STUDENTS’ UNDERSTANDING OF LIMIT CONCEPT: AN APOS PERSPECTIVE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

İBRAHİM ÇETİN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
COMPUTER EDUCATION AND INSTRUCTIONAL TECHNOLOGY

DECEMBER 2009

Approval of the thesis:

STUDENTS' UNDERSTANDING OF LIMIT CONCEPT: AN APOS PERSPECTIVE

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ABSTRACT

STUDENTS' UNDERSTANDING OF LIMIT CONCEPT: AN APOS PERSPECTIVE

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December 2009, 204 pages

The main purposes of this study is to investigate first year calculus students' understanding of formal limit concept and change in their understanding after limit instruction designed by the researcher based on APOS theory. The case study method was utilized to explore the research questions. The participants of the study were 25 students attending first year calculus course in Middle East Technical University in Turkey. All students were first year mathematics majors. Students attended five weeks instruction in the fall semester of 2007-2008. In each week they met in two hours computer laboratory to study in groups, and then they attended four hours classes. In computer labs they were given some programming activities which give students opportunity to think on limit concept before they were given formal lecture in classes. Limit questionnaire including open-ended questions was administered to students as a pretest and posttest to probe change in students' understanding of limit concept. At the end of the instruction a semi-structured interview protocol developed by the researcher was administered to all of the students to explore students' understanding of limit concept in depth. The students' responses in the questionnaire were analyzed both qualitatively and quantitatively. The interview results were analyzed by using APOS framework. The results of the study showed that constructed genetic decomposition was found to be compatible with student data. Moreover, limit instruction was found to play a positive role in facilitating students' understanding of limit concept.

Keywords: Computer programming in calculus, Constructionism, Cooperative Learning, Limit concept, APOS theory

ÖZ

ÖĞRENCİLERİN LİMİT KONUSUNU KAVRAMALARI: APOS PERSPEKTİFİNDEN

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Bu çalışmanın başlıca amaçları, birinci sınıf analize giriş öğrencilerinin limit konusunu nasıl kavradıklarını incelemek ve bu kavrayışın araştırmacı tarafından APOS Teorisi kullanılarak tasarlanan öğretim ortamının uygulamasından sonra nasıl değiştiğini araştırmaktır. Çalışmanın amacına uygun olarak durum çalışması deseni kullanılmıştır. Çalışmaya Orta Doğu Teknik Üniversitenin Matematik Bölümünde öğrenim gören 25 birinci sınıf analize giriş dersi öğrencisi katılmıştır. Öğrenciler 2007-2008 öğrenim yılının bahar döneminde, 5 hafta boyunca araştırmacı tarafından geliştirilen öğretim ortamına devam etmişlerdir. Öğrenciler her hafta iki saatlik laboratuvar uygulamalarında işbirlikçi bir ortamda kümeler halinde çalışmış daha sonra dört saatlik derslere katılmışlardır. Ders saatlerinden önce, bilgisayar laboratuvarlarında öğrencileri limit konusunda düşünmeye yönlendirici bilgisayar programlama etkinlikleri kullanılmıştır. Öğrencilerin limit kavramını anlama düzeylerindeki değişimi belirlemek için açık uçlu sorular içeren limit anketi öğrencilere ön-test ve son-test olarak uygulanılmıştır. Beş haftanın sonunda, öğrencilerin limit konusunu nasıl kavradıklarını derinlemesine incelemek için, öğrencilerin tümüyle yarı yapılandırılmış görüşmeler düzenlenmiştir. Öğrencilerin limit anketinde verdiği cevaplar nitel ve nicel yöntemler kullanılarak incelenmiştir. Ayrıca görüşme sorularına verilen yanıtlar APOS çerçevesi kullanılarak analiz edilmiştir. Çalışmanın sonuçlarına göre, oluşturulan genetik çözümlemenin bu çalışmadan elde edilen öğrenci verileri ile uyumlu olduğu gözlenmiştir. Ayrıca araştırmacı tarafından geliştirilen öğrenim ortamının öğrencilerin limit konusunu kavramalarına olumlu etkide bulunduğu gözlenmiştir.

Anahtar Kelimeler: Analize giriş dersinde bilgisayar programlama, İşbirlikçi öğrenme, Limit kavramı, APOS teorisi

DEDICATION

To my wife, PINAR SEDA

ACKNOWLEDGEMENTS

I wish to express my deepest gratitude to my supervisor, Prof. Dr. Yaşar Özden, for his guidance, advice, criticism and encouragement and insight throughout the research.

I would like to express my special thanks to thesis co-supervisor Prof. Dr.
Ed Dubinsky for valuable guidance, support, encouragement and patience throughout this long period of study.

I would also like to thank my friends for their support.

Finally, I am sincerely thankful to my parents for their moral support and encouragement

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CHAPTER 1

INTRODUCTION

Throughout last century different practices of teaching and different views of learning has emerged. Mayer (1992) argued three views of learning: learning as response strengthening, learning as knowledge acquisition, and learning as knowledge construction. In the first view, response strengthening, learning occurs as the learner strengthens or weakens associations between stimulus and response. Rewards and punishments were used for the instruction. In the second view, knowledge acquisition, learning occurs through placing new information into long-term memory. Presentation of information in textbooks and lectures was used to transmit information from teachers to learners. In the last view, knowledge construction, learners actively construct their own knowledge. This view is called as constructivism.

Constructivism involves two main principles; psychological and epistemological. Psychological principle explains that knowledge can not be directly transferred from teachers to students. Students do not receive knowledge in a passive way; instead they construct their own meaning. Piaget (1964, p.176) formulated this as follows

> *To know an object is to act on it. To know is modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed. An operation is thus the essence of knowledge; it is an interiorized action which modifies the object of knowledge.*

Epistemological principle is about reality. In constructivism reality is determined in a subjective way. Since individual constructs knowledge in a subjective way, outside reality either does not exist or if exist can not be known by the individual. So reality is determined in a personal or subjective way (von Glasersfeld, 1990). Constructivist puts the notion of viability in place of outside reality. Rather than searching for absolute truth, constructivism searches for usefulness and viability of knowledge in different context.

Constructivist researchers have been trying to find ways to improve instruction so that active and meaningful learning occur. In parallel to these efforts, there has been rapid development in computer and communication technologies. Since the development of computers and communication technologies, researchers have been searching for its uses in education. Depending on their views of learning, researchers proposed different uses of computers and communication technologies for instructional purposes. Among these different proposals, constructivists claimed that electronic technologies can reinforce the pedagogical shift to active learning (Kaput & Thompson, 1994). Moreover, it was suggested that constructivist learning context is thought to facilitate learning with technology (Jonassen & Reeves, 1996).

In the intersection of constructivism and use of technology for the instructional purposes, one specific learning theory has surfaced, namely constructionism. In constructionism, students are seen as natural learners. With guidance, these learners can achieve their potentials. In instructor's guidance, the aim is to help students to learn how to learn, or learn their ways of learning, rather than one global approach for all students. If society provides necessary materials, students can learn mathematics as they learn their mother tongue.

Constructionism appreciates students' constructions of external public entity (Ackerman, 1996). In construction of this sharable entity, constructionism emphasizes the role of cultural artifacts which will be internalized and on the role of artifacts that students construct and share with the others in learning process. In this construction, students internalize what is outside and then externalize what is inside and by this way they shape their ideas. But in what ways does construction of tangible entity help students to construct the subject in question? This is explained by the basic belief of the constructionism that I call as parallel construction. Papert (1993, p.28) put this idea as follows "… Programming the Turtle starts by making one reflect on how one does oneself what one would like the turtle to do. Thus teaching Turtle to act or to "think" can lead one to reflect on one's own actions and thinking".

Constructivist theory and the idea of parallel construction found its place in the advanced mathematical thinking. Dubinsky (1991) proposed to use of reflective abstraction of Piaget (Beth & Piaget (1966) in advanced mathematical thinking. Later, APOS theory has been developed by Dubinsky and his colleagues. **A**ction, **P**rocess, **O**bject, **S**chema are the mental structures that an individual builds by the mental mechanism of reflective abstraction. Dubinsky (1991) and Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas (1996) determined six kinds of reflective abstractions: interiorization, coordination, reversal, encapsulation, thematization, and generalization in undergraduate mathematics education. According to the theory, concept formation begins with transformations of existing mental (or physical) objects. This type of transformation is called action. Then by repeating an action and reflecting on it, an individual might interiorize it as a process. If an individual pays attention to and becomes aware of the fact that actions or operations can act on the process being concerned, she/he might encapsulate process into an object. In order to encapsulate a process, process should be seen as a completed totality on which actions or processes can act. Actions, processes, and objects are organized into mental structures which are called schemas. Once constructed, schemas can be applied to other schemas to give meaning to mathematical situations. In describing students' constructions and mechanisms which are used to produce those constructions, APOS researchers devise a tool called genetic decomposition.
APOS researchers used several programming languages (mostly used one is ISETL) to foster students constructions for specific mathematical subject. Then they introduced cooperative learning into their instruction (Dubinsky, 1995). Johnson, Johnson, and

Holubec (1993, p.9) defined cooperative learning as "the instructional use of small groups so that students work together to maximize their own and each other's learning".

Some researchers argued that whether all small group learning is cooperative or not (Hagelgans, Reynolds, Schwingendorf, Vidakovic, Dubinsky, Shahin, Wimbish 1995). Then research studies (e.g., Johnson and Johnson, 1999) about cooperative learning proposed five components that are essential for successful cooperative learning to occur: individual accountability, positive interdependence, promotive interaction, social skills, and group processing.

From the beginning of research area about cooperative learning, there has been a passionate question "Does cooperative learning work". In general, the answer is yes with the precaution that cooperative learning is useful if certain conditions are satisfied (Slavin, 1995). Slavin (2009) considered the question in two different domains: well-structured and ill-structured. He concluded that although there is less research about cooperative learning in ill-structured domains than in well-structured domains, for both domains favorable results of cooperative learning are consistently reported, but again, with the following precaution. First is that students need to feel to facilitate other members' learning, and second is that there must be individual accountability. Moreover, by reviewing research on cooperative learning Biehler and Snowman (1997, p.421) concluded that "Students who learn cooperatively tend to be more highly motivated to learn because of increased self-esteem, the pro-academic attitudes of group mates, appropriate attribution for success and failure, and greater on-task behavior. They also score higher on tests of achievement and problem solving and tend to get along better with class mates with different racial, ethnic, and social class backgrounds".

Turning back to APOS theory, introduction of cooperative learning strategies into computer activities to facilitate students' constructions resulted in a standard structure called ACE (**A**ctivities, **C**lass, and **E**xercises) cycle.

According to this standard structure (Hagelgans et al., 1995), at the beginning of the lesson, groups including three or four members are formed. Groups meet in the computer laboratory to study on the materials which is prepared by the instructor. They are, usually, required to write code segments or modify given code. Computer activities are designed to foster specific mental constructions suggested by genetic decomposition. This pre-class activity gives students chance of informal introduction to the concept in consideration and of thinking on the concept before class session. After computer laboratory, students meet in the classroom to work on the tasks prepared by the instructor. In classes, all of the members of the groups are required to participate to tasks within their groups. Then, one of the groups presents their report to whole class to be discussed. At the end of the discussion, instructor may clarify certain points, and introduce formal ideas. Lastly, after the class students are given relatively traditional exercise sets as homework. This homework can be done individually or as a group. Aim of the homework is to reinforce and apply what they learned in the computer activities and class.

1.1 Purpose of the Study

Understanding limit concept is crucial for calculus students since it establishes a ground for development of the concepts of continuity, derivative, and integral. Although importance of limit understanding has been recognized, introduction of this concept, because of its complexity, causes serious difficulties.

The purpose of this study is two fold: (1) to explore how students understand limit concept by using APOS framework, (2) to construct a base for the future studies with the aim of making instruction more effective.

1.2 Research Questions

The following research questions guided this study:

1. How do students develop understanding in limit of a function?

   1.1. How do students explain their informal understanding of limit of a function?

   1.2. How do students explain their formal understanding of limit of a function?

   1.3. What kind of difficulties do students encounter in transition from informal understanding to formal understanding of limit of a function?

2. How different is students' understanding of limit of a function after the instruction based on APOS theory?

1.3. Significance of the Study

Notion of limit of a function is fundamental for understanding calculus and the basis of all that follows it. Differentiation and integration, the core of study in calculus, are built on the limit concept. Nevertheless, in literature, it is generally agreed that students have difficulties in understanding limit concept. It is argued that most students have intuitive understanding of limit but very few of them accomplish understanding of the limit definition (Ervynck, 1981; Cottrill, Dubinsky, Nichols, Schwingerdorf, Thomas & Vidakovic, 1996).

Schwarzenberger and Tall (1978) proposed that, in order to convey formal mathematical ideas to students, they are translated into suitable forms. Together with students' prior experience, this translation might cause some conflicts. Monaghan (1991) concluded that daily life meanings of the phrases "approaches", "tends to", "converges", and "limit might become potential cognitive conflict factors when students are exposed to limit instruction, and moreover, such meanings might be continued to be hold after formal instruction.

In addition to daily life meanings attached to words, some of the researchers studied other factors that makes limit concept hard to understand for students. Sierpinska (1985) went in that way and identified five main epistemological obstacles and rearranged this list (1987) as follows. “Obstacles related to four notions seem to be the main sources of epistemological obstacles concerning limits: scientific knowledge, infinity, function, real number”.

In explaining students’ difficulties in understanding limit concept, some researchers (Williams, 1991; Tall & Vinner, 1981) put a dichotomy between dynamic and static notions of limit. For them, “as x goes to a $f(x)$ goes to L” is a process which includes dynamic feeling of motion. Nevertheless, in formal conception, an individual deals with intervals in which x and $f(x)$ values do not move. So, it is dynamical element in informal limit notion that prevents students to move more formal understanding of limit concept. An opposition to this idea came from Cottrill et al. (1996). They considered dynamical notion as a mental process in APOS terms. Actually this is not a single process, rather is coordination of domain and range processes via function in consideration, thus a schema. Contrary to the belief that process conception is easy to understand, they suggested that coordinated process schema is not easily constructed by students. Moreover, they argued that informal process schema of limit concept is necessary in building formal understanding of limit. Formal understanding of limit concept is built on coordinated process of informal limit, rather than hindered by it. Difficulty in moving from informal understanding to formal understanding comes from students’ weak understanding of quantification.

Based on their Action, Process, Object, Schema framework, Cottrill et al. (1996) proposed their first theoretical genetic decomposition. However, data from student interviews did not allow authors to validate some of the steps in the genetic decomposition.

An opposition to APOS theory in general and genetic decomposition of limit concept in specific came from Pinto and Tall (2001). They (2001, p. 57) described two different learning styles: one is formal other is natural. “Formal thinkers attempt to base their work on the definitions… Natural thinkers reconstruct new knowledge from their concept image”. They contended that formal thinkers are compatible with APOS theory, whereas, APOS theory does not explain the way of natural thinkers’ learning.

As seen in the literature, there is not a consensus about how limit concept is learned by students, and what the potential causes for students’ difficulties are. Nevertheless, there is a general agreement that students have difficulties in understanding limit concept. And, mostly cited difficulty is that whether a function can reach its limit or not (Williams, 1991; Tall, 1980a; Tall &Vinner, 1981)

One of the main aims, in this study, is to explore how students understand limit concept. In order to address this question Cottrill et al.’s (1996) genetic decomposition is taken as a primary genetic decomposition. The result of this study might have a contribution to literature to the extent that how students understand limit concept and what type of difficulties do they encounter in learning this concept. In addition, knowing how students learn limit concept and students’ difficulties about it might help instructors in sequencing content, in our case limit, and in designing learning environments.

Designing instruction for helping students to overcome their difficulties is as important as determination of students’ difficulties in limit concept. Some of the researchers tried to develop instruction to facilitate students’ understanding of the concept. However, in literature, it is seen that most of them failed to help students (Buyukkoroglu et al., 2006; Cottrill et al., 1996; Davis & Vinner, 1986; Li & Tall, 1993; Monaghan, Sun & Tall, 1994; Parameswaran, 2007; Sierpinska, 1987; Williams, 1991).

Since the development of computers and communication technologies, researchers have been searching for its uses in education. Some of the researchers claimed that electronic technologies can reinforce the pedagogical shift to active and meaningful learning (Kaput & Thompson, 1994; Jonassen & Reeves, 1996). Moreover, Tall and Ramos (2004) claimed that computers can radically change mathematical learning environment. In this study, researcher designed an instruction based on cooperative learning integrated with technology. The other purpose of this study is to construct a base for the future studies with the aim of making instruction more effective.

1.4 Definition of the Terms

The constitutive and operational definition of important terms will be given in this section.

Instructional Technology: Early definitions viewed instructional technology as media via which instruction is presented to learners. Then it has been evolved to

> *"Instructional technology is the theory and practice of design, development, utilization, management, and evaluation of processes and resources for learning" (Seels & Richey, 1994, p. 1).*

Technology: "In addition to machinery, technology includes processes, systems, management, and control mechanisms both human and non-human" (Finn, 1960, as cited in, Gentry, 1995, p.2).

Informal Limit Definition: "If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, we say that the function f approaches the limit L as x approaches a" (Adams, 1999).

Formal Limit Definition: "We say that $f(x)$ approaches the limit L as x approaches a if the following condition is satisfied:

For every number $\varepsilon>0$ there exists a number $\delta>0$, depending on ε, such that $0<|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$." (Adams, 1999).

CHAPTER 2

REVIEW OF LITERATURE

Present chapter of the study includes relevant theoretical frameworks and studies on which research is grounded. First of all literature review about constructionism is presented. Then cooperative learning literature is provided. In the third part, literature about APOS theory is given. And lastly, students' understanding of limit concept and effectiveness of previously designed instructions are discussed.

2.1 Constructionism

Constructionism is a theory of learning which builds on constructivism and reconstructs it. To dwell on constructionism, first, we need to address principles of constructivism. Constructivism involves two main principles; psychological and epistemological. Psychological principle explains that knowledge can not be directly transferred from teachers to students. Students do not receive knowledge in a passive way; instead they construct their own meaning. Piaget (1964, p.176) formulated this as follows:

> *To know an object is to act on it. To know is modify, to transform the object, and to understand the process of this transformation, and as a consequence to understand the way the object is constructed. An operation is thus the essence of knowledge; it is an interiorized action which modifies the object of knowledge.*

Epistemological principle is about reality. In constructivism reality is determined in a subjective way. Since individual constructs knowledge in a subjective way, outside reality either does not exist or if exist can not be known by the individual. So reality is determined in a personal or subjective way (von Glasersfeld, 1990). Constructivist puts the notion of viability in place of outside reality. Rather than searching for absolute truth, constructivism searches for usefulness and viability of knowledge in different context.

Constructionism shares abovementioned principles of constructivism. For example, Papert (1996) complained that in English there is no word for the art of learning whereas word "pedagogy" used for the art of teaching. He proposed the word "mathetics" for the art of learning to emphasize that students are not passive recipients of the knowledge. Moreover, he saw the role of teacher as a guide rather than transmitter and contended that students often learn best when taught least. The important point is that students need to learn how to learn. "The constructionist attitude to teaching is not at all dismissive because it is minimalist- the goal is to teach in such a way as to produce the most learning for the least teaching. Of course, this cannot be achieved simply by reducing the quantity of teaching while leaving everything else unchanged. The principal other necessary change parallels an African proverb: If a man is hungry you can give him a fish, but it is better to give him a line and teach him to catch the fish himself". (1992, p.139)

The difference between constructivism and constructionism starts with the appreciation of students' constructions of external public entity. In construction of this sharable entity, constructionism put more emphasize on the role of cultural artifacts which will be internalized and on the role of artifacts that students construct and share with the others in learning process. In this construction, students internalize what is outside and then externalize what is inside and by this way they shape their ideas. Papert (1990, p.3) puts this distinction in the following way or it might be more appropriate to say, in constructionist terms, Papert reconstructs constructivism in the following way.

> *We understand "constructionism" as including, but going beyond, what Piaget would call "constructivism". The word with the v expresses the theory that knowledge is built by learner, not supplied by the teacher. The*

word with the n expresses the further idea that this happens especially felicitously when the learner is engaged in the construction of something external or at least sharable… a sand castle, a machine, a computer program, a book. This leads us to a model using a cycle of internalization what is outside, then externalization of what is inside and so on.

Making ideas tangible and sharable are keys to learning in constructionism (Kafai and Resnick, 1996). By doing so, students share these ideas and communicate with others through their own expressions. But in what ways does construction of tangible entity help students to construct the subject in question?

This is explained by the basic belief of the constructionism that I call as parallel construction. It is the belief that transforming and constructing objects on the computer will facilitate students' learning in that they construct corresponding transformations and objects in their mind. Papert (1993, p.28) put this idea as "… Programming the Turtle starts by making one reflect on how one does oneself what one would like the turtle to do. Thus teaching Turtle to act or to "think" can lead one to reflect on one's own actions and thinking".

Another difference between Piagetian constructivism and constructionism is about the role of culture and social interaction in the learning process (Hagelgans et al., 1995). According to Piaget knowledge construction is preceded as the individual interacts with her/his environment. Social interaction and culture are elements of this environment, so, they are important in individual's learning. Nevertheless, when we come to the essential elements of individuals' development, social interaction and culture become secondary factors. Although they may enhance individuals' development, they cannot influence the process of development in crucial ways.

On the other hand, in constructionism, culture plays a major role in individuals' development. Culture provides artifacts for the construction of the knowledge. If these artifacts are provided enough, then they may enhance individuals' development as in

Piagetian theory. But, unlike the Piagetian interpretation, constructionists believe that absence of these artifacts intervene development of the individuals' in essential ways.

> *All builders need materials to build with. Where I am at variance with Piaget is in the role I attribute to the surrounding cultures as a source of these materials. In some cases the culture supplies them in abundance, thus facilitating constructive Piagetian learning. For example, the fact that so many important things (knives and forks, mothers and fathers, shoes and socks) come in pairs is a "material" for the construction of an intuitive sense of number. But in many cases where Piaget would explain the slower development of a particular concept by its greater complexity or formality, I see the critical factor as the relative poverty of the culture in those materials that would make the concept simple and concrete. (Papert, 1993, p.7)*

Thus, for constructionists, knowledge is built actively in a social environment by constructing and reconstructing tangible and sharable entities whose construction helps students to make parallel constructions in their mind. In order to engage in intellectual activities, individuals need to see these activities as personally meaningful. Moreover, social interaction and artifacts supplied by the culture play a crucial role in students' shaping their ideas. Cultural artifacts help individuals to concretize the abstract and artifacts constructed by individuals help them to express and shape their ideas, that is, to owe knowledge rather than being given.

2.2 Cooperative Learning and Its Elements

In this section cooperative learning, its components, teacher's role in cooperative learning, methods of cooperative learning, and lastly research on effectiveness of cooperative learning are addressed.

Johnson, Johnson, and Holubec (1993, p.9) defined cooperative learning as "the instructional use of small groups so that students work together to maximize their own and each other's learning". Some researchers argued that whether all small group learning is cooperative or not (Hagelgans, Reynolds, Schwingendorf, Vidakovic, Dubinsky, Shahin, Wimbish 1995). Then research studies (e.g., Johnson and Johnson, 1999) about cooperative learning proposed five components that are essential for successful cooperative learning to occur.

1. Individual Accountability: Slavin (1995) mentions about free-rider effect. In a group work it is possible that some of the group members do all of the work while others take their free time. Individuals, in cooperative learning, are responsible for their contribution to the end product of the group. End product is not the product of some of the group members, rather, it is constructed by the efforts of each individual member. Teachers can facilitate individual accountability by making individual assessments, giving individual feedback, monitoring students' performances during cooperative learning process, and giving group rewards for individual behavior.

2. Positive Interdependence: Individual performances isolated from the other group members do not ensure successful cooperative learning. Every member has her/his unique contribution, however, achievement of the goal depends on other members' attaining their goals. So, students know that their success depends on the success of other group members as well as their individual success. Teacher can help to improve

positive interdependence by assigning a clearly defined task to the group, by assigning interconnected goals to each group member, and by rewarding them as a group.

3. Promotive Interaction: In order to successful cooperative learning occur, students need to facilitate each others learning. This can be done by providing information, assistance, feedback, and resources needed, explaining their thoughts, negotiating meaning, and challenging each other’s ideas. Teacher can encourage promotive interaction by providing time schedule for group to meet, promoting discussions among group members, and arranging physical conditions to meet and negotiate.

4. Social Skills: Unlike individual work, in a cooperative environment, students need to express their ideas to others, persuade others, respect other group members’ thoughts, know when to interrupt, resolve conflicts by discussions, helping group members. For an effective cooperative learning, students need to have or develop these interpersonal skills. Knowing how to work with others is an essential for cooperative learning, nevertheless as Gillies (2007, p.41) put “Placing children in small groups and telling them that they are to cooperate does not ensure that they will use interpersonal and small group skills needed to work effectively together”. Jacobs, Power, Inn (2002) suggested that for teachers, one important way of the teaching social skills is to be an explicit model for collaborative behavior.

5. Group Processing: This is about group members’ reflection on their learning process. Group processing allows students to discuss how well they are achieving their goals and maintaining effective working relationships (Gillies, 2007). Group members question the usefulness of member actions, effectiveness of their way to accomplish group goal, and consider whether changes are necessary or not. Teachers can enhance group processing by having a contact with each group, monitoring their development, and giving feedback about effectiveness of the individual and group work.

2.2.1 Teachers in Cooperative Learning Environment

Gillies (2007, p.195) developed some strategies for teachers to establish productive cooperative learning environment.

- Establish a cooperative learning environment that is inclusive of all students,
- Negotiate expectations for small group behaviors,
- Develop communication skills that facilitate small group discussion,
- Develop appropriate helping behaviors,
- Choose tasks for small group discussion,
- Monitor students' progress and evaluate outcomes.

Johnson and Johnson (1999) argued the importance of following steps for conducting cooperative lesson. The first step is to make some pre-instructional decisions. These include objective determination, consideration of group size, assigning students to groups, assigning roles to group members, arrange the physical environment. The second step is about task and cooperative structure. In this step students should be informed about the task, criteria for success, and behaviors expected from the students during cooperative learning. The third step happens as cooperative lesson is conducted. Each group is observed and students' behavior and learning are monitored, and if it is needed group work is intervened to give feedback. The last step is about evaluation of group performance. Achievement of group members and members as a group is assessed. They are given feedback about their effectiveness of their group performance. As students reflect on their performance, teacher can facilitate them to devise a plan about future work.

2.2.2 Cooperative Learning Methods

Although there are others, studies about cooperative learning emphasized four main cooperative learning methods: Students Teams-Achievement Divisions, Teams-Games-Tournament, Jigsaw, and Group Investigation. Moreover, one specific method (ACE cycle) for collegiate mathematics education will be addressed.

Student Teams-Achievement Divisions (STAD) (Slavin, 1995): STAD starts with instructor's presentation. This presentation focuses on the unit on which students will take quiz. After teacher presentation, students are assigned to heterogeneous groups including four or five members to study material provided by the instructor. In group work, responsibility of the group members is to master the unit and help other group members to master. Then, all students take quizzes individually and they get individual scores from these quizzes. Individual scores are compared to individual base scores which are determined by the average that they get in the past. Then individual improvements are added to group score. In order to provide team recognition, groups whose scores satisfy certain condition are rewarded. Since STAD is easy to implement, it is recommended for beginning teachers.

Teams-Games-Tournaments (TGT) (DeVries & Slavin, 1978): TGT includes a similar path with STAD, but instead of quizzes there exists some academic games that member of one group competes the member of another group whose past performances are similar. Then group score is given by adding individual scores obtained from academic games.

Jigsaw (Aronson, Blaney, Stephan, Sikes & Snapp, 1978): In Jigsaw method, each member of the group is assigned to a subtopic of main unit to get expertise. Members, who are assigned to same topic, from different groups come together to study on the shared topic. Then, they go back to their home group to teach the topic to other members of the team. At the end, students are given a quiz that covers the main unit, and according to their performance they are graded individually.

Group Investigation (GI) (Sharon and Hertz-Lazarowitz, 1980): This method is different from the previously mentioned ones in that choice of the topic and choice of the group are left to students. Each member in the group chooses a subtopic of an issue. Then they study the issue individually or as a group. For each subtopic a report is prepared by the owner and this report is discusses among group members. And final report of the group is presented to whole class.

ACE cycle (Hagelgans et al., 1995): In addition to above frequently quoted methods, one specific cooperative learning method is generated for collegiate mathematics education. This is called ACE cycle (Asiala et al., 1996) which is closely related with APOS theory which will be touched later. In ACE cycle, cooperative learning is integrated with computer activities. At the beginning of the lesson, groups including three or four members are formed. Groups meet in the computer laboratory to study on the materials which is prepared by the instructor. They are, usually, required to write code segments or modify given code. This pre-class activity gives students chance of informal introduction to the concept in consideration and of thinking on the concept before class session. After computer laboratory, students meet in the classroom to work on the tasks prepared by the instructor. All of the members of the groups are required to participate to tasks within their groups. Then, one of the groups presents their report to whole class to be discussed. At the end of the discussion, instructor may clarify certain points, and introduce formal ideas. Lastly, after the class, students are given a set of exercises to be done by all of the group members.

As seen in methods, some of them (e.g., TGT) include elements of competitive goal structure, in which students compete against each other; and still others (e.g., GI) include elements of individualistic goal structure in which students study alone. Johnson and Johnson (1999) suggested that these two goal structures can be integrated with cooperative goal structure for an effective instruction under strong cooperative context, and contended that “When the three goal structures are used appropriately and in an integrated way, their sum is far more powerful than each one separately” (Johnson and Johnson, 1999, p.10).

2.2.3 Research on Cooperative Learning

From the beginning of research area about cooperative learning, there has been a passionate question “Does cooperative learning work”. In general, the answer is yes with the precaution that cooperative learning is useful if certain conditions are satisfied (Slavin, 1995). Slavin (2009) considered the question in two different domains: well-structured and ill-structured. He concluded that although there is less research about cooperative learning in ill-structured domains than in well-structured domains, for both domains favorable results of cooperative learning are consistently reported, but again, with the following precaution. First is that students need to feel to facilitate other members’ learning, and second is that there must be individual accountability. Moreover, by reviewing research on cooperative learning Biehler and Snowman (1997, p.421) concluded that “Students who learn cooperatively tend to be more highly motivated to learn because of increased self-esteem, the pro-academic attitudes of group mates, appropriate attribution for success and failure, and greater on-task behavior. They also score higher on tests of achievement and problem solving and tend to get along better with class mates with different racial, ethnic, and social class backgrounds”.

2.3 APOS Theory

APOS theory is a constructivist theory that focuses on individual's mental constructions of mathematical knowledge and mental mechanisms that yield these constructions within social context. It builds on Piaget's notion of reflective abstraction. Piaget (Beth & Piaget, 1966) differentiated empirical abstraction in which the focus is on the general characteristics of objects and reflective abstraction in which focus is on the actions or operations done by subject on (mental) objects. Moreover, he paid attention to constructivist aspect of the notion:

> *It is then necessary to suppose that abstraction starting from actions and operations- which we shall call "reflective abstraction"- differs from abstraction from perceived objects- which we shall call "empirical abstraction" (assuming the hypothesis that non-perceptible objects are the product of operations) - in the sense that reflective abstraction is necessarily constructive. In fact, as opposed to empirical abstraction, which consists merely of deriving the common characteristics from a class of objects (by combination of abstraction and simple generalization), reflective abstraction consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflection (in the quasi-physical sense of the term) upon actions or operations of a higher level it guarantees for it is only possible to be conscious of the process of an earlier construction through a reconstruction on a new plane… reflective abstraction proceeds by reconstructions which transcend, whilst integrating, previous constructions* (p. 188-189).

APOS theory extends Piaget's notion of reflective abstraction to undergraduate mathematics education. Asiala et al. (1996, p.32) describe mathematical knowledge in the following way.

> *An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects, by organizing these in schemas to use in dealing with the situations.*

Action, **P**rocess, **O**bject, **S**chema are the mental structures that an individual builds by the mental mechanism of reflective abstraction. Dubinsky (1991) and Asiala et al. (1996) determined six kinds of reflective abstractions: interiorization, coordination, reversal, encapsulation, thematization, and generalization in undergraduate mathematics education.

According to the theory, when an individual encounters a new mathematical concept, concept formation begins with transformations of existing mental (or physical) objects. This transformation is called action. At the action level, transformations are perceived as external. Individuals need external cues to carry out the transformation (Dubinsky, 1991). For example, students who are limited to action conception of function require a formula (as an external cue) and use it to plug numbers in to evaluate.

By repeating an action and reflecting on it, an individual might interiorize it as a process (Asiala et al., 1996). Internalization allows individual to perform transformations in the absence of external cues. Process is being concerned as internal, moreover, she/he can imagine steps of the process without actually performing each step explicitly. For example, students who have process conception of function do not need any external cues, such as formulas, to conceive it as a function. Rather, it is conceived in the way that objects in the domain set are transformed to corresponding objects in range set. Evaluations are not limited to taking a particular value in the domain and finding corresponding value in the range. Actually, they can imagine that objects in the domain set are transformed into objects in the range set without actually doing evaluations step by step for each element. Interiorization is not the only way to construct processes. An individual can construct processes by reversing it or by coordinating it with other previously constructed processes.

If an individual pays attention to and becomes aware of the fact that actions or operations can act on the process being concerned, she/he might encapsulate process into an object (Dubinsky, Weller, Stringer, Vidakovic, 2008). In order to encapsulate a process, process should be seen as a completed totality on which actions or processes can act. Process is a transformation one does, whereas object is a static entity one transforms. For example, students who have the object conception of function can apply derivative on a given function. It is often necessary to de-encapsulate object back to its process in mathematical situations.

