ENTANGLEMENT MEASURES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

KIVANÇ UYANIK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

FEBRUARY 2008

Approval of the thesis

ENTANGLEMENT MEASURES

submitted by **KIVANÇ UYANIK** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Gradute School of **Natural and Applied Sciences** ________________

Prof. Dr. Sinan Bilikmen
Head of Department, **Physics** ________________

Assoc. Prof. Dr. Yusuf İpekoğlu
Supervisor, **Physics** ________________

Examining Committee Members:

Prof. Dr. Namık Kemal Pak
METU, PHYS ________________

Assoc. Prof. Yusuf İpekoğlu
METU, PHYS ________________

Asst. Prof. Dr. Sadi Turgut
METU, PHYS ________________

Assoc. Prof. Dr. M. Özgür Oktel
Bilkent U, PHYS ________________

Assoc. Prof. Dr. Özgür Kişisel
METU, MATH ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: KIVANÇ UYANIK

Signature :

ABSTRACT

ENTANGLEMENT MEASURES

Uyanık, Kıvanç
M.S., Department of Physics
Supervisor : Assoc. Prof. Dr. Yusuf İpekoğlu

February 2008, 40 pages

Being a puzzling feature of quantum mechanics, entanglement caused many debates since the infancy days of quantum theory. But it is the last two decades that it has started to be seen as a resource for physical tasks which are not possible or extremely infeasible to be done classically. Popular examples are quantum cryptography - secure communication based on laws of physics - and quantum computation - an exponential speedup for factoring large integers. On the other hand, with current technological restrictions it seems to be difficult to preserve specific entangled states and to distribute them among distant parties. Therefore a precise measurement of quantum entanglement is necessary. In this thesis, common bipartite and multipartite entanglement measures in the literature are reviewed. Mathematical definitions, proofs of satisfaction of basic axioms and significant properties for each are given as far as possible. For *Tangle* and *Geometric Measure of Entanglement*, which is a multipartite measure, results of numerical calculations for some specific states are shown.

Keywords: Entanglement, Entanglement Measures, Quantum Information, Quantum Computation

ÖZ

DOLANIKLIK ÖLÇÜTLERI

Uyanık, Kıvanç
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Dr. Yusuf İpekoğlu

Şubat 2008, 40 sayfa

Kuvantum mekaniğinin şaşırtıcı özelliklerinden biri olan dolanıklık kuvantum teorisinin ilk günlerinden beri tartışmalara yol açmıştır. Ancak son yirmi yılda klasik olarak yapılamayacak ya da yapılması teknik olarak mümkün olmayan işler için bir kaynak olarak görülmeye başlanmıştır. Bunların popüler örnekleri arasında kuvantum şifreleme - güvenliği fizik yasalarına dayalı şifreleme - ve kuvantum hesaplama - çok büyük tamsayıların çarpanlara ayrılmasında üstel hızlanma - sayılabilir. Diğer taraftan şu anki teknolojik kısıtlamalarla parçacıkların özel dolanık hallerinin saklanması ve birbirinden uzak alıcılara gönderilmesi çok zor görünmektedir. Bu yüzden dolanıklık ölçümünün tam bir tanımı gereklidir. Bu tezde yaygın iki parçalı ve çok parçalı dolanıklık ölçütleri incelenmiştir. Matematiksel tanımlar verilmiş, mümkün olduğunca temel aksiyomların sağlandığı gösterilmiş ve önemli özelliklerine değinilmiştir. Çok parçalı ölçütlerden *Tangle* ve *Geometrik Dolanıklık Ölçütü* için bazı özel örneklerde sayısal hesaplamaların sonuçları gösterilmiştir.

Anahtar Kelimeler: Dolanıklık, Dolanıklık Ölçütleri, Kuvantum İnformasyon, Kuvantum Hesaplama

To the greatest mother of all times

ACKNOWLEDGMENTS

I am very thankful to my supervisor Professor İpekoğlu for his guidance and directions. I would also like to express thanks to Professor Pak for his conceptual advice.

I am so much grateful to Professor Turgut for the discussions with him and for his invaluable suggestions.

Özge Arslan deserves an *honorable mention* for her corrections, editing and especially for her patience and cheer for my work. I am indebted to Gökhan Kiper and Mustafa Devrim Kaba for sharing their vast mathematical knowledge with me.

I would also like to thank Cesim Dumlu for advices on thesis writing, Ali Can Günhan for sharing his thoughts on both the subject and writing style, Halil Gamsızkan and Tolga Yapıcı for LaTeX support.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

1.1 History of Quantum Entanglement and Quantum Information and Computation Science

In the beginning of the twentieth century the nonintuitive characteristics of quantum mechanics forced physicists to revise their understanding of nature. One of the central problems goes back to those early days of discussions. In their famous paper, Einstein Podolsky and Rosen drew attention to one of the puzzling features of quantum mechanics[1]. It is possible that observation on one part of a composite quantum system would immediately affect the other part even if these subsystems are separated millions of light-years apart. This nonlocal property lead EPR to question whether quantum mechanical description of the nature is complete; in other words, is there another complete theory underlying quantum mechanics?

In 1964 John Bell showed that quantum mechanics and local hidden variable theories provide different predictions for the thought experiments like the ones Einstein originally proposed[2]. In fact, he found that hidden variable theories are more restrictive on the statistical outcomes of an “entangled” state, whereas quantum mechanics is free of these extra constraints[3]. This conclusion, which is mathematically described as “Bell Inequalities”, stimulated real experimenters to test which idea explains the Nature better.

Until Aspect made a convincing experiment in 1982 showing that Bell inequalities are violated [4, 5, 6] this debate had continued on a theoretical ground. To date, many other experiments confirming Aspect’s results have been carried out .

Although it is not a direct consequence of entanglement only, the progress related to quantum computation is worth mentioning. Feynman’s idea to use individual quantum systems for computational tasks happens to occur in the same time period[7]. He pointed out that classical computers are insufficient in the "computational power in a reasonable time" sense to simulate quantum mechanical systems, thus he suggested to build up computers working with quantum mechanical principles instead. In 1985, Deutsch wrote a paper pointing out quantum parallelism - inherent parallel computational power of quantum computers [8]. The two famous algorithms that can work on these quantum computers if it is possible to build one are by Shor[9] and Grover [10]. Shor's algorithm is able to factorize large integers which would be a breakthrough in RSA cryptology since this protocol depends on the asymmetry in complexities of the following two tasks: multiplication of two large primes, and finding prime factors of a large integer. No classical algorithm is known up to now that does not require exponential time for the second task. Unlike Shor's, Grover's algorithm is not bringing an exponential speedup, however it has significant implications in database searching. With this algorithm, it is possible to search an element from an unordered set of N elements in $O(\sqrt{N})$ time, which seems to be quite counter-intuitive.

Before passing to the Quantum Information side of the story, we need to recall Shannon's results which founded a basis for classical information theory[11]. *The noiseless channel coding theorem* and *The noisy channel coding theorem* successfully answer the questions concerning information capacity requirements for noiseless channels and the amount of information that can be transmitted through a noisy channel. The quantum part comes into the picture with Schumacher's analogue of the noiseless coding theorem[12]. Unfortunately a *quantum noisy channel coding theorem* is still missing[13].

Important physical applications can be listed as superdense coding, teleportation and quantum cryptography. It was shown by Bennett and Wiesner that two parties initially possessing an entangled pair can send only one bit of quantum information to transmit two bits of classical information[14]. This is what is called superdense coding. Also it is possible to transmit a quantum state of a single particle. Bennett *et al.* showed that this is possible without knowing the state at hand[15]. The last application, quantum cryptography has already begun to be commercialized. Bennett and Brassard designed the BB84 protocol for establishing secure quantum communication[16]. Security of the BB84 protocol depends on laws of physics, thus as far as our knowledge about quantum mechanics is correct, it is unbreakable. We believe that using quantumness of nature in computation we will be able to make the current commercial cryptography fail. It is quite ironic that we are able to employ quantum cryptography as a reliable substitute for it.

1.2 Motivation

Quantum Entanglement is one of the essential resources for the tasks of communication and computation whose main examples were given in the previous section. Since it is a resource (e.g. like energy) we have to quantify it. The problem starts with a fact: the interaction with the environment affects quantum states irreversibly. One can generate EPR pairs and distribute them to distant parties for communication or any other task like quantum teleportation. However, coupling to the environment (decoherence), noise in the quantum channels or any other practical reasons prevent us from keeping these pairs ideal. In the end the distant parties would possess some states still containing entanglement but not the original EPR pairs. Even though they don't have the ideal pairs, is it possible to perform the desired tasks with the states in hand? Or what tasks can they perform? If the answer is not definite, then what is the probability to do a certain task? Or is it possible to calculate this probability? Can we make local operations and use classical communication channels to distill these states to obtain the ones we need to perform certain tasks? A precise quantification of entanglement would be a great help to answer these kinds of questions. Even if we understand bipartite entanglement (entanglement between only two parties) well, it seems to be that there will be much more to explore about multipartite entanglement and entanglement in infinite dimensional systems.

Despite the fact that the motivation for measuring quantum entanglement is concentrated around quantum information tasks, it is possible for it to be beneficial for other areas of physics. A proper measurement theory of multipartite entanglement would be a helpful mathematical tool for open problems like quantification of correlations in quantum many body systems[17].

Bell inequalities may be counted as the first quantification of entanglement. If a state violates Bell inequalities more than another, then it can be said that this state is more entangled. Contrarily, not all the entangled states violate Bell inequalities [18]. Having a rich mathematical structure and being a resource for great technological implications, measuring entanglement became an interesting problem. The theory of entanglement measures has developed in various ways. Mainly, one can split into two categories: geometric measures mostly concentrating on mathematical features of quantum states and operational measures focusing on the applicability of a quantum state to perform certain tasks [19].

In the next section, we will continue the introduction by giving basic definitions and postulates needed for further definitions, proofs and calculations in chapter 2 and chapter 3. In chapter 2, we give definitions of common bipartite entanglement measures. We examine them from different perspectives such as how they satisfy basic axioms of Quantum Information, their computational complexity or what are the useful upper or lower bounds to them. Since the theory of "bipartite measures" seems to be almost complete, while discussing the bipartite setting, we have gone over these measures aiming to capture the ideas underlying the past work. Unlike the bipartite setting, the characterization of multipartite entanglement is a highly nontrivial problem. In addition to some of those that we have conducted in bipartite case, we also examine proposed multipartite measures in the literature by evaluating them numerically for special states.

1.3 Basic Definitions and Postulates

One of the postulates of Quantum mechanics describes how to deal with composite systems:

> The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. If the i'th of n components is prepared in the state $|\psi_i\rangle$ then the joint system is in the state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$[13].

Immediately we can consider a state like $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ where the total system is a composition of two 2-level systems (for example two spin 1/2 systems). Here $|00\rangle$ and $|11\rangle$ are the short forms of $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$. However it can be shown that there is no solution for the complex coefficients if we want to write this state as a tensor product of two component states. Now we are ready to give the definition of entanglement:

Definition 1.3.1 *A state is called* separable *if it can be written as a tensor product of two component states. It is called* entangled *otherwise.*

These definitions of separable and entangled states are given only for pure states, however our physical problems will mostly force us to work with mixed states[20]. So the definitions of pure and mixed states are as follows:

Definition 1.3.2 *Given an ensemble of quantum states of a system, (i.e. given arbitrary - need not to be orthogonal - quantum states $|\psi_i\rangle$ with certain probabilities p_i), if all the states in the ensemble can be characterized by only one quantum state, it is called a* pure state*, if it is a collection of different quantum states with relative probabilities it is called a* mixed state[21].

