A RISK-SENSITIVE APPROACH FOR AIRLINE NETWORK REVENUE MANAGEMENT PROBLEMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

DEMET ÇETİNER

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

SEPTEMBER 2007

Approval of the thesis:

**A RISK-SENSITIVE APPROACH FOR
AIRLINE NETWORK REVENUE MANAGEMENT PROBLEMS**

submitted by **DEMET ÇETİNER** in partial fulfillment of the requirements for the degree of **Master of Science in Department of Industrial Engineering, Middle East Technical University** by,

Prof. Dr. Canan Özgen
Dean, Graduate School of
Natural and Applied Sciences _________________

Prof. Dr. Çağlar Güven
Head of Department, **Industrial Engineering** _________________

Assist. Prof. Dr. Zeynep Müge Avşar
Supervisor, **Industrial Engineering Dept., METU** _________________

Examining Committee Members:

Prof. Dr. Gülser Köksal
Industrial Engineering Dept., METU _________________

Assist. Prof. Dr. Zeynep Müge Avşar
Industrial Engineering Dept., METU _________________

Prof. Dr. Meral Azizoğlu
Industrial Engineering Dept., METU _________________

Assist. Prof. Dr. Ferda Can Çetinkaya
Industrial Engineering Dept., Çankaya University _________________

Dr. Banu Yüksel Özkaya
Industrial Engineering Dept., Hacettepe University _________________

Date: September 7, 2007

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname : **Demet ÇETİNER**

Signature :

ABSTRACT

A RISK-SENSITIVE APPROACH FOR AIRLINE NETWORK REVENUE MANAGEMENT PROBLEMS

Çetiner, Demet
M.Sc., Department of Industrial Engineering
Supervisor: Asst. Prof. Dr. Zeynep Müge Avşar

September 2007, 121 pages

In this thesis, airline network revenue management problem is considered for the case with no cancellations and overbooking. In literature, there exist several approximate probabilistic and deterministic mathematical models developed in order to maximize expected revenue at the end of the reservation period. The aim of this study is to develop models considering also the risks involved in the proposed booking control policies. Two linear programming models are proposed which incorporate the variance of the revenue. The objective of the models is to effectively balance the tradeoff between the expectation and variance of the revenue. The performances of the proposed models are compared to the previous models through a numerical study. The seat allocations resulting from the mathematical models are used in a simulation model working with several booking control policies. The probability distributions of the revenues are investigated and the revenues are compared in terms of expectation, standard deviation, coefficient of variation and probability of poor performance.

It is observed that the use of the proposed models decreases the variability of the revenue and thereby the risk of probability of poor performance. Also, the expected revenues obtained by implementing the solutions of the proposed models with nested booking control policies turn out to be higher than other probabilistic models as long as the degree of variance incorporation is within some interval. When compared with the deterministic models, the proposed models provides for the decision makers with alternative, preferable policies in terms of the expectation and the variability measures.

Keywords: Revenue management, Booking policy, Risk, Variance, Bayesian update.

ÖZ

HAVAYOLU AĞI GELİR YÖNETİMİ PROBLEMLERİ İÇİN RİSK-DUYARLI BİR YAKLAŞIM

Çetiner, Demet
Yüksek Lisans, Endüstri Mühendisliği Bölümü
Tez Yöneticisi: Y. Doç. Dr. Zeynep Müge Avşar

Eylül 2007, 121 sayfa

Bu tezde, kapasite üstü satışın ve bilet iptalinin olmadığı havayolu ağı gelir yönetimi problemi üzerine çalışılmaktadır. Yazında, rezervasyon dönemi sonunda elde edilecek gelirin beklenen değerini maksimize etmek üzere geliştirilmiş olasılıksal ve deterministik matematiksel modeller bulunmaktadır. Bu calışmanın amacı, önerilen rezervasyon kontrol politikalarının içerdiği riski de dikkate alan modeller geliştirmektir. Bu amaçla, elde edilecek gelirin varyansını da içeren iki doğrusal programlama modeli önerilmektedir. Modellerin amacı, gelirin beklenen değeri ile varyansı arasındaki ödünleşimi en etkili şekilde dengelemektir. Önerilen modeller sayısal deneyler yapılarak yazındaki diğer modeller ile karşılaştırılmaktadır. Farklı modellerin yer dağıtım sonuçları çeşitli rezervasyon politikaları ile benzetim modeli kullanılarak karşılaştırılmaktadır. Gelirlerin istatistiksel dağılımı analiz edilmekte ve gelirler beklenen değer, standart sapma, değişkenlik katsayısı ve kötü performans olasılığı bakımından incelenmektedir.

Önerilen modellerin, gelirin değişkenliğini azalttığı ve böylelikle hedeflenen gelir değerinin altına düşme olasılığını düşürdügü gözlenmektedir. Ayrıca, önerilen modellerin çözümlerinin içiçe geçmiş rezervasyon kontrol politikaları ile uygulanmasıyla elde edilen gelirin beklenen değeri, gelir varyansı modellerde belli ölçülerde

içerildiği sürece, diğer olasılıksal modellere kıyasla daha yüksek olmaktadır. Deterministik modeller ile karşılaştırıldığında ise, önerilen modeller karar vericilere beklenen değer ve değişkenlik ölçüleri bakımından tercih edilebilir, alternatif politikalar sunmaktadır.

Anahtar Kelimeler: Gelir yönetimi, Rezervasyon politikası, Risk, Varyans, Bayes güncelleme.

To my family

ACKNOWLEDGEMENT

First and the most important of all, I would like to express my gratitude to my supervisor Asst. Prof. Dr. Zeynep Müge Avşar for her endless support and insight. I am greatly indebted to her enthusiasm, patience and continuous encouragement throughout the study.

I would like to offer my thanks to the members of my examining committee; Prof. Dr. Gülser Köksal, Prof. Dr. Meral Azizoğlu, Assist. Prof. Dr. Ferda Can Çetinkaya and Dr. Banu Yüksel Özkaya for their contributions to this study.

I would like to thank to Özgür Özpeynirci and Bora Kat for all their support and valuable advices in the research and to Erman Terciyanlı for his comments and suggestions during the study.

Also special thanks to my colleges for their understanding during the study.

Murtaza Aşcı accompanied me during my undergraduate and graduate studies. I owe much due to his support and endless patience. Derya, Neslihan, Duygu, Deniz, Yasemin, Mehmet and Veli were with me whenever I had a difficulty. I would like to thank them for being in my life.

My homemates, Emine and Ezgi, deserve great thanks for their understanding and motivating me full of patience throughout the study. Life wouldn't be so pleasurable without them.

Lastly, I would like to thank to my family for the endless love end support they gave me throughout my life.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Revenue management (also called as yield management) is a tool for maximizing revenues by managing the demand. Talluri and Van Ryzin (2005) define the aim of revenue management as to increase revenues by controlling the demand through determining pricing and capacity allocation decisions.

Revenue management appears as a powerful strategy in markets, where the companies must sell a fixed amount of product during the selling season and there are customers willing to pay different amounts of money for the products (Pak and Piersma, 2002). In such an environment, companies can make use of customer segmentation and price discrimination to maximize revenues. They differentiate products by offering different prices for different customer segments and change the available mix of prices during the selling period. In the paper due to Pak and Piersma (2002), revenue management is defined as follows:

> "Revenue management is the art of maximizing profit generated from a limited capacity of a product over a finite horizon by selling each product to the right customer at the right time for the right price."

This thesis focuses on revenue management applications in airline industry. The typical characteristic of the sector is that the seats on a flight are perishable since they have no value after the aircraft departs. Additionally, the main component of the airline operating costs is the fixed cost of the flight, and marginal cost of carrying an additional passenger is very low compared to the fixed cost. Therefore, increasing the filled seat capacity of the flight makes significant contributions to the revenue. Another characteristic of the market is that passengers have different intentions when using airlines and have different travel patterns. While the price of the tickets may be more important for a leisure traveller, the traveling time would be more critical for a business traveller. For these reasons, in order to increase the revenues, airline companies offer different prices for the same seats in the aircraft which provide the same quality of accommodation. They offer lower priced tickets *(discounted fares)* to attract the leisure travellers and thereby to increase the number of seats that are sold. *Load factor* is used to define the filled capacity of the aircraft, and it is the ratio of the number of accepted bookings to the total capacity of the aircraft. By offering lower price tickets, the load factor of the flight would increase and extra revenue would be earned. On the other hand, by offering higher priced tickets *(full fare tickets)*, airline companies aim to earn more revenue per passenger.

In a typical aircraft, there exist different cabins which vary in terms of the quality of accommodation. The prices of these cabins are differentiated for several travel classes (first class, business class, economy class, etc.). Moreover, there are many different *fare classes* defined within a specific travel class (e.g., discounted economy class, full fare economy class, etc.). These fare classes differ in terms of travel restrictions they require or flexibilities they offer. The common restrictions and flexibilities can be classified as advance purchase requirements, length of stay, day-of-week travel, cancellation opportunities and refund options. Usually, a discounted fare class ticket requires purchasing of days or weeks in advance or including a weekend trip. A higher fare class ticket, on the other hand, allows the ticket to be cancelled at any time with full or partial refund. Hence, passengers sitting next to each other in an aircraft receive the same quality of accommodation, but the prices they have paid may be different depending on the fare class of the tickets.

Generally, revenue management can be applied through dynamic pricing or capacity allocation. Although there is no clear difference between these two policies, we can give some explanations. In dynamic pricing, the price of the product is used as a tool for controlling demand. A single product is offered and the price of the product is changed in time according to the realized demand. In capacity allocation applications, multiple products are offered at different prices. These multiple products have small differences like the airline tickets that vary in terms of different restrictions. However, the availability of the products changes over time. That is, some products with associated fares are closed for sale or some others are opened for sale. The type of the application depends on the characteristics of the sector. If the control of prices are flexible, dynamic pricing is usually a preferred option, whereas if the control of the supply is flexible, capacity allocation is more suitable. In the airline market, the ticket prices are not under the control of the airline company itself, it is usually determined by the market. Also, airline companies list the prices in advance of the booking period. On the other hand, an airline company can manage the supply of the seats, and it is also an easier implementation. Because the seats sold to different fare classes have the same quality of accommodation, they can be flexibly allocated to the fare classes. Therefore, airline companies offer several different fare classes and change the set of fare classes that are open for booking rather than the prices throughout the reservation period. These observations are due to Talluri and van Ryzin (2005).

In airline seat capacity allocation applications, the main decision that should be made is whether to accept an arriving request or reject it and keep the seat for customers of higher fare classes to arrive later. As it is stated in the beginning, passengers have different arrival patterns and usually discount fare passengers arrive earlier in the booking period and full fare passengers arrive later. Accepting the discount fare passengers would improve revenues due to increased load factor of the flight. On the other hand, if too many seats are sold to discount fare passengers, the potential higher fare class passengers may be lost due to the lack of seats. This tradeoff is considered in the literature under the heading of "Seat Allocation (Inventory Control) Problem".

