SIMPLE GROUPS OF FINITE MORLEY RANK WITH A TIGHT AUTOMORPHISM WHOSE CENTRALIZER IS PSEUDOFINITE

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ABSTRACT

SIMPLE GROUPS OF FINITE MORLEY RANK WITH A TIGHT AUTOMORPHISM WHOSE CENTRALIZER IS PSEUDOFINITE

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This thesis is devoted to the analysis of relations between two major conjectures in the theory of groups of finite Morley rank. One of them is the Cherlin-Zil'ber Algebraicity Conjecture which states that infinite simple groups of finite Morley rank are isomorphic to simple algebraic groups over algebraically closed fields. The other conjecture is due to Hrushovski and it states that a generic automorphism of a simple group of finite Morley rank has pseudofinite group of fixed points. Hrushovski showed that the Cherlin-Zil'ber Conjecture implies his conjecture. Proving his Conjecture and reversing the implication would provide a new efficient approach to prove the Cherlin-Zil'ber Conjecture.

This thesis proposes an approach to derive a proof of the Cherlin-Zil'ber Conjecture from Hrushovski's Conjecture and contains a proof of a step in that direction. Firstly, we show that John S. Wilson's classification theorem for simple pseudofinite groups can be adapted for definably simple non-abelian pseudofinite groups of finite centralizer dimension. Combining this result with recent related developments, we identify definably simple non-abelian pseudo finite groups with Chevalley or twisted Chevalley groups over pseudofinite fields. After that in the context of Hrushovski's Conjecture, in a purely algebraic set-up, we show that the pseudofinite group of fixed points of a generic automorphism is actually an extension of a Chevalley group or a twisted Chevalley group over a pseudofinite field.

Keywords: Groups of Finite Morley Rank, Pseudofinite Groups.

ÖZ

DURGUN ÖZYAPI DÖNÜŞÜMÜNÜN MERKEZLEYENİ SÖZDESONLU OLAN MORLEY RANKI SONLU BASİT GRUPLAR

Uğurlu, Pınar
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Bu tez Morley rankı sonlu olan gruplar kuramındaki iki büyük sanı arasındaki ilişkinin incelenmesine adanmıştır. Bunlardan biri olan Cherlin-Zil'ber Cebirsellik Sanısı'na göre Morley rankı sonlu olan her basit sonsuz grup cebirsel kapalı bir cisim üzerinde tanımlı bir cebirsel grup ile eş yapılıdır. Diğer sanı Hrushovski'ye aittir ve buna göre Morley rankı sonlu olan basit bir grubun kapsamlı özyapı dönüşümünün sabit bıraktığı noktalar grubu sözdesonludur. Hrushovski, Cherlin-Zil'ber Sanısı'nın kendi sanısını gerektirdiğini gösterdi. Hrushovski'nin Sanısı'nın kanıtlanması ve sözü edilen gerektirmenin zıt yönlüsü üzerinde durulması, Cherlin-Zil'ber Sanısı'nın kanıtlanmasına yeni ve verimli bir yaklaşım sağlayacaktır.

Bu tez Cherlin-Zil'ber Sanısı'nın kanıtını Hrushovski'nin Sanısı'ndan elde etmek için bir yaklaşım ileri sürmekte ve bu yönde atılan bir adımın kanıtını içermektedir. Öncelikle John S. Wilson'ın basit sözdesonlu gruplar için verdiği sınıflandırma Teoremi'nin, tanımsal basit, değişmeli olmayan ve merkezleyen boyutu sonlu gruplar için uyarlanabileceği gösterilmiştir. Bu sonuç, ilgili son gelişmelerle bir araya getirilerek tanımsal basit, değişmeli olmayan ve merkezleyen boyutu sonlu gruplar sözdesonlu cisimler üzerinde tanımlı Chevalley ya da burkulmuş Chevalley gruplarıyla özdeşleştirilmiştir. Bundan sonra, tamamen cebirsel varsayımlar altında, Hrushovski'nin Sanısı bağlamında, kapsamlı özyapı dönüşümünün sözdesonlu sabit noktalar grubunun, sözdesonlu cisim üzerinde tanımlı Chevalley ya da burkulmuş Chevalley grubunun bir genişlemesi olduğu gösterilmiştir.

Anahtar Sözcükler: Morley Rankı Sonlu Gruplar, Sözdesonlu Gruplar.

To my parents

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TABLE OF CONTENTS

CHAPTER 1

Introduction

Morley rank, which was introduced by Michael Morley in 1965 [46], is a notion of dimension which arises in model theory. In the context of groups, this model theoretic notion has an axiomatic description as a ranked group. More generally, a *ranked structure* is a structure equipped with a rank function rk, from its non-empty definable subsets to the non-negative integers. This rank function satisfies some very natural axioms which imitate the behavior of Zariski dimension in algebraic geometry [13]. Although this rank function is not the same as Morley rank in general, in the context of groups, the two rank functions coincide. Indeed, Bruno Poizat proved that a group is ranked if and only if it is a group of finite Morley rank [51]. This axiomatic approach turns a purely model theoretical concept into an abstract algebraic concept which is more convenient to group theorists.

Finite groups and algebraic groups over algebraically closed fields are examples of groups of finite Morley rank and in the latter case Morley rank coincides with the Zariski dimension. Actually, the only known infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields. In this direction, there is a long standing conjecture proposed independently by Gregory Cherlin and Boris Zil'ber in the 1970's. The so-called *Cherlin-Zil'ber Algebraicity Conjecture*, or *Algebraicity Conjecture* in short, states that any infinite simple group of finite Morley rank is isomorphic to an algebraic group over an algebraically closed field. Although this conjecture is still open in its full generality, significant progress has been made by adapting and generalizing ideas from the Classification of Finite Simple Groups. This approach was suggested by Borovik in the 1980's and has been quite useful in this classification project. However, some of the important methods in finite group theory are not applicable in the context of groups of finite Morley rank. For example, counting arguments, which are very important methods in the finite context have no analogues in finite Morley rank context. The importance of this deficiency can be explained by an example once we define simple bad groups. A *bad group* is a non-solvable group of finite Morley rank in which every proper definable connected subgroup is nilpotent. It follows from this definition that infinite simple bad groups are possible counter-examples to the Algebraicity Conjecture since it is well-known that Borel subgroups of simple algebraic groups are not nilpotent (§21.4 in [35]). However, it is currently unknown whether infinite bad groups exist or not. On the other hand, there are several results about the structure of simple bad groups. Namely, a simple bad group G can be written as a union of its Borel subgroups all of which are known to be conjugate and intersect trivially in pairs [13]. When these structural properties of simple bad groups are combined with a simple counting argument one can observe that finite bad groups do not exist. This follows, since finite groups can not be written as a union of conjugates of one of its subgroups whose pairwise intersections are trivial. This example shows the power of the counting arguments in finite group theory. Therefore, the existence of the analogues of these methods in finite Morley rank context would be useful to eliminate one of the biggest obstacles, namely bad groups, on the way to prove the Algebraicity Conjecture.

What can be possible analogues of counting arguments in finite Morley rank context? There has been some work in this direction. It was proven that the fixed points of generic automorphisms of algebraically closed fields are pseudofinite fields [41]. The following nice algebraic characterization of pseudofinite fields, given by Ax in [6], is used in the proof:

> A field F is pseudofinite if and only if it is perfect, has exactly one extension of degree n for each natural number $n > 1$ and every absolutely irreducible variety defined over F has an F-rational point.

Moreover, Hrushovski worked on wider classes of structures, including simple groups of finite Morley rank, with generic automorphisms in [34]. In the particular case of simple groups of finite Morley rank, the aim is to prove that the group of fixed points of a generic automorphism is pseudofinite. Hrushovski suggests that the fixed point subgroups mimic the behavior of pseudofinite groups in some sense, that is, they admit some kind of '*measure*' similar to non-standard probabilistic measure on pseudofinite groups. However, it is not known how to characterize pseudofinite groups. As we mentioned above, Ax's algebraic characterization of pseudofinite fields is very useful when dealing with pseudofinite fields, however, nothing similar to this is known for pseudofinite groups.

As it was mentioned by Hrushovski in [34], the Algebraicity Conjecture implies the following conjecture.

Principal Conjecture. *Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Then, the group of fixed points of α is pseudofinite.*

This thesis is an attempt to construct a bridge between these two conjectures from the other direction. More precisely, we ultimately aim to prove that the Principal Conjecture implies the Algebraicity Conjecture. For this purpose we work around the following conjecture:

Intermediate Conjecture. *Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Assume that the group of fixed points of α is pseudofinite. Then G is isomorphic to a Chevalley group over an algebraically closed field.*

As a first step in the direction of proving this conjecture, we do not use the full strength of the concept of generic automorphisms. We introduce the notion of a *tight automorphism* in order to work in a purely algebraic context. To be more precise, an automorphism α of a group of finite Morley rank is called *tight* if whenever a definable connected subgroup H of G is α-invariant then the definable closure of the group of fixed points of α in H is H. This property of tight automorphism can be considered as some kind of density property which is satisfied by generic automorphisms. Note that throughout the present thesis, we will not use any stronger algebraic properties of generic automorphisms except for tightness.

The main result obtained in this thesis can be stated as follows:

Theorem. *Let G be an infinite simple group of finite Morley rank and α be a tight automorphism of G. Assume that the group of fixed points of α, that is, $C_G(\alpha)$ is pseudofinite. Then there is a definable (in $C_G(\alpha)$) normal subgroup S of $C_G(\alpha)$ such that*

$$S \trianglelefteqslant C_G(\alpha) \leqslant Aut(S)$$

where S is isomorphic to a Chevalley or twisted Chevalley group over a pseudo finite field.

In order to prove this theorem, we analyze the structure of the pseudo finite group which arises in a simple group of finite Morley rank as a group of fixed points of a tight automorphism. The first result we obtain is about the structure of definably simple pseudofinite subgroups of groups of finite Morley rank. More precisely and generally, we classify definably simple pseudofinite groups of finite centralizer dimension. For obtaining this result, we adapt the main ideas in the proof of the classification of simple pseudofinite groups given by John S. Wilson in [64]. In the further analysis of the fixed point subgroup, we use properties of the tight automorphism, some basic facts from the theory of simple groups of finite Morley rank together with some first order properties of finite groups and obtain a proof of the main theorem of this thesis.

With the main result of this thesis at hand, there are several ways for attacking the Intermediate Conjecture. Although, the details of the program will be given in the last chapter, the possible configurations and several paths to be taken in this direction can be summarized as follows.

We begin by introducing *supertight automorphism* which satisfies another property of generic automorphisms which is explained below.

An automorphism α of a group of finite Morley rank is called *supertight* if every positive power of α is tight. In other words, whenever a definable connected subgroup H of G is α^n-invariant for some non-negative integer n, the definable closure of the fixed points of α^n in H is H.

If we work with a supertight automorphism α instead of a tight automorphism, under the assumption that the groups of fixed points of every positive power of α is pseudofinite, then there seem to be two possible ways for identifying G with a Chevalley group over an algebraically closed field.

One approach is to consider the union of the groups of fixed points of powers of α and try to prove that this union is elementarily equivalent to G.

The other approach is to use induction on the Morley rank and the Prüfer 2-rank of G. If the small Prüfer 2-rank cases are handled then the generic case is believed to follow from the results similar to results obtained in the classification project of simple groups of finite Morley rank. Moreover, in some cases the latter can be directly used.

The structure of this thesis can be outlined as follows.

In Chapter 2, we give some definitions and state some results in the theories of algebraic groups and groups of finite Morley rank. Some basic model theoretical notions are defined in this chapter as well.

In the third chapter, we give necessary background information in order to define ultraproducts and pseudofinite structures. We state some results concerning pseudofinite groups. Moreover, as first order statements and the notion of definability are quite important in the theory of pseudofinite groups, we will go into the details of this subject. More precisely, we will explicitly write down some first order formulas and statements related to groups and fields so that we can refer to them in the sequel.

In Chapter 4, we outline the proof of the classification theorem for simple pseudofinite groups which was given by John S. Wilson up to elementary equivalence [64]. Then, we prove an analogue of this classification for definably simple pseudofinite groups of finite centralizer dimension. This result will be beneficial when we start analyzing the structure of the pseudofinite group arising in a group of finite Morley rank since any subgroup of a finite Morley rank group has finite centralizer dimension.

In the last chapter, we start the analysis of the structure of the group of fixed points of a tight automorphism of a simple group of finite Morley rank and we prove the main theorem. Moreover, we outline a research program concerning the future plans for proving the Intermediate Conjecture.

CHAPTER 2

PRELIMINARIES

This chapter covers some background material which will be necessary throughout this thesis.

In the first section, linear algebraic groups are defined and the classification of simple algebraic groups over algebraically closed fields is given without proofs. Moreover, Chevalley and twisted Chevalley groups are introduced and the structure of their automorphism groups are given. Since only some basic properties of algebraic groups and Chevalley groups will be used in this thesis, we introduce them very briefly and refer the reader to the standard books such as [18], [35] and [61] for a detailed discussion of the subject.

In the second section, some basic concepts in model theory are summarized. The books [19], [33] and [45] are among the standard references for this subject.

In the next section, two old conjectures, namely *Ore's Conjecture* and *Thompson's Conjecture*, are stated. The results obtained on the way to prove *Thompson's Conjecture* are summarized briefly and the current status of the conjectures are given. For a detailed discussion of the subject we refer the reader to the survey article by Kappe and Morse [36].

In the last section, the notion of Morley rank and groups of finite Morley rank are introduced briefly. The current status of the classification of simple groups of finite Morley rank is given. Moreover, some important results in the theory of groups of finite Morley rank are listed without proofs. The proofs of these facts and detailed information about groups of finite Morley rank can be found in the book [13]. For a detailed discussion of the classification project, we refer the reader to the book [1].

2.1 Linear Algebraic Groups

Let K be an algebraically closed field and K^n denote the n-dimensional affine space over K. A subset $A \subseteq K^n$ is called an *affine algebraic set* if it is the zero set of some set of polynomials in $K[x_1, \ldots, x_n]$. A topology, namely *Zariski topology*, is defined on K^n by declaring affine algebraic subsets as Zariski closed sets. It can be observed that, a set of polynomials and the ideal generated by this set of polynomials have exactly the same set of zeros. Moreover, it is well-known that the polynomial ring $K[x_1, \ldots, x_n]$ is Noetherian and so, every ideal is finitely generated in $K[x_1, \ldots, x_n]$. Therefore, an affine algebraic set is in fact the zero set of finitely many polynomials.

An *affine algebraic group* is an affine algebraic set in K^n with a group structure such that the group operations, multiplication and inversion

$$\begin{aligned} \mu : G \times G &\longrightarrow G, & \qquad\qquad i : G &\longrightarrow G \\ (g, h) &\longmapsto gh & g &\longmapsto g^{-1} \end{aligned}$$

are given by *morphisms* of algebraic varieties, that is, by polynomial maps with coefficients in K. For example, the general linear group $GL_n(K)$, which is the group of invertible $n \times n$ matrices with coefficients in K, is an algebraic group. To see this, we first identify any $n \times n$ matrix X with an element of K^{n^2}. Then, $GL_n(K)$ can be identified with the set

$$\{(X, \lambda) \in K^{n^2+1} \mid \lambda \det(X) = 1\}$$

Since determinant of an $n \times n$ matrix is a polynomial function in n^2 variables, this set is Zariski closed as the zero set of a polynomial in $n^2 + 1$ variables. Moreover, since multiplication and inversion of $n \times n$ matrices are given by polynomial maps, $GL_n(K)$ is an algebraic group.

Affine algebraic groups are called *linear algebraic groups* and the following fact explains the reason for this:

Fact 2.1.1. (§8.6 in Humphreys [35]) *Any affine algebraic group is isomorphic to a closed subgroup of a general linear group $GL_n(K)$ for some n.*

Note that although linear algebraic groups are not the only class of algebraic groups, throughout this thesis by an algebraic group we mean a linear algebraic group.

Let k be an arbitrary subfield of an algebraically closed field K. A closed set X in K^n is called *k-closed* if it is the zero set of some collection of polynomials with coefficients in k. X is said to be *defined over* k, if it is the zero set of an ideal which is generated by polynomials with coefficients in k. Note that being k-closed is weaker than being defined over k, and the relation between these two notions is given by the following fact:

Fact 2.1.2. (§34.1 in Humphreys [35]) *If X is k-closed in K^n, then X is defined over a finite, purely inseparable extension of k.*

Throughout this thesis, we will be dealing with perfect fields and so, these two notions coincide.

If X is a closed subset of K^n defined over k we can talk about *k-rational points* of X which is denoted by $X(k) = X \cap k^n$. Similarly, an algebraic group G is said to be defined over k if it is defined over k as a variety and multiplication and inverse maps are given by polynomial maps with coefficients in k. If this is the case, we denote k-rational points of G by $G(k)$, which is a subgroup of G.

Let G be a linear algebraic group defined over a field k. The *Zariski closure* of any subset (resp. subgroup) X of G in G is defined as the smallest Zariski closed set (resp. subgroup) in G containing X. If the Zariski closure of X is G then X is called *Zariski dense*. G is said to be *connected* if it has no proper closed subgroups of finite index.

The following important result, which will be referred in the sequel, is due to Rosenlicht.

Fact 2.1.3. (Rosenlicht [54]) *Let G be a connected algebraic group defined over an infinite perfect field k. Then $G(k)$ is Zariski dense in G.*

2.1.1 The Classification of Simple Algebraic Groups

A connected algebraic group is called *simple*, if it has no non-trivial connected closed proper normal subgroups. It follows from this definition that a simple algebraic group G has finite center $Z(G)$ and the quotient group $G/Z(G)$ is simple as an abstract group.