Actions, processes, and objects are organized into mental structures which are called schemas (Asiala et al., 1996). A schema can be thematized into an object that might be included in other schemas. Once constructed, schemas can be applied to other schemas to give meaning to mathematical situations. For example, quantification schema can be applied in the limit case to get understanding of formal definition of limit. When an individual sees that available schema can be applied to wider situations, then the schema is generalized.

In describing students' constructions and mechanisms which are used to produce those constructions, APOS researchers devise a tool called genetic decomposition (Asiala et al., 1996). Genetic decomposition is firstly constructed theoretically. This includes the researchers' understanding of the concept, conclusions drawn from the literature, and historical development of the concept. This preliminary version is needed to be empirically tested. This could be done in two ways. First is done by questioning students qualitatively who already attended a course. Second is done by designing an instruction based on preliminary version of genetic decomposition, and then by questioning those students qualitatively. Second type allows us to empirically test the results of the designed instruction. In both ways, qualitative data is gathered and analyzed based on APOS theory. If the preliminary version of genetic decomposition does not describe students thinking appropriately, then it must be modified. This can result in a new cycle that begins with modified version to be empirically tested with new set of data. Cycles are repeated until reasonable understanding is deduced. If data is in now way compatible with genetic decomposition, then theory should be questioned.

2.4 Limit Literature

In this section literature about learning limit concept and its instructional suggestions will be addressed.

2.4.1 Concept Image

Schwarzenberger and Tall (1978) studied the conflicts in the learning of limits. It was proposed that, in order to convey formal mathematical ideas to students, they are translated into suitable forms. Together with students' prior experience, this translation might cause some conflicts. And these conflicts might lead to greater conflicts, or even totally blocking further learning. Schwarzenberger and Tall (1978) considered, as an example, informal translation of limit as a ground for some conflicts. Word "close" in the translation might infer the meaning near but not coincident with, which can cause one to think that s_n can be close but not equal to s. And "as close … as" might carry the hidden meaning we can get infinitely close. So, when designing curricula, particular attention should be paid to such conflicts.

To explain these conflicts Tall and Vinner (1981) differentiated formal mathematical concept and the way the concept developed in an individual's mind by defining concept image as "… total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes".

At a particular time, a portion of concept image might be activated. Tall and Vinner (1981) called it as evoked concept image. Concept image might include conflicting portions which might be activated at different times. In order to conflict occur, it is needed that conflicting portions are evoked simultaneously in the individuals' mind. Not only portions of concept image can be in conflict each other but also conflict might happen between concept image and its formal definition.

Tall and Vinner (1981), moreover, defined potentiality and actuality of conflict as follows "We shall call a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition, a potential conflict factor. Such factors need never be evoked in circumstances which cause actual cognitive conflict but if they are so evoked the factors concerned will then be called cognitive conflict factors".

Potential conflict factor, which is between concept image and formal concept definition, is very serious. If an individual cannot develop concept image of formal theory and then cannot simultaneously evoke conflicting parts, then there is no way that potential conflict factor becomes actual cognitive conflict. Tall and Vinner (1981) stated that "Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous".

Dynamic Notion of Limit, Infinitesimals and Potential Infinity

Some of the researchers have focused on to figure out potential conflict factors created in learning limit concept. They determined several such factors, but, commonly cited conflict factor is that function does not attain its limit. "A common misinterpretation of the limit definition among these students is that it says that limits are not attainable" (Juter, 2003).

In explaining cause of this potential conflict factor, some researchers (Williams, 1991; Tall & Vinner, 1981) put a dichotomy between dynamic and static notions of limit. For them, "as x goes to a $f(x)$ goes to L" is a process which includes dynamic feeling of motion. Nevertheless, in formal conception, an individual deals with intervals in which x and $f(x)$ values do not move. So, it is dynamical element in informal limit notion that prevents students to see that function can attain its limit value.

Williams (1991) examined students' concept images of the notion of limit. A brief questionnaire applied to 341 second-semester university calculus students and it was detected that students have various conceptions of limit including dynamical notion of limit, static formal like notion of limit, limit as unreachable, limit as a boundary, and limit as an approximation. He then selected 10 students to be interviewed. According to results, he concluded that "…Limits tend to be seen as processes performed on functions, an idealized form of evaluating the function at a series of points successively closer to a given value. The dynamic element here is clear, and because the actual value of the function at the point of interest is irrelevant, the limit is never reached".

Similarly, Tall and Vinner (1981) argued that concept image of dynamic limit definition, which includes definite feeling of motion, is so strong in students. So, their concept image includes the potential conflict factor that function never reaches its limit value.

Tall (1980a) contended that dynamic conception is easy to grasp and natural to develop for students. He came to this conclusion by focusing on the duality of process and object conceptions of limit, followed by a suggestion that students have primary intuition of limit as a process rather than limit as a static object. "… before limiting process had been discussed, the concept image intuitively alights on the infinite nature of the process rather than finite numerical limits". Even, after students are exposed to formal limit instruction, they continue to hold dynamic view of limit.

Williams (1991) echoed same ideas and he, further, concluded that dynamical notion of limit prevents development of formal notion of limit. "It is easily understood, and it is likely the most coherent and well-organized model of limit available to students… such a view of limit does present a cognitive obstacle to further understanding."

Still, another serious set of students' difficulties in understanding limit concept determined by other researchers (Tall, 1980b; Mamona-Downs, 1990; Tall, 1980a). These are infinitesimal quantities and potential infinity.

Tall (1980b) contended that notion of the limit of a function is often interpreted in dynamical sense, never ending process of getting close to limit value rather than the limit value itself. This dynamism often leads students to construct infinitesimal quantities.

Mamona-Downs (1990) came to same conclusion. "Expressions like "as x gets closer and closer to a, $f(x)$ approaches l" do convey a feeling of motion and flow. The question "how close do you mean?" disturbs pupils who give either tautologous answers of the kind "as close as you can" or again infinitesimal arguments such as "x differs from a by an incredibly small amount"".

Corresponding to infinitesimal quantities, it was argued that potential infinity is another conflict factor. Tall (1992b) suggested that students' predominant notion of infinity is potential infinity, which implies process of getting closer to limit value goes n for ever, without being completed. Thus students' idea that 0.999… is less than 1 might be because of their intuitions of potential infinity.

Mamona-Downs (1990) concluded, similarly, about potential infinity in limiting process. "$0.\dot{9}$ must be constructed by an unending process of adding a 9 to what you already have, starting with 0.9. This process is ruled out by time; every step has at least interval of time for it to be performed …The statement is false as although in the limit it may be said that this is true 0.999… would never actually reach 1 but would always be a very small amount less than 1".

In order for proper understanding of limit notion, students need to move from notion of potential infinity to notion of actual infinity. Introduction of sets promote the secondary intuitions of actual infinity. And development of notion of infinity and limit goes hand in hand. Nevertheless, intuition of actual infinity of sets may not be applicable to limit situation. “Many university students go through a stage where they accept the actual infinity of a set but only potential infinity of a process” (Tall, 1980a).

How to Help Students to Overcome Their Difficulties

In order to help students to overcome mentioned difficulties, some researchers suggested alternative ways of teaching. Li and Tall (1993) proposed “function/numeric computer paradigm”. Students made some programs in BASIC programming language to experience that what happens to values of $s(n)$ as n becomes larger and larger. Underlying idea was that for large values of n students computed $s(n)$ and looked for the values of $s(n)$ to see whether stabilization occur for a given accuracy. As the accuracy increases the greater value of n is needed for stabilization. They expected that this idea would lead to the ε-N definition of limit. Nevertheless, the expected transition from informal process view of limit of a sequence to informal object view of sequence did not happen.

Monaghan, Sun, and Tall (1994) proposed that “function/numeric computer paradigm” emphasizes process of computing values of $s(n)$ for larger and larger n so that students hold the belief that limit is a process not an object. They posited another paradigm: “the key stroke computer algebra paradigm”. By using Derive computer algebra system students can evaluate limits with simple key strokes. The underlying idea is that computer handles necessary process internally to evaluate limits so that learner is free to experience properties of object produced by computer before, at the same time, or after studying process itself. To explore possible effects of this paradigm, they constructed two groups. The study revealed that students in both groups were beginning to see the limit sum as an object. But, simultaneously, students in both groups had the well known misconception that limit is never ending process.

Williams (1991) took another route to help students. He selected 10 students to be exposed to carefully designed instruction based on conceptual change approach in which students faced with situations creating cognitive conflict. Results indicated that students had idiosyncratic variations on dynamical notion of limit and notion of limit as unreachable. Nevertheless, students failed to adopt more formal model of limit.

Oppositions to Results of Dynamical Notion of Limit

Two different oppositions came to the idea that dynamical conception of limit is natural for students and hinders their development toward more formal understanding of limit concept from a researcher and group of others (Oehrtman, 2003; Cottrill et al., 1996).

Oehrtman (2003) investigated students' metaphors for limits. Results revealed that students hold five strong metaphors: collapse, approximation, closeness, infinity as a number, and physical limitation. One of the strong metaphors was exemplified: collapse metaphor. For collapse metaphor, students imagined a physical referent for the changing dependent quantity, and this quantity loosed one or more dimensions by collapsing along one of its dimensions in a limiting situation. It is noticeable that strong metaphors did not include motion metaphor, rather it was included in weak metaphors which were the followings: motion, zooming, and arbitrary smallness. And he concluded that although weak metaphors, such as motion, run in the background, strong metaphors, such as collapse, organized ideas and were touchstones for reasoning. Thus, dynamical notion of limit is not as natural as suggested and cannot hinder development of more formal conceptions of limit.

Second opposition came from Cottrill et al. (1996). They considered dynamical notion as a mental process in APOS terms. Actually this is not a single process, rather is coordination of domain and range processes via function in consideration, thus a schema. Contrary to the belief that process conception is easy to understand, they suggested that coordinated process schema is not easily constructed by students. Moreover, they argued that informal process schema of limit concept is necessary in building formal understanding of limit. Formal understanding of limit concept is built on coordinated process of informal limit, rather than hindered by it. Difficulty in moving from informal understanding to formal understanding comes from students' weak understanding of quantification.

Spontaneous Conceptions

When students come to the classroom to learn limit concept, their learning does not begin on blank slate. Rather, students come to the class with some understanding about limit, which they have built from daily life experiences. Cornu (1991) defines conceptions that occur prior to formal education as spontaneous conceptions.

Another source of students' conflicts can be their spontaneous conceptions. Before introduced formal limit instruction, students have some kind of understanding of the language used by the instructor to communicate calculus concepts. Educators use phrases "approaches", "tends to", "converges", and "limit" with a special meaning, whereas, these phrases have different meanings in students' daily language which is created by their experience. Monaghan (1991) explored the effects of language on high school students' understanding of limit concept. Students were given six curves in graphs and required to decide whether each curve tends to 0, has 0 as limit, converges to 0, approaches to 0. Although these four words mathematically equivalent, students often agreed with one of them but disagreed with the other, considering the same curve. Moreover, interviews conducted with the students revealed following results.

Most of the students saw the "limit" as a boundary which cannot be passed (sometimes as a rule, sometimes impossible). In mathematical situations limit is seen as an unreachable boundary point and as a final point: $0.\dot{9}$ is the final point of the sequence 0.9, 0.99, 0.999, 0.9999, …, but limit is also 1 to which sequence never reaches.

"Approaches" was seen in a more dynamic way than limit in students' daily life languages. Majority of the students held the idea that "approaches" involved a movement of one thing toward other things (constant or moving), sometimes with the contention that the thing that is approached will eventually be reached, and sometimes with the contention that it is never reached.

Most of the students used “tends to” in the meaning of personal inclination or of general trend. In mathematical situations both “tends to” and “approaches” were seen as same, representing movement of an object that never reaches point being approached.

For majority of the students “approaches” meant that two continuous objects come together and touch each other. However, in mathematical context, students were often unsure about what “converges” mean.

Based on these results, Monaghan concluded that daily life meanings of these words might become potential cognitive conflict factors when students are exposed to limit instruction, and moreover, such meanings might be continued to be hold after formal instruction.

Similarly, Kim, Sfard and Ferrini-Mundy (2005) studied students’ colloquial and literate use of the word “limit”. Their sample consisted of two elementary school students, two middle school students, two high school students, and two university students. They found that colloquial use of the “limit” (as an upper bound and being unreachable) is in conflict with mathematical definition of limit. Unlike Monaghan (1991), they found that well defined misconception “being unreachable”, was used only by university and high school students who are exposed to formal limit instruction. This leads to conjecture that the misconception “being unreachable” is not because of daily use of the word, but rather related to education. However, we must take their small sample size into consideration, we should be precautious to make generalizations.

2.4.2 Obstacles to Learning

Learning is not a smooth process without breaks. In transition from one mental state to another, unstable behavior is possible. Some of the researchers have focused on students' difficulties in these transitions. Bachelard (Bachelard, 1938, as cited in Cornu, 1991) considered epistemological obstacles in the acquisition of scientific knowledge.

"We must pose the problem of scientific knowledge in terms of obstacles. It is not just a question of considering external obstacles, like the complexity and transience of scientific phenomena, nor the lament the feebleness of the human senses and spirit. It is in the act of gaining knowledge itself, to know, intimately, what appears, as an inevitable result of functional necessity, to retard the speed of learning and cause cognitive difficulties. It is here that we may find the causes of stagnation and even regression, that we may perceive the reasons for inertia, which we call epistemological obstacles."

Sierpinska (1987), Cornu (1991) and Brousseau (1997) applied such obstacles to mathematical learning. Brousseau (1997) defined epistemological obstacle as a piece of knowledge that functions well within a limited context, thus becomes well established in the students' mind, nevertheless, it is not generalizable beyond its specific context. So, this piece of knowledge cause conflicts in another context in which students face new mathematical objects and processes.

Cornu (1991) described three forms of obstacles depending on their origin: genetic and psychological obstacles, didactical obstacles, and epistemological obstacles. Genetic and psychological obstacles are related to mental development of child. Didactical obstacles emerge as a result of instructional sequence. So, there is a possibility that didactical obstacles can be overcome by implementing carefully designed instruction. Nevertheless, origin of epistemological obstacles is nature of mathematical concept itself.

Tall (Tall, 1986 as cited in Tall, 1991) suggested similar explanation to this phenomena by introducing generic extension principle which is defined as "If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts" (p. 8).

Tall (1991) put generic extension principle into practice in the following way. When students first introduced to limit of a sequence, simple sequences given by a formula are used, such as 1/n. This sequence converges to zero, but its terms never equal to the limit. Moreover, colloquial use of the phrases like "gets close to" supports this belief. In the absence of any counter examples, students gradually form the implicit belief that terms of a convergent sequence gets closer and closer to limit without actually attaining it.

The idea of "whether limit is actually attained or not" is determined as an epistemological obstacle, by Cornu (1991), which occurred in the history of the development of the limit concept. Cornu (1991) determined three other epistemological obstacles in the history of development of the limit concept; the failure to link geometry with numbers, the notion of the infinitely large and infinitely small, the metaphysical aspect of the notion of limit.

It is interesting for mathematics education research to detect the epistemological obstacles that occurred in the history also occur in today's classrooms. Sierpinska (1985) went in that way and identified five main epistemological obstacles and rearranged this list (1987) as follows. "Obstacles related to four notions seem to be the main sources of epistemological obstacles concerning limits: scientific knowledge, infinity, function, real number" (p. 371).

Recent research focused on investigating obstacles of students in limit concept. Moru (2009) found that some of the students directly input a into function f to find limit of f at the point a. Using only one value to determine limit was also seen in the context of the limit of sequences, but with a different reasoning. Some of the students substituted one big value for n in the formula of sequence to determine limit. Although the result achieved by these students was the same, the ways that students used to achieve this result were different. For example, in finding $\lim_{n\to\infty}\frac{(-1)^n}{n}$ students substituted a big value for n, and the result obtained was number close to 0. Some of the students take 0 as the limit value because the word "approach" means "nearer to", whereas, others rounded off the result.

Similarly, Parameswaran (2007) detected that students tend to take very small numbers as zero and very large numbers as infinity. This tendency leaded students to face with cognitive obstacles under limiting situations. Introduction of ε-δ definition of limit is supposed to overcome such difficulties, but contrarily after introducing students to ε-δ definition, their difficulties persisted.

Hofe (1999) studied epistemological obstacles of the students who attended computer based learning environment. Two students used Math View CAS software to experience limiting behavior of the difference quotient and of secant lines at the same time. According to results of this study

(i) students held the notion of infinitely small quantities which famous mathematicians used in the development of infinitesimal calculus,
(ii) there is a problem in the interplay between students' process and object conceptions of informal limit, and hence,
(iii) the main problem is in the relationship between intuitive idea and its mathematical specification.
Some others concentrated on teachers' understanding of limit concept, and didactical transmission of epistemological obstacles. Huillet (2005) concluded that teachers have

weak concept images of limit concept; based on mostly algebraic tasks and having difficulty in transition between representations, and dependence on procedural knowledge. Cinvestav and Chavez (1999) found that teacher's dynamical idea of limit was passed to students via instruction and this primitive idea was a cognitive obstacle in the construction of the concept of limit.

How to Help Students to Overcome Their Obstacles

Once the obstacles to learning limit concept is determined, the natural question to pose is that how can educators help students to overcome these obstacles. Schwarzenberger and Tall (1978) claimed that these obstacles can be overcome by carefully designing instructional sequence. In helping students to understand limit concept, the important point is to be aware of the difficulties that students have had, and not to misguide them, which might cause aforementioned obstacles. So, instructional sequence can be designed in a way that the conflicts determined in the literature are avoided. "Mathematics is a difficult enough subject to understand without the additional hazards which are introduced by misguided attempts to provide the wrong sort of motivation or help; the helper conscious of the havoc caused by conflict between concept will try to adopt an approach which conflicts neither with the preconceptions of the pupil nor with neighboring mathematical material" (p. 49).

Davis and Vinner (1986) followed a similar path. In order to avoid cognitive obstacles arising from the colloquial use of "limit", they did not use the word "limit" in the initial stages of the instruction. Nevertheless, they came to the conclusion that there are unavoidable misconception stages that occur in developing understanding of limit concept, and "avoiding appeals to such pre-mathematical mental representation fragments may very well be futile".

Sierpinska (1987) had a different path in dealing with cognitive obstacles. He studied with 17 and 16 year old humanities students. The first attempt was to determine epistemological obstacles that students have about limit concept. It was found that the students' beliefs about mathematical knowledge, in particular, infinity are very serious obstacles. In order to help students to overcome their obstacles, he designed a didactical situation emphasizing cognitive conflict. Results revealed that "It is rather clear from the foregoing that none of the epistemological obstacles has been completely overcome. But mental conflicts were born and this may be starting point".

Szydlik (2000), further, explored the relationship between second-semester calculus students' beliefs about mathematical knowledge and their conceptual knowledge. Students who have external sources of conviction depend on authority to determine mathematical truth, whereas students who have internal sources of conviction consider their intuition, and look for logic and empirical evidence. Results showed that students, seeing calculus as collection of facts and procedures and devalued the underlying theory, held contradictory or primitive limit models with misconceptions "limit as bound", "limit is unreachable". Contrarily, students with internal sources of conviction were able to construct conflict free concept images and more likely held static conception of limit which author believes as an important step to formal limit ideas. So, for students with internal sources of conviction it is appropriate to provide formal structure, but, for students with external sources of conviction this formal structure can cause cognitive obstacles. Thus, instructor should be aware of the fact that they need to make sense of mathematics and are not ready for proofs.

2.4.3 Other Difficulties to Learning Limit Concept

In addition to above studies, other researchers (e.g., Tall, 1992a; Przenioslo, 2004; Elia, Gagatsis, Panaoura, Zachariades & Zoulinaki, 2009) studied high school students' and university students' difficulties in learning limit concept.

Tall (1992a) considered that notion of limit does not rely on simple arithmetic and algebra which causes difficulties for students. "The calculus represents the first time in which the students are confronted with the limit concept, involving calculations that are no longer performed by simple arithmetic and algebra, and infinite processes that can only be carried out by indirect arguments" (p. 13).

Przenioslo (2004) considered university students' (some of them studying limit concept and the others saw the concept in their previous courses) key elements of concept image of limit of functions. He defined key element of the image as "…the element, which a student appeared to perceive as most significant for solving problems…" (p. 105).He determined six key elements with given percentages: neighborhoods (10%), graph approaching (34%), values approaching (16%), being defined at x_0 (18%), limit of f at x_0 is $f(x_0)$ (9%), use of algorithms (13%).

Students who focused on the "neighborhoods" applied formal definition (not always correctly) to solve given problem, and were the most efficient problem solvers. Students who had the key elements "graph approaching" and "values approaching" used dynamical language in limiting situations (mostly involving misconceptions). Students in both groups were less efficient in problem solving than the students in "neighborhoods" group. Moreover, some of the students saw that to limit exist function must be defined at the given point. Some of the students went further and held the idea that limit is $f(x_0)$ at x_0. The last group applied algorithms in problem situations in a schematic fashion to get a result, but their schemes were quite frequently incorrect.

Then, a questionnaire designed by Elia et al. (2009) administered to 222 12-th grade high school students who had mathematics as a major subject and had already been taught elementary calculus. Instruction did not include the formal definition of limit. Results revealed that only 20.7% of the students gave the correct definition of limit, and majority of the students held some misconceptions: including limit is a number that cannot be reached, limit of a function at a point is the evaluation of the function at this point, function must be defined at the point where limit is looked for, limit of the function does not exists only under the condition that left and right limits are not equal, and 1.999… is not equal to 2. Moreover Elia et al. considered students' translation from algebraic to geometric representations and vice versa. Authors concluded that students who had the conceptual understanding of limit were more likely to make such transitions between representation types.

Another study (Cetin, 2009) focused on procedural and conceptual knowledge to explain student difficulties. She asked 63 university students, who completed their first year mathematics study involving limit concept, to find limit of four functions at a point *a* and approximate values of those functions at a point which is near to *a*. For example, students were asked:

$$f(x)=\frac{\sin x}{x},\ f(0.015)\cong ?\ \text{(x variable is in radian)}$$
$$f(x)=\frac{\sin x}{x},\ \lim_{x\to 0} f(x)=?$$

Majority of the students find the limit of given functions, but did not use this data to find approximate values of the given functions, rather they tended to evaluate functions for the given point. Cetin concluded that most of the students used some procedures to find limit of a function for a given point but they did not consider the idea underlying these procedures.

An investigation done by Williams (1991) corroborated this finding, and searched for the reason in didactic contract. "… students often considered ease and practicality of a model more important than mathematical formality. This is particularly true in the sense that models of limit that allow them to deal with the realities of limits in the classroom, the kind they see on tests, tend to be seen as sufficient for the purposes of most students. It was noted by several students that neither formal nor dynamic models of limit figure heavily in the procedures students use to work problems from their calculus classes; their procedural knowledge (e.g., substituting values into continuous functions, factoring and canceling, using conjugates, employing L'Hopital's rule) is largely separate from their conceptual knowledge." (p. 233).

Przenioslo (2004) echoed similar results "… the particular elements of an image were often not conceptually connected in students, who were then unaware of the contradictions among them. Students could correctly state an 'official' definition but would not notice a contradiction between this definition and his or her other, more 'private' conceptions, and, worse, would not try to confront the two parts of his or her knowledge. More importantly still, for the majority of the students the definition was not the most significant element of the image, the criterion of 'significance' being its usefulness in solving problems." (p. 129).

Tall and Vinner (1981) asked first year university students to write down definition of $\lim_{x \to a} f(x) = l$, if they knew one. Results showed that most of the students who give the dynamical definition were able to state it correctly whereas students who recalled formal definition misstated it.

Table 2.1: Dynamic- Formal Responses

	Correct	Incorrect
Formal	4	14
Dynamic	27	4

Cornu (1991) and Cottrill et al. (1996) ascribed students' difficulty in understanding formal limit concept to their weak understanding of the concept quantifiers.

Bloch (2000), also, took the attention on quantifiers. "Many statements of theorem in mathematics have quantifiers in them, sometimes multiple quantifiers. The importance of the quantifiers in the rigorous proofs cannot be overestimated. From the author's experience teaching undergraduate mathematics courses, confusion arising out of either the misunderstanding of quantifiers in complicated definitions and theorems, or the ignoring quantifiers when writing proofs, is the single largest cause of the problems for students who are learning to construct proofs" (p. 42).

Lastly, Fernandez (2004) detected that notation used in the formal definition causes problems in learning formal definition of limit.

2.4.4 APOS Approach

Based on their Action, Process, Object, Schema framework APOS researchers Cottrill et al. (1996) proposed their first theoretical genetic decomposition.

1. The action of evaluating the function *f* at a few points, each successive point closer to *a* than was the previous point.

2. Interiorization of the action of Step 1 to a single process in which *f(x)* approaches *L* as *x* approaches *a*.

3. Encapsulate the process of 2 so that, for example, in talking about combination properties of limits, the limit process becomes an object to which actions (e.g., determine if a certain property holds) can be applied.

4. Reconstruct the process of 2 in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

5. Apply quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.

6. A completed ε-δ conception applied to specific situations.

Then, depending on interview data they revised their primary genetic decomposition. One step was added to the beginning and Step 2 is revised in the way that it is not a single process, rather, it includes coordination of two processes, that is, it is schema. Then they concluded that since it is not a single process, students' difficulties are due to this coordinated process schema whose construction is not easy for students. Revised genetic decomposition is as follows.

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to a.

2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.

3. Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
   (b) Construction of a range process in which y approaches L.
   (c) Coordination of (a), (b) via f.

4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way schema 3 is encapsulated to become an object.

5. Reconstruct the process of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

6. Apply quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.

7. A completed ε-δ conception applied to specific situations.

Actually, data from student interviews did not allow authors to validate the Steps 5 (a few students constructed), 6, and 7 (no student constructed). But they chose not to drop them. "We repeat them in the revised version, although they might be dropped for the present since there is no evidence for them".

As mentioned above, in literature there was a common distinction between dynamic informal (process) limit conception and static (formal) limit conception. Cottrill et al. (1996) opposed this belief by using theoretical Step 6 in secondary genetic decomposition. According to this step, students apply quantification schema. But, application of quantification schema, at this level, is not a static entity, rather it is process. Moreover, it is not the dynamic nature of limit conception that hinders students' transition to more formal ideas. Cottrill et al. conjectured that it is students' weak quantification schema that hinders development of formal limit conception.

Without using APOS framework, Roh (2007) conducted a study about transition from intuitive understanding of limit to more formal understanding of limit in the case of sequences. In his study he developed a tool called ε-strip. He used this tool to assess students understanding of the limit concept, as well as help students to develop concept of limit. Students' use of ε-strip prevents us to see how students relate ε, N, n and a_n, since for a fixed ε value, it automatically produces an N value. Nevertheless, students are free to vary ε value. In this sense, students' use of this tool can give some indications how they understand phrase "any ε". Moreover, in informal models, students first think of index and corresponding term for this index. Then, they consider the difference between each term and the limit value as index increases to infinity. But, in formal definition students are required to, first, consider the error bound, then, proper index for the given error bound. Roh calls this transition as reverse thinking process, and he adds understanding the relation between ε and N to this process to define reversibility. Then to classify students' level of reversibility he made 1-hour interviews with 12 students for 5 weeks. Analysis of the results indicated five level of reversibility as given below.

"Reversibility level 4 includes the case where students conceptualize the following three ideas: (1) the dependency of N on ε, (2) the arbitrary choice of ε, and (3) the dynamic feature of ε to decrease to 0."

"Reversibility level 3, on the other hand, includes the case where students conceptualize the first two ideas, "the dependency on N on ε" and the "arbitrary choice of ε", but not the third one, "such chosen values of ε can be rearranged to decrease to 0"."

"Reversibility level 2 describe the case where students conceptualize that "N can be dependent on ε", but improperly perceive the second notion "the arbitrary choice of ε". To be precise, students in this level can perceive only some positive values for ε."

"At reversibility level 1, students tend to complete the ε- process preferentially so as to fix the value of ε at 0 or infinity. These students assume that any positive value of ε can be ultimately substituted to 0 or infinity, and as a consequence, limit values of a sequence are found by replacing 0 or infinity for ε."

"Students at level 0 of reversibility tend to select a value of N first and then determine the value of ε." (p. 107).

Although limit in sequences and limit of functions are two different subjects, they have commonalities: relationship between ε and N, relationship between ε and δ. Findings of Roh and conjectures of APOS researchers (Asiala et al., 1996) are compatible. In Roh's study, firstly, students could not reverse the relationship between ε and N, which can be taken as corresponding to students' difficulties in transition from 3(c) to 5. Moreover, students, firstly, considered only one fixed ε. Then, some ε values come into consideration. And lastly, some of them come to the conclusion that every ε value should be taken into account in the relationship between ε and N. These levels of understanding are compatible with what APOS theory conjectures.

An opposition to APOS theory in general and genetic decomposition of limit concept in specific came from Pinto and Tall (2001) and Pinto and Tall (2002) respectively. Pinto and Tall (2001) described two different learning styles: one is formal other is natural. "Formal thinkers attempt to base their work on the definitions… Natural thinkers reconstruct new knowledge from their concept image" (p. 57). They contended that formal thinkers are compatible with APOS theory, whereas, APOS theory does not explain the way of natural thinkers' learning. Formal learners' starting point is concept definition. They build their concept image from formal concept definition by focusing on rules and procedures and by routinising them reflectively. Then, they deductively construct formal theory. On the other hand, natural learners' starting point is their concept image. They try to assimilate formal theory into their concept image which results in cognitive conflicts. Then, they proceed by making thought experiments to reconstruct their concept images on which formalism is built. Finally, they build formal theory which is integrated with imagery.

In their later research (Pinto & Tall, 2002) they gave a specific example, Chris, for natural learning style. In this example, Chris started knowledge building with his concept image, that is, he interpreted formal definition in terms of his old knowledge. His concept image included both limit processes and limit objects. Then, Chris made some thought experiments to reconstruct his image. Finally, he built formalism on his modified images to get formal understanding which is integrated with his concept image.

After Cottrill et al. (1996) and Pinto and Tall (Pinto & Tall, 2001; Pinto & Tall, 2002) this subject has not been studied to generate hypothesis and solutions to mentioned discrepancies. However, some researchers used APOS framework as a starting point to further elaborate students' difficulties. For example, Swinyard and Lockwood (2007), by using APOS, contended that in order to understand formal limit concept, requires students to understand purpose of definition, individual components of definition, role of quantifiers.

How to Help Students to Construct Limit Concept

Cottrill et al. (1996) designed an instruction based on first version of genetic decomposition to help students to construct limit concept. ISETL software was used to help students in this construction. In studying groups, they wrote some ISETL code, or modified given code to start to make constructions mentioned in the theoretical version of genetic decomposition of limit concept. This is followed by classroom discussions to reflect on what they do. Nevertheless, this approach was not proved successful.

2.4.5 Instructional Approaches to Facilitate Learning Limit Concept

Understanding limit concept is crucial for calculus students since it establishes a ground for development of the concepts of continuity, derivative, and integral. Although importance of limit understanding has been recognized, introduction of this concept, because of its complexity, causes serious difficulties. So, the question of "How can educators/teachers facilitate students in developing limit concept?" is inevitable.

To investigate whether formal definition should be given from the start or not Mamona-Downs (1990) conducted a comparison study between Greek and English students. "A comparison between the nationals is interesting in that the English have no formal instruction about limits on real line, contrary to Greek case. We find the English use "infinitesimals" which often confounds the completion of a limiting process, whereas Greeks sometimes display difficulties in using formal symbolism and reasoning, suggesting that little insight is given by the strict definition".

Parameswaran (2007) tested the idea that whether the introduction of ε-δ definition of limit helps students in overcoming their difficulties about informal limit concept or not. Negative results were drawn by the author: after introducing students to ε-δ definition, their difficulties persisted. Moreover, author concluded that instructor's emphasis on ε-δ definition as rigorous, mathematically accurate alternative for intuitive notion of limit caused them to distrust informal limit notion. Even for some questions which could be easily solved with informal limit approach, some of the students tended to use (sometimes incorrect) formal ε-δ arguments.

Williams (1991) took another route. He examined second semester calculus students' conceptions of limit. Results indicated that students had idiosyncratic variations on dynamical notion of limit and notion of limit as unreachable. He took these notions as a starting point for the learning of more formal limit notion. Then, he carefully designed an instruction based on conceptual change approach with the focus of creating cognitive conflict to help students in moving from informal dynamic models to more formal models of limit. Nevertheless, students failed to adopt more formal model of limit.

Relatively positive result came from the study of Roh (2006). He developed a tool, called ε-strip, to help students in transition from informal understanding of limit to formal understanding. Students who had not any experience with ε-N proof were chosen for the study. They used ε-strips to explore ε-N relation for monotone bounded, unbounded, constant, oscillating convergent, and oscillating divergent sequences. Finally, students were presented two ε-strip definitions, one of them is correct and the other one is incorrect. Although intention was not to facilitate students' learning of formal limit concept, after 5 weeks of interviews (one hour for each week) students showed better understanding of formal limit concept. Author concluded that ε-strip is a promising tool which might be used to help students in learning limit concept.