To deal with these statistical terms in a better way, an equivalent formalism of quantum mechanics - *the density matrix* or *the density operator* formalism - is introduced[13]. A density operator ρ is defined as follows:

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1.1}$$

where the summation is over all the states $|\psi_i\rangle$ in the ensemble with corresponding probabilities p_i. All the information about the ensemble that can be obtained with measurements can be extracted from the density matrix[21]. A density operator satisfies the following properties[13, 21]:

1. ρ is hermitian
2. $tr\rho = 1$
3. ρ is positive semidefinite.

The trace of the square of the density matrix informs us about whether it corresponds to a pure or a mixed state: Let ρ be a density operator. Then $\mathrm{tr}(\rho^2) \leq 1$, with equality if and only if ρ characterizes a pure state[13, 21]. An important benefit of the density operator formalism is that it becomes a useful tool for describing subsystems of a composite quantum system[13].

Definition 1.3.3 *Let us have a physical system composed of subsystems A and B, whose joint state is described by the density matrix ρ^{AB}. Then the* reduced density matrix *of the subsystem A is defined by*

$$\rho^A = tr_B(\rho^{AB}) \tag{1.2}$$

*where tr_B (*partial trace*) is a linear mapping from operators to operators defined as*

$$tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| tr(|b_1\rangle\langle b_2|) \tag{1.3}$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B.

Extension of this definition linearly to all states and to higher dimensions is straightforward.

Equipped with the tools we needed, we can continue with a definition of entanglement measure. There are different ways to define an entanglement measure but a reasonable approach is given in [17]. Here, we follow the idea given in this reference:

Definition 1.3.4 *An entanglement measure is a mapping E from the set of density matrices into positive real numbers:*

$$\rho \longrightarrow E(\rho) \in \mathbb{R}^+.$$

In the bipartite setting, for convenience, this quantity is often normalized to 1 for maximally entangled states which are defined as

$$|\psi^+\rangle = \frac{|00\rangle + |11\rangle + .. + |d-1, d-1\rangle}{\sqrt{d}}$$

where d is the dimension of the component systems. This definition of maximally entangled states follows from Nielsen's majorization theorem[22]. That is, it is not possible to transform any state with local operations and classical communication(LOCC) to this state, however the reverse is always possible.

Any entanglement measure must be equal to zero for separable states:

$$E(\rho_{separable}) = 0.$$

Entanglement measures are not allowed to increase under LOCC. One can mathematically describe any LOCC operation with generalized measurements A_i, where the only condition on these operators is [1] $\sum_i A_i^\dagger A_i = \mathbb{1}_{n\times n}$:

$$E(\rho) \geq \sum_i p_i E\left(\frac{A_i \rho A_i^\dagger}{\mathrm{tr}(A_i \rho A_i^\dagger)}\right)$$

where p_i is the probability of obtaining outcome i

$$p_i = \mathrm{tr}(A_i \rho A_i^\dagger).$$

[1]($\dagger$) symbol denotes the Hermitian conjugate (Hermitian adjoint), and it is also denoted by($*$) symbol.

The operators satisfying $\sum_i A_i^\dagger A_i = \mathbb{1}_{n\times n}$ are also called Kraus operators.

To call a quantity an entanglement measure we need one extra condition: for a pure state $|\psi\rangle$ $E(|\psi\rangle\langle\psi|)$ should be reduced to *entropy of entanglement*, which we will define in the next chapter

$$E(|\psi\rangle\langle\psi|) = S(tr_B(|\psi\rangle\langle\psi|))$$

where $S(\rho) = -\mathrm{tr}(\rho\log_2\rho)$ is the von Neumann Entropy.

A quantity satisfying these conditions is called an *entanglement measure*.

There is a very similar term *entanglement monotone* in the literature and these are often used interchangeably. Instead of the last condition, Vidal makes another requirement[23]:

Decrease in the information about the system can not increase an entanglement monotone: for any ensemble $\{q_i, \rho_i\}$

$$E(\sum_i q_i\rho_i) \leq \sum_i q_i E(\rho_i).$$

CHAPTER 2

BIPARTITE ENTANGLEMENT MEASURES

The most basic quantum system that demonstrates various aspects of entanglement is a bipartite system. Thus, most of the entanglement measures that has been suggested is a result of a attempts to measure how much entanglement is contained in a bipartite system. This question is rephrased in many different forms, like how much non-local resource can we extract from an entangled state asymptotically, or how much gold standard Bell states does one need to use to create a desired state, or what is the distance from the quantum state that we consider to the set of separable states, etc., each leading to a different kind of definition of measure. In this chapter, we will review the most featured ones, proving some of the key properties that these measures possess.

2.1 Entropy of Entanglement

For a bipartite pure state $|\psi\rangle$, its *entropy of entanglement* is defined as the von Neumann Entropy, $S(\rho)$, of the density matrix which is obtained by taking a partial trace over either one of the subsystems.

$$E(\psi) \equiv S(\sigma) = -\mathrm{tr}(\sigma \log_2(\sigma)) \text{ where } \sigma = \mathrm{tr}_B(|\psi\rangle\langle\psi|) = \mathrm{tr}_A(|\psi\rangle\langle\psi|). \tag{2.1}$$

In fact, a unitarily invariant, concave function $h(\sigma)$ of the partial trace $\sigma = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ is an entanglement monotone[23].

2.2 Entanglement Cost

The motivation behind how entanglement cost is defined is to quantify how much resource - it is the maximally entangled Bell states in this case - that we need to use if we want to prepare a given mixed state. *Entanglement cost* is defined as the minimum of the ratio of the number of maximally entangled input states over the number of the output states we want to produce over all LOCC(Local Operations and Classical Communication) protocols when the number of output states is going to infinity. Formal definition includes infimum instead of minimum:[24, 25]

$$E_C(\rho) \equiv \inf_{\{\lambda_{LOCC}\}} \lim_{n_\rho \to \infty} \frac{n^{in}_{|\phi^+\rangle}}{n^{out}_\rho} \tag{2.2}$$

where $|\phi^+\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $n^{in}_{|\phi^+\rangle}$ is the number of the input state $|\phi^+\rangle$ and n^{out}_ρ is the number of an output state ρ.

Since it involves a minimization over all possible LOCC protocols, this quantity is too difficult to compute, however it may be the case that another entanglement measure having a closed formula for bipartite pure states, *entanglement of formation*, which we will define later, may be equal to entanglement cost[17].

2.3 Entanglement of Distillation

Entanglement of Distillation or *distillable entanglement* quantifies how much non-local resource we can extract from a given state using LOCC only. It is defined as the ratio of the number of maximally entangled output states over the number of given input states maximized over all LOCC protocols when the number of input states is going to infinity: [24, 25]

$$E_D(\rho) \equiv \sup_{\{\lambda_{LOCC}\}} \lim_{n_\rho \to \infty} \frac{n^{out}_{|\phi^+\rangle}}{n^{in}_\rho} \tag{2.3}$$

where $|\phi^+\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $n^{out}_{|\phi^+\rangle}$ is the number of the output state $|\phi^+\rangle$ and n^{in}_ρ is the number of an input state ρ.

Despite its significance, entanglement of distillation is also not easy to compute as it can be seen from the definition. Finding better bounds to distillable entanglement and developing techniques or algorithms to decide whether a state is distillable are still open problems[17].

Entanglement cost and entanglement of distillation are extremal measures in the sense that all entanglement measures should lie between these two. An elementary proof of this fact is given in [26] and it is further improved in [27].

2.4 Entanglement of Formation

Entanglement of formation is defined as the minimum of the average *entropy of entanglement* over all pure state decompositions[24, 25].

$$E_F(\rho) \equiv \inf_{\{decomp.\}} \sum_i p_i S(\sigma_i) \text{ where } \sigma_i = \mathrm{tr}_B(|\psi_i\rangle\langle\psi_i|) \tag{2.4}$$

At the time this thesis is being written, computation of the extremal measures given above are extremely difficult, however this is not the case for *entanglement of formation*. Wootters gave an exact formula to evaluate *entanglement of formation* for bipartite mixed states, defining a new quantity, *concurrence*, which is also an entanglement monotone itself [28]. It is still not known whether $E_F(\rho)$ is additive, however if this is the case, this will make the computation of *entanglement cost* much easier. It will then follow that

$$E_F(\rho) = E_F^\infty(\rho) = E_C(\rho) \tag{2.5}$$

where the regularized version of *entanglement of formation* is defined as [17]

$$E_F^\infty(\rho) \equiv \lim_{n\to\infty} \frac{E_F(\rho^{\otimes n})}{n} \tag{2.6}$$

and a rigorous proof of the second equality 2.5 is given in ref.[29]
The exact formula by Wootters is given below

$$E_F(\rho) = h(\frac{1+\sqrt{1-C^2(\rho)}}{2}) \tag{2.7}$$

where $h(x)$ is defined as

$$h(x) \equiv -x\log_2 x - (1-x)\log_2(1-x) \tag{2.8}$$

and the *concurrence* for two qubits is defined as

$$C(\rho) \equiv \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \tag{2.9}$$

where λ_i's are the eigenvalues of

$$\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{2.10}$$

in a decreasing order and ρ^* is the complex conjugate(not the hermitian conjugate) of ρ, σ_y is the Pauli matrix $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ [28].

2.5 Relative Entropy of Entanglement

Classical relative entropy is a measure of the distance between two probability distributions, namely it is the measure of the inefficiency of assuming the probability distribution to be $\{q_i\}$ when the true distribution is $\{p_i\}$[30]:

$$D(p_i||q_i) \equiv \sum_i p_i \log_2(\frac{p_i}{q_i}) \tag{2.11}$$

The quantum version is [31, 32]

$$S(\rho||\sigma) \equiv \mathrm{tr}(\rho \log_2(\rho) - \rho \log_2(\sigma)) \tag{2.12}$$

and the *relative entropy of entanglement* with respect to a set T introduced by Vedral *et al.* as an entanglement measure [33, 34]

$$E_R(\rho) \equiv \inf_{\sigma \in T} S(\rho||\sigma) = \inf_{\sigma \in T} \mathrm{tr}\{\rho \log_2(\rho) - \rho \log_2(\sigma)\} \tag{2.13}$$

where it measures the distance between the quantum state ρ and a set T. Basically this set can be chosen to be the set of separable states as it was done in the references [33, 34], however there are other options depending upon what we are calling *free states* [17]. Note that even though we said that *quantum relative entropy* measures the distance between two states, neither it nor its classical analog satisfies the metric property; i.e. $S(\rho||\sigma) \neq S(\sigma||\rho)$ [30, 33].

Quantum relative entropy is bounded from above by *entanglement cost* [35] and from below by *distillable entanglement* [36].

Like *distillable entanglement*, *relative entropy of entanglement* requires a minimization over a high dimensional space, thus it is difficult to compute.