The simple algebraic groups over algebraically closed fields of arbitrary characteristic were classified up to isomorphism by Chevalley [23]. The isomorphism type of a simple algebraic group is determined by two invariants, the abstract root system which is determined up to isomorphism by its Dynkin diagram, and the fundamental group (See [35] for details). The algebraic groups of types A_n, B_n, C_n, D_n are called *classical groups* consisting of linear, orthogonal and symplectic groups and the rest E_6, E_7, E_8, F_4, G_2 are called *exceptional groups*. The subscripts denote the *Lie ranks* of the corresponding groups. For each type there are simply connected and adjoint versions. The groups of adjoint type have trivial centers and hence they are *abstractly simple*, that is, they have no non-trivial proper normal subgroups. For example, $SL_{n+1}(K)$ is simply connected and $PSL_{n+1}(K) \cong PGL_{n+1}(K)$ is adjoint versions of algebraic groups of type A_n.

The adjoint versions of the simple algebraic groups of types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ were constructed uniformly by Chevalley as subgroups of automorphism groups of finite dimensional simple Lie algebras over $\mathbb{C}$ ([18], [24], [61]). In this construction, Chevalley also proved that these groups can be defined over arbitrary fields and they are simple as abstract groups except for a few exceptions over small fields, namely, $A_1(\mathbb{F}_2), B_2(\mathbb{F}_2), G_2(\mathbb{F}_2)$ and $A_1(\mathbb{F}_3)$. The groups constructed this way are called *Chevalley groups*. There are also the so-called *twisted Chevalley groups*, denoted by ${}^2A_n, {}^2D_n, {}^3D_4, {}^2E_6, {}^2B_2, {}^2F_4, {}^2G_2$, which can not be obtained by Chevalley's construction. Over appropriate finite fields, the groups of types ${}^2A_n, {}^2D_n, {}^3D_4, {}^2E_6$ were constructed by Steinberg [59] in 1959 and the groups of types ${}^2F_4, {}^2G_2, {}^2B_2$ were constructed by Suzuki and Ree ([52], [53], [62]). Finally, in 1968 Steinberg constructed Chevalley and twisted Chevalley groups over finite fields uniformly [60] and characterized them as groups of fixed points of some special endomorphisms of algebraic groups over the algebraic closures of the finite fields in concern. We will not go into details of their construction methods in this text. Twisted Chevalley groups, over finite and infinite fields (over which they exist) are also simple as abstract groups except for ${}^2A_2(\mathbb{F}_4), {}^2B_2(\mathbb{F}_2), {}^2F_4(\mathbb{F}_2)$ and ${}^2G_2(\mathbb{F}_3)$ (See [18]).

The groups of types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, {}^2A_n, {}^2D_n, {}^3D_4, {}^2E_6, {}^2B_2, {}^2F_4, {}^2G_2$ over finite fields, with the exceptions mentioned in the last paragraph, are called *simple groups of Lie type.* Construction of these groups was an important progress in the classification project of finite simple groups.

Note that Chevalley groups over algebraically closed fields are algebraic groups by construction. In other words, a Chevalley group over an algebraically closed field K coincide with the K-rational points of an algebraic group defined over K.

Throughout this thesis, a Chevalley or twisted Chevalley group over an arbitrary field K will be denoted by $X(K)$ where X denotes one of the types given above.

2.1.2 Automorphism Groups of Chevalley Groups

In this subsection, the structure of automorphism groups of Chevalley groups over arbitrary perfect fields are given without proofs and details. Note that throughout this thesis, $Aut(G)$ will denote the automorphism group of any group G.

For Chevalley groups over algebraically closed fields and finite fields, the structures of the automorphism groups are determined by using conjugacy properties of Sylow subgroups [18], [32], [60]. Since Sylow theorems are not applicable in general, some more complicated methods are needed to obtain the same results for Chevalley groups over perfect fields [18].

There are four types of automorphisms of a Chevalley group $X(K)$. They are called *inner, diagonal, field* and *graph automorphisms.* These automorphisms can be described very briefly as follows.

Inner automorphisms are well-known in group theory and they are induced by conjugation by the elements of the group. More precisely, for each $g \in X(K)$

$$\begin{aligned} \varphi_g : X(K) &\longrightarrow X(K) \\ x &\longmapsto g^{-1}xg \end{aligned}$$

induces an inner automorphism of $X(K)$.

Let $Inn(X(K))$ denote the group of inner automorphisms of $X(K)$. Then, it is routine to check that the map

$$\begin{aligned} \varphi : X(K) &\longrightarrow Inn(X(K)) \\ g &\longmapsto \varphi_g \end{aligned}$$

is an isomorphism as $X(K)$ is a simple group. Therefore we can identify $X(K)$ with $Inn(X(K))$.

Diagonal automorphisms of $X(K)$ are induced by conjugation by some elements which can be represented by diagonal matrices with respect to the Chevalley basis. If K is algebraically closed, then diagonal automorphisms are inner. However, for an arbitrary perfect field there might be outer diagonal automorphisms.

Field automorphisms are induced by automorphisms of the field in concern.

Graph automorphisms are induced by the symmetries of the associated Dynkin diagram.

The following fact, which is due to Steinberg if the field is finite, gives the structure of the automorphism group of a Chevalley group $X(K)$ where K is a perfect field.

Fact 2.1.4. (Gorenstein et al. [32]) *Let α be an automorphism of a Chevalley group $X(K)$ where K is a perfect field. Then, $\alpha = idfg$ where i, d, f, g denote inner, diagonal, field and graph automorphisms respectively. Moreover*

$$Aut(X(K)) = ID \rtimes \Phi\Gamma$$

where I, D, Φ, Γ denote the group of inner, diagonal, field and graph automorphisms of $X(K)$ respectively. The so-called outer diagonal automorphism group*, ID/I, is either cyclic of order bounded in terms of the rank or isomorphic to elementary abelian group of order* 4.

The following fact is known as *Schreier Conjecture* and the Classification of Finite Simple Groups is needed for its proof.

Fact 2.1.5. (Conway et al. [25]) *Let S be a finite simple group. Then $Aut(S)/S$ is solvable.*

2.2 Basic Model Theory

We start with *structures* which are the main objects of mathematics. A *structure* $\mathfrak{M}$ is formed by an underlying set M together with a *signature* which consists of function, relation and constant symbols that are to be interpreted. For example, $(G, \cdot, ^{-1}, e)$ denote the group structure $\mathcal{G}$ where G is the underlying set, $\cdot$ is a binary function symbol interpreted as the group operation, $^{-1}$ is a unary function symbol denoting the inversion operation and e is a constant symbol denoting the identity element of the group. A *formula* in a structure $\mathfrak{M}$ is a finite string of symbols which is formed, with respect to some well-known rules, by symbols of the signature, variables $v_1, v_2, \ldots, v_n$ denoting the elements of the underlying set, equality symbol $=$, predicates $\vee$ (or), $\wedge$ (and), $\neg$ (not), quantifiers $\forall$ (for all), $\exists$ (there exists) and parentheses (,). Moreover, the elements of the underlying set can be used as parameters in the formula. The collection of all of these symbols which are used in order to write formulas is called *language*. Note that languages are distinguished by the signature and therefore throughout this thesis when we deal with a language $\mathcal{L}$, notationally we will not include the symbols which are common in all languages. For example the language of groups can be taken as $\{\cdot, ^{-1}, e\}$ or $\{\cdot, ^{-1}, 1\}$ or $\{\cdot, ^{-1}, 0\}$ when we are dealing with abelian groups. We emphasize that, throughout this thesis we will stay in the borders of the first order logic, that is, quantification will be over individual variables.

A *sentence* is a formula without free variables, that is, a formula in which all of the variables are preceded by quantifiers. For a sentence or a formula σ, we write $\mathfrak{M} \models \sigma$ to mean that the sentence σ holds or the formula σ is satisfied in the structure $\mathfrak{M}$.

A *theory* $\mathcal{T}$ in a language $\mathcal{L}$ is just a set of $\mathcal{L}$-sentences and if $\mathcal{T}$ has a model then it is called *consistent theory.* A theory T is *complete* if it is a maximal consistent set of sentences and T has *quantifier elimination* if for every formula there is an equivalent quantifier free formula. A *model* $\mathfrak{M}$ *of a theory* $\mathcal{T}$ in a language $\mathcal{L}$ is a structure in which all of the $\mathcal{L}$-sentences of $\mathcal{T}$ hold and we write $\mathfrak{M} \models \mathcal{T}$. The *theory of a model* $\mathfrak{M}$ in the language $\mathcal{L}$ is the set of $\mathcal{L}$-sentences which hold in the structure $\mathfrak{M}$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two structures in a common language $\mathcal{L}$ with underlying sets M and N respectively. $\mathfrak{M}$ and $\mathfrak{N}$ are said to be *elementarily equivalent* if they satisfy the same $\mathcal{L}$-sentences and we write $\mathfrak{M} \equiv \mathfrak{N}$. An embedding π of $\mathfrak{M}$ and $\mathfrak{N}$ is called *elementary* if for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_k)$ and elements $m_1, \ldots, m_k$ from M we have

$$\mathfrak{M} \models \varphi(m_1, \ldots, m_k) \Longleftrightarrow \mathfrak{N} \models \varphi(\pi(m_1), \ldots, \pi(m_k))$$

$\mathfrak{M}$ is a *substructure* of $\mathfrak{N}$ if $M \subseteq N$ and the interpretations of relation, function and constant symbols in $\mathfrak{M}$ are just the restrictions of the corresponding interpretations in $\mathfrak{N}$. A substructure $\mathfrak{M}$ is *existentially closed* in $\mathfrak{N}$ if whenever a quantifier free formula $\varphi(x, m_1, \ldots, m_k)$, with parameters $m_1, \ldots, m_k$ from M, is satisfied in $\mathfrak{N}$ then it is satisfied in $\mathfrak{M}$ as well. A model $\mathfrak{M}$ of a theory $\mathcal{T}$ is called *existentially closed* if $\mathfrak{M}$ is existentially closed in every model of $\mathcal{T}$ which contains $\mathfrak{M}$ as a substructure.

A class of $\mathcal{L}$-structures is called *axiomatizable* if there is a set Σ of $\mathcal{L}$-sentences such that an $\mathcal{L}$-structure $\mathfrak{M}$ belongs to that class if and only if all of the sentences in Σ holds in $\mathfrak{M}$. If Σ is a finite set, then the class is called *finitely axiomatizable*, that is, the class can be axiomatized by a single sentence. A *definable set* in a structure $\mathfrak{M}$ is a subset $X \subseteq M^n$ which satisfies a first order formula in the language of the structure. Throughout this thesis when we say definable, we mean definable possibly with parameters. Therefore finite sets will always be definable.

In the next chapter, we will include several examples which will make these definitions clearer.

2.3 Ore's Conjecture and Thompson's Conjecture

Commutator subgroup of a group G is defined as a subgroup generated by the *commutators*, namely by the elements of the form $a^{-1}b^{-1}ab$. Since the product of two commutators need not to be a commutator, the set of commutators does not form a group in general. Oystein Ore worked on the groups in which all of the elements of the commutator subgroups are commutators [49]. In the article [49], Ore proved that every element of the alternating groups $Alt(n)$, for $n \geq 5$, is a commutator. In the same article, he claimed that this result could be extended to all non-abelian finite simple groups. This claim is known as *Ore's Conjecture*. This old conjecture was followed by a stronger conjecture, which is attributed to John Thompson by Arad and Herzog in [5], stating that for every non-abelian finite simple group G, there exist a conjugacy class C such that $G = CC$. For a detailed discussion of these conjectures see [29] and [36].

It can be observed that Thompson's Conjecture implies Ore's Conjecture. More precisely, if $g \in G = CC$ where $C = \{c^G\}$ then $g = c^{g_1}c^{g_2}$ for some $g_1, g_2 \in G$. Moreover, as $1 \in G$ and $c \neq 1$ there are $x_1, x_2 \in G$ such that $1 = c^{x_1}c^{x_2}$, that is, $c = (c^{-1})^{x_2{x_1}^{-1}}$.

Now, the following argument shows that g is a commutator:

$$\begin{aligned} g &= c^{g_1}c^{g_2} && (2.1) \\ &= c^{g_1}(c^{-1})^{x_2{x_1}^{-1}g_2} && (2.2) \\ &= {g_2}^{-1}x_1{x_2}^{-1}(x_2{x_1}^{-1}g_2{g_1}^{-1})c(g_1{g_2}^{-1}x_1{x_2}^{-1})c^{-1}x_2{x_1}^{-1}g_2 && (2.3) \\ &= {g_2}^{-1}x_1{x_2}^{-1}[g_1{g_2}^{-1}x_1{x_2}^{-1}, c^{-1}]x_2{x_1}^{-1}g_2 && (2.4) \\ &= [g_1{g_2}^{-1}x_1{x_2}^{-1}, c^{-1}]^{x_2{x_1}^{-1}g_2} && (2.5) \end{aligned}$$

Although the conjectures are still open, several mathematicians obtained partial results and the current status of both of the conjectures are the same. For the details of these results we refer the reader to the survey article by Kappe and Morse [36].

The current status of Thompson's Conjecture can be stated briefly as follows:

- In 1965, Thompson's Conjecture for alternating groups $Alt(n)$ where $n \geq 5$ was proved by Cheng-Hao in [21].
- In 1984, Thompson's Conjecture for sporadic groups was proved by Neubüser et al. in [48].
- Several mathematicians have obtained partial results for finite simple groups of Lie type. Finally, in 1998 Ellers and Gordeev [29] verified Thompson's Conjecture for finite simple groups of Lie type over a field with more than 8 elements. (For $^2B_2(q^2), {}^2G_2(q^2)$ and $^2F_4(q^2)$ it is enough to assume $q^2 > 8$)

By the classification of finite simple groups, the only unverified cases are the simple groups of Lie type over some small fields. Therefore, for finite simple groups of Lie type over big fields, Thompson's Conjecture can be used as a fact. For the purposes of this thesis, the importance of these conjectures lies in the fact that they can be stated in a first order way and allows one to get some definability results. These will become clearer in the following chapters.

2.4 Groups of Finite Morley Rank

This section starts with a brief explanation of the notion of Morley rank in the context of groups. Then, the possible types of infinite simple groups of finite Morley rank according to the structures of the connected components of their Sylow 2-subgroups are listed and the current status of the Algebraicity Conjecture is given. Moreover, basic properties of groups of finite Morley rank are given without proofs. For the details and proofs we refer the reader to the books [1] and [13].

2.4.1 Morley Rank and Simple Groups of Finite Morley Rank

A *universe* $\mathscr{U}$ is defined to be a collection of sets satisfying certain conditions such as being closed with respect to some set theoretic operations. We will not go into details of these conditions, for the details we refer the reader to the book [13].

For any $\mathcal{L}$-structure $\mathfrak{M}$ there is a *minimal universe* which contains the underlying set M as well as some subsets of M^n which are interpretations of the relation symbols and graphs of the interpretations of function symbols in $\mathcal{L}$. Actually, this minimal universe associated to $\mathfrak{M}$ coincide with the collection of all *interpretable sets*, that is, equivalence classes of definable sets with respect to definable equivalence relations in the $\mathcal{L}$-structure $\mathfrak{M}$.

A universe $\mathscr{U}$ is called *ranked* if a rank function rk can be defined from the non-empty sets in $\mathscr{U}$ to the natural numbers satisfying the following axioms for all $A, B \in \mathscr{U}$:

- *Monotonicity of rank.* $rk(A) \geq n + 1$ if and only if there are infinitely many pairwise disjoint, non-empty, definable subsets of A each of rank at least n.

- *Definability of rank.* If f is a definable function from A into B, then, for each integer n, the set $\{b \in B \mid rk(f^{-1}(b)) = n\}$ is definable.

- *Additivity of rank.* If f is a definable function from A onto B and if $rk(f^{-1}(b)) = n$ for all $b \in B$ then $rk(A) = rk(B) + n$.

- *Boundedness of finite preimages.* For any definable function f from A into B there is an integer m such that for any $b \in B$ the preimage $f^{-1}(b)$ is infinite whenever it contains at least m elements.

A structure $\mathfrak{M}$ is said to be a *ranked structure* if $\mathscr{U}(\mathfrak{M})$ is a ranked universe or more generally if $\mathfrak{M}$ is definable in a ranked universe. For example, an algebraically closed field K is known to be a ranked structure in the language of fields since the definable sets in $\mathscr{U}(K)$ are just Boolean combinations of

Zariski closed sets and Zariski dimension can be attached to these definable sets as a rank function. As a result, every algebraic group G over an algebraically closed field K is a ranked group since G is known to be definable in K, that is, definable in the ranked universe $\mathscr{U}(K)$.

Note that the rank function defined above does not necessarily coincide with Morley rank in the case of arbitrary structures. More precisely, the universe of a structure of finite Morley rank need not be ranked in the sense defined above (See [51]). However, Bruno Poizat proved that in the case of groups these two notions coincide, that is, ranked groups are exactly the groups of finite Morley rank. This characterization gives an axiomatic description of groups of finite Morley rank which are model theoretical objects defined as ω-stable groups having finite Morley rank.

The result obtained by Poizat can be stated as follows:

Fact 2.4.1. (Poizat [51]) *Let G be an ω-stable group of finite Morley rank. Then $\mathscr{U}(G)$ is a universe with rank. Conversely, if $\mathscr{U}$ is a universe with rank and a group G is definable in $\mathscr{U}$ then G is an ω-stable group of finite Morley rank.*

The discussion above makes both group theoretical and model theoretical approaches possible in the theory of groups of finite Morley rank. Group theoretical approach, more precisely, adaptation of some ideas from the classification of finite simple groups has been quite useful in the classification project of infinite simple groups of finite Morley rank. As mentioned in the introduction, this project is based on the Algebraicity Conjecture which claims that infinite simple groups of finite Morley rank are isomorphic to algebraic groups over algebraically closed fields. Although, the results obtained up to now suggest that simple groups of finite Morley rank and simple algebraic groups over algebraically closed fields have many properties in common, the conjecture is still open. The current status of the conjecture can be explained best after introducing the possible types of simple groups of finite Morley rank.