Because of the uncertainty in the demand behaviour and the diversity of the arrival patterns of various fare classes, it is not easy to estimate the revenue and obtain a dynamic allocation policy. The fundamental approach to the seat allocation problem is to find optimal booking limits for each fare class in order to maximize the expected revenue that is earned during the reservation period. The *booking limit* is defined as the maximum number of seats that can be sold to a specific fare class on a flight. Static and dynamic models are developed for determining optimal booking limits for *single-leg* flights, which consist of only one takeoff and landing without any intermediate stop. These models give optimal booking limits for each fare class at any point in time during the reservation period depending on the updated demand distributions and the available seat capacity. An allocation policy with separate booking limits for each fare class is a *partitioned allocation policy*. If the realized demand for a fare class is lower than its booking limit, the aircraft departs with empty seats under a partitioned allocation policy. On the other hand, these empty seats can be sold to even higher fare class passengers whenever there is sufficient demand. Therefore, partitioned allocation policies are not effective and more revenue would be earned if the booking limits are determined in a nested manner. A *nested booking limit control policy* is such that seats available for a particular fare class are also available for higher fare classes. Hence, booking limit for a fare class is an upper bound for the number of seats that can be sold to that fare class and any lower fare classes. The remaining number of seats above that booking limit are protected for higher fare classes. A nested control policy gives higher revenues compared to a partitioned policy for any given allocation, and it is a common approach used by the airline companies.

The seat allocation problem is a complicated problem not only because of the uncertainty in demand but also the structure of the real airline networks to be considered. Airlines do not offer direct flights for every origin-destination pair unlike the example in Figure 1.1. In a point-to-point network, each origin-destination pair is served by a single-leg flight.

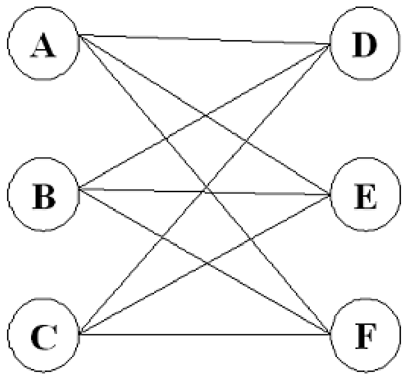

Figure 1.1: A point-to-point network.
(Williamson, 1992)

Instead, they offer many *itineraries* that are connected through several *flight legs* as in the example network illustrated in Figure 1.2. For some of the origin-destination pairs, travel is through multiple flight legs. Therefore, an allocation decision made for a flight affects the available seat capacity of the other flights connected to that flight. Hence, optimizing each individual flight leg separately cannot guarantee that the revenue earned from the whole flight network is optimized.

There are optimal booking control policies developed for single-leg problems, but it is impossible to find an optimal policy for network problems as pointed out by Talluri and Van Ryzin (2005). That is, the scale and complexity of the network does not allow to solve an optimal dynamic control model.

For network revenue management problems, approximate mathematical models are considered to maximize the expected revenue. The resulting approximate allocations are used with different booking control policies. The major drawback

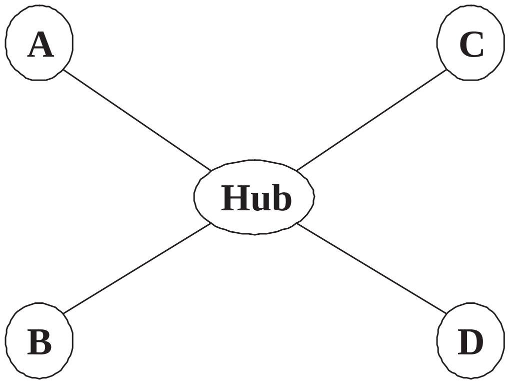

Figure 1.2: An example airline network .
(Talluri and van Ryzin, 2005)

of the mathematical models is that they do not consider nesting of the origin-destination and fare class combinations (ODFs). They divide the available seat capacity into several portions each of which is allowed to be sold to only one particular ODF. However, since the demand is random, such an allocation policy may cause some of the seats to be unsold. Therefore, the aircraft might depart with empty seats. Despite this drawback, mathematical models are used to determine the allocations for different ODFs on the network. The resulting allocations are, then, used with nested policies.

Nested booking limit control policy and *bid price control policy* are two common control policies proposed for making seat allocation decisions for network problems. Although partitioned booking control policy can also be used for seat allocation decisions, it is not as common as the other two. In order to obtain an effective dynamic control policy, the mathematical models are solved several times during the reservation period according to the realized bookings; and, this way, the control policies are updated. Different than the partitioned and nested booking limit control policies, the main idea in a bid price control policy is to approximate the opportunity cost of each itinerary and accept an arriving booking request as long as its fare exceeds this opportunity cost. For this purpose, threshold values (*bid prices*) are determined for each flight leg and the opportunity cost of the itinerary is approximated by the sum of the bid prices of the flight legs that form the itinerary. Bid price of a flight leg is the dual variable of the corresponding capacity constraint in the mathematical model. The main advantage of the bid price control policy is that the ODFs are automatically nested since, once a seat is available for one fare class, it is also available for classes with higher fares as well. For an effective control policy, bid prices are computed at the beginning of the reservation period and updated several times during the reservation period by taking the accepted bookings into account. Bid prices are updated solving the mathematical models with the updated available seat capacities.

The mathematical models that are developed for network revenue management problems can be classified as deterministic and probabilistic models. The probabilistic models consider the random behaviour of the demand and determine seat allocations accordingly, whereas the deterministic models assume that the demand is equal to its expected value. Williamson (1992) tests the mathematical models in the literature using a simulation model with nested booking limit and bid price control policies. The observation is that the deterministic models usually perform better than the probabilistic models in spite of the expectation that the probabilistic models would give better results because random nature of the demand is taken into account in these models. The explanation due to de Boer *et al.* (2002) for better performance of the deterministic models as compared to the probabilistic models is ignoring nesting in these mathematical models. Since the models do not consider nesting, they allocate many seats to higher fare classes and result in overprotection of seats for higher fare classes. The situation gets worse in the probabilistic models, especially when the variance of high fare class demand is high. In that case, there is an increased potential for earning higher revenue from high fare class demand. Since the probabilistic model considers the demand distributions of each ODF, it considers this potential. As a result, the probabilistic models assign even more seats to higher fare classes which worsens the resulting expected revenue when using the allocations obtained by the model with nested booking control policies.

In this thesis, we consider the network seat allocation problem where the prices of the fare classes are assumed to be fixed and constant throughout the reservation period. The demands of the origin-destination pairs do not change for the set of available fares. In other words, the preferences of the passengers do not shift from one fare class to another. Also, cancellations and no shows are not allowed. Previous studies in the literature aim to maximize the expected total revenue attained during the reservation period or the expected total marginal revenue of the seats sold. Both concepts are equivalent as it is explained in Section 3.3. Our objective, on the other hand, is not restricted to maximizing the expected revenue. It is usually necessary to provide the decision makers with more information about behaviour of the revenue as a function of the feasible policies to be used as pointed out by Filar *et al.* (1989). Expectation is not sufficient when the decision makers are concerned about the risks of the proposed policies. It is likely that the decision makers have a revenue target and do not want to have a high risk of attaining revenues below that target. A policy might give a high expected revenue, but at the same time, the probability that the revenue falls below the target level might be high because of the high variance of the revenue under this policy. Another policy, on the other hand, might give a lower expected revenue with lower variance and the revenue might fall below the target value with a tolerable probability. In such a case, the decision maker may choose the second alternative which is more likely to result in revenues above a certain desired level. However, these comparisons can be made if the probability distribution of the revenue is known under every feasible policy. When the probability distribution is known, any information about the risk probabilities and variability measures can be obtained.

With this perspective, our main aim in this thesis is to analyze the problem in terms of specified risk measures in addition to the expected revenue. For this purpose, we use the following risk measures: standard deviation and coefficient of variation of the revenue and probability of poor performance. The *probability of poor performance* is defined as the probability that revenue is below some predetermined target level. The main contribution of this thesis is the development of two mathematical models to incorporate the expectation and variance of the revenue. Also, probability distributions of the revenues are investigated under alternative allocation policies in order to gain information about the any relevant risk measure.

Two probabilistic mathematical models developed are called as EMVLP and CVLP models. Both of the models are linear programming formulations. EMVLP is to incorporate variance of the total marginal revenue into the objective function of the existing models in the literature and CVLP is to consider a constraint on a ratio similar to coefficient of variation. The first model we propose (EMVLP) penalizes variance of the revenue by a given factor while, at the same time, maximizing the expected revenue. The objective in EMVLP is to find the control policy that gives the best balance between expectation and variance of the total marginal revenue with respect to the penalty specified for the variance. When the variance penalty is equal to zero, the proposed model EMVLP is equivalent to SLP model in the literature. Since the proposed objective function does not only maximize the expected revenue, the optimal expected value of the revenue obtained by the proposed model turns out to be less than the ones obtained by the other models existing in the literature. However, when the optimal allocations resulting from the EMVLP model are used in a nested booking limit control or in a bid price control policy, we expect that the expected revenue might be higher as compared to using the optimal allocations of the other existing models. The probabilistic models existing in the literature show poor performance as compared to deterministic models especially when the variance of demand for higher fare classes is high. By penalizing variance to some degree, we aim to improve the probabilistic approach and obtain better results in terms of expected revenue. When variance of the revenue is penalized, we expect the number of seats allocated to fare classes with highly variable demand to decrease. Then, the level of overprotection would not be as much as in the other existing probabilistic models and the effect of ignoring nesting in the mathematical models would diminish. In the second model we propose (CVLP), the total expected marginal revenue is maximized under a constraint on the ratio of the expectation and variance of the total marginal revenue. That ratio is very similar to coefficient of variation. In order to eliminate nonlinearity in the models, we consider this ratio. However, it is very effective in decreasing the variance. Although each of the EMVLP and CVLP can be used, in this study we propose using them together. The EMVLP model is solved first and, then, the CVLP model is solved by taking the ratio of the expectation and variance of the total marginal revenue corresponding to the optimal EMVLP solution as a constraint in the CVLP model. The dual prices of the capacity constraints in the CVLP model are used in bid price control policy.

By assuming the demands of different ODFs are independent, we derive the probability distribution of the revenue under given seat allocations. Then, the distributions are fitted using the method proposed by Hahn and Shapiro (1967) in order to decide the type of the probability distribution. That revenue provides a lower bound for the revenue that will be obtained by nested booking control policies, because we do not integrate nesting of the ODFs. The probability distribution provides the decision makers with the behaviour of the revenue under any feasible allocation policy. With this information, the decision makers would have the flexibility of choosing any feasible allocation policy by comparing these lower bound revenues. The detailed information about the derivation of the revenue distributions are presented by a numerical study.

Organization of this thesis is as follows. In Chapter 2, an overview of the airline revenue management literature is presented relating our study to the previous ones. In Chapter 3, the existing models that have been developed for network seat allocation problems are reviewed and the models we propose are given. Then, the booking control policies are explained. Finally, we analytically derive and numerically evaluate the probability distributions of the revenue under given allocation policies. Chapter 4 is dedicated to a numerical study for comparison of the models we propose and the other existing models in the literature. The mathematical models are solved and the results (the seat allocations and bid prices) obtained by the mathematical models are used in simulation models. The probability distributions of the revenues under different policies derived analytically are compared with the simulation results. For a part of the simulation experiments, we allow nesting and use Bayesian updating method for demand estimations. Finally, Chapter 5 concludes the thesis with comments on further research directions.

