublets is removed in the presence of the tensor potential $(D \neq 0)$. From Table (4.4), one can observe that there are degeneracies between the pair of orbitals $(ns_{1/2}, (n-1)p_{1/2})$, $(np_{3/2}, (n-1)d_{3/2})$ and $(nd_{5/2}, (n-1)f_{5/2})$ for $D \neq 0$. It is also interesting to note that all $p_{1/2}$ states considered as the pseudospin singlets $(\widetilde{s}_{1/2})$ may have pseudospin partner for $D = 0.5$. To investigate these degeneracies, we change the parameter D and keep all other parameters same. The pair of orbital $(1d_{5/2}, 0f_{5/2})$ is considered as an example. From Table (4.4), one can observe that these two states in doublet are degenerate $E_{1,-3} =$

Table 4.4: The bound state energy eigenvalues in units of fm^{-1} in the pseudospin symmetry case for $D = 0$ and $D = 0.5$

$\widetilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0(D=0)}$	$E_{n,\kappa<0(D=0.5)}$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0(D=0)}$	$E_{n-1,\kappa>0(D=0.5)}$
0	-	-	-	-	0,1	$0p_{1/2}$	-0.169462	-0.168217
0	-	-	-	-	1,1	$1p_{1/2}$	-0.382147	-0.380728
0	-	-	-	-	2,1	$2p_{1/2}$	-0.752435	-0.750789
0	-	-	-	-	3,1	$3p_{1/2}$	-1.310610	-1.308650
1	1,-1	$1s_{1/2}$	-0.166143	-0.168217	0,2	$0d_{3/2}$	-0.166143	-0.163243
1	2,-1	$2s_{1/2}$	-0.378364	-0.380728	1,2	$1d_{3/2}$	-0.378364	-0.375059
1	3,-1	$3s_{1/2}$	-0.748050	-0.750789	2,2	$2d_{3/2}$	-0.748050	-0.744221
1	4,-1	$4s_{1/2}$	-1.305380	-1.308650	3,2	$3d_{3/2}$	-1.305380	-1.300820
2	1,-2	$1p_{3/2}$	-0.159519	-0.163243	0,3	$0f_{5/2}$	-0.159519	-0.154976
2	2,-2	$2p_{3/2}$	-0.370818	-0.375059	1,3	$1f_{5/2}$	-0.370818	-0.365645
2	3,-2	$3p_{3/2}$	-0.739309	-0.744221	2,3	$2f_{5/2}$	-0.739309	-0.733321
2	4,-2	$4p_{3/2}$	-1.294960	-1.300820	3,3	$3f_{5/2}$	-1.294960	-1.287840
3	1,-3	$1d_{5/2}$	-0.149617	-0.154976	0,4	$0g_{7/2}$	-0.149617	-0.143448
3	2,-3	$2d_{5/2}$	-0.359547	-0.365645	1,4	$1g_{7/2}$	-0.359547	-0.352532
3	3,-3	$3d_{5/2}$	-0.726268	-0.733321	2,4	$2g_{7/2}$	-0.726268	-0.718159
3	4,-3	$4d_{5/2}$	-1.279440	-1.287840	3,4	$3g_{7/2}$	-1.279440	-1.269800

$E_{1,3} = -0.154976$ for $D = 0.5$. However, if we take $D = 0.8$, we find out that $E_{1,-3} = -0.157800$ and $E_{1,3} = -0.151858$. This means that degeneracies between the pair of orbitals $(ns_{1/2}, (n-1)p_{1/2})$, $(np_{3/2}, (n-1)d_{3/2})$ and $(nd_{5/2}, (n-1)f_{5/2})$ are formed accidentally. Then, our results displayed that degeneracy between the two states in doublet is removed by tensor interactions, and results presented are coincidence with ones obtained in Refs. [36, 79, 76, 77]. In Table (4.4), we can also see that energy levels of the pseudospin unaligned ($\kappa < 0$) and aligned ($\kappa > 0$) states in the doublets move in the opposite directions for $D \neq 0$. This can be explained by considering Eq. (4.119). The energy eigenvalues equation (4.119) depends on the term $2\kappa D$ through $\widetilde{\gamma}$. The pseudospin-dependent term $2\kappa D$ takes negative and positive values for the pseudospin unaligned and aligned states with respect to values of κ, respectively.