In the theory of groups of finite Morley rank, Sylow 2-subgroups are conjugate [13]. Moreover, the structure of Sylow 2-subgroups are well-known. More precisely, a Sylow 2-subgroup S of a group G of finite Morley rank is a finite extension of $U * T$ where U is a definable connected 2-group of bounded exponent, T is a divisible abelian 2-group and $*$ denotes the central product of them, that is, $[U, T] = 1$ and $U \cap T$ is finite [1]. If T is non-trivial then it is isomorphic to a direct sum of finitely many copies of *Prüfer* 2*-groups* which can be defined as

$$\mathbb{Z}_{2^\infty} = \{x \in \mathbb{C} \mid x^{2^n} = 1 \text{ for some } n \in \mathbb{N}\}$$

with the usual multiplication of complex numbers. Since all Sylow 2-subgroups of G are conjugate, the number of isomorphic copies of Prüfer 2-groups in T is an invariant of G and this number is called the *Prüfer* 2*-rank* of G.

Depending on the structure of the connected components of Sylow 2-subgroups, the Algebraicity Conjecture breaks up into four cases:

(a) $U \neq 1$ and $T = 1$, Even type

(b) $U = 1$ and $T \neq 1$, Odd type

(c) $U \neq 1$ and $T \neq 1$, Mixed type

(d) $U = 1$ and $T = 1$, Degenerate type

Note that in the category of algebraic groups, even and odd types correspond to simple algebraic groups over algebraically closed fields of even and odd characteristics respectively. Moreover, there are no simple algebraic groups of mixed and degenerate types. Therefore, the aim of the classification project is to identify even and odd type groups with simple algebraic groups over algebraically closed fields of appropriate characteristics and to prove non-existence of mixed and degenerate types. Some important results have been obtained in this direction. The current status of the Algebraicity Conjecture can be stated as follows:

(a) Simple groups of finite Morley rank of even type have been identified with Chevalley groups over algebraically closed fields of characteristic 2 (See [1]).

(b) There are some important restrictions on the structure of potential non-algebraic simple groups of finite Morley rank of odd type [15].

(c) There are no simple groups of finite Morley rank of mixed type [1].

(d) Degenerate type is the most difficult case which needs to be handled [14].

2.4.2 Basic Properties of Groups of Finite Morley Rank

The proofs of the following facts can be found in the books [1] or [13] as well as in the indicated articles.

Fact 2.4.2. (Macintyre [42]) *An infinite field of finite Morley rank is algebraically closed.*

Fact 2.4.3. (Macintyre [43]) *Definable subgroups of a group of finite Morley rank satisfy* descending chain condition*, that is, every proper descending chain of definable subgroups stabilizes after finitely many steps.*

Descending chain condition on definable subgroups is a strong property which allows one to define the notions of *definable closure* and *connected component* in the context of groups of finite Morley rank.

Fact 2.4.4. (Borovik and Nesin [13]) *Let X be any subset of a group of finite Morley rank G. Then, there is a smallest definable subgroup of G containing X, denoted by $d(X)$.*

$d(X)$ is called the *definable closure* of X and it is the intersection of all definable subgroups of G containing X. The definability of this intersection is guaranteed by the descending chain condition on definable subgroups.

Some properties of definable closure in a group of finite Morley rank G are listed in the following fact:

Fact 2.4.5. (Borovik and Nesin [13]) *Let G be a group of finite Morley rank. Then*

(a) If elements of a subset X of G commute with each other then $d(X)$ is abelian.

(b) *For any subset X of G we have $C_G(X) = C_G(d(X))$.*

(c) *If a subgroup A normalizes a set X then $d(A)$ normalizes $d(X)$.*

(d) *Let $A \leqslant B \leqslant G$. If A has finite index in B then $d(A)$ has finite index in $d(B)$. Moreover, if $A \trianglelefteqslant B$ then $d(A) \trianglelefteqslant d(B)$ and $d(B) = d(A)B$.*

(e) *$[d(A), d(B)] = d([A, B])$ for subgroups A, B of G.*

Fact 2.4.6. (Zil'ber [66]) *If A is solvable (resp. nilpotent) subgroup of class n in a group of finite Morley rank, then $d(A)$ is also solvable (resp. nilpotent) of class n.*

Fact 2.4.7. (Borovik and Nesin [13]) *Let G be a group of finite Morley rank. Then, G contains a unique minimal definable subgroup of finite index denoted by G°.*

G° is called the *connected component* of G and it is the intersection of all definable finite index subgroups of G. If $G = G^\circ$ then G is called *connected*. There is a closely related notion, the so-called *irreducibility*. A definable subset of Morley rank n in a group of finite Morley rank is called *irreducible* if it has no disjoint definable subsets of rank n. As in the algebraic group context, a group of finite Morley rank is connected if and only if it is irreducible. However, while an algebraic group can be decomposed into its irreducible components in a unique way, there is no such a well-defined decomposition for a group G of finite Morley rank. More precisely, if a group G with Morley rank n can be written as unions of definable irreducible subsets of rank n in two ways then the number of components in each union are equal. Moreover, for each component in one of the unions there is a component in the other union such that this pair of components intersects in a set of rank n.

Fact 2.4.8. (Baldwin and Saxl [7]) *For any subset X of a finite Morley rank group G, the centralizer $C_G(X)$ of X in G is a definable subgroup. Moreover, for any $X \subseteq G$ there is $X_0 \subseteq X$ with $|X_0| \leq n$ such that $C_G(X) = C_G(X_0)$.*

Remark 2.4.1. The second part of Fact 2.4.8 states that there is a uniform bound for every proper descending chain of centralizers in a group of finite

Morley rank. In the literature, a group with this property is said to have *finite centralizer dimension*. More precisely, for any integer $k \geq 0$, a group has centralizer dimension k if it has proper descending chain of centralizers of length k and has no proper descending chain of centralizers of length greater than k. By a centralizer chain of length k we mean a proper descending chain of centralizers which has the following form:

$$G = C_G(1) > C_G(x_1) > C_G(x_1, x_2) > \cdots > C_G(x_1, \ldots, x_k) = Z(G)$$

It follows from this definition that the centralizer dimension of an abelian group is 0. Moreover, it is not difficult to observe that any group of centralizer dimension 1 is forced to be abelian, that is, it has centralizer dimension 0. It is clear that groups of finite centralizer dimension satisfy descending chain condition on centralizers. However, the converse does not necessarily hold since Roger Bryant constructed an example of a group which satisfies descending chain condition on centralizers but does not have finite centralizer dimension [16]. It is well-known that the class of groups with finite centralizer dimension is closed under taking subgroups and finite direct products [47]. Moreover, for any integer $k \geq 0$, the property of having centralizer dimension k can be expressed in the first order language of groups (See [28] and [40]). In other words, there is a first order sentence in the language of groups such that this sentence holds in a group if and only if the group has centralizer dimension k. We will write this sentence in the next chapter.

The following four facts are non-trivial corollaries of an important result due to Boris Zil'ber, the so-called *Zil'ber's Indecomposability Theorem*. This theorem generalizes the well-known Indecomposability Theorem in the theory of algebraic groups (Humphreys §7.5 in [35]).

Fact 2.4.9. (Zil'ber [67]) *Let $\{X_i \mid i \in I\}$ be a family of connected and definable subgroups of a group of finite Morley rank. Then the subgroup $\langle X_i \mid i \in I\rangle$ is definable connected and it is the setwise product of finitely many of them.*

Fact 2.4.10. (Altınel et al. [1]) *If H is a definable and connected subgroup of G then the subgroup $[H, X]$ is definable and connected for any subset $X \subseteq G$.*

Fact 2.4.11. (Altınel et al. [1]) *The commutator subgroup of a group of finite Morley rank is definable.*

Fact 2.4.12. (Poizat [51]) *A definably simple non-abelian infinite group of finite Morley rank is simple.*

CHAPTER 3

Ultraproducts and Pseudofinite Structures

In this chapter, pseudofinite structures will be defined after introducing ultrafilters and ultraproducts. Some well-known results about ultraproducts and pseudofinite groups will be given without proofs. The proofs of these results can be found in the indicated references. For a detailed information about ultraproducts we refer the reader to the books by Bell and Slomson [8] and Chang and Keisler [19].

In Section 3.3, we write explicit formulas in the language of groups which define some subgroups and subsets of a group. Moreover, some first order properties of groups and fields are stated in the language of groups and rings so that we can refer to them in the following chapters. We also include some known results about first order expressibility of some notions in several classes of structures. More precisely, we consider some axiomatizable classes and check whether these classes can be finitely axiomatizable or not. We include proofs of some of these results to illustrate the power and beauty of Łoś's Theorem.

3.1 Ultraproducts

Ultraproducts are model theoretic objects which are constructed by defining an equivalence relation, with respect to an ultrafilter, on the cartesian product of some $\mathcal{L}$-structures. The significance of this construction lies in the fact that first order properties of '*most*' of the structures in the cartesian product are transferred to their ultraproduct. As ultrafilters are the main ingredient of this construction we start by defining ultrafilters.

Let I be a non-empty set and $P(I)$ be the set of all subsets of I. An *ultrafilter* $\mathcal{U}$ on I is a set $\mathcal{U} \subset P(I)$ satisfying the following properties:

(a) $I \in \mathcal{U}$

(b) If $X, Y \in \mathcal{U}$ then $X \cap Y \in \mathcal{U}$

(c) If $X \in \mathcal{U}$ and $X \subseteq Y \subseteq I$ then $Y \in \mathcal{U}$

(d) For all $X \in P(I)$, $X \in \mathcal{U}$ if and only if $(I \backslash X) \notin \mathcal{U}$

The first three conditions together with the assumption $\emptyset \notin \mathcal{U}$ define a *filter*. Note that this assumption about the empty set becomes superfluous once we have the last condition.

It follows from the definition that ultrafilters are maximal filters. It can be checked that the union of any chain of filters is again a filter. Therefore, the existence of ultrafilters is guaranteed by the existence of filters together with Zorn's Lemma.

If the index set I is infinite then it is possible to define different ultrafilters on I. For example, the collection of all *co-finite subsets* of I, that is, the subsets with finite complements forms a filter which is called *Fréchet filter* and this filter can be extended to an ultrafilter by using Zorn's Lemma. For another example, fix some $i \in I$ and consider the set $\{X \subseteq I \mid i \in X\}$. It is routine to check that this set forms an ultrafilter on I which is called the *principal ultrafilter* generated by i. If the index set I is finite or more generally if an ultrafilter $\mathcal{U}$ contains a finite set then it is not difficult to observe that $\mathcal{U}$ is a principal ultrafilter generated by some element in I. Therefore, every non-principal ultrafilter contains Fréchet filter, that is, it contains all co-finite sets. As a result of this, a non-principal ultrafilter can be thought of as a collection of '*huge sets*'.

Now, we are ready to define the ultraproduct.

Let $\{X_i \mid i \in I\}$ be a collection of non-empty structures in the same language and $\mathcal{U}$ be an ultrafilter on I. A relation $\sim_{\mathcal{U}}$ can be defined on the cartesian product $\prod_{i \in I} X_i$ as follows

$$x \sim_{\mathcal{U}} y \quad \text{if and only if} \quad \{i \in I \mid x(i) = y(i)\} \in \mathcal{U}$$

where $x, y \in \prod_{i \in I} X_i$ and $x(i)$, $y(i)$ denote the i^{th} coordinate of x, y respectively. The quotient of the cartesian product with respect to $\sim_{\mathcal{U}}$ is called the *ultraproduct*.

It is routine to check that $\sim_{\mathcal{U}}$ is an equivalence relation. In this equivalence relation, we identify two elements of the cartesian product if their coordinates agree *almost everywhere*, that is, on a set belonging to the ultrafilter. Elements of the ultraproduct are equivalence classes of the elements of the cartesian product with respect to $\sim_{\mathcal{U}}$. We denote the equivalence class of $x \in \prod_{i \in I} X_i$ by $[x]_{\mathcal{U}}$, more precisely,

$$[x]_{\mathcal{U}} = \left\{ y \in \prod_{i \in I} X_i \mid \{i \in I \mid y(i) = x(i)\} \in \mathcal{U} \right\}$$

Note that the algebraic operations on the cartesian product, which are defined componentwise, induce well-defined algebraic operations on the ultraproduct. This can be checked directly by applying the properties of the ultrafilters mentioned above (See (b), (c) in the definition of ultrafilters). Similarly, each relation $\mathcal{R}$ on the cartesian product induces a well-defined relation on the ultraproduct as follows:

$$[x]_{\mathcal{U}}\ \mathcal{R}\ [y]_{\mathcal{U}} \text{ if and only if } \{i \in I \mid x(i)\ \mathcal{R}\ y(i)\} \in \mathcal{U}$$

Throughout this thesis, the ultraproduct will be denoted by $\prod_{i \in I} X_i / \mathcal{U}$ or by $(X_i)_{\mathcal{U}}$ when we need to simplify the notation. If $X_i = X$ for all $i \in I$, then their ultraproduct, which is called an *ultrapower*, will be denoted by $X^I / \mathcal{U}$.

The *fundamental theorem of ultraproducts* is due to Jerzy Łoś and can be stated as follows:

Łoś's Theorem. Let $X = \prod_{i \in I} X_i / \mathcal{U}$ where $\mathcal{U}$ is an ultrafilter on I and X_i is a non-empty $\mathcal{L}$-structure for each $i \in I$. Then for any first order $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and for all elements $a_1, \ldots, a_n \in \prod_{i \in I} X_i$

$$X \models \varphi([a_1]_{\mathcal{U}}, \ldots, [a_n]_{\mathcal{U}}) \text{ if and only if } \{i \in I \mid X_i \models \varphi(a_1(i), \ldots, a_n(i))\} \in \mathcal{U}$$

In the proof of this theorem a well-known proof method in model theory, namely induction on the complexity of the formula, is used. The fact that $\mathcal{U}$ is an ultrafilter and not just a filter is needed for the negation step of the proof (See [8] or [19] for a proof).

As a particular case of the Łoś's Theorem, it follows that a first order sentence holds in the ultraproduct if and only if it holds in *almost all* of the structures, that is, in the structures indexed by a set belonging to the ultrafilter. This means that, first order properties of the ultraproduct is determined by the first order properties of the structures in the ultraproduct together with the choice of the ultrafilter. Therefore, in some sense, the ultraproduct construction is more powerful than some other ways of getting new structures out of several structures. For example, if we take cartesian product of fields we get a commutative ring with identity which is not a field since, for example, the non-zero element (1,0,0, $\ldots$) has no inverse. However, by taking ultraproduct of fields we get a field. More generally, first order axiomatizable structures such as groups, fields, algebraically closed fields are closed under taking ultraproducts. Moreover, ultraproduct construction guarantees the existence of infinite models of any theory with finite language which has infinitely many non-isomorphic finite models.

As another consequence of Łoś's Theorem, it can be observed that any structure can be elementarily embedded in its ultrapower. More precisely, there is a natural diagonal embedding of any structure A into its ultrapower $A^I/\mathcal{U}$ via the map:

$$a \longmapsto [f_a]_{\mathcal{U}}$$

where $f_a(i) = a$ for all $i \in I$ and this embedding is elementary by Łoś's Theorem. Moreover, this embedding is not onto unless A is finite. This argument clearly explains that cardinality need not be preserved under elementary equivalence and therefore elementary equivalence is weaker than isomorphism.

Now, we recall, without proofs, some important facts in the theory of ultraproducts.

The following two facts follow from the definition of the ultraproduct together with the properties of ultrafilters and they will be used in the sequel even without mention.

Fact 3.1.1. *Any set of maps* $\{\varphi_i : A_i \longrightarrow B_i \mid i \in I\}$ *induces a well-defined map*

$$\varphi_{\mathcal{U}} : \prod_{i\in I} A_i/\mathcal{U} \longrightarrow \prod_{i\in I} B_i/\mathcal{U}$$

for any ultrafilter $\mathcal{U}$.

Example. Let I be the set of all prime numbers and $\mathcal{U}$ be a non-principal ultrafilter on I. Let

$$\varphi_{p_i} : \ x_i \longmapsto {x_i}^{p_i}$$

be the standard Frobenius automorphism of $\overline{\mathbb{F}}_{p_i}$ for each prime $p_i \in I$. Then the following map is well-defined by Fact 3.1.1:

$$\varphi_{\mathcal{U}} : \ \prod_{p_i\in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U} \longrightarrow \prod_{p_i\in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U}$$

$$[x_i]_{\mathcal{U}} \longmapsto [{x_i}^{p_i}]_{\mathcal{U}}$$

It is routine to check that $\varphi_{\mathcal{U}}$ is an automorphism of $\prod_{p_i\in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U}$. This automorphism is called the *non-standard Frobenius automorphism* and it was studied by Macintyre in the context of generic automorphisms of fields in [41].

Let $Fix(\varphi_{\mathcal{U}})$ denote the elements of $\prod_{p_i\in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U}$ which are fixed by the automorphism $\varphi_{\mathcal{U}}$. Then, by *Łoś's Theorem* we get

$$Fix\,(\varphi_{\mathcal{U}}) = \prod_{p_i\in I} Fix(\varphi_{p_i})/\mathcal{U} = \prod_{p_i\in I} \mathbb{F}_{p_i}/\mathcal{U}$$

Fact 3.1.2. *For any ultrafilter* $\mathcal{U}$ *on* I*, we have*

$$\prod_{i\in I} (A_i \times B_i)/\mathcal{U} \cong \prod_{i\in I} A_i/\mathcal{U} \times \prod_{i\in I} B_i/\mathcal{U}$$

This isomorphism easily generalizes to finite direct products.

An algebraic characterization of elementary equivalence is given by *Keisler-Shelah's Ultrapower Theorem* which can be stated as follows:

Fact 3.1.3. *Two structures A and B are elementarily equivalent if and only if there is a set I and an ultrafilter $\mathcal{U}$ on I such that $A^I/\mathcal{U} \cong B^I/\mathcal{U}$.*

This fact was first proved by Keisler in 1964 in the article [37] by assuming the generalized continuum hypothesis and then Shelah provided a proof in [56] which is free from the generalized continuum hypothesis.

The following fact is about the cardinalities of ultraproducts of countably many finite structures. We denote the cardinality of a set X by $card(X)$.