CHAPTER 2

LITERATURE REVIEW

The research on airline seat allocation control problem can be classified as single-leg and multi-leg (network) problems. The earliest studies are especially on single-leg problems. For the network problems with high number of itineraries having connected flights, the development of approaches that take network effects into account become important. In this chapter, firstly the literature on single-leg problems is summarized. Solutions for single-leg problems form the basis for approaches to solve network problems. Then, the research on network problems is presented briefly. The approaches for network seat inventory control problem will be analyzed in detail in Chapter 3.

2.1 Single-Leg Problems

The methods proposed for solving single-leg problems result from static and dynamic control approaches. Static control approaches assume that passenger requests arrive sequentially in such a way that a passenger from a lower fare class book before all of the passengers from higher fare classes. This assumption makes modeling of the problem simpler because it is sufficient to know the total demand for each fare class that will come throughout the reservation horizon. However, in dynamic control approach, no assumption is made about the arrival pattern of the reservation requests and the demand is modeled as a stochastic time dependent process. The details of these models are presented in this section.

The first study in the area of airline seat inventory control problem is due to Littlewood (1972), where the optimal control policy for a single-leg problem with two fare classes is investigated with the assumption that lower fare class customers book before higher fare class customers. In that study it is suggested that a request of a lower fare class customer should be rejected, when the expected revenue of selling the seat to a higher fare class customer exceeds the lower fare.

Suppose that there are two fare classes with associated fares f_1 and f_2, such that $f_1 > f_2$. Random variable D_i denotes the demand of for class i. According to Littlewood's rule, customers of class 2 are accepted as long as f_2 exceeds the expected marginal revenue of a seat reserved for class 1. That is the rule to accept the customer of class 2 if

$$f_2 \geq f_1 Pr(D_1 \geq x),$$

where x is the number of seats reserved for customers of class 1. The seats allocated to class 1 will be sold if the demand for class 1 exceeds x. The revenue that will be earned from allocating an additional seat to class 1 is equal to f_1. Hence, $f_1 Pr(D_1 \geq x)$ is the expected marginal revenue of a seat reserved for class 1. Littlewood's rule characterizes the optimal policy. It is easily observed that the expected marginal revenue is monotonically decreasing in x. The optimal policy is such that the reservation requests of customers of class 2 are accepted as long as there are at least x^* seats available. They are rejected, if the number of available seats is less than x^*, where x^* satisfies

$$f_2 < f_1 Pr(D_1 \geq x^*) \text{ and } f_2 \geq f_1 Pr(D_1 \geq x^* + 1).$$

If D_1 is a continuous random variable, x^* is such that $f_2 = f_1 Pr(D_1 \geq x^*)$.

Belobaba (1987) extends Littlewood's study for the single-leg problem with multiple fare classes. He develops the heuristic called as Expected Marginal Seat Revenue (EMSR) for determining booking limits for each fare class. It is observed that the heuristic gives booking limits that are much different than the optimal booking limits. However, the simulation studies of Curry (1990), Wollmer

(1992), Brumelle and McGill (1993) and Robinson (1995) show that the revenue that would be generated with those booking limits is very close to optimal revenue.

After Belobaba (1987), the single-leg, multiple fare class problem is studied by Curry (1990), Wollmer (1992), Brumelle and McGill (1993). They derive optimal booking limits for the different fare classes while assuming a low before high arrival pattern. The difference between these studies is the assumption about the distribution of the demand for the fare classes.

Curry (1990) focuses on the problem by assuming a continuous demand distribution and develops a recursive equation for determining optimal booking limits for each fare class. Wollmer (1992) assumes a discrete probability distribution for demand of each fare class and derives similar equations for determining the optimal protection levels and booking limits. In the study of Brumelle and Mc Gill (1993), a booking limit control policy is developed that can be used both with discrete and continuous demand distributions.

Robinson (1995) develops optimal control policy for single-leg multiple fare class problem by relaxing the assumption of low before high arrival pattern, but assumes that all customers of a fare class book before any customers of another fare class.

In dynamic models, no assumption is made about the arrival of the different fare classes. The first dynamic programming formulation for single-leg problems is due to Lee and Hersh (1993). They divide the reservation period into decision periods in which at most one request can arrive and formulate the expected revenue function recursively. In their study, it is proposed that a request in period t with the available capacity x should be accepted as long as its fare exceeds the expected marginal revenue of the seat. They show that for each fare class there exist an optimal booking limit or an optimal decision period, after which the requests of these fare classes will not be accepted. In that study *batch bookings* are also included, where the number of bookings that a customer requests is not restricted to one. The optimal policy can be based on optimal decision periods in the presence of batch bookings.

Lautenbacher and Stidham (1999) present a review of static and dynamic control policies for single-leg problems. In that study, the static and dynamic models developed for single-leg problems are presented highlighting the similarities and differences between them. Moreover, it is shown that both approaches can be included with a Markov Decision Process (MDP) model formulation.

Subramanian *et al.* (1999) extend the study of Lee and Hersh (1993) by incorporating cancellations, overbooking and no-shows. They find out the equivalence of the problem to the optimal control of arrivals to a queueing system and obtain optimal booking limits for each fare class, when the cancellation probabilities are independent of the fare class. It is shown that the optimal booking limits are not monotonic in fare class, when the cancellation probabilities depend on the fare class. In that case, nesting of fare classes would not be wise. However, we do not consider these extensions in this study.

2.2 Network Problems

In Section 3.2, the deterministic and mathematical approximate models developed for network revenue management problems are presented in detail. The deterministic model assumes that demand is equal to its expectation, whereas the probabilistic models incorporates the uncertainty in the demand.

The first network formulation of the seat inventory control problem is due to Glover *et al.* (1982). In that study, a network flow formulation is proposed for the problem by assuming deterministic demand.

Wollmer (1986) is the first to propose a linear probabilistic model which is used when the demands for origin-destination pairs are discrete random variables. The model is based on the expected marginal revenues of the seats and,it is called expected marginal revenue (EMR) model. However, the large number of decision variables makes the model impractical to solve.

Simpson (1989) and Williamson (1992) develop the concept of bid price control for making seat allocation decisions. Bid prices are threshold values used for approximating the opportunity cost of each flight leg. The opportunity cost of an itinerary is found by the sum of bid prices of the flight legs it crosses. Bid price control policy is a heuristic for network seat inventory control decisions. The policy implies that an arrival request should be accepted if its fare exceeds the sum of the bid prices of the flight legs that are on the route of it. Bid prices are the dual prices of capacity constraints of the flight legs in the deterministic and probabilistic approximate models.

Williamson (1992) further investigates the network seat inventory control approaches based on the probabilistic and deterministic approximate mathematical models. The extensive simulation studies in that research show that deterministic approximate models perform often better than probabilistic models. Furthermore, nested booking control policy and bid price control policy perform similarly. In that sense, the study of Williamson (1992) leads significant improvements in network seat inventory control problem.

Talluri and van Ryzin (1998) derive the structure of the dynamic optimal control policy for network problems. However, due to the large scale of the airline networks it is not possible to derive and implement a dynamic optimal control policy. The theoretical background of bid price controls is analyzed and it is shown that bid prices are asymptotically optimal for large networks.

Talluri and van Ryzin (1999) suggest a different method for computing bid prices which they call as Randomized Linear Programming Method (RLP). In that method, a deterministic model is used with expected demand, but in a different way. A set of demand realizations is simulated and the deterministic model is solved for every realization of demand. The resulting bid prices are averaged and used in booking control policies. They state that RLP makes a small but significant improvement as compared to the deterministic model, but needs to be further analyzed.

McGill and van Ryzin (1999) present an overview of airline revenue management research. They classify airline revenue management in four ares: forecasting, overbooking, seat inventory control and pricing. An extensive literature review is presented emphasizing the interactions of these 4 areas.

An overview of solution methods developed for single-leg and network problems is due to Pak and Piersma (2002). In that review, the bid price control policy and nesting booking limit control policy are also included.

The closest research to our study is due to de Boer *et al.* (2002). In that study, it is argued that deterministic models result in higher expected revenues, because both probabilistic and deterministic models ignore nesting of different origin-destination and fare class combinations. They test the mathematical models for different problem scenarios and observe their performance with nested booking limit control and bid price control policies. They develop a nesting heuristic used in simulation of the booking process. The simulation studies in de Boer *et al.* (2002) indicate that generally deterministic models perform better, but there are cases when probabilistic models are better. Our study is different from de Boer in the sense that we develop new models for incorporating variance of the revenue and simulate the booking process in order to compare the proposed models with the other mathematical models existing in the literature.

Higle (2007) develops a two-stage stochastic programming model to compute bid prices. That study is not included in this thesis, because it does not consider each ODF separately. Seats on the flight legs are allocated to fare classes, and according to realized demand they are assigned to the routes subject to the leg based allocations. The model allows nesting of the fare classes of different origin-destination pairs, but the origin-destination and fare class combinations are nested according to the fare class not the net contribution to network revenue.

CHAPTER 3

APPROACHES FOR NETWORK SEAT INVENTORY CONTROL PROBLEM

In this chapter, the network seat inventory control problem is introduced and the existing and proposed approaches to solve this problem are presented. The structure of the optimal dynamic booking control policy for network problems is given in Section 3.1. However, optimal control is impossible for network problems due to the large scale of the airline networks. Therefore, the fundamental approach for network problems is to use approximation methods. In literature, approximate mathematical models have been developed for maximizing the expected revenue for the whole network. These models are classified as probabilistic and deterministic approximation models and they are given in Section 3.2.

All of the mathematical models developed so far find optimal seat allocations with the objective of maximizing expected revenue. However, decision makers may want to have information about the variability of the revenue obtained by those policies. Variability is a risk measure which is important when the decision maker do not want the revenues to vary severely between realized problem instances. Since demand is random, the realized revenue may differ from expectation, but the probability distribution of revenue provides information about the behaviour of the revenue. When the probability distribution is known, information about the measures like, standard deviation and coefficient of variation of the revenue, the probability that the revenue is below or above a certain level can be obtained.

The proposed two models, which take the variance of the revenue into account are presented in Section 3.3. In Section 3.4, the probability distribution of the revenue that would be obtained by given seat allocation policies are derived analytically.

The solutions of these models are used with seat inventory control policies, in order to make the allocation decisions during the reservation period. Partitioned booking limit control policy, nested booking limit control policy and bid price control policy are control policies used for network problems. However, partitioned booking limit control policy is not an effective policy and for this reason it is seldom used in practice. The other two policies, on the other hand, are commonly used for making allocation decisions. These three policies and the use of the solutions of the mathematical models with these policies are presented in Section 3.5. To cope with the dynamic nature of the problem, the models are solved several times throughout the reservation period, using revised capacities and updated demand estimations. Then, these solutions are used with updated booking control policies.