Then, Li and Tall (1993) proposed "function/numeric computer paradigm" to help students in transition from informal understanding of limit to formal understanding of limit in the context of sequences. Students learned, first, BASIC programming language. They use it to experience what happens to values of $s(n)$ as n becomes larger and larger. Then their attention was shifted from index, namely n, to the given accuracy within $s(n)$ values stabilized. As the accuracy increases the greater value of n is needed for stabilization. They expected that this idea would facilitate students' learning in transition from informal view of limit to formal view of limit. Nevertheless, the expected transition from informal process view of limit of a sequence to formal, ε-N, view of sequence did not happen.

Similarly, Oehrtman (2004) addressed the approximation metaphor to see how it affects students' learning of limit of a function and a sequence. Unlike Li and Tall (1993) who stated that designing instruction depending on approximation metaphor did not help students to move from informal process view to more formal view, Oehrtman found that approximation metaphor is a productive tool for the calculus students' understanding and application of limit concept. Moreover, he concluded that approximation metaphor can help students to move from informal view of limit to formal ε-δ and ε-N view of limit with guidance from instructor.

Monaghan, Sun, and Tall (1994) used the idea that introduction of computers into school environments give students opportunity to shift from emphasis on routine work to bigger picture. They criticized "function/numeric computer paradigm" in that it emphasizes process of computing values of $s(n)$ for larger and larger n so that students hold the belief that limit is a process not an object. They posited another paradigm "the key stroke computer algebra paradigm". The underlying idea is that computer handles necessary process internally to evaluate limits so that learner is free to experience properties of object produced by computer before or at the same time studying process itself. To explore possible effects of this paradigm, they constructed two groups: experimental (Derive) group in which students were exposed to the key stroke computer algebra paradigm and control group in which students had similar backgrounds but without computer algebra system. The study revealed that students in both groups were beginning to see the limit sum as an object. But, simultaneously, students in both groups had the well known misconception that limit is never ending process. Thus, in comparison to students who are exposed to "function/numeric computer paradigm" in the previous study of Li and Tall (1993), students in Derive group made more progress in transition from process view of informal limit to object view of informal limit. However, Derive group students showed similar performance with control group in this transition.

In another study (Lauten, Graham, and Ferrini-Mundy, 1994), students are exposed to graphics calculator-based environment. Researchers interviewed 7 students, but then decided to focus on interview of one student: Amy. Analysis showed that formal definition of limit seemed to have no meaning for her and considering informal definition she had the view that points moved along a curve and never quite reached the limit point, which is well known misconception. It is found that "Amy seemed comfortable with her view that points moved along a curve and never quite reached the limit point" and it is stated that "In fact, the formal definition seemed to have no meaning for her" (p. 234).

Buyukkoroglu et al. (2006) constructed two groups; control group was instructed by classical methods and experimental group was instructed by using computer support. Using MATLAB applet, researchers designed computer laboratory sessions for experimental group students. MATLAB applets were used mainly for the visualization of the informal limit idea. To assess students' performance in informal limit concept, researchers prepared a questionnaire including four open ended questions. Study revealed that there is no significant difference between two groups.

In the study about how students learn limit concept, Cottrill et al. (1996) commented that "We have not, however, found any reports of success in helping students to overcome these difficulties" (p. 171). In my literature review I either came to same conclusion (for most of the studies), or found some promising approaches in helping students to learn limit concept. Nevertheless to see their real potential, these approaches must be, further, systematically investigated. Contrary to all studies mentioned above, one systematic study that produced positive result is the following.

Fernandez (2004) decided to take students' viewpoints into consideration for lesson planning in teaching formal limit concept. Students' difficulties in understanding formal definition of limit were detected as "definition contained too much notation and that the need for this notation should be motivated" (p. 45). Depending on this result, Fernandez designed a 100-minute lesson. At the beginning of the lesson distinction between informal and formal approach was discussed. Then for a given specific epsilon, students are required to find appropriate delta. This continued until students began to see the pattern in the formal definition of limit and became tired of calculation. Then, by making discussions, students were introduced to formal definition of limit with interval notation rather than absolute value notation, since students were more familiar and comfortable with interval notation. After this lesson, students' understanding was evaluated with two questions: first one required students to show that limit of a given function is L, and for the same function and the same point, last one required students to find appropriate delta for a given specific epsilon. 34 of the 48 students successfully solved first question while 22 of the 48 students responded second question correctly. However, only 15 of the 48 students responded both of the questions correctly. Comparing the results with his previous teaching experiences, Fernandez concluded that this new approach helped students to improve their performance in problem solving.

As a result of using more familiar notation, of motivating students for the necessity of formal definition, and of discussing what the formal definition of limit means, students' showed relatively better performance in solving these two questions. However, we do not have enough evidence for conceptual understanding, since it is possible that students solved questions without having conceptual understanding of the formal limit concept.

CHAPTER 3

METHOD

In this chapter, the methodology utilized in the study is presented in detail. These include design of the study, pilot study, participants and context, instruments of the study, procedures of the study, analysis of the data, reliability and validity issues, assumptions of the study, and lastly limitations of the study.

3.1 Design of the Study

The purpose of the study is two folds: (1) to explore how students develop understanding of limit of a function, (2) to explore how students' understanding of limit of a function differ after the instruction. The research problems and sub-problems that guide this study are listed below:

1. How do students develop understanding in limit of a function?

    1.1. How do students explain their informal understanding of limit of a function?

    1.2. How do students explain their formal understanding of limit of a function?

    1.3. What kind of difficulties do students encounter in transition from informal understanding to formal understanding of limit of a function?

2. How different is students' understanding of limit of a function after the instruction based on APOS theory?

In order to address these research questions case study is utilized. Stake (1995, p. xi) defined case study as "the study of the particularity and complexity of a single case, coming to understand its activity within important circumstances".

Hitchcock and Hughes (1995, p.317) described the characteristics of case studies as follows.

- It is concerned with rich and vivid description of events relevant to the case.
- It provides a chronological narrative of events relevant to the case.
- It blends a description of events with the analysis of them.
- It focuses on individual actors or groups of actors, and seeks to understand their perception of events.
- It highlights specific events that are relevant to the case.
- The researcher is integrally involved in the case.
- An attempt is made to portray the richness of the case in writing up the report.

In case studies, generally, researchers gather rich data from multiple sources to address the complexity of the case. So, triangulation of the data is the issue in case studies to address complexity of the case and quality of the research. Bogdan and Biklen (1998, p.104) defined triangulation as "It came to mean that many sources of data were better in a study than a single source because multiple sources lead to a fuller understanding of the phenomena you were studying".

Gall, Gall, and Borg (2003, p.472) determined some advantages and disadvantages of case study as follows:

Advantages

- Case studies provides flexibility,
- Case studies provides thick descriptions,
- Thick description helps readers to compare cases with their own situations,
- Case study reports may have better basis for developing theories, designing educational interventions,
- Researchers' perspective is included in case study, thus it enables readers to determine whether researcher and reader shares same perspective or not.

Disadvantages

- Case studies are labor intensive,
- Ethical problems may arise,
- Little or no generalization can be made from the findings of the case study.

Simons (2009) opposed to the last item of the disadvantages of a case study, and stated that generalization is possible in case studies. He proposed five ways of generalization in case studies: cross-case generalization, naturalistic generalization, concept generalization, process generalization, situated generalization.

Yin (2003) provided another kind of generalization called analytic generalization. Analytic generalization differs from statistical generalization in that it does not generalize findings from sample to population. Rather, aim is generalizing from case study to theory. In analytic generalization (Yin, 2003, p. 32-33) "… previously developed theory is used as a template with which to compare the empirical results of the case study. If two or more cases are shown to support the same theory, replication may be claimed. The empirical results may be considered yet more potent if two or more cases support the same theory but do not support an equally plausible, *rival* theory".

In order to address the questions of this research two instruments were developed. First one is the limit questionnaire and the second one is interview protocol. This study included two phases. In the first phase pilot study was done. Depending on the pilot study results main study was conducted. In the main study, limit questionnaire was administered as a pretest. After pretest, students attended five weeks instruction. In each week they meet in two hours computer laboratory to study in groups, and then they attended four hours traditional classroom instruction. In computer labs they were given some programming activities which give students opportunity to think on limit concept before they are given formal lecture in classes. At the beginning of computer labs, small discussions about previou week's activities were done. After the classroom instruction, students were given relatively classical question sets about week's concept as homework. At the end of the instruction limit questionnaire was administered as a posttest. And then a semi-structured interview protocol developed by the researcher administered to all of the students.

3.2 Pilot Study

In this research two instruments were developed: limit questionnaire and interview protocol. During the summer school of 2008, the semester prior to the actual study, the researcher conducted a pilot study that lasted five weeks. This study was done at Middle East Technical University (METU) in which 37 students attended classes of Math153. Math153 is a calculus course for mathematics majors. Aims of the students to take this course were to increase the grade that take in regular classes, because of they failed in their regular class, and lastly to gain more information about calculus.

Nine students volunteered to attend the computer lab sessions in addition to regular classes. They attended two hour labs and four hour regular classes. In lab hours, researcher was the guide for students. And in classes, instructor was from Department of Mathematics. Students studied in three groups each including three members in lab sessions. They studied on lab sheets and were responsible to complete the tasks in the week's lab sheets and to give them to the researcher. In lab sessions, researcher observed students in order to assess the quality of the activities in lab sheets. Moreover, researcher examined the student responses to tasks in lab sheets to assess the quality of the activities. In this assessment, five criteria were used: (i) whether the instructions before activities are easily understandable by students or not, (ii) whether activities are suitable for students' level or not, (iii) whether given activities work or not, (iv) whether activities are motivating or not, (v) whether activities help students to construct limit concept or not. As a result of this assessment, some of the activities were modified and some new activities were added.

In the preparation process of lab sheets researcher used different resources. Activities in the lab sheets were either taken or modified from the suggested lab activities of the book "Calculus, Concepts, and Computers" (Dubinsky, Schwingendorf, & Mathews, 1995) or of Cottrill et al.'s limit study (Cottrill et al., 1996). In addition to these, some of the activities were prepared originally by the researcher.

After five weeks of instruction, limit questionnaire was administered to all 37 students. Depending on the results 7 students were chosen for the interview phase. Four of 7 students were from regular class, and three of 7 students attended lab hours and regular class. The criterion of selection of students for interview phase was their performance on limit questionnaire. Two of 7 students were low achievers, three of 7 students were average achievers, and two of 7 students were high achievers.

All 7 interviews were audio-taped, and their analysis was done by considering APOS framework (Asiala et al, 1996). Genetic decomposition of Cottrill et al. (1996) was taken as primary genetic decomposition which is as follows:

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to a.

2. The action of evaluating the function *f* at a few points, each successive point closer to *a* than was the previous point.
3. Construction of a coordinated schema as follows:
   (a) Interiorization of the action of Step 2 to construct a domain process in which *x* approaches *a*.
   (b) Construction of a range process in which *y* approaches *L*.
   (c) Coordination of (a), (b) via *f*.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way schema 3 is encapsulated to become an object.
5. Reconstruct the process of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed ε-δ conception applied to specific situations.

According to data analysis no students found above the fifth step of the genetic decomposition. One of the aims of the pilot study was to gain experience in interviewing and asking appropriate follow up question. Nevertheless, none of the interviewed students showed progress after fifth step of the genetic decomposition. Then, two research assistants in Mathematics Department in METU were interviewed to get more experience in interviewing. They showed understanding of the seventh step of the primary genetic decomposition.

The two purposes of the pilot study were to test questionnaire and interview protocol for their effectiveness in gaining information about students' limit conception. All of the interview questions were maintained. But one of the questions in the questionnaire is removed because it did not give necessary information about students' understanding of limit concept. No new question was added to questionnaire because it sufficiently provided information about students' understanding of the concept. The other aims of the pilot study were for researcher to gain experience in analyzing student responses in the interview and lastly, by using student responses on the limit concept, to help refine primary genetic decomposition.

3.3 Participants and the Context

In this section participant selection procedure, demographic information of the participants, detailed description of the instruction and the context that study took place are given.

3.3.1 Participants

In this study, sample selection was based on purposeful strategies, in particular criterion, and convenience sampling were used. Vaughn, Schumm, and Singagub (1996, p.58) defined purposeful sampling as "a procedure by which researchers select a subject or subjects based on predetermined criteria about the extent to which the selected subjects could contribute to the research study". In purposeful sampling, particular participants are chosen because they are believed to facilitate the expansion of developing theory (Bogdan & Biklen, 1998). Miles and Huberman (1994) identified 16 purposeful sampling strategies including extreme or deviant case sampling, intensity sampling, typical case sampling, maximum variation sampling, stratified purposeful sampling, homogeneous sampling, critical case sampling, snowball or chain sampling, criterion sampling, theory-based sampling, confirming and disconfirming case sampling, random purposeful sampling, convenience sampling, and opportunistic sampling.

In criterion sampling, selection of cases is done considering some pre-determined criterion, and all individuals meet this criterion. Cases chosen depending on predetermined criterion are assumed to yield rich information about the concept in consideration. Miles and Huberman (1994) contended that criterion sampling can help in assurance of the quality of the research. Moreover, convenience sampling is the strategy for selection of cases because participants are easily available for the researcher. This strategy saves time, money, and effort (Miles & Huberman, 1994).

The major aim of this study was to explore students' understanding of informal and formal limit concept. In Turkey, students are first introduced the limit concept at 12^{th} level in high school. Nevertheless, in high schools formal definition of limit is not covered. They start to study formal limit in universities. Since both informal and formal understanding of limit concept of the students is investigated, high school students were not appropriate subjects for this study. Then researcher considered university students as subjects of this study. In universities, limit concept is studied in calculus or introductory analysis courses in their first year of the study. But, there is no uniform approach for all departments. Mostly, non-mathematics and non-mathematics education majors are introduced the limit of functions quickly: within two or three weeks. The instruction prepared for this research study lasts in five weeks. So, subjects of this study are needed to be limited mathematics and mathematics education majors.

Researcher contacted with five departments in different universities, namely, Gazi University, Ankara University, METU, Hacettepe University, and Abant İzzet Baysal University. First four of these universities are in the city of Ankara and the last one is located in the city of Bolu. All five departments were volunteered to provide facilities for the researcher. Then researcher applied additional criteria for the selection of the research site. First of all, instructor of the course should volunteer to design the instruction for the limit concept with the researcher of this study. In each part of the instruction, researcher is involved in decision making process. Permission taken from the university does not guarantee that instructor of the course collaborate with the researcher of this study. Second criterion was the availability of the computer labs for the lab hours and availability of the computers in general for the use of the students. Activities in lab hours, in general, are long enough to be not completed in two lab hours. So, students need to study in front of the computer after dedicated lab hours. Thus, researcher used the following criteria for sample selection:

1. Introduction of both informal and formal limit concept,
2. Five weeks of instruction time,
3. Acceptance of researcher by the instructor of the course in instructional decisions,
4. Availability of computer facilities.

In addition to above criteria, researcher considered easy accessibility of the subjects for classes and for interviews. Thus, researcher chose the students who attend Math153 course at METU among other possibilities. The participants of this study included 25 volunteer students who attended Math153 course in fall semester of 2008. All students were freshmen mathematics majors who are taking their first formal course about limit of functions Table 3.1 presented demographic information of participants of the study.

Table 3.1: Demographic Information of the Participants of the Study

Sex	**Number**	**Percentage (%)**
Male	7	28
Female	18	72
Age		
18	5	20
19	15	60
20	3	12
21	2	8
Type of High School		
Anatolian H.S.	16	64
Science H.S.	6	24
Regular H.S.	3	12
Location of High School		
Ankara	16	64
İzmir	2	8
İstanbul	2	8
Kayseri	1	4
Samsun	1	4
Bursa	1	4
Denizli	1	4
Zonguldak	1	4

3.3.2 Context

METU is one of the biggest universities in Turkey located in Ankara which is the capital of Turkey. Mathematics department is one of the central units of METU that provides service courses for other departments and compulsory and elective courses for mathematics majors. In Turkey there is a central university entrance exam, called OSS. After high school graduation, students who want to have university degree enter OSS to pursue their education. Mathematics department in METU is one of the top departments in Turkey. According to 2007 OSS statistics, mathematics department students in METU were among the top 11.344 students who entered OSS. Likewise, in 2008 they were among the top 17.853 students.

In METU students have computer facilities in dormitories. They can access computers with internet connection in these computer labs 24 hours/ 7 weeks. Moreover, in Mathematics department, there is a computer lab including 40 computers with internet connection. This computer lab serves students 24 hours/ 7 weeks. The computes program that is used in this study is ISETL which is freely distributed in the internet. Once ISETL is copied into hard disk, it runs immediately without any installation procedures. In Mathematics Department, ISETL was installed before the study began, and students can install ISETL easily to computer labs in their dormitories and to their personal computers.

Math-153 is a compulsory course for the first year mathematics majors in their first semester. Its content mainly includes functions, limit, continuity, and derivative of functions of one variable. In addition to Math-153, students are given another compulsory course called Fundamentals of Mathematics (Math-111) in their first semester. Math-111 includes the contents of symbolic logic, proof techniques, set theory, Cartesian product, relations, functions, and induction. Before studying formal limit concept in Math-153, students have been introduced to Quantifiers in Math-111.

Before the beginning of the instruction, instructor of the course and teaching assistants including researcher of this study met to discuss content of the course, sequencing, and design of instruction. In addition, each week, previous to classes and lab hours, instructor of the course and research assistants met to discuss about how to instruct week’s content, and what questions to give as homework.

Students attended five weeks instruction. In each week they met in two hours computer laboratory to study in groups, and then they attended four hours classical classes. In computer labs they were given some programming activities which give students opportunity to think on limit concept before they are given formal lecture in classes. Researcher of this study participated to laboratory hours as a teaching assistant. In general two hours were not enough to complete lab activities. Students were required to complete lab activities after the laboratory hours till next laboratory meeting. At the beginning of computer labs, small discussions (approximately 30 minutes) about last week's activities were done. In classes teacher used chalk and board to instruct. After the class students were given relatively classical question sets about week's concept as homework.

In computer laboratory students were required to either write a code or modify given code. ISETL, stands for Interactive SET Language, was used as a programming language. In the following figure, an example of ISETL window is shown.

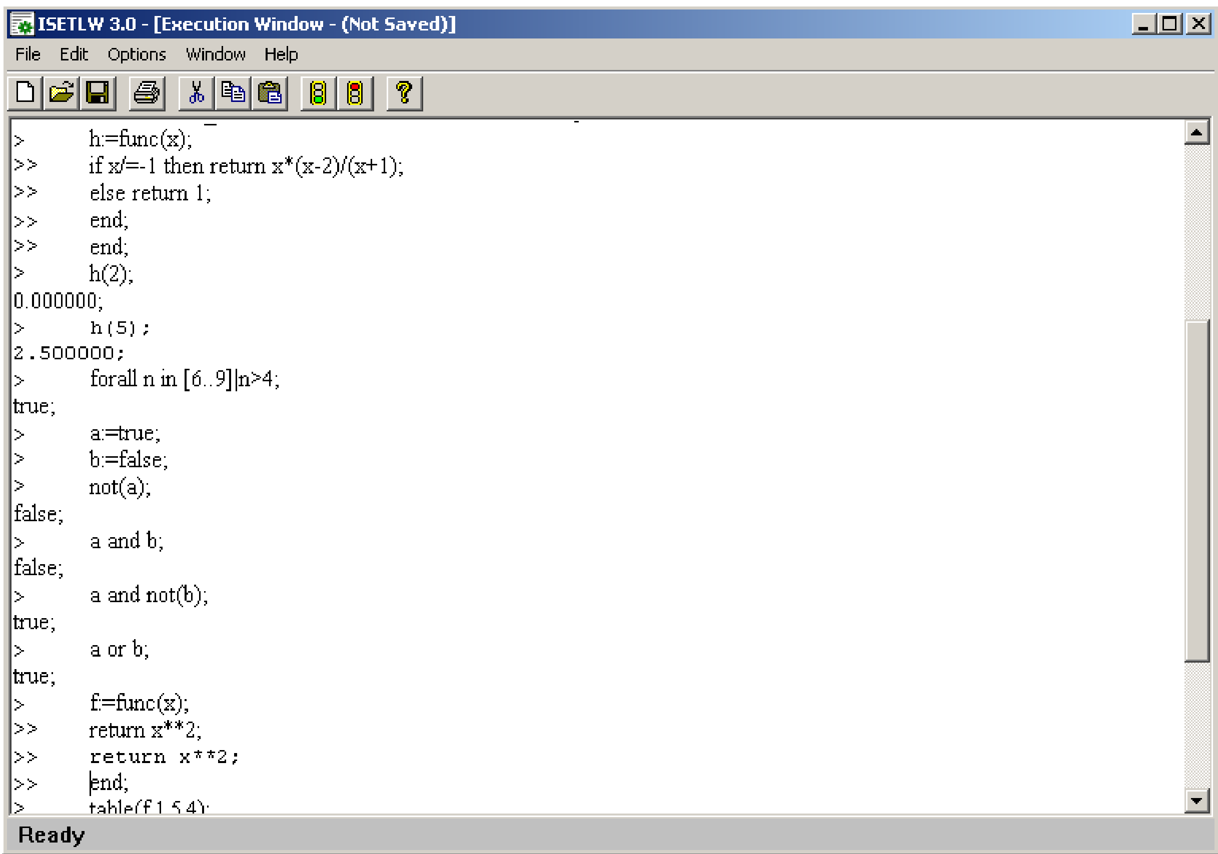

Figure 3.1: ISETL Window

The advantage of ISETL for mathematical programming is that its syntax is similar to mathematical language. Consider following examples.

- The set $A = \{2,3,5,7,11\}$ is expressed in ISETL as $A := \{2,3,5,7,11\};$.
- The set $B = \{x \mid x \in C \ and \ x \text{ is even}\}$ is expressed in ISETL as

  $B := \{x : x \text{ in } C \mid \text{even(x)}\};$
- The statement "For all x in A, x>0" is translated into ISETL language as "forall x in A| x>0;"
- The statement "For all x in A, there exists y in B such that x>y" is translated into ISETL language as "forall x in A| exists y in B| x>y"

In the first lab session, students were introduced to ISETL syntax and required to use table function of ISETL so that intuitive introduction of limit of a function is done. For example following question was one of the activities in the first session.

Q. Write an ISETL function *f* that accepts *x* and returns $\frac{\sin(x)}{x}$. Use command "table(f,-0.99,0.01,20);" to examine function values around *x*=0. And answer the following.

$f(x) = \frac{\sin(x)}{x}$, what happens with *f*(*x*) if *x* values is close to 0?

In the second lab session, students were given a code called LimitProcess. By using this function they studied on domain process, range process, coordination of domain and range process via function, and representation of these processes by inequalities.

In the third lab session, students were given the following code:

```
lim:=func(f,a);
s:=[a+((-1)**n)/(10**n): n in [1..6]];
for i in [2..6] do
if abs(f(s(i))-f(s(i-1)))<0.0001 then return f(s(i));
end;
end;
return "Unable to find limit";
end;
```

This code finds approximate value of limit. By using “lim” function, students found approximate limit value of several functions. Then they were required to write a code that finds approximate value of addition or multiplication of two functions, namely “limadd” and “limprod”. In order to do that, students needed to input “lim” function to the function that they are required to write. Lastly, students studied on “lim” function to produce a function called “limatinf” that finds approximate value of limit at infinity. In order to do that, they needed to modify “lim” function.

In the fourth lab session, students studied on *ε-δ* window activities. They were given vertical dimensions of a graph on computer screen, and asked to find domain scale for the graph so that graph resides in the computer window, or does not leave computer window. They studied on several different functions to find domain scale.

In the last lab session, students started to study on truth table of implication and its negation. Then, for fixed ε and δ they studied on formal definition. For example, they were asked to translate the following expression into ISETL language and to test its truth by using ISETL.

$$0<|x-1|<0.1 \text{ implies } |2x-2|<0.2$$

After this, for a fixed ε, they were allowed to vary δ to study on formal limit definition. For example, they were asked to translate the following statement into ISETL syntax and to test its truth by using ISETL.

There exist d in D={0.05,0.005,0.0005,0.00005} such that for all x in X={0.9,0.99,0.999,1.001,1.01,1.1}, 0<|x-1|<d implies |2x-2|<0.01.

And then they were allowed to vary both ε and δ to study on formal definition. For example, they were asked to translate the following statement into ISETL syntax and to test its truth by using ISETL.

For all e in E, there exist d in D, such that for all x in X, 0<|x-1|<d implies |2x-2|<e.

Moreover, at each step they considered the negation of given statement by expressing it in ISETL and testing its truth in ISETL. Lastly, they were required to study on negation of formal definition of limit by expressing it in ISETL and testing its truth in ISETL.

After computer laboratories, students met in classes. There were four 50 minutes sessions per week, and treatment was conducted over five weeks. In classes, they studied individually. The content of the class was parallel to content of the lab sessions. In general, 50 minutes divided into 15 minutes discussions and 35 minutes lecture. In 15 minutes discussions, teacher asked questions to be discussed as a class. Questions were either parallel to the lab activities or for about content of lecture. Moreover, instructor let students to ask questions to be discussed. In 35 minutes lecture, instructor was dominant. He used chalk and board to write definitions, prove theorems, and solve problems. Students listened to the instructor, asked questions about what is written on the board when necessary.

After classes, students were given, except first week, question sets as homework to be solved till next week. They studied on questions as a group and they gave the homework individually, that is, although they solved questions as a group, they wrote the solutions in their own words and homework was graded individually. Homework questions were prepared by researcher. Questions were prepared to reinforce the concepts being considered and challenge students' thinking. Moreover, question sets were another base for discussion, so reflections, for students.

Some of the questions were drill and practice problems. Following question is an example for such questions.

E1. Find the limit if exists. If not explain why.

a) $\lim_{x \to 0} \frac{|x|}{x}$

b) $\lim_{x \to 2} \frac{|x^2 + x - 6|}{x - 2}$

c) $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$

Some of the questions were more challenging. Following question is an example for such questions.

E2. Prove that

a) If $\lim_{x \to a} f(x) = L$ then $\lim_{x \to a} | f(x) |=| L |$,

b) Converse does not hold (by giving counter example).

On the other hand some of the questions required reflection on what is learned in laboratory and class sessions. Following question is an example for such questions.

E3. A friend of yours was given homework about "limit" concept. She/he knows only limits of simple functions. To start her/his homework she/he needs your help. What short explanations and examples would you use to explain to your friend what the "limit" is all about?

3.4 Instruments of the Study

3.4.1 Limit Questionnaire

A questionnaire is a research instrument that asks same questions to all participants of the study with the purpose of gathering information about participants' thoughts, values, feelings, attitudes, beliefs, and personality. Questionnaires can be in the form of structured, semi-structured, and unstructured. "Between a completely open questionnaire that is akin to an open invitation to 'write what one wants' and a completely closed, completely structured questionnaire, there is a powerful tool of semi-structured questionnaire" (Cohen, Manion, Morrison, 2007, p. 321). All of the items in the questionnaire were open ended items.

In this study, the limit questionnaire (see Appendix B) was prepared by the researcher. As a pilot, it was administered to 37 first year mathematics major who were taking introductory calculus course in METU in the summer school of 2008. The aim of the pilot was to

- check the clarity of the questionnaire items,
- eliminate ambiguities in wording,
- determine redundant and irrelevant items,
- consider the time taken to complete the questionnaire,
- check the difficulty level,
- check the effectiveness of questionnaire items in gaining information about students' limit conception.
- check commonly misunderstood items.

As a result of the pilot testing, one of the items in the questionnaire was found redundant and removed completely. Other questionnaire items were found useful and kept.

After pilot testing, peer review of the limit questionnaire was done by two experts whose subject areas were mathematics and mathematics education. Then to address the reliability of the questionnaire, evaluation of student responses was done both researcher of the study and research assistant of the course. Then results were compared and discussed. 100 % agreement was achieved.

Limit questionnaire was then applied as both pretest (before the treatment) and posttest (after the treatment) to 25 first year mathematics majors who are taking Math153 course in the main study in autumn semester of 2009.

Then, to address the reliability of the questionnaire, evaluation of student responses was done both researcher of the study and two research assistants of the course. Then results were compared and discussed. 100 % agreement was achieved.

Limit questionnaire was used to probe the difference between students' understanding of the concept of limit of a function before and after attending the instruction based on APOS theory. It was composed of seven open ended questions. First item was about students' understanding of the limit concept. They were asked to describe what it means to say that the limit of the function f at the point a is L. The aim of this item is to address students' concept images of limit of a function. Second item was about whether a function attains its limit value at the point limit is looked for and at other points. In literature (e.g., Williams, 1991), the mostly cited misconception is that "a limit is a number or point the function gets close to but never reaches". The aim of the second item is to check, in practice, whether students hold this misconception or not. In the third item, students were given graph of a function, and asked to determine limit value, if exists, for continuous, removable discontinuous, and jump discontinuous points. The aim of this item was to address how students apply their limit knowledge in a graphic situation where formula of the function is not available. Fourth item was again about finding limit of a given function for a given point. But this time, algebraic formula of the function was given, rather than its graph. Given functions were oscillatory discontinuous, removable discontinuous, or continuous for given points. The aim of this item was to address how students apply their limit knowledge in situations where formula of the function is available, but not its graph.

5th, 6th, and 7th items were about formal definition of limit. In the fifth item, students were given a function that has limit for the given point. Then, students were asked for given epsilon values, whether there is delta value that satisfies formal definition of limit or not. Here the intention is to deal with how students relate epsilon and delta values. In sixth item, students were asked to prove, by using formal definition of limit, that the limit of the function is the proposed value for a given point. The aim of this item was to tackle with how students apply formal definition of limit. In seventh item, students were asked to prove, by using formal definition of limit, that limit of a function is not the proposed limit value for a given point. In this item, students needed to negate the formal definition of limit. So again, the aim of this item was to address how students apply formal definition of limit in the situation where its negation is needed.

3.4.2 Interview

An interview is a data collection method in which purposeful conversation between two or more people happens so that interviewer obtains information about interviewee by asking question or questions about topic in consideration. We cannot observe everything, for example, we cannot observe thoughts or feelings. If we cannot observe such things, reasonable strategy is to ask people about points in consideration. So, the purpose of the interview is to find out what is in the interviewee's mind, or more appropriately, what interviewee thinks or feels about topic in consideration. Interviews can take variety of forms in the degree to which they are structured. Fraenkel and Wallen (2006) determined three types of interviews: structured, semi-structured, and informal interviews. In structured interviews questions were determined beforehand. Interviewer asks the same questions to all interviewees. In semi-structured interviews interviewer determines interview questions beforehand, but can ask follow up questions, and can ask new questions depending on the situation. In informal interviews, interview questions were not determined beforehand. Questions emerge from the immediate context.

In this study semi-structured interview protocol (see Appendix C) prepared by the researcher was used. After preparation peer review of it was done by an expert. Before applying interview protocol in the main study, as a pilot, it was applied to 7 first year mathematics major who were taking introductory calculus course in METU in the summer school of 2008 and to two research assistants in Mathematics Department in METU. The aim of the pilot was to

- check whether there is a need for rephrasing questions or not,
- check whether there is a question that interviewee's are uneasy to answer,
- check whether questions are ambiguous or not,
- check effectiveness of interview questions in gaining information about students' limit conception
- check whether questions are leading or not.

Above considerations were held after pilot. And all of the questions were kept, because they satisfied the above conditions. In addition to above aims about interview questions, researcher aimed to gain experience in interviewing, in asking appropriate follow up question, in analyzing student responses in the interview, and lastly, to help refine primary genetic decomposition by using student responses on the limit concept.

First part of the interview deals with the demographic information about students. Second part of the interview probes the students' understanding of the concept of limit of a function. Second part composed of eight questions which are parallel to the items in questionnaire. First question was about students' general understanding of the limit concept. Students made general comments about how to determine limit of the function f at the point a in the context where no formula or graph is present. Questions 2, 3, and 4 were about to find limit of a three different type of functions for a given point: removable discontinuous, oscillatory discontinuous and continuous functions. The aim of these questions was to address students' knowledge about informal limit concept. Fifth question was about how to determine limit of addition of two functions for a given point. The aim of this question was to address how students treated limit concept in the given condition. Questions 6, 7, and 8 were about the use of formal definition under different context. The aim of these questions was to address students' knowledge about formal limit concept.

3.5 Procedures of the Study

The study included 17 main steps: (1) Designing and developing questionnaire, interview protocol, and laboratory activities, (2) Expert review, (3) Piloting questionnaire, interview protocol, and laboratory activities, (4) Analyzing data in pilot study, (5) Expert review, (6) Revisions, (7) Finding participants, (8) Taking necessary permissions to make the main study, (9) Administrating limit questionnaire as a pretest to 25 volunteer students, (10) 5 weeks instruction including computer laboratory activities within groups, (11) Administrating limit questionnaire to same students as a post test, (12) Analyzing students' responses in questionnaire, (13) Applying interview protocol, (14) Analyzing students' responses in the interview, (15) Expert review, (16) Interpreting and writing the results, (17) Expert review. Time table of the study was as follows:

Table 3.2: Timetable of the study

Stages/Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
April, May 2008	✓	✓															
June, July 2008			✓	✓													
August 2008				✓	✓		✓	✓									
September 2008				✓	✓	✓			✓								
October 2008										✓							
November 2008										✓	✓	✓	✓				
December 2008													✓				
January-June 2009														✓	✓		
July 2009														✓	✓	✓	
August 2009																✓	
September 2009																✓	
October 2009																✓	✓

(1) Designing and developing questionnaire, interview protocol, and laboratory activities,
(2) Expert review,
(3) Piloting questionnaire, interview protocol, and laboratory activities,
(4) Analyzing data in pilot study,
(5) Expert review,
(6) Revisions,
(7) Finding participants,
(8) Taking necessary permissions to make the main study,
(9) Administrating limit questionnaire as a pretest to 25 volunteer students,
(10) 5 weeks instruction including computer laboratory activities within groups,
(11) Administrating limit questionnaire to same students as a post test,
(12) Analyzing students' responses in questionnaire,
(13) Applying interview protocol,
(14) Analyzing students' responses in the interview,
(15) Expert review,
(16) Interpreting and writing the results,
(17) Expert review.