2.6 Squashed Entanglement

Squashed entanglement is defined as [37, 38, 39]

$$E_{sq}(\rho^{AB}) \equiv \frac{1}{2} \inf_{\{\rho^{ABE}\}} I(A;B|E) \tag{2.14}$$

where the infimum is taken over all extensions ρ^{ABE} satisfying $\rho^{AB} = \mathrm{tr}_E(\rho^{ABE})$. $I(A;B|E)$, the quantum conditional mutual information [40] of ρ^{ABE}, which takes place in the formula, is given as

$$I(A;B|E) \equiv S(\rho^{AE}) + S(\rho^{BE}) - S(\rho^{ABE}) - S(\rho^{E}), \tag{2.15}$$

and as previously defined in (2.1) $S(\rho^{AB})$ is the von Neumann entropy of the *joint state*, $S(\rho^{AB}) \equiv -\mathrm{tr}(\rho^{AB}\log_2(\rho^{AB}))$. One desired property of being an entanglement measure is being equal to the *entropy of entanglement* for pure states. Here it can be easily proven [39] that this is the case: Let $\rho^{AB} = |\psi\rangle\langle\psi|$ be a pure state. Any extension of ρ^{AB} should be separable: $\rho^{ABE} = \rho^{AB} \otimes \rho^{E}$. It follows from the additivity of von Neumann entropy that,

$$\tfrac{1}{2}I(A;B|E) = \tfrac{1}{2}\{S(\rho^{A}) + S(\rho^{E}) + S(\rho^{B}) + S(\rho^{E}) - S(\rho^{AB}) - S(\rho^{E}) - S(\rho^{E})\}$$

$$= \tfrac{1}{2}(S(\rho^{A}) + S(\rho^{B})) = S(\rho^{A}) = E(\psi)$$

Proof of the *squashed entanglement* being an entanglement monotone is as follows. In [23] it has been shown that the following two conditions for a mapping $\mu(\rho)$ to be, on average, non-increasing under local transformations are necessary and sufficient:

1. For any quantum state ρ and local, trace preserving, completely positive quantum operations $\mathcal{E}_{i,k}$ performed by either of the observers i,

$$\mu(\rho) \geq \sum_k p_k \mu(\rho_k), \tag{2.16}$$

where

$$p_k = \mathrm{tr}(\mathcal{E}_k(\rho)) \text{ and } \rho_k = \frac{1}{p_k}\mathcal{E}_k(\rho). \tag{2.17}$$

2. Decrease in the information about the system can not increase an entanglement monotone: for any ensemble $\{q_i, \rho_i\}$

$$\mu(\sum_i p_i \rho_i) \leq \sum_i p_i \mu(\rho_i) \tag{2.18}$$

Proof of *statement 2* is given as follows: Consider an ensemble with k elements $\{p_i, \rho_i^{AB}\}$, let ρ_i^{ABE} be arbitrary extensions of ρ_i^{AB}. To form a mixture of these k states having a formal extension define

$$\tau^{ABEE'} \equiv \sum_i p_i \rho_i^{ABE} \otimes |i^{E'}\rangle\langle i^{E'}| \tag{2.19}$$

where the mixture state is $\tau^{AB} = \sum_i p_i \rho_i^{AB}$. Note that without loss of generality, extensions can be defined on identical systems E. Up to now, quantum mutual conditional information

$I(A;B|E)$ was defined on a single arbitrary underlying density operator, henceforth whenever needed, underlying density matrices will be denoted by a subscript such as $I(A;B|E)_{\rho_i}$.

To prove *statement 2* showing equality 2.20 below would be sufficient. Since the extensions ρ_i^{ABE} are arbitrary, one can choose the minimizing ones, so the left hand side would be equal to twice the sum of the squashed entanglements of the density matrices in the ensemble. Also it is guaranteed that $I(A;B|EE')_\tau \geq 2E_{sq}(\tau^{AB})$ by definition.

$$\sum_i p_i I(A;B|E)_{\rho_i} = I(A;B|EE')_\tau \tag{2.20}$$

Writing the definitions and using the additivity property of von Neumann entropy, we will prove equation 2.20:

$$\begin{aligned}
\sum_i p_i I(A;B|E)_{\rho_i} &= \sum_i p_i\{S(\rho_i^{AE}) + S(\rho_i^{BE}) - S(\rho_i^{ABE}) - S(\rho_i^{E})\} \\
&= \sum_i p_i S(\rho_i^{AE}) + \sum_i p_i S(\rho_i^{BE}) - \sum_i p_i S(\rho_i^{ABE}) - \sum_i p_i S(\rho_i^{E}) \\
&= \sum_i p_i S(\rho_i^{AEE'}) + \sum_i p_i S(\rho_i^{BEE'}) - \sum_i p_i S(\rho_i^{ABEE'}) - \\
&\sum_i p_i S(\rho_i^{EE'}) \\
&= S(\tau^{AEE'}) + S(\tau^{BEE'}) + S(\tau^{ABEE'}) + S(\tau^{EE'}) \\
&= I(A;B|EE')_\tau
\end{aligned} \tag{2.21}$$

For the proof of the first condition, the reader may refer to the original paper [39].

Another important property of the *squashed entanglement* is that it is bounded from above by the *entanglement of formation*:

$$E_{sq}(\rho^{AB}) \leq E_F(\rho^{AB}). \tag{2.22}$$

Here, we will add more explanatory steps to the proof which is given in [39]. Let $\{p_k, |\psi_k\rangle\}$ be a pure state ensemble of ρ^{AB},

$$\sum_k p_k |\psi_k^{AB}\rangle\langle\psi_k^{AB}| = \rho^{AB}. \tag{2.23}$$

Consider an arbitrary extension of ρ^{AB}:

$$\rho^{ABE} \equiv \sum_k p_k |\psi_k^{AB}\rangle\langle\psi_k^{AB}| \otimes |k\rangle\langle k|^E, \tag{2.24}$$

it follows then

$$\frac{1}{2} I(A;B|E) = \sum_k p_k S(\sigma_k^A) \tag{2.25}$$

where $\sigma_k^A \equiv \text{tr}_B(|\psi_k^{AB}\rangle\langle\psi_k^{AB}|)$. To show the last equality, we first start with the definition of $I(A;B|E)$

$$I(A;B|E) = S(\rho^{AE}) + S(\rho^{BE}) - S(\rho^{ABE}) - S(\rho^{E}).$$

Evaluating these entropies one by one

$$\begin{aligned}
\rho^{AE} &= -\text{tr}_B(\sum_k p_k|\psi_k^{AB}\rangle\langle\psi_k^{AB}| \otimes |k\rangle\langle k|^E) \\
&= \sum_k p_k \text{tr}_B(|\psi_k^{AB}\rangle\langle\psi_k^{AB}| \otimes |k\rangle\langle k|^E)) \\
&= \sum_k p_k \text{tr}_B(|\psi_k^{AB}\rangle\langle\psi_k^{AB}|) \otimes |k\rangle\langle k|^E \\
&= \sum_k p_k \sigma_k^A \otimes |k\rangle\langle k|^E
\end{aligned}$$

The von Neumann entropy of ρ^{AE} is a function of its eigenvalues. Let $\lambda_{k,i}^A$ be the i'th eigenvalue of the k'th reduced density operator σ_k^A. From the definition of the extension ρ^{ABE}, eigenvalues of ρ^{AE} are found to be the probability weighted composition of these $\lambda_{k,i}^A$:

$$\lambda_{k,i}^{AE} = p_k \lambda_{k,i}^A.$$

The von Neumann entropy of ρ^{AE} and, by symmetry, of ρ^{BE} are found to be:

$$\begin{aligned}
S(\rho^{BE}) = S(\rho^{AE}) &= -\sum_{k,i} \lambda_{k,i}^{AE} \log_2 \lambda_{k,i}^{AE} \\
&= -\sum_{k,i} p_k \lambda_{k,i}^A \log_2 p_k \lambda_{k,i}^A \\
&= -\sum_{k,i} p_k \lambda_{k,i}^A (\log_2 \lambda_{k,i}^A + \log_2 p_k) \\
&= -\sum_{k,i} p_k \lambda_{k,i}^A \log_2 \lambda_{k,i}^A - \sum_i \lambda_{k,i}^A \sum_k p_k \log_2 p_k \\
&= -\sum_{k,i} p_k \lambda_{k,i}^A \log_2 \lambda_{k,i}^A - \sum_k p_k \log_2 p_k.
\end{aligned}$$

By a reasoning similar to above, the entropies of the other terms are found to be

$$S(\rho^{ABE}) = S(\rho^{E}) = -\sum_k p_k log_2 p_k, \tag{2.26}$$

thus we have proven the desired equality

$$\begin{aligned}
\frac{1}{2} I(A;B|E) &= -\sum_{k,i} p_k \lambda_{k,i}^A \log_2 \lambda_{k,i}^A \\
&= -\sum_{k,i} p_k \lambda_{k,i}^A \log_2 \lambda_{k,i}^A \\
&= \sum_k p_k S(\sigma_k^A).
\end{aligned} \tag{2.27}$$

Therefore, both of them being evaluated for the same quantity $\frac{1}{2}I(A;B|E)$, we have seen that the only difference between entanglement of formation and squashed entanglement is that $E_F(\rho)$ can be regarded as an infimum over a certain class of extensions and $E_{sq}(\rho)$ is an infimum over all extensions of ρ^{AB}, so $E_F(\rho^{AB})$ is smaller than or equal to $E_{sq}(\rho^{AB})$.

Now we will follow some basic steps to show that the *entanglement cost* is an upper bound for *squashed entanglement*, under the assumption that the squashed entanglement is additive.

$$E_{sq}(\rho^{AB}) \leq E_C(\rho^{AB}) \tag{2.28}$$

As it was mentioned before (eqn. 2.5), equality of entanglement cost and regularized entanglement of formation has been proved [29]. Accepting squashed entanglement as an additive entanglement measure, one will immediately see that

$$E_C(\rho^{AB}) = \lim_{n\to\infty} \frac{1}{n} E_F((\rho^{AB})^{\otimes n}) \leq \lim_{n\to\infty} \frac{1}{n} E_F((\rho^{AB})^{\otimes n}) = E_{sq}(\rho^{AB}). \tag{2.29}$$

As it is bounded from above by the entanglement cost, it is also shown to be bounded from below by the distillable entanglement [39].

2.7 Negativity and Logarithmic Negativity

Before attempting to measure how much entanglement a state contains, one naturally tries to answer the question if a state is separable or not. It has been shown that a state is separable if and only if the partial transpose of its density matrix ρ with respect to either of its subsystems is also a positive operator [41, 42]. Since partial transposition of a density matrix with respect to one subsystem is the transpose of its partial transpose with respect to the other subsystem, the spectra of the partially transposed states are the same. Partial transposition with respect to subsystem B of a bipartite state $\rho \equiv \sum \rho_{ij,kl}|i\rangle\langle j| \otimes |k\rangle\langle l|$ is defined as

$$\rho^{T_B} \equiv \sum_{i,j,k,l} \rho_{ij,kl}|i\rangle\langle j| \otimes |l\rangle\langle k| \tag{2.30}$$

where the state ρ is expanded in a given local orthonormal basis. However this criterion is shown to be valid only for 2×2 and 2×3 systems [42]. From this statement, we understand that if a bipartite state is separable, the eigenvalues of the partial transposed density matrix are all non-negative. For the inseparable case however, some of the eigenvalues are negative.

The idea is to use the sum of these negative eigenvalues to quantify entanglement, or in other words to measure how much the given state fails to satisfy Peres' criterion [41]. For this purpose *negativity* is defined as twice the sum of the negative negative eigenvalues of the partial transposed density matrix [43]

$$E_N(\rho) \equiv \sum_i |\lambda_i^{T_B}| - 1 \tag{2.31}$$

where the $\lambda_i^{T_B}$ are the eigenvalues of the partial transpose ρ^{T_B}. A mathematically equivalent definition is given in [17, 44] as

$$E_N(\rho) \equiv \|\rho^{T_B}\| - 1 \tag{2.32}$$

where $\|X\| \equiv \text{tr}\sqrt{X^\dagger X}$ is the trace norm.