The existing and proposed models are numerically compared in terms of expected revenue, standard deviation of revenue, coefficient of variation and probability of poor performance in Chapter 4.

3.1 Network Seat Inventory Control Problem

Airlines operate on networks with many itineraries that are connected through several flight legs. Direct flights do not exist between every origin-destination pair. That is, the itineraries are composed of one or several flight legs. If a flight leg is crossed by several itineraries, the sale of all the itineraries crossing that flight leg is dependent on the availability of a seat on this flight leg. Hence, it becomes necessary to make allocation decisions considering the interdependencies between connected flight legs. Although the employment of leg-based control policies is a possible option for network problems, simulation experiments show that significant improvements in expected revenue can be obtained with a network approach. In leg-based control, each flight leg is considered individually and the seats on the flight leg are allocated to the fare classes by using the single-leg optimization methods. The drawback of the leg-based control policies in optimizing the network revenue is obviously disregarding the interdependencies of the itineraries. This is explained for the multi-leg flight shown in Figure 3.1. In this figure, there is a flight network consisting of 3 itineraries AB, BC and AC with two flight legs. Suppose that a high fare class request for itinerary AB arrives. When a leg-based control policy is applied, the request would be accepted since it is the request for the highest fare class. However, if there is available capacity on the flight from B to C, it may be more profitable to sell the seat to a low fare customer willing to fly from A to C. If the price of low fare class ticket for itinerary AC is higher than the price of a high fare class ticket for itinerary AB, then accepting the low fare request for itinerary AC would be more profitable.

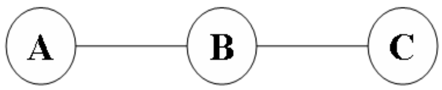

Figure 3.1: An illustration of a multi-leg flight network.

Network seat inventory control problem is more than just differentiating only the fare classes. As suggested by Williamson (1992) airline companies can develop better booking control policies with an approach that differentiates both the itineraries and the fare classes competing for the same seats on the flight legs. In single-leg problems, it is obvious that a higher fare class would contribute to the revenue more than a lower class. Thus, the main decision that should be made is whether to sell the seat or keep it for higher fare class customers. However, determining the contributions of the various origin-destination and fare class combinations is not trivial in network problems. In other words, it is difficult to calculate the opportunity cost of the seats that each ODF uses resolving the tradeoff between selling a seat to a higher fare class customer of a single-leg itinerary or to a lower fare class customer of a multi-leg itinerary.

The structure of the optimal control policy for the network problems is investigated by Talluri and van Ryzin (1998). The dynamic programming formulation the authors work with is given next. Consider an airline network having n itineraries operating over m flight legs. The reservation period T is divided into time periods and index t shows the current time period. Multiple bookings are not allowed and it is assumed that at most one reservation request can arrive in each time period. In other words, the time intervals are so small that the probability of more than one request is negligible. Let a_{ij} denote whether flight leg i is used by ODF j; that is,

$$a_{ij} = \begin{cases} 1 & \text{if flight leg } i \text{ is on the route of ODF } j, \\ 0 & \text{otherwise,} \end{cases}$$

and the matrix $\mathbf{A} = [a_{ij}]$ represents the relation between the flight legs and the ODFs. Accordingly, $\mathbf{A}_j$ is the (row) vector showing the flight legs used by ODF j and it is the transpose of the j^{th} column of matrix $\mathbf{A}$. The vector $\mathbf{x} = (x_1, ..., x_m)$ shows the available seat capacities on each flight leg i. If a request for ODF j is accepted, then the updated available capacity will be $\mathbf{x} - \mathbf{A}_j$.

The fares of each ODF j is assumed to be random and the demand for period t is modeled by the vector $\mathbf{F}(t) = (F_1(t), ..., F_n(t))$. That is, $F_j(t) = f_j \geq 0$ shows that a request for itinerary j arrives in period t and its fare is f_j and $F_j(t) = 0$ means that there is no arrival request for ODF j in period t. Since there is at most one arrival in each period, at most one component of the vector $F_j(t)$ is greater than zero. Suppose that a reservation request arrives at time t, the main decision that should be made is whether to accept or reject this request. This decision depends on the available seat capacity and the fare of the ODF. Define U($\mathbf{x}$)denoting the accept/reject decisions for ODFs, and $u_j(t, \mathbf{x}, f_j)$ shows the decision for ODF j at time t. That is,

$$u_j(t, \mathbf{x}, f_j) = \begin{cases} 1 & \text{if the request for ODF } j \text{ with fare } f_j \text{ is accepted at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

Let $V_t(\mathbf{x})$ denote the maximum expected revenue in period t, which must satisfy the Bellmann equation

$$V_t(\mathbf{x}) = max_{\mathbf{u} \in U(\mathbf{x})} \; E[\{\mathbf{F}(t)^T \mathbf{u}(t, \mathbf{x}, \mathbf{f}) + V_{t+1}(\mathbf{x} - \mathbf{A}\mathbf{u})\}]$$

with the boundary condition $V_{T+1}(\mathbf{x}) = 0$, for all $\mathbf{x}$. $F(t)^T$ is the transpose of $F(t)$. The optimal expected revenue function $V_t(\mathbf{x})$ is finite for all x as shown by Talluri and van Ryzin (1998). This observation leads to the existence of an optimal control policy. The optimal $\mathbf{u}^*(.)$ satisfies the following equation

$$u_j^*(t, \mathbf{x}, f_j) = \begin{cases} 1 & \text{if } f_j \geq V_{t+1}(\mathbf{x}) \text{ - } V_{t+1}(\mathbf{x} - \mathbf{A}_j) \text{ and } \mathbf{A}_j \leq \mathbf{x}, \\ 0 & \text{otherwise.} \end{cases}$$

The optimality condition given above implies that a reservation request is accepted as long as its fare exceeds the opportunity cost of the seats it uses and there is available capacity. That is,

$$f_j \geq V_{t+1}(\mathbf{x}) - V_{t+1}(\mathbf{x} - \mathbf{A}_j) \text{ and } \mathbf{A}_j \leq \mathbf{x}.$$

Although the optimal policy is characterized theoretically, it is very difficult to evaluate $V_t(\mathbf{x})$ for all $\mathbf{x}$ and t. This is because of the large state space with all possible $\mathbf{x}$ and t. Airlines usually have hundreds of origin-destination and fare class combinations. Due to the large scale of the airline network, it is not possible to derive and implement an optimal policy as noted by Talluri and van Ryzin (2005). The major difficulty for the implementation is that information about the demand estimations for each ODF, available seat capacities and the fare structures should be monitored and stored and processed dynamically throughout the reservation period. Accordingly, the optimal policy should be updated. Thus, the major criterion that needs to be considered in choosing a booking control policy is the applicability of the policy in practice for a given large-scale airline network. For this purpose, several heuristic solutions are proposed for network problems. Leg-based control policy, which considers each flight leg separately and allocates seats on the flight leg to different fare classes, is a possible option for network problems. Although the implementation of a leg-based control policy is easy, taking a network approach would be important in maximizing network revenue. In the following parts of this chapter, the mathematical models that have been developed for making allocation decisions for network problems are presented. Note that these models are approximate and do not give optimal policies. Solutions of these are used with booking control policies.

3.2 Mathematical Programming Formulations

Several mathematical models have been developed to determine the optimal seat allocations for different combinations of itineraries and fare classes. These models give probabilistic and deterministic approximations of the optimal expected revenue function. Exactness is claimed for the dynamic programming formulation given in the previous section. The main advantage of the approximate mathematical models is that each itinerary from an origin to a destination pair with an associated fare class is regarded as one product and denoted by an ODF. The drawback, on the other hand, is that the models give static one-time allocations and nesting of the ODFs is not considered.

The nested control policy allows a higher fare class customer to buy the seat as long as there are available seats allocated to lower classes. This policy is used by many airlines and increases the revenues compared to partitioned policy for a given allocation. Figure 3.2 shows a partitioned allocation policy, where 20 seats are allocated to fare class 3, 20 seats to fare class 2 and 10 seats for fare class 1. After the 10 seats available for fare class 1 are sold, this policy will reject the reservation requests for fare class 1. However, it is obvious that the highest fare class is the most profitable class and rejecting customers of this class and selling the remaining seats to lower fare classes would not make sense.

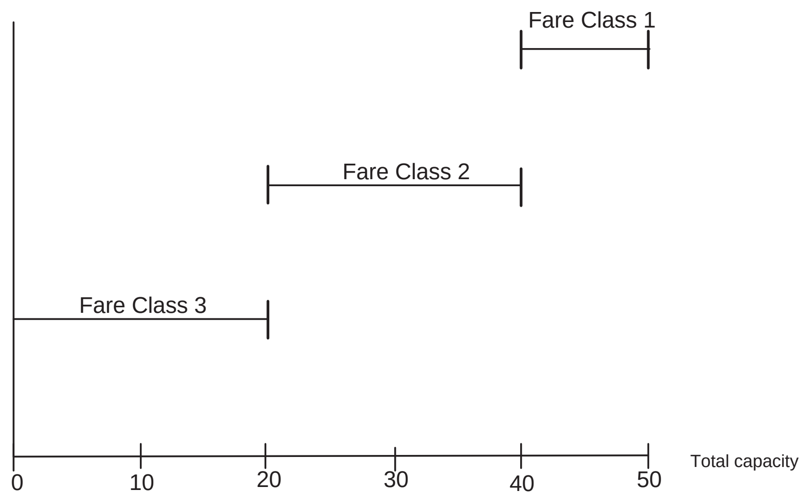

Figure 3.2: An illustration of partitioned allocation policy.
(Williamson, 1992)

Therefore, instead of partitioned allocations, booking limits (BL) are defined for the fare classes as it is illustrated in Figure 3.3. The figure represents the nesting structure for the same single-leg flight. The booking limit for the highest fare class is 50 implying that customers of this class will not be rejected until all of the seats are occupied. Similarly, the booking limit for the next highest class, class 2, is 40, which means that class 2 customers will be accepted as long as the number of seats sold do not exceed 40. If there are already 40 seats sold, the remaining number of seats are open only for customers of fare class 1. Lastly, the booking limit for fare class 3 is 20 and class 3 customers will not be allowed for bookings after these 20 seats are sold. When, there are less than 30 number of seats left, seats are closed for customers of fare class 3.