In the first stage of the study, limit questionnaire, interview protocol, and laboratory activities were designed and developed. Then, expert review of them was done by an independent researcher. In the third stage, pilot study was done. Gained data in pilot study was analyzed at the fourth stage. Following data analysis, expert review was done. And depending on results of data analysis and expert review, revisions were considered in instruments and lab activities.

At the seventh and eight stages, participants of the study were determined and necessary permissions to conduct the study were taken. After this, main study started with the administration of the limit questionnaire as a pre test to 25 students. Following pretest, five weeks of instruction was done with same 25 students. The same questionnaire administered to same 25 students after the instruction. After the administration of the limit questionnaire, researcher of this study interviewed all of the 25 five students attended to this study. Researcher administered interviews in which only researcher and interviewee were present. The interviews were conducted in a non-formal friendly environment, in the building of Mathematics Department in which students attended their classes. During the interviews, eight open ended questions were asked to the participants in an order. Interviews were recorded with voice recorder. Approximate interview length was about one hour. Analysis of the questionnaire responses and administration of interview protocol were done in parallel.

After this, the part of the study that took the longest time (analysis of student responses to interview questions) happened. This part was check by an expert as the analysis went on. Lastly, researcher of this study interpreted and wrote the results which were accompanied by the expert review.

3.6 Analysis of the Data

In this study, two instruments, namely limit questionnaire and interview protocol, were used to gather both quantitative and qualitative data. At each step of the process- design, development, administration, data analysis- peer review was done by different researchers from different areas.

From the questionnaire both quantitative and qualitative results were gained. First item of the questionnaire, that yielded qualitative results, addressed the students' concept images of limit of a function. The responses to this item were grouped in three categories: informal, formal, and others. Informal response is related with informal conception of limit, formal response is related with formal conception of limit, and remaining responses are categorized as others. Informal category has three subcategories: pre-action, action, and process. Responses, including evaluation of only one point to determine limit, were put under pre-action. Responses, in which several values used to determine limit, were categorized as action. Lastly responses, including dynamism in domain, or in range, or in both in determination or description of limit, were put under process. Moreover, formal category has three sub-categories: incorrect, lack of quantifiers, and correct. Responses, that stated formal limit definition incorrectly, were categorized under incorrect. Responses including lack of quantifiers over one (or more) of x, δ, ε, were categorized as lack of quantifiers. And lastly, responses, that stated formal limit definition correctly, were categorized under correct.

Remaining items of the questionnaire yielded quantitative scores. Rubric to evaluate student responses was adapted from Asiala, Cottrill, Dubinsky, and Schingendorf (1997) by the researcher as follows:

- 0 for empty or irrelevant response,
- 1 for responses that showed some progress toward solution but far from the correct solution,
- 2 for almost correct responses with minor flaws in the solution,
- 3 for totally correct responses.

All of the remaining items, both in pre and posttest, were analyzed by using this rubric. After analysis of the responses of all items by the researcher, categories and rubric were explained to two research assistants in the Department of Mathematics in METU. Then they analyzed questionnaire responses depending on the same criteria. Researcher of this study and two research assistants met to discuss discrepancies in analysis results. After discussion, 100 % agreement was accomplished. Data gained from the analysis was entered into SPSS software to construct descriptive statistic. This form of data, then, was used in the conclusion and interpretation part of this research. Results of the first item and remaining items were used to triangulate qualitative data gathered from interviews that will be explained in the following paragraphs.

Second instrument of the study was interview protocol. Analysis of the data gathered from interviews was done according to the framework suggested by Asiala et al. (1996). The data recorded during the interviews were transcribed by the researcher. Then, by using Word 2003 software, this transcribed text was segmented and put into two-column format. Segments were meaningful units in which interviewee discusses about a specific mathematical point. Each segment was numbered so that it is easy to handle the text later. The first column included original transcribed text and the second column included the brief statement about what is going on at the first column as shown in the following figure.

S: Hı hı evet. ************************	
I: Peki onu nasıl belirledin? S: Çünkü şey sinx periyodik bir fonksiyon. Devamlı belirli değerler alıyor. Bu durumda limiti yok. I: Periyodik olması limitinin olmamasını nasıl gerektiriyor? S: (8 saniye bekledi). Periyodik olması sonsuza giderken limitin olmamasını gerektiriyor. Çünkü devamlı belirli aralıkta değerler alacak. Yani ben ne kadar büyük değerler verirsem vereyim. Hani belirli bir değere yaklaşmayacak. Devamlı farklı değerler alacak. Sinus belirli bir değere yaklaşmayacak. I: 4. soruya geçelim. ***	7. Since sinus is a periodic function, it will get different values for each x, then limit does not exist.

Figure 3.2: Segmented Transcript

Then researcher started to read each transcribed text carefully to produce issues for each individual. Issues were very specific mathematical point of view. Also, we can call them as subcategories. When producing issues, researcher noted the segment number of it. These lists of issues of each individual were rearranged as a single list of issue including issues of each individual and its segment number. Then depending on list of issues (or subcategories) researcher created categories. At this point, an independent researcher peer reviewed of the all process.

Researcher had categories, issues, and segment numbers of each issue in his hand. At this point, each issue was again read carefully by the researcher. In this reading, the researcher considered the performances of each individual on the issue. Then, by comparing these performances of individuals, the researcher tried to explain differences in terms of actions, processes, objects, and schemas. In this process, whether they constructed a specific mental construction or not were considered. This part of the analysis was peer reviewed by an independent researcher who is expert on APOS theory.

3.7 Reliability and Validity Issues

Yin (2003) described reliability as "… if a later investigator followed the same procedures as described by an earlier investigator and conducted the same case study all over again, the later investigator should arrive at the same findings and conclusions… The goal of reliability is to minimize the errors and biases in the study" (p. 37). Lincoln and Guba (1985) suggested dependability in place of reliability in qualitative research. In order to produce dependable research, researchers need to address the changing context within which research occurs. Following measures were taken to address reliability issues in this research.

- In this research, triangulation was done by using both questionnaire and interview to address research questions.

- Students attended treatment, questionnaire, and interview sessions in the same building in which they take their mathematics courses throughout their undergraduate studies.

- Researcher of this study and two independent experts analyzed the questionnaire results individually. And they came together to discuss to discrepancies. 100 % agreement was achieved.

- Researcher of this study and an independent expert produced subcategories and categories depending on the subset of interviews. Then they come together to discuss to discrepancies. 100 % agreement was achieved on the subcategories and categories.

Internal validity is defined by Cohen, Manion, and Morrison (2007) as "the explanation of a particular event, issue or set of data which a piece of evidence provides can actually be sustained by the data" (p. 135). Lincoln and Guba (1985) suggested credibility in place of internal validity in qualitative research. In order to meet the credibility researcher needs to identify and describe the subject and to take into account subject's point of view, hence multiple realities. Following measures were taken to address internal validity issues in this research.

- In this research, triangulation was done by using both questionnaire and interview to address research questions. And results of responses of students to two instruments were compared.

- In each step of instrument (questionnaire and interview protocol) design and development, experts made peer review.

- In each step of data analysis, experts made peer review.

- In determining research methodology for this research, members of thesis follow up committee made peer review.

- Low level descriptors were used in presenting results.

- Researcher used APOS theory framework to interpret the student responses in the interview. An expert peer reviewed the interpretations of the researcher.

- Researcher considered rival theories in interpreting the results.

External validity refers to the degree to which the results of the study can be generalized to other cases, settings, to the wider population. In this study, subjects were limited to 25 first year first semester mathematics major who were taking introductory calculus course in METU. So, the researcher of this study did not aim to generalize the result of this study to whole population. Nevertheless, researcher provided rich descriptions about the context and participants and their responses in instruments so that other researchers who want to transfer the results of this research to their situations have a rich base of information. Moreover, another issue is the analytical generalization. The term introduced by Yin (2003, p.32-33) as "…previously developed theory is used as a template with which to compare the empirical results of the case study". In this research, APOS theory and its implications in the form of genetic decomposition (Cottrill et al., 1996) were used with the aim of analytic generalization.

In addition to above concerns, in order to address the quality of the research, researcher made a pilot study to test effectiveness of instruments and laboratory activities used in the main study and to get experience in data collection and analysis.

3.8 Assumptions of the Study

For this study, the following assumptions were made:

1. The participants responded accurately to all instruments used in this study.

2. The data were truthfully recorded and analyzed.

3. Reliability and validity of all the measures used in this study were accurate enough to permit accurate conclusions.

3.9 Limitations of the Study

Limitations of this study were the followings:

1. Number of participants was limited to 25 first semester mathematics majors in METU.

2. Validity of this study was limited to the reliability of the instruments used in this study.

3. Validity and reliability were limited to the honesty of the participants of this study.

4. Generalizations are only possible for similar situations and contexts.

CHAPTER 4

RESULTS AND CONCLUSIONS

In this chapter analysis of results of research problems is given. Analysis of students' responses to interview protocol is presented. Also students' responses to limit questionnaire is stated.

4.1 Students' Understanding of Limit Concept

In this section following research questions will be addressed.

1. How do students develop understanding in limit of a function?

    1.1. How do students explain their informal understanding of limit of a function?

    1.2. How do students explain their formal understanding of limit of a function?

In describing students' constructions and mechanisms which are used to produce those constructions, APOS researchers devise a tool called genetic decomposition. This includes the researchers' understanding of the concept, conclusions drawn from the literature, historical development of the concept, and previously constructed versions. This preliminary version is needed to be empirically tested. Genetic decomposition of Cottrill et al. (1996) is taken as primary genetic decomposition for this study which is as follows:

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to a.

2. The action of evaluating the function *f* at a few points, each successive point closer to *a* than was the previous point.
3. Construction of a coordinated schema as follows:
   (a) Interiorization of the action of Step 2 to construct a domain process in which *x* approaches *a*.
   (b) Construction of a range process in which *y* approaches *L*.
   (c) Coordination of (a), (b) via *f*.
4. Perform actions on the informal limit concept by talking about, for example, limits of combinations of functions. In this way schema 3 is encapsulated to become an object.
5. Reconstruct the process of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed ε-δ conception applied to specific situations.

In order to investigate students' understanding of limit concept, a semi-structured interview protocol was administered. Results are represented, as follows, parallel to the form of genetic decomposition. First four steps of the genetic decomposition are related with the sub-problem 1.1 and remaining steps of genetic decomposition are related with sub-problem 1.2.

1. Action of evaluating *f* at a single point *x* that is considered to be close, or even equal to *a*.

There were no students who were limited to step1 considering students' responses in the interview. This step can be dropped from the genetic decomposition. Nevertheless, as explained in the coming section, in analysis the of responses to limit questionnaire as a pretest it was observed that, before students were given instruction, some of them responded what it means limit to them as determining limit by evaluating *f* at the point *a* or at very close point to *a*. This might indicate that there were some students who were limited to this step before the instruction. Moreover, in the literature, there are some studies that show examples for this step. For example, Cottrill et al. (1996, p.11) reported the following excerpt. Here Jean clearly shows the elements of this step.

I: So when you're actually looking at a limit situation and trying to determine if a limit does or does not exist, what are you doing?

Jean: First I plug the negative 2 into my function $f(x)$ to see it is defined.

I: Mhm.

Jean: If it is, then that is the limit. If negative 2 does not exist, or if it's not defined at negative 2,…

I: Mhm.

Jean: … then I would take a point very close to negative 2, may be on each side of negative 2

I: Mhm.

Jean: to see if those two values are very close to the same number.

In addition to above example, Przenioslo (2004) showed that some of the students see the limit of *f* at x_0 as $f(x_0)$. And Elia et al. (2009) reported that some of the students see the limit of a function at a point as evaluation of the function at this point.
Similar to step 1, no students were found to be limited to steps 2 and 3(a) in the interview, after the instruction, in the main study. However, unlike step 1, some students

were observed to limited to steps 2 and 3(a) in the pilot study done previous to main study. In the coming steps 2 and 3(a), example excerpts are from the pilot study, not from the main study. Absence of observation of steps 1, 2 and 3(a) in the main study might be accounted for the positive role played by the instruction. At the same time, because of the absence of observation of these steps in the main study, they can be dropped from the genetic decomposition. However, in questionnaire (as a pretest, before the instruction) some of the students showed elements of step 1 and in the literature there are some studies (Cottrill et al., 1996; Przenioslo, 2004; Elie et al., 2009) that show examples for this step. Moreover, in the pilot study of this research some students were observed to be limited to steps 2 and 3(a). Thus, although it is possible to remove them from the genetic decomposition, it seems more appropriate to hold them in the genetic decomposition.

2. Action of evaluating function *f* at a few points, each successive point closer to *a* than was the previous point.

In the following excerpt the discussion is about what it means to say that limit of the function *f* at the point *a* is *L*. Hazan was talking about smallest number (sometimes numbers) which is very close to *a*. When prompted by the researcher, she showed the awareness that there is no closest number to 3. But she insisted on to use such closest number. Hazan took several closest numbers to *a* to determine the limit. This indicates that she did not construct domain process rather she had static evaluation conception.

I: What do you mean by left and right approach, can you explain it more?

Hazan: You take a so small value, how to say, … for example we take a value from the left which is smallest but is not equal to *a*.

I: What type of number is this?

Hazan: Numbers which are not equal to a, but very close to it.

I: Number or numbers?

Hazan: It is not always possible to find smallest. Since there is no such determined value it will be close to *a* as much as possible.

Later in the case of "how to find limit of the function given by $f(x)=2x$, at the point 3" she explains that smallest number is the closest one to *a*.

Hazan: We are not taking 3, we are taking the closest number to it …

I: Can you exemplify the closest number to 3.

Hazan: I cannot give.

I: Then what is closest to 3.

Hazan: Numbers.

I: What do you do after you take closest numbers to 3.

Hazan: We are looking at outputs of these values to determine limit.

3. Construction of coordinated process schema as follows

3-a Domain Process

Similarly, Gizem used closest number argument to determine limit value. She used two points, namely 3-*a* and 3+*a*, to determine the limit, but after prompt from researcher, she showed some indication of domain process by saying that "there is always closer one".

Gizem: From the right and left, at the closest points to 3, function values must be equal to each other. For example, let *a* be very very very small number. If values of f at the points 3+*a* and 3-*a* are equal to each other, limit exists at the point 3.

I: Can you give such a number for this function?

Gizem: I can say… but whatever I say there is always closer one.

Later in the case of how to find limit of $\cos 1/x$ at 0, she showed some progression toward domain process. By saying "As I approach 0, there are many numbers… I find closer numbers to 0 by making interval around 0 narrower and narrower" she showed that in the domain she considers a process, rather than static values. But she did not show same progress in the case of range process, she used only closest numbers to find limit.

Gizem: As I approach 0, there are many numbers. I make the interval around 0 narrower. I find closer numbers to 0 by making interval around 0 narrower and narrower. I need their closest ones.

I: What do you do after finding closest ones?

Gizem: I look their function outputs…

I: When do you say limit exists?

Gizem: If they are equal.

3-b Range Process

According to primary genetic decomposition, once the domain process is constructed, one can construct range process. In the following excerpt, when explaining limit of $f(x)=2x$ at 3, Ozlen constructed two domain and range processes. Although she used word "function", she did not use it to coordinate domain and range sequences. The sequence of numbers in the range was not constructed by applying function to domain process. This is so apparent when she took 3.009 and 3.008 for domain but 6,2 and 6.1 for the range.

Ozlen: How do I determine the limit at the point 3? When x approaches from left and right, function gets closer to 6. So, limit is 6.

I: Well. How did you find 6?

Ozlen: I get closer to 3 by taking x as 2.99, 2.999. And function will get the values like 5.9, 5.99, 5.999. It is not important whether function gets 6 or not, but it continuously gets closer to the constant 6. That is, it points 6. Similarly, by taking values greater than 3, for example, 3.008, 3.009, function will get values a bit greater than 6 like 6.1, 6.2.

I: When you look from the left, you said 5.999 as a last number. But there is a difference between 6 and 5.999. How do you say that limit is 6?

Ozlen: I feel, I think on it… I give many values other than 5.999. All of them point the same value. I feel what happens between 5.999 and 6.

A few minutes later, when considering limit of *f*, defined as $f(x)=\begin{Bmatrix} 1-x, x \le 1 \\ x, x>1 \end{Bmatrix}$, at the point 1, Ozlen started to use function to coordinate domain and range sequences. She, first, considered domain sequence 0.99, 0.999, 0.9999 and then constructed range sequence 0.01, 0.001, 0.0001. For right limit, she constructed another domain and range sequences. In her explanation and writings in Figure 4.1, it is clear that domain and range sequences were coordinated via the function. This is similar to phenomenon observed by Dubinsky, Elterman and Gong (1988). One student, named as REI, was not able to deal with two-level quantification, however, she was able to, in the same interview, coordinate two quantifiers in a different context after a few minutes. In some situations students are not able to show some structure, but in another situation they can show indication of this structure even after a few minutes. This is why Asiala et al. (1996) called mathematical knowledge as tendency.

Ozlen: If I take *x* values as 0.99, 0.999, 0.9999 and continue like this, function will get the values 0.01, 0.001, 0.0001 and continue like this. In the same manner, from the right, it will get 1.9, 1.8, 1.7, 1.1. There are many values between 1.1 and 1, I cannot write all of them. But, values of the function point 1 by decreasing.

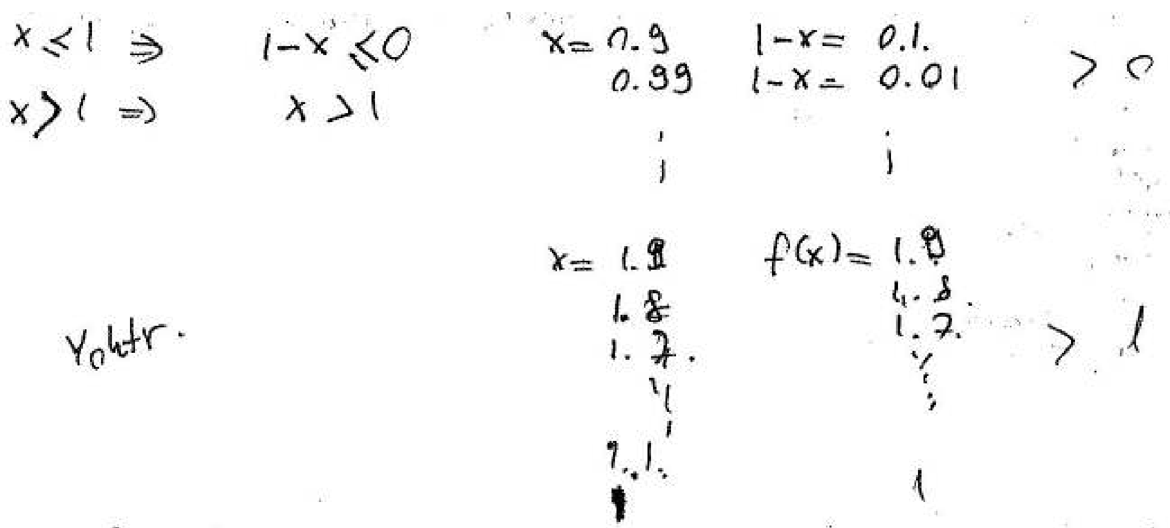

Figure 4.1: Coordination via Function- Ozlen

3-c Process Conception of Informal Limit

All of the students showed indication of this step. There were two type of response. In the first type, students determined elements of domain sequence first, and then by applying function to this sequence of values, they determined range sequence to figure out limit. Whereas, in the second type, students determined one element for domain sequence, and by applying function, they determined corresponding element for range sequence. After, they determined second element for domain sequence, and then by applying function they determined corresponding element for range sequence. This goes on until the realization of the limit. I call this as simultaneous construction of domain and range sequences. Nevertheless, the point is that students might determine domain sequence first and applied function one by one to elements of this sequence as in the first case. We do not know whether they constructed first domain sequence and applied function one by one to elements of this sequence in their mind or they really constructed simultaneously domain and range sequence in their mind. But, in practice we know that there is such a distinction.

Whatever route they took, function was used to coordinate domain and range sequences. Following excerpt is an example for the former type. Selin constructed, first, domain sequence 2.9, 2.99, 2.999, 2.9999, which yielded corresponding range sequence: 5.8, 5.98, 5.998, 5.9998. Then to determine right limit, she took the following values in the domain 3.1, 3.01, 3.02 (she incorrectly wrote 3.02; correct value should be 3.001), which yielded corresponding function values 6.2, 6.02, 6.002. It is clear, from her explanations and what she wrote in Figure 4.2, that she constructed range sequences by applying function to domain sequences.

I: Let f be a function given by $f(x)=2x$. How do you determine the limit of this function at the point 3?

Pelin: By looking at values that it takes when approaching.

I: Let's look then.

Pelin: Let's give values so close to 3: 2.9, 2.99, 2.999, 2.9999. For these values, as we see, function will get values 5.8, 5.98, 5.998, 5.9998 which are getting close to 6.

I: How do you understand that these are getting close to 6?

Pelin: 5.998, 5.9998 it will keep going like this.

I: How will it go on?

Pelin: It will take values which are getting closer to 6 constantly. Again at the same manner, I can get closer to 3 from the right. I can give values like 3.1, 3.01, 3.02. For these values, function will get values which are slightly bigger than 6: 6.2, 6.02, 6.002. These values are getting closer to 6 by decreasing. So limit is 6.

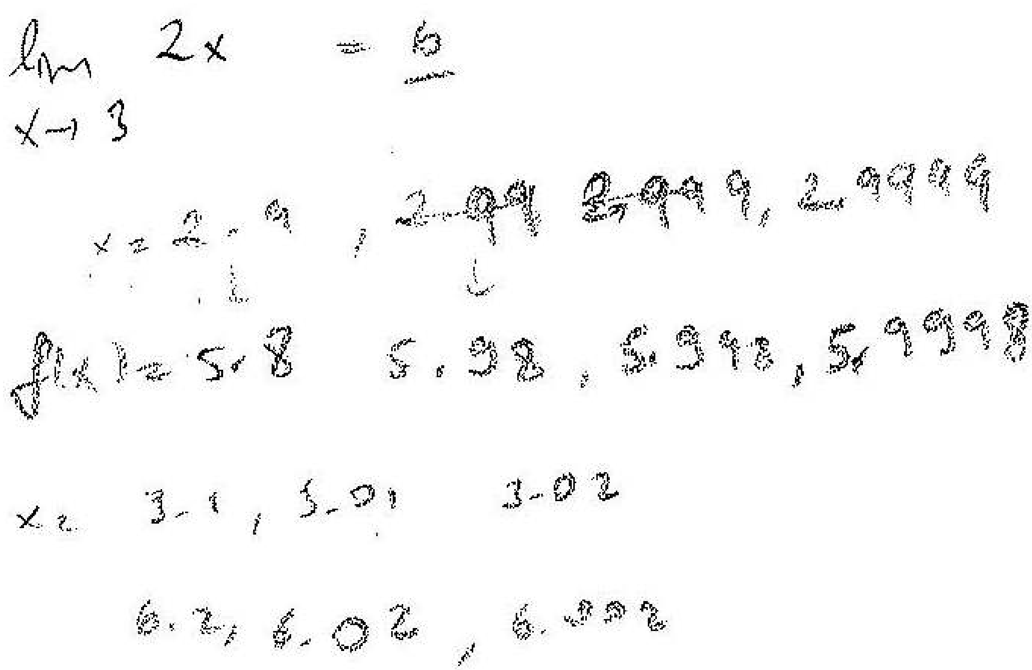

Figure 4.2: Coordination via Function-Pelin

Following excerpt is an example for the latter type. Mert, simultaneously, constructed domain and range sequences for left and right limit. And since left and right limits are not equal, he concluded that limit does not exist. By saying, “When approaching from left, I will use $1-x$. When approaching from right I will use x.” he gave first indication for the application of function. Then by saying “I gave 0.8, function returned 0.2…” he explicitly used function to construct domain and range sequences simultaneously. Moreover, it is noticeable that in the middle of his explanation, without any question from the researcher, he mentioned his experience with ISETL in such questions.

I: Let f be a function given by $f(x)=\begin{Bmatrix} 1-x, x\leq 1 \\ x, x>1 \end{Bmatrix}$. How do you determine the limit of this function at the point 1?

Mert: I approach both from right and left. When approaching from left, I will use $1-x$. When approaching from right I will use x. Then I figure out that if I approach to 1 from the left, function will go to 0.

I: How did you figure out?

Mert: How I figured out… First I gave 0.8, function returned 0.2; I gave 0.9, 0.1 was returned; I gave 0.99, 0.01 was returned; I gave 0.999999…9, 0.000000…1 was returned. Actually we did something like this before in ISETL (he mentions lab activities). Then I figured out that function goes to 0 when approaching from the left. When approaching from the right, first I gave 1.2, since function is defined as $f(x)=x$, it will return 1.2 for 1.2, 1.1 for 1.1, 1.01 for 1.01. Then I took 1.0000…1, and I understood that function goes to 1. Left limit is 0 and right limit is 1. So, since left and right limits are not equal, limit does not exist.

Beside how function is applied in the construction of range sequences, there was another factor common in students’ explanations with two types. In the first type, students coordinated domain and range sequences with finite terms, and then after seeing the pattern "as finite terms in the domain sequence get closer to *a*, corresponding terms of the range sequence get closer to *L*" they contemplated other values in the informal limit process. Similarly, in the other type, students imagined the steps of informal limit process after considering finite steps, nevertheless, they went further and see the informal limit process as a single action or as a whole. In the single action, generally students see domain process as $a \pm 10^{-n}$ (as *n* increases expression approaches to *a*) and range process as $f(a \pm 10^{-n})$ (as *n* increases function approaches to *L*). So, increasing *n* forces function to approach to *L*. We can see this distinction from the following two students' excerpts: Basak is an example for the former type whereas Erhan is an example for the latter type.

In the following excerpt Basak explains how she finds limit of 2*x* at 3. She constructed domain sequences (4, 3.5, 3.2) and (2, 2.2, 2.5, 2.7), and correspondingly range sequences (8, 7, 6.4) and (4, 4.4, 5, 5.4). She concluded that since the terms of the both range sequences approach to 6, limit is 6. Interviewer asked that depending on the sequences she formed, is it possible that limit is 6.1 rather than 6? She showed awareness that number of the terms in the sequences actually was not finite, there were infinitely many terms that she contemplated in her mind. And those numbers were ultimately approaching to 6 rather than 6.1. Thus, by constructing finite sequences she saw the pattern that for each closer term to 3 in the domain sequences, terms of the range sequences get closer to 6. After seeing this pattern she imagined remaining infinitely many terms in the sequences to conclude that as *x* goes to 3, 2*x* goes to 6, so limit is 6.

Basak: Function gives 6 at the point 3, but this is not important for the limit. What I want to do is to get closer to 3. For *x*=4 it gives 8, for *x*=3.5 it gives 7, for *x*=3.2 it gives 6.4. For *x*=2, it gives 4, for x=2.2 4.4, for 2.5 5, for 2.7 5.4. As I approach from the right it decreases in the way 8, 7, 6.4. As I approach from the left it increases in the way 4, 4.4, 5, 5.4. That is, from both sides function approaches to 6. So, limit is 6.

I: From the right, last value that you got is 6.4, and from the left you got 5.4. Cannot we say that function approaches to 6.1 by looking numbers that you produced?

Basak: If I gave a smaller number for *x*, function could get 6.1. But I can give smaller *x* than the previous one, so, for example, function can get 6.001. So it approaches to 6 rather than 6.1. I can give smaller and smaller numbers. Here we have infinitely many numbers, but I know where those numbers go ultimately. I know they approach to 6.

Erhan constructed domain sequence x=(3.01, 3.001, 3.0001, …, $3+10^{-(n+2)}$,…), then he applied function to terms of this sequence to get range sequence f=(6.02, 6.002, 6.0002, …, $6+2*10^{-(n+2)}$,…). By taking index, *n*, to infinity he completed informal limit process. Further, although he did not mention explicitly, he saw the informal right limit process as single coordinated action: increasing 0s in 3.0001 causes to increase 0s in 6.0002. Nevertheless, we can see explicitly this single coordinated process when he considered left limit: "As I increase the number of 9s in 2.9999, number of 9s in 5.9998 will also increase".

Erhan: We want *x* to go 3, accordingly, we need to find where $f(x)$ goes. Firstly I take x_0=3.01, then $f(x)$=6.02. Then take x_1=3.001, then $f(x)$=6.002. Take x_2=3.0001, $f(x)$=6.0002. Values of $f(x)$ are decreasing. If 2 goes to infinity (he means sub-index of *x* goes to infinity), *x* approaches 3 and $f(x)$ goes to 6. This is the right limit. Now, take x_0=2.99, then $f(x)$=5.98. Take x_1=2.999, then $f(x)$=5.998. Take x_2=2.9999, $f(x_2)$=5.9998. As I increase the number of 9s in 2.9999, number of 9s in 5.9998 will also increase. That is, function will approach to 6. Since from both sides *f* goes to 6, I can conclude that as *x* goes to 3, $f(x)$ goes to 6.

4. Object Conception of Informal Limit

Once the students reflect on the process in 3(c), and can see this process as a totality that an action or process can act on it, then process of 3(c) might be encapsulated as an object. To observe this object conception, students were asked following type of questions.

1. Assume that f and g have limit at the point a. How do you determine limit of $f + g$ at the point a?

2. Does the following limit exist? If yes, explain why. If no, explain why not.

$$\lim_{x \to 1} f(x), \text{ where } f(x) = \begin{Bmatrix} \dfrac{x}{2}, x \neq 1 \\ 1, x = 1 \end{Bmatrix}$$

For the first question, in their explanation, it is expected that students show the awareness of three distinct objects, namely, $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$, and $\lim_{x \to a}(f + g)(x)$. Moreover they are expected to show that these objects are produced by application of coordinated process of 3(c). For the second type of question, it is expected that students see left limit as a single action and also right limit as a single action. Then they are expected to compare left and right limit processes which are seen as a single action. Here, the comparison is an action that students apply to two processes which are seen as a whole. In order to apply an action or a process on a process it is needed to encapsulate this process into object. In our situation, objects are left limit object and right limit object.

In the following excerpt, Nur used limit property to explain limit of sum of two functions at a point. In this way she treated limits as objects. But she did not show any indication for the de-encapsulated process.

Nur: Assume that as x goes to a $f(x)$ goes to K, and similarly, as x goes to a $g(x)$ goes to L. According to the limit rule, I can write as x goes to a, limit of $(f+g)$ is equal to as x goes to a limit of f plus as x goes to a limit of g. We said that as x goes to a limit of $f(x)$ is K and as x goes to a limit of $g(x)$ is L. And we also said according to limit rule we can write $\lim_{x\to a}(f+g)=\lim_{x\to a} f+\lim_{x\to a} g$. I know individual limits were K and L. So I can say that limit of $f+g$ as x goes to a, is $K+L$.

In the following excerpt, Gokhan took two functions and find their sum. Then he applied informal limit process to these three functions to find their limit values.

Gokhan: Let's take $f(x)=x$ and $g(x)=2x$. I will consider their limits at the point 1. $(f+g)(x)=3x$. As I did in previous question, I look at the function values of finitely many points from the left and right of 1. And then I consider remaining infinitely many function values. If I do so limit of $f(x)=x$ is 1, and limit of $g(x)$ is 2. If I look at limit of $f+g$, I see that its limit is 3.

In the following excerpt Seval explains why limit of f defined as $f(x)=\begin{Bmatrix} 1-x, x\le 1 \\ x, x>1 \end{Bmatrix}$ does not exist at $x=1$. By saying "I can increase 0s in 1.001, then $f(x)$ goes to 1" and "As I increase 9s, 0s will increase", she saw left limit process and right limit process as a single action, or as a totality. Then by comparing single action from left and single action from right she concluded that limit does not exist. Here, comparison is an action applied to two "left" and "right" processes which are seen as totality. If an individual sees process as a totality on which an action or process can act, then process might be encapsulated as an object. Then, need to apply action to this process makes it encapsulated. So, Seval encapsulated left and right limits to compare them to determine limit.

Seval: I need to look at left and right limits. On the x axis, I take values greater than and go to 1, for example, 1.1, 1.01, 1.001. Since $f(x)=x$, these values are also y values. Those values are getting closer to 1. I can increase 0s in 1.001, then $f(x)$ goes to 1. So, right limit is equal to 1. Now I will look at from the left. I will take $f(x) = 1 - x$ for this part. I take values less than 1, for example, 0.9, 0.999. If we look at $f(x)$, it gets 1-0.9=0.1 for x=0.9. For 0.999, it gets 1-0.999=0.001. Similarly, $f(x)$ is 0.0001 for x=0.9999. As I increase 9s (she means 9s in 0.9999), 0s (she means 0s in 0.0001) will increase. Then it will go to 0. Therefore left limit is equal to 0. And since left limit is not equal to right limit, limit does not exist.