Negativity is shown to be an entanglement monotone in several papers [44, 45, 46, 47]. Here, we will briefly present the proof of Eisert [44]. Since separable states do satisfy Peres' criterion and the partial transposes of these states have no vanishing eigenvalues, we only need to check two conditions: eqn.(2.16) and eqn.(2.18). The second condition (eqn.2.18) is satisfied, because the trace norm satisfies the triangle inequality. We can show this by using corollary 3.4.3 in [48] The trace norm can be equivalently defined as the sum of absolute values of the eigenvalues. Since the sum of the k largest eigenvalues is a convex function [48],

$$\sum \lambda_i^{(A+B)\downarrow} \leq \sum \lambda_i^{A\downarrow} + \sum \lambda^{B\downarrow}, \tag{2.33}$$

the sum of the inequalities for positive eigenvalues of the original matrix and the negated matrix leads to the triangle inequality for the trace norm.

To prove the first condition, consider the generalized measurements represented by the Kraus operators $\{A_i\}$, which can be applied by party A on her subsystem. Being Kraus operators, A_i's satisfy

$$\sum_i^k A_i^\dagger A_i = \mathbb{1}_A \tag{2.34}$$

and the final states are of the form

$$\rho_i = \frac{A_i \rho A_i^\dagger}{p_i} \tag{2.35}$$

with

$$p_i = \text{tr}(A_i \rho A_i^\dagger).$$

Let

$$\rho^{T_B} = \rho_+^{T_B} + \rho_-^{T_B} \tag{2.36}$$

be the *Hahn - Jordan Decomposition* of the partially transposed state ρ [49]. By definition of the decomposition, both $\rho_+^{T_B}$ and $\rho_-^{T_B}$ are positive and Hermitian. Then

$$\sum_i^k p_i E_N(\rho_i) = \sum_i^k p_i \left(\frac{\|(A_i \rho A_i^\dagger)^{T_B}\|}{p_i} - 1 \right) = \sum_i^k \|A_i \rho^{T_B} A_i^\dagger\| - 1. \tag{2.37}$$

Definition of negativity gives the first equality, and the second equality follows from the fact that partial transposition with respect to party B has no effect on the operations applied by party A. For each $i = 1, .., k$, the triangle inequality of trace norm and the positivity and Hermiticity of $\rho_+^{T_B}$ and $\rho_-^{T_B}$ imply

$$\begin{aligned} \|A_i(\rho_+^{T_B} - \rho_-^{T_B})A_i^\dagger\| \leq \|A_i \rho_+^{T_B} A_i^\dagger\| + \|A_i \rho_-^{T_B} A_i^\dagger\| &= \mathrm{tr}(A_i \rho_+^{T_B} A_i^\dagger) + tr(A_i \rho_-^{T_B} A_i^\dagger) \\ &= \mathrm{tr}(A_i \rho_+^{T_B} A_i^\dagger + A_i \rho_-^{T_B} A_i^\dagger) \\ &= \mathrm{tr}(A_i(\rho_+^{T_B} + \rho_-^{T_B})A_i^\dagger) \end{aligned} \tag{2.38}$$

Then, using equation 2.34 we obtain

$$\sum_i^k \mathrm{tr}(A_i(\rho_+^{T_B} + \rho_-^{T_B})A_i^\dagger) = \mathrm{tr}(\rho_+^{T_B} + \rho_-^{T_B}) \tag{2.39}$$

Finally,

$$\sum_i^k p_i E_N(\rho_i) \leq \mathrm{tr}(\rho_+^{T_B} + \rho_-^{T_B}) - 1 = \|\rho^{T_B}\| - 1 = E_N(\rho) \tag{2.40}$$

Unlike the other measures that we have reviewed except entanglement of formation, negativity does not require a computationally difficult minimization procedure on a high dimensional space [44]. This makes negativity a useful entanglement monotone even though it does not agree with von Neumann entropy evaluated on pure states. Negativity is a convex entanglement monotone but it is not additive. One can consider the logarithm of the trace norm and define *logarithmic negativity* [50]

$$E_{LN}(\rho) \equiv \log_2 \|\rho^{T_B}\| \tag{2.41}$$

in order to overcome this difficulty[47]. This monotone is, by construction, additive:

$$\begin{aligned}
E_{LN}(\rho\otimes\rho) &= \log_2\left(\mathrm{tr}\sqrt{[(\rho\otimes\rho)^{T_B}]^\dagger[(\rho\otimes\rho)^{T_B}]}\right)\\
&= \log_2\left(\mathrm{tr}\sqrt{(\rho^{T_B}\otimes\rho^{T_B})^\dagger(\rho^{T_B}\otimes\rho^{T_B})}\right)\\
&= \log_2\left(\mathrm{tr}\sqrt{((\rho^{T_B})^\dagger\rho^{T_B})\otimes((\rho^{T_B})^\dagger\rho^{T_B})}\right)\\
&= \log_2\left(\mathrm{tr}\sqrt{((\rho^{T_B})^\dagger\rho^{T_B})}\otimes\sqrt{((\rho^{T_B})^\dagger\rho^{T_B})}\right)\\
&= \log_2\left(\left(\mathrm{tr}\sqrt{((\rho^{T_B})^\dagger\rho^{T_B})}\right)^2\right)\\
&= 2\log_2 E_{LN}(\rho).
\end{aligned}$$

However it is not convex, so does not satisfy equation 2.18. Thus, it was believed not to be an entanglement monotone. On the other hand, in [47] a proof of *logarithmic negativity* being an entanglement monotone despite its non-convexity is given by discussing the relationship between convexity requirements and physical operations regarding loss of information highlighting the importance of continuity.

Although it is only an entanglement monotone (not an entanglement measure), their ease of calculation for a given state make negativity and logarithmic negativity very advantageous. Furthermore, logarithmic negativity is shown to be an upper bound for distillable entanglement, $E_D(\rho) \leq E_{LN}(\rho)$, and for the teleportation capacity of quantum channels [46].

2.8 Robustness of Entanglement

A different idea to quantify entanglement is to measure how much an entangled state can endure mixing separable states. The minimum amount of a suitable separable state that should be mixed with the given density matrix to make the combination separable is defined as the *robustness of entanglement*[51] of that density matrix. The state which has to be mixed in order to destroy entanglement can be chosen from different sets each leading to a different kind of robustness measure, such as *generalized robustness*, *random robustness* or *Schmidt robustness*[19].

Let us first define the *relative robustness* of one state with respect to another. Given a state ρ, the minimum positive real number s such that one obtains a separable state, ρ_s^+, after mixing with an appropriate density operator ρ_s^-

$$\rho_s^+ \equiv \frac{1}{1+s}(\rho + s\rho_s^-) \tag{2.42}$$

is defined as the *relative robustness* $R(\rho\|\rho_s^-)$ of ρ with respect to the state ρ_s^-. Why it has been defined in this way will become clear soon. If we impose the condition that ρ_s^- is separable, we obtain the original definition of Vidal and Tarrach [51]. They also defined *random robustness* by setting ρ_s^- to be the normalized maximally mixed state

$$\rho_s^- = \frac{1}{n}\mathbb{1}_{n\times n} \tag{2.43}$$

where n is the dimension of the Hilbert Space. One can remove the restriction on the density matrix ρ_s^- to be separable and consider the most general case since it is possible to obtain separable states by mixing entangled states. This case is studied independently by Harrow and Nielsen [52] and Steiner [53]. They showed that *generalized robustness* (also known as *global robustness*) and *robustness* give the same value. In a recent work, *Robustness* is generalized one step further to *Schmidt-k robustness* by restricting ρ_s^- to the set of Schmidt-k rank states [19].

Before showing that *robustness* is an entanglement monotone, for simplification, introducing the concept of pseudomixtures would be useful [51, 54]. Writing equation (2.42) in terms of ρ_s^- and ρ_s^+, we obtain a decomposition for ρ

$$\rho = (1+s)\rho_s^+ - s\rho_s^-. \tag{2.44}$$

Here it should be noted that, s being a positive real number, equation (2.44) does not represent a physical decomposition. In the original case of Vidal and Tarrach these states are emphasized to be *local*[51].

To show that the *robustness of entanglement* is an entanglement monotone, again we have to prove that it satisfies the monotonicity conditions, namely, equation (2.16) and equation (2.18) [23] It is enough to show the convexity of a mixture of two states [51] since it can be extended to the case of more than two states by induction. Let ρ be a mixture of two states and $p \in (0,1)$ be the mixing probability

$$\rho = p\rho_1 + (1-p)\rho_2. \tag{2.45}$$

Let ρ_1 and ρ_2 have optimal decompositions

$$\rho_1 = [1 + R(\rho_1)]\rho_{s,1}^{+} - R(\rho)\rho_{s,1}^{-} \tag{2.46a}$$

$$\rho_2 = [1 + R(\rho_2)]\rho_{s,2}^{+} - R(\rho)\rho_{s,2}^{-} \tag{2.46b}$$

Substituting equations 2.46a and 2.46b into equation 2.45 we get:

$$\begin{aligned}
\rho &= p\{[1 + R(\rho_1)]\rho_{s,1}^{+} - R(\rho_1)\rho_{s,1}^{-}\} + (1-p)\{[1 + R(\rho_2)]\rho_{s,2}^{+} - R(\rho_2)\rho_{s,2}^{-}\} \\
&= p[1 + R(\rho_1)]\rho_{s,1}^{+} + (1-p)[1 + R(\rho_2)]\rho_{s,2}^{+} - pR(\rho_1)\rho_{s,1}^{-} - (1-p)R(\rho_2)\rho_{s,2}^{-} \\
&= (1+t)\frac{p[1 + R(\rho_1)]\rho_{s,1}^{+} + (1-p)[1 + R(\rho_2)]\rho_{s,2}^{+}}{p[1 + R(\rho_1)] + (1-p)[1 + R(\rho_2)]} - t\frac{pR(\rho_1)\rho_{s,1}^{-} + pR(\rho_2)\rho_{s,2}^{-}}{pR(\rho_1) + (1-p)R(\rho_2)} \\
&= (1+t)r_s^{+} - tr_s^{-}
\end{aligned}$$

where r_s^{+}, r_s^{-} and t are defined as

$$\begin{aligned}
r_s^{+} &= \frac{p[1 + R(\rho_1)]\rho_{s,1}^{+} + (1-p)[1 + R(\rho_2)]\rho_{s,2}^{+}}{1 + pR(\rho_1) + (1-p)R(\rho_2)} \\
r_s^{-} &= \frac{pR(\rho_1)\rho_{s,1}^{-} + pR(\rho_2)\rho_{s,2}^{-}}{pR(\rho_1) + (1-p)R(\rho_2)} \\
t &= pR(\rho_1) + (1-p)R(\rho_2).
\end{aligned}$$

Thus we have obtained a decomposition with a distance t not necessarily being optimal:

$$\begin{aligned}
&R(\rho) \leq t \\
&R(\rho) \leq pR(\rho_1) + (1-p)R(\rho_2)
\end{aligned}$$

Next, we have to prove that *robustness* does not increase under local trace preserving quantum operations. These operations can be represented by generalized local measurements which satisfy equation (2.34):

$$\sum_i^k A_i^{\dagger} A_i = \mathbb{1}_A$$

Let the initial state ρ have an optimal pseudomixture

$$\rho = [1 + R(\rho)]\rho_s^{+} - R(\rho)\rho_s^{-} \tag{2.47}$$

and let us apply general local measurements on this state

$$\rho_k = \frac{A_k \rho A_k^{\dagger}}{\mathrm{tr}(A_k \rho A_k^{\dagger})} = \frac{[1 + R(\rho)]A_k\rho_s^{+}A_k^{\dagger} - R(\rho)A_k\rho_s^{-}A_k^{\dagger}}{\mathrm{tr}(A_k \rho A_k^{\dagger})}. \tag{2.48}$$