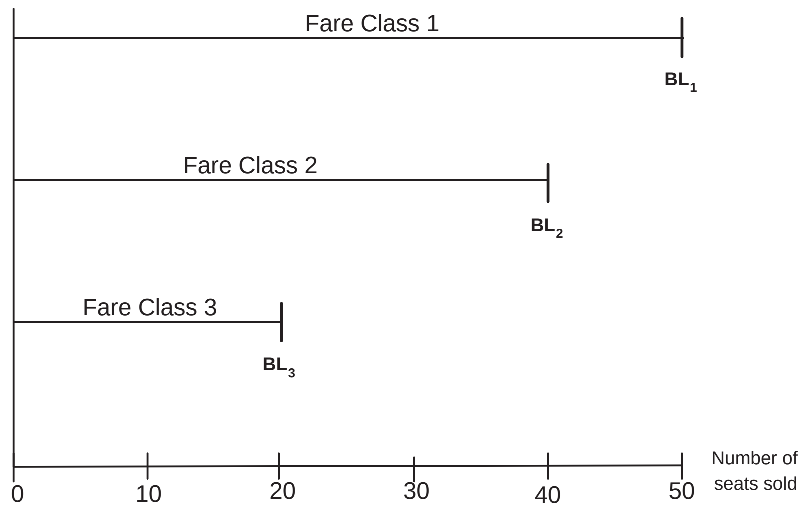

Figure 3.3: An illustration of nested allocation policy .
(Williamson, 1992)

Hence, in nested booking limit control policy, the booking limit for a fare class is an upper bound for the number of seats that can be sold to that fare class and any lower fare classes. The remaining number of seats above that booking limit are protected for higher fare classes. However, for the network problems, determining the nesting order of the ODFs is not trivial. Since opportunity cost of each ODF is difficult to estimate, the net contributions of the ODFs to the revenue is not so obvious. In Section 3.5, we present a nesting heuristic proposed by de Boer *et al.* (2002).

The mathematical models determine separate allocations for each ODF such that a seat allocated to an ODF can be sold only to customers of that ODF. If there is no demand for that particular ODF, the seats would be unsold. However, these seats can be sold to other customers with higher fares and extra revenue is earned from the seat which, otherwise, would remain empty. The resulting allocations are, then, used with nested allocation policies. The revenue of the nested allocation policy is always higher than the corresponding partitioned allocation policy.

As noted above, the mathematical models give one-time static allocations but not dynamic allocations. This drawback is handled by solving the models more than once during the reservation period according to the revised capacities and updated demand estimations. As bookings are realized, the remaining capacities of the flight legs are updated and, as new information about the demand becomes available, the demand estimations for the remaining part of the reservation period are updated. Through observation of the booking process, the models are solved again and the updated allocations are used for the remaining part of the reservation period.

Next, the assumptions and the notation used for the mathematical models are introduced. In the mathematical models, each itinerary from an origin to a destination pair with an associated fare class is regarded as one product and denoted by an ODF. The demand distribution assumptions are made for each ODF separately and they are updated throughout the reservation period. Demand for each ODF is assumed to be independent and switching of passengers from one fare class to another is not allowed. The fares of the ODFs are assumed to be known and constant throughout the reservation period. The objective of the models is to allocate optimal number of seats to these ODFs considering both the available seat capacity and the random demand behaviour. The notation used in the mathematical models is as follows:

j: index for the ODFs in the airline network, $j = 1, ..., n$,

l: index for flight legs in the network, $l = 1, ..., m$,

S_l: set of ODFs that use flight leg l,

T_j: set of flight legs that are on the route of ODF j,

C_l: available seat capacity for flight leg l,

f_j: fare of each ODF j,

D_j : random demand for ODF j during the whole reservation period.

3.2.1 Probabilistic Mathematical Programming Models

The most general problem developed for network seat inventory control problem is called as Probabilistic Mathematical Programming (PMP) Model and its formulation is as follows.

$$\begin{aligned} PMP: \ & Maximize\ E(\sum_j f_j min(D_j, x_j)) \\ & subject\ to \\ & \sum_{j \in S_l} x_j \leq C_l\ for\ l = 1, ..., m, \\ & x_j \geq 0\ and\ integer\ for\ j = 1, ..., n. \end{aligned}$$

In this model, x_j is the decision variable defined for the number of seats allocated to ODF j. This definition indicates that each seat is reserved for only one particular ODF. If the demand for an ODF turns out to be at least as many as the number of seats allocated to this ODF, all the x_j number of seats will be sold. However, if the demand turns out to be less than the allocation x_j, the number of seats sold will be equal to the demand of the ODF and the remaining seats will be unsold. The objective function is to maximize the expected total revenue. Here, the uncertainty in demand is included in the model. However, working with the probability distribution of the demand explicitly in the objective function causes the objective function to be nonlinear. In order to simplify the model, the LP relaxation of PMP is considered. The resulting model is called as Probabilistic Nonlinear Programming ($PNLP$) Model. PNLP is the same as PMP except that x_js are not restricted to be integer. That is, by the LP relaxation we mean relaxing the integrality constraints for only x_js, the objective function stays nonlinear. It is shown that PNLP is tight for multiple-leg flights and single-hub networks where the capacities of the legs are integer and, it gives integer solutions. The related references for this observation are due to de Boer *et al.* (2002).

Wollmer (1986) develops a linear model, which can be used when the demand is a discrete random variable. The formulation is to maximize the expected total marginal revenues of the seats and it is called as Expected Marginal Seat Revenue (EMR) Model.

$$EMR: \; Maximize \sum_{j=1}^{n} \sum_{i=1}^{C_l} f_j Pr(D_j \geq i) x_j(i)$$

$$subject \; to$$

$$\sum_{j \in S_l} \sum_{i=1}^{C_l} x_j(i) \leq C_l, \; for \; l = 1, ..., m,$$

$$x_j(i) \in \{0, 1\} \; for \; j = 1, ..., n \; and \; i = 1, ..., C_l,$$

where the decision variable $x_j(i)$ is defined as follows:

$$x_j(i) = \begin{cases} 1 & \text{if } i \text{ or more seats are allocated to ODF } j, \\ 0 & \text{otherwise.} \end{cases}$$

The coefficient of $x_j(i)$ in the objective function shows the expected marginal seat revenue obtained by allocating the i^{th} seat to ODF j.

Remark 1. *Note that $Pr(Dj \geq i)x_j(i)$ is a monotonically decreasing function of i; for this reason, expected marginal revenues are monotonically decreasing in i. This ensures that if $x_j(i+1)$ is equal to 1, then $x_j(i)$ will be definitely 1. In other words, $x_j(i+1)$ cannot be 1 unless $x_j(i)$ is 1. Hence, the model would not assign fractional values to $x_j(i)$ and $x_j(i+1)$. This property enables us to relax the integrality assumption for $x_j(i)s$ and the $x_j(i)s$ take the values of 0 or 1 in the LP Relaxation of the model. These observations are due to Williamson (1992).* □

The major disadvantage of this model is the large number of decision variables. Although the integrality constraints can be relaxed, working with the EMR model is not practical for large networks.

de Boer *et al.* (2002) propose another probabilistic linear model reducing the number of decision variables as compared to EMR. Rather than considering each possible value of demand in the model, they partition demand into intervals. That is, the total demand is equal to the sum of the demands in each interval. This model is called as Stochastic Linear Programming (SLP) Model.

$$SLP: \ Maximize \ \sum_j f_j x_j - \sum_j f_j \sum_{i=1}^{k_j} Pr(D_j < d_j(i)) x_j(i)$$

$$subject \ to$$

$$\sum_{j \in S_l} x_j \leq C_l \ for \ l = 1, ..., m,$$

$$x_j = \sum_{i=1}^{k_j} x_j(i) \ for \ j = 1, ..., n,$$

$$x_j(1) \leq d_j(1),$$

$$x_j(i) \leq d_j(i) - d_j(i-1) \ for \ j = 1, ..., n \ and \ i = 2, ..., k_j,$$

$$x_j(i) \geq 0 \ and \ integer \ for \ j = 1, ..., n \ and \ i = 2, ..., k_j.$$

In this model, the demand of each ODF j is divided into k_j parts and each allocation $x_j(i)$ represents the amount of seats allocated to the i^{th} partition of the demand of ODF j and $d_j(i)$ is the maximum value of demand for ODF j that corresponds to the partitions i, $i-1$, ..., 1. The sum of $x_j(i)$s over all partitions show the total amount of seats allocated to ODF j.

The SLP Model is equivalent to the EMR Model when each demand interval is of unit size. This equivalence can easily be seen between the solution spaces of the two models. The first term in the objective function in SLP shows the revenue that would be generated if all the aircrafts depart fully loaded, and the second term is a correction for the uncertainty in demand. Rewriting the objective function of SLP, one can see the equivalence also between the objective functions of SLP and EMR (de Boer *et al.*, 2002). Therefore, as in the EMR model, the SLP model also would not assign fractional values to x_js and $x_j(i+1)$ cannot be 1 unless $x_j(i)$ is 1.

When at least one of the demand intervals is not of unit size, SLP is an approximation to reduce the number of decision variables as compared to EMR. The model can be simplified by decreasing the number of demand portions, but the resulting allocations may result in smaller expected revenues. Hence, (the LP relaxation of) EMR is just a special case of SLP when the demand partition is such that $d_j(i+1) - d_j(i) = 1$ and $d_j(1) = 1$. Analogous to the numerical observations for EMR, de Boer *et al.* (2002) show that, for multiple-leg flights and single-hub networks, the LP relaxation of SLP is tight and the solutions consist of integer allocations. LP Relaxation of SLP is such that integrality constraints for $x_j(i)$s in SLP are relaxed.

3.2.2 Deterministic Mathematical Programming (DMP) Model

The deterministic model is obtained by assuming that the demand for an ODF is deterministic and equal to the expected demand. It is a special case of SLP, where the only demand scenario to be considered is the expected demand.

$$
\begin{aligned}
DMP:\ & Maximize \sum_j f_j x_j \\
& subject\ to \\
& \sum_{j \in S_l} x_j \leq C_l\ for\ l = 1, ..., m, \\
& x_j \leq E(D_j)\ for\ j = 1, ..., n, \\
& x_j \geq 0\ and\ integer\ for\ j = 1, ..., n.
\end{aligned}
$$

$E(D_j)$ is the expected value of the demand for ODF j. This integer programming model is not practical to work with large numbers of decision variables and constraints. In this case, use of LP relaxation of this model, which is called as Deterministic Linear Programming DLP, is preferred. It is shown that, for multiple-leg flights or single-hub networks, when the demand expectations and the flight leg capacities are integers DLP formulation result in integer allocations (e.g., Bertsimas and Tsitsiklis, 1997).

The DLP is advantageous because it is easy to solve. The drawback of the model is that all the uncertainty in demand is ignored. At first glance, it is usual to expect that ignoring the random nature of demand would lead to poor results. However, the simulation studies of Williamson (1992) and de Boer *et al.* (2002) show that DLP gives better results compared to probabilistic models in terms of expected revenue.

de Boer *et al.* (2002) argue that the main factor leading to better performance of DLP as compared to the probabilistic models is that the models do not consider nesting of ODFs. It is numerically observed that although this is a drawback of both deterministic and probabilistic models, the probabilistic models are affected more as compared to the deterministic model. The numerical experiments show that both models allocate a large number of seats to higher fare classes because they are more profitable. Then, the arriving requests of lower fare class customers are rejected and the load factor of the flights turns out to be low. However, the seats allocated to lower fare classes can also be used by higher fare classes when nesting is allowed. When the variance of high fare class demand is high, the probabilistic models reserve large number of seats for higher fare classes. Since the revenue loss resulting from the decreased load factor of the flights would be larger than the increase in the revenue obtained from higher fare class customers, the probabilistic model performs worse. Solutions of the deterministic models, on the other hand, do not change with the demand distributions, as long as the expected demand remains constant. Then, the impact of ignoring nesting in the deterministic models is not as much as in the probabilistic models. However, for partitioned allocations, the probabilistic model solutions perform better. Note that partitioned allocation policy is not a common policy in airline operations.

