REAL LEFSCHETZ FIBRATIONS

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ABSTRACT

REAL LEFSCHETZ FIBRATIONS

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In this thesis, we present real Lefschetz fibrations. We first study real Lefschetz fibrations around a real singular fiber. We obtain a classification of real Lefschetz fibrations around a real singular fiber by a study of monodromy properties of real Lefschetz fibrations. Using this classification, we obtain some invariants, called real Lefschetz chains, of real Lefschetz fibrations which admit only real critical values. We show that in case the fiber genus is greater then 1, the real Lefschetz chains are complete invariants of directed real Lefschetz fibrations with only real critical values. If the genus is 1, we obtain complete invariants by decorating real Lefschetz chains.

For elliptic Lefschetz fibrations we define a combinatorial object which we call necklace diagrams. Using necklace diagrams we obtain a classification of directed elliptic real Lefschetz fibrations which admit a real section and which have only real critical values. We obtain 25 real Lefschetz fibrations which admit a real section and which have 12 critical values all of which are real. We show that among 25 real Lefschetz fibrations, 8 of them are not algebraic. Moreover, using necklace diagrams we show the existence of real elliptic Lefschetz fibrations which can not be written as the fiber sum of two real elliptic Lefschetz fibrations. We define refined necklace diagrams for real elliptic Lefschetz fibrations without a real section and show that refined necklace diagrams classify real elliptic Lefschetz fibrations which have only real critical values.

Keywords : Lefschetz fibrations, real structure, real Lefschetz fibrations, real Lefschetz chains, necklace diagrams.

ÖZ

REEL LEFSCHETZ LİFLENMELERİ

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Bu tezde reel Lefschetz liflenmeleri sunulmaktadır. Öncelikle reel Lefschetz liflenmelerini herhangi bir reel singuler lif yakınında çalıştık. Monodromy özelliklerini kullanarak, singuler bir lif etrafında ki reel Lefschetz liflenmelerinin bir sınıflandırmasını verdik. Bu sınıflandırmayı kullanarak, sadece reel kritik değerleri olan reel Lefschetz liflenmeleri için reel Lefschetz zincirleri diye adlandırdığımız bir değişmez tanımladık. Liflerin genus sayısı 1' den büyük ise, bu değişmezin sadece reel kritik değerleri olan yönlü reel Lefschetz liflenmelerinin tam değişmezi olduğunu gösterdik. Lif genus sayısı 1 ise reel Lefschetz zincirlerini decore ederek tam değişmezler elde ettik.

Elipsel reel Lefschetz liflenmeleri icin kolye diyagramları diye adlandırdığı mız kombinatoryal nesneleri tanımladık. Kolye diyagramlarını kullanarak, reel kesit kabul eden ve sadece reel kritik değerleri olan reel Lefschetz liflenmelerinin bir sınıflandırmasını elde ettik. Bu sınıflandırmanın sonucu olarak 25 tane, reel kesit kabul eden ve sadece reel kritik değerleri olan ve toplam kritik değer sayısı 12 olan elipsel reel Lefschetz liflenmeleri elde ettik. Elde ettiğimiz 25 tane liflenmelerin 8'i hariç hepsinin cebirsel olduğunu gösterdik. Bununla birlikte, kolye diyagramlarını kullanarak iki elipsel reel Lefschetz liflenmesinin lif eklenmesi olarak yazılamayan elipsel reel Lefschetz liflenmeleri örneklerini bulduk. Son olarak, reel kesit teşkil etmeyebilen liflenmeler için rafine kolye diyagramlarını tanımladık ve sadece reel kritik değerleri olan elipsel reel Lefschetz liflenmelerinin rafine kolye diyagramları ile sınıflandırıldığını gösterdik.

Anahtar kelimeler : Lefschetz liflenmeleri, reel yapı, reel Lefschetz liflenmeleri, reel Lefschetz zincirleri, kolye diyagramları.

To my sisters, ***Yasemin and Nesrin****, who let me learn how to share selflessly...*

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CHAPTER 1

INTRODUCTION

The richness of complex manifolds is mainly due to the existence of two important maps: multiplication by i and complex conjugation. To be able to obtain smooth manifolds which resemble complex manifolds as much as possible, generalizations of these maps to smooth even-dimensional manifolds are introduced. The generalization of multiplication by i is called an *almost complex structure* and of complex conjugation is called a *real structure*.

In this thesis, we study Lefschetz fibrations which admit a real structure. Let us recall that a Lefschetz fibration of a smooth 4-manifold is a fibration by surfaces such that only a finite number of fibers are allowed to have a nodal type of singularity. Lefschetz fibrations naturally appear on complex surfaces in complex projective 3-space as blow ups of a pencil of planes, generic with respect to surfaces. It is known that the monodromy of Lefschetz fibrations around a singular fiber is given by a single (positive) Dehn twist along a simple closed curve (called the *vanishing cycle*) [K] and that decompositions of the monodromy (up to Hurwitz moves and conjugation by an element of the mapping class group) into a product of Dehn twists classify Lefschetz fibrations over D^2. One important property of the Lefschetz fibrations is that they form the topological counterpart of symplectic 4-manifolds (see S. Donaldson [Do], R. Gompf [GS]).

The study of real Lefschetz fibrations is motivated by the work of S. Yu. Orevkov [O1] in which he presented a method of reading the (braid) monodromy of a fibration, $\pi : C \to \mathbb{C}P^1$, of a (complex) curve C (which is invariant under complex conjugation) in $\mathbb{C}P^2$ from the part $\mathbb{R}P^2 \cap C \to \mathbb{R}P^1$ where the fibration, π, of C is obtained from a real pencil of lines in $\mathbb{C}P^2$, generic with respect to C. He observed that the total monodromy is quasipositive (product of conjugations of positive twists) if the curve C is algebraic and used this observation to show that certain distributions of ovals in $\mathbb{R}P^2$ are not algebraically realizable. It is not hard to see that if his construction is applied to surfaces in $\mathbb{C}P^3$, what we obtain is nothing but a Lefschetz pencil which commutes with the standard complex conjugation of $\mathbb{C}P^3$. This gives a prototype of the real Lefschetz fibrations.

We define a *real structure* on a smooth $2k$-dimensional manifold as an orientation reversing involution if k is odd and an orientation preserving involution if k is even. We also require that the fixed point set, if it is not empty, has dimension k to make the situation as similar as possible to that of an honest complex conjugation. A manifold together with a real structure is called a *real manifold* and the set of points fixed by the real structure is called the *real part*. Although, naturally, we cannot talk about a real structure on an odd dimensional manifold, we also use the term *real* for odd dimensional manifolds which appear as the boundary of real manifolds.

A real structure on a Lefschetz fibration, $\pi : X \to B$, is a pair, (c_X, c_B), of real structures, $c_X : X \to X$ and $c_B : B \to B$, such that $\pi \circ c_X = c_B \circ \pi$. We study Lefschetz fibrations up to equivariant diffeomorphisms. We assume that fibrations are relatively minimal (that is none of the vanishing cycles bounds a disc on the fiber) and that the genus of the regular fibers is at least 1. We consider also real fibrations over S^1 which are boundaries of real Lefschetz fibrations over a disc.

In this thesis, we treat mainly the cases $B = D^2$ and $B = S^2$. In both cases, we consider real structures which have nonempty real part. By abuse of notation, we denote both real structures by *conj*. Indeed, one can identify S^2 with $\mathbb{C}P^1$ in a way such that *conj* becomes the standard complex conjugation on $\mathbb{C}P^1$. Similarly, $(D^2, conj)$ can be identified with a 2-disc in $\mathbb{C}P^1$ which is invariant under complex conjugation. Most of the time, we assume that the real part of $(D^2, conj)$ is oriented. We call such fibrations *directed real Lefschetz fibrations*.

The first chapter of the thesis gives some basic definitions. In Chapter 2 we examine monodromies of real Lefschetz fibrations in terms of monodromies of real fibrations over S^1. Note that there are two real points, $r_\pm$, of $(S^1, conj)$ and the fibers over them, $F_\pm$, inherit a real structure, $c_\pm$, from the real structure of X. The main observation is that these two real structures are related by the monodromy, f, of the fibration: namely, $c_+ \circ c_- = f$. This decomposition property is fundamental for the results obtained in this thesis, so it is discussed in detail. In the last section of Chapter 2, we give a classification of real fibrations over S^1, whose fiber genus is 1, using the decomposition property of their monodromy.

Chapter 3 is devoted to the classification of real Lefschetz fibrations over a disc with a unique nodal singular fiber, we call such fibrations *elementary real Lefschetz fibrations*. Such fibrations give a local model for real Lefschetz fibrations around a real singular fiber. Note that the compatibility of real structures with the fibration forces the critical value and the critical point of the elementary real Lefschetz fibration to be real.

We mostly work with *marked Lefschetz fibrations*. This means that we fix a base point b and an identification, $\rho : \Sigma_g \to F_b$, of the fiber over b with an abstract genus-g surface, Σ_g. On real Lefschetz fibrations, we consider two types of markings: $\mathbb{R}$-marking, (b, ρ), where b is a real boundary point and $\mathbb{C}$-marking, $(\{b, \bar{b}\}, \{\rho, \rho \circ c_X\})$, where $\{b, \bar{b}\}$ is a pair of complex conjugate points on the boundary. In the case of $\mathbb{R}$-marking, Σ_g has a real structure $c : \Sigma_g \to \Sigma_g$ obtained as the pull back of the inherited real structure on F_b, so we require that ρ satisfies $c_X \circ \rho = \rho \circ c$. For $\mathbb{C}$-markings, F_b and hence Σ_g, have no real structure; however, one can obtain a real structure by pulling back a real structure on a real fiber. This way we obtain a real structure defined up to isotopy.

Let us choose a simple closed curve, $a \subset \Sigma_g$, representing the vanishing cycle on Σ_g such that $c(a) = a$. We call the pair (c, a) with $c(a) = a$ a *real code*. Two real codes (c, a) and (c', a') are called isotopic if there exists a smooth family of orientation preserving diffeomorphisms $\phi_t : \Sigma_g \to \Sigma_g$ such that $\phi_0 = id$ and $\phi_1(a) = a'$, $c \circ \phi_1 \circ c = c'$. We denote by $[c, a]$ the isotopy class of the real code (c, a). Similarly, two real codes (c, a) and (c', a') are called conjugate if there is an orientation preserving diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi(a) = a'$ and $\phi \circ c = c' \circ \phi$. The conjugacy class of the real code is denoted by $\{c, a\}$.

The main theorem of Chapter 3 is the following.

Proposition 1.0.1. *Up to equivariant diffeomorphisms preserving the marking, directed $\mathbb{C}$-marked elementary real Lefschetz fibrations are classified by the isotopy classes, $[c, a]$.*

Up to equivariant diffeomorphisms, directed elementary real Lefschetz fibrations are classified by the conjugacy classes, $\{c, a\}$.

By enumerating possible classes $\{c, a\}$, we have obtained the classification of directed elementary real Lefschetz fibrations.

In Chapter 4, we generalize the classification of elementary real Lefschetz fibrations to a classification of real Lefschetz fibration over D^2 whose critical values are all real. For this purpose we define a boundary fiber sum for real Lefschetz fibrations over D^2. Let us note that unlike the boundary fiber sum of Lefschetz fibrations the boundary fiber sum of two real Lefschetz fibrations is not always defined since one needs the compatibility of real structures on fibers to be glued. We have shown that the boundary fiber sum (when it is defined) of two directed $\mathbb{C}$-marked genus-g real Lefschetz fibrations over D^2 is well-defined if $g > 1$. In case of $g = 1$ (in this case we call the fibration *elliptic Lefschetz fibration*), the boundary fiber sum is well-defined provided fibrations admit a real section.

Let $\pi : X \to D^2$ be a $\mathbb{C}$-marked real Lefschetz fibration with only real critical values, $q_1 < q_2 < \cdots < q_n$. We divide D^2 into smaller (topological) discs, each containing a single critical value (see Figure 1.1). Let $r_0 = r_-, r_1, \ldots, r_{n-1}, r_n = r_+$ denote the real boundary points of the obtained smaller discs.

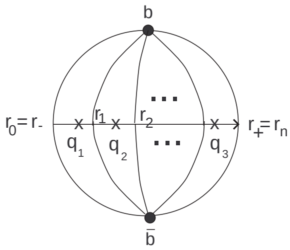


Fig. 1.1.

Each fibration over such discs is determined by the pair $[c_i, a_i]$ such that $c_i(a_i) = a_i$. And each pair of real structures c_{i-1}, c_i are related by the monodromy t_{a_i}; $c_i \circ c_{i-1} = t_{a_i}$ where c_i is the real structure carried over from the real structure on the fiber F_{r_i} and a_i is the vanishing cycle corresponding to the critical value q_i.

If $g > 1$ the classes $[c_i, a_i]$ can be carried over to Σ_g canonically. Thus, we get a sequence $[c_1, a_1]$, $[c_2, a_2], \ldots, [c_n, a_n]$ on Σ_g such that $c_i(a_i) = a_i$ and $c_i \circ c_{i-1} = t_{a_i}$. We call this sequence the *real Lefschetz chain*. In the case of $g = 1$, we can apply the same idea for real Lefschetz fibrations which admit a real section, then the real structures are determined up to isotopy relative to the points determined by the section. Let us denote the relative isotopy class by $[c, a]^*$. We call the sequence $[c_1, a_1]^*, [c_2, a_2]^*, \ldots, [c_n, a_n]^*$ such that $c_i(a_i) = a_i$ and $c_i \circ c_{i-1} = t_{a_i}$ the *pointed real Lefschetz chain*.

Theorem 1.0.2. *If $g > 1$, there is a one-to-one correspondence between the real Lefschetz chains, $[c_1, a_1], [c_2, a_2], \ldots, [c_n, a_n]$ on Σ_g and the isomorphism classes of directed $\mathbb{C}$-marked genus-g real Lefschetz fibrations over D^2 with only real critical* *values. If $g = 1$, there is a one to one correspondence between the pointed real Lefschetz chains, $[c_1, a_1]^*, [c_2, a_2]^*, \ldots, [c_n, a_n]^*$, on Σ_1 and the isomorphism classes of directed genus-g $\mathbb{C}$-marked real Lefschetz fibrations over D^2 with a real section and with only real critical values.*

Moreover, in the both cases, if the total monodromy is isotopic to the identity, one can extend the fibration to a fibration over S^2. We will show that such an extension is unique in both cases.

A similar result can be obtained for directed real elliptic Lefschetz fibrations which do not admit a real section. However, for such fibrations there is no canonical way to carry the classes $[c_i, a_i]$ to the fiber Σ_g. Thus, we consider the boundary fiber sum of non-marked fibrations and work with the conjugacy classes $\{c_i, a_i\}$ of real codes. We see that the boundary fiber sum is not uniquely defined for certain cases and hence the chain $\{c_1, a_1\}$, $\{c_2, a_2\}, \ldots, \{c_n, a_n\}$ of conjugacy classes of real codes, called the *weak Lefschetz chain*, is not sufficient for a correspondence theorem.

On Σ_1, for certain real structures a special phenomenon may occur: two invariant curves can be isotopic without being equivariantly isotopic. When we glue two elementary real Lefschetz fibrations at real fibers where the vanishing cycles are such invariant curves, the boundary sum depends on whether or not we switch the two such vanishing cycles while identifying the fibers. We mark such a gluing point if we switch the two vanishing cycles. We consider the weak real Lefschetz chain, $\{c_1, a_1\}, \{c_2, a_2\}, \ldots, \{c_n, a_n\}$ and mark the real codes corresponding to marked gluing points by $\{c_i, a_i\}^R$ (where R refers to the rotation exchanging the vanishing cycles). The resulting chain is called the *decorated weak real Lefschetz chain.*

Theorem 1.0.3. *There exists a one-to-one correspondence between the decorated weak real Lefschetz chains and the isomorphism classes of directed (non-marked) real elliptic Lefschetz fibrations over D^2 with only real critical values.*

(Let us note that if on the weak real Lefschetz chain, none of the real structures c_i has no real component and none of the real codes $\{c_i, a_i\}$ is marked then the corresponding real elliptic Lefschetz fibration admits a real section.)

If the total monodromy is the identity then we can talk about the extension of the fibration over D^2 to a fibration over S^2. We show that such an extension is unique, if the point of infinity does not require a marking; otherwise, the extension is uniquely determined by the marking of infinity.

The remaining part of the thesis is devoted to the classification of real elliptic Lefschetz fibrations over S^2 with only real critical values. We see that elliptic Lefschetz fibrations, $\pi : X \to S^2$, with only real critical values are determined by their real locus, $\pi_{\mathbb{R}} : X_{\mathbb{R}} \to S^1$, where $X_{\mathbb{R}} = Fix(c_X)$ and $\pi_{\mathbb{R}} : \pi|_{Fix(c_X)}$. In fact, under the assumption that there is a real section, one can control the isotopy types of the real structures over the regular fibers of $\pi_{\mathbb{R}}$. By encoding the types of the real structure on the fibers (singular or nonsingular) on S^1, we obtain a decoration. We introduce a combinatorial object called *necklace diagrams* related to the decorated S^1. When the fibration is directed the associated necklace diagram is naturally oriented.

As was shown by B. Moishezon, [Mo] (non-real) elliptic Lefschetz fibrations are classified by the number of critical values. The latter is divisible by 12 and one denotes by $E(n)$ the class of elliptic Lefschetz fibrations with $12n$ critical values. In Chapter 5, we respond to the following question: how many real structures does the fibration $E(n)$ admit, for each n, such that all critical values are real? We give the answer to the above question in terms of necklaces diagrams.

An oriented necklace diagram is an oriented circle, called the *necklace chain* on which we have finitely many elements of the set $S = \{\square, \bigcirc, >, <\}$. The elements of S are called the *necklace stones.* Two necklace diagrams will be considered identical if their stones go in the same cyclic order.

An example of an oriented necklace diagram is shown in Figure 1.2.

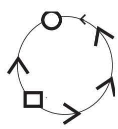


Fig. 1.2.

There is a way to assign a matrix in $PSL(2, \mathbb{Z})$ to each stone of S. We call such a matrix the *monodromy of the stone.* The *necklace monodromy* is by definition the product of the monodromies of the stones where the product is taken in accordance with the orientation and relative to a base point on the necklace chain.

Clearly, the necklace monodromy relative to another base point is conjugate to the previous one.

Theorem 1.0.4. *There exists a one-to-one correspondence between the set of oriented necklace diagrams with* $6n$ *stones whose monodromy is the identity and the set of isomorphism classes of real directed fibrations* $E(n)$*,* $n \in \mathbb{N}$*, which have only real critical values and admit a real section.*

A non-directed real elliptic Lefschetz fibration corresponds to a pair of oriented necklace diagrams, in which one is the mirror image of the other. By using an algorithm which takes into account such symmetry equivalence to enumerate all possible such necklace diagrams we obtain the following result for $n = 1$.

Theorem 1.0.5. *There exist precisely 25 isomorphism classes of real non-directed fibrations $E(1)$ having only real critical values and admitting a real section. These classes are characterized by the non-oriented necklace diagrams presented in Figure 1.3.*

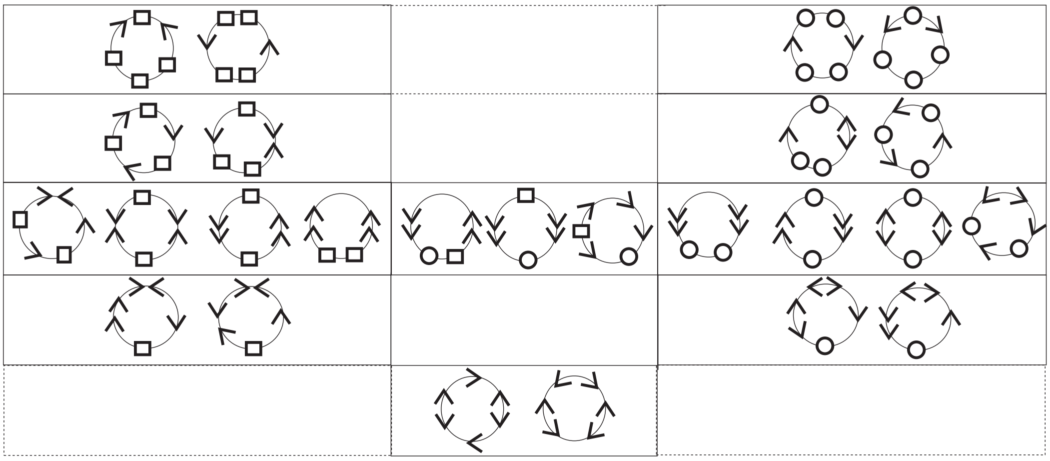


Fig. 1.3.

In Appendix, we will show that among the 25 isomorphism classes which we obtain, there are 8 which are not algebraic. The proof uses the real dessins d'enfants introduced by S. Yu. Orevkov [O2].

Using necklace diagrams, we found some interesting examples. For example, there are real elliptic Lefschetz fibrations of type $E(n)$ with only real critical values which can not be decomposed into a fiber sum of a real $E(n-1)$ and a real $E(1)$ both with only real critical values. Note that for fibrations (non-real) without real structure we have $E(n) = E(n-1)\#_\Sigma E(1)$, [Mo].

Necklace diagrams can be modified to cover the case of fibrations without a real section. Namely, one needs to replace each ○-type stone by one of ○, ⊖, ●, ⊖ without changing the monodromy in $PSL(2, \mathbb{Z})$. The resulted necklace diagrams are called *refined necklace diagrams*. (Refined necklace diagrams whose circle-type stones are all ○-type correspond to fibrations admitting a real section.)

Theorem 1.0.6. *There is a one-to-one correspondence between the set of oriented refined necklace diagrams with* $6n$ *stones whose monodromy is the identity and the set of isomorphism classes of directed real fibrations* $E(n)$*,* $n \in \mathbb{N}$*, whose critical values are all real.*

CHAPTER 2

PRELIMINARIES

2.1 Lefschetz fibrations

Throughout the present work X will stand for a compact connected oriented smooth 4-manifold and B for a compact connected oriented smooth 2-manifold.

Definition 2.1.1. A Lefschetz fibration is a surjective smooth map $\pi : X \to B$ such that:

- $\pi(\partial X) = \partial B$ and the restriction $\partial X \to \partial B$ of π is a submersion;
- π has only a finite number of critical points (that is the points where $d\pi$ is degenerate), all the critical points belong to $X \setminus \partial X$ and their images are distinct points of $B \setminus \partial B$;
- around each of the critical points one can choose orientation-preserving charts $\psi : U \to \mathbb{C}^2$ and $\phi : V \to \mathbb{C}$ so that $\phi \circ \pi \circ \psi^{-1}$ is given by $(z_1, z_2) \to {z_1}^2 + {z_2}^2$.

We will often address a Lefschetz fibration by its initials $\mathcal{LF}$.

Let $\Delta \subset B$ denote the set of critical values of π. As a consequence of the definition above the restriction, $\pi|_{\pi^{-1}(B \setminus \Delta)} : \pi^{-1}(B \setminus \Delta) \to B \setminus \Delta$, of π to $B \setminus \Delta$ is a fiber bundle whose fibers are closed oriented surfaces of the same genus; inheriting a canonical orientation from the orientations of X and B. At critical values, the fibers have nodal singularities.

When we want to specify the genus of the nonsingular fibers, we prefer calling them *genus-g Lefschetz fibrations*. In particular, we will use the term *elliptic Lefschetz fibrations* when the genus is equal to one. For each integer g, we will fix a closed oriented surface of genus g, which will serve as a model for the fibers, and denote it by Σ_g.

In what follows we will always assume that a Lefschetz fibration is *relatively minimal*, that is none of its fibers contains a self intersection -1 sphere. This is not restrictive (if $g \geq 1$) since any self intersection -1 sphere can be blown down while preserving the projection a Lefschetz fibration.

Definition 2.1.2. A *marked genus-g Lefschetz fibration* is a triple (π, b, ρ) such that $\pi : X \to B$ is an $\mathcal{LF}$, $b \in B$ is a regular value of π (if $\partial B \neq \emptyset$ then $b \in \partial B$) and $\rho : \Sigma_g \to F_b = \pi^{-1}(b)$ is a diffeomorphism. (Later on, when precision is not needed, we will denote F_b simply as F.)

Definition 2.1.3. Two Lefschetz fibrations, $\pi : X \to B$ and $\pi' : X' \to B'$, are called *isomorphic* if there exist orientation preserving diffeomorphisms $H : X \to X'$ and $h : B \to B'$, such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{H} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow{h} & B'. \end{array}$$

Two marked Lefschetz fibrations, say (π, b, ρ) and (π', b', ρ'), are called isomorphic if H, h also satisfy $h(b) = b'$ and $H \circ \rho = \rho'$.

Let $Map(S)$ denote the *mapping class group* of a compact closed orientable surface S, that is the group of isotopy classes of orientation preserving diffeomorphisms $S \to S$.

Definition 2.1.4. The *monodromy homomorphism* $\mu : \pi_1(B \setminus \Delta, b) \to Map(\Sigma_g)$ of a marked Lefschetz fibration (π, b, ρ) is defined as follows: pick an element $\gamma \in \pi_1(B \setminus \Delta, b)$, represent it by a smooth map $\tilde{\gamma} : (S^1, *) \to (B \setminus \Delta, b)$, and consider the pull back $\tilde{\gamma}^*(X)$, which is a fiber bundle over S^1 with fibers Σ_g. This fiber bundle does not depend on the choice of $\tilde{\gamma} \in \gamma$ and can be obtained from the trivial bundle $F_b \times I$ over an interval I by identifying both ends by a diffeomorphism $f_\gamma : F_b \to F_b$, that is $\gamma^*(X) = F_b \times I \diagup_{(f_\gamma(x),0)\sim(x,1)}$. The latter diffeomorphism is well defined up to isotopy and the image of γ is defined as the isotopy class $[\rho^{-1} \circ f_\gamma \circ \rho]$ which is called the *monodromy of* π *along* γ *relative to the marking* ρ.

Obviously, if $\rho : \Sigma_g \to F$ is replaced by $\rho' = \rho \circ \phi$, where $\phi \in Map(\Sigma_g)$, we get the monodromy $\mu'(\gamma) = \phi^{-1} \circ \mu(\gamma) \circ \phi$, which is ϕ-conjugate to the previous one.

Therefore, for Lefschetz fibrations without marking the monodromy is defined up to conjugation.

Let us give an example of $\mathcal{LF}s$ obtained by blowing up the pencil of cubics in $\mathbb{C}P^2$.

Example 2.1.5. Take two generic cubics C_1, C_2 defined by degree three polynomials Q_1, Q_2. Let $\{p_1, \ldots, p_9\}$ denote the intersection points of C_1 and C_2.

The pencil $t_0C_1+t_1C_2$, $[t_0 : t_1] \in \mathbb{C}P^1$, defines a projection $\pi : \mathbb{C}P^2 \backslash \{p_1, ..., p_9\} \to \mathbb{C}P^1$ where $\pi^{-1}([t_0 : t_1])$ is the cubic $t_0Q_1+t_1Q_2 = 0$. By blowing up $\mathbb{C}P^2$ at $p_1, .., p_9$ we obtain a Lefschetz fibration $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ whose nonsingular fibers are smooth cubics, which are topologically closed genus-1 surfaces, while singular fibers are nodal cubics. We will denote the manifold $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ considered with such a Lefschetz fibration by $E(1)$. The Lefschetz fibration $E(1)$ that we obtain does not depend, up to isomorphism, on the choice of C_1, C_2, due to the fact that the space of generic pencils of cubics in $\mathbb{C}P^2$ is connected (cf. [KRV]).

We have $\chi(\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}) = 12$ and $\chi(\Sigma_1) = 0$ while $\chi(\text{Nodal } \Sigma_1) = 1$. Therefore, applying to $E(1)$ the additivity and multiplicativity of the Euler characteristic, we find that $E(1)$ has 12 singular fibers.

Notice that $E(1)$ is also unique, up to isomorphism, as a marked Lefschetz fibration.

Definition 2.1.6. Let us take two marked genus-g Lefschetz fibrations, $(\pi : X \to B, b, \rho)$ and $(\pi' : X' \to B', b', \rho')$, such that $\partial B = \partial B' = \emptyset$. We consider small neighborhoods of the fibers F and F' over b and b', respectively, and identify them both with $\Sigma_g \times D^2$. The *fiber sum*, $X\#_\Sigma X' \to B\#B'$, is the Lefschetz fibration obtained by gluing $X \setminus (\Sigma_g \times D^2)$ and $X' \setminus (\Sigma_g \times D^2)$ along their boundaries by a map $\Phi : \partial(\Sigma_g \times D^2) \to \partial(\Sigma_g \times D^2)$ given by $\Phi = (id, conj)$ where *conj* stands for the usual complex conjugation.

In order to define a fiber sum for $\mathcal{LF}s$ without marking, one can pick a diffeomorphism ϕ between two arbitrary chosen regular fibers F and F' of $\pi : X \to B$ and $\pi' : X' \to B'$ respectively, then we will employ $\Phi = (\phi, conj)$, and will proceed in the same manner as we have done in the definition above. Note that the diffeomorphism type of the 4-manifold $X\#_\Sigma X'$ and the fibration depend, in general, on the choice of the diffeomorphism $\phi : F \to F'$. We denote the fiber sum as $X\#_{\Sigma,\phi}X'$ when the gluing diffeomorphism ϕ is not the identity.

Let us take a fiber sum of $E(1)$, n times with itself. The fibration we obtain, $E(n) = \#nE(1)$, has got $12n$ singular fibers. It follows from the theorem of

B. Moishezon and R. Livne [Mo] that elliptic Lefschetz fibrations over S^2 are classified by their number of singular fibers, which is a multiple of 12. As a consequence, $E(n)$ is well defined up to isomorphism and each elliptic $\mathcal{LF}s$ over S^2 is isomorphic to $E(n)$ for suitable n.

Definition 2.1.7. The notion of Lefschetz fibration can be slightly generalized to cover the case of fibers with boundary. Then X turns into a manifold with corners and its boundary, ∂X, becomes naturally divided into two parts: the *vertical boundary* $\partial^v X$ which is the inverse image $\pi^{-1}(\partial B)$, and the *horizontal boundary* $\partial^h X$ which is formed by the boundaries of the fibers. We call such fibrations *Lefschetz fibrations with boundary.*

2.2 Real Lefschetz fibrations

Definition 2.2.1. A *real structure* on a smooth 4-manifold X is an orientation preserving involution $c_X : X \to X$, $c_X^2 = id$, such that the set of fixed points, $Fix(c_X)$, of c_X is empty or of the middle dimension.

Two real structures, c_X and c'_X, are said to be *equivalent* if there exists an orientation preserving diffeomorphism $\psi : X \to X$ such that $\psi \circ c_X = c'_X \circ \psi$. A real structure, c_B, on a smooth 2-manifold B is an orientation reversing involution $B \to B$. Such structures are similarly considered up to conjugation by orientation preserving diffeomorphisms of B.

The above definition mimics the properties of the standard complex conjugation on complex manifolds. In fact, around a fixed point, every real structure defined as above, behaves like the complex conjugation.

We will call a manifold together with a real structure a *real manifold* and the set $Fix(c)$ the *real part* of c.

It is well known that for given g there is a finite number of equivalence classes of *real genus-g surfaces* (Σ_g, c), which can be distinguished by their *types* and the number of real components. Namely, one distinguishes two types of real structures: separating and nonseparating. A real structure is called *separating* if the complement of its real part has two connected components, otherwise we call it *nonseparating* (in fact, in the first case the quotient surface Σ_g/c is orientable, while in the second case it is not). The number of real components of a real structure (note that the real part forms the boundary of Σ_g/c), can be at most $g + 1$. This estimate is known as *Harnack inequality* [KRV]. By looking at the possible number of connected components of the real part, one can see that on Σ_g there are $1+[\frac{g}{2}]$ separating real structures and $g+1$ nonseparating ones. Let us also note that, in the case of genus 1, the number of real components, which can be 0, 1, or 2, is enough to distinguish the real structures.

Definition 2.2.2. A real structure on a Lefschetz fibration $\pi : X \to B$ is a pair of real structures (c_X, c_B) such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{c_X} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{c_B} & B. \end{array}$$

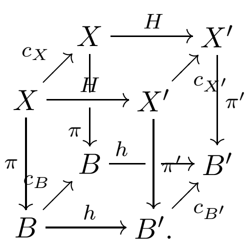
A Lefschetz fibration equipped with a real structure is called a *real Lefschetz fibration*, and is referred as $\mathcal{RLF}$.

When the fiber genus is 1, we call it *real elliptic Lefschetz fibration*, or abbreviated $\mathcal{RELF}$.

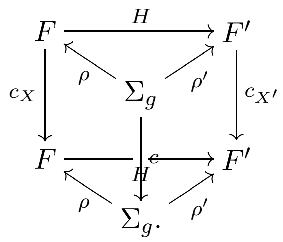
Definition 2.2.3. An $\mathbb{R}$-*marked* $\mathcal{RLF}$ is a triple (π, b, ρ) consisting of a real Lefschetz fibration $\pi : X \to B$, a real regular value b and a diffeomorphism $\rho : \Sigma_g \to F_b$ such that $c_X \circ \rho = \rho \circ c$ where $c : \Sigma_g \to \Sigma_g$ is a real structure. Let us note that if $\partial B \neq \emptyset$ then b will be chosen in ∂B.

A $\mathbb{C}$-*marked* $\mathcal{RLF}$ is a triple $(\pi, \{b, \bar{b}\}, \{\rho, c_X \circ \rho\})$ including an $\mathcal{RLF}$, $\pi : X \to B$, a pair of complex conjugate regular values $b, \bar{b}$, and a pair of diffeomorphisms $\rho : \Sigma_g \to F_b$, $\bar{\rho} = c_X \circ \rho : \bar{\Sigma}_g \to \bar{F}_{\bar{b}}$ where F_b, $\bar{F}_{\bar{b}} = c_X(F_b)$ are the fibers over b and $\bar{b}$, respectively. As in the case of $\mathbb{R}$-marking, if $\partial B \neq \emptyset$ then we choose b in ∂B. Later on, when precision is not needed we will denote $F_b, \bar{F}_{\bar{b}}$ by $F, \bar{F}$, respectively.

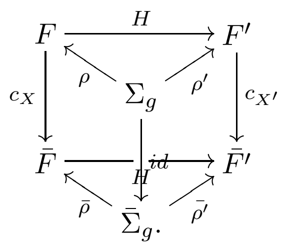
Two real Lefschetz fibrations, $\pi : X \to B$ and $\pi' : X' \to B'$ are said to be *isomorphic* if there exist orientation preserving diffeomorphisms $H : X \to X'$ and $h : B \to B'$, such that the following diagram is commutative



Two $\mathbb{R}$-marked $\mathcal{RLF}s$, are called isomorphic if they are isomorphic as $\mathcal{RLF}s$, $h(b) = b'$, and the following diagram is commutative



Two $\mathbb{C}$-marked $\mathcal{RLF}s$ are called isomorphic if they are isomorphic as $\mathcal{RLF}s$ and the following diagram is well defined and commutative



Definition 2.2.4. A real Lefschetz fibration $\pi : X \to B$ is called *directed* if the real part of (B, c_B) is oriented.