We investigate the tensor potential dependence of the energies of the two states in the pseudospin doublets. These dependencies are displayed in Fig. (4.6) for the pseudospin-orbital partners $(2s_{1/2}, 1d_{3/2})$ and $(1p_{3/2}, 0f_{5/2})$. From Fig. (4.6), one can observe that the magnitude of the energy difference between the two states in the pseudospin doublets increases while D increases. The reason is that although energies of the pseudospin unaligned states increase as D increases, energies of the pseudospin aligned states decrease with increasing D due to the term $2\kappa D$ in the energy eigenvalues equation.

Keeping all other parameters fixed, we change the diffusivity to investigate how the

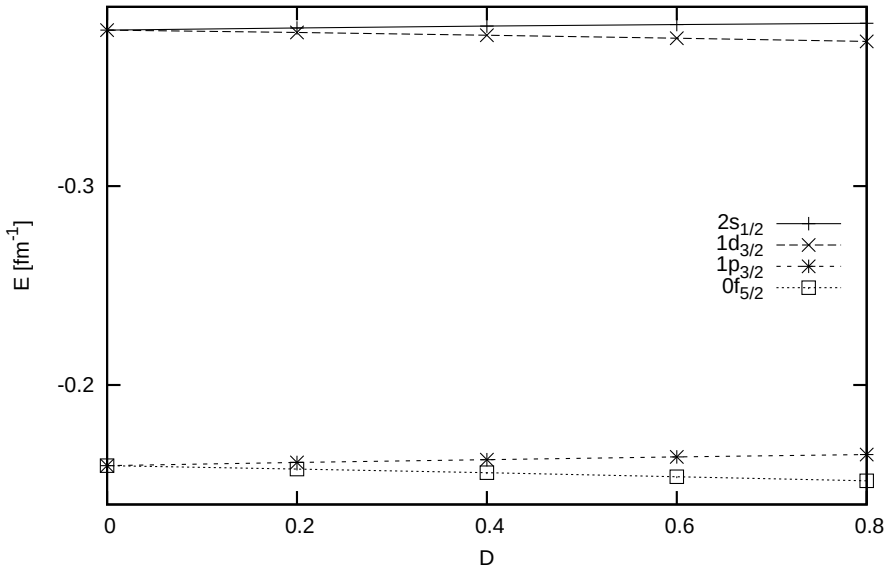

Figure 4.6: Energy values computed as a function of D for the pseudospin doublets $(2s_{1/2}, 1d_{3/2})$ and $(1p_{3/2}, 0f_{5/2})$.

Table 4.5: Effects of the parameter a and C_{ps} on the energy levels of the two states in the pseudospin doublets

	$E(D=0)$ $E(D=0.5)$	$E(D=0)$ $E(D=0.5)$	$E(D=0)$ $E(D=0.5)$	$E(D=0)$ $E(D=0.5)$
state	$a=0.1$	$a=0.4$	$a=0.7$	$a=1.0$
$1s_{1/2}$	-2.11613 -2.11871	-0.20566 -0.20765	-0.13161 -0.13395	-0.11266 -0.11554
$0d_{3/2}$	-2.11613 -2.11253	-0.20566 -0.20289	-0.13161 -0.12835	-0.11266 -0.10871
	$C_{ps}=-10$	$C_{ps}=-12$	$C_{ps}=-14$	$C_{ps}=-16$
$1s_{1/2}$	-0.06567 -0.06773	-2.07668 -2.07909	-4.09242 -4.09536	-6.11725 -6.12107
$0d_{3/2}$	-0.06567 -0.06279	-2.07668 -2.07330	-4.09242 -4.08832	-6.11725 -6.11193

energy levels are sensitive to the a in the presence $(D \neq 0)$ and absence $(D = 0)$ of the tensor potential. The sensitiveness is given in Table (4.5) for the pseudospin doublet $(1s_{1/2}, 0d_{3/2})$. From Table (4.5), we can observe that energies of these states decrease while a increases both in the presence and absence of the tensor potential. The reason of this can be explained by considering the derivative of the Woods-Saxon potential given in Eq. (4.93). From Eq. (4.93), one can easily see that the derivative of the Woods-Saxon potential decreases when the diffusivity increases.