Lastly, in the following excerpt, Mehmet reasoned on his limit concept and deduced that if a function has a limit then it is bounded on an interval around a. His reasoning on his limit schema to conclude such a statement is an indication for object conception of informal limit. If his schema was limited to process conception, since process is an action one does, it was not possible to reason on this dynamic action to conclude that function must be bounded. In order to act on (reason about) informal limit process, what needed is to encapsulate informal limit process to get static object of it. Once the process is encapsulated, one can act on it to deduce another result. Moreover, since type of functions were not known, Mehmet determined randomly domain interval and range bound (notice that in order to be mathematically correct point a in the domain interval should be excluded). Then by saying "By making 0.001 smaller and smaller, we can see that x goes to a and $f(x)+g(x)$ goes to $L+M$.", he de-encapsulated this object, and applied resulting process to conclude that limit of $f+g$ is $L+M$.

Mehmet: Let's say limit of $f(x)$ is L and limit of $g(x)$ is M. And let's take $a - 0.001 < x < a + 0.001$. We do not know whether the functions f and g are increasing, decreasing, or constant. But, anyway, we know that functions have limits. Since both functions have limits, there will be a bound for them. For the present purpose, I can pick a random bound for them. Let's say $L - 0.000001 < f(x) < L + 0.000001$ and $M - 0.000001 < g(x) < M + 0.000001$. If we add inequalities, we get $L + M - 0.000002 < f(x) + g(x) < L + M + 0.000002$, for $a - 0.001 < x < a + 0.001$. So

$f(x)+g(x)$ is in the interval of $L+M$ with a small size. I took 0.001 randomly. By making 0.001 smaller and smaller, we can see that x goes to a and $f(x)+g(x)$ goes to $L+M$.

5. Reconstruction of informal limit process conception in terms of intervals or inequalities

16 of 25 students showed indications of this step. Students reconstruct the process of 3(c) in terms of intervals and inequalities. This reconstruction includes construction of proposition valued function of variable x. Given fixed positive numbers ε and δ, the value of this function is the truth or falsity of the following statements

$0< |x-a|< \delta$ *implies* $|f(x) - L| < \varepsilon$ or

If x *is in* $(a- \delta, a+ \delta)-\{a\}$, $f(x)$ *is in* $(L- \varepsilon, L+ \varepsilon)$ where L is the possible limit value.

Let's represent this function as

$$P_1 : x \to \{T, F\} .$$

This new reconstructed process includes universal quantification on x. That is, P_1 is not checked for only single x, rather it is true for all x. The underlying process is that variable x is iterated and P_1 is evaluated at each point then iteration is controlled by universal quantification.

In the following excerpt Veli used both words epsilon and delta. But he used these words to determine domain and range sequences, rather than constructing intervals or inequalities. Here, Veli makes a discussion about what the formal definition of limit means by considering $f(x)=5x$ at the point 2.

Veli: I say that from here to 2, I started to give values starting from 2-δ and I started to give values that go to 2 and this goes towards 10 from 10-ε. And then I started to give values starting from 2+δ, I see that this goes towards 10 from 10+ε.

Veli assimilated formal limit concept into informal limit schema. Assimilation is not proper way to move towards understanding of formal limit concept. What students need is accommodation. In the following excerpt, Sevda might begin to make such an accommodation. She took fixed epsilon and delta intervals, and just like Veli she used words epsilon and delta to determine domain and range sequences. But after the question from the interviewer she started to consider epsilon and delta as intervals. Then she said "*y* values of *x* values in the $(a\text{-}\varepsilon, a+\varepsilon)$ interval must be in $(L\text{-}\delta, L+\delta)$ interval". This might indicate that she checked the truth value of implication, but she did not iterate for all *x* values.

Sevda: Here is $L+\delta$, here is $L\text{-}\delta$. Starting from $a\text{-}\varepsilon$ if we give values getting closer to *a*, on the *y* axis these values get closer to *L* starting from $L\text{-}\delta$. And as we approach *a* from right, starting from $a+\varepsilon$ if we give values getting closer to *a*, starting from $L+\delta$ it gets closer to *L* on the graph. And limit is *L* at the point *a*.

I: When can we say that limit exists?

Sevda: Hımmm when does limit exist? Those delta values…, when delta interval satisfies epsilon interval, when these two intervals satisfy each other.

I: What do you mean by saying that these two intervals satisfy each other?

Sevda: These two intervals satisfy each other. Epsilon and delta should be grater than 0 and *x* minus, ııı one second, so ııı $(a\text{-}\varepsilon, a+\varepsilon)$ interval should go in $(L\text{-}\delta, L+\delta)$ interval. So it could be closer … as it gets closer … these two intervals, how to say, should satisfy each other.

I: So how, what does satisfy mean?

Sevda: Satisfy means, *y* values of *x* values in the (a-ε, a+ε) interval must be in (L-δ, L+δ) interval.

Following excerpt is another example in which epsilon and delta are taken fixed. By saying that "for increasing or decreasing *x* values", Alev showed awareness of iteration over *x* and then she checked truth value of implication. Moreover by saying "all values that I take in the (a-δ, a+δ)", she controlled iteration over *x* with universal quantification. She put relation between epsilon and delta incorrectly by saying "for all delta there must be an epsilon". Nevertheless her explanation was based on specific epsilon and delta. She did not show any iteration over epsilon and/or delta in practice. This might suggest that in her schema she had static epsilon and delta values with incorrectly remembered relationship "for all delta there must be an epsilon".

Alev: $f(x)$ has a value at the point a, has a value at the point a-δ, has a value at the point a+δ. These values are at the certain distance at $f(x)$. I call this interval as epsilon. If I take $f(x)$ as L, it becomes ε+L and L-ε. That is, if I choose x in the (a-δ, a+δ) interval, my $f(x)$ values will be in the (L- ε, L+ε) interval. This gives the formal definition. So, if limit of $f(x)$ is L, for increasing or decreasing x values that I take, $f(x)$ will take values like L+ε or L-ε. But here we have a requirement that for all delta there must be an epsilon. So, the values that I take in this interval, that is, all values that I take in the (a-δ, a+δ) interval will be at the ε distance at $f(x)$.

6. Applying quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.

10 of 25 students showed indication of this step. In (5) variable x is iterated and P_1 is evaluated at each point. Now to go towards understanding of formal limit concept, it is needed that this process is encapsulated by means of action of universal quantification to obtain object T or F, one for each fixed value of ε and δ. Let's call this object as Q_1. They are then varied, first δ and then ε, to get processes in the following way.

With fixed ε, students iterate over δ. And value of Q_1 is considered until finding appropriate δ. Let's call this proposition valued function of δ as P_2 and express it in the following way.

$$P_2 : \delta \to \{T, F\} .$$

Then this process is controlled by existential quantification to obtain object true or false for fixed value of ε. Let's call this object as Q_2.

Lastly, epsilon is varied and for all epsilon, value of Q_2 is considered. Let's call this proposition valued function of ε as P_3 and express it in the following way.

$$P_3 : \varepsilon \to \{T, F\}$$

In the following excerpt, Sude discussed what the formal definition of limit means. She says "For epsilon interval here, I will find a delta interval here" and "All values will go into epsilon interval". This might indicate that she took epsilon fixed and find a proper delta value among possible delta values. That is, she iterated over delta and controls this iteration with existential quantification. This shows us that Sude reconstructed process conception of informal limit in terms of intervals and correctly varied first x, and then δ. Nevertheless she did not go beyond this to apply iteration over ε. She only considered one epsilon value to conclude that limit exists.

Sude: For epsilon, we will find a delta interval, (a-δ, a+δ). For epsilon interval here, I will find a delta interval here. That is, in the epsilon interval, or let's say in this way, all values that I take in the delta interval will go into epsilon interval, go into epsilon interval here. That is, $f(x)$ values of the all values in the delta interval will go into epsilon interval.

I: Well, you considered an epsilon interval.

Sude: Yes.

I: You find a delta interval for it. Then the values you take in the delta interval (Sude intervenes)

Sude: All values will go into epsilon interval.

I: Well. Is it enough to show this for one epsilon value? For example, let's say epsilon $1/2$ and you find delta $1/4$, can you say that limit exists?

Sude: I can.

I: So, for an epsilon finding a delta is enough to say that limit exists.

Sude: Yes, it is enough. Otherwise it does not satisfy definition.

Cihan previously found δ as $\varepsilon/5$ to prove that limit of $5x$ is 10 at point 2. In the following excerpt, he explained on the graph what the formal definition of limit means with this specific example. He first took an epsilon interval and finds an appropriate delta. Then since he found δ as $\varepsilon/5$ in his previous work, for $\varepsilon=10$ he finds δ as 2. After that, last two lines might indicate that he varied epsilon and controls it by universal quantification.

Cihan: Let's draw the graph (he draws graph of $f(x)=5x$). When we draw the graph of function $f(x)=5x$, here for all epsilon we need to find delta. Let's take an epsilon first, we restricted $f(x)$ in the $(10-\varepsilon, 10+\varepsilon)$ interval. Now we need to find a delta on the x axis such that function values of all x values that we take in the $(2-\delta, 2+\delta)$ interval must go into $(10-\varepsilon, 10+\varepsilon)$ interval. Previously, we found the general relationship between epsilon and delta to see what delta should be chosen for all epsilon. For example, if we choose epsilon as 10, if we want function to be in 0 and 20, then it should be between 0 and 4. Since we take δ as $\varepsilon/5$, we take δ as 2.

I: Is this true for one epsilon or for all?

Cihan: This is true for all epsilon. Indeed we found the general relationship between epsilon and delta.

Eda is given $f(x) = 5x$ and asked to prove that limit is 10. She found $\delta=(\varepsilon/5)$ in her previous work, and here she explained what the formal definition of limit means with this specific example. For a given epsilon, Eda found an appropriate delta among possible delta values. This is an indication for existential quantification over delta. Then for $\varepsilon=100$ she found $\delta=20$, and she says that this is a general relationship. So, it might be concluded that she iterates over epsilon and this iteration is controlled by universal quantification.

Eda: I am given epsilon and I am asked to find a delta. As I found here, I can take δ as $\varepsilon/5$. If I input all x values here (showing delta interval) into function, they will go into this interval (showing epsilon interval).

I: Well, is this valid for 2?

Eda: Yes, it is valid for this question.

I: So if we think limit in general.

Eda: No, function would take another value for 2. But the important thing for me is what the function points for the values around 2.

I: You wrote delta in terms of epsilon. Why did you write this?

Eda: Because to make it general. I do not know epsilon. I need to find delta depending on epsilon. So, by writing it in terms of epsilon I made it general. If I took epsilon 100, delta would be 20 and satisfy it, for this reason, for making it general.

I: What did you generalize? I could not understand.

Eda: Because I do not know epsilon… Epsilon can take all values greater than 0. So, I need to find such a delta that images of all x values in this delta interval will reside in epsilon interval. So, by writing it in terms of epsilon I guarantee this.

Here is example in which for an epsilon value Nur found an appropriate delta among possible delta values which is an indication for existential quantification over delta. For this delta value she iterated over x and checked truth of implication for all x values. Then she said "epsilon can be given as 5, or as 1/2. I want this for all epsilon values". This might indicate that she iterated over epsilon values and this iteration was controlled by universal quantification.

Nur: What I want to find is delta. I can change this delta. That is, since epsilon is given to me I can find a delta depending on it. Then I choose x values from $(a-\delta, a+\delta)$ interval. My aim is that I want these x values to go in $(L-\varepsilon, L+\varepsilon)$. I know this from the definition. And I look for the values in $(a-\delta, a+\delta)$ (interviewer intervenes)

I: All values or some values?

Nur: Indeed, I need to choose such an interval that all values will go into this interval (she shows epsilon interval). That is, for example, if all values satisfy but only one value does not, if this one goes out of $(L-\varepsilon, L+\varepsilon)$ interval, then this means that this delta does not run … does not satisfy. That is, I will pick such an interval that all x values in $(a-\delta, a+\delta)$ interval, except a, will go into epsilon interval which is considered previously… Absolutely, no value will go out of this interval.

I: Well, is it sufficient that for one epsilon value you find one delta? For example, let's say you for ε=3 you find δ=2. And what you said is satisfied, values in (a-2, a+2) interval goes into (L-3, L+3) interval. Then, can you say that limit exists?

Nur: No, it is not sufficient that it runs for only ε=3. Since I said for all epsilon greater than 0, I do not only want that for all x values in (a-2, a+2) interval goes into (L-3, L+3). For example epsilon can be given as 5, or as 1/2. I want this for all epsilon values.

I: Well, do you take one delta value, or does delta changes depending on epsilon?

Nur: Delta changes depending on epsilon. It is not correct to take it as constant. Why? If I take it as constant… let's say I take δ=3. That is, my interval on the x axis is (a-3, a+3) interval. Now, I can take epsilon whatever I want. This does not satisfy for all epsilon values. Let's say this satisfies for ε=6, but this may not satisfy for ε=5. Eventually, there could be an interval which it does not satisfy.

Here is another example in which for a given epsilon appropriate delta was found and truth value of implication was checked for all x values. Moreover, process of iteration over epsilon and controlling this iteration with universal quantification was apparent. Ali iterated over epsilon by taking it as 1000, 500, 100, 50, 1, 4, 3.5, and 3. Then by saying "Then this must be true for all" he controlled this iteration with universal quantification.

Ali: If as x goes to a $f(x)$ goes to L, then for given any epsilon we find a delta. And for the delta that we found, all the values between a-δ and a+δ -except a, because we do not consider a- all these values will go into (L-ε, L+ε) interval.

I: Did you consider this for one epsilon value?

Ali: The epsilon value that I take may not be this one, we can take it from here (showing another epsilon interval on the graph). That is, whenever I am given an epsilon value, if limit exists, I can find an appropriate delta.

I: Then, can we say that limit exists if for some epsilon values we find delta?

Ali: Rather than taking 3 values, we take decreasing epsilon values at each time, and if for each one we can find satisfying delta, we can say that limit exists?

I: Well. If you take finite number of decreasing epsilon values and for these you find delta values, then can you say that limit exists?

Ali: Iıı you give finite number of epsilon values… But let's say in this way. Let's take epsilon starting from 1000, decrease it, 500, 100, 50, 10, 4, 3.5, 3. That is, I take finite number, for example from 1000 to 3 there are finite number of values, and for each one I find a delta. But still limit might not exist.

I: So, if we consider formal definition, is it for all epsilon values or for some?

Ali: Whatever epsilon value I take, next one can be smaller than previous one. And this is of course true for… if limit exists, for each epsilon value I can find delta. But if given value is not correct limit (interviewer intervenes)

I: Let's assume that limit exists.

Ali: Then this must be true for all.

7. A completed ε-δ conception is applied to specific situations

Epsilon-delta conception can be used to solve problems, following three are such examples.

(a) Showing that limit of a function at the point *a* exists and is equal to *L*, formal limit conception is needed,
(b) Showing that limit of sum of two functions that have limit is equal to sum of limit of these functions, formal limit conception is needed,
(c) Showing that limit of a function at a given point is not equal to proposed limit value, formal limit conception is needed.

6 of 25 students showed indications of this step. In the interview, students were given last situation: to show that $\lim_{x \to 2} 2x \neq 5$. Negation is needed to solve this question, and negation is transformation of limit concept. In order to negate limit statement, process conception of formal limit which is called P_3 in (6) is needed to be encapsulated as object: let's call this object as Q_3. In order to respond effectively this situation, it is proposed that students must show two developments: (i) Q_3 is transformed to its negation, let's say $\overline{Q_3}$ and (ii) $\overline{Q_3}$ is properly interpreted. For the latter, in order to interpret $\overline{Q_3}$, it is needed to be de-encapsulated to get its process, let's say P_4. For the former, students used two different kinds of negation methods which are defined (Dubinsky et al., 1988, p.60) as follows:

> *Negation by rules: The most mechanical method of negating a proposition is to express it in formal language and apply rules such as DeMorgan's law from memory.*
>
> *Negation by reasoning: The student has a mental representation of a set of situations that correspond to the statement being true and can then take the complementary set of situations which corresponds to its falsity.*

In the following excerpt, Aysenur used epsilon-delta argument to get a relation between epsilon and delta: $\delta = \varepsilon - 1/2$. She took epsilon equal to 1 that makes delta equal to 0 which is contradiction and enough to conclude that limit is not 5. This is a kind of assimilation of new condition (here it is showing that limit is not 5) into epsilon-delta argument. But assimilation is not a proper way to construct necessary mental structures to handle this new condition. What needed is special reflective abstraction that can be named as encapsulation of process conception of formal limit concept into an object. Following excerpt is such an example.

Aysenur: Let's write $|2x-5|$ as $|2x-4-1|$, then we have $2|x-2-1/2|$. We know that $-\delta < x - 2 < \delta$, then we have $-\delta - 1/2 < x - 2 - 1/2 < \delta - 1/2$. I need to write $x - 2 - 1/2$ within the absolute value, that is, I need to find absolute value of $x - 2 - 1/2$ is less than what. Is it $\delta - 1/2$, no! We can say that $|x-2-1/2| < \delta + 1/2$. Then this (showing $2|x-2-1/2||$) becomes less than $2(\delta + 1/2)$, let's take $2(\delta + 1/2)$ as epsilon.
Then $\delta = \varepsilon/2 - 1/2$. But if epsilon is equal to 1 then delta is 0. Delta must be greater than 0. Thus limit is not 5.

In the following excerpt, Yasemin chose epsilon as ½ and x as 1.8 to make $|2x-5|$ greater than ε=1/2. The reason is that in negation of formal definition it is said that $|2x-5| \geq \varepsilon$. So, for her, this makes negation of formal definition true, which means formal definition is false. Then she says limit cannot be 5. Taking negation of formal limit and concluding that if its negation is correct, formal limit statement must be false might be considered as indications of seeing formal limit concept as object. But after applying negation over formal limit process we get another process which is called P_4 above. What was absent in Yasemin's explanations was this process. She fixed epsilon as 1/2. Now, if process P_4 was correctly interpreted, the thing to do would be to consider fixed delta value and for this delta to find an x value which makes $0 < |x-2| < \delta$ true but $|2x-5| < 1/2$ false. Then it is needed that for another delta, an x value is found, and this is done for all delta values which are greater than 0. However, rather than having this process she uses intuitive idea "if I take an x value which is so close to 2 and is at the left of 2, I feel that its image is not close to 5" to determine only one x value by disregarding delta values.

Yasemin: As x goes to 2, function does not go to 5, but 4. If I approach 2 from the left, function values does not go beyond 4, function values will never approach to 5. Then I choose such an x value that its image does not go into epsilon interval around 5.

I: Let's do then.

Yasemin: Let's choose ε=1/2, then we have the interval (4.5, 5.5). Considering graph of the function, if I take an x value which is so close to 2 and is at the left of 2, I feel that its image is not close to 5. Then let's choose x as 1.8. According to formal definition of limit $|f(x)-L|$ must be less than epsilon, that is. If I input 1.8 into function, I get 3.6. Then this (showing $|3.6-5|$) must be less than epsilon, if limit is 5. But I see that this ($|3.6-5|$) is greater than epsilon. So, I proved that limit is not 5.

I: Well. Considering formal definition of limit what was your plan at the beginning to solve this question?

Yasemin: In the formal definition it is said that for all epsilon there exists delta. I need to find its negation. There exists epsilon for all delta such that $0<|x-2|<\delta$ and $|f(x)-L|\geq\varepsilon$. If I find that $|f(x)-L|$ is greater than epsilon, since in formal definition it is said that $|f(x)-L|<\varepsilon$, I prove that negation of formal definition of limit is correct. That is, limit is not 5.

In the following excerpt, Mehmet used negation by reasoning and started to do so by choosing epsilon as 1/2. Then in order to determine δ for ε=1/2, he found x values whose images are between 4.5 and 5.5: $2.25<x<2.75$. But he figured out that since delta interval should be in the form of (2-δ, 2+δ) and delta must be greater than 0, there is no possible delta interval. Nevertheless, this is not complete negation of formal limit statement.

Mehmet: I will show that limit is not 5. If we look at the graph (he draws graph of $f(x)=2x$), we see that its limit is 4, but you said it is 5. Let's choose epsilon equal to 1/2. If we look at the x values whose images are between 4.5 and 5.5 we get an interval which is at the right of 2. This is the reason why I choose epsilon as 1/2. Now in the formal definition of limit it is said that if x is between 2-δ and 2+δ then function must me between 5-ε and 5+ε. Our function is 2x and we choose epsilon as 1/2, then $4.5 < 2x < 5.5$. Let's look for x, by dividing both sides with 2, we get $2.25 < x < 2.75$. Now we have two inequalities, $2.25 < x < 2.75$ and $2-\delta < x < 2+\delta$. There is no delta value such that both of these two inequalities are satisfied, because delta must be greater than 0.

Following two excerpts are good examples for understanding of formal limit concept. In the first example, Erhan used negation by reasoning. Just like Mehmet, he started to reason on quantifications over epsilon and delta, and Erhan chose ε=1/2 and then he considered x values whose images are between 9/2 and 11/2: $9/4 < x < 11/4$. But unlike Mehmet, he took an x_0 between 2-δ and 2. Fixing epsilon as 1/2 among possible epsilon values might be considered as starting to interpret P_4, and choosing an x_0 might be an indication for negation of quantification over x. Following these, he figured out that x_0 must be between 9/4 and 11/4, which is impossible for any delta value since intersection of (2-δ, 2) and (9/4, 11/4) is empty. Then he specifically chose x_0 as 2-δ/2. 2-δ/2 which is always between 2-δ and 2 but f(2-δ/2) is not in (5-1/2=4.5, 5.5= 5+1/2) for any delta value. By doing so he fixed an epsilon and delta. Variable x is iterated and statement "x_0 is in (2-δ, 2+δ), but $f(x_0)$ is not in (5-ε, 5+ε)" is evaluated at each point then iteration is controlled by existential quantification. This might suggest us two things: one is that he completed the transformation of Q_3 into its negation $\overline{Q_3}$ and the other is that he continued to interpret $\overline{Q_3}$ correctly. Moreover, he said “there is no delta value that satisfies this”. This might be an indication for iteration over delta, and for controlling this iteration with universal quantification. And lastly, choosing one epsilon among possible epsilon values might be an indication for iteration over epsilon and controlling it with existential quantification which means that he completed interpretation of $\overline{Q_3}$ correctly.

Erhan: If limit of it is 5, in the line of definition of limit, for given any epsilon interval I can find at least one delta interval. If it was correct this would be satisfied for all epsilon values, for all epsilon values I could find a delta. But I know that it is false, actually I guess it is false. And in order to show that it is false, I choose an epsilon. If for this epsilon I cannot find any delta, then I show that limit is not 5. For example let’s choose ε=1/2. In the light of limit definition, if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$. Then $|2x-5|$ is less than 1/2. Let’s find the interval: $-1/2<2x-5<1/2$, $9/2<2x<11/2$, $9/4<x<11/4$. For epsilon 1/2 the delta that we need to find should be in the following form: $0<|x-2|<\delta$, that is, $-\delta<x-2<\delta$, $2-\delta<x<2+\delta$. Now If I choose an x_0 value… I want to choose x_0 between 2-δ and 2. For epsilon, the x_0 that I choose should be satisfied, that is, it should go between 4.5 and 5.5. I can take x_0 between 2- δ and 2. For ε=1/2 whatever delta value I choose, x_0 should be between 9/4and 11/4. But whatever delta I take, this condition is not satisfied for x_0 that I choose. Since delta is greater than 0 and $2-\delta<x_0<2$, x_0 is always less than 2. We can take x_0 as 2- δ/2 which is always between 2- δ and 2. Whatever delta value I choose, x_0 is never between 9/4 and 11/4. I can input x_0 value here (showing $|2x-5|<1/2$): $|2(4-\delta/2)-5|<1/2$, $|-1-\delta|<1/2$, since delta is greater than 0 we can take -1- δ out of absolute value as 1+ δ. Then we have $\delta<-1/2$. This is contradiction, since in the light of limit definition delta must be greater than 0.

Erhan assumed that limit of $2x$ at the point 2 is 5, then by negating limit statement and interpreting negation of limit statement simultaneously he got contradiction to conclude that what is assumed at the outset is false. In the last example, Yasir employed a different strategy. He first negated limit statement and then showed that negation of limit statement is true which implies that limit statement is false. Thus limit is not 5. He started to negate statement from quantification part and at the end he negated implication. He made universal quantification existential, existential universal and used equivalent form of negation of implication to get $\overline{Q_3}$. This is an example for negation by rules. Then he took epsilon 1/2 and x=2-(δ/3) which is always in the delta interval for any delta. Since x=2-(δ/3) satisfies $0<|x-a|<\delta$ and $|2x-5|\geq 1/2$ for all delta values, $\overline{Q_3}$ is true. So Q_3 must be false which implies that limit is not 5. This solution is an indication for correct interpretation of $\overline{Q_3}$, but to get more information interviewer asked him to explain all these by using a graph. Yasir fixed epsilon to 1/2 and determined epsilon interval: (4.5, 5.5). Then by choosing x=2-(δ/3) he guaranteed that x is in the delta interval for all delta values. He fixed delta to 3 which means x is 1 and $f(x)$ is 2. And he checked whether $f(1)$ is in the epsilon interval or not. Finding one x value for fixed epsilon and delta values is an indication for existential quantification over x. Then he iterated over delta by taking two more delta values and checked whether $f(x)$ is in the epsilon interval or not for fixed epsilon and for each delta iteration. Moreover by saying that "we can choose delta any number which is greater than 0" he controlled this iteration with universal quantification. After interviewer's prompt he explained why he chose epsilon 1/2. He, first, restricted epsilon values to (0, 1) interval by using properties of function f. Secondly, by saying that "for epsilon, for example, 1/2, 1/3, 1/5 can be taken. But I am looking for only one epsilon value.", he iterated over epsilon values and controlled this iteration with existential quantification.

Yasir: I think this is false. If I show that negation of limit statement is true, then I show that limit statement is false. If limit was 5 we would say that for all ε>0 there exists a δ>0 such that for all x, $0<|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$. Let's negate this statement:

there exists an epsilon>0 such that for all delta>0 there exists an x in the domain of f, - not (p implies q) is equivalent to (p and not q) then- $0<|x-a|<\delta$ and $|f(x)-L|\geq\varepsilon$. Now, since one epsilon is enough, I take epsilon 1/2. And for given any delta, I need to choose such x values that they must be in the delta interval. Then, I choose x=2-(δ/3) for all delta. Since 2-(δ/3) is always in the delta interval, $0<|x-a|<\delta$ is satisfied (interviewer intervenes)

I: What is a?

Yasir: a is equal to 2.

I: What about $f(x)$?

Yasir: It is 2x, and L is 5. Then we have $|2(2-\delta/3)-5|\geq\varepsilon=1/2$. Then we get $|1+2(\delta/3)|$. Since the expression within the absolute value is positive, we can take it out as $1+2(\delta/3)$. Then, $1+2(\delta/3)\geq 1/2$. This is true for all delta values, since delta is greater than 0. So, since its negation is correct, limit statement must be false. And then limit of 2x is not 5 at the point 2.

I: Well. Can you explain all this by using a graph?

Yasir: Okay. (he draws graph of y=2x correctly). Since we choose ε=1/2, epsilon interval is (4.5, 5.5). We choose x=2-(δ/3). x values are always in the delta interval for all delta values. If I take delta 3, x will be 1, and it is in the delta interval. Whatever delta I take, that is for all delta (interviewer intervenes)

I: Can you exemplify these delta values?

Yasir: If I take delta=3, then x is 2-(3/3) =1. $f(1)=2$, but 2 is not in (4.5, 5.5). Choose $\delta=6$, x is 2-(6/3)=0, $f(0)=0$, again 0 is not in (4.5, 5.5). Or let's choose $\delta=30$, 2-10=-8, $f(-8)=-16$, again this is not in (4.5, 5.5). In this way, we can choose delta any number which is greater than 0. The x values that I choose are always in the delta interval, but image of x is never in (4.5, 5.5), that is, it is never in the epsilon neighborhood of L.

I: Why did you choose $\varepsilon=1/2$?

Yasir: I can take any value less than 1 for epsilon, for example, 1/2, 1/3, 1/5 can be taken. But I am looking for only one epsilon value.

I: Can you explain why you can take any value less than 1 for epsilon?

Yasir: If I take epsilon less than 1, lower bound of epsilon neighborhood of 5 is always greater than 4. Since delta is greater than 0, 2-(δ/3) is always less than 2, and the x that I choose is always at the left of 2. Then image of it is always less than 4. Thus I guarantee that image of the x is not in the epsilon neighborhood of 5.

4.2 Students' Difficulties in Transition from Informal to Formal Understanding

In this section following sub-question will be considered:

1.3 What kind of difficulties do students encounter in transition from informal understanding to formal understanding of limit of a function?

In the previous section, students' understanding of limit of a function was explored with the help of genetic decomposition. Student responses under genetic decomposition proposed some difficulties that students encounter in transition from informal to formal understanding of limit. In this section, these difficulties will be explored explicitly by considering students' responses in the interview. Since they are representative, excerpts used in the previous section will also be used in this section.

Reverse Thinking Process

In informal models of limit of a sequence, students first think of index and corresponding term for this index. Then, they consider the difference between each term and the limit value as index increases to infinity. But, in formal definition students are required to, first, consider the error bound, then, proper index for the given error bound. Roh (2007) calls this transition as reverse thinking process. Reverse thinking process is necessary in the case of limit of a function. One of the difficulties that some students had, in transition from informal to formal understanding of limit of a function, is this reverse thinking process. Some of the students in this study could not achieve reverse thinking process.

In the following excerpt Veli makes a discussion about what the formal definition of limit means by considering $f(x)=5x$ at the point 2. Rather than reconstructing his informal understanding of limit concept he assimilated formal limit concept into informal limit schema. This shows us that he was not able to achieve reverse thinking process.

Veli: I say that from here to 2, I started to give values starting from $2-\delta$ and I started to give values that go to 2 and this goes towards 10 from $10-\varepsilon$. And then I started to give values starting from $2+\delta$, I see that this goes towards 10 from $10+\varepsilon$.

In the following excerpt, Sude considered first epsilon then delta. This shows us that she achieved reverse thinking process

Sude: For epsilon, we will find a delta interval, $(a-\delta, a+\delta)$. For epsilon interval here, I will find a delta interval here. That is, in the epsilon interval, or let's say in this way, all values that I take in the delta interval will go into epsilon interval, go into epsilon interval here. That is, $f(x)$ values of the all values in the delta interval will go into epsilon interval.

Weak Understanding of Quantifiers

Cornu (1991) and Cottrill et al. (1996) ascribed students' difficulty in understanding formal limit concept to their weak understanding of the concept quantifiers. Bloch (2000), also, took the attention on quantifiers. "Many statements of theorem in mathematics have quantifiers in them, sometimes multiple quantifiers. The importance of the quantifiers in the rigorous proofs cannot be overestimated. From the author's experience teaching undergraduate mathematics courses, confusion arising out of either the misunderstanding of quantifiers in complicated definitions and theorems, or the ignoring quantifiers when writing proofs, is the single largest cause of the problems for students who are learning to construct proofs" (p. 42).

Another difficulty, detected in this study in transition from informal to formal understanding of limit, was caused by weak conception of quantifiers. Following excerpt is such an example. In explaining what formal definition of limit means to her, Alev considered fixed epsilon and delta values. She did not show any iteration over epsilon and/or delta in practice. Moreover, she put relation between epsilon and delta incorrectly by saying "for all delta there must be an epsilon". She had difficulty in explaining quantifiers in the formal definition of limit. This might be because of her weak conception of quantifiers.

Alev: $f(x)$ has a value at the point a, has a value at the point a-δ, has a value at the point a+δ. These values are at the certain distance at $f(x)$. I call this interval as epsilon. If I take $f(x)$ as L, it becomes ε+L and L-ε. That is, if I choose x in the (a-δ, a+δ) interval, my $f(x)$ values will be in the (L-ε, L+ε) interval. This gives the formal definition. So, if limit of $f(x)$ is L, for increasing or decreasing x values that I take, $f(x)$ will take values like $L+\varepsilon$ or $L-\varepsilon$. But here we have a requirement that for all delta there must be an epsilon. So, the values that I take in this interval, that is, all values that I take in the ($a-\delta$, $a+\delta$) interval will be at the epsilon distance at $f(x)$.

In the following excerpt, Sude believed that consideration of only one epsilon satisfies what is said in formal definition. In other words, she took phrase "for any epsilon" or "for all epsilon" as "for one epsilon". This suggests us that she had difficulty in three-level quantification in the context of formal conception of limit.

I: Well, you considered an epsilon interval.

Sude: Yes.

I: You find a delta interval for it. Then the values you take in the delta interval (Sude intervenes)

Sude: All values will go into epsilon interval.

I: Well. Is it enough to show this for one epsilon value? For example, let's say epsilon 1/2 and you find delta 1/4, can you say that limit exists?

Sude: I can.

I: So, for an epsilon finding a delta is enough to say that limit exists.

Sude: Yes, it is enough. Otherwise it does not satisfy definition.

4.3 Change in Students' Understanding after the Instruction

Limit questionnaire was administered both as a pretest and posttest. Responses of students in the pretest and posttest and their difference is addressed in this section to address following research problem.

2. How different is students' understanding of limit of a function after the instruction based on APOS theory?

4.3.1 Analysis of Responses to Item 1 in the Limit Questionnaire

In first item of the limit questionnaire, students were asked to describe what limit means to them rather than definition of limit concept. So, the aim was to address students' concept images of limit concept. First item of the limit questionnaire was the following.

I-1. Describe in your own words what it means to say that the limit of the function f at the point a is L.