For example, if c_B is separating then we consider an orientation on the real part inherited from one of the halves $B \setminus Fix(c_B)$.

Two directed $\mathcal{RLF}s$ are isomorphic if they are isomorphic as $\mathcal{RLF}s$ with the additional condition that the diffeomorphism $h : B \to B$ preserves the chosen orientation on the real part.

Example 2.2.5. The construction given in Example 2.1.5 can be made equivariantly to obtain an $\mathcal{RLF}$. Namely, we pick out two generic real cubics C_1, C_2 in $(\mathbb{C}P^2, conj)$ given by real degree three polynomials Q_1, Q_2 and consider, following Example 2.1.5, the associated elliptic Lefschetz fibration $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$. The set of 9 blown up points and the fibration are clearly $conj$-invariant. In this way we obtain a real $E(1)$. Note that unlike in the complex case the real fibration does depend on the choice of real cubics C_1, C_2 already since any even number of the 9 blown up points can happen to be imaginary.

The fiber sum of two directed $\mathbb{R}$-marked $\mathcal{RLF}s$ is defined as the fiber sum of two marked $\mathcal{LF}s$. Notice that by definition the gluing diffeomorphism is equivariant once D^2 is chosen equivariant. Evidently, the ultimate $\mathcal{RLF}$ is directed.

For $\mathcal{RLF}s$ without marking, one can start from choosing equivariantly diffeomorphic regular real fibers and then follow the construction with markings.

Remark 2.2.6. The construction of Example 2.2.5 can be applied to pencils of curves of arbitrary degree d. In this way, we obtain $\mathcal{RLF}s$ over $\mathbb{C}P^1 \cong S^2$ with regular fibers diffeomorphic to a genus $g = \frac{(d-1)(d-2)}{2}$ surface.

Definition 2.2.7. Let $\pi : X \to B$ be an $\mathcal{LF}$. We define the *conjugate* $\mathcal{LF}$ as the fibration $\bar{\pi} : X \to \bar{B}$ which coincides with π as a map and differs from the initial $\mathcal{LF}$ only by changing the orientation of the base and the fibers.

To introduce a conjugate of a marked $\mathcal{LF}$, we preselect an orientation reversing diffeomorphism $j : \Sigma_g \to \Sigma_g$ and define the conjugate marked $\mathcal{LF}$ as $(\bar{\pi}, b, \rho \circ j)$.

Remark 2.2.8. It is obvious that two conjugate Lefschetz fibrations have the same set of critical points and critical values. Indeed, let $\psi : U \to \mathbb{C}^2$ and $\phi : V \to \mathbb{C}$ be the local charts of an $\mathcal{LF}$ such that $\phi \circ \pi \circ \psi^{-1}$ is $(z_1, z_2) \to {z_1}^2 + {z_2}^2$. Then local charts of the conjugate $\mathcal{LF}$ can be chosen as $conj \circ \psi : U \to \bar{\mathbb{C}}^2$ and $conj \circ \phi : V \to \bar{\mathbb{C}}$ with $(\bar{z}_1, \bar{z}_2) \to \bar{z}_1^2 + \bar{z}_2^2$.

Definition 2.2.9. An $\mathcal{LF}$ is called *weakly real* if it is equivalent to its conjugate, or in other words if there exist an orientation reversing diffeomorphism, h, of B and an orientation preserving diffeomorphism, H, of X such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{H} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{h} & \bar{B}. \end{array}$$

In particular, every $\mathcal{RLF}$ is weakly real. At this point, one can naturally doubt if the converse is true or not. In case of $g = 1$, a partial answer will be given in Section 3.7.

CHAPTER 3

FACTORIZATION OF THE MONODROMY OF REAL LEFSCHETZ FIBRATIONS

3.1 Fundamental factorization theorem for real Lefschetz fibrations

We will discuss below decomposability of the monodromy of real Lefschetz fibrations over a 2-disc into a product of two involutions, presenting the real structures of the two real fibers. This is a well-known fundamental fact, which we generalize to weakly real Lefschetz fibrations in Theorem 3.1.2. The restriction of a Lefschetz fibration to the boundary of the 2-disc is a usual fibration over a circle, and it will be convenient to extend the terminology from the previous chapter to such fibrations.

More precisely, let $\pi : Y \to S^1$ be a fibration whose fiber is a compact connected oriented smooth 2-manifold F. Shortly, such π will be called an *F-fibration.* In particular, when the genus of F is equal to 1, we call π an *elliptic F-fibration*

Definition 3.1.1. An F-fibration $\pi : Y \to S^1$ is called *weakly real* if there is an orientation preserving diffeomorphism $H : Y \to Y$ which sends fibers into fibers reversing their orientations. If $H^2 = id$, then H will be called a *real structure* on the F-fibration $Y \to S^1$. An F-fibration equipped with a real structure will be called *real.*

Note that H induces an orientation reversing diffeomorphism $h_{S^1} : S^1 \to S^1$ such that the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{H} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S^1 & \xrightarrow{h_{S^1}} & S^1. \end{array}$$

It is not difficult to see that the set of orientation reversing involutions form a single conjugacy class in the diffeomorphism group of S^1 (the crucial observation is that any such involution has precisely two fixed points). So, any real F-fibration is equivariantly isomorphic to an F-fibration whose involution h_{S^1} is standard. Let it be the complex conjugation $c_{S^1} : S^1 \to S^1$, $z \mapsto \bar{z}$, $z \in S^1 \subset \mathbb{C}$.

In the case of a weakly real F-fibration, h_{S^1} may be not an involution, however, it also has precisely two fixed points and can be changed into an involution by an isotopy. It is not difficult to see that this isotopy can be lifted to an isotopy of H. Thus, by modification of H we can always make h_{S^1} an involution. So, it is not restrictive for us to suppose always that $h_{S^1} = c_{S^1}$ both for real and weakly real F-fibrations.

The restrictions of H to the invariant fibers $F_\pm = \pi^{-1}(\pm 1)$ will be denoted $h_\pm : F_\pm \to F_\pm$. In the case of real F-fibrations, we will prefer to use notation c_Y for the involution H, and $c_\pm$ for the involutions $h_\pm$.

It is well known that any F-fibration $\pi : Y \to S^1$ is isomorphic to the projection $M_f \to S^1$ of a mapping torus $M_f = F \times I \big/ _{(f(x),0)\sim(x,1)}$ of some diffeomorphism $f : F \to F$. More precisely, if we fix a particular fiber $F = F_b = \pi^1(b)$, $b \in S^1$, then an isomorphism $\phi : M_f \to Y$ can be chosen so that $F \times 0$ and $F \times 1$ are identified with the fiber F_b, so that $x \times 0 \mapsto x$ and $x \times 1 \mapsto f(x)$.

An F-fibration π determines a diffeomorphism f up to isotopy and thus provides a well-defined element in the mapping class group $[f] \in Map(F)$ called the *monodromy of* π (relative to the fiber $F = F_b$). A map f representing the class $[f]$ will be also often called *monodromy*, or more precisely, a *monodromy map.*

In some cases, we fix a marking $\rho : \Sigma_g \to F_b$. Then the diffeomorphism $\rho^{-1} \circ f \circ \rho : \Sigma_g \to \Sigma_g$ (the pull-back of f) as well as its isotopy class $[\rho^{-1} \circ f \circ \rho] \in Map(\Sigma_g)$ will be called the *monodromy of* π *relative to the marking* ρ.

In what follows, we choose the point b in the upper semi-circle, S^1_+. The restriction $Y_+ = \pi^{-1}(S^1_+) \to S^1_+$ of π admits a trivialization $\phi_+ : Y_+ \to F \times S^1_+$ which is identical on the fiber $F = F_b$. This allows us to consider the pull-back of $c_\pm$ via ϕ, namely, the two involutions $x \mapsto \phi_+(c_\pm(\phi_+^{-1}(x \times \pm 1)))$ on the same fiber F. We will preserve notation $c_\pm$ for these involutions.

Theorem 3.1.2. *Let $\pi : Y \to S^1$ be a weakly real F-fibration with a distinguished fiber $F = F_b$, $b \in S^1_+$.*

Then the two product diffeomorphisms of the fiber F, $(h_+)^{-1} \circ h_-$, and $h_+ \circ (h_-)^{-1}$ are isotopic and describe the monodromy of π relative to the fiber F. In particular, if π is a real F-fibration, then the monodromy can be factorized as $f = c_+ \circ c_-$.

Proof. Consider a trivialization $Y_- \to F \times S^1_-$ of the restriction $Y_- = \pi^{-1}(S^1_-) \to S^1_-$ of π over the lower semi-circle, S^1_-, which is the composition of $\phi_+ \circ H : Y_- \to S^1_+ \times F$, with the map $F \times S^1_+ \to F \times S^1_-$, $(x, z) \mapsto (x, c_{S^1}(z))$.

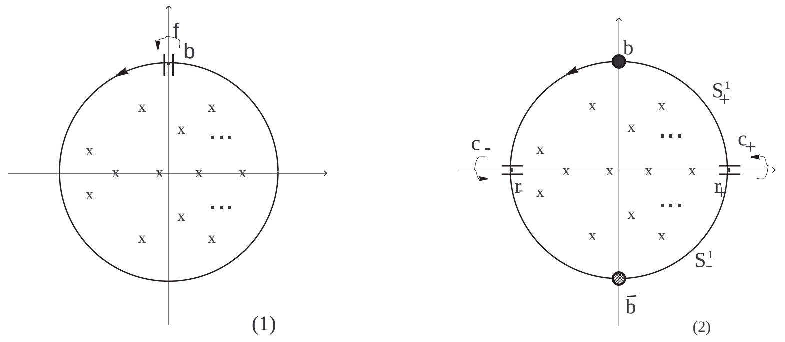


Fig. 3.1.

If S^1 is split into several arcs and a fibration over S^1 is glued from trivial fibrations over these arc, then the monodromy is clearly the product of the gluing maps of the fibers over the common points of the arcs, ordered in the counter-clockwise direction beginning from a marked point $b \in S^1$. In our case, the arcs are S^1_+, S^1_-, their common points follow in the order -1, $+1$, and the corresponding gluing maps, are h_-^{-1} and h_+. This gives monodromy $h_+ \circ (h_-)^{-1}$. If we consider another trivialization $Y_- \to F \times S^1_-$ replacing in its definition H by H^{-1}, then the gluing maps will be h_- and h_+^{-1}, and the monodromy is factorized as $(h_+)^{-1} \circ h_-$. □

Remark 3.1.3. It follows from Theorem 3.1.2 that the diffeomorphisms $h^{-1} \circ f \circ h$ as well as $h \circ f \circ h^{-1}$, where h stands either for h_+, or for h_-, are all isotopic to the inverse f^{-1} of the monodromy f of a weakly real F-fibration π (note that f^{-1} is the monodromy map of the *conjugate F-fibration*). In particular, if π is a real F-fibration, then $f^{-1} = c_+ \circ f \circ c_+ = c_- \circ f \circ c_-$.

Corollary 3.1.4. *Consider a weakly real F-fibration $\pi : Y \to S^1$, fix a trivialization of $\pi_+ : Y_+ \to S^1_+$, and consider the associated diffeomorphisms $h_\pm : F \to F$. Let h stands for any of the four maps $h_\pm$, $h_\pm^{-1}$. Then there exists a diffeomorphism $f : F \to F$ representing the monodromy class $[f] \in Map(F)$ of π, such that $f^{-1} = h \circ f \circ h^{-1}$.*

In particular, if F-fibration π is real, then one can choose a monodromy map f such that $f^{-1} = c \circ f \circ c$.

Definition 3.1.5. A diffeomorphism $f : F \to F$ as well as its isotopy class $[f] \in Map(F)$ will be called *real* (*weakly real*) if it is a monodromy of a real (weakly real, respectively) F-fibration.

Proposition 3.1.6. *An F-fibration is real (weakly real) if and only if its monodromy f is real (weakly real).*

Proof. We give the proof for real F-fibrations; the proof for weakly real ones is analogous. Necessity of the condition in the Proposition is trivial. For proving the converse, let $\pi : Y \to S^1$ be an F-fibration with the monodromy class $[f] \in Map(F)$, and f its representative such that $f^{-1} = c \circ f \circ c$, where c is some real structure on F. Presenting Y as $F \times I \diagup_{(f(x),0)\sim(x,1)}$, we obtain a well-defined involution $c_Y : Y \to Y$ induced from the involution $(x,t) \mapsto (c(x), 1-t)$ in $F \times I$. It preserves the fibration structure and acts as c and $f \circ c$ on the real fibers $F \times \frac{1}{2}$ and $F \times 0 = F \times 1$ respectively. □

3.2 Homology monodromy factorization of elliptic F-fibrations

We will characterize all *real elliptic F-fibrations* by answering the question: which elements in $Map(F)$ are real in the case of torus, $F = T$?

It is well known that $Map(T) = SL(2,\mathbb{Z})$, due to the fact that every diffeomorphism $f : T \to T$ is isotopic to a *linear diffeomorphism.* The latter diffeomorphisms by definition are induced on $T = \mathbb{R}^2/\mathbb{Z}^2$ by a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by a matrix $A \in SL(2,\mathbb{Z})$. Note that we can naturally identify $T = H_1(T,\mathbb{R})/H_1(T,\mathbb{Z})$, and interpret matrix A as the induced automorphism f_* in $H_1(T,\mathbb{Z})$. The latter automorphism is called the *homology monodromy.* Since isotopic diffeomorphisms have the same homology monodromy in $H_1(T,\mathbb{Z})$, we obtain well defined homomorphisms

$Map(T) \to Aut_+(H_1(T,\mathbb{Z})) \to SL(2,\mathbb{Z})$ which are in fact isomorphisms (here Aut_+ stand for the orientation preserving automorphisms).

Let a denote the simple closed curve on T represented by the equivalence class of the horizontal interval $I \times 0 \subset \mathbb{R}^2$, and b is similarly represented by the vertical interval $0 \times I$. We have $a \circ b = 1$ hence, the homology classes represented by these curves are integral generators of $H_1(T,\mathbb{Z})$. The mapping class group of T is generated by the Dehn twists t_a and t_b, which can be characterized by their homology monodromy homomorphism matrices $t_{a*} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and $t_{b*} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$.

Therefore, for elliptic Lefschetz fibrations, the question of characterization of real monodromy classes $[f] \in Map(T)$ can be interpreted as the question on the decomposability of their homology monodromy $f_* \in SL(2,\mathbb{Z})$ into a product of two *linear real structures.* The latter structures by definition are linear orientation reversing maps of order 2 defined by integral (2×2)-matrices. Such decomposability is equivalent to the property that f_* is conjugate to its inverse by a linear real structure. Hence a necessary condition for a matrix A to be real is that both A and A^{-1} lies in the same conjugacy classes in the group $GL(2,\mathbb{Z})$.

Recall that there are three types of real structures on T distinguished by the number of their real components: 0, 1, or 2. We will say that a real structure on T is *even* if it has 0 or 2 components, and *odd* if it has 1 component. Note that the automorphisms of $H_1(T,\mathbb{Z})$ induced by even real structures are diagonalizable over $\mathbb{Z}$, namely, their matrices are conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $GL(2,\mathbb{Z})$. So, we cannot determine if the number of components 0 or 2 knowing only the matrix representing the homology action of the real structure. The homology action of an odd real structure is presented by a matrix conjugate to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3.3 The modular action on the hyperbolic half-plane

Let $\mathbb{C}^2$ be considered as the vector space of 2×1 matrices over $\mathbb{C}$. Then a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $GL(2,\mathbb{Z})$ acts on $\mathbb{C}^2$ from the left as matrix multiplication.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 & bz_2 \\ cz_1 & dz_2 \end{pmatrix}$$

This action can be extended to $\mathbb{C}P^1 = \mathbb{C}^2 \setminus \{(0,0)\} \diagup_{(z_1,z_2)\sim(\lambda z_1,\lambda z_2)}$ since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda z_1 \\ \lambda z_2 \end{pmatrix} = \begin{pmatrix} a\lambda z_1 & \lambda b z_2 \\ c\lambda z_1 & \lambda d z_2 \end{pmatrix} = \lambda \begin{pmatrix} az_1 & bz_2 \\ cz_1 & dz_2 \end{pmatrix}.$$

Let us identify $\mathbb{C}P^1 \cong \{(z_1, z_2) \in \mathbb{C}^2, z_2 \neq 0\} \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\}$ and rewrite the action of $GL(2, \mathbb{Z})$. We obtain a linear fractional transformation $z \to \frac{az+b}{cz+d}$ where $z = \frac{z_1}{z_2}$. In particular, if $A \in SL(2, \mathbb{Z})$, then the transformation preserves the orientation of $\mathbb{C}$ and takes $\mathbb{R} \cup \{\infty\}$ to itself preserving its orientation. Hence, it gives rise to a diffeomorphism of the upper half plane $\mathbb{H}$ which can be seen as a model for the hyperbolic plane where the geodesics are the semi-circles centered at a real point or vertical half-lines which can also be considered as arcs of infinite radius. By identifying the upper half plane with lower half plane by complex conjugation, one extends the action of $SL(2, \mathbb{Z})$ to an action of $GL(2, \mathbb{Z})$. The standard fundamental domain of the action is the set $\{z | \, |Re(z)| \leq \frac{1}{2}, |z| \geq 1\}$ which is shown in the Figure below.

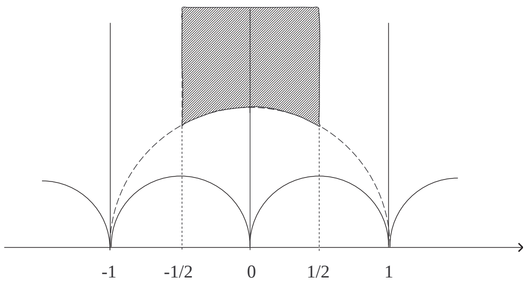


Fig. 3.2. The upper half plane model of hyperbolic space, and the standard fundamental domain of $SL(2, \mathbb{Z})$.

3.4 The Farey Tessellation

Let us identify the upper half plane model with the Poincaré disk model $\mathbb{D}$. We will consider the disk $\mathbb{D}$ together with its boundary $\mathbb{R} \cup \infty$ and define a tessellation on $\mathbb{D}$ as follows:

Set ∞ as $\frac{1}{0}$ and consider the two fractions $\frac{0}{1}$ and $\frac{1}{0}$, spot them on $\mathbb{D}$ as the south and the north poles respectively and connect them with a line which will be the vertical diameter. Consider their mediant $\frac{0+1}{1+0} = \frac{1}{1}$ and connect each of them with a geodesic to the mediant. Apply the same to the fractions $\{\frac{0}{1}, \frac{1}{1}\}$ and $\{\frac{1}{1}, \frac{1}{0}\}$. Iterating this process one obtains a tessellation of the right semi-disk. By taking the symmetry one extends the tessellation to $\mathbb{D}$. (See Figure 3.3).

In the literature this tessellation is called the *Farey tessellation*. Let us denote the disk together with the Farey tessellation by $\mathbb{D}_F$. Note that Farey tessellation is

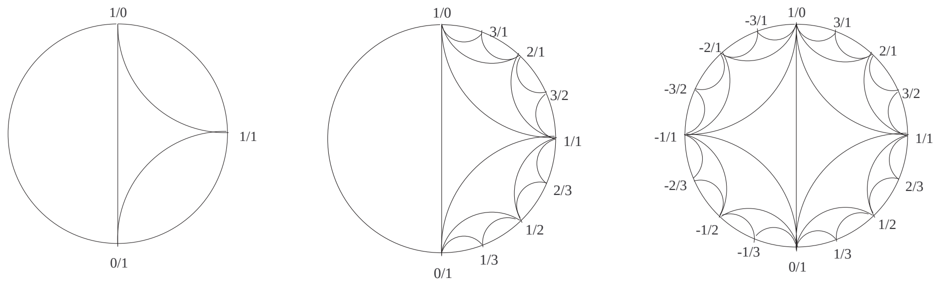


Fig. 3.3. Tessellation of $\mathbb{D}$.

a tessellation of $\mathbb{D}$ by ideal triangles (i.e. triangles with vertices on the boundary $\mathbb{D}_F$). In fact, the set of vertices of the triangles is exactly $\mathbb{Q} \cup \{\infty\}$. Moreover, two fractions $\frac{m_1}{n_1}, \frac{m_2}{n_2}$ are connected by a line iff $m_1 n_2 - m_2 n_1 = \pm 1$. Hence the action of $GL(2,\mathbb{Z})$ on $\mathbb{D}$ induces an action on $\mathbb{D}_F$ which is transitive on the geodesics of $D_{\mathbb{F}}$. Only $\pm I$ acts as the identity hence the modular group $PGL(2,\mathbb{Z}) = GL(2,\mathbb{Z})/\pm I$ is the symmetry group of $\mathbb{D}_F$ where the subgroup $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$ gives the orientation preserving symmetries. In what follows we denote by Γ the triangle with vertices $\{0, 1, \infty\}$. Note that Γ splits in 3 copies of a fundamental region.

3.5 Elliptic and parabolic matrices

The fixed points of the modular action of a matrix $A \in PSL(2,\mathbb{Z})$, $A \neq I$, in $\mathbb{D}_F$ are solutions of $z = \frac{az+b}{cz+d}$. This gives a quadratic equation $cz^2 + (d-a)z - b = 0$ with the discriminant $(d-a)^2 + 4bc = (d-a)^2 + 4(ad-1) = (a+d)^2 - 4$, and we have 3 cases. If the trace $|tr(A)| < 2$ then the discriminant is negative and the modular action is a rotation around an imaginary point (an interior point of $\mathbb{D}_F$). Such matrices are called *elliptic*. If $|tr(A)| = 2$, then the discriminant vanishes, and A acts as a translation with one fixed rational point, $\frac{d-a}{2}$ (on the boundary of $\mathbb{D}_F$). Such matrices are called *parabolic*. The *hyperbolic matrices* have $|tr(A)| > 2$ and define a translation of $\mathbb{D}_F$ with two fixed quadratically irrational real points (on the boundary of $\mathbb{D}_F$).

Elliptic Matrices: As mentioned above an elliptic matrix, $A \in PSL(2,\mathbb{Z})$ act on $\mathbb{D}_F$ as rotation around a point in the interior of $\mathbb{D}_F$. The center of the rotation belongs to one of the triangles of the tessellation. Without loss of generality let us assume that the fixed point belongs to the triangle Γ. If the fixed point belongs to an edge of Γ, then A rotates Γ by an angle π. The other possibility is rotation by angle $\pm\frac{2\pi}{3}$ around the center of Γ. Note that the pair of rotations by angles $\pm\frac{2\pi}{3}$ are conjugate to each other via an orientation reversing matrix from $PGL(2,\mathbb{Z})$.

Since $PGL(2,\mathbb{Z})$ acts transitively on the triangles of the tessellation rotation by π around the center of an edge of Γ and rotation by $\frac{2\pi}{3}$ around the center of Γ defines the conjugacy classes in $PGL(2,\mathbb{Z})$ of elliptic matrices of $PSL(2,\mathbb{Z})$.

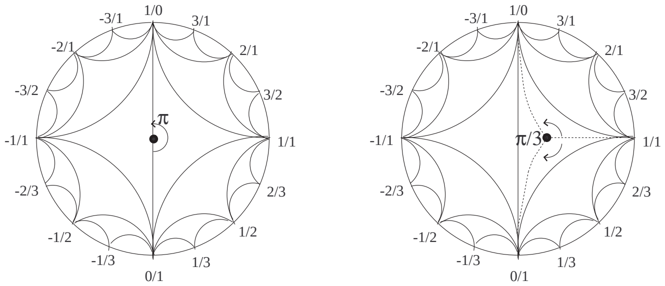


Fig. 3.4. Modular actions of elliptic matrices, $E_\pi, E_{\frac{2\pi}{3}}$.

With respect to the triangle Γ, we can consider following matrices representing these two conjugacy classes. $E_\pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $E_{\frac{2\pi}{3}} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

Each matrix A in $PSL(2,\mathbb{Z})$ defines two matrices $\pm A$ in $SL(2,\mathbb{Z})$. It is not hard to see that the matrices $\pm E_\pi$ are conjugate to each other via reflection with respect to the edge containing the fixed point while $\pm E_{\frac{2\pi}{3}}$ are not, simply by the fact that they have different traces. Hence, there are three conjugacy classes in $GL(2,\mathbb{Z})$, $E_\pi, \pm E_{\frac{2\pi}{3}}$, of matrices in $SL(2,\mathbb{Z})$ where $E_{\frac{2\pi}{3}}$ gives the clockwise rotation while $-E_{\frac{2\pi}{3}}$ is conjugate to the clockwise rotation of $\mathbb{D}_F$ with respect to the center of the triangle Γ.

Parabolic Matrices: The fixed point of the action of a parabolic matrix in $PSL(2,\mathbb{Z})$ is rational, thus it is a common vertex of an infinite set of triangles of $\mathbb{D}_F$. Since $PGL(2,\mathbb{Z})$ acts transitively on the rational points, it is not restrictive to assume that the fixed point of the translation is 0.

Hence, a parabolic element can shift the triangle Γ by arbitrary number n triangles to the right or to the left(Figure 3.5) fixing 0. The left shift is conjugated to the right shift by the reflection with respect to the vertical line. Hence the equivalence classes in $PGL(2,\mathbb{Z})$ are determined by the number n of shifts. Such a shift can be represented by the matrix $P_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, $n \in \mathbb{N}$.

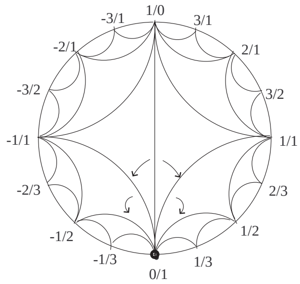


Fig. 3.5. Modular actions of parabolic matrices P_n.

The matrix $P_n \in PSL(2,\mathbb{Z})$ corresponds to matrices $\pm P_n \in SL(2,\mathbb{Z})$. Note that $\pm P_n$ can not be in the same conjugacy class since they have different traces. Thus the conjugacy classes in $GL(2,\mathbb{Z})$ of parabolic matrices in $SL(2,\mathbb{Z})$ are determined by the integer $\pm n$. A representative of conjugacy classes can be chosen as $\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, $n \in \mathbb{N}$.

3.6 Hyperbolic matrices

A hyperbolic matrix $A \in PSL(2,\mathbb{Z})$ acts on $\mathbb{D}_F$ as translation fixing two irrational points. The geodesic (a semicircle), l_A, connecting these fixed points, oriented in the direction of translation, remains invariant under the translation, so A preserves also the set of the triangles of $\mathbb{D}_F$ which are cut by l_A.

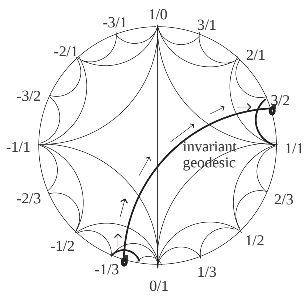


Fig. 3.6. Modular action of a hyperbolic matrix.

With respect to the orientation of l_A, such triangles are situated in two different ways: a set of triangles with a common vertex lying on the left of l_A followed by a set of triangles with common vertex lying on the right of l_A, see Figure 3.7.

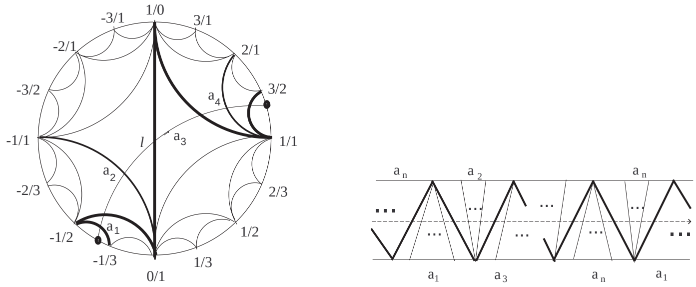


Fig. 3.7. Periodic pattern of the truncated triangles of the Farey tessellation.

Let us label right and left triangles by R and L, respectively. Then we encode the arrangement of left and right triangles with respect to l_A as an infinite word, $\ldots LL\ldots LRR\ldots RLL\ldots L\ldots$, of 2 letters. This word is called the *cutting word* of l_A. Let us fix a point p at the intersection of l_A with an edge separating two types of triangles. Relative to this point, we obtain a sequence, $(a_1, a_2, a_3, \ldots)_p$, from the cutting word where a_{2i-1} stands for the number of consecutive triangles of one type while a_{2i}, $i = 1, 2, \ldots$ is the number of consecutive triangles of the other type. For example, if the cutting word with respect to p reduced to the word $\underbrace{LL\ldots L}_{a_1}\underbrace{RR\ldots R}_{a_2}\underbrace{LL\ldots L}_{a_3}\ldots = L^{a_1}R^{a_2}L^{a_3}\ldots$, then we obtain $(a_1, a_2, \ldots)_p$. This sequence is called the *cutting sequence* relative to the point p.

Left and right triangles form a periodic pattern and the action of A is a shift by the period, so the cutting sequence has a period of even length. Note that choice of the point p is not canonical, hence we can encode the period only as a cycle, $[a_1 a_2 \ldots a_{2n-1} a_{2n}]_A$, which we call the *cutting period-cycle associated to the matrix* A.

Because of the fact that $PGL(2,\mathbb{Z})$ is the full symmetry group of $\mathbb{D}_F$, the cutting period-cycle of a hyperbolic matrix $A \in PSL(2,\mathbb{Z})$ gives the complete invariant of the conjugacy class in $PGL(2,\mathbb{Z})$ of A. In other words, two matrices $A, B \in PSL(2,\mathbb{Z})$ are in the same conjugacy class in $PGL(2,\mathbb{Z})$ if and only if $[a_1 a_2 \ldots a_{2n}]_A = [a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(2n)}]_B$ for a cyclic permutation σ. Hence we will denote the conjugacy classes in $PGL(2,\mathbb{Z})$ of hyperbolic matrices of $PSL(2,\mathbb{Z})$ by the cycle $[a_1 a_2 \ldots a_{2n}]$ (defined up to cyclic ordering).

It can be seen geometrically that with respect to the triangle Γ a matrix representing a translation corresponding to the cutting period-cycle $[a_1, a_2, \ldots, a_n]$ can be chosen as the following product of parabolic matrices.

$$\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_{2n-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{2n} & 1 \end{pmatrix}.$$

For the sake of simplicity, let us denote $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then the above product is written as $U^{a_1} V^{a_2} \ldots V^{a_{2n}}$. Note that U is conjugate to V in $PGL(2, \mathbb{Z})$ but not in $PSL(2, \mathbb{Z})$.

Let us note that in certain cases, namely if l_A intersects the vertical line of $\mathbb{D}_F$, (since the action of $PGL(2, \mathbb{Z})$ is transitive on the geodesics of $\mathbb{D}_F$, up to conjugation this property is always satisfied), the cutting sequence of l_A with respect to the point of intersection of l_A with the vertical line is related to the continued fraction expansion of the fixed point, ξ, which is the "end point" of l_A with respect to the orientation. The corresponding theorem is due to C. Series [S1, S2].

Theorem 3.6.1 ([S1, S2])**.** *Let $x > 1$, and let l be any geodesic ray joining some point p on the vertical line of $\mathbb{D}_F$ to x, oriented from p to x. Suppose that cutting word of l with respect to p is $L^{a_1} R^{a_2} L^{a_3} \ldots$. Then $x = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}$.*

Note that if $0 < x < 1$ then the sequence starts with R and $x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$.

If $x < 0$ everything applies with x replaced by $-x$ and with R and L interchanged.

A matrix $A \in PSL(2, \mathbb{Z})$ corresponds to $\pm A \in SL(2, \mathbb{Z})$. Since $\pm A$ have different traces the cutting period-cycle $[a_1 a_2 \ldots a_{2n}]_A$, together with the sign determine the conjugacy of $\pm A$ in $GL(2, \mathbb{Z})$. A representative of the conjugacy classes of $\pm A$ can be chosen as $\pm U^{a_1} V^{a_2} \ldots V^{a_{2n}}$.

3.7 Real factorization of elliptic and parabolic matrices

Let us first recall that the modular action of linear real structures $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the hyperbolic plane $\mathbb{D}$ is $z \mapsto -\bar{z}$ and $z \mapsto \frac{1}{\bar{z}}$ respectively. Geometrically, these are reflections with respect to the vertical and, respectively, the horizontal lines, see Figure 3.8. In particular, the first reflection takes our basic triangle Γ with vertices $\{0, 1, \infty\}$ to the triangle with vertices $\{0, -1, \infty\}$, and the second one takes Γ to itself.

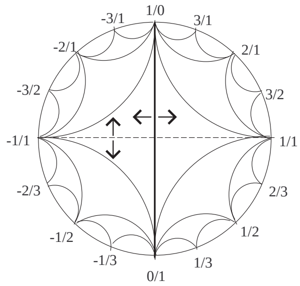


Fig. 3.8. Modular actions of linear real structures.

Theorem 3.7.1. *Every elliptic and parabolic matrices in $SL(2,\mathbb{Z})$ is a product of two linear real structures.*

Proof. The explicit real decomposition for each conjugacy class of elliptic matrices is given below.

$$\begin{aligned} E_{\frac{2\pi}{3}} &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ -E_{\frac{2\pi}{3}} &\cong \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ E_{\pi} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Figure 3.9 illustrates geometrically the above decompositions in terms of the corresponding modular action of the matrices.

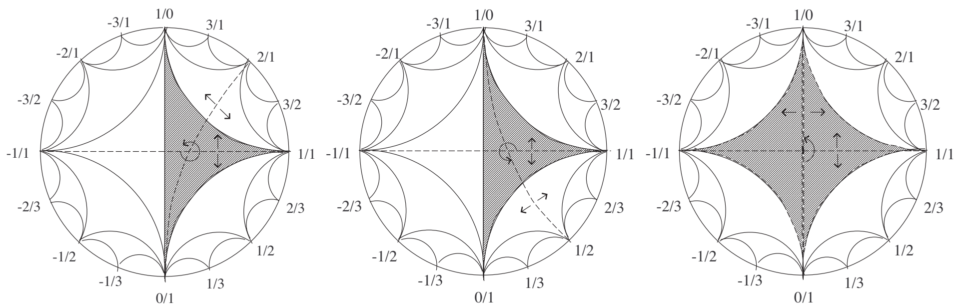


Fig. 3.9. Decompositions of modular actions of elliptic matrices.

A real decomposition for each conjugacy class of parabolic matrices can be given as follows.

$$P_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$-P_n = \begin{pmatrix} -1 & 0 \\ -n & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Example 3.7.2. Figure 3.10 shows the real decomposition of the modular action of matrices $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for $n = 1, 2$.