The sensitivity of the energy levels to the width of the Woods-Saxon potential is given in Fig. (4.7) in the presence and absence of the tensor potential. In Fig. (4.7), we vary R and keep all other parameters fixed. We can see in Fig. (4.7) that though energies of the bound states increase with increasing R, the sensitivity of the energy levels

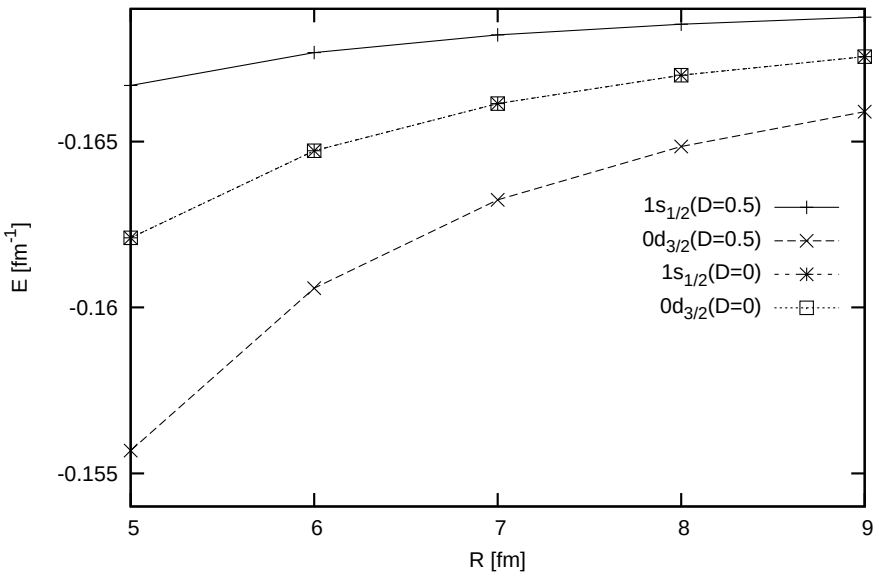

Figure 4.7: Energy values computed as a function of R for the pseudospin doublet $(1s_{1/2}, 0d_{3/2})$ with $D = 0$ and $D = 0.5$.

to the width of the Woods-Saxon potential decreases with increasing R. This can be also explained by considering the derivative of the Woods-Saxon potential. From Eq. (4.93), one can see that the derivative of the Woods-Saxon potential increases while the width R increases.

Finally, we investigate the dependence of the bound state energy levels to the parameter $C_{ps} = V(r) - S(r) = Const$ in the presence of the tensor potential. In Table (4.5), we fix all parameters except for the C_{ps}. From Table (4.5), one can see that energies of the pseudospin doublet $(1s_{1/2}, 0d_{3/2})$ increase almost linearly with the C_{ps}.

4.2.5 Pseudoharmonic Potential together with Tensor Potential

Energy eigenvalues equation for the Dirac particles presence of the tensor potential under the condition of the pseudospin symmetry has been obtained in section 4.1.5. By setting $\sigma = 0$ and making some arrangements on Eq. (4.136), it becomes

$$A\left(2 + 4n + \sqrt{(2\kappa - 1)^2 + 4(E_{n\kappa} - M - C_{ps})B}\right)^2 - (E_{n\kappa} - M - C_{ps})(E_{n\kappa} + M - C)^2 = 0 \tag{4.148}$$

Table 4.6: The bound state energy eigenvalues in units of fm^{-1} for $C_{ps} = -11.5fm^{-1}$ in the presence and absence of the tensor potential