Here, separable states ρ_s^{+} and ρ_s^{-} remain separable under local measurements A_k, so it is possible to obtain a not necessarily optimal pseudomixture for each k. Normalizing these separable states one would have

$$\begin{aligned}\rho_k &= \frac{[1+R(\rho)]\mathrm{tr}(A_k\rho_s^+A_k^\dagger)}{\mathrm{tr}(A_k\rho A_k^\dagger)}\frac{A_k\rho_s^+A_k^\dagger}{\mathrm{tr}(A_k\rho_s^+A_k^\dagger)} - \frac{R(\rho)\mathrm{tr}(A_k\rho_s^-A_k^\dagger)}{\mathrm{tr}(A_k\rho A_k^\dagger)}\frac{A_k\rho_s^-A_k^\dagger}{\mathrm{tr}(A_k\rho_s^-A_k^\dagger)} \\ &= (1+\tau)\frac{A_k\rho_s^+A_k^\dagger}{\mathrm{tr}(A_k\rho_s^+A_k^\dagger)} - \tau\frac{A_k\rho_s^-A_k^\dagger}{\mathrm{tr}(A_k\rho_s^-A_k^\dagger)}.\end{aligned} \tag{2.49}$$

Considering optimal pseudomixtures for each ρ_k, one would obtain the following inequalities without solving for τ explicitly:

$$R(\rho_k) \leq \frac{R(\rho)\mathrm{tr}(A_k\rho_s^-A_k^\dagger)}{\mathrm{tr}(A_k\rho A_k^\dagger)} \tag{2.50}$$

thus,

$$\begin{aligned}\sum_k p_kR(\rho_k) &\leq \sum_k R(\rho)\mathrm{tr}(A_k\rho_s^-A_k^\dagger) \\ \sum_k p_kR(\rho_k) &\leq \sum_k R(\rho)\mathrm{tr}(A_k^\dagger A_k\rho_s^-) \\ \sum_k p_kR(\rho_k) &\leq R(\rho)\mathrm{tr}(\mathbb{1}_A\rho_s^-) \\ \sum_k p_kR(\rho_k) &\leq R(\rho),\end{aligned} \tag{2.51}$$

and the proof is complete.

Relative robustness has the nice property that it is convex over the set of separable states [51].

$$R(\rho\|\sum_i p_i\rho_{s,i}) \leq \sum_i p_iR(\rho\|\rho_{s,i}) \tag{2.52}$$

Even though most of the measures defined up to now require a minimization over a space, inequality (2.52) distinguishes *robustness of entanglement* in such a way that a numerical calculation to minimize robustness of entanglement would not suffer from the local minima problem.

As an important note, for two party pure states, the robustness of entanglement is evaluated as [51]

$$R(|\psi\rangle) = \left(\sum_i \lambda_i\right)^2 - 1 \tag{2.53}$$

where λ_i are the Schmidt coefficients of $|\psi\rangle$:

$$|\psi\rangle = \sum_i \lambda_i|i\rangle\otimes|i\rangle.$$

CHAPTER 3

MULTIPARTITE ENTANGLEMENT MEASURES

It is natural to consider the many party extensions of bipartite entanglement. However, most of the measures that are defined for bipartite systems are inadequate for the multipartite setting. For *entanglement cost* and *distillable entanglement*, a standard of entanglement like Bell states is missing. One can extend the definition of the *relative entropy of entanglement*, *negativity* or *robustness*, but these suffer from the problem of defining suitable unentangled sets. One obtains different multipartite measures for each of these definitions capturing different aspects of multipartite entanglement. In this chapter we will continue with analyzing multipartite entanglement measures.

3.1 Entanglement Cost and Distillable Entanglement

In the bipartite case, *entanglement cost* was defined as the minimum asymptotical rate of transforming maximally entangled Bell states into the state we wish for using only local operations and classical communication. Its dual measure, *distillable entanglement* was the maximum LOCC conversion rate from the state in hand to the gold standard Bell states in the asymptotic limit. A direct generalization to multiparty setting for both measures lacks a common standard of entangled states like Bell states. Bell states possessed the essential property that it was possible to convert them to any desired state [22]. On the other hand, for the entanglement shared by more then two parties there are more then one distinct classes of entanglement for which conversion of states from one class to another is not possible even with a small probability. The simplest example is given by Dur *et al.* [55]. $|GHZ\rangle \equiv 1/\sqrt{2}(|000\rangle + |111\rangle)$ and $|W\rangle \equiv 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$ states are the representatives of two distinct classes of tripartite entanglement.

One may continue to employ Bell states as a target for distilling multipartite states or one may aim for multipartite entangled states such as $|GHZ\rangle$ state [56] or $|W\rangle$ state. Similarly, it is possible to start with a representative of any class of entangled states as a resource or as a target to distill. Each of these quantities would give rise to a measure that reveals a different entanglement characteristics [17].

3.2 Tangle

Many new features appear when one increases the size of a quantum system from two to three. One of them is the *monogamy of entanglement*, which can be described as follows: the amount of entanglement between party A and party B is reduced if one tries to increase the amount of entanglement between party A and a third party C. This property is intrinsically quantum, unlike the classical case in which one can correlate as many systems as desired to add to an already correlated system [57]. Coffman, Kundu and Wootters provided a quantitative expression of this property and obtained a locally invariant measure called *residual entanglement* or *3-tangle* [58] to measure genuine 3-partite entanglement [57]. They observed that concurrences (eqn.2.9) for reduced density matrices of a 3-partite system satisfy the inequality

$$C^2(\rho^{AB}) + C^2(\rho^{AC}) \leq C^2(\rho^{A(BC)}) \tag{3.1}$$

which leads to the definition of *3-tangle*

$$\tau_{ABC} \equiv \tau_{A(BC)} - \tau_{AB} - \tau_{AC} \tag{3.2}$$

where τ_{AB}, τ_{AC} and $\tau_{A(BC)}$ are *2-tangle*s (squared concurrences).

For the well known *GHZ* state [59], this measure is equal to 1 since

$$\begin{aligned}
\tau_{ABC}(|GHZ\rangle) &= \tau_{A(BC)}(|GHZ\rangle) - \tau_{AB}(|GHZ\rangle) - \tau_{AC}(|GHZ\rangle) \\
&= C^2_{A(BC)}(|GHZ\rangle) - C^2_{AB}(|GHZ\rangle) - C^2_{AC}(|GHZ\rangle) \\
&= 1 - 0 - 0.
\end{aligned}$$

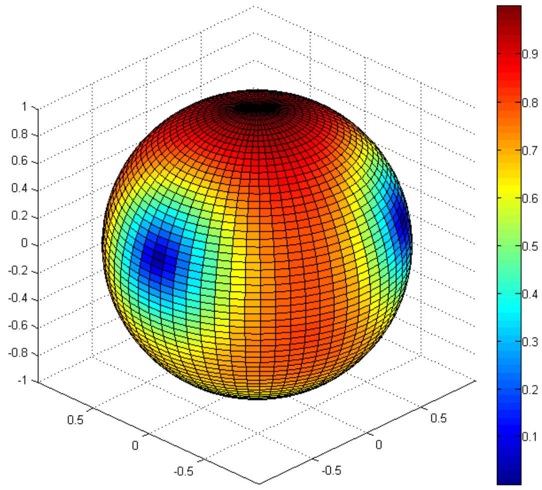

Figure 3.1: Upper view of the tangle of $|GW(\theta, \phi)\rangle$ on a Bloch sphere. The vector $\hat{k}$ represents the $|GHZ\rangle$ state and the vector $-\hat{k}$ represents the $|W\rangle$ state.

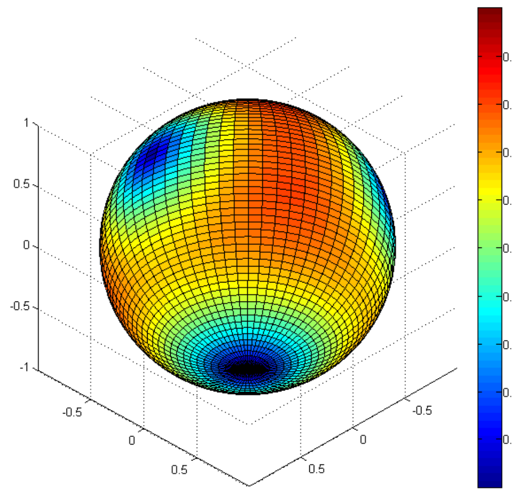

Figure 3.2: Lower view of tangle of the $|GW(\theta, \phi)\rangle$ on a Bloch sphere. The vector $\hat{k}$ represents the $|GHZ\rangle$ state and the vector $-\hat{k}$ represents the $|W\rangle$ state.

On the other hand, for the *W-state* [55] it is evaluated as

$$\begin{aligned}\tau_{ABC}(|W\rangle) &= \tau_{A(BC)}(|W\rangle) - \tau_{AB}(|W\rangle) - \tau_{AC}(|W\rangle) \\ &= C^2_{A(BC)}(|W\rangle) - C^2_{AB}(|W\rangle) - C^2_{AC}(|W\rangle) \\ &= \frac{8}{9} - \frac{4}{9} - \frac{4}{9} \\ &= 0.\end{aligned}$$

Finally, consider the superpositions of the $|GHZ\rangle$ and $|W\rangle$ states:

$$|GW(\theta, \phi)\rangle \equiv \cos(\theta/2)|GHZ\rangle + \sin(\theta/2)e^{i\phi}|W\rangle. \tag{3.3}$$

The values of *tangle* are plotted on the surface of a Bloch sphere and shown in two different views of the sphere in figures 3.1 and 3.2. The values of tangle are evaluated by a function included in the QLIB package [60] running on MATLAB. These results are consistent with the results shown in a recent paper by Lohmayer *et al.* [61].

An n-qubit generalization of the *3-tangle* (n-tangle) is defined, proved to be an entanglement monotone and shown to be equal to the square of a generalized concurrence for even n in [58]. However, the *n-tangle* is not a quantifier of genuine n-partite entanglement. A geometrical approach to generalize tangle and concurrence to multi-qubits using hyperdeterminants is given by Miyake [62]. He showed that the absolute value of the higher order hyperdeterminants are also entanglement monotones[63].

Generalization to the mixed states is done by convex roof extension [57, 58],

$$\tau(\rho) = \inf_{S_\rho}\{\sum_i p_i \tau(|\psi_i\rangle)\} \tag{3.4}$$

where S_ρ is the set of all possible realizations $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ of the mixed state characterized by the density operator ρ.

3.3 Relative Entropy of Entanglement

In the bipartite case, the *relative entropy of entanglement* of a state was defined as its minimum distance to a disentangled set. The most general choice for this set was the set of separable states. The idea for the bipartite case can be directly generalized to the multipartite case [33].

$$E_R(\rho) \equiv \inf_{\sigma\in T} S(\rho\|\sigma) = \inf_{\sigma\in T} \mathrm{tr}\{\rho\log_2(\rho) - \rho\log_2(\sigma)\} \tag{3.5}$$

where T is the set of n-partite fully separable states [34, 35]

$$\sigma = \sum_i s_i \sigma_i^A \otimes \sigma_i^B \otimes ... \otimes \sigma_i^Z. \tag{3.6}$$

This set can also be selected as the set of mixtures of k-partite entangled states where $k < n$ [33, 35]. The simplest example of this is for tripartite states and the states are of the form

$$\sigma = \sum_i s_i \sigma_i^{AB} \sigma_i^C + t_i \sigma_i^{AC} \sigma_i^B + u_i \sigma_i^{BC} \sigma_i^A. \tag{3.7}$$

In the latter case the measure will be sensitive to truly n-partite entanglement whereas in the former case it can confuse n-partite entanglement with combinations of k-partite entanglements [3]. Of course one can evaluate the relative entropy distance with respect to any convex set such as PPT states or non-distillable states[3].