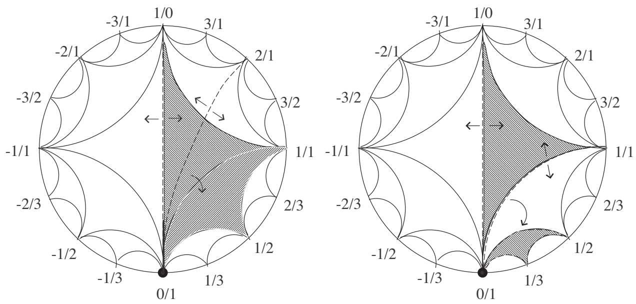


Fig. 3.10. Decompositions of modular actions of parabolic matrices P_1, P_2.

3.8 Criterion of factorizability for hyperbolic matrices

Lemma 3.8.1. *If the cutting period-cycle of a hyperbolic matrix A is $[a_1 a_2 \ldots a_{2n}]_A$, then the cutting period-cycle of A^{-1} is $[a_{2n} a_{2n-1} \ldots a_1]_{A^{-1}}$.*

Proof. Note that $l_A = l_{A^{-1}}$ with opposite orientation. So, the cutting word of A^{-1} can be obtained from the cutting word of A by taking the mirror image of the word and interchanging L with R. Interchanging L and R does not effect the cutting period-cycle, hence the cutting period-cycle of A^{-1} is the reverse $[a_{2n} a_{2n-1} \ldots a_1]_{A^{-1}}$ of the cutting period-cycle $[a_1 a_2 \ldots a_{2n}]$ of A. □

Definition 3.8.2. A finite sequence $(a_1 a_2 \ldots a_k)$ is called *palindromic* if it is equal to the reversed sequence $(a_k a_{k-1} \ldots a_1)$. We call k the *length of the sequence.*

Definition 3.8.3. A cutting period-cycle is called *bipalindromic* if there is a cyclic permutation of it such that the permuted period can be subdivided into two palindromic sequences.

In particular, if the cutting period-cycle is subdivided into two palindromic sequences of odd length (even length) we call it *odd-bipalindromic* (respectively, *even-palindromic*).

For example, if the period [1213] is odd-bipalindromic, while the period [1122] is even-bipalindromic.

If $A^{-1} = Q^{-1}AQ$ for some $Q \in PGL(2, \mathbb{Z})$ then by Lemma 3.8.1 we get that $[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(2n)}] = [a_{2n}, a_{2n-1}, \ldots, a_1]$ for some cyclic permutation σ. This implies that the cutting period-cycle $[a_1 a_2 \ldots a_{2n}]$ is bipalindromic.

Note that when the cutting period-cycle is odd-bipalindromic then the symmetry of palindromic pieces lifts to a symmetry of left/ right triangles corresponding to cutting period-cycle. This is not true for even-bipalindromic periods. For example, for [1213] we have $121 \sim LR^2L = LRRL$ and $3 \sim R^3 = RRR$ while for [1122] we have $11 \sim LR$ and $22 \sim L^2R^2 = LLRR$.

Theorem 3.8.4. *A hyperbolic matrix A is a product of two linear real structures if and only if its cutting period-cycle $[a_1 a_2 \ldots a_{2n}]_A$ is odd-bipalindromic.*

Lemma 3.8.5. *Let $A \in PSL(2, \mathbb{Z})$ such that $A^{-1} = Q^{-1}AQ$ for some $Q \in PGL(2, \mathbb{Z})$ and let l_A be the geodesic invariant under the action of A. Then $Q(l_A) = l_A$.*

Proof. Clearly, if $A(l_A) = l_A$ then $A^{-1}(l_A) = l_A$. Hence,

$$A^{-1}(l_A) = Q^{-1}AQ(l_A) \Leftrightarrow Q(l_A) = A(Q(l_A)).$$

By the uniqueness of the invariant geodesic we get $Q(l_A) = l_A$. $\square$

Lemma 3.8.6. *Let A, Q, l_A as above. If the cutting period-cycle $[a_1 a_2 \ldots a_{2n}]_A$ of A is even-bipalindromic, then Q is orientation preserving.*

Proof. By Lemma 3.8.5 we have $Q(l_A) = l_A$, hence Q preserves triangles meeting l_A. The action of Q on $\mathbb{D}_F$ is a linear fractional transformation, so it preserves the angles. An analysis on the angles at meeting points of l_A and the edges of the triangles will forbid the existence of the orientation reversing map in the case that the cutting period-cycle is even-bipalindromic. Let us assume that the cutting period-cycle has the form

$$[\underbrace{a_1a_2\dots a_ka_k\dots a_2a_1}_{P}\underbrace{a'_1a'_2\dots a'_sa'_s\dots a_2a'_1}_{P'}]$$

where $s+k=n$ and P and P' are two palindromic pieces. Substituting the pieces P, P' to the cutting sequence we obtain a sequence of P and P' of the form $PP'PP'\dots$. Clearly, the action of the matrix A on the sequence we obtain, corresponds to a shift by two: it takes P to P, P' to P'. Let us call edges which separate the triangles corresponding to P from the triangles corresponding to P' as *boundaries*. There are two types of boundaries: if we go in the direction of translation along l_A we encounter boundaries where we pass from P to P' and boundaries where we pass from P' to P. Let us denote such boundaries by e_i and e'_i respectively.

Each triangle of $\mathbb{D}_{\mathbb{F}}$ which is cut by l_A splits into two pieces one of which is a triangle. Let τ_i (τ'_i) be triangles having one edge e_i (e'_i, respectively) and obtained as the union of triangle-pieces of triangles of $\mathbb{D}_F$ with a common vertex of one side of l_A, see Figure 3.11. Let α_i, (α'_i) be the interior angles of τ_i (τ'_i, respectively) between the edges e_i (e'_i, respectively) and l_A. Let β_i and β'_i be the other interior angles of τ_i and τ'_i corresponding to edges on l_A.

Note that since A shift triangles by the period PP', A takes α_i to α_{i+1} (similarly α'_i to α'_{i+1}). Hence all α_i (similarly all α'_i) are equal. Let $\alpha=\alpha_i$ for all i (and $\alpha'=\alpha'_i$ for all i.)

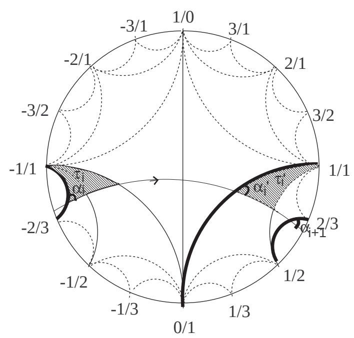


Fig. 3.11.

Moreover, there is an elliptic matrix in the conjugacy classes of E_π which fixes the point of intersection of l_A with the middle edge of P or P'(such edge exists since the pieces have even length). Such matrix interchanges the edges e_i to e'_i. Hence $\alpha = \alpha'$. (In the same way we obtain $\beta = \beta_i = \beta'_i$ for all i.)

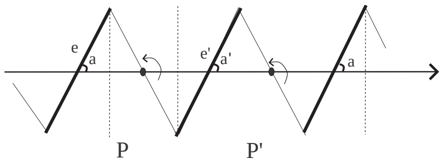


Fig. 3.12.

Let us assume that $\alpha < \frac{\pi}{2}$. (If it is not so, we can replace α with β. Being two interior angles of a triangle, α and β can not be both grater then $\frac{\pi}{2}$.)

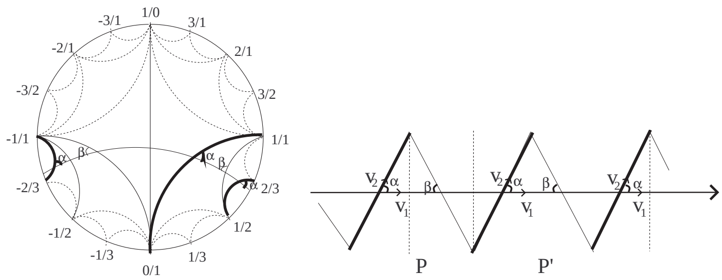


Fig. 3.13.

Let us chose an orientation of $\mathbb{D}_F$ by specifying (v_1, v_2) where v_1 is a tangent vector of l_A and v_2 is the tangent vector of e_i or e'_i such that the angle α between v_1 and v_2 is $\alpha < \frac{\pi}{2}$, see Figure 3.13. The proof follows from the following observation. The matrix Q takes (v_1, v_2) to itself since it preserves l_A and the set of boundaries of P and P' hence the angles between them. However, an orientation reversing map can not preserve the angle both $\alpha < \frac{\pi}{2}$ between the vectors (v_1, v_2) and the vectors at the same time.□

Proof of the Theorem 3.8.4 (⇒) The matrix A is a product of two liner real structures, which implies that the cutting period-cycle is odd-bipalindromic by Lemma 3.8.6.

($\Leftarrow$) If the cutting period-cycle is odd-bipalindromic, then up to cyclic ordering, the period cutting-cycle has two palindromic pieces of odd length. Let us assume that the cutting period-cycle is of the form

$$[a_1a_2\dots a_ka_{k+1}a_k\dots a_2a_1a'_1a'_2\dots a'_sa'_{s+1}a'_s\dots a'_2a'_1]$$

where $(2k+1)+(2s+1)=2n$. Then for some $Q\in PGL(2,\mathbb{Z})$, we have $B=Q^{-1}AQ$ such that $B=U^{a_1}V^{a_2}\dots U^{a_2}V^{a_1}U^{a'_1}V^{a'_2}\dots U^{a'_2}V^{a'_1}$. Matrices U^{a_i} and V^{a_i} have the following real decompositions,

$$U^{a_i}=\begin{pmatrix}1 & -a_i\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\text{and}\,V^{a_i}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ -a_i & -1\end{pmatrix}.$$

Hence the product $U^{a_1}V^{a_2}\dots U^{a_2}V^{a_1}$ can be rewritten in the form

$$\begin{pmatrix}1 & -a_1\\ 0 & -1\end{pmatrix}\cdots\begin{pmatrix}1 & 0\\ -a_k & -1\end{pmatrix}\begin{pmatrix}1 & -a_{k+1}\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ -a_k & -1\end{pmatrix}\cdots\begin{pmatrix}1 & -a_1\\ 0 & -1\end{pmatrix}.$$

This gives a linear real structure, since it is a conjugate of $\begin{pmatrix}1 & -a_{k+1}\\ 0 & -1\end{pmatrix}$. Similarly, the product $U^{a'_1}V^{a'_2}\dots U^{a'_2}V^{a'_1}$ gives a linear real structure conjugate to $\begin{pmatrix}1 & -a_{s+1}\\ 0 & -1\end{pmatrix}$. □

Theorem 3.8.7. *Every elliptic F-fibration is real if and only if it is weakly real.*

Proof. Theorem follows from the following observations

(1) $\pi:Y\to S^1$ is real if and only if the monodromy f is real. i.e. $f^{-1}=c\circ f\circ c$, where c is a real structure. (Proposition 3.1.6).

(2) $\pi:Y\to S^1$ is weakly real if and only if the monodromy f is weakly real. i.e. $f^{-1}=h\circ f\circ h^{-1}$, where h is an orientation reversing diffeomorphism. (Proposition 3.1.6).

(3) $f^{-1}=c\circ f\circ c$ iff $f^{-1}=h\circ f\circ h^{-1}$.

We only need to prove the observation (3).

Obviously f is real $\Rightarrow$ f is weakly real.

For the converse note that, if $f^{-1}=h\circ f\circ h$, where h is orientation reversing, then the cutting period-cycle $[a_1a_2\dots a_{2n-1}a_{2n}]_{f_*}$ of the corresponding homology monodromy f_* is odd-bipalindromic by Lemma 3.8.6. Then by Proposition 3.8.4, we have $f^{-1}=c\circ f\circ c$, for a real structure c. □

CHAPTER 4

REAL LEFSCHETZ FIBRATIONS AROUND SINGULAR FIBERS

It is well known that a singular fiber of a Lefschetz fibration is obtained from a nearby regular fiber, F, by pinching a simple closed curve, $a \subset F$, the so-called *vanishing cycle.* In a neighborhood of a singular fiber, a Lefschetz fibration is determined by the monodromy, which is a *positive Dehn twist*, t_a, along the vanishing cycle [K]. Recall that t_a is a homeomorphism of F obtained by cutting F along a and gluing back after one full twist in the positive direction.

In this chapter we classify and enumerate the real structures in a neighborhood of a real singular fiber of a real Lefschetz fibration. Such a neighborhood can be viewed as a Lefschetz fibration over a disc D^2 with a unique critical value $q = 0 \in D^2$. Without loss of generality, we may assume that the complex conjugation, *conj* in D^2 is the standard one, induced from $\mathbb{C} \supset D^2$. We call such fibrations *elementary Lefschetz fibrations* (real or not). We start with an exposition of the techniques giving the (well known) classification of Lefschetz fibrations in the non-real setting and then generalize it for the real setting.

4.1 Elementary Lefschetz fibrations

Let $(\pi : X \to D^2, b, \rho : \Sigma_g \to F_b)$ be an elementary marked $\mathcal{LF}$. By definition there exist local charts (U, ϕ_U), (V, ϕ_V) around the critical point $p \in \pi^{-1}(0)$ and the critical value $0 \in D^2$, respectively, such that U, V are closed discs and $\pi|_U : U \to V$ is isomorphic (via ϕ_U and ϕ_V) to $\xi : E \to D_\epsilon$, where $E = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq$

$\sqrt{\epsilon}$, $\left|z_1^2+z_2^2\right| \le \epsilon^2\}$ and $D_\epsilon = \{t \in \mathbb{C} : |t| \le \epsilon^2\}$, $0<\epsilon<1$ with $\xi(z_1,z_2) = z_1^2+z_2^2$. Replacing the Lefschetz fibration by an isomorphic one over a smaller base, we can assume that $D_\epsilon = D^2$ and $b \in \partial D_\epsilon$ and the critical value $q = 0 \in D_\epsilon$.

The projection $(z_1,z_2) \to z_1$ maps each fiber $\xi^{-1}(t) = \{(z_1,z_2) : z_1^2+z_2^2 = t\}$ of ξ to the disc $|z_1| \le \sqrt{\epsilon}$. This mapping represents the fiber $\xi^{-1}(t)$ as a two sheeted covering ramified at $z_1 = \pm\sqrt{t}$. Therefore, topologically the regular fibers $\xi^{-1}(t)$, $t \ne 0$, are cylinders and the fiber $\xi^{-1}(0)$ is a cone obtained from a nearby fiber by pinching a simple closed curve, a, the vanishing cycle. Furthermore, such a curve a realizes a non-trivial homology class in $\xi^{-1}(t)$ and, hence, it is unique up to isotopy in $\xi^{-1}(t)$.

Recall that ∂E is naturally divided in two parts, $\partial^v E$ and $\partial^h E$, see Definition 2.1.7. Let us fix a marking $s : S^1 \times I \to \xi^{-1}(b)$, $I = [0,1]$. Then, using the double sheeted coverings of V ramified at $z_1 = \pm\sqrt{t}$, the vertical boundary $\partial^v E = \xi^{-1}(\partial D_\epsilon) \to \partial D_\epsilon$ can be identified with $S^1 \times I \times [0,1]\big/_{(t_a(x),0)\sim(x,1)} \to [0,1]\big/_{0\sim 1}$ and the horizontal boundary $\partial^h E \to D_\epsilon$ with $S^1 \times D_\epsilon \to D_\epsilon$.

The complement of U in $\pi^{-1}(V)$ does not contain any critical point. Therefore, X can be written as union of two $\mathcal{LF}s$ with boundary: one of them, $U \to V$, is isomorphic to $E \to D_\epsilon$, and the other one is isomorphic to the trivial fiber bundle $R \to D_\epsilon$ whose fibers are diffeomorphic to the complement of an open regular neighborhood of the vanishing cycle a in F_b.

Let $\mathcal{A}_g$ be the set of isotopy classes of simple closed non-contractible (non-oriented) curves on Σ_g, and let $\mathcal{V}_g$ be the set of isotopy classes of non-contractible embeddings $\nu : S^1 \times I \to \Sigma_g$. We denote by $\mathcal{L}_g$ the set of isomorphism classes of elementary marked genus-g Lefschetz fibrations and define $\hat{\Omega} : \mathcal{V}_g \to \mathcal{L}_g$ such that $\hat{\Omega}([\nu]) = [L_\nu]$ where $[L_\nu]$ stands for the isomorphism class of the Lefschetz fibration L_ν. The construction of L_ν is as follows.

Let us choose a representative ν of $[\nu]$, and let Σ_g^ν denote the closure of $\Sigma_g \backslash \nu(S^1 \times I)$. Consider the trivial fibration $R_\nu = \Sigma_g^\nu \times D_\epsilon \to D_\epsilon$ with horizontal boundary $\partial\Sigma_g^\nu \times D_\epsilon \to D_\epsilon$. We take $(\xi : E \to D_\epsilon, b, s : S^1 \times I \to \xi^{-1}(b))$ as above, switch the marking to $s \circ \nu^{-1} : \nu(S^1 \times I) \to \xi^{-1}(b)$, and denote by $E_\nu \to D_\epsilon$ the marked Lefschetz fibration $(\xi : E \to D_\epsilon, b, s \circ \nu^{-1} : \nu(S^1 \times I) \to \xi^{-1}(b))$. Then $L_\nu \to D_\epsilon$ and its marking $\rho_\nu : \Sigma_g \to F_b$ is obtained by gluing $R_\nu \to D_\epsilon$ and $E_\nu \to D_\epsilon$ along their trivial horizontal boundaries.

Lemma 4.1.1. *$\hat{\Omega} : \mathcal{V}_g \to \mathcal{L}_g$ is a well defined map.*

Proof. Let $\nu, \nu' : S^1 \times I \to \Sigma_g$ be two isotopic embeddings, and let $\psi_t : S^1 \times I \to \Sigma_g$, $t \in [0,1]$, be a continuous family of embeddings such that $\psi_0 = \nu$ and $\psi_1 = \nu'$. Then, there exists an ambient isotopy $\Psi_t : \Sigma_g \to \Sigma_g$ such that $\Psi_0 = id$ and $\psi_t = \Psi_t \circ \psi_0$. Clearly, Ψ_1 induces diffeomorphisms $R_\nu \to R_{\nu'}$ and $E_\nu \to E_{\nu'}$, which respects the gluing and the fibrations, so that it gives an equivalence of $L_\nu \to D_\epsilon$ and $L_{\nu'} \to D_\epsilon$ as marked fibrations. Hence $[L_\nu] = [L_{\nu'}]$. $\square$

We consider the map $o : \mathcal{V}_g \to \mathcal{A}_g$ such that $o([\nu]) = [\nu(S^1 \times \{\frac{1}{2}\})] = [a]$. Due to the uniqueness of regular neighborhoods, the mapping o is a two sheeted covering: the two elements of a fiber $o^{-1}([a])$ corresponding to opposite orientations of a. Since the automorphism $(z_1, z_2) \to (z_1, -z_2)$ of $E \to D_\epsilon$ is reversing the orientation of the vanishing cycle (or, equivalently, since the Dehn twist does not depend on the orientation on the vanishing cycle), the map $\hat{\Omega}$ descends to a well defined map Ω and the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_g & \xrightarrow{\ o\ } & \mathcal{A}_g \\ {\scriptstyle\hat{\Omega}}\downarrow & \swarrow {\scriptstyle\Omega} & \\ \mathcal{L}_g. & & \end{array}$$

Remark 4.1.2. The above diagram implies that the isomorphism class of resulting fibration $L_\nu \to D_\epsilon$ does only depend on $[a] = o([\nu])$. From now on we will denote L_ν by L_a.

Theorem 4.1.3. *$\Omega : \mathcal{A}_g \to \mathcal{L}_g$ is a bijection.*

Proof. The surjectivity is already shown at the beginning of this section. Let us show that Ω is injective. Consider $[a], [a'] \in \mathcal{A}_g$ such that $\Omega_g([a]) = \Omega_g([a'])$. We will show that $[a] = [a']$. Since Ω is well defined, for some representatives a, a' of $[a], [a']$ respectively, $(L_a \to D_\epsilon, b, \rho_\nu : \Sigma_g \to F_b)$ is isomorphic to $(L_{a'} \to D_\epsilon, b', \rho_{\nu'} : \Sigma_g \to F'_{b'})$.

Then there exist orientation preserving diffeomorphisms $H : L_a \to L_{a'}$ and $h : D_\epsilon \to D_\epsilon$ such that we have the following commutative diagram

$$\begin{array}{ccc} L_a & \xrightarrow{H} & L_{a'} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ D_\epsilon & \xrightarrow{h} & D_\epsilon \end{array}$$

where $h(b) = b'$ and $H \circ \rho_\nu = \rho_{\nu'}$.

The diffeomorphism H necessarily takes the critical point to the critical point hence it takes the corresponding vanishing cycle a to a curve in a regular neighborhood of a'. Since in a cylinder all non-contractible closed curves are isotopic, $H(a)$ is isotopic to a'. Moreover, since $H \circ \rho = \rho'$, we have $H(\rho_\nu(a)) = \rho_{\nu'}(a)$ and hence $\rho_{\nu'}(a)$ is isotopic to $\rho_{\nu'}(a')$.

Let $\psi_t : F'_{b'} \to F'_{b'}$, $t \in [0,1]$ such that $\psi_0 = id$ and $\psi_1(\rho'(a)) = \rho'(a')$. Then $\Psi_t = \rho'^{-1} \circ \psi_t \circ \rho' : \Sigma_g \to \Sigma_g$ provides an isotopy from a to a'. □

To deal with Lefschetz fibrations without marking we introduce the following definition. Two simple closed curves, a and a', on Σ_g are called *conjugate* if there is an orientation preserving diffeomorphism of Σ_g which carries a to a'. Note that isomorphic $\mathcal{LF}s$ give conjugate vanishing cycles by the following evident lemma.

Lemma 4.1.4. *If there exists a diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi(a)$ is isotopic to a then there exists a diffeomorphism ψ of Σ_g with $\psi(a) = a'$.* □

Proposition 4.1.5. *There is a one-to-one correspondence between the classes of elementary Lefschetz fibrations (non-marked) and the set of conjugacy classes of non-contractible simple closed curves on Σ_g.*

Proof. The proposition follows from Lemma 4.1.4 and Theorem 4.1.3. □

Corollary 4.1.6. *There are $1+[\frac{g}{2}]$ isomorphism classes of elementary (non-marked) genus-g Lefschetz fibrations.*

Proof. Topologically, there are two types of simple closed curves on Σ_g: separating and nonseparating. Up to diffeomorphism there exists only one nonseparating curve.

The separating curves are determined by how they divide the genus in two positive integer summands (the summands are positive because we should exclude the case when the curve bounds a disc in Σ_g, since pinching such a curve creates a sphere with self intersection -1). Hence, totally we obtain $1 + [\frac{g}{2}]$ many local models. □

4.2 Elementary Real Lefschetz fibrations

Let ($\pi : X \to D^2$, b, $\rho : \Sigma_g \to F_b$) be an $\mathbb{R}$-marked elementary real Lefschetz fibration. We classify such fibrations up to isomorphism then obtain a classification of $\mathbb{C}$-marked and non-marked $\mathcal{RLF}s$.

As in the non-real case, there exist equivariant local charts (U, ϕ_U), (V, ϕ_V) around the critical point $p \in \pi^{-1}(0)$ and the critical value $0 \in D^2$, respectively, such that U and V are closed discs and $\pi|_U : (U, c_U) \to (V, conj)$ is equivariantly isomorphic (via ϕ_U and ϕ_V) to either of $\xi_\pm : (E_\pm, conj) \to (D_\epsilon, conj)$, where

$$E_\pm = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq \sqrt{\epsilon}, \left|z_1^2 \pm z_2^2\right| \leq \epsilon^2\}$$

$$D_\epsilon = \{t \in \mathbb{C} : |t| \leq \epsilon^2\}, \ 0 < \epsilon < 1$$

with $\xi_\pm(z_1, z_2) = z_1^2 \pm z_2^2$,

The above two real local models $\xi_\pm : E_\pm \to D_\epsilon$ can be seen as two real structures on $\xi : E \to D_\epsilon$. These two real structures are not equivalent. The difference can be seen already at the level of the singular fibers: in the case of ξ_+ the two branches are imaginary and they are interchanged by the complex conjugation; in the case of ξ_- the two branches are both real (see Figure 4.1 where the two halves of the cone correspond to the two branches so that the real structure becomes a corresponding reflection).

To understand the action of the real structures on the regular real fibers of $\xi_\pm$, we can use the branched covering defined by the projection $(z_1, z_2) \to z_1$. Thus, we obtain that:

- in the case of ξ_+, there are two types of real regular fibers; the fibers F_t with $t < 0$ have no real points, their vanishing cycles have invariant representatives (that is $c(a_t) = a_t$ set-theoretically), and in this case, c acts on the invariant vanishing cycles as an antipodal involution; the fibers F_t with $t > 0$ has a circle as their real part and this circle is an invariant, pointwise fixed, representative of the vanishing cycle;

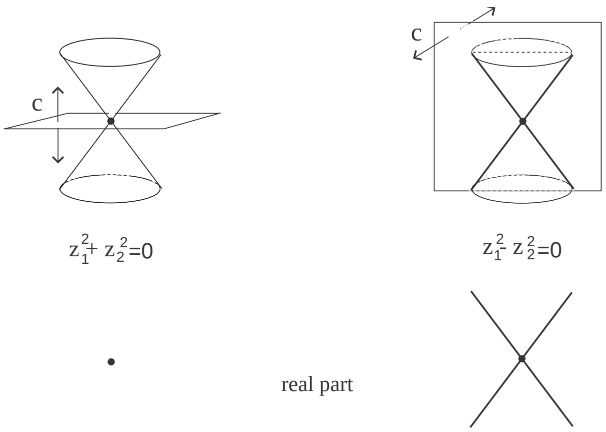


Fig. 4.1. Actions of real structures on the singular fibers of $\xi_{\pm}$.

- in the case of ξ_{-}, all the real regular fibers are of the same type and the real part of such a fiber consists of two arcs each having its endpoints on the two different boundary components of the fiber; the vanishing cycles have still invariant representatives and c acts on them as a reflection.

(In Figure 4.2, all types of the real regular fibers and vanishing cycles of $\xi_{\pm}$ are shown.)

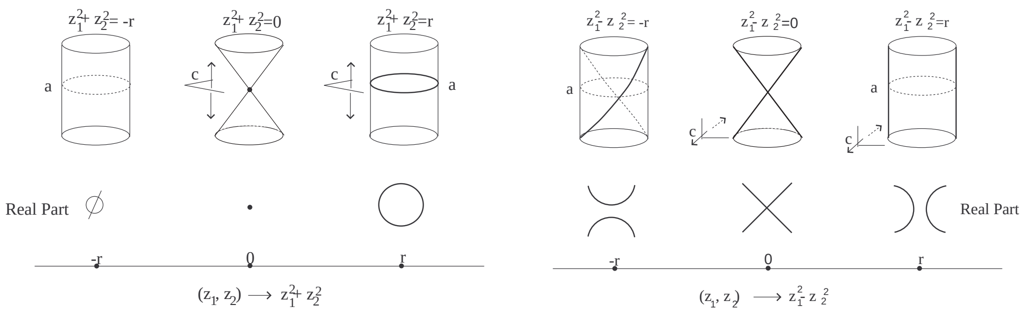


Fig. 4.2. Nearby regular fibers of $\xi_{\pm}$ and vanishing cycles.

Using once more the ramified covering $(z_1, z_2) \to z_1$, we observe that the horizontal part of the fibration $\xi_{\pm}$ is equivariantly trivial and, moreover, has a distinguished equivariant trivialization. On the other hand, since the complement of U in $\pi^{-1}(V)$ does not contain any critical point, X can be written as union of two $\mathcal{RLF}s$ with boundary: one of them, $U \to V$, is isomorphic to $\xi_{\pm} : E_{\pm} \to D_{\epsilon}$, and the other one is isomorphic to the trivial real fiber bundle $R \to D_{\epsilon}$ whose fibers are equivariantly diffeomorphic to the complement of an open regular neighborhood of the vanishing cycle a in F_b. The two types of models, with ξ_+ and with ξ_-, can also be distinguished by the action of the complex conjugation on the boundary components of the real fiber of $R \to D_\epsilon$: in the case of ξ_+ it switches the boundary components, and in the case of ξ_- they are preserved (and the complex conjugation acts as a reflection on each of them).

Let $\mathcal{A}_g^c$ denote the set of equivariant isotopy classes of non-contractible curves on the real surface (Σ_g, c) such that $c(a) = a$, and $\mathcal{V}_g^c$ the set of equivariant isotopy classes of non-contractible embeddings $\nu : S^1 \times I \to \Sigma_g$ such that $c \circ \nu = \nu$ and $\mathcal{L}_g^{\mathbb{R},c}$ the set of classes of directed $\mathbb{R}$-marked elementary genus-g real Lefschetz fibrations.

Let $[\nu]_c \in \mathcal{V}_g^c$. We consider the map $\hat{\Omega}^c : \mathcal{V}_g^c \to \mathcal{L}_g^{\mathbb{R},c}$ such that $\hat{\Omega}^c([\nu]_c) = [L_\nu^{\mathbb{R}}]_c$, where $[L_\nu^{\mathbb{R}}]_c$ denote the isomorphism class of directed $\mathbb{R}$-marked real Lefschetz fibration $L_\nu^{\mathbb{R}}$. The construction of $L_\nu^{\mathbb{R}}$ is the equivariant version of the construction of L_ν. Let ν be a representative of $[\nu]_c$, we consider Σ_g^ν which is the closure of $\Sigma_g \setminus \nu(S^1 \times I)$. Since $c \circ \nu = \nu$, the surface Σ_g^ν inherits a real structure from (Σ_g, c). On the boundary of Σ_g^ν the real structure acts in two ways, either it switches two boundary components or acts as reflection on each boundary components. We consider a trivial real fibration $R_\nu = \Sigma_g^\nu \times D_\epsilon \to D_\epsilon$ where $c_{R_\nu} = (c, conj) : R_\nu \to R_\nu$ is the real structure. Let $E_{\nu\pm} \to D_\epsilon$ denote the model $\xi_\pm : E \to D_\epsilon$ whose marked fiber is identified with $\nu(S^1 \times I)$. Depending on the real structure on the horizontal boundary $S^1 \times D_\epsilon \to D_\epsilon$ (where the real structure on $S^1 \times D_\epsilon$ is taken as $(c_{\partial \Sigma_g^\nu}, conj)$) of $R_\nu \to D_\epsilon$, we choose either of $E_{\nu\pm} \to D_\epsilon$ and then glue $R_\nu \to D_\epsilon$ and the suitable model $E_{\nu\pm} \to D_\epsilon$ along their horizontal trivial boundaries.

Lemma 4.2.1. *$\hat{\Omega}^c : \mathcal{V}_g^c \to \mathcal{L}_g^{\mathbb{R},c}$ is well defined.*

Proof. Let $\nu, \nu' : S^1 \times I \to \Sigma_g$ be two c-equivariant isotopic embeddings, and let $\psi_t : S^1 \times I \to \Sigma_g$, $t \in [0,1]$, be a continuous family of equivariant embeddings such that $\psi_0 = \nu$ and $\psi_1 = \nu'$. Then, there exists an equivariant ambient isotopy $\Psi_t : \Sigma_g \to \Sigma_g$ such that $\Psi_0 = id$ and $\psi_t = \Psi_t \circ \psi_0$ with $\Psi_t \circ c = c \circ \Psi_t$ for all t. Hence Ψ_1 induces an equivariant diffeomorphisms $R_\nu \to R_{\nu'}$ and $E_{\nu\pm} \to E_{\nu'\pm}$, which respects the fibrations, and the gluing thus it gives an equivalence of $L_\nu^{\mathbb{R}} \to D_\epsilon$ and $L_{\nu'}^{\mathbb{R}} \to D_\epsilon$ as $\mathbb{R}$-marked fibrations. □

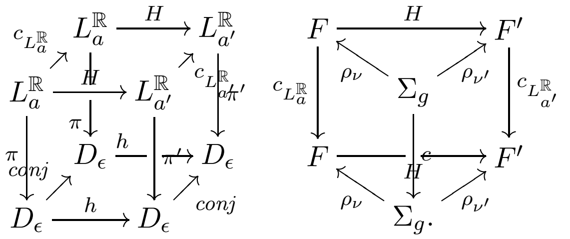
Since $c \circ \nu = \nu$, we have $c(\nu(S^1 \times \{\frac{1}{2}\})) = \nu(S^1 \times \{\frac{1}{2}\}))$. Hence we can define $o^c : \mathcal{V}_g^c \to \mathcal{A}_g^c$ such that $o([\nu]_c) = [\nu(S^1 \times \{\frac{1}{2}\})]_c = [a]_c$. As in the case of $\mathcal{LF}s$ the mapping o^c is two-to-one. Since the monodromy does not depend on the orientation of the vanishing cycle, there exists a well defined mapping, Ω^c, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_g^c & \xrightarrow{o^c} & \mathcal{A}_g^c \\ \hat{\Omega}^c \big\downarrow & \swarrow \Omega^c & \\ \mathcal{L}_g^{\mathbb{R},c}. & & \end{array}$$

Theorem 4.2.2. $\Omega^c : \mathcal{A}_g^c \to \mathcal{L}_g^{\mathbb{R},c}$ *is a bijection.*

Proof. The proof is the equivariant version of the proof of 4.1.3. Let us denote the image of Ω^c by $[L_a^{\mathbb{R}}]_c$. As it is discussed in the beginning of the section, any elementary $\mathcal{RLF}$ can be divided equivariantly into two $\mathcal{RLF}s$ with boundary: an equivariant neighborhood of the critical point (isomorphic to one of the models, $\xi_\pm$), and the complement of this neighborhood (isomorphic to a trivial real Lefschetz fibration). Such a decomposition defines the equivariant isotopy class of the vanishing cycle. This gives the surjectivity of Ω^c.

To show that Ω^c is injective let us consider $[a]_c, [a']_c \in \mathcal{V}_g^c$ such that $\Omega^c([a]_c) = \Omega^c([a']_c)$. We will show that $[a]_c = [a']_c$. Since Ω^c is well defined we have $[L_a^{\mathbb{R}}]_c = [L_{a'}^{\mathbb{R}}]_c$ hence there exist equivariant orientation preserving diffeomorphisms $H : L_a^{\mathbb{R}} \to L_{a'}^{\mathbb{R}}$ and $h : D_\epsilon \to D_\epsilon$ such that we have the following commutative diagrams



Clearly, $H(\rho_\nu(a))$ is equivariantly isotopic to $\rho_{\nu'}(a')$ where a and a' are representatives of $[a]_c$ and $[a']_c$ respectively. Moreover, we have $H \circ \rho_\nu = \rho_{\nu'}$ which gives $H(\rho(a)) = \rho'(a)$, so $\rho'(a)$ is equivariant isotopic to $\rho'(a')$. Let $\psi_t : F' \to F'$, $t \in [0,1]$ such that $\psi_0 = id$ and $\psi_1(\rho'(a)) = \rho'(a')$, $\psi_t \circ c' = c' \circ \psi_t$. Then $\Psi_t = \rho'^{-1} \circ \psi_t \circ \rho' : \Sigma_g \to \Sigma_g$ is the required isotopy. $\square$

Theorem 4.2.2 shows that c-equivariant isotopy classes of vanishing cycles classify the directed $\mathbb{R}$-marked elementary $\mathcal{RLF}s$. To obtain a classification for directed $\mathbb{C}$-marked $\mathcal{RLF}s$ we study the difference between two markings. We will be also interested in the classification of non-marked $\mathcal{RLF}s$.