$\widetilde{l}$	n, κ	(l, j)	$E_{n,\kappa_{<0}}$ $\sigma = 0$	$E_{n,\kappa_{<0}}$ $\sigma \neq 0$	$n-1, \kappa$	$(l+2, j+1)$	$E_{n,\kappa_{>0}}$ $\sigma = 0$	$E_{n-1,\kappa_{>0}}$ $\sigma \neq 0$
1	1,-1	$1s_{1/2}$	-0.985845	0.058942	0,2	$0d_{3/2}$	-0.985845	-0.637548
2	1,-2	$1p_{3/2}$	-0.876620	0.345040	0,3	$0f_{5/2}$	-0.876620	-0.772562
3	1,-3	$1d_{5/2}$	-0.737799	0.643012	0,4	$0g_{7/2}$	-0.737799	-0.853108
4	1,-4	$1f_{7/2}$	-0.577439	0.948574	0,5	$0h_{9/2}$	-0.577439	-0.893433
1	2,-1	$2s_{1/2}$	-0.424705	0.714893	1,2	$1d_{3/2}$	-0.424705	0.074972
2	2,-2	$2p_{3/2}$	-0.319515	0.982156	1,3	$1f_{5/2}$	-0.319515	-0.058697
3	2,-3	$2d_{5/2}$	-0.179754	1.264342	1,4	$1g_{7/2}$	-0.179754	-0.148071
4	2,-4	$2f_{7/2}$	-0.015172	1.556767	1,5	$1h_{9/2}$	-0.015172	-0.198569

Eq. (4.148) is just expression (4.20) obtained for the pseudoharmonic potential in the pseudospin limit by using the Nikiforov-Uvarov method absence of the tensor potential. We have shown that there are degeneracies between each pair of eigenstates $(ns_{1/2}, (n-1)d_{3/2})$, $(np_{3/2}, (n-1)f_{5/2})$, *etc.*. Actually, these states which have same pseudo-orbital angular momentum $\widetilde{l}$ and pseudospin $\widetilde{s} = 1/2$ are pseudospin doublet. Now, we investigate the degeneracy between the two states in pseudospin doublet and radial nodes of the wave functions in the presence of the tensor potential.

Energy eigenvalue equation obtained in the presence of the tensor potential (4.136) is a very complicated equation. So, one can obtain the numerical solution of this equation with the help of the computer. In Table (4.6), we take a set of parameter used in Ref. [72]. From Table (4.6), we can observe that all degeneracies between members of pseudospin doublet are removed in the presence of the tensor potential. It can be helpful to investigate the pseudospin doublet splitting. As mentioned before, levels should have negative energy solutions in the pseudospin symmetry. In this context, it is interesting to emphasize that some levels with negative energy obtained in the absence of the tensor potential replace with positive energy state in the presence of the tensor potential, which means that tensor potential damages the negative bound state energy solutions in the pseudospin symmetry limit. For instance, although $1s_{1/2}$ state is a negative energy bound state $E_{1,-1} = -0.985845$ for $\sigma = 0$, it turns into positive energy state $E_{1,-1} = 0.058942$ for $\sigma \neq 0$.

We investigate how the pseudospin doublet splitting is sensitive to the parameters D_e, r_e, σ and C_{ps}. In our calculations, we use the same set of pseudospin doublets :