In the next section, the models we propose are presented. The variance of the revenue is incorporated in the models. They result in seat allocations which give the best balance between the expected revenue and variance of the revenue.

3.3 The Proposed Models

The main idea that motivates us to develop new models is to give the decision makers more information about the revenue that is obtained by an allocation policy. For the decision makers, knowing the behaviour of the revenue in terms of risk measures may be critical. With this perspective, we build two new models that incorporate the variance of the revenue. Our first model penalizes the variance of the revenue by a given factor and the penalty term is subtracted from the expected revenue in the objective function. In the second model we propose, the objective function maximizes expected revenue subject to an additional constraint that limits the ratio of the variance and expectation of marginal revenue.

With the proposed models, the variance induced by the policies would be decreased and we also expect to increase the expected revenue for some values of the penalty factor.

Recall that the PMP model gives optimal seat allocations x_j that maximize the expected revenue function $E(\sum_j f_j min(D_j, x_j))$. Our aim is to build a model which incorporates the variance of the revenue in the objective function. Note that for a given allocation scheme (given x_js) and discrete random variables D_js, $min(D_j, x_j)$ can be calculated, and the expectation and variance of the revenue can be obtained. However, the values of x_js are not known beforehand, and our aim is to determine the optimal x_j for each ODF. Let $Z_j = min(D_j, x_j)$. First, the derivations for the expectation and variance of Z_j are given. Then, the expectation and variance of the revenue are formulated as functions of the random variable Z_j.

Derivation 1. *Recall that demand for each ODF j, D_j, is assumed to be a discrete random variable. Then, for a given integer allocation $\boldsymbol{x}$,*

$$Pr(Z_j = z_j) = \begin{cases} Pr(D_j = Z_j) & \text{if } Z_j < x_j, \\ Pr(D_j \geq x_j) & \text{if } Z_j \geq x_j. \end{cases}$$

The expected value of Z_j is formulated as

$$E(Z_j) = \sum_{z_j=0}^{x_j-1} z_j Pr(D_j = z_j) + x_j Pr(D_j \geq x_j).$$

The variance of Z_j is

$$
\begin{aligned}
Var(Z_j) &= E((Z_j)^2) - (E(Z_j))^2 \\
&= \sum_{z_j=0}^{x_j-1} (z_j)^2 Pr(D_j = z_j) + (x_j)^2 Pr(D_j \geq x_j) \\
&\quad - [\ \sum_{z_j=0}^{x_j-1} (z_j)^2 (Pr(D_j = z_j))^2 + (x_j)^2 (Pr(D_j \geq x_j))^2 \\
&\quad + 2 \sum_{z_j=0}^{x_j-1} \sum_{z_j^{'}=1}^{x_j-2} z_j z_j^{'} Pr(D_j = z_j) Pr(D_j = z_j^{'}) \\
&\quad + 2 \sum_{z_j=0}^{x_j-1} z_j x_j Pr(D_j = z_j) Pr(D_j \geq x_j)\ ].
\end{aligned}
$$

Then by using the derivations above, the expectation and variance of the revenue $R = \sum_j f_j Z_j$ for a given allocation $\boldsymbol{x}$ and given f_j values are

$$
\begin{aligned}
E(R) &= \sum_{j=1}^{n} f_j E(Z_j) \\
&= \sum_{j=1}^{n} f_j \sum_{z_j=0}^{x_j-1} z_j Pr(D_j = z_j) + x_j Pr(D_j \geq x_j),
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
Var(R) &= \sum_{j=1}^{n} f_j^2 Var(Z_j) \\
&= \sum_{j=1}^{n} f_j^2 \sum_{z_j=0}^{x_j-1} (z_j)^2 Pr(D_j = z_j) + (x_j)^2 Pr(D_j \geq x_j) \\
&\quad - [\ \sum_{z_j=0}^{x_j-1} (z_j)^2 (Pr(D_j = z_j))^2 + (x_j)^2 (Pr(D_j \geq x_j))^2 \\
&\quad + 2 \sum_{z_j=0}^{x_j-1} \sum_{z_j^{'}=1}^{x_j-2} z_j z_j^{'} Pr(D_j = z_j) Pr(D_j = z_j^{'}) \\
&\quad + 2 \sum_{z_j=0}^{x_j-1} z_j x_j Pr(D_j = z_j) Pr(D_j \geq x_j)\ ].
\end{aligned}
$$

where $Z_j = min\{D_j, x_j\}$ *for* $j = 1, ..., n.$ $\square$

In order to determine the optimal allocation policy, we define the decision variable $u_j(x_j)$ such that

$$u_j(x_j) = \begin{cases} 1 & \text{if } x_j \text{ seats are allocated to ODF } j, \\ 0 & \text{otherwise.} \end{cases}$$

By using the derivations above and setting a variance penalty factor θ, we can consider the following formulation:

$$\begin{aligned}
&Maximize \sum_{j=1}^{n} f_j \sum_{x_j=0}^{h_j} u_j(x_j) \sum_{z_j=0}^{x_j-1} (z_j Pr(D_j = z_j) + x_j Pr(D_j \geq x_j)) \\
&- \theta \sum_{j=1}^{n} f_j^2 \sum_{x_j=0}^{h_j} u_j(x_j) [\sum_{z_j=0}^{x_j-1} ((z_j)^2 Pr(D_j = z_j) + (x_j)^2 Pr(D_j \geq x_j)) \\
&\quad - (\sum_{z_j=0}^{x_j-1} (z_j)^2 (Pr(D_j = z_j))^2 + (x_j)^2 (Pr(D_j \geq x_j))^2 \\
&\quad + 2 \sum_{z_j=0}^{x_j-1} \sum_{z_j^{'}=z_j+1}^{x_j-2} z_j z_j^{'} Pr(D_j = z_j) Pr(D_j = z_j^{'}) \\
&\quad + 2 \sum_{z_j=0}^{x_j-1} z_j x_j Pr(D_j = z_j) Pr(D_j \geq x_j))] \\
&subject\ to
\end{aligned}$$

$$\sum_{x_j=0}^{h_j} u_j(x_j) = 1 \ for\ j = 1, ..., n, \tag{3.2}$$

$$\sum_{j \in S_l} \sum_{x_j=0}^{h_j} x_j u_j(x_j) \leq C_l \ for\ l = 1, ..., m, \tag{3.3}$$

$$u_j(x_j) \in \{0, 1\} \ for\ j = 1, ..., n \ and\ x_j = 1, ..., h_j. \tag{3.4}$$

In this model, θ is used as a penalty for the revenue variance. The first term of the objective function is the expected revenue and the second term represents the variance of the revenue penalized by the factor θ. Constraint (3.2) is to assure that only one value of $u_j(x_j)$ is equal to 1 for each ODF j. In this constraint h_j is the upper limit for x_j and it represents the maximum number of seats that can be allocated to ODF j which is equal to the number seats that are available on all of the flight legs that the ODF crosses. The value of h_j is determined as follows: $h_j = min_{l \in T_j}\{C_l\}$. Constraint (3.3) is the capacity constraint of the flight legs. Constraint (3.4) sets the decision variables $u_j(x_j)$ to binary values. The major drawback of this model is the large number of binary decision variables, which makes the model impractical.

We build a new model which is an extension of the SLP model developed by de Boer *et al.* (2002). In this model, we take a slightly different approach and set a penalty factor for the variance of the marginal revenue rather than the variance of the total revenue. This approach works well as seen in Chapter 4 on the numerical studies.

After the derivations used for the model are given, the formulation of the model is presented.

Derivation 2. *Let the random variable $F_j(i)$ denote the marginal seat revenue that is obtained when the additional i^{th} seat is allocated to ODF j. That is,*

$$F_j(i) = \begin{cases} f_j & \textit{if } D_j \geq i, \\ 0 & \textit{otherwise,} \end{cases}$$

and the marginal revenue to be denoted by MR is given as follows: $MR = \sum_j \sum_i F_j(i)x_j(i)$, where $x_j(i)$ is as it is defined in Section 3.2.1. The expected marginal revenue of the network is formulated as

$$\begin{aligned} E(MR) &= \sum_j \sum_i E(F_j(i))x_j(i) \\ &= \sum_j \sum_i f_j Pr(D_j \geq i)x_j(i), \end{aligned} \tag{3.5}$$

which is equal to the objective function in the EMR model. Note that $f_jPr(D_j \geq i)$ is the expected marginal revenue of the i^{th} seat allocated to ODF j. variance of the marginal revenue of the network is formulated as follows:

$$\begin{aligned} Var(MR) &= \sum_j \sum_i (x_j(i)^2) Var(F_j(i)) \\ &= \sum_j \sum_i (x_j(i)^2)[E(F_j(i)^2) - (E(F_j(i)))^2] \\ &= \sum_j \sum_i (x_j(i)^2) f_j^2 (Pr(D_j \geq i) - (Pr(D_j \geq i))^2) \\ &= \sum_j \sum_i (x_j(i)^2) f_j^2 Pr(D_j \geq i) Pr(D_j < i). \qquad \square \end{aligned} \tag{3.6}$$

Note that the expectation of marginal value and the expected revenue are equal, but the variance of total marginal revenue is not equal to the variance of the total network revenue as it is shown in the remark below.

Remark 2. *For any given allocation $\boldsymbol{x}$:*

$$\begin{aligned} E(R) &= \sum_j f_j \sum_{z_j=0}^{x_j-1} z_j Pr(D_j = z_j) + x_j Pr(D_j \geq x_j) \\ E(MR) &= \sum_j f_j \sum_{i=0}^{x_j} Pr(D_j \geq i). \end{aligned} \tag{3.7}$$

Since we assume that demands are discrete random variables,

$$Pr(D_j \geq i) = Pr(D_j \geq x_j) + \sum_{i}^{x_{j-1}} Pr(D_j = i). \tag{3.8}$$

Substituting equation (3.8) into the equation (3.7), we obtain,

$$E(MR) = \sum_j f_j \sum_{i=0}^{x_j - 1} iPr(D_j = i) + x_j Pr(D_j \geq x_j), \tag{3.9}$$

which is equivalent to the expected revenue denoted by E(R). However, Var(MR) and Var(R) are not equal to each other. The formulation of Var(MR) is given in equation 3.6.