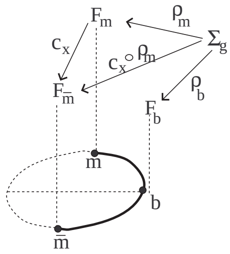


Fig. 4.3.

A $\mathbb{C}$-marking on a directed elementary $\mathcal{RLF}$ defines an $\mathbb{R}$-marking up to isotopy. Let $(\{m, \bar{m}\}, \{\rho_m, c_X \circ \rho_m\})$ be a $\mathbb{C}$-marking on a directed $\mathcal{RLF}$, $\pi : X \to D^2$. The complement, $\partial D^2 \setminus \{m, \bar{m}\}$, has two pieces $S_\pm$ (upper/ lower semicircles) distinguished by the direction. By considering a trivialization of the fibration over the piece of S_+ connecting m to the real point, b, (the trivialization over the piece connecting $\bar{m}$ to the real point obtain by the symmetry), we can pull the marking, $\rho_m : \Sigma_g \to F_m$, to F_b to obtain a marking, say $\rho_b : \Sigma_g \to F_b$ and a real structure $c = \rho_b^{-1} \circ c_X \circ \rho_b : \Sigma_g \to \Sigma_g$. Any other trivialization results in an other marking isotopic to ρ_b and a real structure isotopic to $c : \Sigma_g \to \Sigma_g$.

Hence directed elementary $\mathbb{C}$-marked $\mathcal{RLF}s$ defines a vanishing cycle defined up to c-equivariant isotopy where the real structure c is considered up to isotopy.

Definition 4.2.3. A pair (c, a) of a real structure $c : \Sigma_g \to \Sigma_g$ and a non-contractible simple closed curve $a \in \Sigma_g$, is called a *real code of an elementary* $\mathcal{RLF}$ if $c(a) = a$. Two real codes, (c_0, a_0), (c_1, a_1), will be called isotopic if there exist an isotopy (c_t, a_t), $t \in [0, 1]$ such that $c_t(a_t) = a_t$, $\forall t$. Moreover, two real codes, (c_0, a_0) and (c_1, a_1), will be called *conjugate* if there is an orientation preserving diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi \circ c_0 = c_1 \circ \phi$ and that $[\phi(a_0)]_{c_1} = [a_1]_{c_1}$. We denote the isotopy class of the real code, (c, a), by $[c, a]$ and the conjugacy class by $\{c, a\}$.

Proposition 4.2.4. *There is a one-to-one correspondence between the isomorphism classes of directed* $\mathbb{C}$*-marked elementary* $\mathcal{RLF}s$ *and the isotopy classes of real codes.*

Proof. Let $\mathcal{L}_g^{\mathbb{C},[c]}$ denote the set of classes of directed $\mathbb{C}$-marked elementary genus-g real Lefschetz fibrations and $\mathcal{A}_g^{[c]}$ denote the isotopy classes, $[c, a]$, of real codes. We consider the map $\omega : \mathcal{L}_g^{\mathbb{C}} \to \mathcal{A}_g^{[c]}$. As it is discussed above, a directed $\mathbb{C}$-marked elementary $\mathcal{RLF}$ determines an isotopy class of a directed $\mathbb{R}$-marked elementary $\mathcal{RLF}$. By Theorem 4.2.2 we obtain a vanishing cycle up to c-equivariant isotopy. Since the real structure c is also determined up to isotopy we obtain the real code $[c, a]$. Evidently, isomorphic directed $\mathbb{C}$-marked elementary $\mathcal{RLF}s$ give isotopic real codes. Hence ω is well-defined. Surjectivity of ω is also clear.

For the injectivity, we consider two isotopy classes $[c_i, a_i]$, $i = 1, 2$ such that $[c_1, a_1] = [c_2, a_2]$. Let $(\pi_1 : X_1 \to D^2, \{m_1, \bar{m}_1\}, \{\rho_{m_1}, \bar{\rho}_{m_1}\})$ and $(\pi_2 : X_2 \to D^2, \{m_2, \bar{m}_2\}, \{\rho_{m_2}, \bar{\rho}_{m_2}\})$ be two directed $\mathbb{C}$-marked elementary $\mathcal{RLF}s$, associated to the classes $[c_1, a_1]$ and $[c_2, a_2]$, respectively. We need to show that π_1 and π_2 are isomorphic as directed $\mathbb{C}$-marked $\mathcal{RLF}s$.

Note that we can always choose a representative c for both $[c_1]$ and $[c_2]$ such that $[a_1]_c = [a_2]_c$. Then by Theorem 4.2.2, π_1 is isomorphic to π_2 as $\mathbb{R}$-marking $\mathcal{RLF}s$. An isomorphism of $\mathbb{R}$-marked $\mathcal{RLF}s$ may not preserve the $\mathbb{C}$-markings. However, it can be modified to preserve the $\mathbb{C}$-markings:

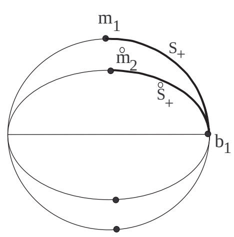


Fig. 4.4.

Up to homotopy, one can identify X_2 with a subset of X_1. Let $\mathring{\pi}_2 : \mathring{X}_2 \to D^2$ be the corresponding fibration. Then, one can transform $\mathring{m}_2$ to m_1 preserving the real marking and the trivializations over the corresponding paths, S_+ and $\mathring{S}_+$, see Figure 4.4 to obtain an isomorphism of $\mathbb{C}$-marked $\mathcal{RLF}s$, preserving the isomorphism of $\mathbb{R}$-marked $\mathcal{RLF}s$. Since the difference $X_1 \setminus \mathring{X}_2$ has no singular fiber. □

For fibrations without marking we allow to change $[c, a]$ by an equivariant diffeomorphism. Hence we have the following proposition.

Proposition 4.2.5. *There is a one-to-one correspondence between the set of conjugacy classes, $\{c, a\}$, of real codes and the set of classes of directed non-marked elementary real Lefschetz fibrations.* □

4.3 Vanishing cycles of real Lefschetz fibrations

By definition any real code, (c, a), of directed elementary $\mathcal{RLF}$ satisfies $c(a) = a$. Hence, the real structure acts on the vanishing cycle a. Such an action can be either the identity, or an antipodal map, or a reflection. In the latter case, there are two points fixed by c. They either belong to the same or different real components of c.

We call the curves on which c acts as an antipodal map *totally imaginary* and those curves on which c acts as a reflection *real-imaginary.* (Recall that the curves on which c acts as the identity are called *real.*)

In Figure 4.5 we show an invariant curve a together with the action of c. When necessary, on figures, we will distinguish invariant curves by showing the action of the real structure c.

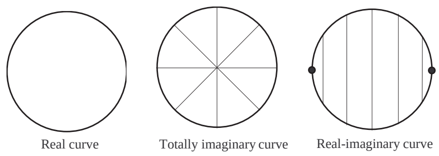


Fig. 4.5. Invariant curves together with the action of real structures.

Lemma 4.3.1. *Let c be a real structure on a closed surface Σ_g, let a be an embedded simple closed curve on Σ_g such that $c(a) = a$ then $c' = t_a \circ c$ (as well as $c'' = c \circ t_a$) is a real structure on Σ_g.*

Moreover, if a is real with respect to c then a is totally imaginary with respect to c', and vice versa. On the other hand, a is real-imaginary with respect to c if and only if a is real-imaginary with respect to c'.

Proof. Clearly $t_a \circ c$ is an orientation reversing diffeomorphism of Σ_g. Since c is orientation reversing, the conjugation $c \circ t_a \circ c$ coincides with $t^{-1}_{c(a)}$. Then we have $(t_a \circ c)^2 = t_a \circ c \circ t_a \circ c = t_a \circ t^{-1}_{c(a)} = t_a \circ t^{-1}_a = id$. This shows that $t_a \circ c$ is a real structure on Σ_g. (The proof of the case $c \circ t_a$ is analogous.)

As for the second part, let us first recall the definition of the Dehn twist on Σ_g along a. Let $\nu(a)$ be a regular neighborhood of a. We choose an orientation preserving diffeomorphism $\phi : S^1 \times [0, 1] \to \nu(a)$ such that $\phi(S^1 \times \{\frac{1}{2}\}) = a$ and consider $\tau : S^1 \times [0, 1] \to S^1 \times [0, 1]$ such that $\tau(\theta, t) = (\theta + 2\pi t, t)$. The Dehn twist t_a along a is the diffeomorphism obtained by taking $\phi \circ \tau \circ \phi^{-1} : \nu(a) \to \nu(a)$ on $\nu(a)$ and extending it to Σ_g by the identity. In particular, t_a rotates a by an angle of π. Hence, $c|_a$ is the identity if and only if $(t_a \circ c)|_a$ is the antipodal map and $c|_a$ is reflection if and only if $(t_a \circ c)|_a$ is reflection. See Figure 4.6. $\square$

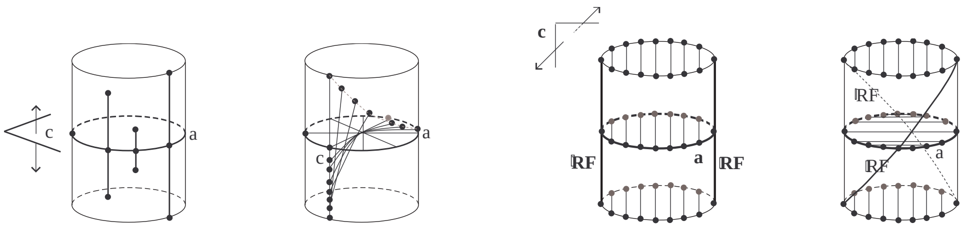


Fig. 4.6. Actions of the real structure on nearby regular fibers of $\xi_{\pm}$.

The next example shows a real surface together with some non-contractible c-invariant curves.

Example 4.3.2. Let c' be a reflection on a genus-5 surface whose real part is the set of curves $\{a_1, a_2, a_3, a_4\}$ shown in Figure 4.7. We set $c = t_{a_1} \circ c'$ and consider the real surface (Σ_5, c). Figure 4.7 shows some examples of invariant curves on the real surface (Σ_5, c). Lemma 4.3.1 implies that c acts on a_1 as the antipodal map, hence the curve a_1 is totally imaginary, while a_2, a_3, a_4 are real. The curves, a_5 and a_6 are real-imaginary. The real points of a_5 belong to two different real curves, a_2 and a_3, whereas the real points of a_6 belong to the real curve a_4. Note that the curves $a_1, a_2, a_3, a_4, a_5, a_6$ are nonseparating. While the curve a_7 is an example of separating real-imaginary curve.

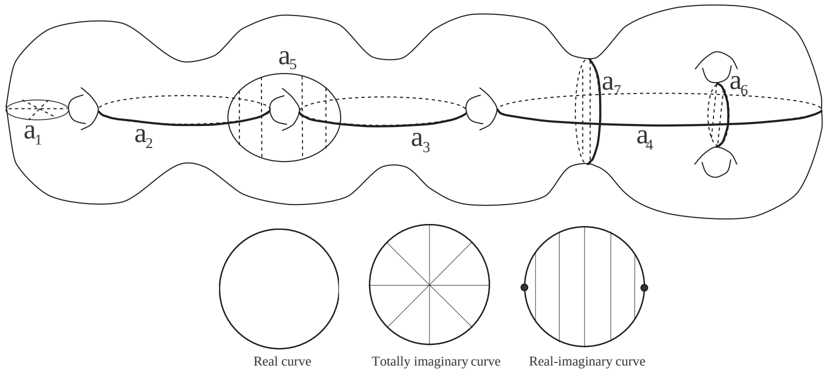


Fig. 4.7. c-invariant curves on (Σ_5, c). We showed explicitly the action of c on $a_1, a_2, a_3, a_4, a_5, a_6, a_7$.

4.4 Classification of elementary real Lefschetz fibrations with nonseparating vanishing cycles

Let $\mathcal{S}_g^*$ be the set of classes of real closed genus-g surfaces ($g \geq 1$) with two marked points which are, as a set, invariant under the action of real structure and let $\mathcal{L}_g^c$ be the set of classes of directed non-marked elementary genus-g $\mathcal{RLF}s$. We assume that the vanishing cycle is nonseparating and define a map $e : \mathcal{L}_g^c \to \mathcal{S}_{g-1}^*$ as follows.

Given a directed elementary $\mathcal{RLF}$, we consider the associated real code (c, a). We take a c-invariant regular closed neighborhood, $\nu(a)$, of a in (Σ_g, c). The complement $\Sigma_g^{\nu(a)} = \Sigma_g \setminus \nu(a)$ inherits the real structure from Σ_g and can be seen as a real surface with two punctures. Let us consider the punctures as marked points on the closed surface and define the image of e as the closed marked surface we obtain. By construction the pair of marked points is invariant under the action of the real structure. Clearly, equivalent real codes give equivalent real genus-$(g-1)$ surfaces, hence e is well defined.

Lemma 4.4.1. *The map $e : \mathcal{L}_g^c \to \mathcal{S}_{g-1}^*$ is surjective.*

Proof. Given (Σ_{g-1}, c_{g-1}), a representative of a class in $\mathcal{S}^*{}_{g-1}$, by Proposition 4.2.5 it is enough to assign to it, a real code (c, a). Let $\{s_1, s_2\}$ be the marked points on Σ_{g-1}, consider open neighborhoods $\nu(s_1)$, $\nu(s_2)$ of s_1 and s_2, respectively such that,

- if s_1 and s_2 are real then we have $c_{g-1}(\nu(s_i)) = \nu(s_i)$ for $i = 1, 2$,
- if one is the conjugate of the other then we set $\nu(s_2) = c_{g-1}(\nu(s_1))$.

The complement, Σ_{g-1}^{ν}, of the neighborhoods $\nu(s_i)$, $i = 1, 2$, in Σ_{g-1} is a real surface with two boundary components. We consider $S^1 \times [0, 1]$ and glue it to Σ_{g-1}^{ν} along the boundary components. The resulting surface has genus g.

The real structure of Σ_{g-1}^{ν} can be extended to $S^1 \times [0, 1]$ to obtain a real structure c on Σ_g such that $a = S^1 \times \{\frac{1}{2}\}$ is a c-invariant curve. Thus, we obtain $c : \Sigma_g \to \Sigma_g$ and $a \subset \Sigma_g$ such that $c(a) = a$.

Clearly, any other representative $(\Sigma'_{g-1}, c'_{g-1})$ give another code which is conjugate to (c, a). □

Lemma 4.4.2.

$$|\mathcal{S}^*_{g-1}| = \begin{cases} \frac{9g-5}{2} & \textit{if g-1 even,} \\ \frac{9g-6}{2} & \textit{if g-1 odd.} \end{cases}$$

Proof. Note that an invariant pair of marked points on a real surface can be chosen:

- as a pair of complex conjugate points,
- as real points on a real component, if there is at least one real component,
- as real points on two different real components, if there are at least 2 real components.

Up to equivariant diffeomorphisms such choices are unique. Thus, for each real structure which has at least two real components we have 3 choices. When there is only one real component, we get 2 choices and lastly if there are no real component, we get only 1 choice for marked points. Recall that for each genus there is only one real structure with no real component. There is one real structure with one real component, if genus is odd and there are two such real structures if genus is even.

Since on Σ_{g-1} there are $g+1+[\frac{g-1}{2}]$ real structures, we obtain

$$|\mathcal{S}^*_{g-1}| = 3(g+1+[\frac{g-1}{2}]) - k \text{ where } \begin{cases} k=4 & \text{if g-1 even,} \\ k=3 & \text{if g-1 odd.} \end{cases}$$

□

Proposition 4.4.3.

$$|\mathcal{L}^c_g| = \begin{cases} 6 & \textit{if g=1,} \\ 8g-3 & \textit{if g>1 odd,} \\ 8g-4 & \textit{if g>1 even.} \end{cases}$$

Proof. Since e is surjective we will count the inverse images of $(\Sigma_{g-1}, c_{g-1}) \in \mathcal{S}^*{}_{g-1}$. By Proposition 4.2.5, it is enough to count the real codes of elementary $\mathcal{RLF}s$.

Case 1: Let (Σ_{g-1}, c_{g-1}) be a real surface with a pair of conjugate marked points, say s_1, s_2. As we discussed above we obtain the genus-g surface by gluing a cylinder to the surface Σ^{ν}_{g-1}. Note that if marked points are conjugate pairs the real structure switches the boundary components Σ^{ν}_{g-1}. Hence on the cylinder $S^1 \times [0,1]$ we consider a real structure which exchanges the boundaries. There are two such real structures. One has a real component which is the central curve the other has no real component. Hence, we have two inverse images for each real surface Σ_{g-1}.

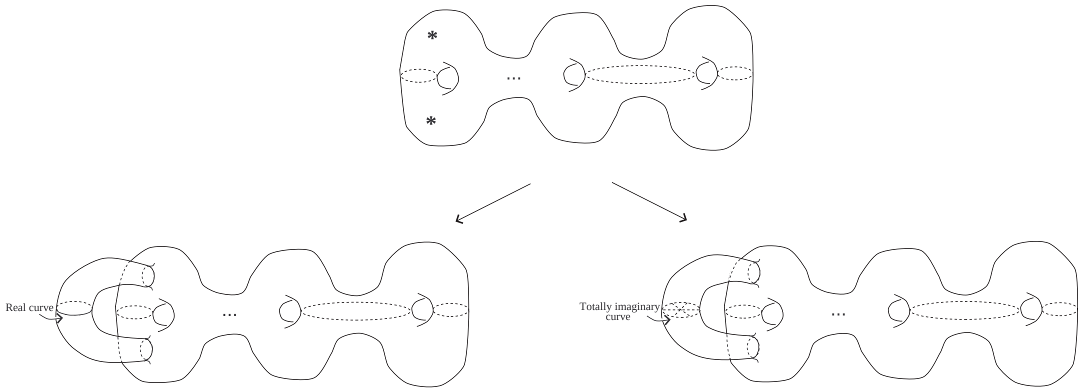


Fig. 4.8. Gluing neighborhood of the vanishing cycle to a real genus-$(g-1)$ surface with two complex conjugated marked points.

Since the points, s_1, s_2 are not real, there is no condition on the number of real components, so there are exactly $2(g+1+[\frac{g-1}{2}])$ directed elementary $\mathcal{RLF}s$.

Case 2: Let us assume that two marked points are chosen on a real component of the real genus-$(g-1)$ surface. In this case, the real structure on the boundary components of Σ^{ν}_{g-1} is reflection hence each component has two real points. Recall that there is a unique real structure up to diffeomorphism on the cylinder where the action on the boundary is reflection. However, if we extend the real structure of Σ^{ν}_{g-1} to the cylinder we have two choices to connect the real points. These choices result in different real structures since their number of real components are not the same.

Excluding the case when the real structure has no real component we obtain $2(g+[\frac{g-1}{2}])$ many local models.

Case 3: Finally, let us assume that the marked points are real points belonging to different real components. This case can occur only if $g-1>0$. As in the case 2, boundary components of Σ^{ν}_{g-1} have two real points. Unlike the previous case, the way we connect the real points does not effect the number of real components, see Figure 4.10. However, it may change the type of the real structure.

Namely, if c_{g-1} is separating then we may obtain either separating or nonseparating real structure. When c_{g-1} is nonseparating the resulting real structure is nonseparating regardless of how we connect the real points.

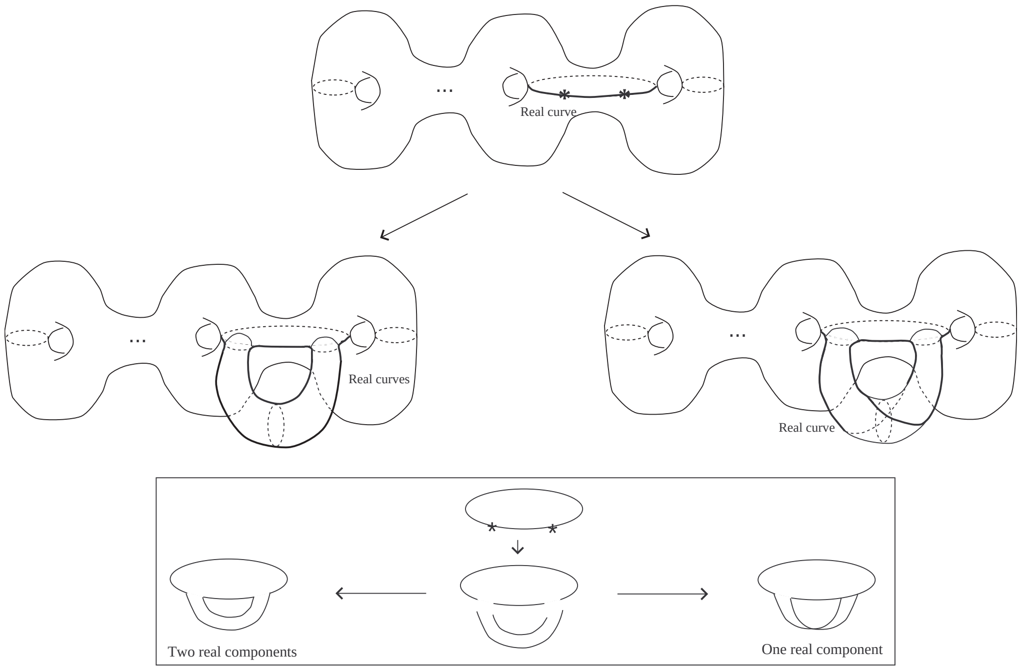


Fig. 4.9. Gluing neighborhood of the vanishing cycle to a real genus-$(g-1)$ surface with two real marked points belonging to the same real component.

There are exactly g nonseparating real structures on a genus-$(g-1)$ surface. Among nonseparating real structures there is one without real component and one with a unique real component. The number of separating real structures on a genus-$(g-1)$ surface whose real part has at least two real components is $1+[\frac{g-1}{2}]$ if $g-1$ is odd and $[\frac{g-1}{2}]$ if $g-1$ is even.

Hence, totally we have

$$g-2+2(1+[\tfrac{g-1}{2}]) \text{ real structures if } g-1 \text{ is odd and}$$
$$g-2+2[\tfrac{g-1}{2}] \text{ real structures if } g-1 \text{ is even.}$$

Therefore,

- If $g=1$, we have only cases 1 and 2, hence there are $4+2=6$ directed non-marked elementary $\mathcal{RLF}s$ with nonseparating vanishing cycle,
- if $g>1$, is even then we have $2(g+1+[\frac{g-1}{2}])+2(g+[\frac{g-1}{2}])+2(1+[\frac{g-1}{2}])+g-2 = 8g-4$ directed non-marked elementary $\mathcal{RLF}s$ with nonseparating vanishing cycle,

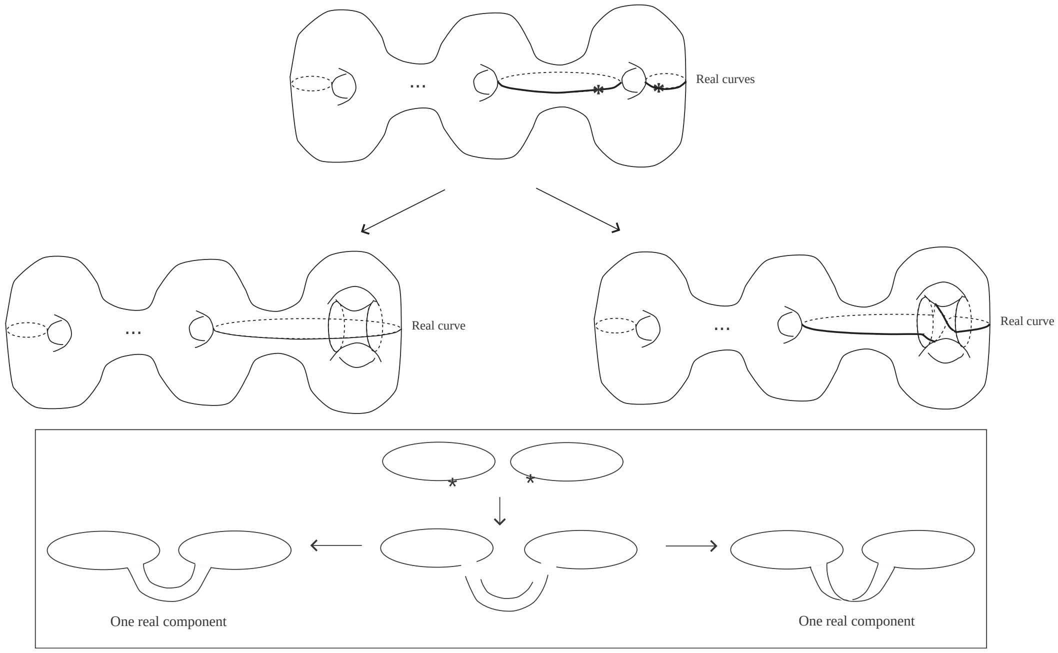


Fig. 4.10. Gluing neighborhood of the vanishing cycle to a real genus-$(g-1)$ surface with two real marked points belonging to different real components.

- if $g > 1$, odd we have $2(g+1+[\frac{g-1}{2}]) + 2(g+[\frac{g-1}{2}]) + 2[\frac{g-1}{2}] + g - 2 = 8g-3$ directed non-marked elementary $\mathcal{RLF}s$ with nonseparating vanishing cycle.

□

4.5 Classification of elementary real Lefschetz fibrations with separating vanishing cycles

In this section, we consider the real code (c, a) of an elementary $\mathcal{RLF}$ such that $a \subset \Sigma_g$ is a separating curve. Recall that we restrict ourselves to the study of relatively minimal $\mathcal{LF}s$. That is no fiber contains an exceptional sphere. Such phenomenon corresponds to the case when the vanishing cycle bounds a disc. Hence, we will assume that the vanishing cycle a does not bound a disc.

As before c acts on a. This action can be the identity, the antipodal map or reflection. However, since a is separating if c acts on a as a reflection then two real points of a necessarily belong to the same real component.

Lemma 4.5.1. *If g is even then there exists a real structure c and a separating invariant simple closed curve a on (Σ_g, c) such that a is real or totally imaginary with respect to c.*

Proof. Clearly, a real curve separates the surface if and only if the real structure is separating and has only one real component, see Figure 4.11. Such phenomenon appears only in the case of even genus. Evidently, up to diffeomorphism there exists unique such pair (c, a).

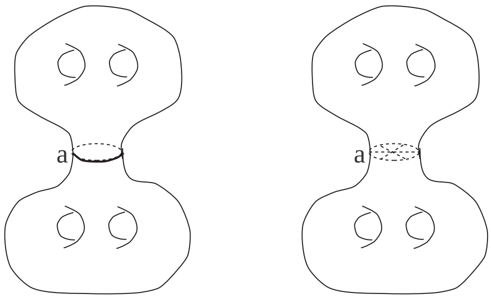


Fig. 4.11. Real and totally imaginary separating curves.

Recall that there is a strong relation between the real curves and the totally imaginary curves. Namely, one can change the real structure by a Dehn twist along a (see Lemma 4.3.1) to obtain a totally imaginary curve from a real curve and vice versa. Hence, a totally imaginary separating curve a appears only in the case of even genus and the real structure is nonseparating without real component. □

Unlike real and totally imaginary curves, there are many separating real-imaginary curves on a real surface. They are distinguished by how they separate the real surface.

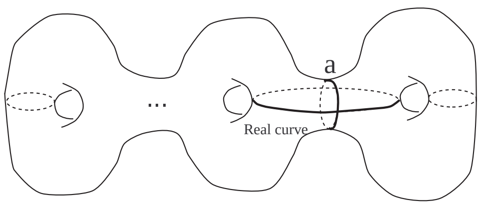


Fig. 4.12. Real-imaginary separating curve.

Note that if there is a real-imaginary curve then the real structure has necessarily at least one real component. Let us fix a real surface (Σ_g, c) of genus $g \geq 1$ such that c has at least one real component. Then to calculate the possible separating curves we will make use of the quotient Σ_g/c. For a nonseparating real structure c on a genus-g surface with $k > 0$ real components, the quotient Σ_g/c is a disc with $k-1$ holes and $l = g - k + 1$ cross caps see Figure 4.13.

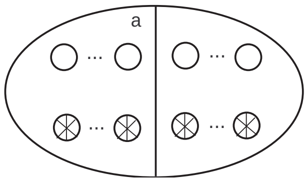


Fig. 4.13.

If the real structure is separating, the quotient Σ_g/c is an orientable genus $\frac{g+1-k}{2}$ surface with k boundary components, see Figure 4.14. By abuse of notation we will denote $\frac{g+1-k}{2}$ also by l.

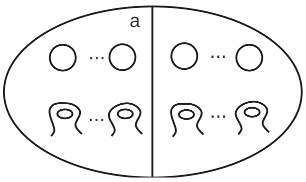


Fig. 4.14.

Hence in either case we have the following calculations.

Lemma 4.5.2. *If both $k-1$ and l are even numbers then we have $[\frac{k(l+1)}{2}]$ separating curves. Otherwise there are $[\frac{k(l+1)}{2}] - 1$ separating curves.*

Proof. This is a counting problem. A separating curve on Σ_g gives an arc on Σ_g/c with endpoints lying on one of the boundary components. We count how many different ways we can divide Σ_g/c by a such an arc.

When both $k-1$ and l are even the arc can divide the Σ_g/c into two symmetric pieces, Figure 4.15. Excluding such case we have $\frac{(k-1+1)(l+1)-1}{2}$ choices. Hence, totally we obtain $\frac{(k-1+1)(l+1)-1}{2} + 1$. Finally, by subtracting the case when the curve bounds a disc we obtain $\frac{(k-1+1)(l+1)-1}{2} + 1 - 1 = [\frac{(k)(l+1)}{2}]$ such arc.

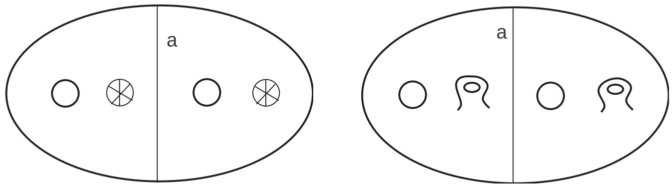


Fig. 4.15. Examples of $k = 3, l = 2$.

When $k-1$ or l is odd, we repeat the same idea. Note that in this case, such an arc can not divide Σ_g/c symmetrically, hence we get $\frac{(k-1+1)(l+1)}{2} - 1 = [\frac{(k)(l+1)}{2}] - 1$. □

Proposition 4.5.3. *The number of conjugacy classes of real codes $\{c, a\}$ where a is a separating curve is given as follows. By Proposition 4.2.5 this gives the number of classes of directed $\mathbb{R}$-marked elementary $\mathcal{RLF}$s whose vanishing cycle is separating.*

$$g \text{ even}, g > 0 \begin{cases} 1 + \sum\limits_{\substack{k \in \{1,3,\ldots,g+1\} \\ l \text{ even}}} [\frac{k(l+1)}{2}] + \sum\limits_{\substack{k \in \{1,3,\ldots,g+1\} \\ l \text{ odd}}} ([\frac{k(l+1)}{2}] - 1) & \text{if } c \text{ is separating,} \\ 1 + \sum\limits_{\substack{k \in \{1,2,\ldots,g\} \\ l \text{ even}}} [\frac{k(l+1)}{2}] + \sum\limits_{\substack{k \in \{1,2,\ldots,g\} \\ l \text{ odd}}} ([\frac{k(l+1)}{2}] - 1) & \text{if } c \text{ is nonseparating,} \end{cases}$$

$$g \text{ odd} \begin{cases} \sum\limits_{k \in \{2,4,\ldots,g+1\}} ([\frac{k(l+1)}{2}] - 1) & \text{if } c \text{ is separating,} \\ \sum\limits_{k \in \{1,2,\ldots,g\}} ([\frac{k(l+1)}{2}] - 1) & \text{if } c \text{ is nonseparating.} \end{cases}$$

Proof. The proposition follows from Lemma 4.5.1 and Lemma 4.5.2.
Note that if c is nonseparating then $k - 1 + l = g$. Thus,

if $g > 0$ is even: $(k-1, l) = (\text{even, even})$ or $(k-1, l) = (\text{odd, odd})$
if g is odd: $(k-1, l) = (\text{even, odd})$ or $(k-1, l) = (\text{odd, even})$.

If c is separating then $k-1+2l=g$. Thus,

$$\begin{array}{llllll} \text{if } g \text{ is even:} & (k-1,l) & = & (\text{even, even}) \text{ or } & (k-1,l) & = & (\text{even, odd}) \\ \text{if } g \text{ is odd:} & (k-1,l) & = & (\text{odd, even}) \text{ or } & (k-1,l) & = & (\text{odd, odd}). \end{array}$$

□

CHAPTER 5

INVARIANTS OF REAL LEFSCHETZ FIBRATIONS WITH ONLY REAL CRITICAL VALUES

The classification of elementary $\mathcal{RLF}s$ can be used to obtain certain invariants for $\mathcal{RLF}s$ over a disc with only real critical values. For this reason we introduce boundary fiber sum of real directed Lefschetz fibrations over D^2. We will study separately the cases of the fiber genus $g > 1$ and $g = 1$, since they are of different nature with respect to the boundary fiber sum. On the other hand, if we assume that fibration admits a real section then the case of $g = 1$ can be treated similar to the case $g > 1$.

5.1 Boundary fiber sum of genus-g real Lefschetz fibrations

Let $\pi : X \to D^2$ be a directed real Lefschetz fibration. Following the notation of previous sections, we denote by $S_\pm$ the upper/ lower semicircles of ∂D^2. We consider also left/ right semicircles, denoted by $S^\pm$, and the quarter-circles $S^\pm_\pm = S^\pm \cap S_\pm$. (Here directions right/ left and up/ down are determined by the orientation of the real part.) Let $r_\pm$ be the real points of $S^\pm$, and $c_\pm$ the real structures on $F_\pm = \pi^{-1}(r_\pm)$.

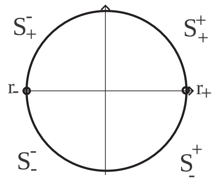


Fig. 5.1.

Definition 5.1.1. Let $(\pi' : X' \to D^2, \{b', \bar{b}'\}, \{\rho', \bar{\rho}'\})$ and $(\pi : X \to D^2, \{b, \bar{b}\}, \{\rho, \bar{\rho}\})$ be two directed $\mathbb{C}$-marked real Lefschetz fibrations such that the real structures c'_+ on F'_+ and c_- on F_- induce (via the markings) isotopic real structures on Σ_g. Then we define the *boundary fiber sum*, $X' \natural_{\Sigma_g} X \to D^2 \natural D^2$, of $\mathbb{C}$-marked $\mathcal{RLF}s$ as follows.