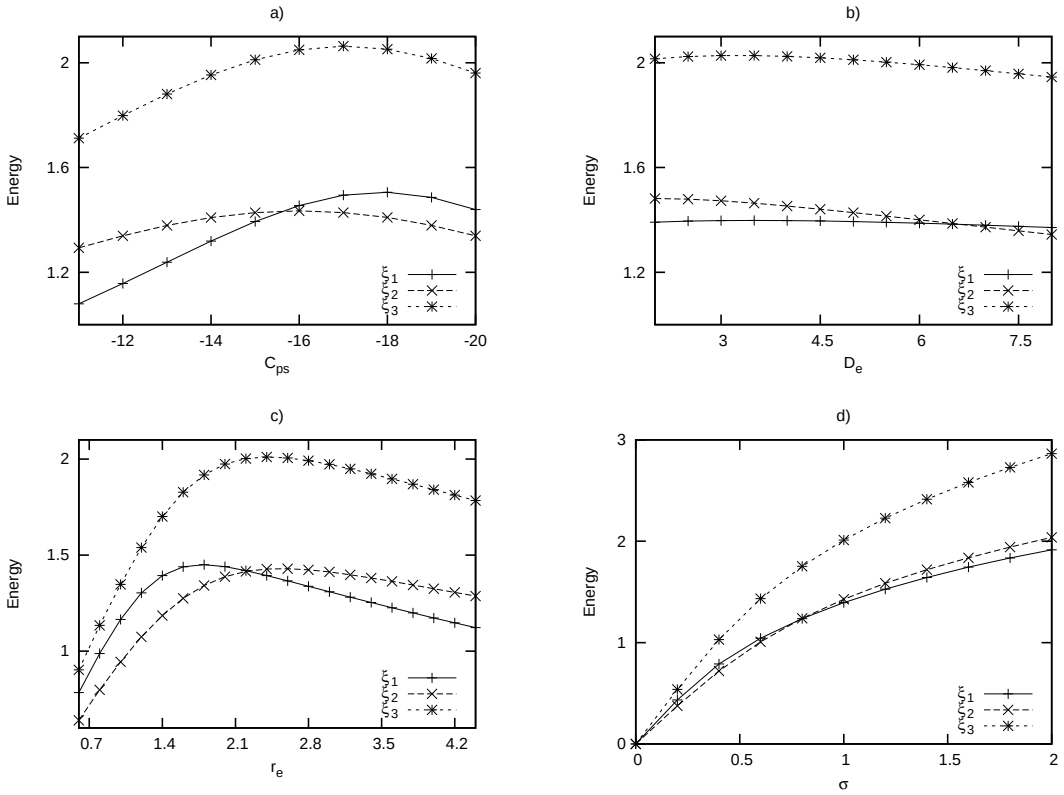

Figure 4.8: Effect of the potential parameter D_e, r_e, σ as well as C_{ps} on pseudospin doublet splitting.

$\xi_1 \equiv (1p_{3/2}, 0f_{5/2})$, $\xi_2 \equiv (3d_{5/2}, 2g_{7/2})$, $\xi_3 \equiv (2f_{7/2}, 1h_{9/2})$. The results are given in Fig. (4.8). From Fig. (4.8), we observe the following cases:

1. For all parameters, ξ_3 is the most splitting pseudospin doublet.

2. For all parameter, ξ_1 and ξ_2 cross each other by increasing D_e, r_e, σ and C_{ps}.

3. Splitting of the pseudospin doublets increases with increasing r_e, C_{ps} and D_e. However, the pseudospin doublet splitting decreases by further increasing r_e, C_{ps} and D_e. In this case, sign of the energy splitting is inverted.

4. In the absence of the tensor potential, there is no any energy splitting between the two states in pseudospin doublet, this means that each state in the pseudospin doublet has same energy with its partners. However, pseudospin doublet splitting is formed in the presence of the tensor potential. In addition, variation of the energy splitting of the pseudospin doublet is quickly changed by increasing σ.

In Fig. (4.9), radial node structures of the upper and lower components of the Dirac spinor are investigated. In calculations, we take following parameters: $M = 10.0fm^{-1}$, $r_e = 2.4fm$, $D_e = 5.0fm^{-1}$ and $C_{ps} = -15fm^{-1}$. Then, the following energies for the each member of the pseudospin doublet $(2s_{1/2}, 1d_{3/2})$ are obtained : $E_{2,-1}(\sigma = 0) = E_{2,2}(\sigma = 0) = -3.07812$, $E_{2,-1}(\sigma \neq 0) = -1.8036$ and $E_{2,2}(\sigma \neq 0) = -2.53539$. We know that the generalized Laguerre polynomials of degree n have n distinct zero, thus one can reach the following conclusion from Eqs. (4.134) and (4.135): lower component $G_{n\kappa}$ has n nodes while upper component $F_{n\kappa}$ has n and $n+1$ nodes for $\kappa > 0$ and $\kappa < 0$, respectively [79, 142]. The radial node structure of the pseudospin doublet $(2s_{1/2}, 1d_{3/2})$ is given in Fig. (4.9). One can see in Fig. (4.9a) and (4.9c) that lower and upper radial wave functions of the state $1d_{3/2}$ ($\kappa = 2$) have 1 node. Besides, lower radial wave function of the state $2s_{1/2}$ ($\kappa = -1$) has 1 node while upper component has 2 nodes. This means that, number of nodes of the radial wave functions is that $n_F = n_G$ for $\kappa > 0$ and $n_F = n_G + 1$ for $\kappa < 0$. These results are in good agreement those available in literature [36]. We can conclude from Fig. (4.9b) and (4.9d) that tensor interaction does not change the node structure of radial wave functions. However, it effects the shape of the radial wave functions.