As we mentioned in chapter 2, computing the *relative entropy of entanglement* requires to solve a difficult optimization problem. In the multipartite case, its complexity even increases with the number of subsystems [33].

For the $|GHZ\rangle$ like pure states *relative entropy of entanglement* is evaluated as[17]

$$E_R(\alpha|000\rangle + \beta|111\rangle) = -|\alpha|^2 \log_2 |\alpha|^2 - |\beta|^2 \log_2 |\beta|^2$$

and for the $|W\rangle$ states [64]

$$E_R((|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}) = \log_2(9/4).$$

3.4 Geometric Measure of Entanglement

Another quantification that can be seen as a *distance based* measure is the *geometric measure of entanglement*. This distance is "geometric" in the sense that this measure is defined first by Shimony [65] as the maximum angle between a bipartite pure state $|\psi\rangle$ and a separable state. This definition is generalized to n-partite pure states by Barnum and Linden [66] and extended to multipartite mixed states by Wei and Goldbart [67]. Formally, the distance from the state $|\psi\rangle$ to a separable state $|\phi\rangle$ is minimized

$$d = \min \| |\psi\rangle - |\phi\rangle | \tag{3.8}$$

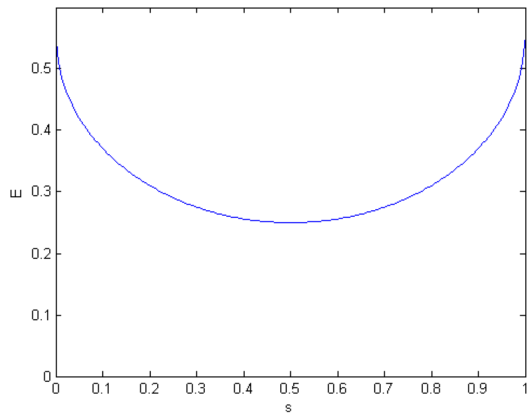

Figure 3.3: Geometric measure of entanglement of $\sqrt{s}|W\rangle + \sqrt{1-s}|\tilde{W}\rangle$ vs. s.

Eventually it boils down to finding the maximal angle between these states as we follow the derivation in Wei and Goldbart's paper [67]

$$\Lambda_{max} = \max_{\phi} |\langle\phi||\psi\rangle| \tag{3.9}$$

where Λ_{max}, which they call the *entanglement eigenvalue*, corresponds to the cosine of the maximal angle. For future purposes, geometric measure is defined as the sine of this angle:

$$E_G(orE_{sin^2}) \equiv 1 - \Lambda_{max}^2 \tag{3.10}$$

If one tries to make this minimization for the bipartite case, finding Λ_{max} would correspond to the largest Schmidt coefficient of the state $|\psi\rangle$[65].

Even though finding the maximum angle for a given multipartite state is a nonlinear problem with no general analytical solution found so far, some symmetric cases enable us to evaluate the *geometric measure* on these states. For example, to evaluate permutation symmetric states, (i.e. the states $|\psi\rangle = \sum_{i_1..i_n} \chi_{i_1..i_n} |i_1\rangle^1 |i_2\rangle^2 .. |i_n\rangle^n$ whose coefficients $\chi_{i_1..i_n}$ are invariant under permutations of i_k's) following Wei and Goldbart's reasonable assumption that the closest separable state $|\phi\rangle$ would also possess a symmetry. They assumed that $|\phi\rangle$ is of the form

$$|\phi\rangle \equiv \otimes_{k=1}^{n} \left(\sum_l c_l |i_l\rangle^k \right) \tag{3.11}$$

thus the minimization task greatly simplifies. To give examples, the *geometric measure of entanglement* of the $|GHZ\rangle$ state for any dimension is found to be $1/2$ and of $|W\rangle$ state is $5/9$ [67]. Considering superpositions of these states does not break the permutation symmetry.

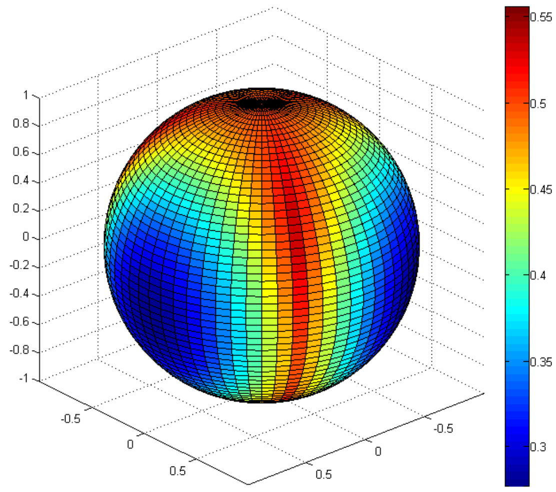

Figure 3.4: Upper view of geometric measure of entanglement of $|GW(\theta, \phi)\rangle$ on a Bloch sphere. The vector $\hat{k}$ represents the $|GHZ\rangle$ state and $-\hat{k}$ vector represents $|W\rangle$ state. Colors are adjusted by interpolating between the maximum value and the minimum value of E to give a high contrast.

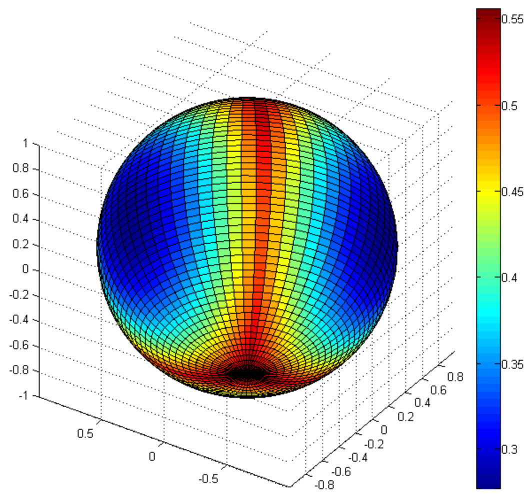

Figure 3.5: Lower view of the geometric measure of entanglement of $|GW(\theta, \phi)\rangle$ on a Bloch sphere. The vector $\hat{k}$ represents the $|GHZ\rangle$ state and the vector $-\hat{k}$ represents the $|W\rangle$ state. Colors are adjusted by interpolating between the maximum value and the minimum value of E to give a high contrast.

First let's define the $|\tilde{W}\rangle$ state as

$$|\tilde{W}\rangle \equiv (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3} \tag{3.12}$$

Any superposition of $|W\rangle$ and $|\tilde{W}\rangle$ is of the form

$$|W\tilde{W}(\theta, \phi)\rangle \equiv \cos(\theta/2)|W\rangle + \sin(\theta/2)e^{i\phi}|\tilde{W}\rangle \tag{3.13}$$

Observing that the result is ϕ independent [67] and solving the equation for θ, we can obtain the entanglement value for a θ corresponding to the angle of superposition of the given state. In figure 3.3 geometric measure of entanglement versus cosine square of the superposition angle ($\sqrt{s} = \cos(\theta/2)$) is given.

This result is consistent with the one in the Wei's paper [67]. In the next example, the values of *geometric measure of entanglement* is illustrated before is the superposition of $|GHZ\rangle$ and $|W\rangle$ states (eqn. 3.3):

$$|GW(\theta, \phi)\rangle \equiv \cos(\theta/2)|GHZ\rangle + \sin(\theta/2)e^{i\phi}|W\rangle.$$

In Wei's paper the entanglement characteristics is given again as an E vs. s graph with random ϕ values[67]. Here we give the same result by coloring the surface of a bloch sphere spanned by the $|GHZ\rangle$ and the $|W\rangle$ states in figures 3.4 and 3.5.

Here, it is interesting to observe the parallelism between *tangle* and the *geometric measure of entanglement*. Numerical results in [61, 67] and our results show that both are symmetric with respect to $\phi \longrightarrow \phi + 2\pi/3$ transformation. This can be derived from the invariance under the transformation given in eqns. 3.14a and 3.14b applied to all three qubits[67].

$$|0\rangle \longrightarrow |0\rangle, \tag{3.14a}$$

$$|1\rangle \longrightarrow e^{i\frac{2k\pi}{3}}|1\rangle \tag{3.14b}$$

where $k = 1, 2$ or 3. As one can predict, extension to mixed states is given by convex roof extension:

$$E_G(\rho) \equiv \inf_{p_i, |\psi_i\rangle} \sum_i p_i E_G(|\psi_i\rangle) \tag{3.15}$$

where the minimization is over all possible decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Wei and Goldbart also proved that this convex hull construction to be an entanglement monotone satisfying the essential conditions [23, 67]. Although it is relatively easy to find analytic formulas for some special states such as symmetric, antisymmetric pure states [68] and Werner states [69], no analytical formula for the most general case is given in the literature up to date. Still, it can be evaluated with numerical computation methods.

3.5 Entanglement of Assistance and Localizable Entanglement

Consider three observers Alice, Bob and Charlie possessing the pure state $|GHZ\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle)$ such that as seen by Alice and Bob the state is described by the density matrix $\rho^{AB} = \mathrm{tr}_C(|GHZ\rangle\langle GHZ|) = \frac{1}{2}\mathbb{1}$. If Charlie makes a measurement in the $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ basis on his local subsystem and inform Alice and Bob about his result, this process would immediately turn the completely separable state Alice and Bob have into one of the maximally entangled states $(|00\rangle + |11\rangle)/\sqrt{2}$ or $(|00\rangle - |11\rangle)/\sqrt{2}$. DiVincenzo *et al.* made use of this fact to quantify entanglement; they call this the *entanglement of assistance* (EoA) between Alice and Bob [70].

Entanglement of assistance is defined for a pure tripartite state of parties A, B and C, as the maximum average bipartite entanglement that could be obtained for A and B, after C helping them by performing a local measurement on his subsystem and report his outcome to the others [70]. Let us denote the pure state they possess by $|\psi^{ABC}\rangle$ and let $\{p_i, |\psi_i^{AB}\rangle\}$ be an ensemble constituting ρ^{AB}:

$$\rho^{AB} = \sum_i p_i |\psi^{AB}\rangle\langle\psi^{AB}|. \tag{3.16}$$

Following the definition of *entanglement of assistance* one naturally aims to maximize the average bipartite entanglement. Moreover this maximization can be given directly as the formula for this quantity[70]:

$$E_A(|\psi^{ABC}\rangle = E_A(\rho^{AB}) \equiv \sup_{\{p_i^{AB}, |\psi_i^{AB}\rangle\}} \sum_i p_i S(|\psi_i^{AB}\rangle\langle\psi_i^{AB}|), \tag{3.17}$$

where $S(\rho)$ is the von Neumann Entropy defined for the bipartite pure state entanglement. We are able to write eqn. 3.17, because as a result of the Hughstone-Jozsa-Wootters theorem [71] one can be sure that C can find an appropriate basis to make measurements such that the state obtained by A and B after C's measurements is ψ_i^{AB} with probability p_i [13].

Entanglement of assistance is defined for three parties, one of which is the "assistant" who tries to choose the most efficient basis to make measurements in order to maximize the entanglement between the other parties. If more than one "assistants" are considered, one comes up with *localizable entanglement* (LO)[72]. *Localizable entanglement* is the maximal entanglement that two parties can obtain by the help of n other parties performing only LOCC operations on their particles.

Both *entanglement of assistance* and *localizable entanglement* are not LOCC monotones. Gour and Spekkens showed that, including classical data transfer from A and B to the other parties may increase these quantities deterministically [73]. Thus, they generalized EOA and LE further to *entanglement of collaboration* and *collaborative localizable entanglement* respectively where the definitions of the generalized version include two sided classical communication between all parties.