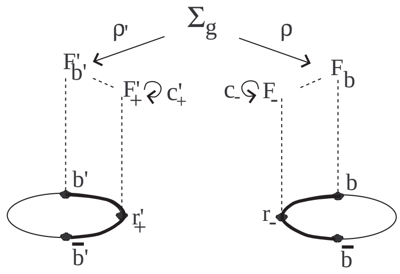


Fig. 5.2.

We choose trivializations of $\pi'^{-1}(S_+^+)$ and $\pi^{-1}(S_+^-)$ such that the pull backs of c'_+ and c_- give the same real structure c on Σ_g. Then the trivialization of $\pi'^{-1}(S^+)$ can be obtained as a union $\Sigma_g \times S_+^+ \cup \Sigma_g \times S_-^+ \diagup_{(x,1_+)\sim(c(x),1_-)}$ and similarly $\pi^{-1}(S^-) = \Sigma_g \times S_+^- \cup \Sigma_g \times S_-^- \diagup_{(x,-1_+)\sim(c(x),-1_-)}$. Then the boundary fiber sum $X' \natural_{\Sigma_g} X \to D^2 \natural D^2$ is obtained by gluing $\pi'^{-1}(S^+)$ to $\pi^{-1}(S^-)$ via the identity map.

Remark 5.1.2. 1. In fact, the construction described above creates a manifold with corners but there is a canonical way to smooth the corners, hence the boundary fiber sum is the manifold obtained by smoothing the corners.

2. By definition, the boundary fiber sum is associative but not commutative.

3. The boundary fiber sum of $\mathbb{C}$-marked $\mathcal{RLF}s$ is naturally $\mathbb{C}$-marked.

4. Note that $D^2 \natural D^2 = D^2$ so when the precision is not needed we use D^2 instead of $D^2 \natural D^2$.

Proposition 5.1.3. *If $g > 1$, then the boundary fiber sum, $X' \natural_{\Sigma_g} X \to D^2$, of directed $\mathbb{C}$-marked genus-g real Lefschetz fibrations is well-defined up to isomorphism of $\mathbb{C}$-marked $\mathcal{RLF}$s.*

Proof. Note that the boundary fiber sum does not effect the fibrations outside a small neighborhood of the intervals where the gluing is made. Let us slice a topological disc D, a neighborhood (which does not contain a critical value) of the gluing interval on $D^2 = D^2 \natural D^2$. Let c'_+ and c_- denote the real structures on the real fibers over the real boundary points of D, see Figure 5.3. Since D contains no real critical value, real structures c'_+, c_- induce isotopic real structures on Σ_g. Hence each real fibration over a disc without a critical value defines a path in the space of real structures on Σ_g. Therefore, the difference of two boundary fiber sums gives a loop in this space. The proof follows from contractibility of such loops discussed in the next section, see Proposition 5.2.4. □

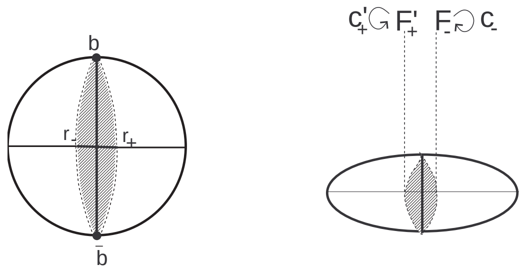


Fig. 5.3.

5.2 Equivariant diffeomorphisms and the space of real structures

Let $\mathcal{C}^c(\Sigma_g)$ denote the space of real structures on Σ_g which are isotopic to a fixed real structure c, and let $Diff_0(\Sigma_g)$ denote the group of orientation preserving diffeomorphisms of Σ_g which are isotopic to the identity. We consider the subgroup of $Diff_0(\Sigma_g)$, denoted $Diff_0^c(\Sigma_g)$, consisting of those diffeomorphisms which commute with c and the subgroup $Diff_0(\Sigma_g, c)$ of $Diff_0(\Sigma_g)$ consisting of diffeomorphisms which are c-equivariantly isotopic to the identity. Note that the group $Diff_0(\Sigma_g)$ acts transitively on $\mathcal{C}^c(\Sigma_g)$ by conjugation. The stabilizer of this action is the group $\mathit{Diff}_0^c(\Sigma_g)$. Hence $\mathcal{C}^c(\Sigma_g)$ can be identified with the homogeneous space $\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0^c(\Sigma_g)$.

Lemma 5.2.1. *If $g > 1$ then $\mathit{Diff}_0^c(\Sigma_g)$ is connected for all $c : \Sigma_g \to \Sigma_g$. However, for $g = 1$, the space $\mathit{Diff}_0^c(\Sigma_g)$ is connected if c is an odd real structure (i.e. it has 1 real component).*

Proof. (We will use different techniques for $g > 1$ and $g = 1$.) Let us first discuss the case of $g > 1$. To show that $\mathit{Diff}_0^c(\Sigma_g)$ is connected, we consider the fiber bundle description of conformal structures on Σ_g, introduced in [EE]. Let Conf_{Σ_g} denote the space of conformal structures on Σ_g equipped with C^∞-topology. The group $\mathit{Diff}_0(\Sigma_g)$ acts on Conf_{Σ_g} by composition from right. This action is proper, continuous, and effective hence $\mathit{Conf}_{\Sigma_g} \to \mathit{Conf}_{\Sigma_g}/\mathit{Diff}_0(\Sigma_g)$ is a principle $\mathit{Diff}_0(\Sigma_g)$-fiber bundle, (cf. [EE]). The quotient is the Teichmuller space of Σ_g, denoted $\mathit{Teich}_{\Sigma_g}$. Note that conformal structures can be seen as equivalence classes of Riemannian metrics with respect to the relation that two Riemannian metrics are equivalent if they differ by a positive function on Σ_g. Let Riem_{Σ_g} denote the space of Riemannian metrics on Σ_g then we have the following fibrations

$$\begin{array}{ccc} \{u : \Sigma_g \to \mathbb{R} : u > 0\} & \longrightarrow & \mathit{Riem}_{\Sigma_g} \\ & & \downarrow p_2 \\ \mathit{Diff}_0(\Sigma_g) & \longrightarrow & \mathit{Conf}_{\Sigma_g} \\ & & \downarrow p_1 \\ & & \mathit{Teich}_{\Sigma_g}. \end{array}$$

The real structure c acts on $\mathit{Diff}_0(\Sigma_g)$ by conjugation. This action can be extended to Conf_{Σ_g} and Riem_{Σ_g} as follows. We fix a section $s : \mathit{Teich}_{\Sigma_g} \to \mathit{Conf}_{\Sigma_g}$ of the bundle p_1 and we consider a family of diffeomorphisms $\phi_\zeta^s : \mathit{Diff}_0(\Sigma_g) \to p_1^{-1}(\zeta)$ parametrized by $\mathit{Teich}_{\Sigma_g}$ such that $\phi_\zeta^s(id) = s(\zeta)$. Let $[\mu_x]$ denote a conformal structure where μ_x is a Riemannian metric on Σ_g. Then we have $\phi_\zeta^s(f(x)) = [\mu_{f(x)}]$ for all $f \in \mathit{Diff}_0(\Sigma_g)$, in particular $\phi_\zeta^s(id) = s(\zeta) = [\mu_x]$. The action of real structure, then, can be written as $c.[\mu_{f(x)}] = [\mu_{c \circ f \circ c(x)}]$. Clearly the definition does not depend on the choice of the representative of the class $[\mu_{f(x)}]$ so the action extends to Riem_{Σ_g}.

Let $Fix_{\mathit{Conf}_{\Sigma_g}}(c)$ denote the set of fixed points of the action of c on Conf_{Σ_g} and $Fix_{\mathit{Riem}_{\Sigma_g}}(c)$, the set of fixed points on Riem_{Σ_g}. Note that $s(\zeta)$ is in $Fix_{\mathit{Conf}_{\Sigma_g}}(c)$, $\forall \zeta \in \mathit{Teich}_{\Sigma_g}$. In fact each $[\mu_{f(x)}]$ where $f \in \mathit{Diff}_0^c(\Sigma_1)$ is in $Fix_{\mathit{Conf}_{\Sigma_g}}(c)$. Our aim is the show that $Fix_{\mathit{Conf}_{\Sigma_g}}(c)$ is connected.

Note that if $Fix_{Conf_{\Sigma_g}}(c)$ is disconnected then the inverse image (which is $Fix_{Riem_{\Sigma_g}}(c)$) is also disconnected in $Riem_{\Sigma_g}$. It is known that $Riem_{\Sigma_g}$ is convex and hence $Fix_{Riem_{\Sigma_g}}(c)$ is convex. However this contradicts to disconnectivity, therefore $Fix_{Conf_{\Sigma_g}}(c)$ is connected. Then $Fix_{Conf_{\Sigma_g}}(c) \cap \mathit{Diff}_0(\Sigma_g) = \mathit{Diff}_0^c(\Sigma_g)$ is connected since $Fix_{Conf_{\Sigma_g}}(c)$ is a union of sections.

For the case of $g = 1$, we consider the quotient Σ_1/c which is a Möbius band (MB) when c is an odd structure. It is known that the space of diffeomorphisms of Möbius band has two components: the identity component $\mathit{Diff}_0(\Sigma_1/c)$, and the component of diffeomorphisms isotopic to the reflection h shown in Figure 5.4. Note that when the Möbius band is obtained by from $I \times I$ by identifying appropriate points of $I \times 0$ with the points of $I \times 1$, the diffeomorphism h can be seen as the diffeomorphism induced from the reflection of $I \times I$ with respect to the $I \times \frac{1}{2}$. The diffeomorphism h is not isotopic to the identity, since before identifying the ends it reverses the orientation of $I \times I$.

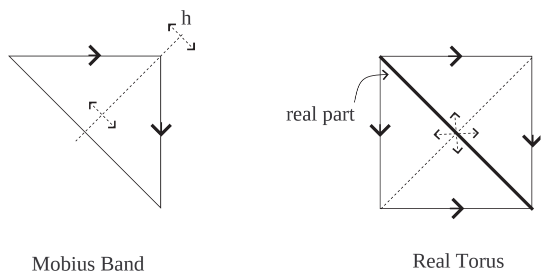


Fig. 5.4.

The diffeomorphism h lifts to the central symmetry $\hat{h} : \Sigma_1 \to \Sigma_1$ of Σ_1. Central symmetry is not isotopic to the identity on Σ_1 since it reverses the orientation of the real curve. Hence, we have

$$\{f : \Sigma_1/c \to \Sigma_1/c : \ \hat{f} : \Sigma_1 \to \Sigma_1 \text{ is isotopic to } id\} = \{f : \Sigma_1/c \to \Sigma_1/c : f \cong id\}.$$

The former is identified by $\mathit{Diff}_0^c(\Sigma_1)$ and the latter is connected, hence $\mathit{Diff}_0^c(\Sigma_1)$ is connected. $\square$

Lemma 5.2.2. *For any real structure $c : \Sigma_g \to \Sigma_g$*

$$\pi_1(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c), [id]) = \begin{cases} 0 & \text{if } g > 1 \\ \mathbb{Z} & \text{if } g = 1 \end{cases}$$

Proof. (When it is not needed we will omit the base point from the notation.) Note that the subgroup $Diff_0(\Sigma_g, c)$ acts from the left on $Diff_0$ by composition.

$$\begin{array}{ccc} Diff_0(\Sigma_g, c) \times Diff_0(\Sigma_g) & \to & Diff_0(\Sigma_g) \\ (f, g) & \to & f \circ g \end{array}$$

Such action is free so $Diff_0(\Sigma_g) \to Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c)$ is a $Diff_0(\Sigma_g, c)$-fiber bundle. We consider the following long exact homotopy sequence of this fibration

$$\begin{array}{c} ... \to \pi_2(Diff_0(\Sigma_g)) \to \pi_2(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c)) \to \pi_1(Diff_0(\Sigma_g, c)) \to \\ \pi_1(Diff_0(\Sigma_g)) \to \pi_1(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c)) \to \pi_0(Diff_0(\Sigma_g)) \to ... \end{array}$$

Case of $g > 1$: it is known that $Diff_0(\Sigma_g)$ is contractible if $g > 1$, so we have $\pi_k(Diff_0(\Sigma_g), id) = 0$ for all k [EE]. Using the exact homotopy sequence we obtain

$$\pi_1(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c), [id]) \cong \pi_0(Diff_0(\Sigma_g, c), id).$$

Note that the group $Diff_0(\Sigma_g, c)$ is isomorphic to $Diff_0(\Sigma_g/c)$. Moreover, since for any real structure c the Euler characteristic of Σ_g/c is negative, $Diff_0(\Sigma_g/c)$ is contractible [ES]. Hence, $\pi_1(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c), [id]) = 0$.

Case of $g = 1$: it is known that Σ_1 is deformation retract of $Diff_0(\Sigma_1)$. Hence up to homotopy we consider $Diff_0(\Sigma_1)$ as a group generated by two rotations, shown in Figure 5.5.

• If c has 2 real components, we consider an identification of Σ_1 by $\mathbb{C}/\mathbb{Z}^2$ such that the real structure c is induced from the standard complex conjugation on $\mathbb{C}$. Let $\varrho : \mathbb{C}/\mathbb{Z}^2 \to \Sigma_1$ be such an identification.

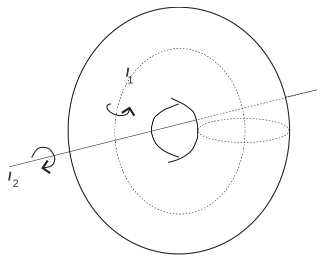


Fig. 5.5. Rotations generating $\mathit{Diff}_0(\Sigma_1)$.

We consider the following family of diffeomorphisms for $t \in [0,1]$,

$$\begin{array}{rrcl} R_1'(t): & \mathbb{C}/\mathbb{Z}^2 & \to & \mathbb{C}/\mathbb{Z}^2 \\ & (x+iy)_{\mathbb{Z}^2} & \to & (x+t+iy)_{\mathbb{Z}^2} \\ R_2'(t): & \mathbb{C}/\mathbb{Z}^2 & \to & \mathbb{C}/\mathbb{Z}^2 \\ & (x+iy)_{\mathbb{Z}^2} & \to & (x+i(y+t))_{\mathbb{Z}^2}. \end{array}$$

where $(x+iy)_{\mathbb{Z}^2}$ denotes the equivalence class of $x+iy$ in $\mathbb{C}/\mathbb{Z}^2$.

Note that $R_i'(0) = R_i'(1) = id$ and each $R_i'(t)$, $t \in [0,1]$ is isotopic to identity. Hence $R_i(t) = \varrho \circ R_i'(t) \circ \varrho^{-1}$, $i = 1,2$ form a bases of $\mathit{Diff}_0(\Sigma_1)$.

To understand $\mathit{Diff}_0(\Sigma_1, c) \subset \mathit{Diff}_0(\Sigma_1)$ we consider the quotient Σ_1/c which is topologically an annulus. It is known that $\pi_k(\mathit{Diff}_0(\Sigma_1/c), id) = \pi_k(\mathit{Diff}_0(S^1), id)$ [I]. Hence, using the fact $\mathit{Diff}_0(\Sigma_g, c) = \mathit{Diff}_0(\Sigma_g/c)$ we get

$$\pi_k(\mathit{Diff}_0(\Sigma_1, c), id) = \begin{cases} 0 & \text{if } k > 1 \\ \mathbb{Z} & \text{if } k = 1 \end{cases}$$

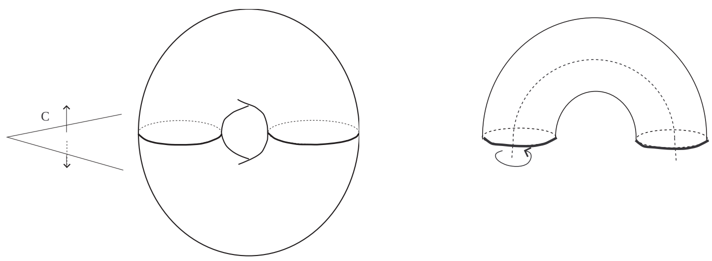


Fig. 5.6.

Note that with respect to the identification ϱ, diffeomorphisms $R_1(t) \in Diff_0(\Sigma_1, c)$, $\forall t \in [0,1]$, hence $R_1(t)$ gives a loop in $\pi_1(Diff_0(\Sigma_1, c), id)$. Thus we choose $R_1(t)$ as a generator of $\pi_1(Diff_0(\Sigma_1, c), id) = \mathbb{Z}$. Then from the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \pi_1(Diff_0(\Sigma_g, c)) & \xrightarrow{f'} & \pi_1(Diff_0(\Sigma_g)) & \xrightarrow{g'} & \pi_1(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c)) & \to & 0 \\ 0 & \to & \mathbb{Z} & \xrightarrow{f'} & \mathbb{Z}+\mathbb{Z} & \xrightarrow{g'} & \pi_1(Diff_0(\Sigma_g)/Diff_0(\Sigma_g, c)) & \to & 0 \\ & & 1 & \to & (1,0) & & & & \end{array}$$

we get $Im(f') = ker(g') = \pi_1(Diff_0(\Sigma_1)/Diff_0(\Sigma_1, c), [id]) = \mathbb{Z}$.

• If c has no real component, we consider $\varrho : \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma_1$ such that the real structure c is induced from the real structure c' where

$$\begin{array}{rccc} c' = \varrho^{-1} \circ c \circ \varrho : & \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}/\mathbb{Z}^2 \\ & (x,y)_{\mathbb{Z}^2} & \to & (x+\frac{1}{2}, -y)_{\mathbb{Z}^2}. \end{array}$$

Then we consider the family of diffeomorphisms $R_i'(t) : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, $t \in [0,1]$ such that

$$\begin{array}{rcclrccl} R_1'(t) : & \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 & R_2'(t) : & \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ & (x,y)_{\mathbb{Z}^2} & \to & (x+t, y)_{\mathbb{Z}^2} & & (x,y)_{\mathbb{Z}^2} & \to & (x, y+t)_{\mathbb{Z}^2}. \end{array}$$

Hence $R_i(t) = \varrho \circ R_i'(t) \circ \varrho^{-1}$, $i = 1, 2$ form a bases of $Diff_0(\Sigma_1)$. As above to understand $Diff_0(\Sigma_1, c)$ we consider the quotient Σ_1/c is a Klein bottle (KB). It is known that $Diff_0(KB) \cong S^1$, [EE]. Hence we consider $Diff_0(\Sigma_1/c)$ as a group generated by the rotation which lifts to a translation in the universal cover of Klein bottle. Such translation in the lattice and corresponding rotation shown in Figure 5.7.

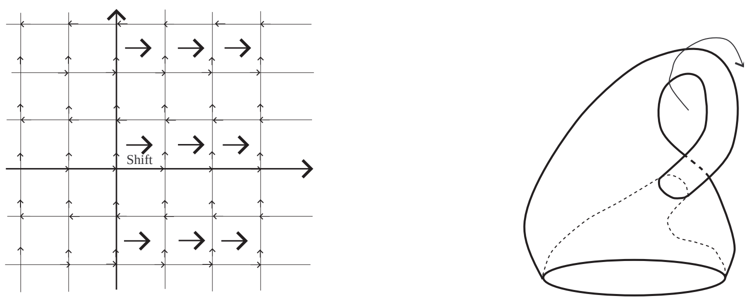


Fig. 5.7.

Since $\mathit{Diff}_0(\Sigma_1, c) \cong \mathit{Diff}_0(\Sigma_1/c) \cong S^1$ we have $\pi_1(\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$. With respect to the identification ϱ and the real structure c, $R_1(t)$ gives a generator of $\pi_1(\mathit{Diff}_0(\Sigma_1, c), id) = \mathbb{Z}$. Then from the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \pi_1(\mathit{Diff}_0(\Sigma_1, c)) & \to & \pi_1(\mathit{Diff}_0(\Sigma_1)) & \to & \pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c)) & \to & 0 \\ & & 1 & \to & (1, 0) & & & & \end{array}$$

we get $\pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c), [id]) = \mathbb{Z}$.

• If c is an odd real structure, Σ_1 has unique real component, denoted C. The restriction of $f \in \mathit{Diff}_0(\Sigma_1, c)$ to C defines a diffeomorphism of C. This restriction gives a fibration with fibers isomorphic to

$$\mathit{Diff}_0(\Sigma_1, C) = \{f \in \mathit{Diff}_0(\Sigma_1, c) : f|_C = id\}.$$

Note that $\mathit{Diff}_0(\Sigma_1, C) \cong \mathit{Diff}_0(\overline{\Sigma_1 \setminus C}, \partial)$ where $\overline{\Sigma_1 \setminus C}$ denote the closure of $\Sigma_1 \setminus C$ and $\mathit{Diff}_0(\overline{\Sigma_1 \setminus C}, \partial)$ diffeomorphisms of $\overline{\Sigma_1 \setminus C}$ which are the identity on the boundary. Note that $\Sigma_1 \setminus C$ is an annulus. It is known that $\mathit{Diff}_0(\overline{\Sigma_1 \setminus C}, \partial)$ is contractible [I]. Hence from the exact sequence of the fibration

$$\begin{array}{ccc} \mathit{Diff}_0(\Sigma_1, C) & \longrightarrow & \mathit{Diff}_0(\Sigma_1, c) \\ & & \downarrow \\ & & \mathit{Diff}_0(C) \end{array}$$

we get $\pi_k(\mathit{Diff}_0(\Sigma_1, c), id) \cong \pi_k(\mathit{Diff}_0(C), id)$, $\forall k$.

Let us choose the identification $\varrho : \mathbb{C}/\Lambda \to \Sigma_1$ where Λ is the lattice generated by $v_1 = (\frac{1}{2}, \frac{1}{2})$ and $v_2 = (\frac{1}{2}, -\frac{1}{2})$, see Figure 5.8. Then the real structure c can be taken as the one induced from the complex conjugation on $\mathbb{C}$.

We consider $R_i'(t) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, $t \in [0, 1]$ such that

$$\begin{array}{llcl} R_1'(t) : & \mathbb{C}/\mathbb{Z}^2 & \to & \mathbb{C}/\mathbb{Z}^2 \\ & (x+iy)_\Lambda & \to & (x+t+iy)_\Lambda \end{array} \qquad \begin{array}{llcl} R_2'(t) : & \mathbb{C}/\mathbb{Z}^2 & \to & \mathbb{C}/\mathbb{Z}^2 \\ & (x+iy)_\Lambda & \to & (x+i(y+t))_\Lambda . \end{array}$$

Clearly, $R_i(t) = \varrho \circ R_i'(t) \circ \varrho^{-1}$ gives a bases for $\mathit{Diff}_0(\Sigma_1)$, since $R_1(t)$ commutes with the real structure gives a generator for $\pi_1(\mathit{Diff}_0(\Sigma_1, c)) = \mathbb{Z}$.

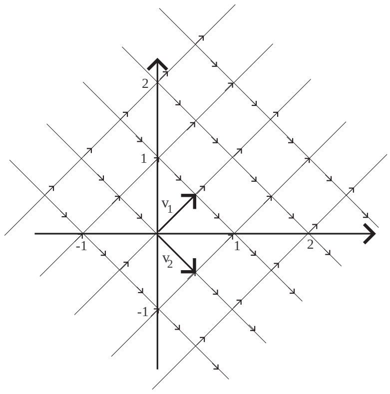


Fig. 5.8.

Therefore, we have

$$\begin{array}{ccccccccc} 0 & \to & \pi_1(\mathit{Diff}_0(\Sigma_g, c)) & \xrightarrow{f} & \pi_1(\mathit{Diff}_0(\Sigma_g)) & \xrightarrow{g} & \pi_1(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0(\Sigma_g, c)) & \to & 0 \\ 0 & \to & \mathbb{Z} & \xrightarrow{f} & \mathbb{Z}+\mathbb{Z} & \xrightarrow{g} & \pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c)) & \to & 0 \\ & & 1 & \to & (1,0) & & & & \end{array}$$

Since the sequence is exact

$$Im(f) = ker(g) = \pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0(\Sigma_1, c), [id]) = \mathbb{Z}.$$

□

Definition 5.2.3. A rotation in $\mathit{Diff}_0(\Sigma_1)$ is called *real rotation* if it is in the subgroup $\mathit{Diff}_0(\Sigma_g, c)$, otherwise it will be called *imaginary rotation.*

Proposition 5.2.4. *For any real structure* $c : \Sigma_g \to \Sigma_g$

$$\pi_1(\mathit{Diff}_0(\Sigma_g)/\mathit{Diff}_0^c(\Sigma_g), [id]) = \begin{cases} 0 & \text{if } g > 1 \\ \mathbb{Z} & \text{if } g = 1 \end{cases}$$

Proof. If $g > 1$, then $\mathit{Diff}_0^c(\Sigma_g)$ is connected $\forall c$; if $g = 1$, then $\mathit{Diff}_0^c(\Sigma_g)$ is connected for the real structures c which have 1 real component. Therefore, in these cases we have $\mathit{Diff}_0^c(\Sigma_g) = \mathit{Diff}_0(\Sigma_1, c)$ and thus the result follows from Lemma 5.2.2.

If $g = 1$ and $c : \Sigma_1 \to \Sigma_1$ has 2 real components, then we consider the identification $\varrho : \mathbb{C}/\mathbb{Z}^2 \to \Sigma_1$ and the diffeomorphism $R_2(\frac{1}{2})$ induced from $(x+iy)_{\mathbb{Z}^2} \to$

$(x + i(y + \frac{1}{2}))_{\mathbb{Z}^2}$. Since $y + \frac{1}{2} = y - \frac{1}{2}$ modulo $\mathbb{Z}$, the diffeomorphism $R_2(\frac{1}{2})$ is equivariant, however it is not equivariantly isotopic to the identity.

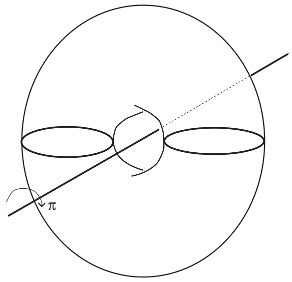


Fig. 5.9.

Similar construction can be made for real structure with no real component by considering $\varrho : \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma_1$. Therefore, if c is an even real structure (has either 2 or no real components) on Σ_1, then $\mathit{Diff}_0^c(\Sigma_1)$ has two components: $\mathit{Diff}_0(\Sigma_1, c)$ and the group of diffeomorphisms generated by the imaginary rotation $R_2(\frac{1}{2})$. (In what follows we denote $R_2(\frac{1}{2})$ by $R_{\frac{1}{2}}$.)

The quotient $\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0^c(\Sigma_1)$ contains only imaginary rotations up to composition by $R_{\frac{1}{2}}$. By letting $\{(x + iy)_{\mathbb{Z}^2} \to (x + i(y + t))_{\mathbb{Z}^2}\} \longrightarrow 2\pi t$, we identify imaginary rotations by S^1. Then, rotations in $\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0^c(\Sigma_1)$ are identified by $S^1/_{\alpha\sim(\alpha+\pi)} \cong S^1$. Thus, we have $\pi_1(\mathit{Diff}_0(\Sigma_1)/\mathit{Diff}_0^c(\Sigma_1), [id]) = \mathbb{Z}$. □

5.3 Real Lefschetz chains

Let us consider a directed $\mathcal{RLF}$ over D^2 with only real critical values. We slice D^2 up into smaller discs, D_i, shown in Figure 5.10 such that over each D_i, we have an elementary $\mathbb{C}$-marked $\mathcal{RLF}$.

Let $r_0, r_1, r_2, \ldots, r_n$ be the real points on the boundaries of D_i (ordered with respect to the orientation of the real part of $(D^2, conj)$). We denote by c_i the real structure on Σ_g which is the pulled back from the real structure on F_{r_i}. Then we have $c_i \circ c_{i-1} = t_{a_i}$ where a_i denotes the corresponding vanishing cycle. As we have seen in the previous section that each $\mathbb{C}$-marked elementary $\mathcal{RLF}$ over D_i is determined by the isotopy class, $[c_i, a_i]$, of a real code. Hence, an $\mathcal{RLF}$ over D^2 with only real critical values gives a sequence of real codes $[c_i, a_i]$ satisfying $c_i \circ c_{i-1} = t_{a_i}$.

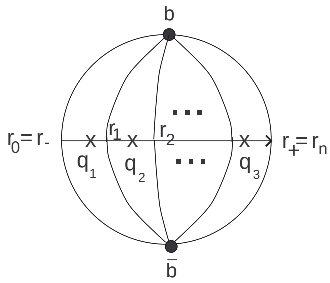


Fig. 5.10.

Definition 5.3.1. A sequence $[c_1, a_1], [c_2, a_2], ..., [c_n, a_n]$ of isotopy classes of real codes is called the *real Lefschetz chain* if we have $c_i \circ c_{i-1} = t_{a_i}$ for all $i = 2, ..., n$.

Theorem 5.3.2. *If $g > 1$, then there is a one-to-one correspondence between the real Lefschetz chains, $[c_1, a_1], [c_2, a_2], ..., [c_n, a_n]$ on Σ_g and the isomorphism classes of directed $\mathbb{C}$-marked genus-g real Lefschetz fibrations over D^2 with only real critical values.*

Proof. Above we have discussed how to associate a real Lefschetz chain to a class of directed $\mathbb{C}$-marked $\mathcal{RLF}$. As for the converse, we consider a real Lefschetz chain $[c_1, a_1], [c_2, a_2], ..., [c_n, a_n]$, by Theorem 4.2.4, we know that each code $[c_i, a_i]$ determines a unique isomorphism class of $\mathbb{C}$-marked elementary $\mathcal{RLF}s$. Using the boundary fiber sum, we glue these fibrations from left to right respecting the order determined by the chain. By Proposition 5.1.3 the boundary fiber sum is unique up to isomorphism if $g > 1$. □

When the total monodromy of a fibration $\pi : X \to D^2$ is the identity then we can consider the extension of it to a fibration $\hat{\pi} : \hat{X} \to S^2$. Two such extensions, $\hat{\pi} : \hat{X} \to S^2$ and $\check{\pi} : \check{X} \to S^2$, will be considered *isomorphic* if there is an equivariant orientation preserving diffeomorphism $H : \hat{X} \to \check{X}$ such that $\hat{\pi} = \check{\pi} \circ H$.

Proposition 5.3.3. *In $g > 1$ and $c_0 = c_1 \circ t_{a_1}$ is isotopic to c_n, then the fibration $\pi : X \to D^2$ can be extended uniquely up to isomorphism to a real Lefschetz fibration over S^2.*

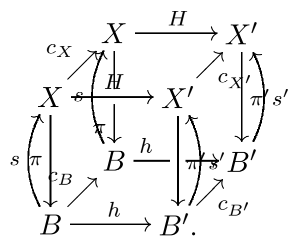
Proof. The real structure c_n is isotopic to c_0 if and only if the total monodromy, $c_n \circ c_0$, is isotopic to the identity hence we can glue to $\pi : X \to D^2$ a trivial real

Lefschetz fibration $\Sigma_g \times D^2$ (with the real structure $(c_n, conj)$) along their boundaries. This gives an extension of π over S^2. A trivial fibration glued to $\pi : X \to D^2$ defines an isotopy between c_0 and c_n hence an extension gives a path in the space of real structures connecting c_0 and c_n. The difference of two extensions give a loop in this space. Thus, the result follows from Proposition 5.2.4. □

5.4 Real elliptic Lefschetz fibrations with real sections and pointed real Lefschetz chains

Definition 5.4.1. Let $s : B \to X$ be a section of a real Lefschetz fibration $\pi : X \to B$. The section s is said to be *real* if $s \circ c_B = c_X \circ s$.

Two real Lefschetz fibrations $(\pi : X \to B, s)$ and $(\pi' : X' \to B', s')$ with a real section are called *isomorphic as fibrations with a real section* if there are orientation preserving diffeomorphisms $H : X \to X'$ and $h : B \to B'$ such that the following diagram commutes



If r denotes a real point on B, then we have $c(s(r)) = s(r)$ where c denotes the real structure on the fiber F_r.

Let us consider a directed $\mathbb{C}$-marked elementary $\mathcal{RELF}$ $(\pi : X \to D^2, \{b, \bar{b}\}, \{\rho, \bar{\rho}\})$ with a real section s. The section s defines a point $*$ (the pull back of the point $s(b)$) on Σ_1 such that if (c, a) is a real code then $c(*) = *$ and $*$ is disjoint from a. Such a real code will be called the *pointed real code.* Recall that the real code is determined up to an isotopy on Σ_1. Let $[c, a]^*$ denote the isotopy class of a pointed real code $(c, a)^*$, where the isotopy is taken relative to the point marked by the section. In other words, the pointed real code considered up to the action of the group $Diff_0{}^*(\Sigma_g)$, which is the connected component of the identity of the group $Diff^*(\Sigma_g)$ formed by the orientation preserving diffeomorphisms of Σ_g which keep fixed a marked point $*$.

Lemma 5.4.2. *The isotopy classes of pointed real codes $[c, a]^*$ classify the directed $\mathbb{C}$-marked elementary $\mathcal{RELF}s$ endowed with a real section.*

Proof. Above we have shown how we assign a pointed class $[c, a]^*$ to a given directed $\mathbb{C}$-marked elementary $\mathcal{RELF}$ (considered up to isomorphism of directed $\mathbb{C}$-marked $\mathcal{RELF}s$).

As for the converse, let us consider $[c, a]^*$ on Σ_1 with a distinguished point $*$. Let us consider the directed $\mathbb{C}$-marked elementary $\mathcal{RELF}$, $\pi : X \to D^2$, associated to the underlying isotopy class $[c, a]$. We will construct the section $s : D^2 \to X$ as follows. Let us consider a continuous family of paths $\alpha_r(t)$ on the upper half-disc of D^2 connecting the base point b to regular real points r of (D^2, c), see Figure 5.11.

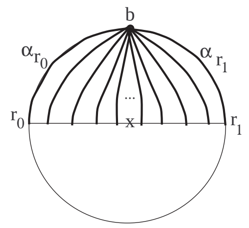


Fig. 5.11.

Using these paths we obtain a family of identifications $\rho_r : \Sigma_1 \to F_r$. Then by setting $s(r) = \rho_r(*)$ we obtain a section over the real part of D^2 except the singular fiber. Since the vanishing cycle a does not contain the distinguished point $*$, this section extends to the singular fiber.

The section s can be extended to real section over small neighborhood of the real part. This finishes the proof because the fibration over a small neighborhood of the real part of D^2, is homotopically the same as $\pi : X \to D^2$ as π has only real critical values. Note that changing the paths α_r up to homotopy, defines a directed $\mathbb{C}$-marked elementary $\mathcal{RELF}$ with a section associated to a real code $[c', a']^*$ such that $[c, a]^* = [c', a']^*$. $\square$

With a Lefschetz fibration over D^2 which has only real critical values and is endowed with a section, we associate a sequence $[c_1, a_1]^*, [c_2, a_2]^*, ..., [c_n, a_n]^*$ of isotopy classes of pointed real codes, such that $c_i \circ c_{i-1} = t^*_{a_i}$ for all $i = 2, ..., n$. Here $t^*_{a_i}$ denotes a Dehn twist as an element of $\mathit{Diff}^*(\Sigma_g)$. This kind of sequence is called *pointed real Lefschetz chain.*

Let us consider the subgroup $\mathit{Diff}_0^{c*}(\Sigma_g) \subset \mathit{Diff}_0{}^*(\Sigma_g)$ consisting of those diffeomorphisms which commute with c.