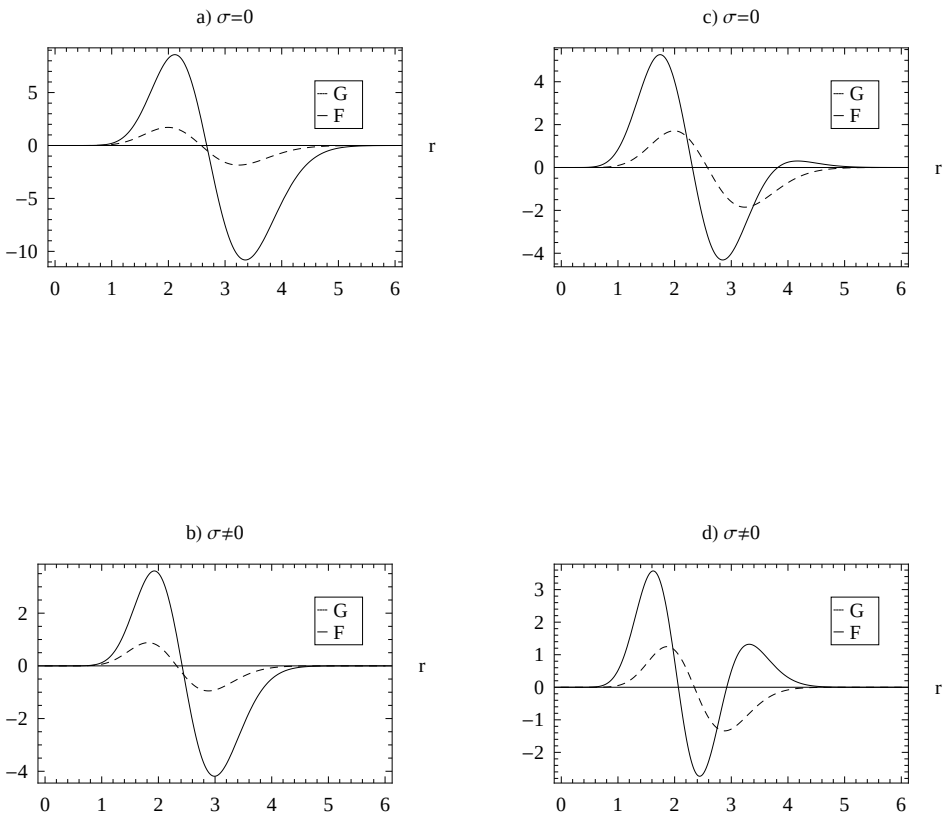

Figure 4.9: Lower and upper radial wave functions for $1d_{3/2}$ *a)* $\sigma = 0$ *b)* $\sigma \neq 0$ and for $2s_{1/2}$ *c)* $\sigma = 0$ *d)* $\sigma \neq 0$.

CHAPTER 5

CONCLUSION

In the present thesis, we have investigated the pseudospin symmetric solution of the Dirac equation for the exactly solvable potentials, such as pseudoharmonic potential, Mie-Type potential, Hulthén plus ring-shaped potential, Woods-Saxon together with Coulomb-like tensor potential and pseudoharmonic together with linear tensor potential, in terms of different methods. In the calculations, we have used the NU method, AI method and functional analysis method.