Here will be presented a simplified proof of how collaboration with A and B increases EOA [73]. Suppose A, B and C possess the state of dimensions $4 \times 4 \times 2$

$$|\phi^{ABC}\rangle = \frac{[(|00\rangle + |11\rangle)|0\rangle + (|00\rangle - i|11\rangle)|0\rangle + (|22\rangle + |33\rangle)| + i\rangle + (|22\rangle - |33\rangle)| - i\rangle]}{2\sqrt{2}}, \tag{3.18}$$

where $| \pm i\rangle \equiv (|0\rangle \pm i|1\rangle) / \sqrt{2}$. First, A makes a measurement with operators $|0\rangle\langle 0| + |1\rangle\langle 1|$ and $|2\rangle\langle 2| + |3\rangle\langle 3|$ and sends her result to C. C, then, depending on the outcome, makes a measurement in the $|0\rangle, |1\rangle$ or $| + i\rangle, | - i\rangle$ basis. In each of the cases A and B are left with a maximally entangled state, hence the *entanglement of collaboration* of this state is 2.

Let us now try to evaluate the entanglement of assistance of this state. The state $|\phi^{ABC}\rangle$ can be written as

$$\begin{aligned} |\phi^{ABC}\rangle &= \frac{1}{2\sqrt{2}}\Big((|00\rangle + |11\rangle + \sqrt{2}|22\rangle)|0\rangle + (|00\rangle - i|11\rangle + i\sqrt{2}|33\rangle)|1\rangle\Big) \\ &= \frac{1}{2\sqrt{2}}\left(|\alpha\rangle|0\rangle + |\beta\rangle|1\rangle\right). \end{aligned} \tag{3.19}$$

Thus, tracing out C, the state that A and B would be left with is an equal mixture of $|\alpha\rangle$ and $|\beta\rangle$. By the Hughstone-Jozsa-Wootters theorem [71], we have a unitary freedom to mix the states in the ensemble.That is, if there is any linear combination $a|\alpha\rangle+b|\beta\rangle$, with normalization on $a, b \in \mathbb{C}$, that gives a maximal bipartite entanglement, then C can choose an appropriate basis for his measurements to leave A and B with that pure state with a nonzero probability p. However it can be shown that no such combination would give a maximally entangled state like the previous collaborative case [73].

For $a|\alpha\rangle + b|\beta\rangle$ to be maximally entangled, the magnitudes of its Schmidt coefficients should be equal [22]. In our case, the linear combination is already Schmidt decomposed

$$\begin{aligned} a\left(|00\rangle + |11\rangle + \sqrt{2}|22\rangle\right) + b\left(|00\rangle - i|11\rangle + i\sqrt{2}|33\rangle\right) = \\ (a+b)|00\rangle + (a-ib)|11\rangle + \sqrt{2}a|22\rangle + i\sqrt{2}b|33\rangle \end{aligned} \tag{3.20}$$

So, equating magnitudes we get

$$|a+b| = |a-ib| = \sqrt{2}|a| = \sqrt{2}|b| \tag{3.21}$$

with the normalization condition

$$|a|^2 + |b|^2 = 1. \tag{3.22}$$

One can easily show that this set of equations has no solutions for complex a and b.

The definition of *Entanglement of Collaboration* includes all possible local measurements and classical communication between all parties to maximize the value so it is an entanglement monotone [17]. There are other possible modifications to make *entanglement of assistance* an entanglement monotone. The regularized version of *entanglement of assistance*

$$E_A^{\infty}(|\psi^{ABC}\rangle) \equiv \lim_{n\to\infty} \frac{E_A\left(|\psi^{ABC}\rangle^{\otimes n}\right)}{n} \tag{3.23}$$

is shown to become an entanglement monotone [74, 75, 76], even though it is not a monotone for finite n [73]. Also, one can replace *entropy of entanglement* with *concurrence* and define *concurrence of assistance* [77] as

$$C_A(|\psi^{ABC}\rangle) \equiv \sup_{\{p_i^{AB}, |\psi_i^{AB}\rangle\}} \sum_i p_i C(|\psi_i^{AB}\rangle) \tag{3.24}$$

where the *concurrence* of a pure state can be given as [57]

$$C(|\phi^{AB}\rangle) = 2\sqrt{\det \rho^A}. \tag{3.25}$$

Here ρ^A is the marginal density matrix of the pure state $|\phi^{AB}\rangle\langle\phi^{AB}|$. Defining assistance this way rewards us with an explicit formula for two qubits [77]:

$$C_A(|\psi^{ABC}\rangle) = F(\rho^{AB}, \tilde{\rho}^{AB}) \tag{3.26}$$

where *Fidelity* $F(\rho, \sigma)$ is defined as

$$F(\rho, \sigma) \equiv \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho 1/2}$$

and *tilde* ($\tilde{}$) operation is the same spin flip operation that we used in eqn.(2.10)of Wootters [28]:

$$\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{3.27}$$

CHAPTER 4

CONCLUSION

Despite the recent progress after the pioneering papers by Bennett *et al.* [78, 79, 24], there is still much to be explored about entanglement. How to measure the amount of entanglement that a given state possesses is an important question that has been attempted to be answered from different points of view. A convenient measure (or multiple measures in some cases) capturing the essential characteristic features of entanglement would be useful for evaluating probabilities of transformation of states, calculating the amount of non-local resources that we have in hand, or calculating quantum channel capacities. In this thesis, we have tried to review the most common entanglement measures briefly.

For bipartite pure states, the measure of entanglement is well established. This is mostly because any bipartite state can be Schmidt decomposed. Thereafter, von Neumann entropy gives the exact information content of entanglement. However, while considering mixed states, optimization of construction, extraction or manipulation of nonlocal resources is not straightforward. This leads to two types of entanglement measures: geometrical or operational [19]. Geometric measures are mostly related with the distance to a set of states like separable states or PPT states. On the other hand operational measures find their meaning in physical operations such as state transformations. Among the ones we have studied, *relative entropy of entanglement*, *robustness*, *geometric measure* and *negativity* might be counted as the former type whereas *entanglement of distillation* and *entanglement cost* can be included in the latter type.

Many applications in the field quantum information and computation require the treatment of entangled states shared by more than two parties. Unfortunately, it is not possible to give direct generalizations of all the approaches we reviewed for the bipartite case to the n-partite case. Absence of a standard maximum entangled state like Bell states is one of the problems that one encounters at first. Another difficulty is the characterization of the genuine n-partite entanglement. One might be interested in the maximum amount of entanglement that can be distilled in EPR pairs or one may only be interested in the entanglement between all the parties involved. Therefore a single measure of multipartite entanglement even for pure states is not likely to appear.

We tried to present most of the promising entanglement measures for bipartite mixed states in Chapter 2, and for multipartite states in Chapter 3. The measures existing in the literature are not limited to the ones that are analyzed here. Among those, Schmidt measure [44], hyperdeterminants [80, 62] and the multipartite version of *squashed entanglement*[81, 82] might be added to our list at a first glance. A pure geometric view like Miyake's hyperdeterminants may be also helpful in understanding entanglement. Although a new entanglement measure is not introduced in Mosseri and Dandoloff's paper [83], their approach generalizing the Bloch sphere to two dimensions using Hopf fibrations and making use of quaternions to simplify calculations is noteworthy. This approach is further explored for three dimensions in another article [84].

In Table 4.1, a summary of the properties of bipartite measures are given. One can see from the table that there seems to be a tradeoff between the satisfaction of desired properties and the ease of calculation. The first five measures satisfy most of the properties, but up to now, for only *entanglement of formation* of two qubits, a simple formula has been found [28]. It seems that to evaluate the other four, one needs to solve a difficult optimization problem. On the contrary, *negativity* requires a one step calculation and *robustness* has some good properties simplifying numerical calculations.

We hope that this thesis was helpful for both the readers aiming to get a feeling about entanglement measures and for the ones who will continue their research on this topic. The major open problems in this area require mostly sophisticated mathematics, however both analytical and numerical calculations of some measures or at least lower or upper bounds to these will be beneficial. Setting aside *negativity* and *entanglement of formation*, a closed formula has not been found yet for any of these measures. Having a clear operational meaning and being an extremal measure, *entanglement cost* is shown to be equal to *entanglement of formation* depending on the additivity conjecture of *entanglement of formation*[29]. Thus, proving this conjecture would be great help. Considering multipartite entanglement, a complete characterization is still missing [17]. Absence of a single state of entanglement standard creates difficulties in extending the definitions of *entanglement cost* and *distillable entanglement* to the multipartite case. For specific target states, optimum distillation protocols are present in the literature, however it would be better to have more general results [17].

Table 4.1: A summary of properties

	Additivity	Convexity	Continuity	Reduces to Entropy	Easy to Calculate
E_{Cost}	YES	YES	?	YES	NO
E_{Dist}	YES	NO?	?	YES	NO
E_{Form}	?	YES	YES	YES	Easy for 2 qubits
E_{Rel}	NO	YES	YES	YES	NO
E_{Sq}	YES	YES	YES	YES	NO
E_{Neg}	NO[1]	YES[1]	?	NO	YES
E_{Rob}	NO[2]	?	?	NO	Relatively Easy

Measuring entanglement is more or less related to defining invariants in the state space of quantum systems. However, mixed states or multipartite pure states have a nontrivial geometry. Understanding geometry of quantum states would be a great advancement in both visualization of problems and obtaining more general results. The interested reader may find the book by Bengtsson useful [85].

[1]One can define *Logarithmic Negativity* but then this quantity fails to be convex[50]

[2]*Robustness* itself is not additive however a function of it can be additive [51]

REFERENCES

[1] A. Einstein, B. Podolsky, and N. Rosen, “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?,” *Phys. Rev.* **47** (1935), no. 10, 777–780.

[2] J. S. Bell, “On the Einstein Rosen Podolsky paradox,” *Physics (Long Island City, N.Y.)* **1** (1964) 195.

[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” `arXiv:quant-ph/0702225`.

[4] A. Aspect, P. Grangier, and G. Roger, “Experimental Tests of Realistic Local Theories via Bell’s Theorem,” *Phys. Rev. Lett.* **47** (1981), no. 7, 460–463.

[5] A. Aspect, P. Grangier, and G. Roger, “Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell’s Inequalities,” *Phys. Rev. Lett.* **49** (1982), no. 2, 91–94.

[6] A. Aspect, J. Dalibard, and G. Roger, “Experimental Test of Bell’s Inequalities Using Time- Varying Analyzers,” *Phys. Rev. Lett.* **49** (1982), no. 25, 1804–1807.

[7] R. P. Feynman *Int. J. Theor. Phys.* **21** (1982) 467.

[8] D. Deutsch, “Quantum-Theory, The Church-Turıng Prıncıple And The Unıversal Quantum Computer,” *Proceedıngs Of The Royal Socıety Of London Serıes A-Mathematıcal Physıcal And Engıneerıng Scıences* **400** (1985) 97–117.

[9] P. W. Shor, “Algorithms for Quantum Computation: Discrete Logarithms and Factoring,” in *IEEE Symposium on Foundations of Computer Science*, pp. 124–134. 1994.

[10] L. K. Grover, “Quantum Mechanics Helps in Searching for a Needle in a Haystack,” *Phys. Rev. Lett.* **79** (1997), no. 2, 325–328.

[11] C. E. Shannon, “A mathematical theory of communication,” *Bell System Technical Journal* **27** (1948) 379–423,623–656.