This item yielded qualitative data. The responses to first item were group in three categories: informal, formal, and others. Informal response is related with informal conception of limit, formal response is related with formal conception of limit, and remaining responses was categorized as others. Informal category has three subcategories: pre-action, action, and process. Responses, including evaluation of only one point to determine limit, were put under pre-action. Responses, in which several values used to determine limit, were categorized as action. Lastly responses, including dynamism in domain, or in range, or in both in determination or description of limit, were put under process. Moreover, formal category has three sub-categories: incorrect, lack of quantifiers, and correct. Responses, that stated formal limit definition incorrectly, were categorized under incorrect. Responses including lack of quantifiers over one (or more) of x, δ, ε, were categorized as lack of quantifiers. And lastly, responses, that stated formal limit definition correctly, were categorized under correct.

4.3.1.1 Pretest Results

1) Category "Others"

Examples to this category included the following responses:

1. The slope of the function is equal to L at the point a.
2. There is a function and this function has a point which has limits.
3. It is a kind of boundary. It is used for finding result of $f(x)$ function for limited values.

In addition to above responses, seven of 25 students gave no response to first item, and one student graphed two functions, first one had removable discontinuity and the second one had jump discontinuity at the point a. She then said that since left and right limit is equal for first one, it has limit. But since left and right limit are not equal for the second one, limit does not exist.

As a result, seven of 25 students left first item blank, two of 25 students mentioned about derivative instead of limit (first response above), one of the students gave self referential answer (second response above), one of the students saw limit as boundary but without clear explanation (third response above), and lastly one of the students determined two conditions, namely, removable discontinuity and jump discontinuity at the point a, and concluded that for the former condition there is a limit, but for the latter there is not. Thus, totally twelve of 25 were categorized under "Others".

2) Category "Informal"

Under this category we have three subcategories: pre-action, action, and process. Nine of 25 responses were categorized under pre-action. Following four were examples for pre-action subcategory.

1. At the point a, the value of the function *f* is equal to *L* or it is very close to *L*.

2. $f(a) = L$

3. If we replace *x* with *a* in function *f*(*x*), the result is *L*.

4. When a number which is very close to *a* is placed in the function *f*, the result is very close to *L*.

Only one response was categorized as action, which is as follows.

1. Numbers that are slightly less than *a* correspond *L*, numbers that are slightly greater than *a* correspond *L*.

There were three responses which were categorized under process. These were as follows:

1. Closest points to *a* in x axis gets close to *L* in y axis under the function *f*. It does not matter if *a* refers to *L*, but, both left and right of *a* should be *getting closer* to *L*.

2. When we *come to* the point *a* from right and left we get the same value for *f*.

3. *L* is the nearest value, whether *f*(*a*) is equal to or not, that the function can *reach*.

3) Category "Formal"

In high school curriculum in Turkey, only informal limit conception is covered. Students need to wait to deal with formal limit conception until university education. So, in the pretest students were not expected to give responses under formal category. Parallel to this expectation, no students used formal ideas to describe what limit is in item one of the questionnaire. Thus no response was categorized as formal in pretest.

4.3.1.2 Posttest Results

1) Category "Others"

Only one response was put under category "Others" in posttest. It was the following:

1. It is about how function behaves at near a and is not interest in value of $f(x)$ at a.

2) Category "Informal"

Under this category, three sub-categories were determined in pretest: pre-action, action, and process. Nevertheless, in posttest only one of these sub-categories was observed, namely, process sub-category under which thirteen responses was categorized. Followings are examples for process sub-category:

1. When x approaches to a from right and left side, the value of function approaches to L.

2. When we approach from left and right of a, value which $f(x)$ approaches is the limit.

3. When the variable of function *f* approaches to point *a* from left and right, the function approaches to the point *L*.

4. In *f* function, when *x* approaches to *a* on both sides, *f*(*x*) approaches to *L*.

In addition to above responses in this sub-category, there was one different type including dynamism but within delta and epsilon intervals. In this type, two responses were observed in which students assimilated delta and epsilon intervals into informal conception of limit in a way that *x* approaches to *a* in the interval $(a - \delta,\ a + \delta),\ x \neq a$, and correspondingly *f*(*x*) approaches to *L* in the interval $(L - \varepsilon,\ L + \varepsilon)$. Following is an example for such response.

1. When *x* approaches to *a* in the interval $(a - \delta,\ a + \delta),\ x \neq a$, *f*(*x*) approaches to *L* in the interval $(L - \varepsilon,\ L + \varepsilon)$.

3) Category "Formal"

Responses, including formal definition of limit, were included in this category. Formal category has three sub-categories, namely, incorrect, lack of quantifiers, and correct. There was one response that stated limit incorrectly as follows:

1. For given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

There was another response that did not include quantifiers over ε and δ of the formal definition but the remaining without excluding point *a* in the domain for the definition. It was as follows:

1. Every point in $(a - \delta,\ a + \delta)$ goes to $(L - \varepsilon,\ L + \varepsilon)$.

Remaining nine students correctly stated formal definition of limit. Followings are examples:

1. $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D(f)$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

2. For given any $\varepsilon > 0$, we can find $\delta > 0$ such that
   if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

3. $\lim_{x \to a} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

4. $\lim_{x \to a} f(x) = L$ means that $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $\forall x \in D(f)$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

4.3.1.3 Comparison of results

Three categories were determined in the analysis of first item of limit questionnaire. These were Informal, Formal, and Others. Others was mainly comprised of incorrect responses, no response, and irrelevant response. In pretest, there were twelve responses under the Others category, whereas in posttest there were only 1 response.

The second category was Informal. This category included three subcategories, namely, pre-action, action, and process. In pre-action, responses included evaluation of only one value (either *a* or a number close to *a*) to determine limit of the function at the point *a*. In action, responses included evaluation of several values close to *a* to determine limit of the function at the point *a*. In process, responses included dynamism in the domain, or in the range, or in both in the consideration of informal limit. In pretest, nine of 25 responses were categorized under pre-action, one response was categorized under action, and three responses were categorized under process conception. Whereas, in posttest, there were 13 responses, categorized under only process. Moreover, comparing the results under process category of pretest and posttest, it was seen that statements in posttest were more mathematically mature than the statements in pretest.

The last category was Formal. Formal category was composed of three sub-categories, namely, incorrect, lack of quantifiers, and correct. Responses under incorrect was comprised of incorrect statement of formal definition of limit. Responses under lack of quantifiers included statement of formal definition of limit with some of the quantifiers in it omitted. And lastly, correct included responses with correct statement of formal definition. In pretest, no students were detected under Formal category, whereas, in posttest one of 25 responses was categorized under incorrect, one was categorized under lack of quantifiers, and lastly nine responses were categorized under correct. Summary of categories in pretest and posttest was given in the following tables.

Table 4.1: Responses to "what limit means"- before instruction

Others	Informal			Formal			Total Number of students
	Pre-Action	Action	Process	Incorrect	Lack of Quantifiers	Correct	
12	9	1	3	0	0	0	25

Table 4.2: Responses to "what limit means"- after instruction

Others	Informal			Formal			Total number of students
	Pre-Action	Action	Process	Incorrect	Lack of Quantifiers	Correct	
1	0	0	13	1	1	9	25

As seen in the tables, mainly, responses under Others and pre-action of pretest moved to process and correct of posttest. Movement from Others or pre-action to process might be counted for indication of improvement in students' understanding of informal limit concept. Moreover, comparing the results under process category of pretest and posttest, it was seen that statements in posttest were more mathematically mature than the statements in pretest. And lastly, movement from Others or pre-action to correct might be counted for indication of improvement in students' understanding of limit concept.

4.3.2 Analysis of Responses to Item 2 in the Limit Questionnaire

Remaining items of the questionnaire including second item was evaluated with the following rubric adapted from Asiala, Cottrill, Dubinsky, and Schingendorf (1997):

- 0 for empty or irrelevant response,
- 1 for responses that showed some progress toward solution but far from the correct solution,
- 2 for almost correct responses with minor flaws in the solution,
- 3 for totally correct responses.

Second item of the limit questionnaire was about whether a function attains its limit value at the point limit is looked for and at other points. In literature (e.g., Williams, 1991), the mostly cited misconception on limit concept is that "a limit is a number or point the function gets close to but never reaches". The aim of the second question is to check, in practice, whether students hold this misconception or not. Second item of the questionnaire was the following.

I-2. Suppose $\lim_{x \to a} f(x) = L$, is it possible that the value of $f(x)$ is equal to L at a, and at some values of x other than a? Explain your answer.

Responses to this item in the pretest and in the posttest are given in the following table.

Table 4.3: Item-2, Pretest- Posttest

Item-2 Pretest				Item-2 Posttest			
0	1	2	3	0	1	2	3
16	2	5	2	1	1	12	11

Students, whose response graded as 2 or 3, correctly stated "it is possible that the value of $f(x)$ is equal to L at a" and explained why it is the case or gave an example case. In the literature (e.g., Williams, 1991), the reason for the misconception "a limit is a number or point the function gets close to but never reaches" is attributed to the dynamical character or feeling of motion in informal conception of limit. In pretest, only three students were categorized under process, in which dynamical element was evident, as seen in Table 4.1. So, correct examples that students used to explain in pre-test might be because of incorrect reasons. For example, one of the students in pre-test gave a constant function as an example and wrote $\lim_{x \to a} f(x) = f(a) = L = \lim_{x \to b} f(x)$. This was a correct example, but same student explained what limit means to him as $\lim_{x \to a} f(x) = f(a)$. So, seeing limit as evaluation of function at the limit point might cause him to give correct example because of incorrect reasons. Thus, seven responses graded as 2 or 3 in pre-test might be irrelevant to deduce a result about mentioned misconception.

On the other hand, from the responses to item 1 in the posttest and to interview questions, it can be concluded that all of 25 students had strong process conception of informal limit in which mentioned dynamism or movement is evident. So, this question addresses mentioned misconception in the posttest. And as seen in above table only two students' responses were graded as 0 or 1. This shows us that most of the students in the practice of item 2 did not show the mentioned misconception. Then, it might be concluded that instruction might facilitate student understanding of informal limit concept so that they did not show mentioned misconception in the context of item 2. Nevertheless, when drawing conclusions from analysis of responses to this item, it is needed to be careful. In the literature, it is known that students might respond similar questions differently under different situations (Tall, 1991; Vinner, 1991). This has an implication for analysis of responses to item 2 in that students might show mentioned misconception under different situations although they did not show in item 2. To gain full understanding of whether students had this misconception or not, interviewing is more appropriate, but it is beyond the scope of this study. To sum up, 23 students did not show mentioned misconception in item 2 in posttest, and it might be concluded that instruction might facilitate students' understanding of informal limit concept so that they did not show mentioned misconception in the context of item 2.

4.3.3 Analysis of Responses to Item 3 in the Limit Questionnaire

Third item of the limit questionnaire required students to find limit of a function, given in graphical form, at given points. Third item of the questionnaire was the following.

I-3. Let f be a function whose graph is shown below. In each of the following situations find the indicated limit. If not possible, explain why not.

(a) $\lim_{x \to -2} f(x)$ (b) $\lim_{x \to -1} f(x)$ (c) $\lim_{x \to 0} f(x)$ (d) $\lim_{x \to 1} f(x)$

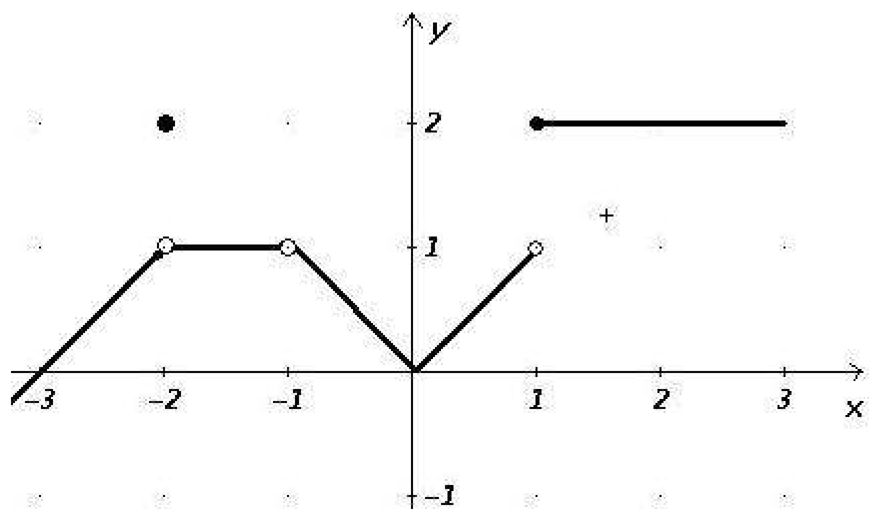

Figure 4.3: Item-3, Given Function

4.3.3.1 Pretest Results

Responses to this item in the pretest are given in the following table.

Table 4.4: Item-3, Pretest

3 (a)				3 (b)				3(c)				3 (d)			
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
12	1	0	12	11	1	0	13	3	2	0	20	9	3	3	10

Analysis of the responses to sub-items 3(a), 3(b), and 3(d) yielded that about twelve of 25 responses were evaluated as fully correct. However, there was an increase in the correct responses for sub-item 3(c); twenty of 25 students solved it correctly. This increase worth mentioning, but, first of all, responses to item 1 and item 3 (including all sub-items) will be compared, then discrepancy between number of correct responses between sub-items of item 3 will be addressed.

In item-1 only three students were categorized under process subcategory. This category is the only category that students showed elements of dynamism, or in APOS terms of process. Responses, under different category and sub-categories, did not show any indication of mentioned dynamism. Rather, these responses either included static evaluation of function at the point a or some points close to a, or were incorrect, confused, left blank. As indication of dynamism in responses does not guarantee that students had process conception of informal limit as mentioned in genetic decomposition, lack of dynamism does not prove that students did not have process conception of limit. Nevertheless, in order to determine limit, evaluation of f at the point a or some points close to a is a strong indication for weak concept images, or in APOS terms weakly formed schemas. Thus, if most of the students had weakly formed schemas, how could they achieve to solve sub-items of item 3? One of the possible explanations is that although they had weak conceptual understanding their procedural understanding helped them to solve these sub-items.

Turning back the discrepancy between number of correct responses between sub-items of item 3, we need to consider those of students' responses to sub-items of item 3. Following four were example responses to item 3. Example N represents the same student under different sub-items, e.g., responses called Example 2 under 3(a), 3(b), 3(c) 3(d) were from the same student.

Responses to 3(a)

Example 1. $\lim_{x\to 2^-} f(x) = 2$, $\lim_{x\to 2^+} f(x) = 2$ so $\lim_{x\to 2} f(x) = 2$

Example 2. $f(-2) = 2$ but at the same time $f(-2)$ is not defined, so limit does not exist.

Example 3. $\lim_{x\to 2} f(x) = 2$

Example 4. Left the question empty.

Responses to 3(b)

Example 1. $\lim_{x\to -1^-} f(x) = 1$, $\lim_{x\to -1^+} f(x) = 1$, and since function is not defined at -1, limit does not exist.

Example 2. Since, $f(-1)$ is not defined, limit does not exist.

Example 3. Left the question empty.

Example 4. Left the question empty.

Responses to 3(c)

Example 1 $\lim_{x\to 0^-} f(x) = 0$, $\lim_{x\to 0^+} f(x) = 0$, so $\lim_{x\to 0} f(x) = 0$.

Example 2. Since $f(0) = 0$, $\lim_{x\to 0} f(x) = 0$

Example 3. $\lim_{x \to 0} f(x) = 0$

Example 4. $\lim_{x \to 0} f(x) = 0$

Responses to 3(d)

Example 1. Limit does not exist.

Example 2. $f(1) = 2$ but at the same time $f(1)$ is not defined, so limit does not exist.

Example 3. $\lim_{x \to 1} f(x) = 2$

Example 4. Left the question empty.

Considering above four examples and others responses in the questionnaire, it can be concluded that in order to determine limit of the function at the point *a* some of the students, first, considered whether function defined at the point *a* or not. If it is defined, most probable strategy is to look for the value of $f(a)$ to determine limit. This might explain why correct responses to 3(c) were more than the correct responses in other sub-items. In 3(c), function was continuous at the given point, so abovementioned students achieved to correctly find limit 3(c) by directly evaluating function at the given point to determine limit. But, since most of them just wrote the solution, e.g., Example 1, 3, 4 in 3(c), they got the full point. On the other hand, in 3(a), 3(b), and 3(d), function was continuous at given points, and in their explanations students tried to evaluate function at given points to determine limit. This leaded to incorrect solutions or correct answers with incorrect explanations, which were assigned to grade 0.

By considering above discussions, it might be concluded that before taking instruction some of the students had and used their procedural understanding to solve questions. On the other hand, some of the students had weak schemas as mentioned others, but they were persistent to use these weak schemas rather than procedures to determine limit of a function given in the graphical form.

4.3.3.2 Posttest Results

Responses to third item in the posttest are given in the following table.

Table 4.5: Item-3, Posttest

3 (a)				3 (b)				3(c)				3 (d)			
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0	0	25	1	0	0	24	0	0	0	25	0	0	0	25

As seen in the above table, almost all of the students gave correct responses to all four items. This might indicate that instruction helped students to improve their performance in the questions where function is given in the graphical form.

4.3.4 Analysis of Responses to Item 4 in the Limit Questionnaire

Fourth item of the limit questionnaire required students to find limit of a function, given in algebraic formulas, at given points. This item included four sub-items as follows:

I-4. Do the following limits exist? If yes, explain why. If no, explain why not.

(a) $\lim_{x\to 0} x\cos\frac{1}{x}$

(b) $\lim_{x\to 0}\frac{|x^2-4|}{x+2}$

(c) $\lim_{x\to 1} f(x)$, where $f(x)=\begin{Bmatrix} \frac{x}{2}, x\neq 1 \\ 1, x=1 \end{Bmatrix}$

(d) $\lim_{x\to\infty}\sqrt{x^2+x-1}-x$

4.3.4.1 Pretest Results

Analysis of responses to this item in the pretest is given in the following table.

Table 4.6: Item-4, Pretest

4 (a)				4 (b)				4 (c)				4 (d)			
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
25	0	0	0	7	12	0	6	18	1	1	5	24	0	0	1

As seen in the above table, none of the students' responses was graded greater than equal to 2 in the item 4(a), similarly in item 4 (d), only one response graded as 3 whereas all the other responses graded as 0. In 4(b), six of 25 responses were graded as 3 and nineteen responses were graded less than 1. Similar to 4(b), in 4(c) five responses were graded as 3 and eighteen responses were graded as 0. Different from other sub-items, in 4(b), it is seen that almost half of the responses were graded as 1. This needs to be more closely addressed. Most of the responses graded as 1 included the one of the following reasoning:

1. $\lim\limits_{x\to 0}\dfrac{|x^2-4|}{x+2}=\dfrac{|x^2-4|}{x+2}=\dfrac{|0^2-4|}{0+2}=2$
2. $\lim\limits_{x\to 0}\dfrac{|x^2-4|}{x+2}=\dfrac{|0^2-4|}{0+2}=2$.

In both type of responses, it is seen that students evaluated function at the point 0 to find its limit at the point 0. This is especially evident in response type 1, in which student equated limit with the function, and then evaluated function at the given point. In response type 2, if students point out that function is continuous at the point 2, so it is legitimate to evaluate function at the point 0 to determine limit, then they would be graded as 3. However, none of them mentioned about continuity of the function, but directly evaluated function at the point 2. So, response types one and two were compatible with how students described limit and how they evaluated limit in the previous item. Thus, this might be taken as another indication for weak schemas included static evaluation of function.

4.3.4.2 Posttest Results

Analysis of responses to this item in the posttest is given in the following table.

Table 4.7: Item-4, Posttest

4 (a)				4 (b)				4 (c)				4 (d)			
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
4	0	10	11	7	1	1	16	2	0	0	23	9	0	5	11

In all of the sub-items, there was an increase in the totally correct or almost correct responses compared to pretest. Twenty-one of 25 responses were graded as greater than or equal to 2 in 4(a). Similarly, in 4 (c), twenty-three of the responses were graded as 3. Seventeen of 25 responses were graded as greater than or equal to 2 in 4(b). Similarly, in 4(d), sixteen of the responses were graded as greater than or equal to 2. Increase in the totally correct or almost correct response rates is evident if we compare pretest and posttest results. This increase might be attributed to effectiveness of the instruction. However there are some points in the posttest results whose consideration might be helpful for gaining more information students' understanding.

More information about students' understanding can be gained by comparing the posttest results of 4(b) and 4(c) with 3(c) and 3 (a) correspondingly. Although represented in different forms, namely graphical and algebraic, students were asked about same mathematical phenomena in 4(b) and 3(c). In both of 4(b) and 3(c) students needed to consider limit of a function at the point where function is continuous. Similarly, in 4 (c) and 3(a) students were asked about same mathematical phenomena under different representations. In both of the sub-items, students needed to consider limit of a function at the point where function has removable discontinuity. Considering the responses to 4(c) and 3(a), we see that all of the 25 responses were graded as 3 in 3(a), similarly twenty-three of the responses were graded as 3 in 4(c). Nevertheless, if we look at results of 4(b) and 3(c) there is no such parallelism as in 4(c) and 3(a). In 3(c) all of the 25 responses were graded as 3, whereas, 16 responses were graded as 3 and one response was graded as 2 in 4(b). To dwell more into this discrepancy, seven responses that were graded as 0 in 4(b) is needed to be addressed. Considering such responses, there was a systematic error in six of seven of them. It was the following:

$$\lim_{x\to 0^+}\frac{|x^2-4|}{x+2}=\lim_{x\to 0^+}\frac{|x-2||x+2|}{x+2}=\lim_{x\to 0^+}x-2=-2$$

$$\lim_{x\to 0^-}\frac{|x^2-4|}{x+2}=\lim_{x\to 0^-}\frac{|x-2||x+2|}{x+2}=\lim_{x\to 0^-}2-x=2$$

Since $\lim_{x\to 0^+} f(x)\neq \lim_{x\to 0^-} f(x)$, limit does not exist.

There might be two possible explanations for this systematic error. One is just misuse of the absolute value function in the solution. The other is that students confused $\lim_{x\to 0}\frac{|x^2-4|}{x+2}$ with the case $\lim_{x\to -2}\frac{|x^2-4|}{x+2}$. In the original question 0 is not a critical point so that consideration of left and right limit produces the same result. Whereas, in the latter limit, -2 is critical point, and consideration of left and right limits are crucial and produces different results so that limit does not exist. It might be the case that students who made mentioned systematic mistake perceived original item as former case.

4.3.5 Analysis of Responses to Item 5 and 6 in the Limit Questionnaire

In this part, analysis of the responses to items 5 and 6 will be done together. But, contrary to questionnaire, in this part firstly analysis of responses to item 6 is reported, and then item 5 is reported. In sixth item, students were asked to prove, by using formal definition of limit, that the limit of the function is the proposed value for a given point.

The aim of this item was to tackle with how students apply formal definition of limit. Sixth item of the questionnaire was the following.

I-6. Use definition of limit to establish followings.

(a) $\lim_{x\to 2} 3x = 6$ (b) $\lim_{x\to 1} x^2 + 5 = 6$

Before discussing results in the posttest, it is needed to consider how grading was done. In responses that were graded as 1, students only considered relationship between epsilon and delta, but did not use this data to prove what was asked. Following was an example for grade 1 for sub-item (a).

Step 1.

For given any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 <| x - 2 |< \delta$ implies $| 3x - 6 |< \varepsilon$,

Step 2.

$$0 <| x-2 |< \delta,\ \ | 3x-6 |= 3 | x-2 |< \varepsilon \text{ so choose } \delta = \frac{\varepsilon}{3}.$$

In responses that were graded as 2, students determined relation between epsilon and delta, and used this to prove what was asked, but in final statement $0<|x-2|<\delta$ implies $|3x-6|<\varepsilon$ is not explicitly given. Following was an example for grade 2 for sub-item (a).

Step 1.

For given any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 <| x - 2 |< \delta$ implies $| 3x - 6 |< \varepsilon$.

Step 2.

$$0 <| x-2 |< \delta,\ \ | 3x-6 |= 3 | x-2 |< \varepsilon \text{ so choose } \delta = \frac{\varepsilon}{3}$$

Step 3.

$$0<|x-2|<\frac{\varepsilon}{3},$$

$$|3x-6|=3|x-2|<3\frac{\varepsilon}{3}=\varepsilon$$

$$\downarrow<\delta$$

In responses that were graded as 3, students determined relation between epsilon and delta, and used this to prove what was asked and final statement is explicitly given. Following was an example for grade 3 for sub-item (a).

Step 1.

For given any $\varepsilon>0$, there exists $\delta>0$ such that $0<|x-2|<\delta$ implies $|3x-6|<\varepsilon$

Step 2.

$$0<|x-2|<\delta,\ |3x-6|=3|x-2|<\varepsilon,\ \text{so choose}\ \delta=\frac{\varepsilon}{3},$$

Step 3.

For given any ε>0, choose $\delta=\frac{\varepsilon}{3}>0$

assume $0<|x-2|<\delta$ then

$$3|x-2|<3\delta=3\frac{\varepsilon}{3}=\varepsilon$$

$$\downarrow<\delta$$

Analysis of responses to item 6 in the pretest and in the posttest is given in the following table.

Table 4.8: Item-6, Pretest- Posttest

6 (a) Pretest				6 (b) Pretest				6(a) Posttest				6(b) Posttest			
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
25	0	0	0	25	0	0	0	0	4	4	17	0	2	9	14

As seen in the above table, in the pretest, all of the responses were graded as 0 for both sub-items (a) and (b). The reason for this was that all of the 25 responses were empty responses. Formal limit concept is not covered in high schools in Turkey, they first encounter formal limit concept at their first semester in the university. So, since pretest was applied before limit instruction, these results were expected. Considering posttest results, twenty-one of 25 responses to sub-item 6(a) were graded as greater than or equal to 2, and similarly, twenty-three of 25 responses to sub-item 6(b) were graded as greater than or equal to 2. This is a clear improvement in student performances.

In the fifth item, students were given a function that has a limit for the given point. Then, students were asked, for given specific epsilon values, whether there is delta value that satisfies formal definition of limit or not. There is a mixture of general and specific in this situation. Specific epsilon value is given, but function and delta values are needed to be considered in a general way. Students can find specific delta values for a specific function and given epsilon values, but may not handle general definition. This was the case found by Pinto and Tall (2002). Fifth item of the questionnaire was the following.

I-5. Assume that $\lim_{x \to 3} f(x) = 5$. Determine whether following statements true or not. Explain your answer.

(a)*One can find a δ_0 such that* $0 < |x-3| < \delta_0 \Rightarrow |f(x)-5| < 0.05$,

(b)*One can find a δ_1 such that* $0 < |x-3| < \delta_1 \Rightarrow |f(x)-5| < 0.001$,

(c)*One can find a δ_2 such that* $0 < |x-3| < \delta_2 \Rightarrow |f(x)-5| < 0.0001$.

Analysis of responses to item 5 in the pretest and in the posttest is given in the following table.

Table 4.9: Item-5, Pretest- Posttest

Item-5 Pretest				Item-5 Posttest			
0	1	2	3	0	1	2	3
25	0	0	0	2	5	5	13

As in item 6, in the pretest all of the 25 responses were graded as 0. This was an expected result, since students first encountered formal limit definition in limit instruction in this study. Considering posttest responses, eighteen of 25 responses were graded as greater than or equal to 2 and 7 of 25 responses were graded as less than or equal to 1. This might be counted as an improvement in student performances related with formal conception. Nevertheless, we can gain more information about students' understanding by considering their responses. When five responses that were graded as 1 considered, it was found that four of them were graded as greater than or equal to 2 in 6(a) and/or 6(b). Then, these four responses were analyzed and following common misinterpretation was found:

Response to 6 (a)

True, because of the formal definition of limit:

For all ε>0 there exists δ>0 such that $0 < |x-3| < \delta_0 \Rightarrow |f(x)-5| < 0.05 = \varepsilon$

In the above response, it is seen that, in the statement, "for all epsilon" and "0.05= ε" were taken together. One of the possible explanations is that for those students "for all" mean "only for one" or "for some". This is parallel to the findings of Roh (2007). In his study, it was stated that for some of the students "for all" meant "for one" or "for some".

In addition to above analysis, one more point should be addressed. Eighteen students gave correct and almost correct response to item 5 whereas this number increased for the item 6. In the fifth item, students were given a function that has a limit for the given point. Then, they were asked, for given specific epsilon values, whether there is delta value that satisfies formal definition of limit or not. There is a mixture of general and specific in this situation. Specific epsilon value is given, but function and delta values are needed to be considered in a general way. In the item 6(b), students were given a function and asked to prove that limit of this function is L at the given point. They were not given a specific epsilon value, needed to consider all epsilon values, and for all these epsilon values they needed to find a delta value. In this sense, item 6(b) is more general than item 5, so students who gave correct response to item 6(b) were also expected to give correct response to item 5. Nevertheless, eighteen of the students gave correct and almost correct answer to item 5 whereas twenty-three of the students gave correct or almost correct answer to item 6(b). To dwell more into this discrepancy, researcher of this study considered five students who gave correct answer to item 6(b), but incorrect answer to item 5. It was found that all of these five students were counted below step 5 of genetic decomposition, which means they were limited to informal conception of limit. So, from conceptual understanding point of view, they were not expected to give correct answers to item 5 and 6(b). Questions like item 6 can be found at the exercise section of most of the calculus text-books, but item 5 is more rarely found in such books. Thus, it is possible that since these five students did not show progress toward formal understanding of limit, or since they were limited to informal understanding of limit concept, they focused on the procedural aspects of the application of formal definition of limit in the questions like item 6.

4.3.6 Analysis of Responses to Item 7 in the Limit Questionnaire

In seventh item, students were asked to prove, by using formal definition of limit, that limit of a function is not the proposed limit value for a given point. In this item, students needed to negate the formal definition of limit. So again, the aim of this item was to address how students apply formal definition of limit as in item 5 and item 6.

Nevertheless, unlike item 5 and 6, in item 7 students need to negate formal definition of limit to solve the question. There are two difficulties about negation. First one is that according to APOS theory in order to apply, conceptually, negation over a statement, students need to encapsulate formal limit definition into object. And in some cases (Dubinsky, Weller, McDonald & Brown, 2005) it is reported that encapsulation of process in to an object is very difficult to handle. So, compared with item 5 and 6, to correctly solve item 7, students need to higher level conceptual knowledge, or in APOS terms, students need to reflectively abstract from a lower level plane to higher one. In addition to mentioned cognitive difficulty, there is another difficulty reported in the literature (Dubinsky et al., 1988) that students need to handle to correctly respond to this item. Dubinsky et al., (1988) showed that students have difficulties in negating a statement, especially in the negation of mathematical implication. So, compared with item 5 and 6, item 7 is more challenging. Seventh item of the questionnaire was the following.

I-7. Prove that the statement $\lim_{x \to 2} 4x = 8.2$ is false by using ε-δ definition of limit.

Analysis of responses to item 7 in the pretest and in the posttest is given in the following table.

Table 4.10: Item-7, Pretest- Posttest

Item-7 Pretest				Item-7 Posttest			
0	1	2	3	0	1	2	3
25	0	0	0	3	15	2	5

As in item 5 and item 6, all of the students left blank item 7 in the pretest. This is expected, since this item is related with formal conception also. As seen in the above table, seven of 25 responses were graded as greater than or equal to 2, whereas, eighteen of the responses were graded less than or equal to 1. Compared to results of item 5 and item 6, there is a reverse tendency. Students' lower performance in item 7 might be because of higher order knowledge needed to be constructed to handle with the question. In addition, consideration of students' responses to item 7 can give more information about the reason for lower performance. Nine of 18 responses, which were graded as less than or equal to 1, included errors in negation of formal limit statement. Three of nine responses included error in the negation of quantifiers. And in all remaining six, students correctly negated quantifiers but negation of implication was not done. Following is a typical example:

$$\exists\varepsilon > 0, \forall\delta > 0, \exists x \in D(f) \text{ such that } 0 < |x-2| < \delta \;\Rightarrow\; |4x-8.2| \geq \varepsilon$$

Moreover, in four of 18 responses, that were graded as less than or equal to 1, it was found that negation of formal statement was done correctly, but in the solution either chosen epsilon or chosen delta was not appropriate. For example, in one of the responses, student chose epsilon as 1 and x as 3 and let delta vary. Choosing x as 3 is not correct according to negation of definition. This might show us that for a fixed epsilon ($\varepsilon = 1$), in the negated statement, "$\forall\delta > 0, \exists x \in D(f)$" was understood by student as only one x value is enough to consider. In another response, a student took epsilon as $\frac{1}{2}$ and x as $\frac{4-\delta}{3}$ to show that $0 < |x-2| < \delta$ and $|4x-8.2| \geq \varepsilon = \frac{1}{2}$. It is correct that $|4(\frac{4-\delta}{3}) - 8.2| \geq \frac{1}{2}$, but choice of x as $\frac{4-\delta}{3}$ does not guarantee that $0 < |x-2| < \delta$.

Lastly in nine of 18 responses, that were graded as less than or equal to 1, it was found that students did not try to negate the formal statement. Rather, they tried to use the truth $\lim_{x\to 2} 4x = 8$ to show that $\lim_{x\to 2} 4x \neq 8.2$.

As a result, majority of the students either incorrectly negated definition or negation was correctly done but it was used incorrectly. Thus, students' lower performance might be attributed to, first, their difficulties in negation of formal statement. Second, in order to apply, conceptually, negation over a statement, students need to encapsulate formal limit definition into object. This requires higher level of conceptual knowledge. It is possible that students who did not have such knowledge incorrectly solved the question.

4.4 Conclusions

In the light of the findings in this research followings can be deduced:

Findings of this research study support the genetic decomposition suggested by Cottrill et al. (1996). Data from students' interviews were compatible with the following description of how students learn limit concept:

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to a.