The energy spectra and corresponding wave functions of the Dirac equation for the pseudoharmonic potential with pseudospin symmetry have been obtained by using the Nikiforov-Uvarov method. For any spin-orbit coupling term κ, we have found out the closed forms of the energy eigenvalues. Some numerical solutions of the energy eigenvalues equation have been given in the pseudospin limit for the pseudoharmonic potential. We have also investigated the energy eigenvalues of harmonic oscillator potential which is the special case of the general form of the pseudoharmonic potential. To get the energy spectra of linear and non-linear systems, solution of the Dirac equation for pseudoharmonic potential is significant. Thus, our result can be used to evaluate the energy spectra of linear and non-linear systems. We have also showed that our results can be reduced well-known solution of the harmonic oscillator potential by setting $A = \frac{1}{2}Mw_1^2$, $B = C = C_{ps} = 0$ and recalling $\kappa(\kappa - 1) = \widetilde{l}(\widetilde{l} + 1)$ in the pseudospin symmetry case. Finally, Eq. (4.20) can be used to evaluate the binding energies of the pseudoharmonic potential for diatomic molecules such as CH, CO and N_2 in the relativistic framework with pseudospin symmetry case.

We have obtained the exact solution of the Dirac equation for the nuclei with Mie-

type potentials under the condition of pseudospin symmetry limit by using asymptotic iteration method. We have obtained the bound state energy equation and the corresponding two-component spinor wave functions of the Dirac equation for the nuclei. The Kratzer-Fues and modified Kratzer potentials which are two examples for the Mie-type potential have been also investigated. We have obtained that there are degeneracies between the pair of orbital such as $(s_{1/2}, d_{3/2})$, $(p_{3/2}, f_{5/2})$ and $(d_{5/2}, g_{7/2})$ *etc.*. This means that each pair of orbital forms pseudospin doublets. By considering particular cases of the Mie-type potentials, the energy eigenvalues equations for the Dirac particle with relevant potentials have been given.

Analytic solution of the Dirac equation for the generalized Hulthén plus ring-shaped potential with pseudospin symmetry has been investigated by using Nikiforov-Uvarov method. We have obtained the energy eigenvalues equation and the corresponding wave functions of the Dirac particles for the pseudospin limit. Radial and angular wave functions have been also obtained in terms of special orthogonal functions in the pseudospin symmetry limit. Besides, we have investigated the contribution of the angle-dependent potential to the relativistic energy spectra. Finally, it has been displayed that our results are compatible with those available in the literature.

We have obtained approximately the energy eigenvalues equation and corresponding upper and lower radial functions of the Dirac particles for the Woods-Saxon potential and Coulomb-like tensor potential under the condition of the pseudospin symmetry. We have showed that tensor interaction removes the energy degeneracy between the two states in the pseudospin doublets. These results agree with previous ones obtained by Lisboa *et. al* and Akcay *et. al.*. We have also obtained that energy levels of the pseudospin aligned and unaligned states move in the opposite directions due to the pseudospin-dependent term. The sensitiveness of the energy levels to the parameters a, R, D, C_{ps} has been investigated in the case of the pseudospin symmetry limit.

We have investigated the exact solution of the Dirac equation for the pseudoharmonic potential in the presence of the tensor potential under the condition of pseudospin symmetry. Energy eigenvalue equation and corresponding radial wave functions have been obtained for any spin-orbit coupling quantum number κ. Degeneracy between members of the pseudospin doublet has been explored in the presence and absence of

the tensor potential. We have seen that that tensor interaction removes this degeneracy. Effects of the potential parameters as well as C_{ps} on the energy levels of the each state of the pseudospin doublet have investigated. We have analyzed the upper and lower radial wave functions and their node structures. We have showed that results presented agree with those available in the literature. Finally, we have concluded that tensor interaction can be useful to study the pseudospin doublet splitting and to remove the degeneracy between the members of the pseudospin doublet.

Consequently, our results will facilitate future discussions on pseudospin symmetry in the Dirac phenomenology. Especially, effect of the tensor potential on the members of the pseudospin doublets can be discussed in detail by considering different physical systems.

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