Lemma 5.4.3. $\pi_1(\mathit{Diff}_0{}^*(\Sigma_1)/\mathit{Diff}_0^{c*}(\Sigma_1), [id]) = 0.$

Proof. Basically we repeat the idea of the proof of Lemma 5.2.2. Note that $\mathit{Diff}_0{}^*(\Sigma_1)$ can be identified with $\mathit{Diff}_0(\Sigma_1 \setminus \{pt\})$. The latter is known to be contractible by [EE]. Moreover, $\mathit{Diff}_0^{c*}(\Sigma_1)$ is a connected subgroup of $\mathit{Diff}_0{}^*(\Sigma_1)$. □

Theorem 5.4.4. *If $g = 1$, then there is a one to one correspondence between the pointed real Lefschetz chains, $[c_1, a_1]^*, [c_2, a_2]^*, \dots, [c_n, a_n]^*$, on Σ_1 and the isomorphism classes of directed $\mathbb{C}$-marked real Lefschetz fibrations over D^2 endowed with a real section and having only real critical values.*

Proof. The proof is analogous to the proof of Theorem 5.3.2 and it follows from Lemma 5.4.2 and Lemma 5.4.3. □

Proposition 5.4.5. *If $c_0 = c_1 \circ t_{a_1}$ is isotopic to c_n then there is a unique extension of $\pi : X \to D^2$ to a fibration with a section over S^2.*

Proof. The proof is analogous to the proof of Proposition 5.3.3. The result follows from Lemma 5.4.3. □

Remark 5.4.6. In fact, if two real Lefschetz fibrations with only real critical values and with a real section are isomorphic then they are isomorphic as fibrations with a real section. The result follows from the observation that any two sections can be carried to each other (without changing the isomorphism type of the fibration) by the twist transformations, T_N and *double* $T_{N_{sing}}$ which we introduce in the next section.

5.5 Real elliptic Lefschetz fibrations without real sections

Let us recall that the boundary fiber sum of two $\mathbb{C}$-marked $\mathcal{RELF}s$ without a real section is not well-defined already because there is no canonical way to carry real codes $[c_i, a_i]$ to the surface Σ_g. So, in this section, we consider the boundary fiber sum of directed non-marked $\mathcal{RLF}s$. We show that for some elementary $\mathcal{RLF}s$ the boundary fiber sum is well-defined.

Definition 5.5.1. Let $\pi' : X' \to D^2$ and $\pi : X \to D^2$ be two directed non-marked $\mathcal{RLF}s$. We consider fibers, F'_+ and F_- of π' and π over the real points r'_+ and r_-, respectively. Let us assume that the real structure $c'_+ : F'_+ \to F'_+$ is equivalent to $c_- : F_- \to F_-$, or in the other words, there is an orientation preserving equivariant diffeomorphism $\phi : F'_+ \to F_-$. Then we define the *boundary fiber sum of non-marked* $\mathcal{RLF}s$, $X'\natural_{F,\phi}X \to D^2$, using the identification of the fibers F'_+ and F_- via ϕ.

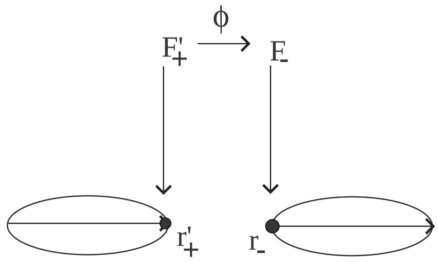


Fig. 5.12.

The boundary fiber sum does depend on the choice of ϕ, however, there is the following (well-known and simple) criterion for a pair of such diffeomorphisms ϕ and ψ to give isomorphic fibrations.

Lemma 5.5.2. *The boundary fiber sums defined via equivariant diffeomorphisms* $\phi, \psi : F'_+ \to F_-$ *are isomorphic, if* $\psi \circ \phi^{-1} : F_- \to F_-$ *can be extended to an equivariant diffeomorphism of* $X \to D^2$*, or if* $\phi^{-1} \circ \psi : F'_+ \to F'_+$ *can be extended to an equivariant diffeomorphism of* $X' \to D^2$*.* □

We will call these two cases the *right extendibility* and the *left extendibility* respectively.

The results in the previous chapter yield a condition for the right (and similarly, for the left) extendibility in the case of elementary $\mathcal{RLF}s$. Namely, $\psi \circ \phi^{-1} : F_- \to F_-$ can be extended to an equivariant diffeomorphism of an elementary $\mathcal{RLF}$, $X \to D^2$, if and only if $\psi \circ \phi^{-1}$ takes the vanishing cycle, a, of X to a curve which is equivariantly isotopic to a.

Lemma 5.5.3. *Let $g(F) = 1$. Then,*

- *if a real structure c on F has 1 real component, then F contains a unique c-equivariant isotopy class of totally imaginary curves, a unique c-equivariant isotopy class of non-contractible real-imaginary curves, and one real curve,*
- *if c has 2 real components, then there is a unique c-equivariant isotopy class of non-contractible real-imaginary curves, no totally imaginary curves, and two real curves,*
- *if c has no real components, then there exist two c-equivariant isotopy classes of totally imaginary curves, but no real and real-imaginary curves.*

Proof. If c has 1 real component, then the quotient F/c which is a Möbius band. The quotient of a totally imaginary curve is a simple closed curve in F/c homologous to the central curve of the band. Such curve has to be isotopic to the central curve. The quotient of a real-imaginary curve is an arc connecting two boundary points on F/c. There is a unique isotopy class of such arcs which are not contractible. Namely, such arcs are isotopic to the fibers of the standard fibration of the Möbius band, $F/c \to S^1$ (see 5.13).

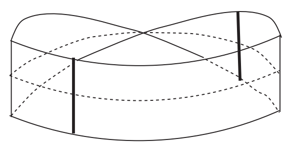


Fig. 5.13.

If c has 2 real components, then F/c is an annulus and the quotient of a real-imaginary curve is a simple arc. It connects the opposite boundary components of F/c if the curve is non-contractible. Such arcs are also obviously all isotopic.

If c has no real component, then F/c is the Klein bottle which can be viewed as a pair of Möbius bands glued along their boundaries. The two central curves of these two Möbius bands represent the quotients of the two c-equivariantly non-isotopic totally imaginary curves in F. □

Lemma 5.5.3 implies that the boundary fiber sum of elementary non-marked $\mathcal{RELF}s$ may be not well-defined only in two cases: if c has 2 real components and a is real, or if c has no real components and a is totally imaginary. In these cases there are two c-equivariant isotopy classes of curves a, and we will be calling a pair of representatives of different classes *c-twin curves*. Note that the imaginary rotation $R_{\frac{1}{2}}$ (introduced in the proof of Proposition 5.2.4) switches the c-twin curves. Hence, c-twin curves can be carried to each other via equivariant diffeomorphisms, although they are not equivariantly isotopic. Thus a diffeomorphism on a real fiber which switches the c-twin curves can not be extended to a fibration over D^2. This shows that in the above two cases there is an ambiguity in the definition of the boundary fiber sum $X' \natural X$: it can be defined in two ways, and to resolve the ambiguity we should specify how we identify the c-twin curves in the fiber F'_+ in X' with the c-twin curves in the fiber F_- in X.

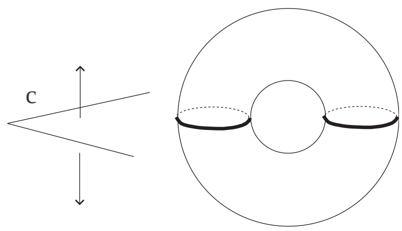


Fig. 5.14.

However in certain cases the problem of switching c-twin curves can be eliminated. For this reason we consider the following definition.

Definition 5.5.4. Let $\pi : X \to D^2$ be a directed $\mathcal{RELF}$. We consider a real slice N of D^2 which contains no critical value, shown in Figure 5.15.

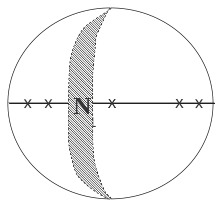


Fig. 5.15.

Let $\xi : I \times I \to N$, $I = [0, 1]$ be an orientation preserving diffeomorphism such that first interval correspond to the real direction on N. The fibration over N has no singular fiber hence it is trivializable.

Let us consider a trivialization $\Xi : \Sigma_1 \times I \times I \to \pi^{-1}(N)$ such that the following diagram commutes

$$\begin{array}{ccc} \Sigma_1 \times I \times I & \xrightarrow{\ \Xi\ } & \pi^{-1}(N) \\ \downarrow & & \downarrow \pi \\ I \times I & \xrightarrow{\ \xi\ } & N. \end{array}$$

Note that since N has no critical value the isotopy type of the real structure on the fibers over the real part of N is constant. If the real structure c has 2 real components then we consider the model $\varrho : \mathbb{C}/\mathbb{Z}^2 \to \Sigma_1$ and set

$$\bar{\varrho} = (\varrho, id) : \mathbb{C}/\mathbb{Z}^2 \times I \times I \to \Sigma_1 \times I \times I$$

then we consider the map,

$$T' : \mathbb{C}/\mathbb{Z}^2 \times I \times I \to \mathbb{C}/\mathbb{Z}^2 \times I \times I$$

such that $T'((x + iy)_{\mathbb{Z}^2}, t, s) = ((x + t + iy)_{\mathbb{Z}^2}, t, s)$. Then let

$$T_N = \Xi \circ (\bar{\varrho} \circ T' \circ \bar{\varrho}^{-1}) \circ \Xi^{-1} : \pi^{-1}(N) \to \pi^{-1}(N).$$

Since at $t = 0, 1$, T_N is the identity we can extend T_N to X by the identity outside of $\pi^{-1}(N)$. The map T_N is called a *twist of an $\mathcal{RELF}$ over N*.

If c has 1 real component then we can construct the twist T_N using $\varrho : \mathbb{C}/\Lambda \to \Sigma_1$; similarly if c has no real component then we repeat the same using $\varrho : \mathbb{R}^2/\mathbb{Z}^2 \to \Sigma_1$ (introduced in the previous section).

Remark 5.5.5. 1. Since the twist T_N is defined by a real rotation, T_N preserves the isomorphism class of the real Lefschetz fibration.

2. The map T_N depends only on the isotopy type of $\pi^{-1}(N)$.

One can define an equivariant twist for a slice N_{sing} which contains only one critical value where the corresponding vanishing cycle is real-imaginary. Let us divide the boundary of N_{sing} into to two pieces: left and right boundaries (left/ right being determined by the direction). Note that since the vanishing cycle is real-imaginary, the real structures on the fibers over real boundary points of N_{sing} have 1 real component on one side and 2 real components on the other side. Let us assume that the real structure on the fiber over the left boundary point has 1 real component.

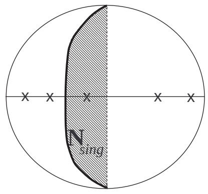


Fig. 5.16.

To construct $T_{N_{sing}}$ we consider the following well-known model for elementary elliptic fibrations. Let $\hat{\Omega} = \{z|\,|Re(z)| \leq \frac{1}{2}, Im(z) \geq 1\} \cup \infty$, (the subset bounded by $Im(z) \geq 1$ of the one point compactification of the standard fundamental domain $\{z|\,|Re(z)| \leq \frac{1}{2}, |z| \geq 1\}$ of the modular action on $\mathbb{C}$, see Figure 5.17.)

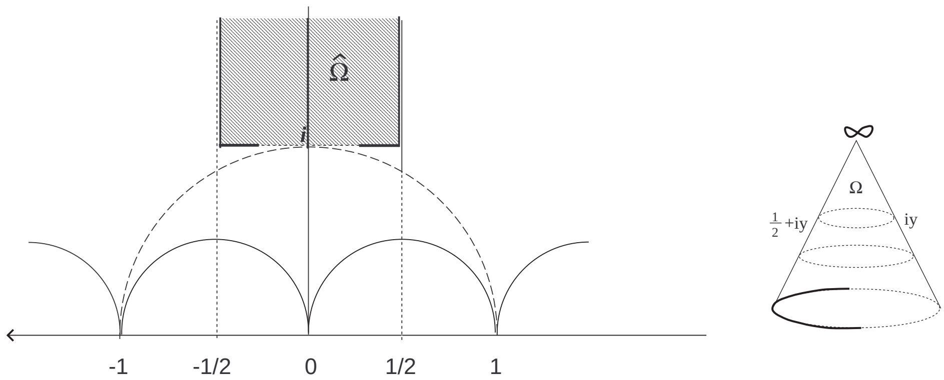


Fig. 5.17.

We consider the real structure $c_{\hat{\Omega}} : \hat{\Omega} \to \hat{\Omega}$ such that $c_{\hat{\Omega}}(\omega) = \overline{-\omega}$. Let Ω denote the quotient $\hat{\Omega}\diagup_{\frac{1}{2}+iy\sim-\frac{1}{2}+iy}$. The real structure $c_{\hat{\Omega}}$ induces a real structure on Ω. Note that Ω is a topological real disc and can be identified with N_{sing} so that the real part of N_{sing} corresponds to the union of the half-lines iy and $\frac{1}{2} + iy$ where $y \geq 1$. For any $\omega \in \Omega$, the fiber over ω is given by $F_\omega = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, where the fiber F_∞ has a required nodal type singularity.

Let $\pi_\Omega : X_\Omega \to \Omega$ denote the fibration such that $\forall \omega \in \Omega$ we have $\pi_\Omega^{-1}(\omega) = F_\omega = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$. Then we consider the translation T'_{sing} defined by

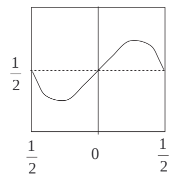
$$\begin{array}{rlcl} T'_{sing}: & X_\Omega & \to & X_\Omega \\ & z_\omega \in F_\omega & \to & (z + \tau(w))_\omega \in F_\omega \end{array}$$

where z_ω denotes the equivalence class of z in $\mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$ and $\tau : \Omega \to \Omega$ such that

$$\tau(\omega) = -\frac{1}{2} + (\frac{1}{2} - f(Re(\omega)) + i)exp(-Im(\omega) + 1)$$

for some smooth mapping $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ satisfying the following properties:

- $f(0) = \frac{1}{2}$ (modulo $\mathbb{Z}$),
- $f(1-x) = 1 - f(x)$, ($\Rightarrow f(\frac{1}{2}) = \frac{1}{2}$) (modulo $\mathbb{Z}$),
- f is linear on $[\frac{1}{4}, \frac{3}{4}]$ (modulo $\mathbb{Z}$).



The graph of f

Note that τ has the following properties. (Equations are considered modulo the relation $-\frac{1}{2} + iy \sim \frac{1}{2} + iy$, $y \geq 1$.)

- $\tau(\overline{-\omega}) = \overline{-\tau(\omega)}$,
- $\tau(\infty) = \frac{1}{2}$,
- $\tau(\frac{1}{2} + iy) = -\frac{1}{2} + iexp(-y+1) = \frac{1}{2} + iexp(-y+1)$,

in particular, if $y = 1$ then $\tau(\frac{1}{2} + i) = \frac{1}{2} + i$,

- $\tau(iy) = -\frac{1}{2} + iexp(-y+1) = \frac{1}{2} + iexp(-y+1)$,

in particular, if $y = 1$ then $\tau(i) = \frac{1}{2} + i$.

Let $T_{N_{sing}}$ denote the twist on $\pi^{-1}(N_{sing})$ induced from the twist T'_{sing} on X_Ω. By definition $T_{N_{sing}}$ is equivariant and is the identity over the left boundary and half rotation on the right boundary component of N_{sing}.

Lemma 5.5.6. *Let $\pi' : X' \to D^2$ and $\pi : X \to D^2$ be two non-marked elementary $\mathcal{RELF}$s such that both c'_+ and c_- have 2 real components. We assume that the vanishing cycle a of π is real with respect to c_-. Then boundary fiber sum $X' \natural_F X \to D^2$ is well-defined if the vanishing cycle a' of π' is real-imaginary with respect to c'_+.*

Proof. Let ϕ and ψ be two equivariant diffeomorphism of F_+ such that

$$\phi \in \mathit{Diff}_0(F_+, c) \text{ and } \psi = \phi' \circ R_{\frac{1}{2}} \text{ where } \phi' \in \mathit{Diff}_0(F_+, c).$$

As we have discussed in the beginning of this section that the boundary fiber sums $X'\natural_{F,\phi}X \to D^2$ and $X'\natural_{F,\psi}X \to D^2$ obtained using diffeomorphisms ϕ and ψ may not give isomorphic fibrations, since two gluing diffeomorphisms belong to different components of $Diff_0^c(F_+)$.

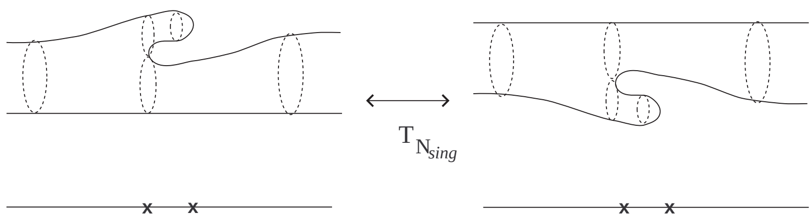


Fig. 5.18. The action of $T_{N_{sing}}$ on the real part.

As the vanishing cycle of π' is real-imaginary we can apply $T_{N_{sing}}$ to X'. At the singular fiber $T_{N_{sing}}$ acts as half rotation, hence the fiber $T_{N_{sing}}(F')_-$ differs from the fiber F'_- by the rotation $R_{\frac{1}{2}}$. Therefore, $X'\natural_{F,\phi}X$ is isomorphic to $T_{N_{sing}}(X')\natural_{F,\phi\circ R_{\frac{1}{2}}}X$ which is isomorphic to $X'\natural_{F,\psi}X \to D^2$. $\square$

Remark 5.5.7. Let $\pi : X \to S^2$ be a real elliptic Lefschetz fibration with only real critical values. Let s and s' be two real sections on $X \to D^2$. Using the twists T_N and *double* $T_{N_{sing}}$ we can modify the section s, over the intervals where s' differs from s, see Figure 5.20. The *double twist* operation is defined for real Lefschetz fibrations with two critical values where the corresponding vanishing cycles are both real-imaginary. The model we use to define the *double twist* is obtained as follows. Let us consider the disc with two critical values as the double cover of a disc with one critical value (where the corresponding vanishing cycle is real-imaginary) branched at a regular real point. Let N_{sing-} and N_{sing+} denote the two corresponding copies of N_{sing} on the branched cover. By pulling back the fibration X_Ω over N_{sing}, we obtain a model fibration over $N_{sing-} \cup N_{sing+}$ where the vanishing cycles are real-imaginary. Thus, we can apply $T_{N_{sing}}$ at the same time to fibrations over N_{sing-} and N_{sing+}. This way we obtain a twist which is identity over the boundary of $N_{sing-} \cup N_{sing+}$ and a half twist over the common boundary of N_{sing-} and N_{sing+}.

We use *double* $T_{N_{sing}}$ to modify the section around the two neighboring singular fibers with real-imaginary vanishing cycles. Possible modification on the real part is shown in Figure 5.20.

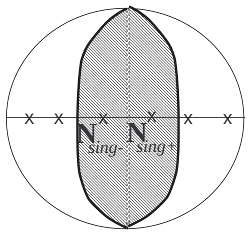


Fig. 5.19.

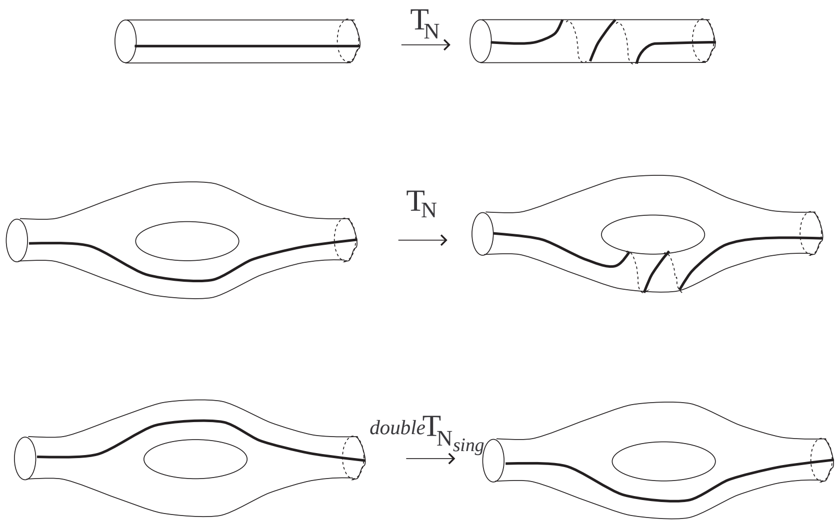


Fig. 5.20. Modification of the real section over the real part of D^2.

Using T_N and *double* $T_{N_{sing}}$, we obtain an isomorphism (as fibrations with a section) of $(\pi : X \to D^2, s)$ and $(\pi : X \to D^2, s')$. Since T_N and *double* $T_{N_{sing}}$ do not change $(\pi : X \to D^2, s')$ outside some slices of D^2, if π can be extended to a fibration over S^2, then extensions of s and s' match, by Lemma 5.4.3.

5.6 Weak real Lefschetz chains

Let us now consider a directed non-marked $\mathcal{RELF}$ over D^2 with only real critical values, $q_1 < q_2 < ... < q_n$. Around each critical value q_i we choose a real disc D_i such that $D_i \cap \{q_1, q_2, ..., q_n\} = \{q_i\}$ and each $D_i \cap D_{i+1} = \{r_i\} \subset [q_i, q_{i+1}]$. Each (non-marked) fibration over D_i is classified by the conjugacy class $\{c_i, a_i\}$ of the real code. Thus we obtain a sequence $\{c_1, a_1\}, \{c_2, a_2\}, ..., \{c_n, a_n\}$ such that $c_i \circ t_{a_i}$ is conjugate to c_{i-1} for all $i = 2, ..., n$. We will call this sequence the *weak real Lefschetz chain*. Clearly, weak real Lefschetz chains are invariants of directed non-marked $\mathcal{RELF}s$ over disc with only real critical values.

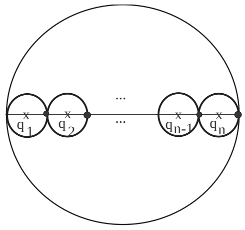


Fig. 5.21.

The discussion about well-definedness of boundary fiber sum shows that weak Lefschetz chains are not sufficient for the classification of the directed $\mathcal{RELF}s$ over D^2 with only real critical values. An additional information is needed if for some i, the real structure c_i has no real component or c_i has 2 real components and vanishing cycles corresponding to the critical values q_i and q_{i+1} are real with respect to c_i.

We fix the fiber F_{r_i} over a real point r_i and consider the vanishing cycles a_i and a_{i+1} on F_{r_i}, corresponding to critical values q_i and q_{i+1}, respectively. When the real structure c_i has no real component then both a_i and a_{i+1} are necessarily totally imaginary with respect to c_i. Either these curves are the same or they are the c_i-twin curves, see Figure 5.22.

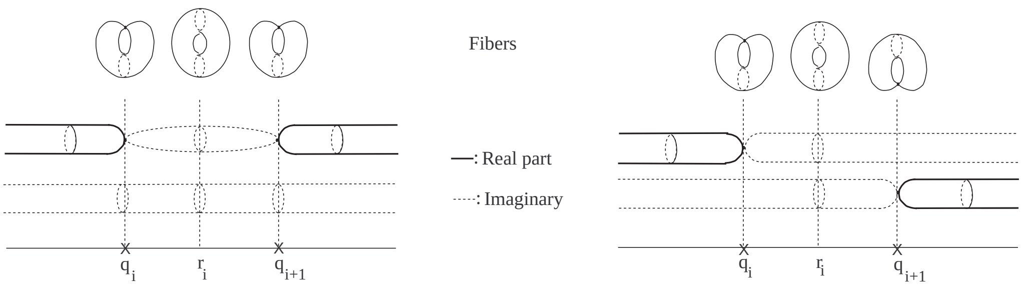


Fig. 5.22.

Similarly, if c_i has 2 real components and both a_i and a_{i+1} on F_{r_i} are real with respect to c_i then either a_i and a_{i+1} are the same curve or they are the c_i-twin curves on F_{r_i}. Note that when both vanishing cycles are the same curve on F_{r_i} then the fibration admits a section over $[q_i, q_{i+1}]$, otherwise there is no such section, see Figure 5.23.

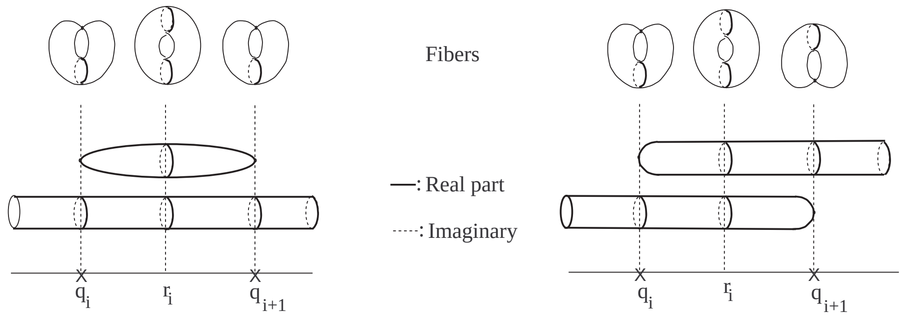


Fig. 5.23.

In the above situations if a_i and a_{i+1} are c_i-twin curves then we mark r_i by r_i^R. (Notation refers to imaginary rotation $R_{\frac{1}{2}}$, since one can switch the vanishing cycle by applying to the imaginary rotation $R_{\frac{1}{2}}$). Then we decorate the weak real Lefschetz chain by marking classes $\{c_i, a_i\}^R$ corresponding to the marked points. The weak Lefschetz chain we obtain is called the *decorated weak real Lefschetz chain.*

Theorem 5.6.1. *There exists a one-to-one correspondence between the decorated weak real Lefschetz chains and the isomorphism classes of directed non-marked real elliptic Lefschetz fibrations over D^2 with only real critical values.*

Proof. Above we discuss how to assign a decorated weak Lefschetz chain to a directed non-marked $\mathcal{RELF}$. As for the converse, we consider a decorated weak real Lefschetz chain. Each real code $\{c_i, a_i\}$ gives a unique class of directed non-marked elementary $\mathcal{RELF}s$ then we consider boundary fiber sums respecting the decoration from left to right with the order determined by the chain. We obtain unique real Lefschetz fibration up to isomorphism since boundary fiber sum is determined uniquely by the decoration. □

If $c_1 \circ t_{a_1}$ is conjugate to c_n then we can consider an extension of $\pi : X \to D^2$ to a fibration over S^2. As before, in case when c_n has 2 real components and neither a_1 nor a_n is a real-imaginary curve or when c_n has no real component a decoration at infinity will be needed.

Proposition 5.6.2. *If c_n has 2 real components and either a_1 or a_n is real-imaginary or if c_n has 1 real component then there exists a unique extension.*

Otherwise, there are two extensions distinguished by the decoration at infinity.

Proof. Let $\pi : X \to D^2$ be the directed $\mathcal{RELF}s$ associated to a given decorated weak real Lefschetz chain. An extension of π to a fibration over S^2 defines a trivialization, $\phi : \Sigma_1 \times S^1 \to \pi^{-1}(\partial D^2)$ over the boundary ∂D^2. Two trivializations ϕ, ϕ' correspond to isomorphic real fibrations if $\phi^{-1} \circ \phi' : \Sigma_1 \times S^1 \to \Sigma_1 \times S^1$ can be extended to an equivariant diffeomorphism of $\Sigma_1 \times D^2$ with respect to the real structure $(c_n, conj) : \Sigma_1 \times D^2 \to \Sigma_1 \times D^2$. Let $\Phi_t = (\phi^{-1} \circ \phi')_t : \Sigma_1 \to \Sigma_1$, $t \in S^1$. Since there is no fixed marking, up to change of marking we assume that $\Phi_t \in \mathit{Diff}_0(\Sigma_1)$.

The real structure splits the boundary into two symmetric pieces, so instead of considering an equivariant map over the entire boundary we consider a diffeomorphism over one the symmetric pieces. Let $\Phi_t, t \in [0, 1]$ denote the family of such diffeomorphisms. The family, Φ_t, $t \in [0, 1]$ defines a path in $\mathit{Diff}_0(\Sigma_1)$ whose end points lie in $\mathit{Diff}_0^{c_n}(\Sigma_1)$, thus Φ_t defines a relative loop in $\pi_1(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_n}(\Sigma_1))$. We will be interested in the contractibility of this relative loop.

As we have calculated in Section 5.2 we have $\pi_1(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_n}(\Sigma_1)) = \mathbb{Z}$. However, there is a way to modify Φ_t without changing the isomorphism class of the $\mathcal{RELF}$ such that Φ_t is transformed to a contractible relative loop. The proposition follows from Lemma 5.6.3 below. □

First, let us consider the exact sequence of the pair $(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_n}(\Sigma_1))$ $\ldots \to \pi_1(\mathit{Diff}_0^{c_n}) \to \pi_1(\mathit{Diff}_0) \xrightarrow{f} \pi_1(\mathit{Diff}_0, \mathit{Diff}_0^{c_n}) \xrightarrow{g} \pi_0(\mathit{Diff}_0^{c_n}) \xrightarrow{h} \pi_0(\mathit{Diff}_0) \to \pi_0(\mathit{Diff}_0, \mathit{Diff}_0^{c_n}) \to 0$.

In case when c_n is an odd real structure, $\mathit{Diff}_0^{c_n}(\Sigma_1)$ is connected so map h is injective hence g is the zero map which implies that f is surjective. Hence any path in $\pi_1(\mathit{Diff}_0(\Sigma_1), \mathit{Diff}_0^{c_n}(\Sigma_1), [id])$ can be seen as a loop in $\pi_1(\mathit{Diff}_0(\Sigma_1), id)$. The following Lemma shows that any loop in $\pi_1(\mathit{Diff}_0(\Sigma_1), id)$ can be written in terms of transformations T_{N_i}, for some regular slices N_i.

In other cases, $\mathit{Diff}_0^{c_n}(\Sigma_1)$ has two components. Let us mark one of the components. Then the map h restricted to the marked component is injective. Hence g is the zero map and f is surjective over the marked component of $\mathit{Diff}_0^{c_n}$. Note that decoration of real Lefschetz chain distinguishes one of the component of $\mathit{Diff}_0^{c_n}(\Sigma_1)$ hence marking one component or other give the two different extension determined by the decoration.

In the case c_n has 2 real components and either a_0 or a_n is real-imaginary, the transformation $T_{N_{sing}}$ changes one marking to other.

Lemma 5.6.3. *Let us assume that $\pi : X \to D^2$ has at least one real-imaginary vanishing cycle. Then there exists a generating set for $\pi_1(Diff_0(\Sigma_1), id) = \mathbb{Z} + \mathbb{Z}$ consisting of transformations T_{N_i} for some nonsingular slices N_i.*

Proof. Let a_i denote the real-imaginary vanishing cycle and q_i corresponding critical value. Let N_-, N_+ be two nonsingular slices of D^2 intersecting the real part (q_{i-1}, q_i) and (q_i, q_{i+1}), respectively. Let r_- and r_+ be left boundary points of N_- and N_+ shown in Figure 5.24, and $c_\pm$ be the real structures on the fibers $\pi^{-1}(r_\pm)$.

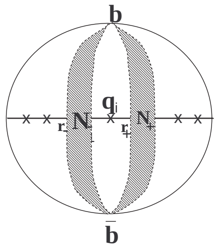


Fig. 5.24.

Since the vanishing cycle is real-imaginary, the real structures on the nearby regular fiber can have either 1 or 2 real components. Let us assume that the real structure over (q_{i-1}, q_i) has 2 real components. (The other case can be treated similarly.)

Let us choose an auxiliary $\mathbb{C}$-marking $(\{b, \bar{b}\}, \{\rho : \Sigma_1 \to F_b, \bar{\rho} : \Sigma_1 \to F_{\bar{b}}\})$. We will also fix an identification $\varrho : S^1 \times S^1 \to \Sigma_1$ of Σ_1 with $S^1 \times S^1$. Since c_- has 2 real components, we can assumed that the induced real structure on $S^1 \times S^1$ is the reflection $(\alpha, \beta) \to (\alpha, -\beta)$. Then real part consists of the curves $C_1 = (\alpha, 0)$ and $C_2 = (\alpha, \pi)$. Since a_i is real-imaginary a representative can be chosen as $(0, \beta)$. By Theorem 3.1.2, we have $c_+ = t_{a_i} \circ c_-$ on $S^1 \times S^1$. Then the real part of c_+ is the curve C_3, given homologically by $2C_1 - a_i$, see in Figure 5.25.

Since C_3 intersects C_1 at one point. We can identify $\Sigma_1 = C_1 \times C_3$. Then rotations along C_1, C_3 generates $Diff_0(\Sigma_1)$. Hence $T_{N_\pm}$ generates $\pi_1(Diff_0(\Sigma_1))$. $\square$

Remark 5.6.4. The assumption that the fibration admits a real-imaginary vanishing cycle is not restrictive. In fact, every real elliptic Lefschetz fibration over S^2 with only real critical values has at least one real-imaginary vanishing cycle. This can be seen easily by analysis of the homology monodromy which will be discussed in next chapter (Corollary 6.10.3).

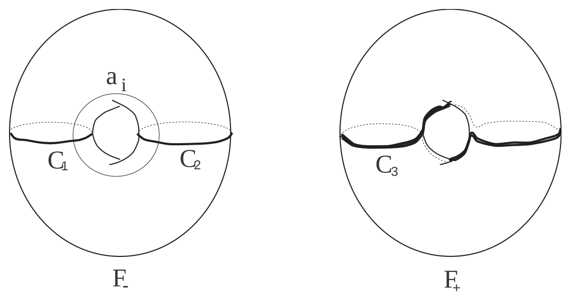


Fig. 5.25.

Theorem 5.6.1 applies naturally to directed non-marked $\mathcal{RELF}s$ over D^2 which admit a real section. Since there is a real section weak Lefschetz chain does not contain a real code $[c_i, a_i]$ with a real structure which has no real component. In addition, if the real structure has 2 real components and the vanishing cycle is real the decoration is not needed, since existence of a real section defines uniquely the gluing of two directed non-marked elementary $\mathcal{RELF}s$ over D^2. Similarly, the extension to a fibration over S^2 is uniquely defined.