2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.

3. Construction of a coordinated schema as follows.
(a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
(b) Construction of a range process in which y approaches L.
(c) Coordination of (a), (b) via f.

4. Perform actions on the informal limit concept by talking about, for example, limits of combinations of functions. In this way schema 3 is encapsulated to become an object.

5. Reconstruct the process of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

6. Apply quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.

7. A completed *ε*-*δ* conception applied to specific situations.

Two difficulties of students in transition from informal to formal understanding were determined. First one is about reverse thinking process. In informal models of limit of a sequence, students first think of index and corresponding term for this index. Then, they consider the difference between each term and the limit value as index increases to infinity. But, in formal definition students are required to, first, consider the error bound, then, proper index for the given error bound. Roh (2007) calls this transition as reverse thinking process. Reverse thinking process is necessary in the case of limit of a function. One of the difficulties that some students had, in transition from informal to formal understanding of limit of a function, is this reverse thinking process.

Second difficulty, detected in this study in transition from informal to formal understanding of limit, was caused by weak conception of quantifiers. Some of the students could not coordinate quantifiers in the formal definition of limit.

Before students were given instruction on limit concept, their limit schemas mainly included static evaluation of function at the point *a* or at a close point to *a*. After the instruction, all of the students showed process understanding of informal limit and sixteen of 25 students showed understanding beyond the informal process conception. In addition to this, in the literature (e.g., Williams, 1991), the mostly cited misconception on limit concept is that "a limit is a number or point the function gets close to but never reaches". Twenty-three of 25 students did not show mentioned misconception in the context of item 2 in limit questionnaire. Thus it might be concluded that limit instruction played a positive role in facilitating students' understanding of limit concept.

CHAPTER 5

DISCUSSION, IMPLEMENTATION AND RECOMMENDATIONS

In this chapter the summary of the study, discussion of the results obtained in Chapter 5 implications of the results and some recommendations for further studies was presented.

5.1. Summary of the Study

Before conducting the main study, limit questionnaire, interview protocol, and laboratory activities were prepared. During the summer school of 2008, the semester prior to the actual study, the researcher conducted a pilot study that lasted in five weeks. This study was done in Middle East Technical University (METU) in which 37 students attended classes of MATH153. MATH153 is a calculus course for mathematics majors. Limit questionnaire was administered to all 37 students. Nine of 37 students attended computer laboratory. Lastly, depending on the results of the questionnaire seven students were chosen for the interview. All 7 interviews were audio-taped, and their analysis was done by considering APOS framework (Asiala et al, 1996). Depending on results, necessary changes were made in limit questionnaire, interview protocol, and laboratory activities.

The main aims of this study were to explore how students understand limit concept by using APOS framework and to construct a base for the future studies with the aim of making instruction more effective. The participants, in the main study, included 25 volunteer students who attended Math-153 course in autumn semester of 2008. All students were first year mathematics majors who were taking their first formal course about limit of functions. Prior to instruction all 25 students were given limit questionnaire as pretest. Then students attended five weeks of instruction. In each week they met in two hours computer laboratory to study in groups, and then they attended four hours classical classes. In computer labs they were given some programming activities which give students opportunity to think on limit concept before they are given formal lecture in classes. Researcher of this study participated to laboratory hours as a teaching assistant. At the beginning of computer labs, small discussions (approximately 30 minutes) about last week's activities were done. After computer laboratories, students met in classes. There were four 50 minutes sessions per week. In classes, they studied individually. The content of the class was parallel to content of the lab sessions. In general, 50 minutes divided into 15 minutes discussions and 35 minutes lecture. In 15 minutes discussions, teacher asked questions to be discussed as a class. Questions were either parallel to the lab activities or for about content of lecture. Moreover, instructor let students to ask questions to be discussed. In 35 minutes lecture, instructor was dominant. He used chalk and board to write definitions, prove theorems, and solve problems. After classes, students were given, except first week, question sets as homework to be solved till next week. They studied on questions as a group and they gave the homework individually, that is, although they solved questions as a group, they wrote the solutions in their own words and homework was graded individually.

At the end of the five weeks all of the 25 students were given limit questionnaire as a posttest. Limit questionnaire was used to probe the difference between students' initial understanding of the concept of limit of a function before attending treatment and after attending the treatment. Following posttest, researcher administered interviews, to all 25 students, in which only researcher and interviewee were present. The aim of the interview was to probe students' understanding of the concept of limit of a function. The interviews were conducted in a non-formal friendly environment, in the building of Mathematics Department in which students attended their classes. Interviews were recorded with voice recorder. Approximate interview length was about one hour.

From the questionnaire both quantitative and qualitative results were gained. Analysis of the data gathered from interviews was done according to the framework suggested by Asiala et al. (1996). Findings of this research study support the genetic decomposition suggested by Cottrill et al. (1996). Moreover, it might be concluded that limit instruction played a positive role in facilitating students' understanding of limit concept. Results of the analysis will be discussed more deeply in the next section.

5.2. Discussion of the Results

Notion of limit of a function is fundamental for understanding calculus and the basis of all that follows it. Differentiation and integration, the core of study in calculus, are built on the limit concept. Nevertheless, in literature, it is generally agreed that students have difficulties in understanding limit concept. It is argued that most students have intuitive understanding of limit but very few of them accomplish understanding of the limit definition (Ervynck, 1981; Cottrill et al., 1996).

One of the aims of this study was to explore how students understand limit concept by using APOS framework. Based on their Action, Process, Object, Schema framework, Cottrill et al. (1996, p.9-10) proposed following genetic decomposition about how students learn limit concept.

> 1. The action of evaluating f at a single point x that is considered to be close to, or even equal to a.
>
> 2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
>
> 3. Construction of a coordinated schema as follows.
> (a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.

(b) Construction of a range process in which *y* approaches *L*.

(c) Coordination of (a), (b) via *f*.

4. Perform actions on the informal limit concept by talking about, for example, limits of combinations of functions. In this way schema 3 is encapsulated to become an object.

5. Reconstruct the process of 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

6. Apply quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.

7. A completed *ε*-*δ* conception applied to specific situations.

In Cottrill et al.'s study (1996), steps 1-4 were observed and explained. There were a few students giving indication of step 5. Moreover, data from student interviews did not allow authors to observe and explain steps 6 and 7 in the genetic decomposition. The findings of this research study support first four steps of the genetic decomposition suggested by Cottrill et al. (1996). In addition to this, some of the steps were explained in a more detailed way. For example, in step-3, there were two kinds of process conception. One was process conception as explained by Cottrill et al. and the other was process conception that is seen as a single action. Furthermore, for steps 5-7, data from students' interviews were compatible with the proposed steps of Cottrill et al.

An opposition to APOS theory in general and genetic decomposition of limit concept of Cottrill et al. (1996) in specific came from Pinto and Tall (2001) and Pinto and Tall (2002) respectively. Pinto and Tall (2001) described two different learning styles: one is formal other is natural. "Formal thinkers attempt to base their work on the definitions…

Natural thinkers reconstruct new knowledge from their concept image”. They contended that formal thinkers are compatible with APOS theory, whereas, APOS theory does not explain the way of natural thinkers’ learning. Formal learners’ starting point is concept definition. They build their concept image from formal concept definition by focusing on rules and procedures and by routinising them reflectively. Then, they deductively construct formal theory. On the other hand, natural learners’ starting point is their concept image. They try to assimilate formal theory into their concept image which results in cognitive conflicts. Then, they proceed by making thought experiments to reconstruct their concept images on which formalism is built. Finally, they build formal theory which is integrated with imagery.

What Pinto and Tall (2001) ascribed for APOS theory seems to be conflicting with genetic decomposition constructed by using APOS framework. Related steps of genetic decomposition, with the above discussion, of this study are the followings:

> 3(c) Process Conception of Informal Limit
> 5. Reconstruction of informal limit process conception in terms of intervals or inequalities
> 6. Applying quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.

According to results of interviews and genetic decomposition, students first constructed coordinated process of informal limit. Then, they reconstructed this schema of coordinated process in terms of intervals and inequalities. And then they applied their quantification schema to previously reconstructed process schema. This is quite different from style of formal thinkers of Pinto and Tall (2001). Formal thinkers build their knowledge on formal concept definition. However, according to genetic decomposition, students’ formal understanding of the limit concept builds on their informal process schema. Moreover, formal thinkers of Pinto and Tall (2001) not only build their concept image from formal concept definition but also focus on rules and procedures and tries to routinise them reflectively. This is also quite different from what is suggested by genetic decomposition. According to genetic decomposition of this study, quantification schema is applied to previously reconstructed process schema. The important point is the individual's level of understanding of quantification. Here, if an individual has understanding of three-level quantification schema, he or she might apply directly her/his schema without any attempt for routinization. But, if an individual does not have proper understanding of three-level quantification, she/he might try to focus on rules and procedures and try to routinise them reflectively. Moreover, an individual might have understanding of three-level quantification, but may not apply it properly in the context of formal definition. So, reconsideration of quantification schema might be necessary. Nevertheless, this is another case which is beyond the scope of this discussion.

In their later research (Pinto & Tall, 2002) they gave a specific example, Chris, for natural learning style. In this example, Chris started knowledge building with his concept image and felt difficulty in reversing focus from first x values then y values to first y values then x values. Then, Chris made some thought experiments to reconstruct his image. After successful reconstruction, he built formalism on his modified images to get formal understanding. But in his first attempts, he considered "for all epsilon" as "for some epsilon". And by reconstructing his schema, he came to the correct understanding of "for all epsilon" and of formal limit definition. Considering steps of genetic decomposition, Chris's development is quite compatible with steps 3(c), 5, and 6. In transition from step 3(c) to 5 what students need is reversing focus from first x values then y values to first y values then x values. And in transition from step 5 to 6, students need to coordinate three-level quantification. In this coordination, as seen in the analysis of students responses in the interview, students might have difficulties like Chris. Thus, it seems that style of natural learners is compatible with genetic decomposition of this research.

Some researchers (Williams, 1991; Tall & Vinner, 1981) put a dichotomy between dynamic and static notions of limit. For them, "as x goes to a $f(x)$ goes to L" is a process which includes dynamic feeling of motion. Nevertheless, in formal conception, an individual deals with intervals in which x and $f(x)$ values do not move. So, it is dynamical element in informal limit notion that prevents students to move towards more formal understanding of limit.

An opposition came to the idea that dynamical conception of limit is natural for students and hinders their development toward more formal understanding of limit concept from a group of researchers (Cottrill et al., 1996). They considered dynamical notion as a mental process in APOS terms. Actually this is not a single process, rather is coordination of domain and range processes via function in consideration, thus a schema. Contrary to the belief that process conception is easy to understand, they suggested that coordinated process schema is not easily constructed by students. Moreover, they argued that informal process schema of limit concept is necessary in building formal understanding of limit. Formal understanding of limit concept is built on coordinated process of informal limit, rather than hindered by it. Difficulty in moving from informal understanding to formal understanding comes from students' weak understanding of quantification.

To address this discussion, results of this study can be used. The question here to ask is that how many of the students who showed understanding of formal limit also do possess, dynamic, notion of limit? If we can find some students who showed understanding of formal limit but didn't show dynamic notion of limit, results would be more in line with former camp of researchers who believed that dynamical element in informal limit notion prevents students to move towards more formal understanding of limit. But, if all of the students who showed understanding of formal limit also had the strong dynamical notion of limit, or in APOS terms process conception of informal limit, then the results would be more in line with the second camp of researchers who believed that strong dynamical notion of limit is necessary for understanding of formal limit concept, and it does not hinder students' development toward formal understanding.

According to results of this study all of the students who showed understanding of formal limit concept, also had strong process conception of limit, or strong dynamical notion of limit. Thus, this study supports the conjecture that strong dynamical notion of limit is necessary for understanding of formal limit concept, and it does not hinder students' development toward formal understanding (Cottrill et al., 1996).

In this study, two difficulties of students in transition from informal to formal understanding were determined. First one is about reverse thinking process. In informal models of limit of a sequence, students first think of index and corresponding term for this index. Then, they consider the difference between each term and the limit value as index increases to infinity. But, in formal definition students are required to, first, consider the error bound, then, proper index for the given error bound. Roh (2007) calls this transition as reverse thinking process. Reverse thinking process is necessary in the case of limit of a function. One of the difficulties that some students had, in transition from informal to formal understanding of limit of a function, is this reverse thinking process. This finding is compatible with the findings of Roh (2007) and Swinyard and Lockwood (2007).

Some researchers proposed that the quantifiers "for every, there is such that and whenever" cause difficulty in understanding formal limit concept (Cornu, 1991; Cottrill et al., 1996). Second difficulty, detected in this study in transition from informal to formal understanding of limit, was caused by weak conception of quantifiers. Some of the students could not coordinate quantifiers in the formal definition of limit. Thus, conjecture of Cornu (1991) and Cottrill et al. (1996) was supported by findings of this study.

It is important to describe how students learn a specific subject. Different learning theories suggested different explanations. Depending on Piaget's notion of reflective abstraction, Dubinsky (1991) and his colleagues (Asiala et al., 1996) developed APOS theory. In this research, students' learning of limit concept is described in the form of genetic decomposition. Nevertheless, in order to bring the work on learning to the stage at which it is useful in practice, developing an instruction facilitating meaningful and higher order learning is necessary.

Some of the researchers tried to develop instruction to facilitate students' understanding of the concept. However, summary of related literature showed that most of them failed to help students (Tall & Vinner, 1981; Davis & Vinner, 1986; Sierpinska, 1987; Williams, 1991; Li & Tall, 1993; Monaghan, Sun & Tall, 1994; Cottrill et al., 1996; Buyukkoroglu et al., 2006; Parameswaran, 2007).

It is argued that most students have intuitive understanding of limit but very few of them accomplish understanding of the limit definition (Ervynck, 1981). Tall & Vinner (1981) asked first year university students to write down definition of $\lim_{x \to a} f(x) = l$, if they knew one. Results showed that most of the students who give the dynamical definition were able to state it correctly whereas students who recalled formal definition misstated it.

Table 5.1: Dynamic-Formal Responses

	Correct	Incorrect
Formal	4	14
Dynamic	27	4

Williams (1991) concluded that "… students often considered ease and practicality of a model more important than mathematical formality. This is particularly true in the sense that models of limit that allow them to deal with the realities of limits in the classroom, the kind they see on tests, tend to be seen as sufficient for the purposes of most students. It was noted by several students that neither formal nor dynamic models of limit figure heavily in the procedures students use to work problems from their calculus classes; their procedural knowledge (e.g., substituting values into continuous functions, factoring and canceling, using conjugates, employing L'Hopital's rule) is largely separate from their conceptual knowledge."

In this study, before students were given instruction on limit concept, their responses in the limit questionnaire indicated that students mainly saw the limit of a function as static evaluation of function at the point *a* or at a close point to *a*. After the instruction, all of the students showed process understanding of informal limit and sixteen of 25 students showed understanding beyond the informal process conception and moved to step 5. Although there was no control group in the study, it was seen from students' pretest and posttest limit questionnaire results that the instruction developed by the researcher had a positive effect on students' understanding of limit concept. In addition to this, in the literature (e.g., Williams, 1991), the mostly cited misconception on limit concept is that "a limit is a number or point the function gets close to but never reaches". Twenty-three of 25 students did not show mentioned misconception in the context of item 2 in limit questionnaire. Thus it might be concluded that limit instruction played a positive role in facilitating students' understanding of limit concept.

In this study there was no control group. It is possible that same students might show similar level of understanding in the context of traditional instruction. Nevertheless, in literature it is common that students have weak conceptual understanding of limit concept. Some of these studies showed that they have difficulties in learning informal process conception of limit (Davis & Vinner, 1986; Monaghan, 1991; Bezuidenhout, 2001; Juter, 2003; Buyukkoroglu et al., 2006; Cetin, 2009). Moreover, transition from informal to formal understanding of limit of a function is rarely observed. The researcher of this study made an extensive literature review and was able to found only two such studies.
In the first one of these, Fernandez (2004) decided to take students' viewpoints into consideration for lesson planning in teaching formal limit concept. Students' difficulties in understanding formal definition of limit were detected as "definition contained too much notation and that the need for this notation should be motivated". Then Fernandez

developed two 100-minutes instruction addressing students' difficulties. After the instruction, students' understanding was evaluated with two questions: first one required students to show that limit of a given function is *L*, and for the same function and the same point, last one required students to find appropriate delta for a given specific epsilon. 34 of the 48 students successfully solved first question while 22 of the 8 students responded second question correctly. However, only 15 of the 48 students responded both of the questions correctly. Fernandez concluded that as a result of using more familiar notation, of motivating students for the necessity of formal definition, and of discussing what the formal definition of limit means, students' showed relatively better performance in these two questions. However, we do not have enough evidence for conceptual understanding, since in the literature it is well documented that students, sometimes, may solve questions without having conceptual understanding.

In the second study, Roh (2007) explored students' intuitive understanding of formal limit definition in the case of sequences. In his study, he developed a tool called ε-strip. He used this tool to assess students understanding of the limit concept, as well as help students to develop concept of limit. Results of this study showed that ε-strip is a promising tool in teaching limit of sequences. Nevertheless, this study focused on students' intuitive understanding of formal definition limit of sequences and did not address formal understanding of limit of sequences. Thus, although ε-strip was seen as promising activity to help students to construct intuitive understanding of formal definition of sequences, it is left unquestioned whether ε-strip activity effective in helping students to form formal understanding of limit of sequences.

In contrast to these two studies, literature is abundant of studies reporting low level understanding of formal limit concept (Tall & Vinner, 1981; Sierpinska, 1987; Williams, 1991; Li & Tall, 1993; Monaghan, Sun & Tall, 1994; Cottrill et al., 1996; Parameswaran, 2007). For example, Cottrill et al. (1996, p.17) reported that "Only a few of the students that we observed gave any indication of passing very far beyond the first four steps of this genetic decomposition. In general, they had only the vaguest notion of the standard inequalities involved in the ε-δ definition of limit… There were no students who progressed to the point where we could ask questions that indicated their thinking relevant to the last two steps of the preliminary genetic decomposition". In this study, all of the students achieved to form process understanding of informal limit concept (corresponding to step 3 of genetic decomposition). Moreover, sixteen of 25 students showed progress toward formal understanding of the concept (corresponding to step 5 of genetic decomposition), and ten of 25 students showed indication of understanding of formal limit concept (corresponding to step 6 of genetic decomposition). Lastly, six of the 25 students applied ε-δ conception to show that limit of a function is not L at the given point (corresponding to step 7 of genetic decomposition). Last step of genetic decomposition requires subtle thinking process to deal with formalism. Considering seventh step of genetic decomposition that six of 25 students achieved to construct, in the literature there was no study that address this type of higher order understanding with one exception. The exception was the study conducted by Pinto and Tall (2002). In their study, they reported a student, named Chris, who was able to deal with such formalism in the context of limit of sequences. Thus, although there was no control group in this study, by taking findings of literature into consideration, we might conclude that instruction in this study was effective in helping students to move from informal understanding to formal understanding of limit of a function.

Then next question to ask is that how can we explain this level of understanding. One possible answer is the characteristics of the students. In Turkey there is a university entrance exam, named OSS. After high school, students who want to pursue university education take OSS. According to results of the OSS exam done in 2007, students who chose METU mathematics department were among the top 11.344 students who took OSS. Similarly in 2008, METU mathematics students were among the top 17.853. Thus it is possible that students' success level in limit concept is because of their characteristics.

On the other hand, as mentioned above, this level of success is not common in the literature. So, other possibility for their success level might be the given instruction. Students attended technology integrated cooperative learning environment. In this environment, laboratory activities designed in a way to facilitate students' constructions, that is, one of the primary interests was to help students to develop constructions of the genetic decomposition. Similar studies using APOS theory were done for other mathematical concepts. Weller et al. (2000, p.1) examined such studies and concluded that "instruction based upon APOS theory is an effective tool in helping students to learn mathematical concepts". Thus, more probable explanation of students' success, which is inline with Weller et al.'s conclusion, might be that instruction based on APOS theory help them to learn the limit concept.

In this study, qualitative data gathered through interviews with the aim of probing students' understanding of limit concept, and quantitative data gathered through limit questionnaire in order to explore difference between students' understanding before the instruction and after the instruction. By checking the compatibility of these two data sets, we can gain more information about students' understanding of limit concept. The expected outcome, in this comparison, is that the higher the genetic decomposition level is, the higher the score in limit questionnaire is.

All of the students showed indication of step 3 (called informal limit process) of genetic decomposition, so they were expected to correctly solve item 3 and 4 in the limit questionnaire. By looking questionnaire results, it was seen that almost all of the students solved these items correctly.

Sixteen of 25 students showed indications of step 5 or more. Twenty-one of the responses to item 6(a) and twenty-three of the responses to item 6(b) were graded as greater than or equal to 2, that is, either solution was almost correct or correct. Fifteen of the 25 students who gave correct and almost correct responses to item 6 were counted at the step 5 or more. Moreover, similarly, eighteen of students' responses to item 5 in the questionnaire were graded as greater than or equal to 2. Considering these correct and almost correct eighteen solutions, fourteen of them were the students who were counted at step 5 or more of the genetic decomposition.

Lastly, seven of 25 students gave correct or almost correct responses to item 7. Correspondingly six of these seven students were counted at the step 7 of the genetic decomposition. Thus, it might be concluded that, in general, the higher the genetic decomposition level is, the higher the score in limit questionnaire is.

5.3. Implications

Based on the findings, the following implications can be offered.

1. Students' prior knowledge is very important in the learning of new concepts. Their weak schemas or misconceptions might hinder the acquisition of new knowledge. Instructor should take students' prior knowledge into consideration in designing instruction. Instructors should reflect on students' prior knowledge before the instruction and prepare instruction accordingly.

2. Use of technology is believed to improve active and meaningful learning. Instructional environment in this study included use of computers and was described in detail. Instructors can use this environment as a beginning point to develop effective instructional environments.

3. In this study, specific aim of the computer laboratories was to help students to develop structures defined in genetic decomposition and to create an environment in which students frequently reflect on what they learn. Instructors can use and develop lab activities used in this study to design instruction.

4. In this study, cooperative learning environment integrated with technology was utilized. And after taking instruction, students showed improvement in their understanding of limit concept. Instructors might use similar strategies in designing instruction about other mathematics concepts.

5. It is believed that cooperative learning environments are more effective than the competitive learning environments in students learning of mathematical concepts. Instructors should create classroom environments that facilitate cooperative learning.

6. Students have many obstacles in learning limit concept. This study determined that students have difficulties in reverse thinking process and quantifiers present in the formal limit definition. Instruction which consciously considers mentioned difficulties might assist students to overcome their difficulties.

7. Knowing students' structures and mechanism to construct these structures are important in both sequencing and designing instruction. This study provided genetic decomposition of limit concept. This genetic decomposition can be used in designing instruction.

8. The aim of mathematics education is to enhance meaningful understanding of mathematics concepts. One way to make instruction meaningful is to take students' way of understanding into account. The genetic decomposition suggested by this study might serve to this purpose.

9. One of the sources of the knowledge is textbooks for students. Textbook authors can make use of genetic decomposition in writing books including limit concept.

5.4 Recommendations

Based on the results, the researcher offered the following future studies:

1. Online collaborative learning environment including similar strategies used in the instruction in this study can be constructed, and, effectiveness of it in students' understanding of limit concept and in other constructs, such as motivation and attitude, can be investigated.

2. Implications of APOS theory can be used to design and develop online learning environments. Different aspects related with online learning environments, such as students' satisfaction and motivation, can be further investigated.

3. Mindtools are defined by Jonassen (2000, p.9) as "computer-based tools and learning environments that have been adapted or developed to function as intellectual partners with the learner in order to engage and facilitate critical thinking and higher order learning". Research on mindtools can be combined with genetic decomposition suggested by this research to investigate students understanding.

4. The study can be conducted at different universities in different countries with larger sample size to increase generalization of results.

5. Further studies can be conducted to investigate students' retention about limit concept.

6. Further research can be conducted to explore effect of cooperative learning environment integrated with technology on students' attitudes toward mathematics.

7. Similar studies can be conducted with high school students to examine students' understanding of informal limit concept.

8. Another idea to be investigated is conceptions of high school mathematics teachers' and university teaching assistants' on limit notion and their practice in the class.

9. The effectiveness of instruction used in this study can be compared with traditional instruction by using control group.

10. Similar studies can be conducted to investigate students' understanding in other mathematics concepts by using APOS framework.

11. Findings of this study can be used to investigate students' understanding in other concepts built on limit notion, such as, integration and derivation.

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APPENDIX A

INSTRUCTIONAL OBJECTIVES

At the end of the instruction students will be able to

- State informal definition of limit
- Evaluate limits by using graphs
- Evaluate limits by using tables
- Differentiate limit of a function at a point and value of a function at a point
- Define algebraic laws for limits
- Use limit laws to calculate limits including polynomials, rational functions, algebraic functions, and transcendental functions
- Apply limit to solve real world velocity problems
- Identify indeterminate forms for limits
- Find the limit of functions in indeterminate forms
- Define a one-sided limit
- Evaluate one-sided limits
- Determine the existence of a limit
- Illustrate where a limits does not exist
- Define limit laws for limits involving infinity
- Evaluate limits involving infinity
- Investigate special limits involving the sine and cosine functions
- Write correctly the epsilon-delta definition for the two-sided limit
- Use definition to prove assertions concerning the limits of simple functions
- Use the epsilon-delta definition of limit to prove simple limit statements

- Use definition to prove assertions concerning nonexistence of limit of simple functions
- Use the squeeze theorem to determine the value of simple indeterminate limits

APPENDIX B

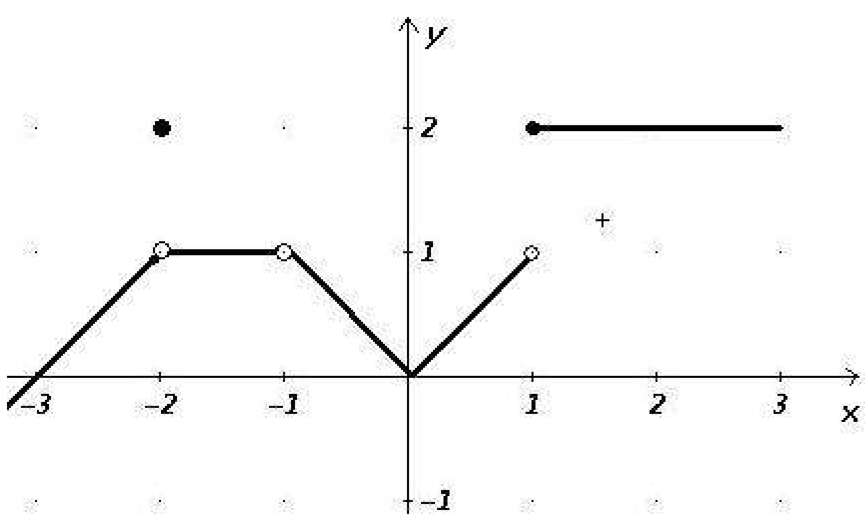
LIMIT QUESTIONNAIRE

1. Describe in your own words what it means to say that the limit of the function *f* at the point *a* is *L*.

2. Suppose $\lim_{x \to a} f(x) = L$, is it possible that the value of *f(x)* is equal to *L* at *a,* and at some values of *x* other than *a*? Explain your answer.

3. Let *f* be a function whose graph is shown below. In each of the following situations find the indicated limit. If not possible, explain why not.

(a) $\lim_{x \to -2} f(x)$ (b) $\lim_{x \to -1} f(x)$ (c) $\lim_{x \to 0} f(x)$ (d) $\lim_{x \to 1} f(x)$



4. Do the following limits exist? If yes, explain why. If no, explain why not.

(a) $\lim_{x\to 0} x\cos\frac{1}{x}$

(b) $\lim_{x\to 0} \frac{|x^2-4|}{x+2}$

(c) $\lim_{x\to 1} f(x)$, where $f(x)=\begin{Bmatrix} \frac{x}{2}, x\neq 1 \\ 1, x=1 \end{Bmatrix}$

(d) $\lim_{x\to\infty} \sqrt{x^2+x-1}-x$

5. Assume that $\lim_{x\to 3} f(x)=5$. Determine whether following statements true or not. Explain your answer.

(a)*One can find a* δ_0 *such that* $0<|x-3|<\delta_0 \Rightarrow |f(x)-5|<0.05$,

(b)*One can find a* δ_1 *such that* $0<|x-3|<\delta_1 \Rightarrow |f(x)-5|<0.001$,

(c)*One can find a* δ_2 *such that* $0<|x-3|<\delta_2 \Rightarrow |f(x)-5|<0.0001$.

6. Use definition of limit to establish followings.

(a) $\lim_{x\to 2} 3x=6$ (b) $\lim_{x\to 1} x^2+5=6$

7. Prove that the statement $\lim_{x\to 2} 4x=8.2$ is false by using ε-δ definition of limit.

APPENDIX C

INTERVIEW SCHEDULE

Research Question: How do students understand limit concept?

Introduction

Hello, my name is İbrahim Çetin, from Computer Education and Instructional Technologies Department. I am here to talk you about your understanding of limit concept. I am interviewing students involved in Math-153 course in this semester in METU. I hope my findings will help instructors who are teaching limit concept. So, I am really interested in your personal thoughts about limit notion. I will ask you some questions about limit concept. Your responses will not be graded. Feel free to answer these questions. What you say to me is completely confidential. Your name will not be used anywhere. Any further questions I can answer? I would like to tape our conversations, Is it OK with you?

Background Questions

1. Sex? : Female ☐ Male ☐
2. How old are you?
3. What type of high school did you graduate from?
4. Where is your high school located?

Questions about Limit Understanding

1. Let *f* be a function. How do you determine limit of the function *f* at the point *a*?

Prompt: Is *a* necessary to find limit?

Evaluation of *f* at the point *a*

Left limit, right limit

2. Let *f* be a function given by *f(x)*=2*x*. How do you determine the limit of this function at the point 3?

Alternative Question: What is meant with the expression "as *x* goes to 3 *f*(*x*) goes to 6"?

3. How do you determine $\lim_{x \to 0^+} \sin(\frac{1}{x})$?

4. Let *f* be a function given by $f(x) = \begin{Bmatrix} 1-x, x \le 1 \\ x, x > 1 \end{Bmatrix}$. How do you determine the limit of this function at the point 1?

Prompt: Is *a* necessary to find limit?

Evaluation of *f* at the point *a*

Left limit, right limit

5. Assume that *f* and *g* have limit at the point *a*. How do you determine limit of *f* + *g* at the point *a*?

6. a) Please state formal definition of limit.

b) By drawing graphics can you explain what is meant by this definition?

Prompt: Relationship between epsilon and delta.

Is one epsilon enough?

7. a) We know that $\lim_{x \to 2} 5x = 10$, how do you prove this by formal definition of limit?

b) Can you explain what you did in part (a) by drawing graphics?

Prompt: Relationship between epsilon and delta.

Is one epsilon enough?

8. Consider the statement $\lim_{x \to 2} 2x = 5$. Determine truth or falsity of the statement. How do you prove your response?

Prompt: How to prove by using formal definition of limit?

APPENDIX D

LAB ACTIVITIES

LAB-1 ACTIVITIES

Name
ID:

1. Use ISETL to make your screen look similar to followings. At the right hand side, there are brief explanations.

Entering Expression In ISETL > 7+10; 17; > 2*5; 10; > 8/4; 2; > (13+14)/(24+12); 0.750000; > 13+14/24+12; 25.583333; > (17+3)*2; 40; > 17+3*2; 23; > 17 >> +3 >> *2 >> ; 23; > 2**3; 8; > sqrt(9); 3.000000; > abs(-4.6);abs(-3+4);abs(0); 4.6; 1;	“>” is ISETL symbol that shows ISETL is ready to accept your code. Each expression must be ended with semicolon “;”. In the left “7+10;” is written and then enter pressed. ISETL responded then answer “17;” “+” is for addition, “–“ is for subtraction, “/” is for division and “*” is for multiplication. “()” has the first precedence, then “*”, then “+”, lastly “-”. That is in an expression, first expression in parentheses will be executed, then “*”, then “/”, then “+”, and lastly “-” will be executed. If you do not end an expression with semicolon “;” and press enter, ISETL will place >> on the screen. This means your command is incomplete. But you can continue to enter commands and end expression with semicolon “;”. To take power “**” is used in ISETL. 2**3 means 2^3 in ISET. To take square root you can use sqrt() function of ISETL. To find absolute value of a number on expression you can use abs() function of ISETL.