Fact 3.1.4. (Bell and Slomson [8]) *Let $\{X_i \mid i \in I\}$ be a countable collection of finite sets and $\mathcal{U}$ be an ultrafilter on I. If for some integer n,*

$$\{i \in I \mid card(X_i) = n\} \in \mathcal{U}$$

then $card(\prod_{i \in I} X_i/\mathcal{U}) = n$, otherwise $card(\prod_{i \in I} X_i/\mathcal{U}) = 2^{\aleph_0}$

The following result is important and it will be used several times throughout the text.

Fact 3.1.5. (Bell and Slomson §6.2 in [8]) *If the ultrafilter $\mathcal{U}$ is defined on a set I which is the disjoint union of finitely many subsets $I_1, \ldots, I_m$, then exactly one of I_j is in $\mathcal{U}$. Moreover, $\mathcal{U}_j = \{X \cap I_j \mid X \in \mathcal{U}\}$ is an ultrafilter on I_j and*

$$\prod_{i \in I} X_i/\mathcal{U} \cong \prod_{i \in I_j} X_i/\mathcal{U}_j$$

Remark 3.1.1. Let $\mathcal{U}$ denote an ultrafilter on I. Throughout this thesis, when we say that *'a property holds for almost all $i \in I$'* we mean that the property holds in the structures indexed by a set J belonging to the ultrafilter $\mathcal{U}$. Moreover, by Fact 3.1.5, the original ultraproduct $\prod_{i \in I} X_i/\mathcal{U}$ is isomorphic to the ultraproduct $\prod_{j \in J} X_j/\mathcal{U}_J$ where $\mathcal{U}_J$ is the ultrafilter $\{X \cap J \mid X \in \mathcal{U}\}$. As an immediate consequence of this, it follows that ultraproducts over principal ultrafilters are isomorphic to one of the structures in the cartesian product. Therefore, throughout this thesis the ultrafilters we consider will always be non-principal. Moreover, whenever we have a situation as mentioned above, that is, whenever a property holds for almost all $i \in I$, we will abuse the language and we will not change the index set and the ultrafilter. More precisely, we will assume, without loss of generality, that the property holds in all of the structures in the ultraproduct.

3.2 Pseudofinite Structures

The motivation for introducing pseudofinite structures stems from field theory, more precisely from the work of James Ax on the first order theory of finite fields in [6]. Ax calls a field F *pseudofinite* if F is perfect, has exactly one extension of degree n for each integer $n > 1$ and every irreducible variety over F has an F-rational point. Note that the second condition implies that the absolute Galois group of F is the profinite completion of $\mathbb{Z}$ denoted by $\hat{\mathbb{Z}}$ (see Chatzidakis [20]). Any field satisfying the third condition is called *pseudo algebraically closed* which is abbreviated as PAC. It can be observed that the three conditions describing pseudofinite fields can be expressed in a first order way in the language of rings [6]. Moreover, first two of them are satisfied by all finite fields while the last condition is not satisfied by any finite field. As it was pointed out by Ax in [6], the last condition can be expressed by infinite collection of elementary statements and each elementary statement holds in all *sufficiently large* finite fields. For each statement, the fields which are sufficiently large can be determined explicitly by a theorem of Lang-Weil which provides a constant for this purpose (see [6] for details). As a result, the three conditions hold in any non-principal ultraproduct of non-isomorphic finite fields and so any infinite field which is elementarily equivalent to such an ultraproduct turns out to be pseudofinite. On the other hand, Ax proved that pseudofinite fields are exactly the infinite models of the theory of finite fields, that is, they are elementarily equivalent to a non-principal ultraproduct of non-isomorphic finite fields.

Motivated by Ax's characterization of pseudofinite fields in [6], Felgner introduced pseudofinite groups as infinite models of the theory of finite groups.

More generally, a *pseudofinite structure* in a language $\mathcal{L}$ is an infinite model of the theory of finite structures in the common language $\mathcal{L}$. By a suitable choice of an ultrafilter, it can be shown that any pseudofinite structure is elementarily equivalent to a non-principal ultraproduct of finite structures with a common language [65]. As a result, an $\mathcal{L}$-structure is pseudofinite if and only if it is elementarily equivalent to a non-principal ultraproduct of non-isomorphic finite $\mathcal{L}$-structures.

For a detailed discussion of the results obtained by Ax as well as some other important results in the theory of finite and pseudofinite fields, we refer the reader to the survey article [20] by Chatzidakis.

Unfortunately, unlike pseudofinite fields, an algebraic characterization is not known for pseudofinite groups. Therefore, it is difficult to determine whether a given group is pseudofinite or not.

We would like to mention three results in the theory of pseudofinite groups which are important for us. One of them is a theorem by Point [50] which can be stated as follows:

Fact 3.2.1. (Point [50]) *Let $\{X(F_i) \mid i \in I\}$ be a family of Chevalley or twisted Chevalley groups of the same type X over finite or pseudofinite fields, and let $\mathcal{U}$ be a non-principal ultrafilter on the set I. Then*

$$\prod_{i \in I} X(F_i)/\mathcal{U} \cong X(\prod_{i \in I} F_i/\mathcal{U})$$

If Fact 3.2.1 is combined with Keisler-Shelah's Ultrapower Theorem then the following result can be obtained:

Fact 3.2.2. (Wilson [64]) *Any group G which is elementarily equivalent to a Chevalley or twisted Chevalley group over a pseudofinite field is pseudofinite.*

In particular, it follows that Chevalley and twisted Chevalley groups over pseudofinite fields are pseudofinite groups. They are examples of simple pseudofinite groups since Chevalley and twisted Chevalley groups are simple except for the ones over some small finite fields. Actually, the next result shows that they are the only examples of simple pseudofinite groups up to elementary equivalence.

Fact 3.2.3. (Wilson [64]) *Every simple pseudofinite group is elementarily equivalent to a Chevalley or twisted Chevalley group over a pseudofinite field.*

This classification was given by John S. Wilson in [64]. In the proof, the classification of finite simple groups and Fact 3.2.1 are used. Wilson's theorem and a version of it for definably simple pseudofinite groups will be the considered in detail in the next chapter.

3.3 Definability and Applications of Łoś's Theorem

Expressibility of some properties of groups in a first order way are important for the purposes of this thesis. Throughout this thesis, in general, we work up to elementary equivalence and use several first order properties. Therefore, this section is devoted to write down explicitly the first order formulas which define some subsets of groups. Moreover we express some first order properties of groups and fields in the language of groups and rings. While, most of these arguments are standard and well-known among model theorists, we include the proofs of the ones which may not be widely known. Moreover, we illustrate several nice applications of Łoś's Theorem.

Throughout the text, we work in the language of groups $\mathcal{L} = \{\ \cdot\ , ^{-1}, 1\}$ or rings $\mathcal{L} = \{\ \cdot\ , +, -, 0, 1\}$ and the language will be clear from the context.

Let x, y denote the elements of the structure (group or field in our context). We use the following abbreviations:

- $p \rightarrow q$ is read as p *implies* q and it is an abbreviation for $\neg p \vee q$.
- $x \neq y$ is an abbreviation for $\neg x = y$.
- We omit $\cdot$ and write xy instead of $x \cdot y$.
- x^n is an abbreviation for $\underbrace{x \cdot x \cdots x}_{n \text{ times}}$ where n is a positive integer.
- nx is an abbreviation for $\underbrace{x + \cdots + x}_{n \text{ times}}$ where n is a positive integer.

- x^y is an abbreviation for $y^{-1}xy$.
- $[x, y]$ is an abbreviation for $x^{-1}y^{-1}xy$.

3.3.1 Definable Subgroups and Subsets

In this subsection, we express some definable subsets and subgroups of a group G in the language of groups.

(a) Let $\varphi_Z(y)$ be the formula

$$\forall x \ \ (xy = yx)$$

It is a formula without parameters and with a free variable y which defines the center of the group G, that is, it defines

$$Z(G) = \{g \in G \mid [x, g] = 1 \text{ for all } x \in G\}$$

(b) Let $\varphi_{C_k}(x, x_1, \dots, x_k)$ be the formula

$$[x, x_1] = 1 \wedge \cdots \wedge [x, x_n] = 1$$

It is a formula with parameters $x_1, \dots, x_k$ and a free variable x. It defines the centralizer of the elements $x_1, \dots, x_k$ which is denoted by $C_G(x_1, \dots, x_k)$.

(c) Let $\varphi_{CC_k}(y, x_1, \dots, x_k)$ be the formula

$$\forall x \ \ (\varphi_{C_k}(x, x_1, \dots, x_k) \to [x, y] = 1)$$

It is a formula with parameters $x_1, \dots, x_k$ and a free variable y. It defines the double centralizer of the elements $x_1, \dots, x_k$ which is denoted by $C_G(C_G(x_1, \dots, x_k))$.

(d) Let $\varphi_K(x, a)$ be the formula

$$\exists y \ (a^y = x)$$

It is a formula with the parameter a and a free variable x which defines the conjugacy class of a in G, that is, it defines the set

$$\{g \in G \mid g = a^y \text{ for some } y \in G\}$$

(e) In a group G, the set of products of two commutators can be defined by the following formula:

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \ \ (x = [x_1, x_2][x_3, x_4])$$

We denote this formula by $\varphi_2(x)$. Clearly, this formula can be generalized for defining the set of products of n commutators which will be denoted by $\varphi_n(x)$.

(f) Let A be a set defined by the formula $\varphi_A(x)$ and let $\varphi_{C(A)}(y)$ be the formula

$$\forall x \ (\varphi_A(x) \to [x, y] = 1)$$

Then $\varphi_{C(A)}(y)$ defines the centralizer of A, that is, it defines

$$C_G(A) = \{g \in G \mid [a, g] = 1 \text{ for all } a \in A\}$$

Similarly, the normalizer of A, that is

$$N_G(A) = \left\{g \in G \mid g^{-1}Ag = A \text{ for all } g \in G\right\}$$

can be defined by the formula

$$\forall x \ \ (\varphi_A(x) \to \varphi_A(x^y))$$

We denote this formula by $\varphi_{N(A)}(y)$.

(g) Let $A_1, \ldots, A_k$ be sets defined by the formulas $\varphi_{A_1}(x), \ldots, \varphi_{A_k}(x)$ respectively. Then the setwise product $A_1 A_2 \cdots A_k$ is defined by the formula

$$\exists y_1 \cdots \exists y_k \ \ (\varphi_{A_1}(y_1) \wedge \cdots \wedge \varphi_{A_k}(y_k) \wedge x = y_1 \cdots y_k)$$

We denote this formula by $\varphi_{P_k}(x)$.

(h) If H is a finite subgroup of order n in a group G, then the multiplication table of H can be expressed by a formula, say $\varphi_H(x_1, \ldots, x_n)$.

Then the sentence

$$\exists x_1, \dots, x_n \,(\varphi_H(x_1, \dots, x_n))$$

holds in a group if and only if that group has a subgroup which is isomorphic to H.

(i) Let $\varphi_I(z, a)$ be the formula

$$\forall y \exists x \ \ ([x, a] = 1 \wedge z = x^y)$$

It is a formula with the parameter a and a free variable z which defines the intersection of all conjugates of $C_G(a)$, that is, $\bigcap_{g \in G} C_G(a)^g$.

(j) In a group G, the commutator subgroup G' is not necessarily definable. Although, any element of G' is a product of finite number of commutators, that is, it has finite *commutator length*, there may not be a uniform bound for the commutator lengths of the elements of G. However, if there is such a bound, that is, if G has finite *commutator width* then we can define the commutator subgroup. We will write the explicit formula defining G' when the commutator width of G is two. This can be easily generalized for groups with finite commutator width.

Every element of the commutator subgroup of a group G is a product of two commutators if and only if the following sentence holds in G:

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 \forall x_5 \forall x_6 \exists y_1 \exists y_2 \exists y_3 \exists y_4 \ \ ([x_1, x_2][x_3, x_4][x_5, x_6] = [y_1, y_2][y_3, y_4])$$

In this case, G' can be defined as the set of products of two commutators. More precisely, the following formula defines G':

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \ \ (x = [x_1, x_2][x_3, x_4])$$

Clearly, the set of products of any fixed number of commutators can be defined similarly.

Note that the commutator subgroups of groups of finite Morley rank are definable by Zil'ber's Indecomposability Theorem (See Fact 2.4.11).

(k) The subgroup generated by the conjugacy class of an element $a \in G$ is not necessarily definable. However, if there is a uniform bound on the number of conjugates of a in order to express each element of the subgroup $\langle a^G \rangle$, then $\langle a^G \rangle$ becomes definable. For example, in the connected groups of finite Morley rank the subgroups generated by the conjugacy class of an element are definable and the bound depends only on the Morley rank by Zil'ber's Indecomposability Theorem.

Another important result follows from these observations. Namely, any group H which is elementarily equivalent to a simple group of finite Morley rank is simple. To see this, let σ be the sentence:

$$\forall x \forall y \exists z_1 \ldots \exists z_k \ \ (x = y^{z_1} \cdots y^{z_k})$$

This sentence holds in a simple group G of finite Morley rank since any conjugacy class in G generates G. Here k is a fixed integer which depends only on the Morley rank of G. Note that any group elementarily equivalent to a group of finite Morley rank n also has Morley rank n [13]. Therefore, if $G \equiv H$ then H is a group of finite Morley rank such that $rk(G) = rk(H)$. Assume that H is not simple. This means that there is a conjugacy class in H which does not generate H. Combining this with Zil'ber's Indecomposability Theorem, it can be observed that the following sentence holds in H:

$$\exists x \exists y \forall z_1 \ldots \forall z_k \ \ (x \neq y^{z_1} \cdots y^{z_k})$$

Obviously this sentence does not hold in G. As a result H is simple.

3.3.2 First Order Properties of Groups

Groups can be axiomatized by the following sentences in $\mathcal{L} = \{\ \cdot\ ,^{-1}, 1\}$:

- $\forall x \forall y \forall z \ ((xy)z = x(yz))$
- $\forall x \ (x \cdot 1 = x \wedge 1 \cdot x = x)$
- $\forall x \ (x^{-1}x = 1 \wedge xx^{-1} = 1)$

We will denote the conjunction of these group axioms by σ_G.

Now, some first order properties of groups can be listed as follows:

(a) $\forall x \forall y \ (xy = yx)$ is a sentence stating commutativity.

(b) The property that a group has finite number of elements can not be expressed by a first order sentence. However, for a fixed n, having exactly n elements can be expressed in a first order way. Let σ_n denote the sentence

$$\exists x_1 \ldots \exists x_n \ \ (x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_{n-1} \neq x_n)$$

This sentence says that there are at least n elements. Therefore, $\sigma_n \wedge \neg \sigma_{n+1}$ holds in a group if and only if the group has exactly n elements.

(c) For any integer $n \geq 2$, let σ_n be the sentence

$$\forall x \exists y \ (y^n = x)$$

Then a group G is divisible if and only if the set of sentences $\{\sigma_n \mid n \geq 2\}$ hold in G.

(d) For any integer $n \geq 2$, let σ_n be the sentence

$$\forall x \ (x^n = 1 \rightarrow x = 1)$$

Then a group G is torsion-free if and only if the set of sentences $\{\sigma_n \mid n \geq 2\}$ hold in G.

(e) Let A be a non-empty definable subset of a group G defined by the formula $\varphi_A(x)$. Then we can express that A is a subgroup of G by the following first order sentence σ_A

$$\forall x \forall y \ (((\varphi_A(x) \wedge \varphi_A(y)) \rightarrow \varphi_A(xy)) \wedge \forall z \ \ (\varphi_A(z) \rightarrow \exists t \ (\varphi_A(t) \wedge zt = 1)))$$

(f) Let A be a non-empty definable subset of a group G defined by the formula $\varphi_A(x)$ and σ_A be the sentence defined as in part (e). Then the property of being a proper normal subgroup can be expressed by the following first order sentence σ_N

$$\forall x \forall y\ (\varphi_A(x) \to \varphi_A(x^y)) \wedge \exists z\ (\neg\varphi_A(z)) \wedge \sigma_A$$

(g) G is a solvable group of derived length at most 2 if and only if the following sentence holds in G

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4\ \ ([[x_1, x_2], [x_3, x_4]] = 1)$$

Being a solvable group of derived length at most n, for a fixed integer n, can also be defined similarly.

(h) G is a nilpotent group of nilpotency class at most 2 if and only if the following sentence holds in G

$$\forall x_1 \forall x_2 \forall x_3\ \ ([[x_1, x_2], x_3] = 1)$$

For any fixed integer n, being a nilpotent group of class n can be expressed similarly.

(i) The property of having no non-trivial abelian normal subgroups can be expressed in the first order language of groups. More precisely, the sentence

$$\forall x\ (x \neq 1 \to \exists y\ [x, x^y] \neq 1)$$

holds in a group G if and only if G has no non-trivial abelian normal subgroups.

To see this, firstly assume that G is a group with an abelian non-trivial normal subgroup N. Then for any $1 \neq n \in N$, $[n, n^g] = 1$ for all $g \in G$ since N is abelian. Therefore the sentence defined above can not hold in G. For the other direction, assume that the sentence above does not hold in G, that is, there is a non-trivial element x such that $[x, x^y] = 1$ for all $y \in G$. As a result, $\langle x^G \rangle$ becomes a non-trivial abelian normal subgroup of G.

(j) Let σ be the sentence

$$\forall x \forall y\ ((x \neq 1 \wedge y \neq 1) \rightarrow \exists z\ [y, x^z] \neq 1)$$

Then, σ holds in a group if and only if the centralizer of any non-trivial normal subgroup is trivial.

A similar argument as in part (i) gives this result. More precisely, assume that G has a non-trivial normal subgroup N whose centralizer has a non-trivial element n. Then, $[n, x^g] = 1$ for any $x \in N$ and for all $g \in G$ and therefore the sentence above does not hold in G. For the other direction, assume that the sentence above does not hold in G, that is, there are non-trivial elements x, y (not necessarily distinct), such that $[y, x^z] = 1$ for all $z \in G$. As a result, we get a non-trivial normal subgroup of G, namely $\langle x^G \rangle$, whose centralizer contains y.