These observations are given numerically in Chapter 4. By using the derivations above, our proposed model is given as follows:

$$\begin{aligned}
&Maximize \ \sum_j \sum_{i=1}^{h_j} x_j(i) f_j Pr(D_j \geq i) \\
&\quad - \theta \sum_j \sum_{i=1}^{h_j} (x_j(i)^2) f_j^2 Pr(D_j \geq i) Pr(D_j < i) \\
&subject \ to \\
&\sum_{j \in S_l} \sum_{i=1}^{h_j} x_j(i) \leq C_l \ for \ l = 1, ..., m, \\
&x_j(i) \in \{0, 1\} \ for \ j = 1, ..., n \ and \ i = 1, ..., h_j,
\end{aligned}$$

where $h_j = min_{l \in T_j}\{C_l\}$. Since the $x_j(i)$ is a binary decision variable, $(x_j(i)^2)$ is also equal to either 0 or 1. Hence, we can replace $(x_j(i))^2$ in the objective function with $x_j(i)$ and obtain a linear model. Then, the model reduces to

$$
\begin{aligned}
EMVR: \ Maximize \ & \sum_{j}\sum_{i=1}^{h_j} x_j(i) f_j Pr(D_j \geq i) \\
& - \theta \sum_{j}\sum_{i=1}^{h_j} x_j(i) f_j^2 Pr(D_j \geq i) Pr(D_j < i) \\
& subject \ to \\
& \sum_{j \in S_l}\sum_{i=1}^{h_j} x_j(i) \leq C_l \ for \ l = 1, ..., m, \\
& x_j(i) \in \{0, 1\} \ for \ j = 1, ..., n \ and \ i = 1, ..., h_j.
\end{aligned}
$$

As in the EMR and SLP models, the proposed model would not assign fractional values to $x_j(i)$, and $x_j(i+1)$ cannot be 1 unless $x_j(i)$ is 1. This is shown in Lemma 1.

Lemma1. *The objective function*

$$
\sum_{j}\sum_{i=1}^{h_j} x_j(i) f_j Pr(D_j \geq i) - \theta \sum_{j}\sum_{i} x_j(i) Pr(D_j \geq i) f_j^2 Pr(D_j < i)
$$

is decreasing in i.

Proof. Rewriting the objective function in the EMVR model,

$$
\begin{aligned}
& \sum_{j}\sum_{i} x_j(i) f_j [Pr(D_j \geq i) - \theta f_j^2 Pr(D_j \geq i)(1 - Pr(D_j \geq i)) \ ] \\
= & \sum_{j}\sum_{i} x_j(i) f_j Pr(D_j \geq i)(1 - \theta(1 - f_j Pr(D_j \geq i))).
\end{aligned}
$$

Since $Pr(D_j \geq i)$ is decreasing in i, both the first term $f_j Pr(D_j \geq i)$ and the second term $(1 - \theta(1 - f_j Pr(D_j \geq i))$ are decreasing in i.

□

The proposed EMVR model is an extension of the EMR Model developed by Wollmer (1986). As in the case of obtaining SLP, working on the EMR model we come up with the following model which we call as the EMVLP Model:

$$
\begin{aligned}
EMVLP: \; & Maximize \sum_j \sum_i x_j(i) f_j Pr(D_j \geq i) \\
& - \theta \sum_j \sum_i x_j(i) f_j^2 Pr(D_j \geq i) Pr(D_j < i) \\
& subject\ to \\
& \sum_{j \in S_l} x_j \leq C_l \ for\ l = 1, ..., m, \\
& x_j = \sum_{i=1}^{k_j} x_j(i) \ for\ j = 1, ..., n, \\
& x_j(1) \leq d_j(1), \\
& x_j(i) \leq d_j(i) - d_j(i-1) \ for\ j = 1, ..., n\ and\ i = 2, ..., k_j, \\
& x_j(i) \geq 0 \ for\ j = 1, ..., n\ and\ i = 2, ..., k_j.
\end{aligned}
$$

In this model, the first term of the objective function is the expected marginal revenue and the second term represents the variance of the marginal revenue penalized by θ. With this formulation, we expect to obtain allocations that would result in more stable booking control policies. Decreasing the variance would decrease also the expected revenue as it is stated in the following remark.

Remark 3. *Recall that EMVLP is equivalent to SLP when $\theta = 0$. Therefore, the optimal solution of EMVLP is a feasible solution for SLP. Hence, the following relation holds for the optimal expected revenue of the SLP model and EMVLP model denoted by $E_{SLP}(MR)$ and $E_{EMVLP}(MR)$ respectively such that $E_{SLP}(MR) \geq E_{EMVLP}(MR)$.*

However, we aim to attain policies giving a reasonable balance between expectation and variance of the revenue. Also, we aim to decrease the probability of poor performance, which can be defined as the probability that the revenue is less than some predetermined level. Even for some degree of θ, we expect to attain not only less variance but also higher expected revenues compared to EMR and DLP. Since the variance is penalized, the model would allocate less number of seats to higher fare classes that usually have higher variances. As a result, the nested control policies would perform better with the allocations of the proposed models even in terms of expected revenue. The question on how the solutions of the proposed models vary as θ changes is addressed in Chapter 4.

Note that, in bid price control policy, the opportunity costs of the flight legs are the dual prices of the capacity constraints in the mathematical models. However, in the proposed model, the objective function does not represent the expected revenue. Therefore, the dual prices of the capacity constraints are not comparable with the fares of the ODFs. In order to implement bid price control policy, we develop a second model, in which an upper limit ξ is considered for the coefficient of variation of the marginal revenue such that

$$\frac{Var(MR)}{(E(MR))^2} \leq \xi^2. \tag{3.10}$$

Recall that $E(MR)$ and $Var(MR)$ are considered in the objective function of the EMVR model. With the formulation in 3.10, variance of the revenue is incorporated into the proposed model using the following constraint: $Var(MR) \leq \xi E(MR)$. In the proposed model, ξ is an upper bound on the coefficient of variation of the marginal revenue, and the objective function aims to maximize expected marginal revenue. Then, the model we propose is given as follows:

$$
\begin{aligned}
&Maximize\ \sum_j \sum_{i=1}^{k_j} x_j(i) f_j Pr(D_j \geq i) \\
&subject\ to \\
&\sum_j \sum_{i=1}^{k_j} x_j(i) f_j^2 Pr(D_j \geq i) Pr(D_j < i) \leq \\
&\quad \xi^2 \sum_j f_j^2 [\sum_{i=1}^{k_j} (x_j(i))^2 (Pr(D_j \geq i))^2 \\
&\qquad + 2 \sum_{i=1}^{k_j} \sum_{i'=i+1}^{k_j-1} x_j(i) x_j(i') Pr(D_j \geq i) Pr(D_j \geq i')] \qquad (3.11) \\
&\sum_{j \in S_l} x_j \leq C_l\ for\ l = 1, ..., m, \\
&x_j = \sum_{i=1}^{k_j} x_j(i)\ for\ j = 1, ..., n, \\
&x_j(1) \leq d_j(1), \\
&x_j(i) \leq d_j(i) - d_j(i-1)\ for\ j = 1, ..., n\ and\ i = 2, ..., k_j, \\
&x_j(i) \in \{0, 1\}\ for\ j = 1, ..., n\ and\ i = 2, ..., k_j.
\end{aligned}
$$

The first constraint represents $Var(MR) \leq \xi(E(MR))^2$, where

$$
\begin{aligned}
E(MR) &= \sum_j \sum_{i=1}^{k_j} f_j Pr(D_j \geq i) x_j(i) \\
Var(MR) &= \sum_j \sum_i (x_j(i)^2) Var(F_j(i)) \\
&= \sum_j \sum_i (x_j(i)^2) [E(F_j(i)^2) - (E(F_j(i)))^2] \\
&= \sum_j \sum_i (x_j(i)^2) f_j^2 (Pr(D_j \geq i) - (Pr(D_j \geq i))^2) \\
&= \sum_j \sum_i (x_j(i)^2) f_j^2 Pr(D_j \geq i) Pr(D_j < i), \qquad \square
\end{aligned}
$$

as they are given in Derivation 2. Since the dual prices of the capacity constraints now show the contributions to the expected revenue, the bid prices that are obtained from the model are comparable with the fares. However, this model is not linear. With this formulation, we aim to obtain bid prices comparable with the fares. The problem with this model is that $(E(MR))^2 = (\sum_j \sum_i x_j(i)Pr(D_j \geq i)f_j)^2$ creates nonlinearity and we cannot replace $(x_j(i))^2$ with $x_j(i)$ because product of the decision variables with different i values as $x_j(i)$ and $x_j(i)'$ are included in the model. We handle this nonlinearity by defining a binary decision variable $w_j(i)$ where

$$w_j(i) = \begin{cases} 1 & \text{if } x_j(i) = x_j(i+1) = 1 \ , \\ 0 & \text{otherwise.} \end{cases}$$

Then, the model becomes

$$Maximize \ \sum_{j}\sum_{i=1}^{k_j} x_j(i) f_j Pr(D_j \geq i)$$

$$subject \ to$$

$$\sum_{j}\sum_{i=1}^{k_j} x_j(i) f_j^2 Pr(D_j \geq i) Pr(D_j < i) \leq$$

$$\xi^2 \sum_{j}\sum_{i=1}^{k_j} f_j^2 [(x_j(i))^2 (Pr(D_j \geq i))^2 + 2w_j(i) Pr(D_j \geq i) Pr(D_j \geq (i+1))],$$

$$w_j(i) \geq x_j(i) + x_j(i+1) - 1, \ for \ j = 1, ..., n, \ and \ i = 1, ..., k_j$$

$$w_j(i) \leq x_j(i+1), \ for \ j = 1, ..., n, \ for \ j = 1, ..., n, \ and \ i = 2, ..., k_j$$

$$w_j(i) \leq x_j(i), \ for \ j = 1, ..., n, \ for \ j = 1, ..., n, \ and \ i = 2, ..., k_j$$

$$\sum_{j \in S_l} x_j \leq C_l \ for \ l = 1, ..., m,$$

$$x_j = \sum_{i=1}^{k_j} x_j(i) \ for \ j = 1, ..., n,$$

$$x_j(1) \leq d_j(1),$$

$$x_j(i) \leq d_j(i) - d_j(i-1) \ for \ j = 1, ..., n \ and \ i = 2, ..., k_j,$$

$$x_j(i) \in \{0, 1\} \ for \ j = 1, ..., n \ and \ i = 2, ..., k_j.$$

Since $x_j(i)$s are binary decision variables, we cannot obtain bid prices working with this model. However, our major aim is to penalize the variance. Thus, we choose to use a different measure which is not equal to the coefficient of variation, but the ratio of the expectation and variance of the revenue. We define this new measure to be bounded as $\frac{Var(MR)}{E(MR)} \leq \rho$ and develop a linear model which we call as the $CVLP$ Model.

$$
\begin{aligned}
CVLP: \ & Maximize \ \sum_j \sum_{i=1}^{k_j} x_j(i) f_j Pr(D_j \geq i) \\
& subject \ to \\
& \sum_{j \in S_l} x_j \leq C_l \ for \ l = 1, ..., m, \\
& x_j = \sum_{i=1}^{k_j} x_j(i) \ for \ j = 1, ..., n, \\
& x_j(1) \leq d_j(1), \\
& x_j(i) \leq d_j(i) - d_j(i-1) \ for \ j = 1, ..., n \ and \ i = 2, ..., k_j, \\
& \sum_j \sum_{i=1}^{k_j} x_j(i) f_j^2 Pr(D_j \geq i) Pr(D_j < i) \leq \\
& \qquad\qquad\qquad \rho \sum_j \sum_i x_j(i) f_j Pr(D_j \geq i), \\
& x_j(i) \geq 0 \ for \ j = 1, ..., n \ and \ i = 2, ..., k_j.
\end{aligned}
$$

In bid price control policy we experiment with in this study, EMVLP and CVLP models are used together. We first solve the EMVLP model, then the value of ρ is determined as the ratio of variance of the marginal revenue and the expected marginal revenue corresponding to the optimal solution of EMVLP. This ρ value is put into the CVLP model and the model is solved. The resulting dual prices of the CVLP model are comparable with the fares and they are used for determining the bid prices. In other words, the optimal seat allocations are found by the EMVLP model, but the contribution of the ODFs to the revenue are found by the CVLP model. The CVLP model results in the same optimal seat allocations, but it gives bid prices which can be used for making accept/reject decisions by comparing them with the fares.