0; > sgn(2.7);sgn(-4);sgn(0); 1.000000; -1; 0;	sgn() function in ISETL gives the sign of a number. If number is positive it returns 1, if number is zero it returns 0, if number is negative it returns -1.
> x:=10; > x; 10; > x:=x+20; > x; 30; > x:=x-4;x; 26; > x:=x*2;x; 52; > x:=3; > print x; 3;	You can assign number to variable. On the left 10 is assigned to x. Notice that ":=" is used for assignment, not "=". When you write "x;" ISETL returns the number that you assigned to x. You can make operations with this variable. For example, on the left 20 is added to x and the result is assigned again to x (10+20=30). You do not have to enter only one expression in ISETL row. For example on the left 4 is subtracted from x, and value of x is asked to ISETL. ISETL responds "26;" Another way of learning which number is assigned to variable print command. "print x" returned 3 which was assigned to x before. Notice that each time you assign a number to x, it is content changed.
Tuples in ISETL > p:=[3,-2,5]; > p; [3, -2, 5];	A tuple is a finite set of elements in a specific order in ISETL. On the left "p" is defined as a tuple with first element 3, second element, -2, third element 5. When you write "p;" ISETL returns tuple in the defined order.
> p(1);p(2);p(3);p(4); 3; -2; 5; OM;	You can ask for individual elements of a tuple. For example on the left, first "p(1)", second "p(2)", third "p(3)", and fourth element "p(4)" of tuple "p" is asked. Notice that for "p(4)" ISETL returned "OM;" which means that p(4) is not defined.
> p(1)*p(2)+p(3); -1;	You can make operations with elements of tuple. For example on the left first element is multiplied with second and third is added.
> q:=[1..8]; > q; [1, 2, 3, 4, 5, 6, 7, 8]; > r:=[1,1.1..2]; > r; [1.000000, 1.100000, 1.200000, 1.300000, 1.400000, 1.500000, 1.600000, 1.700000, 1.800000, 1.900000, 2.000000];	"q:=[1..8]" means the tuple, that contains integers from 1 to 8, is assigned to q. This is **important** notation which may be frequently used later. "r:=[1,1.1..2]" means "1.1-1=0.1" is taken and 0.1 is added to 1 until reaching 2. Notice 1.1-1=0.1 is the increment that ISETL uses to reach from first element 1 to last element 2.

> a:=3; > s:=[a+((-1)**n)/(10**n): n in [1..6]]; > s; [2.900000, 3.010000, 2.999000, 3.000100, 2.999990, 3.000001];	3 is assigned to "a". Then tuple "s" is constructed. Elements of tuple "s" are "a+((-1)**n)/(10**n)" where n represents integers from 1 to 6. That is for n=1 to 6 "a+((-1)**n)/(10**n)" is executed and each result added as an element to tuple "s". This is **important** notation which may be used later.
Sets in ISETL > A:={1,3,4.5,7}; > A; {3, 1, 4.5,7}; > A:={true,false}; > A; {false, true}; > B:={1..8}; > B; {7, 8, 5, 6, 4, 3, 2, 1}; > C:={2,4..16}; > C; {8, 6, 4, 2, 10, 12, 14, 16}; > D:={1,1.1..2}; > D; {1.500000, 1.400000, 1.800000, 1.900000, 1.600000, 1.700000, 1.300000, 1.200000, 2.000000, 1.100000, 1.000000};	Sets in ISETL are constructed in a similar way with tuples. But order is not preserved. Notice when you ask for "A" on the left ISETL returned elements in different order. You can assign logical constants "true" and "false" in tuples and sets.

If Statement
 	
> x:=9;
> if x>5 then
>> print "x is greater than 5";
>> end;
"x is greater than 5";
 	
> x:=4;
> if x>5 then
>> print "x is greater than 5";
>> else
>> print "x is smaller than or equal to 5"
>> end;
"x is smaller than or equal to 5;
	General use of If is the following:
	
	if (boolean expression1) then
	statements1;
	elseif (boolean expression2) then
	statements2;
	else
	statements3;
	end;
	
	If boolean expression 1 is true then statements 1 are implemented (statements 2 and 3 omitted) . If boolean expression 1 is not true and boolean expression 2 is true, then statements 2 are implemented (statements 3 omitted). If neither of boolean expression 1

```
>	x:=-5;
>>	if sgn(x)=1 then
>>	print "x is positive";
>>	elseif sgn(x)=0 then
>>	print "x is zero";
>>	else
>>	print "x is negative";
>>	end;
"x is negative";
```

and boolean expression 2 is true, then statements 3 are implemented.

"elseif" and "else" part of the statement is optional. In the first example on the left neither is used. In the second example on the left "elseif" is not used.

For Loop in ISETL

```
>	for i in [1 ..10] do
>>	print i;
>>	end;
1;
2;
3;
4;
5;
6;
7;
8;
9;
10;
```

A loop is designed to do something over and over. In ISETL it always has three components: the directions on what values to examine, the commands on what to do, and the end. Here is the syntax:

for *variable* in *set or tuple* do
some commands telling the computer what to do
end;

First example prints values of "i" which is determined as [1..10]. In the first execution time "i" is equated to 1 and printed. In the second execution time, "i" is equated to 2, and printed…
In the last execution time "i" is equated to 10 and printed. Notice that in each time "i" is equated to one of the numbers in "[1..10]" orderly. And each time commands between "for … do" and "end" are executed.

```
>	p:=[1..9];
>	for i in [2..9] do
>>	print p(i)-p(i-1);
>>	end;
1;
1;
1;
1;
1;
1;
1;
1;
```

"p" is determined as "p:=[1..9];". For loop prints the p(i)-p(i-1) for 8 times. Notice that "i" is assigned numbers [2..9]. In the first execution time, "i" is equated to 2 and p(2)-p(2-1) is executed. In the second execution time, "i" is equated to 3 and p(3)-p(3-1) is executed… In the last execution time, "i" is equated to 9 and p(9)-p(9-1) is executed.

Functions in ISETL > f:=func(x); >> return 2*x-x**3; >> end; > f(3);f(-2);f(0); -21; 4; 0;	A function in ISETL can include four components: the name of the function, the list of input variables, the directions on how to produce an output, and an end. Here is the syntax: *Name* := func(*list of input variables*); *some commands, which MUST include a* return *statement* end;
> f:=func(x); >> if x>0 then return x**2-1; >> elseif x=0 then return 1; >> else return x+1; >> end; >> end; > f(2);f(5);f(0);f(-1);f(-3); 3; 24; 1; 0; -2;	For the first example name of the function is “f”. And “f” takes “x” as a variable and returns $2x\text{-}x^3$. That is, f is defined as $f(x)= 2x-x^3$. For the second example, following piecewise function is defined. $f(x)=\begin{cases} x^2-1, & if\ x>0 \\ 1, & if\ x=0 \\ x+1, & if\ x<0 \end{cases}$
> f:=func(x,y); >> return abs(x-y); >> end; > f(5,3); 2;	For the last example function “f” with two variables is defined. “f” takes “x” and “y” as varibles and returns |x-y|. That is, f is defined as $f(x,y)=\|x-y\|$.
Logic in ISETL > 5=2+3; true; > 4>=2+3; false; > x:=3; > abs(x-2)<1.001; true; > abs(x-2)=sgn(x-2); true;	“:=” and “=” are different in ISETL. “:=” assigns a value to a variable as you see above. But “=” asks the question “is equal”. On the left example “5=2+3;” means “is 5 equal to 2+3”. ISETL returns “true”. Similarly “<” is for “is less than” question, “>” is for “is greater than” question, “<=” is for “is less than or equal” question, “>=” is for “is greater than or equal” question, “/=” is for “is not equal” question.
> a:=true; > b:=false; > not(a);	You can assign “true” and “false” logical constants to a variable. On the left “true” is

<table>
<tr>
<td>false;

> a and b;
false;
> a and not(b);
true;
> a or b;
true
> a impl b;
false;
> a impl not(b);
true;
> not(a impl b);
true;</td>
<td>assigned to “a”, “false” is assigned to “b”. “not(a)” means negation of “a”. Since “a” is true, ISETL returns “not(a)” as false.

Logical connector and is written as “and”, or written as “or”, implies is written as “impl” in ISETL. Examine commands on the left.</td>
</tr>
<tr>
<td>> forall n in [6..9]|n>4;
true;

> exists n in {1,3..25}| n=8;
false;</td>
<td>Universal quantifier, for all, is represented as “forall”, and existential quantifier, there exists, is represented as “exists” in ISETL.
First statement looks for the truth of “for all n in [6..9], n>4”.

Second statement looks for the truth of “there exist n in {1,3..25} such that n=8”.</td>
</tr>
<tr>
<td>Graphing in ISETL

> f:=func(x);
>> return x+1;
>> end;
> plot(f);
> plot(f,a,b);
> plot(f,a,b,c,d);</td>
<td>To graph predetermined function f, write “plot(f)”. (Note that to draw a graph of a function, first, function that will be graphed should be defined). Your graph will be opened in a new window called “Graph Window”.

Note that after your graphing task is complete, you need to place cursor on the graph window and to enter “q” to quit. After each drawing you need to quit for drawing new graph. If you do not quit, you cannot draw a new graph.

To get “execution window” back (the window that you enter command), you need to minimize “Graph Window”.

“plot(f,a,b)” allows you to set the horizontal scale to [a, b] and uses the default scale for the vertical axis. (a; b must be numbers). That is on the x-axis only “[a,b]” portion of graph is</td>
</tr>
</table>

	drawn. "plot(f,a,b,c,d);" allows you to set both the horizontal [a, b] and vertical [c, d] scales. (a; b; c; d must be numbers). That is on the x-axis, "[a,b]" portion of graph is drawn; and on the y-axis, "[c,d]" portion of graph is drawn. To read the coordinates of a point place the cursor to the point, on the upper left of the window coordinates will appear.
Tabling in ISETL > f:=func(x); >> return x**2; >> end; > table(f,1,5,4); x f 1.00000 1.00000 2.00000 4.00000 3.00000 9.00000 4.00000 16.00000 5.00000 25.00000	Syntax of tabling in ISETL is "table(f,a,b,n)". **f** is the name of function you have constructed, **[a,b]** is the interval over which you wish the table to run, and **n** is the number of points, evenly spaced between **a** and **b** at which you wish the function evaluated. On the left example f is defined as $f(x)=x^2$. And, x values that will be evaluated are defined between 1 and 5. Then interval [1,5] is divided in 4 equal parts ([1,2], [2,3], [3,4], [4,5]. And at the each end point 1, 2, 3, 4, 5 function f is evaluated and printed.

Notice: For the below questions, write your code/answer into the space given under the question.

2. Write an ISETL tuple which includes all integers from 5 to 120.

3.

a) In the below, a function h is written in ISETL notation. Express this function in mathematical notation.

```
>     h:=func(x);
>>    if x/=-1 then return x*(x-2)/(x+1);
>>    else return 1;
>>    end;
```

```
>>	end;
```

b) In the below, an ISETL code is written. Explain what the code is doing and execute it in ISETL. (Notice that the function h used in the below code is defined in 3(a)).

```
>	for i in [1..10] do
>>	print h(-1+(1/10**i));
>>	end;
```

4. Write an ISETL function f that accepts x and returns $\frac{x^2-1}{x-1}$. This code will be used in part b.

b) Execute command "table(f,0.99,1.01,20);" in ISETL. ISETL will give you error "`!Error: Divide by zero`". The reason for this error is that table command tries to execute "f(1)". But evaluation of f at 1 makes the denominator of the function zero and ISETL cannot handle division by zero error. Then you must avoid division by zero error by changing your code of function. Test your modified function with table command in ISETL and write function's code below. (Hint: Consider defined function in 3(a))

c) Use the table that you constructed in ISETL in (b) to answer the following:

$f(x)=\frac{x^2-1}{x-1}$, what happens with $f(x)$ if x values is close to 1?

5. Write an ISETL function f that accepts x and returns $\frac{\sin(x)}{x}$. Use command "table(f,-0.99,0.01,20);" to examine function values around x=0. And answer the following.

$f(x)=\frac{\sin(x)}{x}$, what happens with $f(x)$ if x values is close to 0?

6. For following questions, you can use ISETL to plot to observe what happens.

a) Let *f* be defined as $f(x) = x$, what happens with *f*(*x*) if *x* values is close to 3?

b) Let *f* be defined as $f(x) = x + 2$, what happens with *f*(*x*) if *x* values is close to 3?

c) Let *f* be defined as $f(x) = x^2$, what happens with *f*(*x*) if *x* values is close to 3?

d) Let *f* be defined as $f(x) = x^2 + 2$, what happens with *f*(*x*) if *x* values is close to 3?

e) Let *f* be defined as $f(x) = x^3$, what happens with *f*(*x*) if *x* values is close to 3?

f) Let *f* be defined as $f(x) = x^3 + 2$, what happens with *f*(*x*) if *x* values is close to 3?

g) You may easily deduce that "as x values is close to a, then the values of the function *f*, defined by $f(x) = x$, is close to a. By using this knowledge, answer the following question.

Let *f* be defined as $f(x) = x^n + c$ where n is positive integer and c is real, what happens with *f*(x) if *x* values is close to a?

LAB-2 ACTIVITIES

Name:
ID:

There are some activities below. Do them and write your answers to the gaps that are given below the activity.

1. Using ISETL, enter the following statement to begin for all three parts of this activity.

LimitProcess();

After you type "LimitProcess();" some instructions will appear on the screen. Program is asking for you to type one of **P1;**, **P2;**, or **P3;**. After typing one of them a process will start and require you to type **1** followed by pressing enter-key repeatedly. To continue to process, you need to type **1** repeatedly. To stop process, you need to type **0** followed by pressing enter-key. Now follow the below instructions.

(a) Type **P1;**.When prompted, type 1 repeatedly and **note the output each time**. Continue doing this until you feel that you understand what is going on. Write an explanation of what you think is going on. Explain what would happen ultimately if you continued to type 1.

(b) In order to stop above process, type 0 followed by pressing the enter-key. Now you will implement process called P2. Type **LimitProcess();** again, and then type **P2** when asked. When prompted, type 1 repeatedly and **note the output each time**. Continue doing this until you feel that you understand what is going on. Write an explanation of what you think is going on. Explain what would happen ultimately if you continued to type 1. Compare and contrast what happens with the two processes P1 and P2.

(c) Repeat the above, implementing the process P3. For the following discussion, denote by **x** the current value which appears on the screen when you enter 1, and denote by ***a*** the ultimate value that you would get if you entered 1 indefinitely. Answer the following questions as you repeatedly enter 1.

i. Write down your best guess for the value of *a*.

ii. How many times must you enter 1 before it happens that $|x - a| < 0.01$?

iii. Once this inequality is satisfied, does it continue to hold?

iv. How many times must you enter 1 before it appears that the inequality will always hold?

v. Repeat the previous three questions with 0.01 replaced by 0.0001.

2. Let $f(x)= x^3+x+7$ (You need to first construct this function in ISETL to do following questions). Using ISETL, enter the following statement to begin using LimitProcess for all three parts of this activity.

LimitProcess(f);

(a) Follow the instructions on the screen to implement the process called P1. When prompted, type 1 repeatedly and **note the output each time**. Continue doing this until you feel that you understand what is going on.
In order to stop, type 0 followed by pressing the enter-key. Write an explanation of what you think is going on. Explain what would happen ultimately if you continued to type 1.

(b) Repeat the above, implementing the process P2. Write an explanation of what you think is going on. Explain what would happen ultimately if you continued to type 1. Compare and contrast what happens with the two processes P1 and P2.

(c) Repeat the above, implementing the process P3. For the following discussion, denote by **f(x)** the current value which appears on the screen when you enter 1, and denote by ***L*** the ultimate value that you would get if you entered 1 indefinitely. Answer the following questions as you repeatedly enter 1.

i. Write down your best guess for the value of L.

ii. How many times must you enter 1 before it happens that $|f(x) - L| < 0.01$?

iii. Once this inequality is satisfied, does it continue to hold?

iv. How many times must you enter 1 before it appears that the inequality will always hold?

v. Repeat the previous three questions with 0.01 replaced by 0.0001.

3. Let $f(x)= x^2$ (You need to first construct this function to do following questions). Using ISETL, enter the following statement to begin using LimitProcess for all three parts of this activity.

LimitProcess(f,3);

(a) Follow the instructions on the screen to implement the process called P. When prompted, type 1 repeatedly and **note the output each time**. Continue doing this until you feel that you understand what is going on.
In order to stop, type 0 followed by pressing the enter-key. Write an explanation of what you
think is going on. Explain what would happen ultimately if you continued to type 1.

b) Repeat the above, implementing the process P. For the following discussion, denote by **x** and **f(x)** the current values which appear on the screen respectively when you enter 1, and denote by ***a*** and ***L*** respectively the ultimate values that you would get for x, f(x) if you entered 1 indefinitely. Answer the following questions as you repeatedly enter 1.

i. Set n=2. How many times must you enter 1 before you have the inequality $|x-a|<10^{-n}$ and this inequality appears as if it will continue to hold. Repeat for the values n=3, 4.

ii. Set k=2. How many times must you enter 1 before you have inequality $|f(x)-L|<10^{-k}$ and this inequality appears as if it will continue to hold. Repeat for values k=3, 4.

iii. Try to fill in the dots so as to make statement (as general as possible) of the form,

If |x-a|<….. then |f(x)-L|<…..

4. Let *f* be a function given by $f(x) = \dfrac{x^2-4}{x-2}$ (You need to first construct this function to do following questions). **Note that** when doing this exercise ISETL may give error "`!Error: Divide by zero`", but you learned how to handle with this error last week.

Using ISETL, enter the following statement to begin using LimitProcess.

LimitProcess(f,2);

Follow the instructions on the screen to implement the process called P. When prompted, type 1 repeatedly and note the output each time. For the following discussion, denote by **x** and **f(x)** the current values which appear on the screen respectively when you enter 1, and denote by ***a*** and ***L*** respectively the ultimate values that you would get if you entered 1 indefinitely. Answer the following questions as you repeatedly enter 1. In order to stop, type 0 followed by pressing the enter-key.

i. Set n=2. How many times must you enter 1 before you have the inequality $|x-a|<10^{-n}$ and this inequality appears as if it will continue to hold? Repeat for the values n=3, 4.

ii. Set k=2. How many times must you enter 1 before you have inequality $|f(x)-L|<10^{-k}$ and this inequality appears as if it will continue to hold? Repeat for values k=3, 4.

iii. Try to fill in the dots so as to make statement (as general as possible) of the form,

If $|x-a|<\ldots..$ then $|f(x)-L|<\ldots..$

LAB-3 ACTIVITIES

Name:
ID:

There are some activities below. Do them and write your answers to the gaps that are given below the activity.

1. The code given below is a computer function named **lim**. This computer function accepts as input any other computer function **f** and number **a**, and returns approximate value of the limit at **a** of the function represented by **f** if this limit exists, or the message "Unable to find limit" if there is no limit (or the computer cannot find an appropriate approximation). Examine the code and answer the questions in this activity.

```
lim:=func(f,a);
s:=[a+((-1)**n)/(10**n): n in [1..6]];
for i in [2..6] do
if abs(f(s(i))-f(s(i-1)))<0.0001 then return f(s(i));
end;
end;
return "Unable to find limit";
end;
```

a) What is the line two of code doing? What are the lines four and seven doing?

b) Use the computer function **lim** to approximate and limits of the functions f, g, and h given by $f(x) = \begin{Bmatrix} x^2+2, x \le 0 \\ x+5, x>0 \end{Bmatrix}$ at a=0, $g(x) = x^2 - 9$ at a=2, $h(x) = \dfrac{x^2-9}{x+3}$ at a=3.

(c) Give two examples of functions and a point *a* such that

i) lim(f,a) returns a number

ii) lim(g,a) returns "Unable to find limit"

d) Explain why does lim(f,a) returns a number, and why does lim(g,a) returns “Unable to find limit”?

2. In this activity you are to create two computer functions named **limadd** and **limprod** respectively (Hint you may use computer function **lim**).

a) **limadd** takes as input two functions, namely **f** and **g**, and a point say, **a**. And it finds approximated value of $\lim_{x\to a}(f+g)(x)$. That is, when **limadd(f,g,a);** is written your program finds approximated value of $\lim_{x\to a}(f+g)(x)$.Write code of your program below.

b) **limprod** takes as input two functions, namely **f** and **g**, and a point say, **a**. And it finds approximated value of $\lim_{x\to a}(f\cdot g)(x)$. That is, when **limprod(f,g,a);** is written your program finds approximated value of $\lim_{x\to a}(f\cdot g)(x)$.Write code of your program below.

c) Use computer functions **lim**, **limadd**, and **limprod** to fill the following table

f(x)	g(x)	a	$\lim\limits_{x\to a} f(x) = ?$	$\lim\limits_{x\to a} g(x) = ?$	$\lim\limits_{x\to a}(f+g)(x) = ?$	$\lim\limits_{x\to a}(f\cdot g)(x) = ?$
x^2	5x	2				
x^2	5x	4				
$\frac{x^2}{x-1}$	$-\frac{1}{x-1}$	1				

3. In this activity, you are to create a computer function named **limatinf** which figures out an approximation to a limit of a function at infinity. This computer function should accept as input any other computer function **f** and should give as output an approximate value of the limit at infinity of the function represented by **f** if this limit exists, or the message "Unable to find limit" if the limit does not exist (or the computer cannot find an appropriate approximation).

a) Write your code below (Hint: Modify second line of **lim** function. Is a necessary in limatinf?)

b) Approximate the each of the following limits by using computer function **limatinf**.

(i) $f(x) = \dfrac{2x^3 - 9}{x^2 + 3} =$

(ii) $f(x) = \dfrac{2x^3 - 2.7x + 5.1}{x^3 + 34.7x^2 + 3.6} =$

(iii) $f(x) = \dfrac{7x^2 - 9x + 27}{5x^3 + 3} =$

(iV) $f(x) = \sin(\dfrac{1}{x}) =$

(V) $f(x) = x\sin(\dfrac{1}{x}) =$

(Vi) $f(x) = \dfrac{\sin(x)}{\sqrt{x}} =$

LAB-4 ACTIVITIES

Name:
ID:

There are some activities below. Do them and write your answers to the gaps that are given below the activity.

1. You have learned how to find limit of a function at a given point in your previous activities. For example, limit of f, defined as $f(x)=x^2$, at the point a=2 is L=4, and you used the notation $\lim_{x\to 2} x^2 = 4$ for this. Now we will use this fact and make some activities that will help you in your formal study of limit.

In this activity you will look at specific portions of graphs and investigate the possibility of keeping them within a given window. You know that $\lim_{x\to 2} x^2 = 4$. We will give you an interval around limit L=4 and ask you for an interval around a=2 such that graph of the function will stay within the given bounds on the vertical interval around L=4.

Let's make an example. Suppose you are given function f denoted by $f(x)=x^2$, and we give you an interval around 4 in the form of **(4-ε, 4+ ε)**, namely for ε = 0.9, (3.1, 4.9). The task is to find an interval around 2 in the form of **(2-δ, 2+ δ)** so that graph of the function will stay within (3.1, 4.9). You can use **plot** command to draw graph of the function, but firstly you have to determine horizontal interval that is asked. Let's choose δ=0.6, then your horizontal interval is (2-0.6, 2+0.6), that is (1.4, 2.6). Now use `plot(f,1.4,2.6,3.1,4.9);` command to plot graph which is shown below.

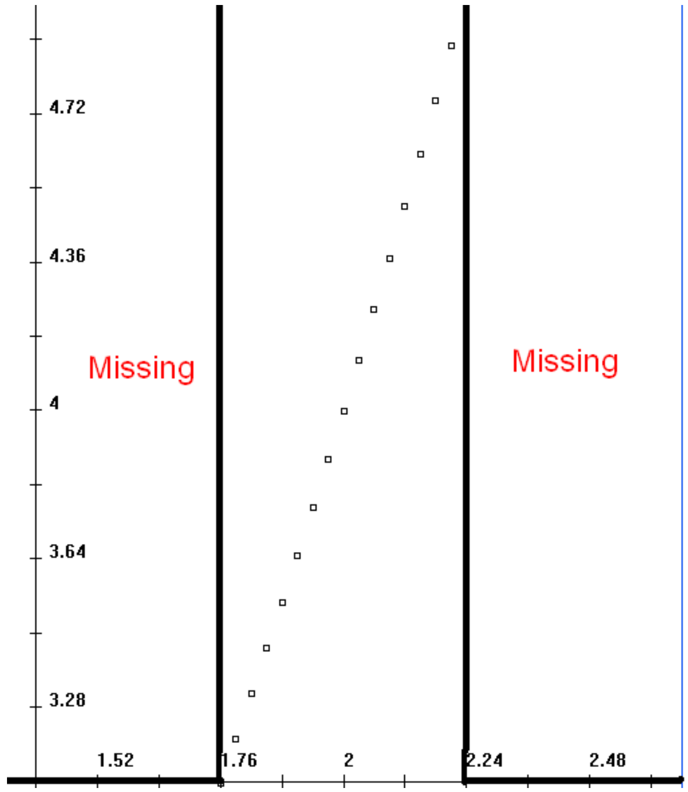


Figure-1

As you have seen in the figure-1, (1.4, 1.76) and (2.24, 2.6) parts of the function is not within the given horizontal interval (3.1, 4.9). But we have necessary information to complete problem now. If we choose horizontal interval within (1.76, 2.24) we are done. Now choose δ=0.2, then your horizontal interval is (2-0.2, 2+0.2), that is, (1.8, 2.2). Now use **`plot(f,1.8,2.2,3.1,4.9);`** command to plot graph which is shown below.

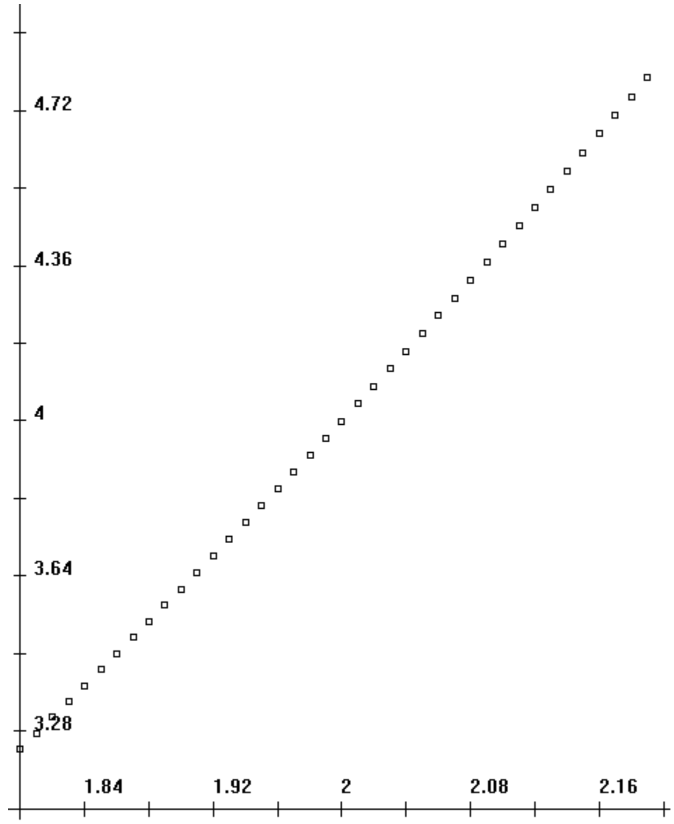


Figure-2

As you have seen figure-2, the curve does not leave the upper or lower boundaries of your window and you can see the entire curve for this interval of the domain variable. However, notice that in general the graph does not have to be within the horizontal interval at the single point where x = a.

Now for the following situations, determine the possible limit value L. Then, for given ε, if possible, choose your δ and write the interval (c-δ, c+ δ), if not possible try to explain why.

a) $f(x) = x^3 + 3$, ε =0.9 and a=5
ε = 0.5 and a=5
ε =0.1 and a=5

b) $g(x)=\begin{cases} \dfrac{x^2-1}{x-1}, & if\ x\neq 1 \\ 5, & if\ x=1 \end{cases}$, ε = 0.9 and a=1

ε = 0.4 and a=1
ε =0.2 and a=1

c) $h(x)=\begin{cases} \dfrac{x^3}{2}, & if\ x\leq 2 \\ 6, & if\ x>2 \end{cases}$, ε = 3 and a=2

ε =2 and a=2
ε = 1 and a=2

2. In the above activities, you find specific δ values for given ε (if possible), but as you know from your logic course, finding some specific δ values does not guarantee that for all ε you can find a δ. To guarantee this, you need to find a relationship between δ and ε. That is you need to find an algorithm for finding δ, once you have ε. Let's make an example together. Take *f* to be function given by $f(x)=2x$, c=2, L=4. Let me **propose** $\boldsymbol{\delta=\frac{\varepsilon}{2}}$. Now suppose that we are given ε=0.8, that is we are given (3.2, 4.8) for the vertical interval. Since we have a formula for δ, we can easily find its value, $\delta=\frac{0.8}{2}=0.4$, (1.6, 2.4). Now if you enter command **plot(f,1.6,2.4,3.2,4.8);** to ISETL, you will see that the curve does not leave the upper or lower boundaries of your window.

You can try this relationship with ε=0.4, ε= 0.2, ε=0.1… In each time you will see that the curve does not leave the upper or lower boundaries of your window. Now do following activities.

a) In this activity, you will take *f* to be function given by $f(x) = 3x$, a=2, L=6. We do not mind telling you that in this case, given any ε>0, that is, given any interval of the form (6- ε, 6+ ε), you can find a δ>0, that is, an interval (2- δ, 2+ δ). Your problem is to verify this for a few decreasing values of ε, namely for 0.9, 0.6, and 0.1 by using ISETL. But then figure out analytically a general formula for δ, given ε.

b) Do the exactly same thing for the function given by $f(x) = 3x + 2$, a=2, L=8.

3. In this activity you are not allowed to use ISETL. You are given $f(x) = x^2$ with a=2. And you are asked to find out analytically a general formula for δ, given ε. You should be able to figure out the value of L. You should also be able to completely solve this problem by making **rough sketch** of the situation and thinking about it, without necessarily using the computer.

LAB-5 ACTIVITIES

Name:
ID:

There are some activities below. Do them and write your answers to the gaps that are given below the activity.

Remember that: implication is represented as **impl** in ISETL.

1.

a) By using ISETL, provide truth table for implication.

p	q	$p \to q$
True	True	
True	False	
False	True	
False	False	

b) By using ISETL, provide truth table for negation of implication.

p	q	$\neg(p \to q)$
True	True	
True	False	
False	True	
False	False	

c) By using ISETL, verify that $\neg(p \to q)$ is equivalent to $p \wedge \neg q$.

p	q	$\neg(p \to q)$	$p \wedge \neg q$
True	True		
True	False		
False	True		
False	False		

Note that: You will use following information in doing some of the activities; $a<b<c$ can be represented as **(a<b) and (b<c)** in ISETL.

2.
a) Express the following statement in ISETL code.

$$0<|x-1|<0.1 \text{ implies } |2x-2|<0.2$$

b) Express its negation in ISETL code.

3.
a) Express the following statements in ISETL code. Test their truth with ISETL, and write whether it is true or false

(i) For all x in X={0.9,0.99,0.999,1.001,1.01,1.1}, $0<|x-1|<0.1$ implies $|2x-2|<0.2$.

(ii) There exist d in D={0.05,0.005,0.0005,0.00005} such that for all x in X={0.9,0.99,0.999,1.001,1.01,1.1}, $0<|x-1|<d$ implies $|2x-2|<0.01$

b) Express the negation of the statement in part (i) in ISETL code without placing "not" in front of it.

c) Express the negation of the statement in part (ii) in ISETL code without placing "not" in front of it.

Note: For the activity-4 and activity-5 use the following sets.
E={1, 0.1,0.01,0.001,0.0001}, D={0.5, 0.05,0.005,0.0005,0.00005},
X={0.9999,0.99999,0.999999,1.000001,1.000001,1.0001}.

4. Express the following statement in ISETL code. Test its truth with ISETL, and write whether it is true or false.

For all e in E, there exist d in D, such that for all x in X, 0<|x-1|<d implies |2x-2|<e.

5.
a) Express the following statement in ISETL code, and test its truth with ISETL.

For all e in E there exist d in D such that for all x in X, 0<|x-1|<d implies |2x-2.1|<e.

b) ISETL will produce false for the above proposition. Can you make statement true by changing values in set E? If yes, explain why. If no, explain why not.

c) Express the negation of the statement in part 5(a) in ISETL code without placing "not" in front of it, and test its truth with ISETL.

d) ISETL returns true for the statement in 4, and false for the statement in 5(a).

(i) What is the difference between these two statements?

(ii) Explain why this difference makes one statement true and other false.

APPENDIX E

WEEK-3 LABORATORY PLAN

Collect Lab-2 Activities
Return Homework-1
Handout Homework-2

State that your expectation is the group work in which each individual has her/his unique contribution to end product, and end product is the product of group not sum of the individual contributions.

During the lab sessions, walk around the computer laboratory and observe what is going on within group: Is there a free-rider, whether group members facilitate each other.

Important points in the laboratory sessions are

- to give appropriate feedback to individuals and groups,
- to monitor their performance,
- to promote discussions among group members,
- to be a model for colloborative behavior.

You can use following phrases in communicating with individuals and with groups:

- Let's think in this way…
- What do you think about…

- What is your opinion about …
- Good point how could you contribute to your friends' thought about…
- Good point, but let's focus on…
- What can you do next to accomplish…
- You can negotiate as a group on…

Beginning Discussion:

Let them close computer monitors and focus on your explanations. Explain the following two tasks, and make aware them about that they will first discuss tasks within the group, then discuss them among groups.

Task 1. Let them use **LimitProcess** to estimate $\lim_{x \to 0} \frac{\sin(x)}{x}$ in groups. Then, let groups discuss their findings. If necessary give some prompts.

Task 2. Let them use **LimitProcess** to estimate $\lim_{x \to 0} \frac{\sin(5x)}{x}$. Then, let groups discuss their findings. If necessary give some prompts.

Handout Lab-3 Activities

Week's Activities:

Let them close computer monitors and focus on your explanations.

Activity 1. Briefly explain what **lim** function is doing and what the question 1 is asking.

Activity 2. Briefly explain what **limadd** and **limprod** are. If they have trouble in constructing these functions, you can give hint that **lim** function can be used to construct **limadd** and **limprod**.

Activity 3. Briefly explain what **limatinf** is. If they have trouble in producing **limatinf** function, you can give hint that **lim** function can be used to construct **limatinf**.

Attendence Sheet

CURRICULUM VITAE

PERSONAL INFORMATION

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