[12] B. Schumacher, “Quantum coding,” *Phys. Rev. A* **51** (1995), no. 4, 2738–2747.

[13] M. A. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information”. Cambridge University Press, Cambridge, New York, 2000.

[14] C. H. Bennett and S. J. Wiesner, “Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states,” *Phys. Rev. Lett.* **69** (1992), no. 20, 2881–2884.

[15] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels,” *Phys. Rev. Lett.* **70** (1993), no. 13, 1895–1899.

[16] C. H. Bennett and G. Brassard., “Quantum cryptography: Public key distribution and coin tossing.,” in *In Proceedings of IEEE, International Conference on Computers, Systems and Signal Processing*, pp. 175–179. 1984.

[17] M. B. Plenio and S. Virmani, “An introduction to entanglement measures,” *QUANT.INF.COMP.* **7** (2007) 1.

[18] S. Popescu, “Bell’s inequalities versus teleportation: What is nonlocality?,” *Phys. Rev. Lett.* **72** (1994), no. 6, 797–799.

[19] L. Clarisse, “On the Schmidt robustness of pure states,” *Journal of Physics A: Mathematical and General* **39** (2006), no. 16, 4239–4249.

[20] M. Horodecki, P. Horodecki, and R. Horodecki, “Mixed-State Entanglement and Distillation: Is there a Bound Entanglement in Nature?,” *Phys. Rev. Lett.* **80** (1998), no. 24, 5239–5242.

[21] J. J. Sakurai, “Modern Quantum Mechanics”. Addison-Wesley Publishing Company, 1993.

[22] M. A. Nielsen, “Conditions for a Class of Entanglement Transformations,” *Phys. Rev. Lett.* **83** (1999), no. 2, 436–439.

[23] G. Vidal, “Entanglement Monotones,” *Journal of Modern Optics* **47** no. 2–3, 355–376.

[24] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, “Mixed-state entanglement and quantum error correction,” *Phys. Rev. A* **54** (1996), no. 5, 3824–3851.

[25] D. Bruss, “Characterizing Entanglement,” *Journal of Mathematical Physics* **43** (2002) 4237.

[26] M. Horodecki, P. Horodecki, and R. Horodecki, “Limits for Entanglement Measures,” *Phys. Rev. Lett.* **84** (2000), no. 9, 2014–2017.

[27] M. J. Donald, M. Horodecki, and O. Rudolph, “The Uniqueness Theorem for Entanglement Measures,” *Journal of Mathematical Physics* **43** (2002) 4252.

[28] W. K. Wootters, “Entanglement of Formation of an Arbitrary State of Two Qubits,” *Phys. Rev. Lett.* **80** (1998), no. 10, 2245–2248.

[29] P. M. Hayden, M. Horodecki, and B. M. Terhal, “The asymptotic entanglement cost of preparing a quantum state,” *Journal of Physics A: Mathematical and General* **34** (2001), no. 35, 6891–6898.

[30] T. M. Cover and J. A. Thomas, “Elements of Information Theory”. A Wiley-Interscience Publication, 1991.

[31] G. Lindblad, “Expectations and entropy inequalities for finite quantum systems,” *Commun. Math. Phys.* **39** (1974), no. 2, 111–119.

[32] G. Lindblad, “Completely positive maps and entropy inequalities,” *Commun. Math. Phys.* **40** (1975), no. 2, 147–151.

[33] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, “Quantifying Entanglement,” *Phys. Rev. Lett.* **78** (1997), no. 12, 2275–2279.

[34] V. Vedral, M. B. Plenio, K. Jacobs, and P. L. Knight, "Statistical inference, distinguishability of quantum states, and quantum entanglement," *Phys. Rev. A* **56** (1997), no. 6, 4452–4455.

[35] V. Vedral and M. B. Plenio, "Entanglement measures and purification procedures," *Phys. Rev. A* **57** (1998), no. 3, 1619–1633.

[36] V. Vedral, "The role of relative entropy in quantum information theory," *Rev. Mod. Phys.* **74** (2002), no. 1, 197–234.

[37] R. R. Tucci, "Quantum Entanglement and Conditional Information Transmission," arXiv:quant-ph/9909041.

[38] R. R. Tucci, "Entanglement of Distillation and Conditional Mutual Information," arXiv:quant-ph/0202144.

[39] M. Christandl and A. Winter, ""Squashed Entanglement" - An Additive Entanglement Measure," *Journal of Mathematical Physics* **45** (2004) 829.

[40] N. J. Cerf and C. Adami, "Negative Entropy and Information in Quantum Mechanics," *Phys. Rev. Lett.* **79** (1997), no. 26, 5194–5197.

[41] A. Peres, "Separability Criterion for Density Matrices," *Phys. Rev. Lett.* **77** (1996), no. 8, 1413–1415.

[42] M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of mixed states: Necessary and sufficient conditions," *Phys. Rev. Lett.* **223** (1996), no. 1–2, 1–8.

[43] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, "Volume of the set of separable states," *Phys. Rev. A* **58** (1998), no. 2, 883–892.

[44] J. Eisert, "Entanglement in quantum information theory," arXiv:quant-ph/0610253.

[45] J. Lee, M. S. Kim, Y. J. Park, and S. Lee, "Partial teleportation of entanglement in a noisy environment," *Journal of Modern Optics* **47** (2000), no. 12, 2151–2164.

[46] G. Vidal and R. F. Werner, "Computable measure of entanglement," *Phys. Rev. A* **65** (2002), no. 3, 032314.

[47] M. B. Plenio, "Logarithmic Negativity: A Full Entanglement Monotone That is not Convex," *Physical Review Letters* **95** (2005), no. 9, 090503.

[48] R. A. Horn and C. R. Johnson, "Topics in Matrix Analysis". Cambridge University Press, 1991.

[49] R. Bhatia, "Matrix Analysis". Springer, 1996.

[50] J. Eisert and M. B. Plenio, "A comparison of entanglement measures," *Journal of Modern Optics* **46** (1999) 145.

[51] G. Vidal and R. Tarrach, "Robustness of entanglement," *Phys. Rev. A* **59** (1999), no. 1, 141–155.

[52] A. W. Harrow and M. A. Nielsen, "Robustness of quantum gates in the presence of noise," *Phys. Rev. A* **68** (2003), no. 1, 012308.

[53] M. Steiner, "Generalized robustness of entanglement," *Phys. Rev. A* **67** (2003), no. 5, 054305.

[54] A. Sanpera, R. Tarrach, and G. Vidal, "Local description of quantum inseparability," *Phys. Rev. A* **58** (1998), no. 2, 826–830.

[55] W. Dür, G. Vidal, and J. I. Cirac, "Three qubits can be entangled in two inequivalent ways," *Phys. Rev. A* **62** (2000), no. 6, 062314.

[56] M. Murao, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight, "Multiparticle entanglement purification protocols," *Phys. Rev. A* **57** (1998), no. 6, R4075–R4078.

[57] V. Coffman, J. Kundu, and W. K. Wootters, "Distributed entanglement," *Phys. Rev. A* **61** (2000), no. 5, 052306.

[58] A. Wong and N. Christensen, "Potential multiparticle entanglement measure," *Phys. Rev. A* **63** (2001), no. 4, 044301.

[59] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities," *American Journal of Physics* **58** (1990), no. 12, 1131–1143.

[60] S. Machnes, "QLib - A Matlab Package for Quantum Information Theory Calculations with Applications," `arXiv:quant-ph/0708.0478`.

[61] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, "Entangled Three-Qubit States without Concurrence and Three-Tangle," *Physical Review Letters* **97** (2006), no. 26, 260502.

[62] A. Miyake, "Classification of multipartite entangled states by multidimensional determinants," *Phys. Rev. A* **67** (2003), no. 1, 012108.

[63] A. Miyake, "Multipartite Entanglement under Stochastic Local Operations and Classical Communication," *International Journal of Quantum Information* **2** (2004) 65.

[64] M. B. Plenio and V. Vedral, "Bounds on relative entropy of entanglement for multi-party systems," *Journal of Physics A: Mathematical and General* **34** (2001), no. 35, 6997–7002.

[65] A. Shimony, "Degree of Entanglement," in *Fundamental Problems in Quantum Theory*, D. M. Greenberger and A. Zelinger, eds., volume 755 of *New York Academy Sciences Annals*, pp. 675–679. 1995.

[66] H. Barnum and N. Linden, "Monotones and invariants for multi-particle quantum states," *Journal of Physics A: Mathematical and General* **34** (2001), no. 35, 6787–6805.

[67] T.-C. Wei and P. M. Goldbart, "Geometric measure of entanglement and applications to bipartite and multipartite quantum states," *Phys. Rev. A* **68** (2003), no. 4, 042307.

[68] S. Bravyi, "Entanglement entropy of multipartite pure states," *Phys. Rev. A* **67** (2003), no. 1, 012313.

[69] R. F. Werner, "Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model," *Phys. Rev. A* **40** (1989), no. 8, 4277–4281.

[70] D. P. DiVincenzo et al., "Entanglement of Assistance," `arXiv:quant-ph/9803033`.

[71] W. K. W. Lane P. Hughston, Richard Jozsa, "A complete classification of quantum ensembles having a given density matrix," *Physics Letters A* **183** (1993) 14–18.

[72] F. Verstraete, M. Popp, and J. I. Cirac, "Entanglement versus Correlations in Spin Systems," *Phys. Rev. Lett.* **92** (2004), no. 2, 027901.

[73] G. Gour and R. W. Spekkens, "Entanglement of assistance is not a bipartite measure nor a tripartite monotone," *Physical Review A (Atomic, Molecular, and Optical Physics)* **73** (2006), no. 6, 062331.

[74] J. A. Smolin, F. Verstraete, and A. Winter, "Entanglement of assistance and multipartite state distillation," *Physical Review A (Atomic, Molecular, and Optical Physics)* **72** (2005), no. 5, 052317.

[75] M. Horodecki, J. Oppenheim, and A. Winter, "Partial quantum information," *Nature* **436** (2005) 673–676.

[76] M. Horodecki, J. Oppenheim, and A. Winter, "Quantum state merging and negative information," *Communications in Mathematical Physics* **269** (2006) 107.

[77] T. Laustsen, F. Verstraete, and S. J. van Enk, "Local vs. Joint Measurements for the Entanglement of Assistance," *Quantum Information and Computation* **3** (2003) 64.

[78] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, "Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels," *Phys. Rev. Lett.* **76** (1996), no. 5, 722–725.

[79] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, "Concentrating partial entanglement by local operations," *Phys. Rev. A* **53** (1996), no. 4, 2046–2052.

[80] A. Miyake and M. Wadati, "Multipartite Entanglement and Hyperdeterminants," *Quantum Information and Computation* **2** (2002) 540, `arXiv:quant-ph/0212146`.

[81] D. Avis, P. Hayden, and I. Savov, "Distributed Compression and Multiparty Squashed Entanglement," `arXiv:quant-ph/0707.2792`.

[82] D. Yang, K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim, and W. Song, "Squashed entanglement for multipartite states and entanglement measures based on the mixed convex roof," `arXiv:quant-ph/0704.2236`.

[83] R. Mosseri and R. Dandoloff, "Geometry of entangled states, Bloch spheres and Hopf fibrations," *Journal of Physics A: Mathematical and General* **34** (2001), no. 47, 10243–10252.

[84] B. A. Bernevig and H.-D. Chen, "Geometry of the three-qubit state, entanglement and division algebras," *Journal of Physics A: Mathematical and General* **36** (2003), no. 30, 8325–8339.

[85] I. Bengtsson and K. Zyczkowski, "Geometry of quantum states : an introduction to quantum entanglement". Cambridge University Press, Cambridge, New York, 2006.