Proposition 5.6.5. *Two directed $\mathcal{RELF}s$ over S^2 admitting a section and having the same weak Lefschetz chain up to cyclic ordering are isomorphic.* □

CHAPTER 6

NECKLACE DIAGRAMS

6.1 Real locus of real elliptic Lefschetz fibrations with real sections

Let $\pi : X \to S^2$ be a directed $\mathcal{RELF}$ admitting a real section, and $\pi_{\mathbb{R}} : X_{\mathbb{R}} \to S^1$ the restriction of π to the real part, $X_{\mathbb{R}}$, of X. Since π has a real section, none of the fibers of $\pi_{\mathbb{R}}$ is empty. As a consequence, topologically regular fibers of $\pi_{\mathbb{R}}$ are either two copies of S^1 (this happens if the real fiber of π has two real components) or a copy of S^1 (this happens if the real fiber π has one real component). There are two types of singular fibers of π_R: topologically either a disjoint union of a circle and an isolated point or a wedge of two circles. In the first case, the singularity, called a *solitary double point*, appear as a local maximum (the local model $-x_1^2 - x_2^2$), or a local minimum (the local model $x_1^2 + x_2^2$) of $\pi_{\mathbb{R}}$, while in the second case, the singularity is called a *crossing double point* and appear as a saddle critical point (the local model $\pm(x_1^2 - x_2^2)$) of $\pi_{\mathbb{R}}$.

The isotopy type of the real structures and in particular the topology of the fibers of $\pi_{\mathbb{R}}$ over its *regular intervals* (between the pairs of neighboring critical points) is constant.

Definition 6.1.1. A regular interval $I \subset S^1$ is called *odd* if the real structure over I is an odd real structure, and otherwise is called *even.*

Lemma 6.1.2. *The topology of the regular fibers of* $\pi_{\mathbb{R}}$ *alternates as we pass through a critical value.*

Proof. Let c_{i-1} and c_i be the real structures on the fibers over the points neighboring a critical value, q_i, and a_i the vanishing cycle corresponding to q_i.

If a_i is real with respect to c_{i-1}, then by Lemma 4.3.1, a_i is totally imaginary with respect to $c_i = c_{i-1} \circ t_{a_i}$, vice versa. Therefore, the number of real components increase or decrease by 1.

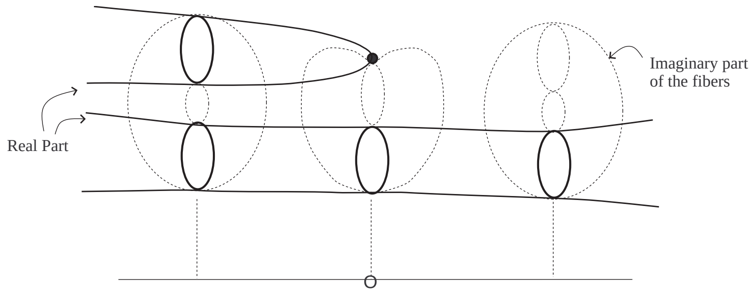


Fig. 6.1.

If a_i is real-imaginary with respect to c_{i-1}, then there are two cases: either c_{i-1} has two real components and a_i intersects each of the real components at one point or c_{i-1} has one component and a_i intersects the real curve at two points. In fact, the latter case can be seen as the inverse of the former case with respect to the direction of S^1. So, it will be sufficient to give a proof for the former case.

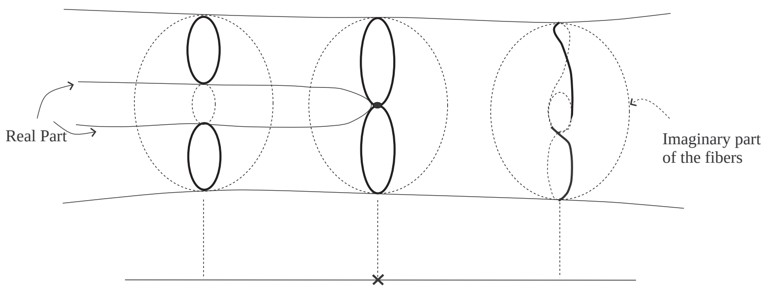


Fig. 6.2.

Note that in the former case, after the Dehn twist along a_i, two real components are connected to each other and form an invariant curve. Since a Dehn twist is the identity map outside a neighborhood of a_i, the real structure c_i acts as the identity on the pieces of this curve, so it should act as the identity on the whole curve. Hence we obtain one real curve which intersects the vanishing cycle a_i at two points. □

On S^1 (the base of $\pi_{\mathbb{R}}$), we will mark the critical values corresponding to the solitary double points by $\circ$ and those corresponding to the crossing double points by $\times$. Moreover, we mark the regular intervals over which fibers of $\pi_{\mathbb{R}}$ have two components by sketching an extra edge, like is shown on Figure 6.3. Evidently, the decoration we obtain is an invariant of real Lefschetz fibrations. We call S^1 together with such a decoration an *uncoated necklace diagram*.

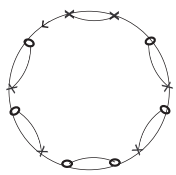
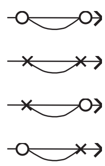


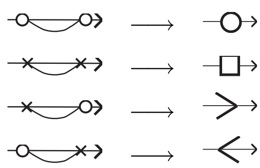
Fig. 6.3.

Remark 6.1.3. Since the decoration of S^1 determines the vanishing cycle and the real structure up to conjugation, uncoated necklace diagrams give a geometric interpretation of weak Lefschetz chains, (up to cyclic ordering).

Let us mark an odd interval on $S^1 \setminus \{\text{critical set}\}$. Then with respect to the marked interval, we have 4 basic positions.



We introduce the following notation for the even intervals.



Thus, we modify the decoration of a circle and call the object we get an *oriented necklace diagram* associated to a directed real elliptic Lefschetz fibration with a real section. We call the elements of the set $\{\bigcirc, \square, >, <\}$ *necklace stones* and the circle *necklace chain* of the necklace diagram. Two oriented necklace diagrams are considered identical if they contain the same types of stones going in the same cyclic order.

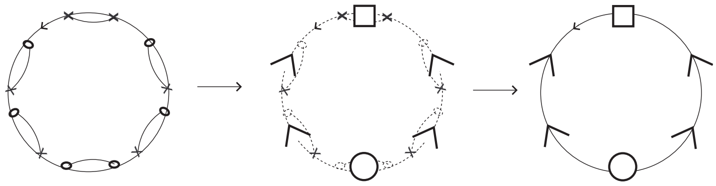


Fig. 6.4. Uncoated necklace diagram.

A necklace diagram is called *non-oriented*, if the orientation of its chain is not fixed. Such diagrams are invariants of non-directed $\mathcal{RELF}s$ admitting a real section. Fixing an orientation, we can obtain a pair of oriented necklace diagrams related by a mirror symmetry. Thus, non-oriented necklace diagrams will be considered up to symmetry.

Note that although (oriented) necklace diagrams can be defined for any directed real elliptic Lefschetz fibration which admits a real section, to be able to obtain a one to one correspondence we will concentrate ourselves on those fibrations whose critical values are all real.

6.2 Monodromy representation of stones

Any real structure, $c : \Sigma_1 \to \Sigma_1$, induces a homomorphism c_* on $H_1(\Sigma_1, \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$ which defines two rank 1 subgroups $H^c_\pm = \{[a] : [a]c_* = \pm[a]\}$ of $H_1(\Sigma_1, \mathbb{Z})$. (Here, $[a]c_*$ denotes $c_*[a]$.) For any real structure c, the subspaces $H^c_\pm$ is nonempty. When the real structure c has two real components, we have $H_1(\Sigma_1, \mathbb{Z}) = H^c_+ + H^c_-$. Otherwise any element of $H_1(\Sigma_1, \mathbb{Z})$ can be written as a linear combination of generators of $H^c_\pm$ where the coefficients taken from the set $\frac{1}{2}\mathbb{Z} = \{\frac{1}{2}m : m \in \mathbb{Z}\}$. Vanishing cycles corresponding to the critical value of type $\circ$ are either real or totally imaginary, hence they give a generator for the subspace H^c_+. On the other hand, vanishing cycles corresponding to the critical values of type $\times$ are real-imaginary so they give a generator of the subspace H^c_-.

Let q be a critical value and c and c' be the real structures on the fibers over $q - \epsilon$ and $q + \epsilon$, respectively, where ϵ is a sufficiently small positive real number. We will call c and c' as *left-hand* and *right-hand* real structure, respectively.

Let $< [a] >= H_+^c$ and $< [b] >= H_-^c$; similarly, $< [a'] >= H_+^{c'}$ and $< [b'] >= H_+^{c'}$. To each critical value, q, we assign the transition matrix, P_q, defined up to sign, such that $([a],[b])P_q = ([a'],[b'])$.

There are two types of critical values. For each type there are two cases distinguished by the direction.

Lemma 6.2.1. *Up to a sign, we obtain the following matrices*

$$\begin{aligned} P_{(-\times<)} &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, & P_{(>\times-)} &= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \\ P_{(-\circ<)} &= \tfrac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, & P_{(>\circ-)} &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}. \end{aligned}$$

Proof. We give the proof for one of the four cases, say $P_{(-\times<)}$. (Calculations for other cases are analogous.)

Recall that, in this case, the vanishing cycle is a real-imaginary curve and hence, gives a generator of H_-^c. Let us denote the vanishing cycle by b, so that we have $< [b] >= H_-^c$. Then, we choose a generator $[a]$ for H_+^c such that $[a] \circ [b] > 0$. Since c is an odd real structure, $[a] \circ [b] = 2$. By Theorem 3.1.2, we have $c' = t_b \circ c$ and thus $c'_* = t_{b*} \circ c_* = c_* t_{b*}$. (To be consistent with the notation $[a]c_*$, in the level of homology we consider the product notation for the composition.) We obtain,

$$\begin{aligned} &[a]c'_* = [a]c_* t_{b*} = [a]t_{b*} = [a] + ([b] \circ [a])[b] = [a] - 2[b] \\ &[b]c'_* = [b]c_* t_{b*} = -[b]t_{b*} = -[b]. \end{aligned}$$

Note that

$$([a] + [a]c'_*)c'_* = [a] + [a]c'_* \text{ and } ([b] - [b]c'_*)c'_* = -([b] - [b]c'_*).$$

Therefore a generator $[a']$ of $H_+^{c'}$ and $[b']$ of $H_-^{c'}$ can be obtained by normalizing $[a] + [a]c'_* = 2[a] - 2[b]$ and $[b] - [b]c'_* = 2[b]$ so that $[a'] \circ [b'] = 1$. We choose $[a'] = \frac{1}{2}([a] - [b])$ and $[b'] = [b]$. Then we get $P_{(-\times<)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$.

We can always replace $([a],[b])$ by $(-[a],-[b])$. Thus, the resulted matrix is well-defined up to a sign. □

Each necklace stone corresponds to a pair of critical values, and the matrices associated to the necklace stones are obtained as the following products (up to an ambiguity of the sign)

$$P_{\square} = P_{(-\times<)}P_{(>\times-)} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

$$P_{\bigcirc} = P_{(-\circ<)}P_{(>\circ-)} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

$$P_{>} = P_{(-\times<)}P_{(>\circ-)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},$$

$$P_{<} = P_{(-\circ<)}P_{(>\times-)} = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

We consider two presentations of $SL(2,\mathbb{Z})$;

$$\begin{aligned} SL(2,\mathbb{Z}) &= \{\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : (\alpha\beta)^6 = id\} \\ &= \{x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} : x^2 = y^3, x^4 = id\}. \end{aligned}$$

One can pass from the first presentation to the second by letting $x = \alpha\beta\alpha = \beta\alpha\beta$ and $y = \alpha\beta$.

Since $x^2 = -id$ we have $PSL(2,\mathbb{Z}) = \{x, y : x^2 = y^3 = id\}$.

Lemma 6.2.2. *Let* $R = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ *and* $\tilde{P} = R^{-1}PR$. *Then for each stone we obtain the following factorization.*

$\tilde{P}_{\square} = yxy$
$\tilde{P}_{\bigcirc} = xyxyx$
$\tilde{P}_{>} = y^2x$
$\tilde{P}_{<} = xy^2$

Proof. We have

$$\begin{aligned} \tilde{P}_{\square} = R^{-1}P_{\square}R = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \tilde{P}_{>} = R^{-1}P_{>}R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ \tilde{P}_{\bigcirc} = R^{-1}P_{\bigcirc}R = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{P}_{<} = R^{-1}P_{<}R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

Note that $\tilde{P}_{\square} = \alpha\beta\alpha^{-1}$, $\tilde{P}_{>} = \alpha$, $\tilde{P}_{\bigcirc} = \alpha^{-1}\beta\alpha$, $\tilde{P}_{<} = \beta$.

Thus, we obtain the following elements in $PSL(2,\mathbb{Z})$ as *monodromies of necklace stones.*

$$\begin{aligned}
\tilde{P}_> &= \alpha = \beta^{-1}\alpha^{-1}\alpha\beta\alpha = y^{-1}x = y^2x \\
\tilde{P}_< &= \beta = \beta\alpha\beta\beta^{-1}\alpha^{-1} = xy^{-1} = xy^2 \\
\tilde{P}_\square &= \alpha\beta\alpha^{-1} = \alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\alpha\beta = yxy \\
\tilde{P}_\bigcirc &= \alpha^{-1}\beta\alpha = \alpha^{-1}\beta^{-1}\alpha^{-1}(\alpha\beta\alpha^{-1})\alpha\beta\alpha = x(yxy)x.
\end{aligned}$$

□

Remark 6.2.3. Note that $\tilde{P}_\bigcirc = x\tilde{P}_\square x$ and $\tilde{P}_< = x\tilde{P}_> x$, hence if a necklace diagram has the identity monodromy, then the necklace diagram obtained from the original by replacing each □-type stone with ○-types stone, and each >-type stone with <-type stones, and vice versa has also monodromy the identity. Such a necklace diagram is called the *dual necklace diagram.*

Lemma 6.2.4. *Let $\pi : X \to S^2$ be a directed real elliptic Lefschetz fibration having only real critical values and admitting a real section. Then the monodromy of the necklace diagram associated to π is the identity in $PSL(2, \mathbb{Z})$.*

Proof. We mark an odd interval on S^1 and denote by $\{q_1, q_2, ..., q_n\}$ the set of critical values, ordered with respect to the orientation and the marked interval. We consider real structures $c_i, i = 1, 2, ..., n$ over regular intervals $I_i = (q_i, q_{i+1}), i = 1, ..., n-1$, and $I_n = (q_n, q_1)$. Since $c_0 = c_1 \circ t_{a_1}$ and c_n are isotopic, we have $c_{0*} = c_{n*}$.

Note that with respect to $([a_0], [b_0])$, such that $[a_0] \in H_+^{c_0}$ and $[b_0] \in H_-^{c_0}$, we can write c_{0*} and c_{n*} as

$$c_{0*} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } c_{n*} = P_{q_1}P_{q_2}...P_{q_n}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}P_{q_n}^{-1}P_{q_{n-1}}^{-1}...P_{q_1}^{-1}.$$

Thus,

$$c_{0*} = c_{n*} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P_{q_1}P_{q_2}...P_{q_n}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}P_{q_n}^{-1}P_{q_{n-1}}^{-1}...P_{q_1}^{-1}.$$

By equating two matrices we see that the latter equality holds if and only if $P_{q_1}P_{q_2}$ $...P_{q_n}$ is the identity $\in PSL(2, \mathbb{Z})$. The product $P_{q_1}P_{q_2}...P_{q_n}$ corresponds to the monodromy of the corresponding necklace diagram. Note that the any other choice of marked odd interval changes the monodromy up to conjugation, which does not effect the result. □

6.3 The Correspondence Theorem

Recall that the elliptic Lefschetz fibrations of type $E(n)$ can be characterized by the number $12n$ of their critical values.

Theorem 6.3.1. *There exists a one-to-one correspondence between the set of oriented necklace diagrams with* $6n$ *stones whose monodromy is the identity and the set of isomorphism classes of directed real fibrations* $E(n)$*,* $n \in \mathbb{N}$*, which have only real critical values and admit a real section.*

Proof. In the previous section we have discussed how to assign an oriented necklace diagram whose monodromy is the identity to a real $E(n)$ which admits a real section and has only real critical values. Since $E(n)$ has $12n$ critical values the corresponding oriented necklace diagram has $6n$ stones.

For a given necklace diagram with $6n$ stones whose monodromy the identity, we consider the underlying uncoated necklace diagram. The underlying uncoated necklace diagram defines a weak Lefschetz chain up to cyclic ordering. Hence by Proposition 5.6.5 there is a unique class of directed non-marked $\mathcal{RELF}$ over S^2 admitting a section and having only real critical values.□

Corollary 6.3.2. *There exists a bijection between the set of symmetry classes of non-oriented necklace diagrams with* $6n$ *stones whose monodromy is the identity, and the set of isomorphism classes of non-directed real* $E(n)$*,* $n \in \mathbb{N}$ *which have only real critical values and admit a real section.* □

6.4 Refined necklace diagrams

One can define a necklace diagram for fibrations not necessarily having a real section. When we discard the condition that the fibration admits a real section, we need to consider also the real structure with no real component. Let us recall that a vanishing cycles with respect to such a real structure can only be totally imaginary. Thus real structure with no real component are associated to the ○-type necklace stones. Recall that ○-type necklace stones define two critical values of type ∘, so corresponding singularities are solitary double points. Therefore, in case when the real Lefschetz fibrations has no real section, with respect to a real structure c on a real fiber F between the corresponding singular fibers, vanishing cycles are both real (if c has 2 real components) or totally imaginary (if c has no real component). As it was discussed in Section 5.6, the isomorphism class of the fibration depends on whether these vanishing cycles are the same curve, or c-twin curves (c-invariant curves which are isotopic but not c-equivariantly isotopic) on F.

Recall that, if c has 2 real components and two vanishing cycles are real, two possible classes of fibrations are already distinguished by whether or not there exists a real section over the interval corresponding to two critical values, as is clear from Figure 6.5.

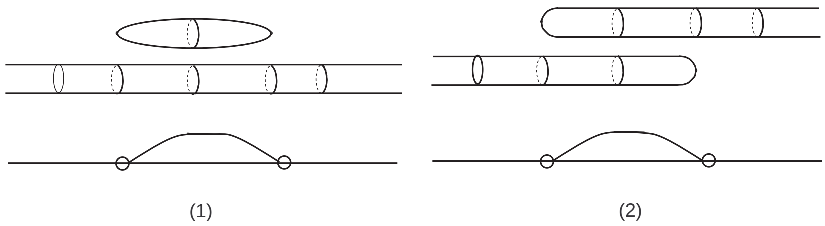


Fig. 6.5.

If c has no real component, as discussed in Section 5.6 we have two non-isomorphic real Lefschetz fibrations although the real part of the fibration does not distinguish two choices of vanishing cycles, see Figure 6.6.

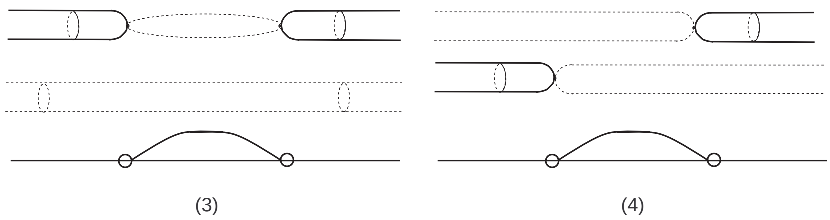


Fig. 6.6.

On the homological level, there is no difference between the real structure with 2 real components and the real structure with no component. As a result, there is no difference in the calculation of the monodromy of the necklaces stones. Thus we assign a refined (oriented) necklace diagram to a (directed) real Lefschetz fibration without real sections by replacing ○-type necklace stones with ○, ⊖, ●, ⊝ corresponding respectively to the four cases discussed above, see Figures 6.5 and 6.6. (Each refined necklace stone corresponds to $xyxyx \in PSL(2, \mathbb{Z})$.)

The necklace diagram which we obtain will be called a *refined necklace diagram.* (Clearly if the refined necklace diagram is identical to the necklace diagram then the corresponding real Lefschetz fibration admits a real section.)

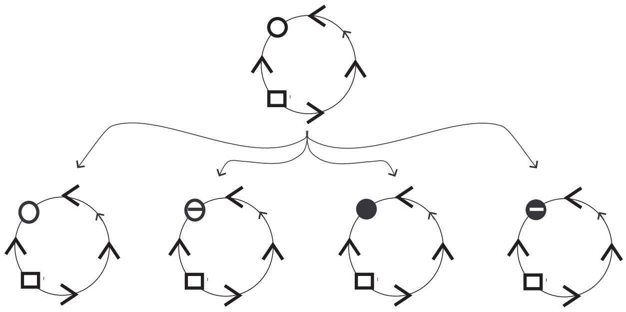


Fig. 6.7. An example of refinements of a necklace diagram.

Theorem 6.4.1. *There is a one-to-one correspondence between the set of oriented refined necklace diagrams with* $6n$ *stones whose monodromy is the identity and the set of isomorphism classes of directed real* $E(n)$, $n \in \mathbb{N}$ *with only real critical values.*

Proof. As we discuss in the beginning of this section, to a given directed real $E(n)$ with only real critical values we can assign an oriented refined necklace diagram.

As for the converse, to an oriented refined necklace diagram, we assign a decorated weak real Lefschetz chain. Note that one can always get a necklace diagram from a refined necklace diagram by forgetting different nuance of ○-type stones. Let us consider the underlying uncoated necklace diagram associated to the necklace diagram obtained from the refined necklace diagram. We get refinement of the uncoated necklace diagram by considering dotted intervals for refined stones of type ●, ⊖, see Figure 6.8. Then the oriented refined uncoated necklace diagram defines a weak real Lefschetz chain up to cyclic ordering, where dotted intervals correspond to a real structure with no real component.

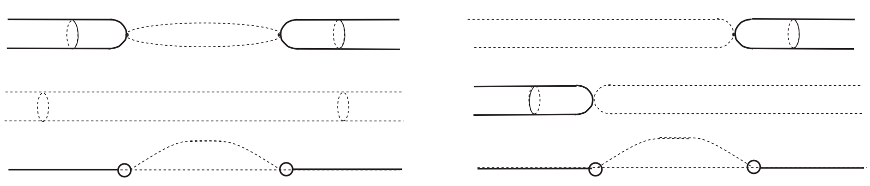


Fig. 6.8. Refinement of uncoated necklace diagram.

Note that by its construction, the refinement of ○-type stones encodes the decoration of the weak Lefschetz chain. Namely, the stone ⊖ (⊖) corresponds to a pair of critical values where the real code $\{c_i, a_i\}$ on a fiber F_i over a real point between the critical values is decorated (corresponding vanishing cycles on F are c_i-twin curves) and c_i has 2 real components (no real components, respectively). On the other hand, the stone ○ (●) corresponds to a pair of critical values where the real code $\{c_i, a_i\}$ on a fiber F_i over a real point between the critical values is not decorated (corresponding vanishing cycles are the same) and c_i has 2 real components (no real components, respectively).

Then by Theorem 5.6.1 and Proposition 5.6.2 we get a unique isomorphism class of directed $\mathcal{RELF}$ with only real critical values. □

6.5 The Euler characteristic and the Betti numbers of necklace diagrams

Proposition 6.5.1. *Let $\pi : X \to S^2$ be a $\mathcal{RELF}$s admitting a real section. Then the Euler characteristic of the real part is*

$$\chi(X_{\mathbb{R}}) = 2(|\bigcirc| - |\square|),$$

and the total Betti number is

$$\beta_*(X_{\mathbb{R}}) = 2(|\bigcirc| + |\square|) + 4.$$

Proof. Each stone of type ○ includes two singular fibers having a solitary double point, and, similarly, each stone of type □ includes two singular fibers having a crossing double point. Regular fibers are either one S^1 or two copies of S^1, hence their Euler characteristics are zero. The Euler characteristic of a singular fiber having a solitary double point is 1, while that of a fiber having a crossing double point is -1. Thus, the result follows by applying Euler characteristic formula for fibrations.

Necklace diagrams determines the topology of the real part of $X_{\mathbb{R}}$. Indeed, each $|\square|$-type stone defines a genus on the real part $X_{\mathbb{R}}$ and since there is a real section each $|\bigcirc|$-type stone defines a sphere component.

Note also that each stone of arrow type does not effect the homology of $X_{\mathbb{R}}$. Hence, we have $\beta_0 = \beta_2 = |\bigcirc| + 1$ and $\beta_1 = 2(|\square| + 1)$. Thus $\beta_* = 2(|\bigcirc| + 1) + 2(|\square| + 1) = 2(|\bigcirc| + |\square|) + 4$. $\square$

Remark 6.5.2. The calculation of the Euler characteristic of the real part of a fibration $\pi : X \to S^2$ using a necklace diagram can be made for a fibration without a real section by replacing $|\bigcirc|$ with |○| + |⊖| + |●| + |⊝|.

Definition 6.5.3. We call the quantity $2(|\bigcirc| - |\square|)$ the *Euler characteristic of the necklace diagram* and $2(|\bigcirc| + |\square|) + 4$ as the *total Betti number of the necklace diagram.*

Definition 6.5.4. Let (X, c) be a real manifold, then the real part $X_{\mathbb{R}}$ is called *maximal* if $\beta_*(X_{\mathbb{R}}) = \beta_*(X)$. (Note that in general we have $\beta_*(X_{\mathbb{R}}) \leq \beta_*(X)$, called *Smith inequality.*)

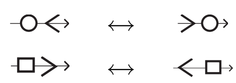
In our case, the total Betti number of $E(n)$ is $\beta_*(E(n)) = 12n$ [GS]. We call a necklace diagram with $6n$ stones *maximal* if its total Betti number is $12n$. This happens when $|\bigcirc| + |\square| = \frac{12n-4}{2}$. In particular, if $n = 1, 2$, then $|\bigcirc| + |\square| = 4$ and $|\bigcirc| + |\square| = 10$, respectively.

6.6 Horizontal and vertical transformations of necklace diagrams

Let $\mathcal{N}_k^{(i,j)}$ denote the set of oriented necklace diagrams with $|\bigcirc| = i$ and $|\square| = j$. We define transformations which allow us to produce new necklace diagrams from the given one.

The transformation h interchanges the pieces as is shown below.

$$h : \quad {\mathcal{N}_k}^{(i,j)} \quad \to \quad {\mathcal{N}_k}^{(i,j)}$$



Clearly, h preserves the Euler characteristic and the total Betti number of the necklace diagram. The transformations v_1 and v_2 are defined as follows.

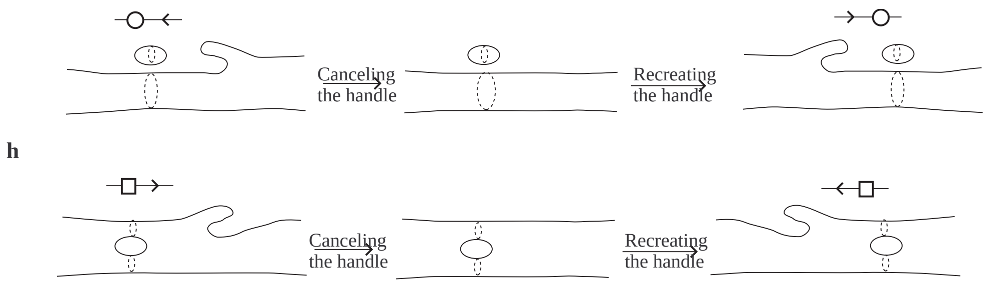
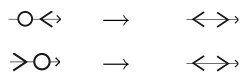


Fig. 6.9. The relation between transformations h and the real part of X.

$$v_1 : \quad \mathcal{N}_k{}^{(i,j)} \quad \to \quad \mathcal{N}_k{}^{(i-1,j)}$$



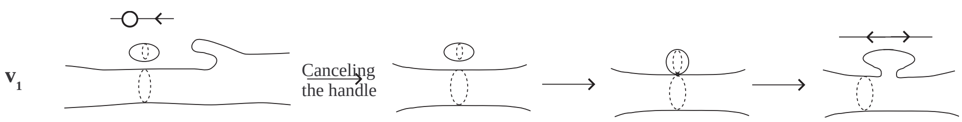
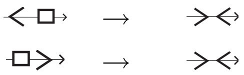


Fig. 6.10. The relation between v_1 and the real part of X.

$$v_2 : \quad \mathcal{N}_k{}^{(i,j)} \quad \to \quad \mathcal{N}_k{}^{(i,j-1)}$$



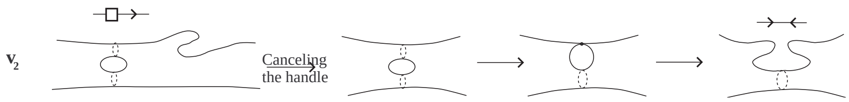


Fig. 6.11. The relation between v_2 and the real part of X.

Note that unlike h, transformations v_1, v_2 change the Euler characteristic and the Betti number of the necklace diagram.

Note that transformations h, v_1, v_2 can be defined for non-oriented necklace diagrams in the same way.

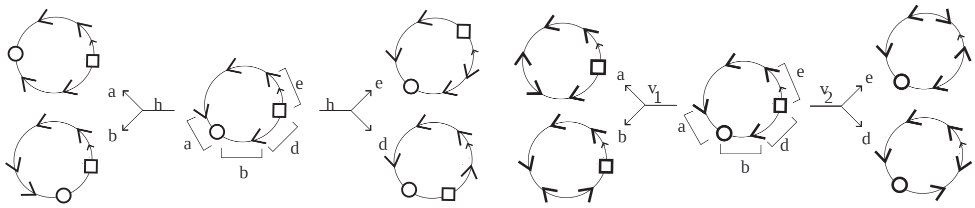


Fig. 6.12. Examples of transformations h, v_1, v_2.

6.7 Producing new necklace diagrams using necklace connected sum

We consider two connected sum operations for oriented necklace diagrams called *odd sum* and *even sum*. Note that even and odd sum of necklace diagrams correspond to fiber sums of real Lefschetz fibration $\pi : X \to S^2$, where the gluing is made on an even or odd interval of S^1. To perform an odd sum, we cut each of two necklaces along an odd interval (piece of chain) and then reglue them crosswise respecting the orientation.

The even sum is obtained by cutting necklace diagrams at a stone (this corresponds to cutting the chain on an even interval) and regluing them according to the table shown in Figure 6.13.

Observe that the Euler characteristic is additive with respect to the odd sum. However, it is not always additive with respect to the even sum.

Example 6.7.1. Examples of odd and even connected sums are given in Figure below.

We can also consider the sum of two non-oriented necklace diagrams by fixing orientations on the necklace chains.

6.8 Classification of real $E(1)$ with real sections via necklace diagrams

Theorem 6.8.1. *There exist precisely 25 isomorphism classes of real non-directed fibrations $E(1)$ admitting a real section and having only real critical values. These classes are characterized by the non-oriented necklace diagrams presented in Figure 6.16.*

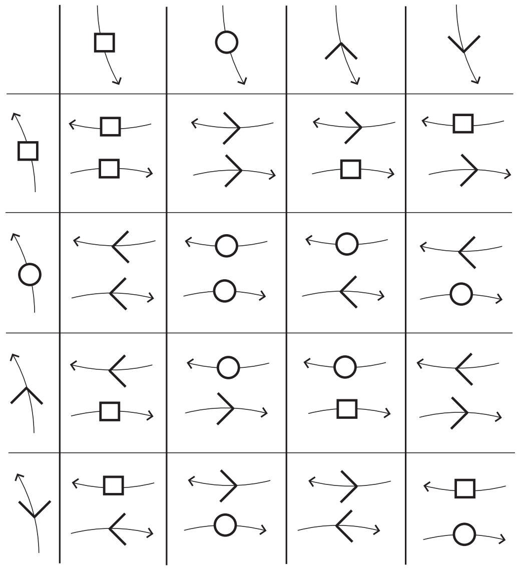


Fig. 6.13.

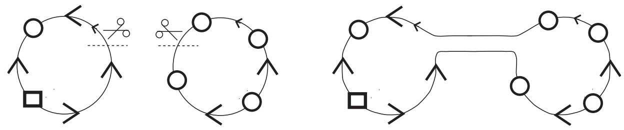


Fig. 6.14. An example of odd connected sum.

Proof. By Theorem 6.3.2, it is enough to find the list of symmetry classes of necklace diagrams of 6 stones whose monodromy is the identity. To find the symmetry classes of necklace diagrams, we consider the following algorithm. Let $S, C, L, R \in PSL(2,\mathbb{Z}) = \{x, y : x^2 = y^3 = [id]\}$, such that $S = yxy$, $C = xyxyx$, $L = xy^2$ and $R = y^2x$. Then,

1. Consider words of length 6 of the letters S, C, L, R.
2. Quotient out the words which are equivalent to each other up to cyclic ordering.
3. Quotient out the symmetry classes. Symmetry classes of necklace diagrams in terms of words can be seen as follows. Two words will be called symmetric if one is obtained from the other by reading from the end to beginning and by changing each letter L by the letter R, and vice versa. For example, $CLLSLL \sim RRSRRC$. □

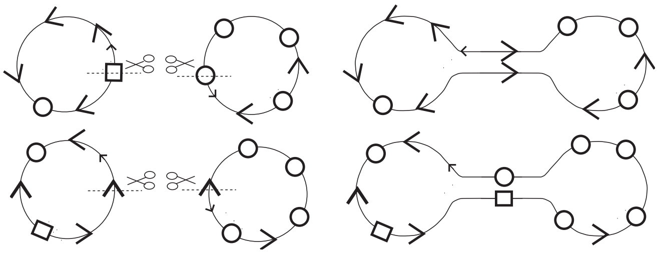


Fig. 6.15. Examples of even connected sums.

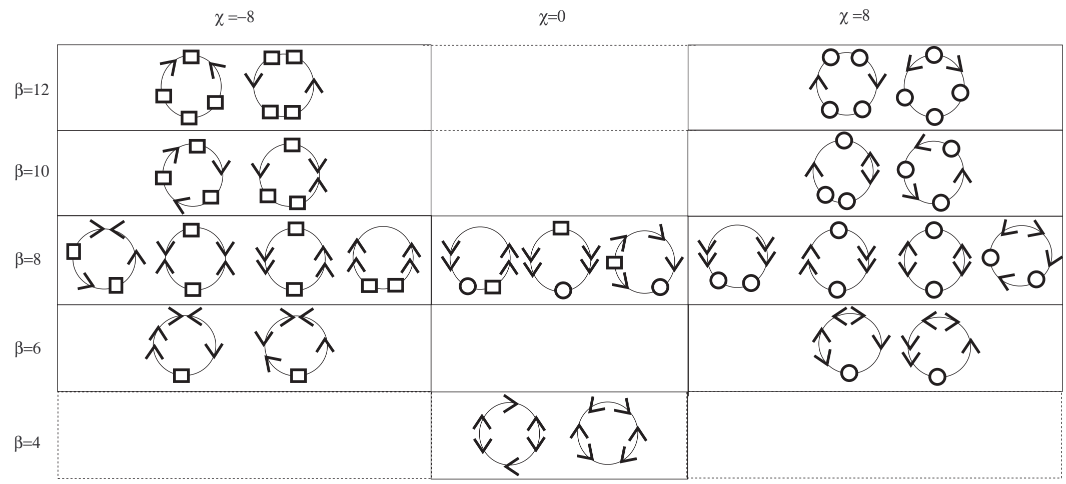


Fig. 6.16. List of necklace diagrams of real $E(1)$ having only real critical values and admitting a real section.