(k) The property of having centralizer dimension k can be expressed in the language of groups. To see this, let $\varphi_{C_k}(x, x_1, \ldots, x_k)$ be the formula defining $C_G(x_1, \ldots, x_k)$ as given in Section 3.3.1 (b). Now, for any $k \geq 2$, let σ_k be the sentence

$$\forall x_1 \cdots \forall x_k \forall y_1 \cdots \forall y_{k-1}\ (\varphi_{C_1}(y_1, x_1) \wedge \neg\varphi_{C_2}(y_1, x_1, x_2) \wedge \varphi_{C_2}(y_2, x_1, x_2) \wedge$$

$$\neg\varphi_{C_3}(y_2, x_1, x_2, x_3) \wedge \cdots \wedge \varphi_{C_{k-1}}(y_{k-1}, x_1, \ldots, x_{k-1}) \wedge \neg\varphi_{C_k}(y_{k-1}, x_1, \ldots, x_k))$$

$$\rightarrow \forall x \forall y\ (\varphi_{C_k}(x, x_1, \ldots, x_k) \rightarrow [x, y] = 1)$$

Note that this statement is written for $k \geq 2$ since the groups with centralizer dimension 0 and 1 are abelian (see Remark 2.4.1) and abelian groups are defined by the sentence $\forall x \forall y\ ([x, y] = 1)$.

Now, we claim that σ_k holds in a group if and only if the group has centralizer dimension less than or equal to k.

Firstly, assume that a group G has centralizer dimension strictly bigger than k. This means that we can find $x_1, \ldots, x_{k+1}$ such that

$$C_G(1) > C_G(x_1) > C_G(x_1, x_2) > \cdots > C_G(x_1, x_2, \ldots, x_{k+1})$$

However, then the sentence above does not hold in G. For the converse, assume that the sentence σ_k does not hold in G. For simplicity, let p denote:

$$\forall x_1 \cdots \forall x_k \forall y_1 \cdots \forall y_{k-1}\ (\varphi_{C_1}(y_1, x_1) \wedge \neg\varphi_{C_2}(y_1, x_1, x_2) \wedge \varphi_{C_2}(y_2, x_1, x_2) \wedge$$

$$\neg\varphi_{C_3}(y_2, x_1, x_2, x_3) \wedge \cdots \wedge \varphi_{C_{k-1}}(y_{k-1}, x_1, \ldots, x_{k-1}) \wedge \neg\varphi_{C_k}(y_{k-1}, x_1, \ldots, x_k))$$

and q denote:

$$\forall x \forall y\ (\varphi_{C_k}(x, x_1, \ldots, x_k) \rightarrow [x, y] = 1)$$

Now, the sentence σ_k can be expressed as $p \rightarrow q$. Then $\neg\sigma_k$ becomes $p \wedge \neg q$ where $\neg q$ is

$$\exists x \exists y\ (\varphi_{C_k}(x, x_1, \ldots, x_k) \wedge [x, y] \neq 1)$$

It is not difficult to observe that this argument guarantees the existence of a centralizer chain of length strictly greater than k.

We conclude that, $\sigma_k \wedge \neg\sigma_{k-1}$ holds in a group if and only if the group has centralizer dimension k.

3.3.3 First Order Properties of Fields

It is well-known that fields can be axiomatized by the following first order sentences in the language $\mathcal{L} = \{\ \cdot\ , +, -, 0, 1\}$ of rings:

- Axioms for additive abelian groups.
- $0 \neq 1$
- $\forall x \forall y \forall z\ ((xy)z = x(yz))$
- $\forall x\ (x \cdot 1 = 1 \cdot x = x)$
- $\forall x \forall y \forall z\ (x(y + z) = xy + xz)$
- $\forall x \forall y \forall z\ ((x + y)z = xz + yz)$

- $\forall x \forall y\ (xy = yx)$
- $\forall x\ (x \neq 0 \rightarrow \exists y\ xy = 1)$

We will denote the conjunction of these field axioms by σ_F.

Now, some first order properties of fields can be listed as follows:

(a) Let σ_p be the sentence $\forall x\ (px = 0)$ for some prime number p. This sentence holds in a field F if and only if F has characteristic p.

(b) Let σ_p denote the sentence $\forall x\ (px = 0)$. Then the set of sentences $\{\neg\sigma_p \mid p > 1\}$ hold in a field F if and only if F has characteristic zero.

(c) Let σ_n be the sentence

$$\forall x_0 \ldots \forall x_n\ (x_n \neq 0 \rightarrow \exists y\ x_n y^n + x_{n-1} y^{n-1} + \ldots + x_1 y + x_0 = 0)$$

This sentence says that every polynomial of degree n has a root. Therefore, a field F is algebraically closed if and only if the infinite set of sentences $\{\sigma_n \mid n \geq 2\}$ hold in F.

(d) Recall that, a field F is *perfect* if either $char(F) = 0$ or $char(F) = p$ and every element is a p^{th} power. For each prime p, let σ_p denote the sentence

$$(p \cdot 1 = 0) \rightarrow \forall x \exists y\ (y^p = x)$$

Then, a field F is perfect if and only if σ_p holds in F for some prime p.

3.3.4 Some Applications of Łoś's Theorem

We start by some well-known classes of structures which are axiomatizable by infinitely many first order sentences and apply Łoś's Theorem to show that these classes are not finitely axiomatizable (See [8]). After that, we present some results concerning the possibility of expressing the notion of simplicity in several classes of groups [65].

Torsion-free groups. The class of torsion-free groups is axiomatizable by the set of sentences $\{\sigma_n \mid n \geq 2\} \cup \{\sigma_G\}$ where σ_n is

$$\forall x \; (x^n = 1 \rightarrow x = 1)$$

as mentioned in Section 3.3.2 (d) and σ_G axiomatizes groups. However, this class is not finitely axiomatizable. This can be observed as follows.

Consider an ultraproduct of cyclic groups of prime orders p_i over an ultrafilter on the set I of all prime numbers. Then for any fixed $n \geq 2$, the set $\{p_i \in I \mid C_{p_i} \models \sigma_n\}$ is co-finite because in a cyclic group of order p_i, the orders of non-trivial elements are p_i and so σ_n holds in C_{p_i} for all $p_i > n$. Therefore, for each $n \geq 2$, σ_n holds in the ultraproduct by Łoś's Theorem. As a result, $\prod_{p_i \in I} C_{p_i}/\mathcal{U}$ is torsion-free. If there were a sentence stating torsion-freeness then that sentence would hold in the ultraproduct and hence in almost all of the factors. However, none of the groups in the ultraproduct is torsion-free.

Divisible groups. The class of divisible groups is axiomatizable by the set of sentences $\{\sigma_n \mid n \geq 2\} \cup \{\sigma_G\}$ where σ_n is

$$\forall x \exists y \; (y^n = x)$$

We will show that this class is not finitely axiomatizable.

Let us consider $\prod_{p_i \in I} C_{p_i}/\mathcal{U}$ as in the previous argument. Then for any fixed $n \geq 2$, σ_n holds in C_{p_i} for all $p_i > n$ since p_i and n are relatively prime. As a result, the ultraproduct is divisible. However, none of the factors is divisible since σ_{p_i} does not hold in C_{p_i} for any prime p_i.

Fields of characteristic zero. The class of fields of characteristic zero can be axiomatized by the set of sentences $\{p_i \in I \mid \neg\sigma_{p_i}\} \cup \{\sigma_F\}$ where I is the set of all primes, σ_{p_i} is the sentence $\forall x \; (p_i x = 0)$ as defined in Section 3.3.3 (a) and σ_F is the sentence that axiomatizes fields. However, this class is not finitely axiomatizable. This can be observed as follows.

Let us consider the field $\prod_{p_i \in I} \mathbb{F}_{p_i}/\mathcal{U}$ where $\mathcal{U}$ is a non-principal ultrafilter on the set of primes I and $\mathbb{F}_{p_i}$ denotes the field with p_i elements. For each fixed prime p_0,

$$\{p_i \in I \mid \mathbb{F}_{p_i} \models \neg\sigma_{p_0}\} \in \mathcal{U}$$

since there is only one field of characteristic p_0 in the factors of the ultraproduct. As a result of Łoś's Theorem, the set of sentences $\{p_i \in I \mid \neg\sigma_{p_i}\}$ hold in $\prod_{p_i \in I} \mathbb{F}_{p_i}/\mathcal{U}$. Therefore the ultraproduct has characteristic zero while none of its factors is of characteristic zero.

Groups with descending chain condition for centralizers. As we showed in Section 3.3.2 (k), having centralizer dimension k is a first order property of groups. However, the descending chain condition for centralizers can not be expressed by a first order sentence. This can be observed by considering the ultraproduct of groups with increasing centralizer dimensions.

Let us consider the alternating groups $Alt(n_i)$ for $n_i \geq 2$. It can be checked that centralizers of products of even number of disjoint transpositions in $Alt(n_i)$ form a centralizer chain of length at least $\lfloor n_i/4 \rfloor$ where $\lfloor n_i/4 \rfloor$ denotes the integer part of $n_i/4$. Let

$$G = \prod_{i \in I} Alt(n_i)/\mathcal{U}$$

be an infinite group where $\mathcal{U}$ is an ultrafilter on the set I. Now, we will construct a descending chain of centralizers of infinite length in G.

For any integer $k \geq 1$, let $[x^k]_{\mathcal{U}}$ denote the image of $x^k \in \prod_{i \in I} Alt(n_i)$ where

$$x^k(i) = (4k-3, 4k-2)(4k-1, 4k)$$

for all $i \in I$ such that $n_i \geq 4k$ and identity elsewhere. We claim that

$$C_G([x^1]_{\mathcal{U}}) > C_G([x^1]_{\mathcal{U}}, [x^2]_{\mathcal{U}}) > \dots$$

forms an infinite proper descending chain of centralizers. To see this, it is enough to prove that for any fixed k

$$C_G([x^1]_{\mathcal{U}}, \dots, [x^k]_{\mathcal{U}}) > C_G([x^1]_{\mathcal{U}}, \dots, [x^{k+1}]_{\mathcal{U}})$$

which means that the chain does not stabilizes after finitely many steps. Fix a positive integer k. It is clear that, for all $i \in I$ such that $n_i \geq 4k+4$, there is $y(i) \in Alt(n_i)$ with the property that

$$y(i) \in C_{Alt(n_i)}\left(x^1(i), \ldots, x^k(i)\right) \quad \text{but} \quad y(i) \notin C_{Alt(n_i)}\left(x^1(i), \ldots, x^{k+1}(i)\right)$$

Therefore,

$$I_m = \{i \in I \mid y(i)x^m(i) = x^m(i)y(i)\} \in \mathcal{U}$$

for all $1 \leq m \leq k$ and

$$J = \left\{i \in I \mid y(i)x^{k+1}(i) \neq x^{k+1}(i)y(i)\right\} \in \mathcal{U}$$

As a result we have

$$[y]_{\mathcal{U}} \in C_G([x^1]_{\mathcal{U}}, \ldots, [x^k]_{\mathcal{U}}) \quad \text{but} \quad [y]_{\mathcal{U}} \notin C_G([x^1]_{\mathcal{U}}, \ldots, [x^{k+1}]_{\mathcal{U}})$$

Now, we consider the expressibility of the notion of simplicity in some classes of groups.

Abelian groups. Simplicity can not be expressed by a first order sentence in the class of abelian groups. To see this, it is enough to consider the ultraproduct of finite cyclic groups of prime order over an ultrafilter on the set I of all prime numbers. It is clear that $\prod_{p_i \in I} C_{p_i}/\mathcal{U}$ is an infinite abelian group and hence not simple. If there were a first order sentence stating simplicity then it would hold in the ultraproduct as all C_{p_i}'s are simple.

Finite abelian groups. ([39]) In the class of finite abelian groups we can not express simplicity by an elementary statement. This can be observed as follows.

Assume that there is a first order sentence σ such that σ holds in a *finite* abelian group G if and only if G is simple. Let

$$G_1 = \prod_{p_i \in I} C_{p_i}/\mathcal{U} \quad \text{and} \quad G_2 = \prod_{p_i \in I} C_{(p_i)^2}/\mathcal{U}$$

where I is the set of all prime numbers and $\mathcal{U}$ is a non-principal ultrafilter on I. Clearly, $G_1 \models \sigma$ and $G_2 \models \neg\sigma$ by Łoś's Theorem. On the other hand, from our previous observations we know that G_1 and G_2 are divisible abelian and torsion-free groups. Moreover, G_1 and G_2 are uncountable by Fact 3.1.4. It is a well-known fact that groups with these properties are isomorphic [33]. As a result, such a σ does not exist as isomorphic groups must satisfy the same first order sentences.

Finite non-abelian groups. Felgner proved that simplicity is an elementary statement in the class of finite non-abelian groups in his article [31]. In other words, he showed that there is a first order sentence τ such that τ holds in a *finite* group G if and only if G is non-abelian simple. He used the classification of finite simple groups in the proof.

His proof can be summarized as follows.

Firstly, Felgner considers the following statement which holds in non-abelian simple groups:

$$\forall x_1 \forall x_2 \ \left(x_1 \neq 1 \wedge C_G(x_1, x_2) \neq 1 \to \bigcap_{g \in G} (C_G(x_1, x_2) C_G(C_G(x_1, x_2)))^g = 1 \right)$$

Remark 3.3.1. Note that the statement given above can be expressed by a first order sentence in the language of groups. The argument goes as follows.

Let $\varphi_{C_2}(x, x_1, x_2)$, $\varphi_{CC_2}(x, x_1, x_2)$ and $\varphi_P(x, x_1, x_2)$ denote the formulas which define $C_G(x_1, x_2)$, $C_G(C_G(x_1, x_2))$ and $C_G(x_1, x_2)C_G(C_G(x_1, x_2))$ respectively as in Section 3.3.1 (b), (c), (g). Now, it can be observed that the formula

$$\forall x \exists y \ (\varphi_P(y, x_1, x_2) \wedge z = y^x)$$

defines the intersection

$$\bigcap_{g \in G} (C_G(x_1, x_2) C_G(C_G(x_1, x_2)))^g$$

We denote this formula by $\varphi_I(z, x_1, x_2)$. The following first order sentence σ expresses the statement above in the language of groups:

$$\forall x_1 \forall x_2 \ (x_1 \neq 1 \wedge \exists y \ (y \neq 1 \wedge \varphi_{C_2}(y, x_1, x_2)) \to \forall z \ (\varphi_I(z, x_1, x_2) \to z = 1))$$

Felgner shows that in a finite group satisfying σ, the subgroup generated by minimal normal subgroups is non-abelian simple. After this point, he uses the following well-known properties of finite non-abelian simple groups which follow from the classification:

- Every finite simple group is generated by two elements.
- For any non-abelian finite simple group S, the structure of $Aut(S)/Inn(S)$ is well-known.
- There is an integer k such that each element of each non-abelian finite simple group G is a product of k commutators.

Note that the last property can be expressed by a first order sentence (Section 3.3.1 (e)), say δ. Felgner proves that $\sigma \wedge \delta$ holds in a finite non-abelian group G if and only if G is simple.

Felgner also proves that this kind of argument can not be generalized to the class of all non-abelian simple groups. This is explained in the following item.

Non-abelian groups. Simplicity can not be expressed by an elementary statement in the class of non-abelian groups. This follows from the following fact:

Fact 3.3.1. (Wilson [65]) *Let $G = \prod_{i \in I} Alt(n_i)/\mathcal{U}$ be an infinite group such that $n_i \geq 5$ for all $i \in I$ and $\mathcal{U}$ is a non-principal ultrafilter on the set I. Then G is not simple.*

We include a proof of this fact in order to illustrate a nice and non-trivial application of Łoś's Theorem.

Proof. Note that $\prod_{i \in I} Alt(n_i)/\mathcal{U}$ is finite when the orders of the alternating groups in the ultraproduct are bounded. Since G is infinite, we assume that n_i's are arbitrarily large in the ultraproduct. Let x be the element of the cartesian product $\prod_{i \in I} Alt(n_i)$ such that $x(i) = (12)(34)$ for all $i \in I$ and let $[x]_{\mathcal{U}}$ denote the image of x in the ultraproduct $\prod_{i \in I} Alt(n_i)/\mathcal{U}$.

Claim. The group generated by the conjugacy class of $[x]_{\mathcal{U}}$ in $\prod_{i\in I} Alt(n_i)/\mathcal{U}$ is a proper normal subgroup of $\prod_{i\in I} Alt(n_i)/\mathcal{U}$.

Let $[y]_{\mathcal{U}}$ be the image of $y \in \prod_{i\in I} Alt(n_i)$ where $y(i) = (123\cdots n_i)$ if n_i is odd and $y(i) = (123\cdots n_i - 2)(n_i - 1, n_i)$ if n_i is even. It is clear that $y(i) \in Alt(n_i)$ has no fixed points. Now, the claim will follow if we prove that $[y]_{\mathcal{U}}$ can not be written as a product of finitely many conjugates of $[x]_{\mathcal{U}}$. Assume that

$$[y]_{\mathcal{U}} = [x]_{\mathcal{U}}^{[d_1]_{\mathcal{U}}} \cdots [x]_{\mathcal{U}}^{[d_k]_{\mathcal{U}}}$$

for some integer $k \geq 1$. Then, by Łoś's Theorem we have

$$\left\{ i \in I \mid y(i) = x(i)^{d_1(i)} \cdots x(i)^{d_k(i)} \right\} \in \mathcal{U}$$

On the other hand, since $x(i)$ moves at most 4 points, the product of k conjugates of $x(i)$ moves at most $4k$ points and so fixes at least $n_i - 4k$ points. As a result, for all $i \in I$ such that $n_i > 4k$, any product of k conjugates of $x(i)$ has fixed points. However, $y(i)$ is fixed-point-free and $n_i > 4k$ for almost all $i \in I$. Therefore, $y(i)$ can not be written as a product of k conjugates of $x(i)$ for almost all $i \in I$. Since this holds for any choice of k, we conclude that $[y]_{\mathcal{U}}$ is not an element of the group generated by the conjugates of $[x]_{\mathcal{U}}$. Therefore, the claim holds and so $\prod_{i\in I} Alt(n_i)/\mathcal{U}$ is not simple.

□

CHAPTER 4

SIMPLE AND DEFINABLY SIMPLE PSEUDOFINITE GROUPS

In the first section of this chapter, we mention the importance of simple groups, especially in the theory of finite groups and we state the Classification of Finite Simple Groups. Moreover, we state Ulrich Felgner's conjecture about the structure of simple pseudofinite groups as well as the result obtained by John S. Wilson which partially answers this conjecture. We summarize Wilson's proof by emphasizing the key points in his argument.

In the second section, we prove a version of Wilson's theorem by combining some of the facts in his proof with some other ideas which are available in our context. More precisely, we classify *definably simple* pseudofinite groups of finite centralizer dimension up to elementary equivalence.

In the last section, we discuss the possibility of replacing elementary equivalence with isomorphism. A brief overview of the literature will give the desired replacement.

4.1 Simple Pseudofinite Groups

In group theory, it is always desirable to classify simple groups in a particular class of groups. As it is mentioned by Solomon in his article [57], the origin of this desire goes back to the last years of nineteenth century when Otto Hölder explained his feelings about the possibility of classifying finite simple groups. The Jordan-Hölder Theorem gives some insight about the importance of simple groups in finite group theory. Namely, simple groups can be considered as building blocks of finite groups.

The Classification of Finite Simple Groups has been completed in the 1980's as a result of incredible efforts of dozens of mathematicians. According to the classification, any finite simple group belongs to one of the following families up to isomorphism:

(a) Cyclic groups of prime orders.

(b) Alternating groups $Alt(n)$, for $n \geq 5$.

(c) Finite simple groups of Lie type

$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, {}^2A_n, {}^2D_n, {}^2E_6, {}^3D_4, {}^2B_2, {}^2F_4, {}^2G_2$

(d) 26 Sporadic groups.

Note that there are infinitely many non-isomorphic groups in each family, except for the sporadic family which has 26 members.

There are some projects to simplify the original proof since it is quite long and complicated. First simplification project, which is called the *second generation classification*, was initiated by Daniel Gorenstein. Several volumes of this second generation proof have been published up to now. Meanwhile, there is another simplification project, called *third generation classification*, carried out by Ulrich Meierfrankenfeld, Bernd Stellmacher, Gernot Stroth et al. These two projects are still in progress.

As mentioned before, in his article [31], Ulrich Felgner introduced pseudofinite groups as infinite models of the theory of finite groups in accordance with Ax's characterization of pseudofinite fields as infinite models of the theory of finite fields. The Classification of Finite Simple Groups has a particular importance in the class of pseudofinite groups since finite and pseudofinite groups are indistinguishable by their first order properties. Felgner believed in the possibility of classifying simple pseudofinite groups by using the Classification of Finite Simple Groups [31] and made the following conjecture:

> Simple pseudofinite groups are isomorphic to Chevalley or twisted Chevalley groups over pseudofinite fields.

Although Felgner obtained important results in the direction of a classification, it was John S. Wilson who classified simple pseudofinite groups, however, up to elementary equivalence [64].

The result obtained by Wilson can be stated as follows:

Fact 4.1.1. (Wilson [64]) *Every simple pseudofinite group is elementarily equivalent to a Chevalley or twisted Chevalley group over a pseudofinite field.*

It is pointed out in Wilson's article that some of the strategy used in his proof is due to Felgner. In the rest of this section, we will summarize the proof of Fact 4.1.1 given by Wilson.

As a first step, by using the ideas in Felgner's article [31], Wilson proves that every simple pseudofinite group is elementarily equivalent to an ultraproduct of non-abelian finite simple groups. The following three facts are used for the proof of this step.

Fact 4.1.2. (Wilson [64]) *There is an integer k such that each element of each finite non-abelian simple group G is a product of k commutators.*

Note that this is a weaker version of Ore's Conjecture which was stated in Section 2.3. The classification of finite simple groups is used in the proof of this fact.

Before stating the next fact used in the proof of the first step, recall that a non-trivial normal subgroup of a group G is called *minimal normal subgroup* if it does not properly contain a non-trivial normal subgroup of G. Throughout this thesis, $Soc(G)$ stands for the subgroup generated by minimal normal subgroups of a group G, the so called *socle* of G.

Fact 4.1.3. (Felgner [31], Wilson [64]) *Let σ denote the following statement*

$$\forall x_1 \forall x_2 \left(x_1 \neq 1 \wedge C_G(x_1, x_2) \neq 1 \rightarrow \bigcap_{g \in G} (C_G(x_1, x_2) C_G(C_G(x_1, x_2)))^g = 1 \right)$$

(a) If G is a non-abelian simple group, then $G \models \sigma$, that is, σ holds in G.

(b) If G is finite and $G \models \sigma$, then $Soc(G)$ is a non-abelian simple group.

As we mentioned in Section 3.3.4, σ is a first order sentence which was defined by Felgner in order to prove that simplicity is definable in the class of finite non-abelian groups. Note that the Classification of Finite Simple Groups is not used in the proof of this fact.

Fact 4.1.4. (Wilson [64]) *Let G be a finite group with a non-abelian simple socle. If G is not simple then $G' \neq G$. Moreover, if every element of $Soc(G)$ is a product of k commutators, then every element of G' is a product of $k+3$ commutators.*

In the proof of this fact, some properties of finite simple groups which follow from the classification are used.

After achieving the first step by using the facts stated above, Wilson considers all possible ultraproduct constructions and eliminates all but one of the cases by following the strategy explained below.

Since there are finitely many families of finite simple groups, without loss of generality, he assumes that just one of the families occur in the ultraproduct by Fact 3.1.5. Sporadic case is eliminated immediately since ultraproduct of groups of bounded order is a finite group.

Then, Wilson proves the following fact:

Fact 4.1.5. (Felgner [31], Wilson [64]) *Let G be an infinite group such that $G \equiv \prod_{i\in I} Alt(n_i)/\mathcal{U}$ where $n_i \geq 5$ for all $i \in I$. Then G is not simple.*

This fact was first proved by Felgner in [31], however, Wilson provides a proof of it in his article [64] as well. Note that this result is stronger than the non-simplicity of an ultraproduct of simple non-isomorphic alternating groups (see Fact 3.3.1). The idea of the proof is similar to that of Fact 3.3.1, however, this time first order characterizations of some elements of alternating groups need to be used rather than the actual elements.

After that, Wilson considers the case where the finite simple groups in the ultraproduct are classical groups. This case is split into two parts by considering the classical groups over fields of odd and even characteristics separately. Wilson characterizes some involutions in classical groups in terms of their first order properties. Combining these with some well-known properties of classical groups, Wilson puts a bound on the Lie ranks of the classical groups in the ultraproduct.

As a result of these reductions, Wilson is left with the case where all of the factors in the ultraproduct are Chevalley or twisted Chevalley groups of the same Lie type and same Lie rank. Therefore, he completes the proof by referring to the result of Point (see Fact 3.2.1).

4.2 Definably Simple Pseudofinite Groups

In the previous section, we mentioned the importance of the classification of simple groups and we concentrated on the classification of finite and pseudo finite ones. One of other well-known classes of simple groups is the class of simple algebraic groups over algebraically closed fields as we mentioned in the introduction. Note that in the context of algebraic groups the terminology is slightly different: Simple groups are allowed to have finite centers such that the quotient is simple as an abstract group. In any case, their classification is well-known [35]. Unfortunately, there are still some classes of simple groups which have not been determined yet. For example, as mentioned in the introduction, there is an ongoing project for classifying infinite simple groups of finite Morley rank.

There is a slightly weaker version of simplicity which arises in model theory. A group is called *definably simple* if it has no non-trivial definable proper normal subgroups. Note that in the category of algebraic groups, definable sets and Zariski closed sets are not the same. However, definable subgroups and Zariski closed subgroups coincide. For the latter claim, one direction is clear since any closed subgroup can be defined as the zero set of finitely many polynomials. For the other direction, that is, for the proof of the fact that definable subgroups of algebraic groups are closed see Lemma 7.4.9 in [45]. Moreover, in the category of non-abelian groups of finite Morley rank, which includes non-abelian algebraic groups over algebraically closed fields, definably simple groups coincide with the simple ones [51]. However, in general, definably simple groups need not be simple. Pseudofinite groups provides an example for this.

Namely, any non-principal ultraproduct of simple non-isomorphic alternating groups is definably simple but not simple (see Fact 3.3.1). Therefore, the distinction between the notions *definably simple* and *simple* becomes important in the theory of pseudofinite groups.

At this point, we would like to analyze the structure of definably simple abelian groups in the context of pseudofinite groups. Clearly, infinite abelian groups can not be simple, however, they can be definably simple. Let A be a definably simple abelian pseudofinite group. First of all, A should be torsion-free since any non-trivial element of finite order would generate a non-trivial proper definable normal subgroup of A. Moreover, A is divisible, because otherwise for some integer $n \geq 2$, there is $y \in A$ such that $y \neq nx$ for all $x \in A$. Therefore, the set $nA = \{nx \mid x \in A\}$ forms a proper definable normal subgroup of A which is not possible as A is definably simple. Therefore A is a divisible abelian torsion-free group. Moreover, we know that $A \equiv A^I/\mathcal{U}$ and $A^I/\mathcal{U}$ is uncountable for any non-principal ultrafilter $\mathcal{U}$ on a countable set I by Fact 3.1.4. On the other hand, it is clear that, as an additive group $\mathbb{Q}^I/\mathcal{U}$ is a divisible abelian torsion-free group of uncountable cardinality which is elementarily equivalent to the additive group of $\mathbb{Q}$, denoted by $\mathbb{Q}^+$. Since torsion-free divisible abelian groups of uncountable cardinality are isomorphic, we have

$$A \equiv A^I/\mathcal{U} \cong \mathbb{Q}^I/\mathcal{U} \equiv \mathbb{Q}^+$$

that is, $A \equiv \mathbb{Q}^+$. Moreover, as we observed in Section 3.3.4, $\prod_{i\in I} C_{p_i}/\mathcal{U}$ is also an uncountable divisible abelian torsion-free group. As a result, we get

$$A \equiv \prod_{i\in I} C_{p_i}/\mathcal{U} \equiv \mathbb{Q}^+$$

Having analyzed the structure of definably simple abelian pseudofinite groups up to elementary equivalence, we will concentrate on definably simple non-abelian pseudofinite groups from now on. Note that in our context, the pseudofinite group in consideration arises as a subgroup of a finite Morley rank group. Since finite Morley rank groups have finite centralizer dimension and this property is inherited by subgroups, it is natural to consider pseudo finite groups with finite centralizer dimension. Therefore, we will deal with definably simple pseudofinite groups with finite centralizer dimension and we will prove the following theorem.

Theorem 4.2.1. *Let G be a definably simple non-abelian pseudofinite group of finite centralizer dimension. Then G is elementarily equivalent to a Chevalley or twisted Chevalley group over a pseudofinite field.*

Proof. As a first step, we show that G is elementarily equivalent to an ultraproduct of non-abelian finite simple groups by applying some methods used by Wilson in his proof. The proofs of the facts given in the previous section are due to Wilson and when it is necessary we give his proof by emphasizing how it works in our situation. In the next step, we take different direction from that of Wilson by using our assumption on centralizer chains.

Step 1. Every definably simple non-abelian pseudofinite group is elementarily equivalent to an ultraproduct of non-abelian finite simple groups.

Let σ be the sentence defined in Fact 4.1.3. We show that the proof of the Fact 4.1.3(a) in Wilson's article [64] works under the weaker assumption that G is definably simple.

Lemma 4.2.1. *Let G be a definably simple non-abelian group. Then, σ holds in G.*

Proof. Assume that σ does not hold in G, that is, there are $x_1, x_2 \in G$ such that for $x_1 \neq 1$ and $C_G(x_1, x_2) \neq 1$ we have

$$\bigcap_{g \in G} (C_G(x_1, x_2) C_G(C_G(x_1, x_2)))^g \neq 1$$

Let

$$N = \bigcap_{g \in G} (C_G(x_1, x_2) C_G(C_G(x_1, x_2)))^g$$

As we observed in Remark 3.3.1, N is a definable subgroup of G defined by the formula $\varphi_I(z, x_1, x_2)$. Moreover, it is clear that N is normalized by G. Since G is definably simple and N is non-trivial by our assumption, we get $G = N$. As a result we have,

$$C_G(x_1, x_2) C_G(C_G(x_1, x_2)) = G$$

Therefore, $C_G(x_1, x_2)$ is normalized by G. Moreover, since $C_G(x_1, x_2)$ is a non-trivial definable subgroup of G, it can not be proper in G, that is, we have $C_G(x_1, x_2) = G$. Clearly, this is not possible since $Z(G) = 1$ as a definable subgroup of the non-abelian group G. Therefore, we get a contradiction to our assumption and we conclude that σ holds in G.

□

Now, let G be a definably simple non-abelian pseudofinite group. Then $G \equiv \prod_{i \in I} X_i/\mathcal{U}$ where each X_i is a non-abelian finite group and $\mathcal{U}$ is a non-principal ultrafilter. Moreover, σ holds in G by Lemma 4.2.1. Therefore,

$$\{i \in I \mid X_i \models \sigma\} \in \mathcal{U}$$

by *Loś's Theorem*, that is, σ holds in almost all of the groups in the ultra-product. Without loss of generality, we may assume that σ holds in all of the groups X_i by Remark 3.1.1. Now, as $X_i \models \sigma$, it follows by Fact 4.1.3(b) that $Soc(X_i)$ is a non-abelian simple group for each $i \in I$. If X_i is not simple then, by Fact 4.1.4, $X_i{}'$ is the set of all products of $k + 3$ commutators where k is the integer given by Fact 4.1.2. Hence, the first order formula $\varphi_{k+3}(x)$ which was defined in Section 3.3.1 (e), defines $X_i{}'$. Moreover, as explained in Section 3.3.2 part (f) there is a first order sentence τ which expresses that $X_i{}'$ is a proper normal subgroup of X_i. Now, if X_i is non-simple for all i in a set belonging to $\mathcal{U}$ then the sentence τ holds in almost all of the X_i's and hence in G. Therefore, in G we get a definable proper normal subgroup. This is not possible as G is definably simple. Therefore, X_i is a non-abelian finite simple group for almost all $i \in I$. Again by referring to Remark 3.1.1 and by abusing the language we can conclude that

$$G \equiv \prod_{i \in I} X_i/\mathcal{U}$$

where X_i is a non-abelian finite simple group for all $i \in I$.

Remark 4.2.1. Note that the Classification of Finite Simple groups is not needed for identifying $Soc(X_i)$'s with non-abelian simple groups. However, classification is used in the rest of the proof.

Remark 4.2.2. If the argument about definably simple abelian pseudofinite groups is combined with the result obtained in Step 1, we get the following result:

> Every definably simple pseudofinite group is elementarily equivalent to an ultraproduct of finite simple groups.

Step 2. In this step, by using the classification of finite simple groups, we analyze all possible ultraproduct constructions by taking into account the finite centralizer dimension property.

Since there are three main families of non-abelian finite simple groups as given in Section 4.1, any ultraproduct of finite non-abelian simple groups is isomorphic to an ultraproduct of members of exactly one family by Fact 3.1.5. Therefore, we have

$$G \equiv \prod_{i \in I} X_i / \mathcal{U}$$

where $\{X_i \mid i \in I\}$ is a collection of non-abelian finite simple groups from the same family. The possibilities are analyzed below.

Case 1. *Sporadic Groups*

Since there are finitely many sporadic groups, without loss of generality, we may assume that all X_i's are the same sporadic group X. However, having exactly n elements is a first order property as explained in Section 3.3.2 part (b). Therefore, $G \equiv X^I/\mathcal{U}$ has the same order as X by Łoś's Theorem. We eliminate this case since G is a pseudofinite group which is infinite by definition.

Case 2. *Alternating Groups*

If there is a bound m on the orders of the alternating groups appearing in the ultraproduct, then

$$J = \{i \in I \mid X_i = Alt(n)\} \in \mathcal{U}$$

for some $n \leq m$ and

$$\prod_{i \in I} X_i/\mathcal{U} \cong (Alt(n))^J/\mathcal{U}_{\mathcal{J}}$$

As in the previous case this is a finite group, actually isomorphic to $Alt(n)$, so it is eliminated.

Therefore, suppose that there are alternating groups with arbitrarily large orders in the ultraproduct. Let us consider the following centralizer chain constructed from products of even number of disjoint transpositions in $Alt(n)$.

$$C_{Alt(n)}(1) > C_{Alt(n)}((12)(34)) > \cdots > C_{Alt(n)}((12)(34), \ldots, (k-3, k-2)(k-1, k))$$

Here, $k = n$ or $k = n - 1$ depending on the parity of n. Clearly, it is a centralizer chain of length $\lfloor n/4 \rfloor$. Since n is not bounded in the ultraproduct, there is no bound on the centralizer dimensions of the alternating groups in the ultraproduct and we get an infinite descending chain of centralizers (See the related argument in Section 3.3). Since G has finite centralizer dimension and this is a first order property, G can not be elementarily equivalent to an ultraproduct of alternating groups of arbitrarily large orders.

Case 3. *Groups of Lie type*

This case can be split into two subcases:

(a) *No bound on Lie ranks*

In this case, all X_i's are from one of the infinite families of Classical groups $A_{n_i}, B_{n_i}, C_{n_i}, D_{n_i}, {}^2A_{n_i}, {}^2D_{n_i}$ and there is no bound on the Lie ranks of the groups occurring in the ultraproduct.

The structure of centralizers of semisimple elements of classical groups are well-known [32]. More precisely, in a classical group of type X, there are some special semisimple elements whose centralizers contain classical group of type X of lower rank. In any classical group, this allows one to construct a descending chain of centralizers whose length increases with the rank. As a result, if there is no bound on the Lie ranks in the ultraproduct, the ultraproduct can not have finite centralizer dimension.