If a necklace diagram has monodromy identity, its dual has also, thus we can always assume $|\bigcirc| \leq |\square|$. One can also proceed by considering words of length 3, then checking the words which are inverses to each other in $PSL(2, \mathbb{Z})$. This way it is possible to get the list without using computer. There is also a computer program written by Andy Wand, which works for the cases $n = 1, 2$.

Proposition 6.8.2. *All necklace diagrams with 6 stones whose monodromy is the identity can be obtained from the maximal necklace diagrams of* 6 *stones by applying the transformations* h, v_1, v_2.

Proof. The proof is obtained by direct analysis of necklace diagrams listed in Figure 6.16. □

By calculating possible refinements of the necklace diagrams (considered up to symmetry) listed in Figure 6.16 we obtain the following results. (Note that refinement concerns only those necklace which have at least one ○-type stone.)

- $(|\bigcirc|, |\square|) = (1, 1)$ there are 4 refined necklace diagrams,
- $(|\bigcirc|, |\square|) = (1, 0)$ there are 4 refined necklace diagrams,
- $(|\bigcirc|, |\square|) = (2, 0)$ there are 46 refined necklace diagrams,
- $(|\bigcirc|, |\square|) = (3, 0)$ there are 84 refined necklace diagrams,
- $(|\bigcirc|, |\square|) = (4, 0)$ there are 251 refined necklace diagrams.

6.9 Real elliptic Lefschetz fibrations of type $E(2)$ with real sections

Using the algorithm written by Andy Wand, we obtain 25263 real $E(2)$ having only real critical values and admitting a real section.

Proposition 6.9.1. *There are 10 maximal necklace diagrams* ($|\bigcirc| + |\square| = 10$) *of 12 stones whose monodromy is the identity. The list is given in Figure 6.17.* □

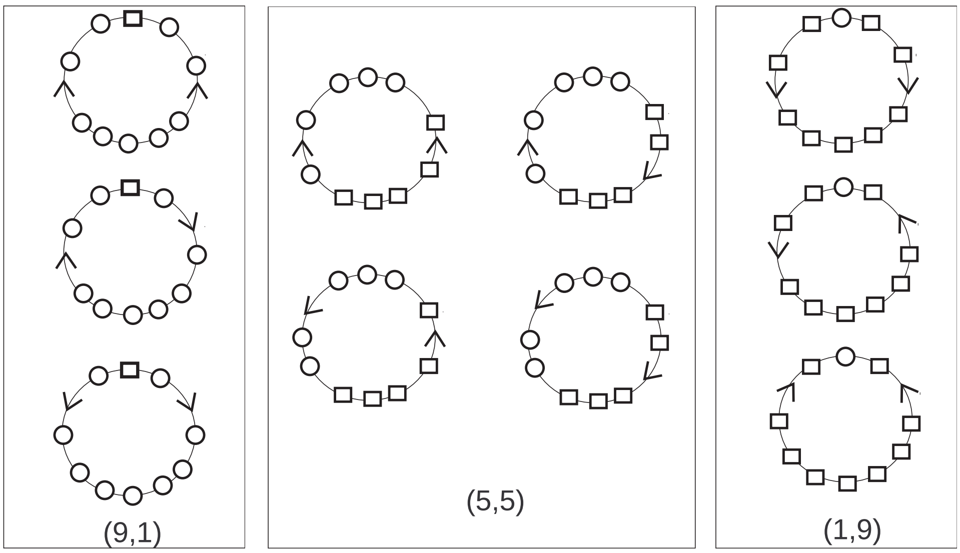


Fig. 6.17. List of necklace diagrams of maximal real $E(2)$ having only real critical values and admitting a real section.

Proposition 6.9.2. *There exist necklace diagrams of* 12 *stones whose monodromy is the identity and which can not be written as a connected sum of two necklace diagrams of* 6 *stones whose monodromy the identity.*

Proof. In Figure 6.18, we construct an example using the necklace connected sum and the operation h of necklace diagrams. Note that neither h nor v_i effects the monodromy of necklace diagram.

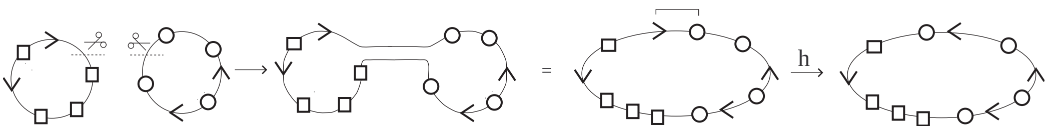


Fig. 6.18. An example of construction of a non-decomposable necklace diagram.

By analyzing possible divisions of the pair $(|\bigcirc|, |\Box|)$, we see that the necklace diagram shown in Figure 6.18 cannot be divided into two necklace diagrams of 6 stones with the identity necklace monodromy, listed in Figure 6.16. □

Remark 6.9.3. The idea of construction can be applied to obtain non-decomposable examples for all n.

Proposition 6.9.4. *There exists a necklace diagram of* 12 *stones which can not be obtained from the maximal necklace diagram by applying the transformations* h, v_1, v_2.

Proof. Examples are given in Figure 6.19, the result is obtain by simple analysis on possible cases. □

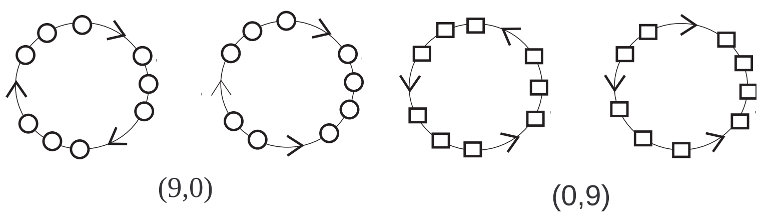


Fig. 6.19. Example of necklace diagrams which can not be obtained from the maximal necklace diagrams using v_1, v_2, h.

6.10 Some other applications of necklace diagrams

Denote by $|\bigcirc|$ (respectively $|\square|$) the number of stones of type $\bigcirc$ (respectively, of type $\square$). By fixing the pair $(|\bigcirc|, |\square|)$, we fix the topology of the real part of $E(n)$, hence we obtain a classification of real parts of $E(n)$ which have only real critical values and admit a real section. Note that, $|\bigcirc|$ is the number of spherical components of the real part and the number of genus of the higher genus component is $|\square| + 1$. In Figure 6.20 and Figure 6.21 we show the corresponding classification for $n = 1$ and $n = 2$, respectively.

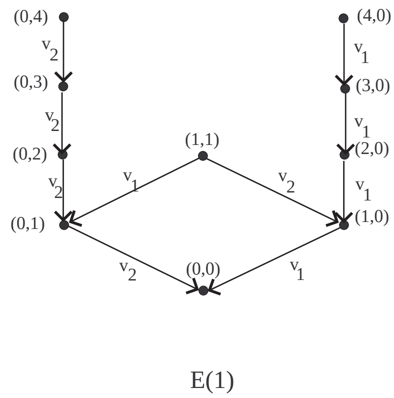


Fig. 6.20. Vertices of the graph correspond to the necklace diagrams of real $E(1)$ whose real part has fixed topological type. Edges correspond to the transformations v_1 or v_2.

Remark 6.10.1. If the real part of real elliptic Lefschetz fibration, $E(n)$, (admitting a section) is disjoint union of 2 tori (happen when n is even) or of 2 Klein bottles (happen when n odd), then $E(n)$ does not admit a real fibration with real critical values.

Proposition 6.10.2. *Each (refined) necklace diagram whose monodromy is the identity contains at least two arrow type stones.*

Proof. Assume that there are necklace diagrams whose monodromy is the identity and which have either no or only one arrow type stones. If there is no arrow type stones then we have only $\square$ and/ or $\bigcirc$. However, there is no cancellation in the product of monodromies of the stones of type $\square$ and $\bigcirc$. Hence, the product can not be the identity. Similarly, if there is one arrow type stone, to be able to obtain the identity the monodromies of rest should give yx or xy. Again it can not be possible

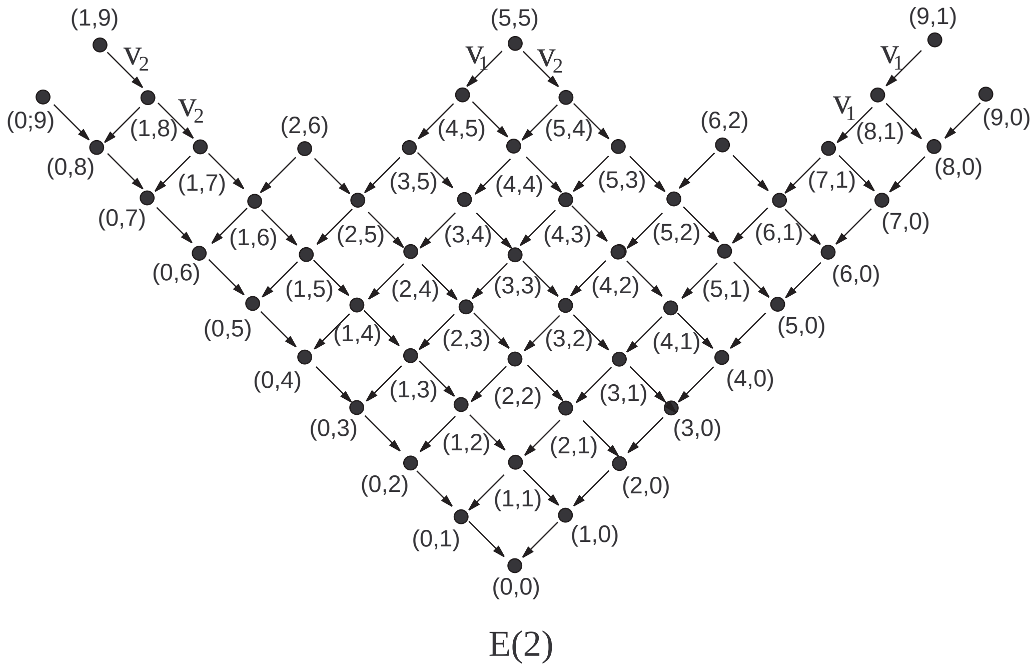


Fig. 6.21. Vertices of the graph correspond to necklace diagrams of real $E(2)$ whose real part has fixed topological type. Edges correspond to the transformations v_1 or v_2.

since there is no cancellation in the product of monodromies of □ and/ or ○. □

Corollary 6.10.3. *Each real elliptic Lefschetz fibration with only real critical values contains at least two critical values of type* $\times$. □

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APPENDIX A

ALGEBRAICITY OF REAL ELLIPTIC LEFSCHETZ FIBRATIONS WITH A SECTION

In this section, we study the algebraicity of the real elliptic Lefschetz fibrations with a real section. We concentrate ourselves mainly on fibrations with 12 real critical values. Note that any algebraic elliptic Lefschetz fibration, $E(n)$, can be seen as the double branched covering of a Hirzebruch surface of degree $2n$, branched at the exceptional section and a trigonal curve disjoint from this section.

S.Yu. Orevkov [O2] introduced a real version of dessins d'enfants for trigonal curves on Hirzebruch surfaces which are disjoint from the exceptional section. We apply his results to determine which of the trigonal curves that appear as the branching set of the covering $E(n) \to H(2n)$, are realizable algebraically and which are not.

A.1 Trigonal curves on Hirzebruch surfaces

The Hirzebruch surface, $H(k)$, of degree k is a complex surface equipped with a projection, $\pi_k : H(k) \to \mathbb{C}P^1$, which defines a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ with a unique exceptional section s such that $s \circ s = -k$. In particular, $H(0) = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $H(1)$ is $\mathbb{C}P^2$ blown up at one point.

Each Hirzebruch surface $H(k)$ can be obtained from $H(0)$ by successive birational transformations, namely, by a sequence of blow ups followed by blow downs at a certain set of points. If these points are chosen to be real, then the resulting Hirzebruch surface has a real structure inherited from the real structure $conj \times conj$ on $H(0)$: this is the real structure which we deal with.

With respect to this real structure, the real part of $H(k)$ is a torus if k is even, otherwise it is a Klein bottle.

In this Appendix, we consider nonsingular curves only, so by a *trigonal curve* on a Hirzebruch surface $H(k)$ we understand a smooth algebraic curve $C \subset H(k)$ such that the restriction to it of the bundle projection, $\pi_k : H(k) \to \mathbb{C}P^1$, is of degree 3. A trigonal curve on $H(k)$ is called *real* if it is invariant under the real structure of $H(k)$.

A.2 Real dessins d'enfants associated to trigonal curves

Let us choose affine coordinates (x, y) for $H(k)$ such that the equation $x = const$ corresponds to fibers of π_k and $y = \infty$ is the exceptional section s. Then, with respect to such affine coordinates any (algebraic) trigonal curve can be given by a polynomial of the form $y^3 + p(x)y + q(x)$ where p and q are real one variable polynomials such that $deg\, p = 2k$ and $deg\, q = 3k$.

The discriminant of $y^3 + p(x)y + q(x) = 0$ with respect to y is $-4p^3 - 27q^2$. Following [O2], we put $D = 4p^3 + 27q^2$. The fraction $f(x) = \frac{D(x)}{q^2(x)}$ defines a rational function whose poles are the roots of q taken with multiplicity 2, zeros are the roots of D, and the solutions of $f = 27$ are the roots of p taken with the multiplicity 3.

Let us color $\mathbb{R}P^1$ as in Figure A.1.

Fig. A.1. Coloring of $\mathbb{R}P^1$.

Then the inverse image $f^{-1}(\mathbb{R}P^1)$ turns naturally into an oriented colored graph on $\mathbb{C}P^1$. Since $f(x)$ is real, the graph is symmetric with respect to the complex conjugation on $\mathbb{C}P^1$.

Sufficient conditions for the realizability of a graph (and the existence of respective polynomials p, q, D) is given by the following theorem.

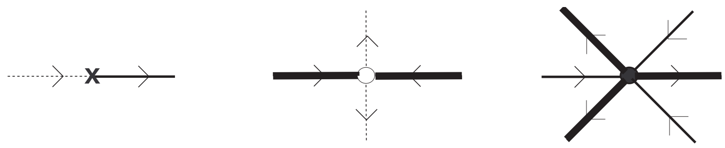

Fig. A.2. The graph around the inverse images of zeros of p, q, D.

Theorem A.2.1. *[O2] Let $\Gamma \subset S^2$ be an embedded oriented graph where some of its vertices are colored by the elements of the set $\{\circ, \bullet, \times\}$ and each of its edges is one of the three kinds: ——, ·······, ▬. Let Γ satisfy the following conditions:*
(1) The graph Γ is symmetric with respect to an equator of S^2, which is included into Γ;
(2) The valency of each vertex "$\bullet$" is divisible by 6, and the incident edges are colored alternatively by incoming ——, and outgoing ▬;
(3) The valency of each vertex "$\circ$" is divisible by 4, and the incident edges are colored alternatively by incoming ▬, and outgoing ·······;
(4) The valency of each vertex "$\times$" is even, and the incident edges are colored alternatively by incoming ·······, and outgoing ——;
(5) The valency of each non-colored vertex is even, and the incident edges are of the same color;
(6) Each connected component of $S^2 \setminus \Gamma$ is homeomorphic to an open disc whose boundary is colored as a covering of $\mathbb{R}P^1$ (colored and oriented as in Figure A.1) and the orientations of the boundaries of neighboring discs are opposite.
Then, there exists a real rational function $f = \frac{4p^3+27q^2}{q^2}$ whose graph is Γ.

Definition A.2.2. A graph on S^2 satisfying the conditions (1)-(6) of the above theorem is called *a real dessin d'enfant.*

A.3 Correspondence between real schemes and real dessins d'enfants

The real scheme of a trigonal curve imposes strong restrictions on the arrangement of the real roots of p, q and D. For example, the zeros of D correspond to the points where the trigonal curve is tangent to the fibers of $\pi_k : H(k) \to \mathbb{C}P^1$. A typical correspondence for certain model pieces of the curve is shown in Figure A.3. (cf [O2] or [DIK])

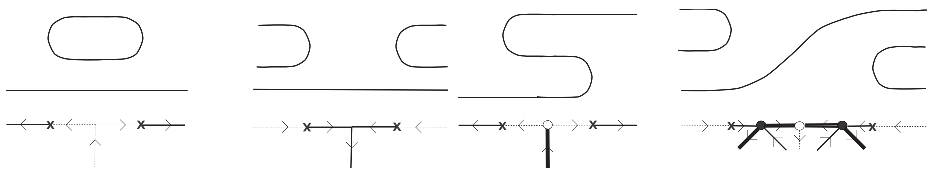

Fig. A.3. Because of the symmetry property we consider only one of the symmetric piece of real dessins d'enfants.

More precisely, fragments of the graph depicted in Figure A.3 determine uniquely the corresponding pieces of the curve.

The topology of the real part of $E(n)$ and hence the real part of the corresponding trigonal curve are determined by the necklace diagrams. Using the correspondences shown in Figure A.3, we obtain a new correspondence between fragments of necklace diagrams and fragments of the graph, see Figure A.4.

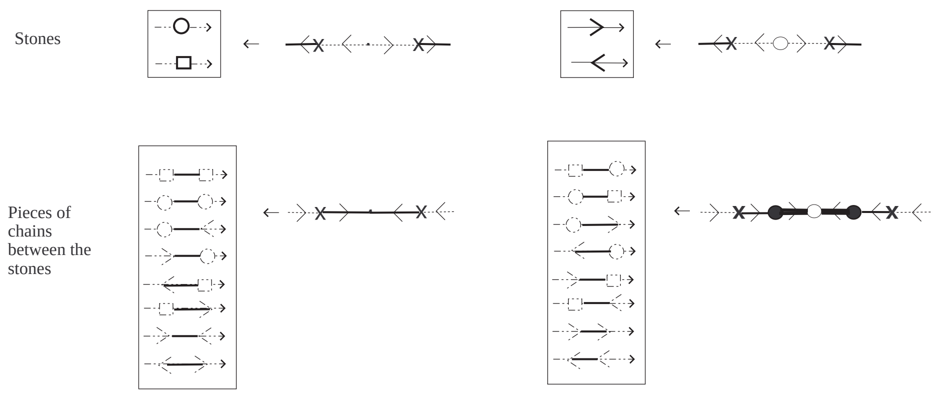

Fig. A.4.

Definition A.3.1. A piece of a chain of a necklace diagram is called a *necklace interval*. We call a necklace interval *essential* if the corresponding fragment of the graph is , see Figure A.4.

Lemma A.3.2. *If a real elliptic Lefschetz fibration, $E(n)$, admitting a real section is algebraic then on the corresponding necklace diagram*

- *the number of essential intervals cannot be more than $2n$,*
- *the sum of the number of essential intervals and the number of arrow type stones cannot be greater then $6n$.*

Proof. For a trigonal curve on $H(2n)$ defined by $y^3 + p(x)y + q(x)$, we have $deg\, p = 2 \cdot 2n$ and $deg\, q = 3 \cdot 2n$. Thus, the real dessin d'enfant can have at most $4n$ vertices colored by "$\bullet$" and at most $6n$ vertices colored by "$\circ$". Each essential interval corresponds to a graph fragment which contains at least two "$\bullet$" type vertices and at least one "$\circ$" type vertex, while each arrow type stones corresponds to a graph fragment having at least one "$\circ$" type vertex. $\square$

For $n = 1$, the number of essential intervals can not be more than 2 and the sum of the number of arrow type stones and the number of essential intervals can not be more than 6. Thus real elliptic Lefschetz fibrations admitting a section, corresponding to the following necklace diagrams can not be algebraic.

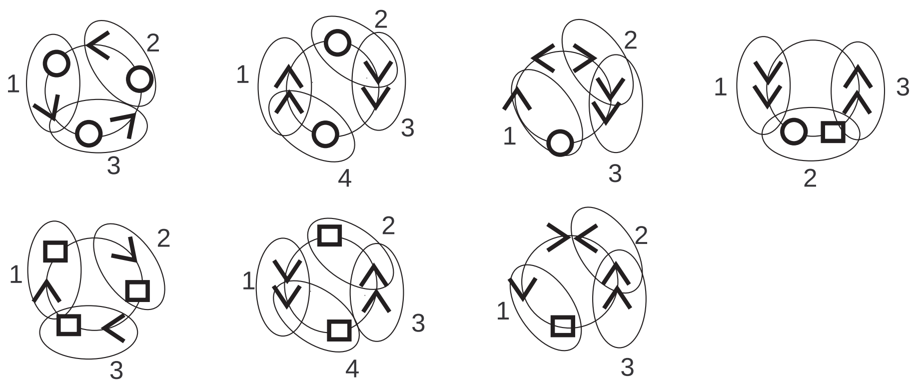

Fig. A.5. Necklace diagrams which contains more than 2 essential intervals.

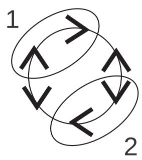

Fig. A.6. The number of essential intervals is 2 and there are 6 arrow type stones.

A.4 Algebraicity of real elliptic Lefschetz fibrations with real sections

Lemma A.4.1. *If a real elliptic Lefschetz fibration admitting a section is algebraic then the real elliptic Lefschetz fibration whose necklace diagram is dual to the necklace diagram of the former is also algebraic.*

Proof. Although the real parts of fibrations associated to the two dual necklace diagrams are different, trigonal curves appearing as the branching curves of coverings $E(n) \to H(2n)$ are the same. Two different real structures on the elliptic fibrations correspond exactly to two different liftings of the real structure of $H(2n)$ to $E(n)$. □

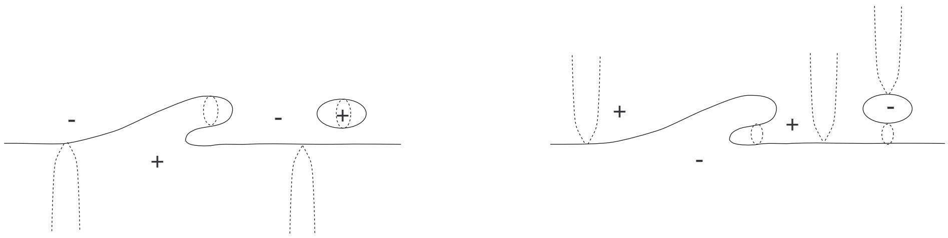

Fig. A.7. For each trigonal curve on $H(2n)$, there are two real structures of $E(n)$.

Theorem A.4.2. *All real elliptic Lefschetz fibrations admitting a real section and having 12 real critical values are algebraic except those whose associated necklace diagram is one of the diagrams shown in Figure A.8.*

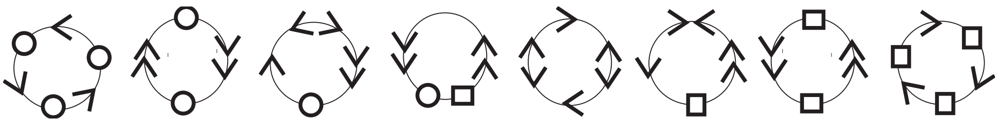

Fig. A.8. Necklace diagrams of non-algebraic real $E(1)$ having only real critical values and admitting a real section.

Proof. We construct real dessins d'enfants corresponding to necklace diagrams which are not prohibited by Lemma A.6. By Lemma A.4.1, we only need to consider necklace diagrams with $|\bigcirc| \geq |\square|$. Figure A.12 and Figure A.13 show such a list of real dessins d'enfants.□

Proposition A.4.3. *Real elliptic Lefschetz fibrations of type $E(2)$ which admit a real section and have only real critical values and which correspond to maximal necklace diagrams are algebraic.*

Proof. Recall that maximal necklace diagrams of $6n$ stones are those with $|\bigcirc| + |\square| = \frac{12n-4}{2}$. In fact, any maximal necklace diagram with 12 stones can be obtained as an even sum of maximal necklace diagrams of 6 stones, where the even sum is made on two arrow type stones of opposite direction. Such a sum increase the number of $|\bigcirc|$-type and $|\square|$-type stones by 1 and $|\bigcirc| + |\square| = 4 + 4 + 2 = 10$.

Fig. A.9.

We have shown that in case $n = 1$, maximal necklace diagrams ($|\bigcirc| + |\square| = 4$) are algebraic. Thus, we need to show that such even sum preserves algebraicity. This follows from the observation that real dessins d'enfants associated to such an even sum can be obtain from the dessins d'enfants of the summands as shown in Figure A.10. $\square$

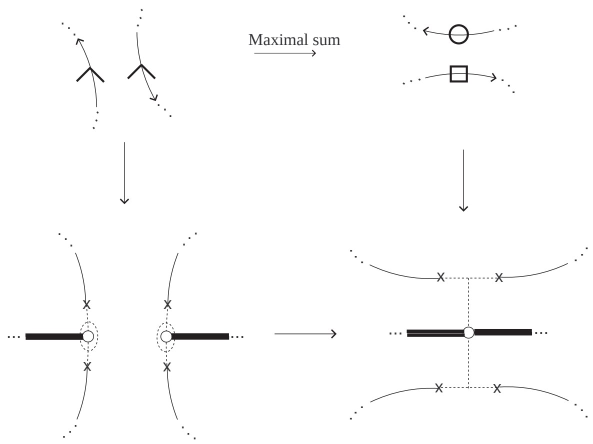

Fig. A.10.

Example A.4.4. An example of even sum which creates maximal necklace diagram and corresponding real dessins d'enfants are given in Figure A.11

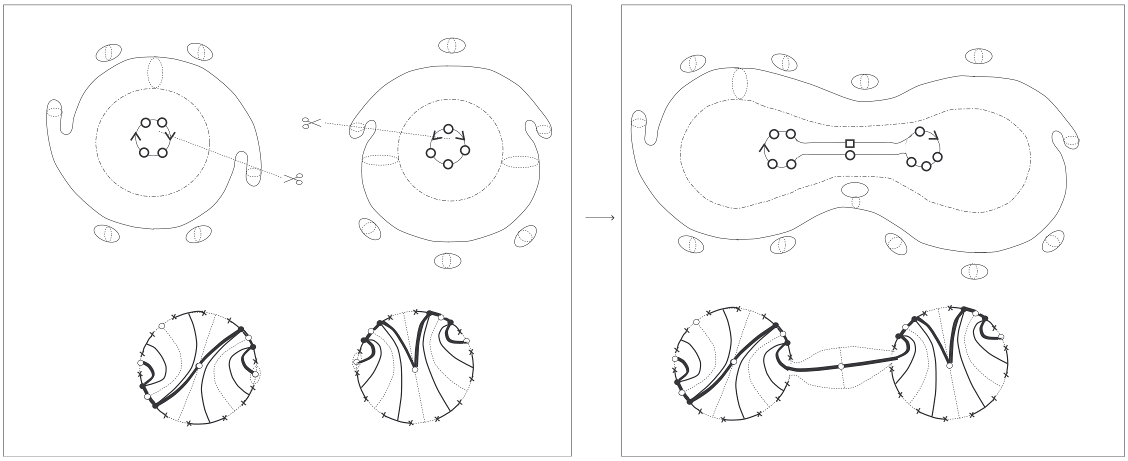

Fig. A.11. An example of even connected sum creating a maximal necklace diagram.

Real dessins d'enfants of real $E(1)$ with real sections.

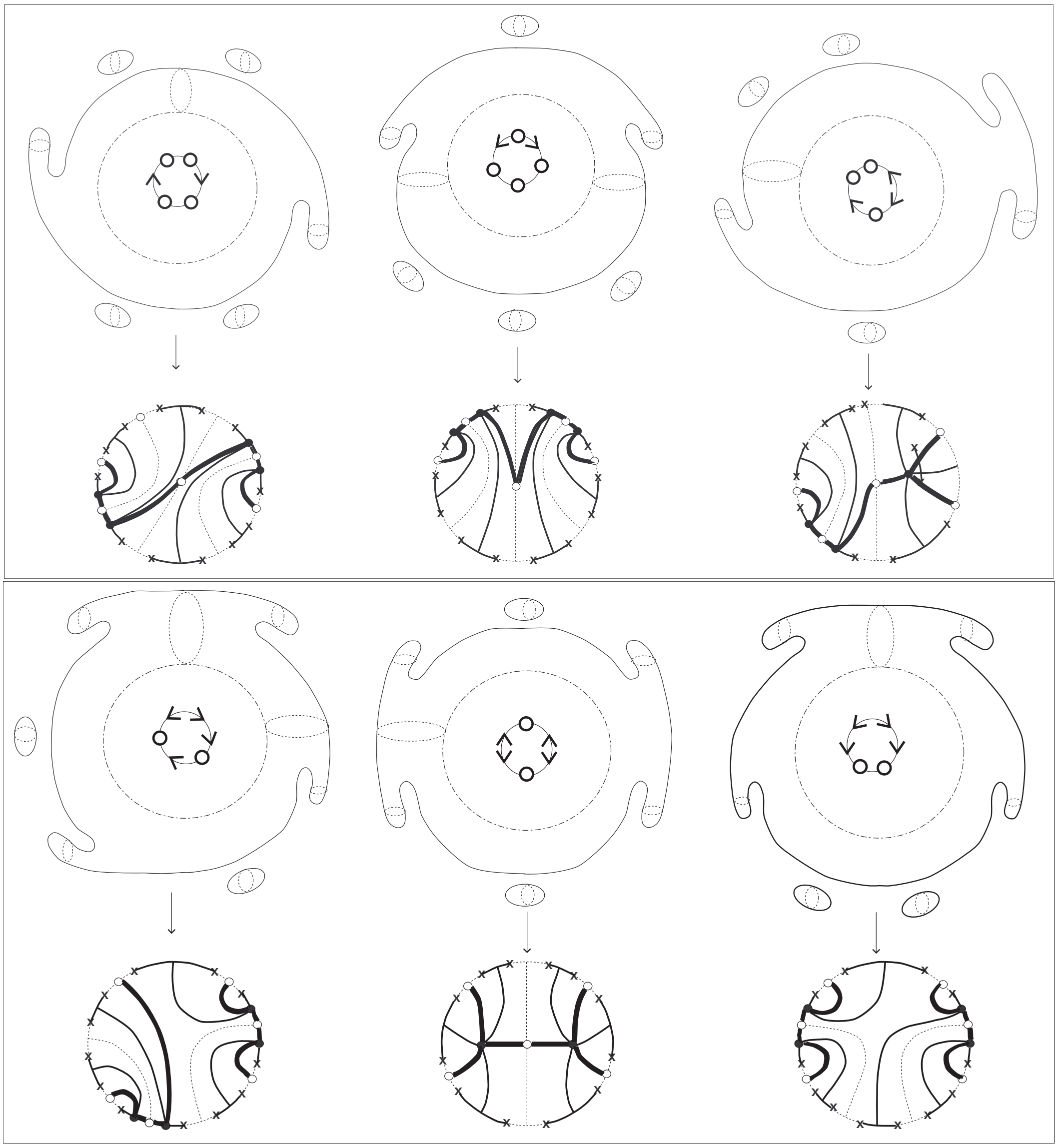

Fig. A.12. Around necklace diagrams, the real part of the corresponding real elliptic Lefschetz fibrations are shown. The dotted inner circle stands for a lift of the exceptional section.

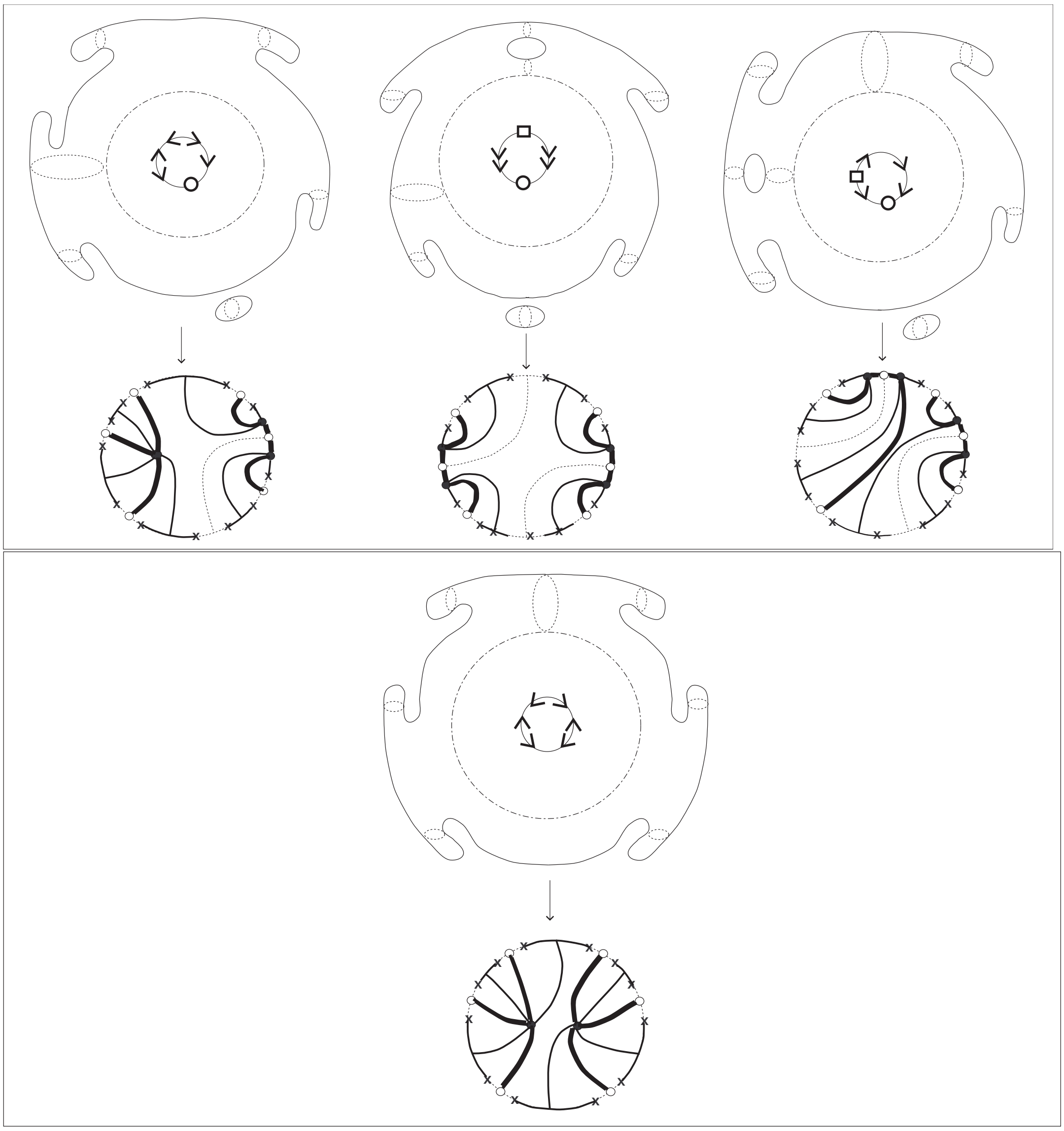

Fig. A.13. Around necklace diagrams, the real part of the corresponding real elliptic Lefschetz fibrations are shown. The dotted inner circle stands for a lift of the exceptional section.

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GRANTS

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