(b) *Lie ranks are bounded*

If there is a bound on the Lie ranks, then without loss of generality, we may assume that all X_i's are the groups of same Lie type with fixed Lie rank n. In this case, we can conclude that G is elementarily equivalent to a Chevalley or twisted Chevalley group over a pseudofinite field by the result of Point (see Fact 3.2.1).

$\square$

4.3 Elementary Equivalence versus Isomorphism

In this section, we will try to clarify the relation between elementary equivalence and isomorphism in some special cases. We will do this by giving a brief survey of the results obtained in this direction in the literature. We will be able to conclude that, any group which is elementarily equivalent to a Chevalley or twisted Chevalley group over a field K is itself a Chevalley or twisted Chevalley group of the same Lie type over a field which is elementarily equivalent to K. As a result, we can replace elementary equivalence with isomorphism in Wilson's theorem as well as in definably simple version of Wilson's theorem.

It is clear that, isomorphic structures are elementarily equivalent, however the converse does not hold in general. This can be seen immediately by considering the ultrapower of any countably infinite group G over a non-principal ultrafilter $\mathcal{U}$ on a countable index set I. We know that, $G \equiv G^I/\mathcal{U}$, however, they are not isomorphic as the ultrapower of G is uncountable (see Fact 3.1.4). We can give another example which is free from ultraproducts. It is a well-known fact in model theory that any two algebraically closed fields K_1, K_2 of the same characteristic are elementarily equivalent (see [45]). However, they need not be isomorphic as in the case of algebraic closure of rational numbers and complex numbers. As mentioned in the Introduction, an algebraic characterization of elementary equivalence is given by Keisler-Shelah Ultrapower Theorem which states that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers.

In the case of Chevalley groups, more can be said about the relation between the notions of isomorphism and elementary equivalence. Simon

Thomas worked on the elementary properties of Chevalley groups in his dissertation [63]. He proved that any group which is elementarily equivalent to a group of the form $X(K)$, where X denotes one of the types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ and K stands for any field, is itself a group of type $X(K)$. In other words, he showed that being a Chevalley group $X(K)$ is first order axiomatizable. However, his work did not include a similar result for twisted Chevalley groups.

For the twisted Chevalley groups over pseudofinite fields, a positive answer to the question of finite axiomatizability has been obtained by M. Ryten [55] very recently. In his dissertation, Ryten obtains some results on the uniform parameter bi-interpretations between groups and families of finite fields which yield finite axiomatizations of Chevalley and twisted Chevalley groups over pseudofinite fields.

CHAPTER 5

PROOF OF THE MAIN THEOREM

In this chapter, we give some information about the origin of the conjecture we work on and try to explain where it sits in the classification project of infinite simple groups of finite Morley rank. Moreover, we analyze an example in the algebraic group context which is useful for having a better understanding of the conjecture. After that, we concentrate on the conjecture and we prove the main theorem of this thesis.

Last section of this chapter is devoted to summarizing possible ways of proving the Intermediate Conjecture by combining the results obtained in this thesis with some results in the theory of groups of finite Morley rank. Moreover, we outline a program which aims to avoid the use of the classification of finite simple groups in the proof.

Throughout this chapter, the notation will be as follows:

For an automorphism α of a group G, we write $C_G(\alpha)$ for the subgroup of G which is fixed elementwise by α, that is,

$$C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$$

The definable closure of a set X is denoted by $d(X)$ when the universe we work in is clear from the context. The centralizer dimension of a group G is denoted by $c\text{-}dim(G)$.

5.1 Principal and Intermediate Conjectures

As mentioned in the introduction, there is a classification project for infinite simple groups of finite Morley rank which is based on the Cherlin-Zil'ber Algebraicity Conjecture. The aim of this project is to identify these groups with algebraic groups over algebraically closed fields, that is, with Chevalley groups over algebraically closed fields. Some ideas and techniques from finite group theory have been quite useful in this project and have led to important results. More precisely, among the four possible types of simple groups of finite Morley rank, the groups of even type were identified with simple algebraic groups over algebraically closed fields of characteristic 2 and the groups of mixed type were proven not to exist [1]. Some important restrictions have been obtained on the structures of hypothetical non-algebraic simple groups of odd and degenerate types ([15], [14]). Meanwhile, an alternative approach for handling this classification project was proposed by Hrushovski in [34]. The works on the fixed points of generic automorphisms of some structures were motivating developments for the birth of this alternative approach. Note that an automorphism α of a structure $\mathfrak{M}$ is called *generic* if $(\mathfrak{M}, \alpha)$ is an existentially closed model of the theory

$$T \cup \{\text{`}\alpha \text{ is an automorphism'}\}$$

where T is a complete theory with quantifier elimination. We would like to mention a few results supporting this alternative approach without going into details.

Firstly, it was proven that the fixed points of a generic automorphism of an algebraically closed field is a pseudofinite field [41], by using the algebraic characterization of pseudofinite fields. Then, Hrushovski worked on some other structures with generic automorphisms in [34]. He showed that the fixed points of generic automorphisms of the structures with some nice properties are PAC with Galois groups $\hat{\mathbb{Z}}$. Note that the notions PAC and *Galois group* which are relatively well-known in field theory can also be defined for arbitrary structures (See [34]). This result allowed Hrushovski to prove that any fixed point subgroup arising this way admits some kind of measure. In particular, these results hold for simple groups of finite Morley rank with generic automorphisms. A conjecture of Hrushovski can be composed from these observations:

Principal Conjecture. *Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Then $C_G(\alpha)$ is pseudofinite.*

Although the fixed point subgroup mimics some properties of pseudofinite groups as mentioned above, there is no known direct way of proving that a group is pseudofinite.

As it was pointed out by Hrushovski in [34], the Algebraicity Conjecture implies the Principal Conjecture. To be more precise, if the Algebraicity Conjecture holds then any infinite simple group of finite Morley rank is isomorphic to a Chevalley group over algebraically closed field. Therefore, the group in the Principal Conjecture can be identified with K-rational points $G(K)$ of an algebraic group G for an algebraically closed field K. Then, as mentioned in [34], the fixed point subgroup of the generic automorphism is $G(k)$ where k is a pseudofinite field.

What is more interesting and what is the motivation for the present work is the reverse direction, that is, derivation of the Algebraicity Conjecture from the Principal Conjecture. Our work is ultimately aimed at a proof of the following conjecture:

Intermediate Conjecture. *Let G be an infinite simple group of finite Morley rank with a generic automorphism of α. Assume that $C_G(\alpha)$ is pseudofinite. Then G is isomorphic to a Chevalley group over an algebraically closed field.*

Since generic automorphism is a model theoretical notion, we introduce the notion of a *tight automorphism* in order to work in a purely algebraic context.

An automorphism α of a group of finite Morley rank G is called *tight* if whenever a connected definable subgroup H of G is α-invariant, then

$$d(C_H(\alpha)) = H$$

In order to give some insight about the conjectures mentioned above, we analyze an example in the context of algebraic groups.

Example. Let G be a simple algebraic group of adjoint type defined over the prime subfield of the algebraically closed field $K = \prod_{p_i \in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U}$. Here, I is the set of all prime numbers p_i and $\mathcal{U}$ is a non-principal ultrafilter on I. We identify G with the group of its K-rational points, that is, with $G(K)$. Let α

be a non-standard Frobenius automorphism of K, that is,

$$\alpha = \prod_{p_i \in I} \varphi_{p_i} / \mathcal{U}$$

where φ_{p_i} is the standard Frobenius map $x \longmapsto x^{p_i}$. Then, α induces an automorphism of G which satisfies the following properties:

(a) $C_G(\alpha)$ is pseudofinite.

(b) If $H \leqslant G$ is a definable connected α-invariant subgroup of G, then $d(C_H(\alpha)) = H$.

Let $Fix_K(\alpha)$ denote the elements of the field K which are fixed by α. Then,

$$C_G(\alpha) = G(Fix_K(\alpha))$$

where

$$Fix_K(\alpha) = \prod_{p_i \in I} \mathbb{F}_{p_i} / \mathcal{U}$$

Moreover, by Łoś's Theorem it is not difficult to observe that

$$C_G(\alpha) = G(\prod_{p_i \in I} \mathbb{F}_{p_i} / \mathcal{U}) \cong \prod_{p_i \in I} G(\mathbb{F}_{p_i}) / \mathcal{U}$$

Therefore, $C_G(\alpha)$ is pseudofinite.

Since definable subgroups coincide with closed subgroups in the context of algebraic groups, H is a closed subgroup of G (See Section 4.2). Therefore, H is an algebraic group itself and $C_H(\alpha) = H(k)$ where k is a pseudofinite field. Since H is also connected, it follows by Fact 2.1.3 that $C_H(\alpha)$ is Zariski dense in H. Since definable closure of $C_H(\alpha)$ is Zariski closed, we have $d(C_H(\alpha)) = H$.

5.2 Observations about the Group of Fixed Points

Throughout this section, unless stated otherwise, G is a simple group of finite Morley rank and α is a tight automorphism of G. Moreover, we assume that $C_G(\alpha)$ is pseudofinite and denote it by P.

Lemma 5.2.1. *Let N be a non-trivial normal or subnormal subgroup of P. Then, $d(N) = G$.*

Proof. Firstly, let N be a non-trivial normal subgroup of P. Since G is α-invariant and connected we have

$$d(P) = d(C_G(\alpha)) = G$$

by the definition of tight automorphism. Moreover, as P normalizes N we get

$$d(N) \trianglelefteq d(P) = G$$

by part (c) of Fact 2.4.5. Since G is simple we have $d(N) = G$.

Now, if N is a non-trivial subnormal subgroup of P which is normalized by a normal subgroup M of P then $d(N)$ is normalized by $d(M)$. However, $d(M) = G$ by the first part of the proof. As a result, $d(N) = G$.

□

Corollary 5.2.1. *P does not contain any non-trivial normal or subnormal subgroups which are definable in G. In particular, P has no non-trivial finite normal or subnormal subgroups.*

Proof. For any non-trivial normal or subnormal subgroup N of P, we have $d(N) = G$ by Lemma 5.2.1. On the other hand, as N is definable in G, we have $d(N) = N$. This is not possible since N is a proper subgroup of G. For the second part, it is enough to emphasize that finite subgroups of a group are always definable since parameters are allowed.

□

Corollary 5.2.2. *G does not contain any non-trivial proper definable subgroups which are normalized by P.*

Proof. Let N be a non-trivial proper definable subgroup of G which is normalized by P. Then $d(N) = N$ is normalized by $d(P) = G$ which is not possible as G is simple.

□

Lemma 5.2.2. *Let N be a non-trivial normal or subnormal subgroup of P. Then, $C_G(N) = 1$.*

Proof. Let $1 \neq N$ be a normal or subnormal subgroup of P. It is clear that $N \leqslant C_G(C_G(N))$. Moreover, $d(N) = G$ by Lemma 5.2.1. Since $C_G(C_G(N))$ is a definable subgroup of G by Fact 2.4.8, we have

$$G = d(N) \leqslant C_G(C_G(N))$$

that is, $C_G(N) \leqslant Z(G)$. However, as a simple group G has trivial center and so $C_G(N) = 1$.

□

Corollary 5.2.3. *P has no non-trivial abelian normal or subnormal subgroups.*

Proof. Follows directly from Lemma 5.2.2

□

Lemma 5.2.3. *$P \equiv \prod_{i \in I} G_i/\mathcal{U}$ where each G_i is a finite group and $\mathcal{U}$ is a non-principal ultrafilter on I. Moreover, the following conditions hold:*

(a) $\prod_{i \in I} G_i/\mathcal{U}$ has finite centralizer dimension and so, there is a uniform bound for the centralizer dimensions of the groups in the ultraproduct.

(b) Each G_i is a finite group without non-trivial abelian normal subgroups.

(c) Centralizers of non-trivial normal subgroups of $\prod_{i \in I} G_i/\mathcal{U}$ are trivial.

Proof. (a) Since P is a pseudofinite group, it is clear that

$$P \equiv \prod_{i \in I} G_i/\mathcal{U}$$

where G_i is a finite group for all $i \in I$ and $\mathcal{U}$ is a non-principal ultrafilter on I. Moreover, as a subgroup of a finite Morley rank group P has finite centralizer dimension, let us say k. Since having centralizer dimension k is a first order property, as explained in Section 3.3.2 part (k), it is preserved under elementary equivalence. Therefore, we get

$$c\text{-}dim(\prod_{i \in I} G_i/\mathcal{U}) = k$$

(b) As explained in Section 3.3.2 part (i), there is a first order sentence σ which holds in a group if and only if the group has no non-trivial abelian normal subgroups. By Corollary 5.2.3, σ holds in P and hence $\prod_{i\in I} G_i/\mathcal{U} \models \sigma$ as well. It follows by *Łoś's Theorem* that

$$\{i \in I \mid G_i \models \sigma\} \in \mathcal{U}$$

Therefore, by Remark 3.1.1, we may assume that $P \equiv \prod_{i\in I} G_i/\mathcal{U}$ where each G_i has no non-trivial abelian normal subgroups.

(c) There is a first order sentence σ, given in Section 3.3.2 part (j), such that σ holds in a group if and only if the centralizers of non-trivial normal subgroups are trivial in that group. Therefore, $P \models \sigma$ by Lemma 5.2.2. As a result, $\prod_{i\in I} G_i/\mathcal{U} \models \sigma$.

□

Lemma 5.2.4. *Let $\bar{x}, \bar{y}$ denote the k-tuples $(x_1, \ldots, x_k), (y_1, \ldots, y_k)$ where x_i, y_i ($1 \leq i \leq k$) are elements of a group G. Then the following statement can be expressed by a first order sentence σ_2:*

If $\bar{x}, \bar{y}$ are k-tuples such that

$$C_G(C_G(\bar{x})) \cap C_G(C_G(\bar{y})) = 1 \quad \text{and} \quad [C_G(C_G(\bar{x})), C_G(C_G(\bar{y}))] = 1$$

then $C_G(C_G(\bar{x}))C_G(C_G(\bar{y}))$ can not be normalized by G.

Proof. For any k-tuple $\bar{x}$, the double centralizer of $\bar{x}$, that is, $C_G(C_G(\bar{x}))$ is defined by the formula $\varphi_{CC_k}(y, \bar{x})$ as in Section 3.3 part (c). Now, $C_G(C_G(\bar{x}))C_G(C_G(\bar{y}))$ can be defined by the formula:

$$\exists x \exists y \ (\varphi_{CC_k}(x, \bar{x}) \wedge \varphi_{CC_k}(y, \bar{y}) \wedge z = xy)$$

and we denote this formula by $\varphi_{PCC_k}(z, \bar{x}, \bar{y})$. Now, the following sentence σ_2 expresses the statement in the lemma:

$$\forall \bar{x} \forall \bar{y} \ (((\forall x \ (\varphi_{CC_k}(x, \bar{x}) \wedge \varphi_{CC_k}(x, \bar{y})) \to x = 1) \wedge$$
$$\wedge \forall y \forall z \ ((\varphi_{CC_k}(y, \bar{x}) \wedge \varphi_{CC_k}(z, \bar{y})) \to yz = zy))$$
$$\to \exists t \exists u \ (\varphi_{PCC_k}(t, \bar{x}, \bar{y}) \to \neg \varphi_{PCC_k}(t^u, \bar{x}, \bar{y})))$$

□

Remark 5.2.1. Lemma 5.2.4 can be generalized for any finite number of k-tuples. More precisely, for each finite number of k-tuples $\bar{x}_1, \cdots, \bar{x}_n$ for $n \geq 2$, there is a first order sentence σ_n which expresses the following statement:

> Let $\bar{x}_1, \cdots, \bar{x}_n$ be a collection of k-tuples. If for any pair of k-tuples $\bar{x}_i$ and $\bar{x}_j$ we have
>
> $C_G(C_G(\bar{x}_i)) \cap C_G(C_G(\bar{x}_j)) = 1$ and $[C_G(C_G(\bar{x}_i)), C_G(C_G(\bar{x}_j))] = 1$
>
> where $1 \leq i, j \leq n$ then $C_G(C_G(\bar{x}_1)) \cdots C_G(C_G(\bar{x}_n))$ can not be normalized by G.

Lemma 5.2.5. *The sentence σ_2 defined in Lemma 5.2.4 holds in P.*

Proof. Assume that σ_2 does not hold in P. This means that, there are k-tuples $\bar{x}, \bar{y}$ such that:

$$CC(\bar{x}) \cap CC(\bar{y}) = 1, \ [CC(\bar{x}), CC(\bar{y})] = 1 \text{ and } CC(\bar{x})CC(\bar{y}) \trianglelefteqslant P$$

It is clear that $CC(\bar{x})$ and $CC(\bar{y})$ are non-trivial as they contain the tuples $\bar{x}$ and $\bar{y}$ respectively. Moreover, they are proper subgroups of P since otherwise their intersection is $CC(\bar{x})$ or $CC(\bar{y})$ which are non-trivial contradicting to condition above. Therefore, $CC(\bar{x})$ is a non-trivial proper subnormal subgroup of P which is centralized by a non-trivial group $CC(\bar{y})$. However, this is contradictory to Lemma 5.2.2. Therefore, σ_2 holds in P.

$\square$

Remark 5.2.2. Note that the same argument in the proof of Lemma 5.2.5 can be generalized to show that σ_n, which was mentioned in Remark 5.2.1, holds in P for any $2 \leq n \leq k$ where k is the centralizer dimension of P.

In the next lemma, we show that P has an involution, that is, an element of order two by using the Feit-Thompson Odd Order Theorem together with the following result which was obtained by Khukhro as a corollary of his theorem in [38]:

Fact 5.2.1. (Khukhro [38]) *A pseudofinite group with finite centralizer dimension which is elementarily equivalent to an ultraproduct of solvable groups is solvable.*

Lemma 5.2.6. *P has an involution.*

Proof. Assume that there is no involution in P. Then, the sentence

$$\forall x\ (x^2 = 1 \rightarrow x = 1)$$

holds in P. Therefore it holds in almost all of the groups in the ultraproduct. By Remark 3.1.1, P is elementarily equivalent to an ultraproduct of groups of odd orders. Moreover, every finite group of odd order is solvable by Feit-Thompson theorem [30]. Since P has finite centralizer dimension and it is elementarily equivalent to an ultraproduct of solvable groups, P is solvable by Fact 5.2.1. Then $d(P)$ is solvable by Fact 2.4.6. This is not possible as $d(P) = G$ and G is a simple group.

□

5.3 Identification Theorem for the Group of Fixed Points

For simplicity of the notation, any ultraproduct $\prod_{i\in I} X_i/\mathcal{U}$ is denoted by $(X_i)_{\mathcal{U}}$ throughout this section. Moreover, for any subset X of $(G_i)_{\mathcal{U}}$, $C(X)$ and $N(X)$ stand for $C_{(G_i)_{\mathcal{U}}}(X)$ and $N_{(G_i)_{\mathcal{U}}}(X)$ respectively.

This section is devoted to prove the following theorem:

Theorem 5.3.1. *Let G be an infinite simple group of finite Morley rank and α be a tight automorphism of G. Assume that $C_G(\alpha)$ is pseudofinite. Then there is a definable (in $C_G(\alpha)$) normal subgroup S of P such that*

$$S \trianglelefteq C_G(\alpha) \leqslant Aut(S)$$

where S is isomorphic to a Chevalley or twisted Chevalley group over a pseudo finite field.

Proof. By Lemma 5.2.3 (b), $P \equiv (G_i)_{\mathcal{U}}$ where each G_i is a finite group without abelian normal subgroups. If G_i is simple for almost all i, then $(G_i)_{\mathcal{U}}$ is isomorphic to a Chevalley group over a pseudofinite field by the argument in

Section 4.2 Step 2 and so does P since we can replace elementary equivalence with isomorphism (See Section 4.3).

Now, assume without loss of generality that G_i is not simple for all i and let M_i denote a minimal normal subgroup of G_i. It is clear by Lemma 5.2.3 (b) that each M_i is non-abelian. Moreover, M_i is a direct product of non-abelian conjugate simple groups. To see this, let S_i be a minimal normal subgroup of M_i. Since $M_i \trianglelefteq G_i$, the subgroup $\langle (S_i)^g \mid g \in G_i \rangle$ generated by the conjugates of S_i is a normal subgroup of G_i contained in M_i. Since M_i is minimal normal subgroup of G_i we get

$$\langle (S_i)^g \mid g \in G_i \rangle = M_i$$

Moreover, as minimal normal subgroups of M_i, any pair of conjugates of S_i are either equal or intersect trivially and in the latter case they commute pairwise. Therefore,

$$M_i = S_i \times (S_i)^{g_{i1}} \times \ldots \times (S_i)^{g_{ik_i}}$$

where S_i is a non-abelian simple group. Note that if S_i has a proper non-trivial normal subgroup N_i then N_i is normalized by $(S_i)^{g_{ij}}$, for all $1 \leq j \leq k_i$ and hence by M_i. This is not possible as S_i is a minimal normal subgroup of M_i.

In the following three lemmas, we aim to show that almost all of the groups G_i in the ultraproduct have non-abelian simple socles.

Firstly, we prove that almost all of the groups in the ultraproduct have unique minimal normal subgroups.

Lemma 5.3.1. *G_i has a unique minimal normal subgroup for almost all i.*

Proof. Assume that G_i has two minimal normal subgroups M_i and N_i for almost all i. Since they are minimal normal,

$$M_i \cap N_i = 1 \quad \text{and} \quad [M_i, N_i] = 1$$

that is, N_i centralizes M_i for almost all i. It follows by Łoś Theorem that, $(N_i)_U$ centralizes $(M_i)_{\mathcal{U}}$ which is a normal subgroup of $(G_i)_{\mathcal{U}}$. However, centralizers of normal subgroups of $(G_i)_{\mathcal{U}}$ are trivial by Lemma 5.2.3 (c). Therefore, G_i has a unique minimal normal subgroup for almost all i.

□

Now, without loss of generality, we may assume that each G_i has unique minimal normal subgroup which is denoted by M_i. We will show that there is a uniform bound for the number of simple direct factors of M_i in the ultraproduct.

Lemma 5.3.2. *There is a bound n on the total number of simple direct factors of M_i in the ultraproduct.*

Proof. Let us consider the chain

$$C_{G_i}(s_i) > C_{G_i}(s_i, s_i^1) > \cdots > C_{G_i}(s_i, s_i^1, \ldots, s_i^{k_i})$$

where $s_i^j \in (S_i)^{g_{ij}} \setminus \{1\}$ for $0 \leq j \leq k_i$ with the abbreviations $s_i^0 = s_i$ and $(S_i)^{g_{i0}} = S_i$. Since each $(S_i)^{g_{ij}}$ has trivial center, the chain above is a proper descending chain whose length increases with the total number of direct factors of M_i. If there is no bound on this number, we get descending chain of centralizers of arbitrarily large length in the ultraproduct. This is not possible as $(G_i)_{\mathcal{U}}$ has finite centralizer dimension. Therefore, there is an integer $n \geq 1$ such that the total number of simple direct factors of M_i is bounded by n for almost all i.

□

As a result of Lemma 5.3.2, for almost all i, the number of simple direct factors of $Soc(G_i)$ is m for some integer $1 \leq m \leq n$. Without loss of generality we may assume that this holds for all G_i. Therefore, we have $P \equiv (G_i)_{\mathcal{U}}$ where each G_i is finite group without abelian normal subgroups and

$$(M_i)_{\mathcal{U}} \cong (S_i)_{\mathcal{U}} \times (S_i)_{\mathcal{U}}^{[g_{i1}]_{\mathcal{U}}} \times \ldots \times (S_i)_{\mathcal{U}}^{[g_{im-1}]_{\mathcal{U}}}$$

In the next lemma, we prove that M_i is a non-abelian simple group for almost all i.

Lemma 5.3.3. *M_i is a non-abelian simple group for almost all i.*

Proof. Assume that the claim is not true, that is,

$$(M_i)_{\mathcal{U}} \cong (S_i)_{\mathcal{U}} \times (S_i)_{\mathcal{U}}^{[g_{i1}]_{\mathcal{U}}} \times \ldots \times (S_i)_{\mathcal{U}}^{[g_{im-1}]_{\mathcal{U}}}$$

where $m > 1$. In order to simplify the notation, let $M_{S_0}, M_{S_1}, \ldots, M_{S_{m-1}}$ denote $(S_i)_{\mathcal{U}}, (S_i)_{\mathcal{U}}^{[g_{i1}]_{\mathcal{U}}}, \ldots, (S_i)_{\mathcal{U}}^{[g_{im-1}]_{\mathcal{U}}}$ respectively. Clearly,

$$M_{S_j} \leqslant CC(M_{S_j})$$

for all $0 \leq j \leq m-1$. Moreover, for any distinct pair M_{S_j}, M_{S_k} we have $CC(M_{S_j}) \cap CC(M_{S_k}) = 1$. To see this, let $x \in CC(M_{S_j}) \cap CC(M_{S_k})$, so that x centralizes both $C(M_{S_j})$ and $C(M_{S_k})$. It is clear that

$$M_{S_0} \times \cdots \times M_{S_{j-1}} \times M_{S_{j+1}} \times \cdots \times M_{S_{m-1}} \leqslant C(M_{S_j})$$

$$M_{S_0} \times \cdots \times M_{S_{k-1}} \times M_{S_{k+1}} \times \cdots \times M_{S_{m-1}} \leqslant C(M_{S_k})$$

Since $j \neq k$, it follows that x centralizes M_{S_j} for all $0 \leq j \leq m-1$ and hence x centralizes $(M_i)_{\mathcal{U}}$. However, $(M_i)_{\mathcal{U}} \trianglelefteq (G_i)_{\mathcal{U}}$ and we know that normal subgroups of $(G_i)_{\mathcal{U}}$ have trivial centralizers, so $x = 1$.

Now, as $M_{S_j} \leqslant C(M_{S_k})$ for $j \neq k$, we have

$$CC(M_{S_k}) \leqslant C(M_{S_j}) \leqslant N(M_{S_j})$$

Therefore, $CC(M_{S_k})$ normalizes M_{S_j} and so it normalizes $CC(M_{S_j})$ as well. Similarly, $CC(M_{S_j})$ normalizes $CC(M_{S_k})$. As a result we get

$$[CC(M_{S_j}), CC(M_{S_k})] \leqslant CC(M_{S_j}) \cap CC(M_{S_k}) = 1$$

for all $0 \leq j \neq k \leq m-1$. Moreover, $(G_i)_{\mathcal{U}}$ permutes the direct factors of the product $M_{S_0} \times \cdots \times M_{S_{m-1}}$ by conjugation, and so $CC(M_{S_0}) \times \cdots \times CC(M_{S_{m-1}})$ is permuted by $(G_i)_{\mathcal{U}}$ as well. Since $(G_i)_{\mathcal{U}}$ is a group with finite centralizer dimension k, there are k-tuples $\bar{x}_0, \ldots, \bar{x}_{m-1}$ such that

$$CC(M_{S_j}) = CC(\bar{x}_j)$$

for all $0 \leq j \leq m-1$. On the other hand, for each $2 \leq n \leq k$ the sentence σ_n which was mentioned in Remark 5.2.1 holds in P by Remark 5.2.2. In particular, σ_m holds in P and hence in $(G_i)_{\mathcal{U}}$. However, σ_m does not allow $(G_i)_{\mathcal{U}}$ to have the k-tuples $\bar{x}_0, \ldots, \bar{x}_{m-1}$ with the properties mentioned above. Therefore, we conclude that $(M_i)_{\mathcal{U}}$ has no non-trivial direct factors, in other words, M_i is a non-abelian finite simple group for almost all i.

□

From now on, without loss of generality, we may assume that

$$Soc(G_i) = M_i = S_i$$

is a non-abelian simple group for all i. Now, the situation can be summarized as follows.

$(G_i)_{\mathcal{U}}$ has a normal subgroup $(M_i)_{\mathcal{U}} = (S_i)_{\mathcal{U}}$ where each S_i is a non-abelian finite simple group. If $(S_i)_{\mathcal{U}}$ were finite then we could express the existence of a finite normal subgroup of $(G_i)_{\mathcal{U}}$ by a first order sentence as in Section 3.3.1 (h). This would yield a finite normal subgroup in P. However, this is not possible by Corollary 5.2.1. Now, by the same argument as in Step 2 of the proof of Theorem 4.2.1 we can conclude that $(S_i)_{\mathcal{U}}$ is isomorphic to a Chevalley group over a pseudofinite field. Actually more can be said about $(S_i)_{\mathcal{U}}$ and this is stated in the following lemma.

Lemma 5.3.4. *$(S_i)_{\mathcal{U}}$ is a definable normal subgroup of $(G_i)_{\mathcal{U}}$.*

Proof. We know that $(S_i)_{\mathcal{U}}$ is a Chevalley group over a pseudofinite field and so, almost all of the groups S_i are Chevalley groups over large fields. Therefore, as we discussed in Section 2.3, *Thompson's Conjecture* is applicable in this context. It follows that, for almost all i, there is a conjugacy class C_i in S_i such that $S_i = C_iC_i$. Let r_i be a representative of the conjugacy class C_i in S_i, that is, $\{(r_i)^{S_i}\} = C_i$. Let D_i denote the conjugacy class of r_i in G_i, that is, $\{(r_i)^{G_i}\} = D_i$. Since $r_i \in S_i$ and $S_i \trianglelefteq G_i$, it is clear that $D_i \subseteq S_i$. Moreover, D_i is a definable subset of G_i with one parameter r_i. Now, S_i can be defined as the set of products of two elements from the definable set D_i. Since this is true for all of the groups S_i in the ultraproduct, every element of $(S_i)_{\mathcal{U}}$ is a product of two elements from the definable set $\{((r_i)_{\mathcal{U}})^{(G_i)_{\mathcal{U}}}\}$ in $(G_i)_{\mathcal{U}}$. In other words, $(S_i)_{\mathcal{U}}$ is the set of products of two elements from the definable set $\{((r_i)_{\mathcal{U}})^{(G_i)_{\mathcal{U}}}\}$.

□

As we have observed in Lemma 5.3.4, the set of products of two elements from the definable set $\{((r_i)_{\mathcal{U}})^{(G_i)_{\mathcal{U}}}\}$, namely $(S_i)_{\mathcal{U}}$, is a proper normal subgroup of $(G_i)_{\mathcal{U}}$ which is isomorphic to a Chevalley group over a pseudofinite field. Therefore, the set S of products of two elements from the corresponding conjugacy class in P forms a normal subgroup of P. Moreover, as we have discussed in Section 4.3, being a Chevalley or twisted Chevalley group is a first order property. Therefore, S is isomorphic to a Chevalley or twisted Chevalley group over a pseudofinite field. Moreover, since

$$S \trianglelefteq P \quad \text{and} \quad C_P(S) = 1$$

P embeds in $Aut(S)$ and the theorem follows.

□

5.4 Outline of the Future Research Program

As briefly mentioned in the Introduction, the result obtained in this thesis can be used for further investigation of simple groups of finite Morley rank with generic automorphisms. In particular, the Intermediate Conjecture is expected to follow when we combine the results of this thesis with some results in the theory of groups of finite Morley rank.

In this section, we give details of the possible approaches mentioned in the Introduction in order to prove the Intermediate Conjecture. Moreover, we discuss the possibility of obtaining a proof without using the Classification of Finite Simple Groups.

Recall that an automorphism α of a group of finite Morley rank is called *supertight* if whenever a definable connected subgroup H of G is α^n-invariant for some non-negative integer n then

$$d(C_H(\alpha^n) = H$$

Assume that we have the following set-up:

> G is a simple group of finite Morley rank with a supertight automorphism α such that $C_G(\alpha^n)$ is pseudofinite for each non-negative integer n.

Can we identify G with a Chevalley group over an algebraically closed field?

First approach can be outlined as follows.

Let P_n denote $C_G(\alpha^n)$ for any integer $n \geq 1$. Then the following observations can be made without much difficulty:

- P_n is a simple by solvable pseudofinite group by the main theorem of this thesis. More precisely, P_n is an extension of a Chevalley or twisted Chevalley group S_n over a pseudofinite field such that P_n/S_n is solvable.

- $\bigcup_{n\geq 1} P_n$ is a Chevalley group over an algebraically closed field.

In this set-up, if it can be shown that $\bigcup_{n\geq 1} P_n$ is elementarily equivalent to G then we can identify G with a Chevalley group over an algebraically closed field. This identification follows as being a Chevalley group over an algebraically closed field can be expressed in a first order way.

The other approach is to use induction on the Morley rank and the Prüfer 2-rank of G. The plan can be outlined as follows.

- Simple groups of finite Morley rank of even type were identified with Chevalley groups over algebraically closed fields of characteristic 2 in the book [1]. Therefore even type case can be eliminated in our context.

- Simple groups of finite Morley rank of mixed type do not exist (See [1]).

- Simple groups of finite Morley rank of degenerate type have finite Sylow 2-subgroups by definition. Moreover, it was proven in [14] that degenerate type groups have no involutions at all. Therefore, if G is a simple group of degenerate type in our set-up, then $C_G(\alpha)$ is a pseudofinite group without involutions. Moreover, as $C_G(\alpha)$ has finite centralizer dimension, an application of the Feit-Thompson Theorem together with a result of Khukhro [38] force $C_G(\alpha)$ to be solvable which is not possible (Lemma 5.2.6). As a result, we can eliminate the degenerate type case from our configuration as well.

- We are left with the case where G is a simple group of finite Morley rank of odd type. We can divide this case into three subcases according to the Prüfer 2-ranks of G.

More precisely, we have the following subcases:

- Prüfer 2-rank of G is 1.

- Prüfer 2-rank of G is 2.

- Prüfer 2-rank of G is ≥ 3.

If the Prüfer 2-rank is 1, then the idea is to identify G with with $PSL_2(K)$, for some algebraically closed field K, by using the following identification theorem by Delehan and Nesin.

Fact 5.4.1. (Delehan and Nesin [26]) *Let G be an infinite split Zassenhaus group of finite Morley rank. If the stabilizer of two distinct points contains an involution then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from* 2.

Note that a doubly transitive permutation group G acting on a set X with at least 3 elements is called *Zassenhaus group* if the stabilizer of any three distinct points is the identity. For any distinct elements $x, y \in X$, let $G_{\{x\}}$, $G_{\{x,y\}}$ denote one-point and two-point stabilizers respectively. G is called *split Zassenhaus group* if one point stabilizer $G_{\{x\}}$ is a split Frobenius group, that is, $G_{\{x,y\}} \neq 1$ and $G_{\{x\}} = A \rtimes G_{\{x,y\}}$ where $G^g_{\{x,y\}} \cap G_{\{x,y\}} = 1$ for all $g \in G_{\{x\}} \backslash G_{\{x,y\}}$.

Some arguments in the articles by Cherlin and Jaligot [22] and Deloro and Jaligot [27] can be applied in our context yielding sufficient results for the application of the Fact 5.4.1.

If the Prüfer 2-rank of G is 2, then we may use ideas from the articles [3], [4] by Altseimer. They are much more workable in our context than in the general setting of groups of finite Morley rank.

For the generic case, we believe that the arguments from the articles by Berkman and Borovik [10] and Berkman et al. [11] can be reproduced without much technical difficulties and yield the result.

In this approach, the generic case can be handled without the Classification of Finite Simple Groups since the articles [10] and [11] are free from the Classification. For the non-generic cases, some results about the structures of Sylow 2-subgroups of non-abelian finite simple groups can be used in an effective way in order to avoid the Classification of Finite Simple Groups.

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CONFERENCE TALKS

(a) Simple Groups of Finite Morley Rank with a Tight Automorphism whose Centralizer is Pseudofinite, Antalya Algebra Days X, Antalya, Turkey, May, 2008.

(b) (Invited) On Pseudofinite Centralizers of Automorphisms in Groups of Finite Morley Rank, Groups and Models: Cherlin Bayramı, İstanbul Bilgi University, Turkey, June, 2009.