3.4 Analysis of the Revenue Distribution

Knowing the distribution of the revenue that will be earned by an allocation policy allows the decision makers to obtain any information about the behaviour of the revenue. That is, they can decide which alternative to choose by making comparisons with respect to several measures instead of only the expected revenue or variance of the revenue. The probability distribution of the revenue gives information about any risk measure that the decision makers would need to know and is an effective tool for comparing alternative feasible policies that are competent in terms of different measures.

Deriving the probability distribution of the network revenue analytically provides the decision makers with the flexibility of knowing the behaviour of the revenue at any time during the reservation period and accordingly changing the booking policy. For example, suppose the company uses a nested booking limit control policy and update the seat allocations throughout the reservation period. The decision maker, at some point in time, may wonder how much revenue will be earned with any feasible allocation policy $\mathbf{x}$ during the remaining part of the reservation period. When the probability distribution of the revenue for any given allocation policy $\mathbf{x}$ is known, the decision maker gain any information about the revenue to attain in the remaining part of the reservation period, which is a lower bound for the revenue that will be obtained by a nested booking control policy.

In this section, we investigate the probability distribution of the revenue under a given allocation **x**. Recall that for given seat allocations x_j and known parameters of total demand distribution for each ODF j, we can derive the probability distribution of the revenue analytically but without using simulation. For each possible demand realization $D_j = d_j$ for $j = 1, ..., n$, we calculate the values of $z_j = min(d_j, x_j)$ and its occurrence probability $Pr(D_j = z_j)$. Let R_j denote the revenue that will be earned from passengers of ODF j by allocating x_j number of seats to ODF j.

$$R_j = f_j Z_j = \begin{cases} d_j f_j & \text{if } d_j < x_j \text{ with probability } Pr(D_j = d_j) \\ x_j f_j & \text{if } d_j \geq x_j \text{ with probability } Pr(D_j \geq x_j). \end{cases} \tag{3.12}$$

Recall that our assumption is that demands for different ODFs are independent. Then, the total network revenue $\sum_j R_j$ to be denoted by R is found as follows:

$$Pr(R = r) = \prod_j Pr(R_j = r_j). \tag{3.13}$$

By using equations (3.12) and (3.13), we write a macro in Excel and find the possible values of the total network revenue R with the corresponding probabilities. Then, we investigate this probability distribution. The numerical analysis is performed for the base problem that is given in Chapter 4 and it is presented in Section 4.1. The base problem is solved working with DLP and SLP models and the optimal seat allocations are obtained. For these optimal seat allocations, the probability distributions of the revenue are derived as explained above. The type of the distribution is investigated by using the method proposed by Hahn and Shapiro (1967). They develop a graph defining the regions for different probability distributions in the plane, wher the x axis and y axis of the graph are skewness and kurtosis of the probability distribution. Some distributions are defined with one point or line instead of a region. The type of the probability distribution is determined by calculating the skewness and kurtosis of the distribution and matching them with the corresponding point or region.

For our numerical problem, we investigate the revenue distribution under optimal DLP and SLP models and it is presented in Section 4.1. The skewness and kurtosis of the distributions show that they fit to the Normal Distribution. Additionally, Normal probability plots of the revenues are drawn and it is observed that revenues obtained by the two models fit well to Normal Distribution.

3.5 Booking Control Policies for Network Problems

There are three different control policies used for network problems. These are partitioned (non-nested) booking limit control policy, nested booking limit control policy and bid price control policy. This section presents these booking control policies. The common characteristic of the policies is that all of them use the results of the mathematical models. However, they show differences in using the results.

3.5.1 Partitioned (Non-nested) Booking Limit Control Policy

In partitioned booking limit control policy, seats are allocated to each ODF separately. The policy uses directly the optimal allocations resulting from the mathematical models. Because of the random nature of the demand, such an allocation policy may cause the aircrafts to depart with empty seats, means that this is not an effective policy. For this reason, partitioned booking limits are not commonly used in practice. However, the revenue obtained using a partitioned booking limit policy provides a lower bound for the network revenue. The reason for this is that the allocations are only feasible solutions for nested policies, but they are not optimal.

3.5.2 Nested Booking Limit Control Policy

In nested booking limit control policies, the ODFs are ranked according to their contributions to network revenue. The highest ranked ODF is allowed to book all the available seats. Similarly, the seats allocated to a fare class are allowed to be be booked by the customers of a higher ranked ODF. The main consideration in nesting the ODFs is the criterion to be used in ranking the ODFs. Although nesting fare classes is trivial for a single-leg problem, the net contributions of the ODFs to the revenue are not easy to determine in network problems.

Williamson (1992) considers three ways of ranking the ODFs. These are ranking by fare class, ranking by fares and ranking by dual price. When ODFs are ranked by fare class, a full fare class of an origin-destination pair is always rated higher than a discount fare class without taking the fare amounts into account. However, a discount fare passenger of a long journey may contribute more than a full fare passenger for a short flight. Ranking by fares is not a preferable way of ranking ODFs. In this case, long itineraries will be ranked higher than the short itineraries because the fares of long itineraries are usually higher as compared to the short itineraries. When ODFs are ranked according to their fares, many seats would be protected for the long itineraries and the requests for the other short itineraries would be rejected. Hence, the load factor of the aircraft may decrease resulting in a revenue loss.

Williamson (1992) proposes ranking the ODFs based on the dual prices of the ODFs. The dual price of an ODF is defined as the additional revenue that is generated by allocating one more seat to that ODF when the other allocations remain unchanged. For the DLP model, Williamson (1992) uses the dual prices of the demand constraints or the reduced costs of the decision variables. In EMR model, on the other hand, there is no demand constraint. Working with EMR model, Williamson (1992) uses the dual price as the incremental revenue that is generated when the mean demand of the ODF is increased by one. However, this is not easy for a network problem. When the mean demand is changed, the distribution parameters and the objective function coefficients of the decision variables, $f_j Pr(D_j \geq i)$, change. Therefore, the EMR model needs to be solved for a second time. The studies performed by Williamson (1992) show that nesting by dual prices gives better performance than nesting by total fare or fare class.

Here, we use the dual prices of the capacity constraints of the flight legs, which is a ranking method proposed by de Boer *et al.* (2002). We approximate the opportunity costs of an ODF with the sum of the dual prices of the flight legs that are connected for the ODF. Then, the net contribution of the ODF to the network revenue is found by subtracting this opportunity cost from the total fare of the ODF and the ODFs are ranked. Namely, the net contribution of an ODF is $f_j - \sum_{l \in T_j} w_l$, where w_l is the dual price of the capacity constraint of flight leg l. Incorporating nesting into the mathematical models is difficult. Therefore, as for the partitioned booking limit policy, the optimal allocations obtained by the mathematical models are used also for nested booking limit control policy. However, these allocations are used with nested heuristics to make accept/reject decisions. Suppose the mathematical models are solved and the ODFs are ranked using the dual prices of the capacity constraints of each flight leg l, and $j \geq i$ means that ODF j is ranked higher than ODF i. In simulating nested booking limit control policy, we use the heuristic algorithm developed by de Boer *et al.*(2002). This algorithm is given below.

Nesting Heuristic

Step 0. *Let $\eta(j)$ be the number of requests of ODF j that have already been accepted and C_l be the remaining capacity of flight leg l. Let C_l to be equal to the initial capacity and let $\eta(j) = 0$ for every ODF j.*

Step 1. *A booking request for an ODF i arrives and should be considered for acceptance.*

Step 2. *Let* $b(j) = max\{x(j) - \eta(j), 0\}$ *for all ODF* j.

Step 3. *Let* $b_l = \sum_{j \in S_l \ni j \geq i} b(j)$ *for each flight leg* $l \in T_i$.

Step 4. *Let* $c_{min} = min_{l \in T_j}\{C_l - b_l\}$.

Step 5. *If* $c_{min} \geq 0$, *accept the booking request, let* $C_l = C_l - 1$ *for all* $l \in T_i$ *and let* $\eta(j) = \eta(j) + 1$. *Otherwise, reject the request.*

Step 6. *Go to step 1 for the next reservation request.*

Step 0 is the beginning of the booking process; the capacities of the flight legs are initialized and there are not any accepted requests. In step 1, the arriving booking request for a particular ODF is considered. The $b(j)$ shows the number of seats that we want to protect against lower ranked ODFs. When a request for ODF is accepted, $b(j)$ is decreased by one because the number of seats to be protected decreases. The b_l values are calculated for the arriving requests. They represent the booking limit of the ODF on each flight leg. In other words, the seats above b_l are protected for fare classes that are ranked higher than the ODF. In step 4, the number of available seats for the ODF i is calculated. In step 5, if there is available seat for the ODF, the request is accepted. The number of accepted requests is increased by one and the capacity of the flight legs that are used by the ODF is decreased by one. Otherwise, the request is rejected.

Throughout the reservation period, when the seat allocations are updated according to the available capacity and estimated demand estimations, the allocation amounts and the nesting order of the ODFs may change. Therefore, the algorithm should be restarted, when the allocations are updated.

3.5.3 Bid Price Control Policy

Bid price control is introduced by Simpson (1989) as an alternative seat inventory control approach for network problems. In bid price control policy, threshold values (bid prices) are determined for each flight leg, which are used to approximate the opportunity cost of an itinerary. The idea behind the policy is that a reservation request should be accepted if its fare exceeds the sum of the bid prices of the flight legs on the path of the itinerary.

Recall the structure of the optimal control due to Talluri and van Ryzin (1998) given in Section 3.1. According to the optimal control policy, a reservation request at time t with corresponding fare f_j will be accepted as long as the fare exceeds the opportunity cost of the flight legs that the itinerary uses and there are seats available on the corresponding flight legs. The optimal decision is denoted by $u_j^*(t, \mathbf{x}, f_j)$ and it satisfies the following equation:

$$u_j^*(t, \mathbf{x}, f_j) = \begin{cases} 1 & \text{if } f_j \geq V_{t+1}(\mathbf{x}) \text{ - } V_{t+1}(\mathbf{x} - \mathbf{A}_j) \text{ and } \mathbf{A}_j \leq \mathbf{x}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. *A control* $u_t(\boldsymbol{x}, f)$ *is said to be a bid price control if there exists real valued functions* $\mu(t, \boldsymbol{x}) = (\mu_1(t, \boldsymbol{x}), \mu_2(t, \boldsymbol{x}), ..., \mu_m(t, \boldsymbol{x}))$, $t = 1, ..., T$ *(called bid prices) such that*

$$u_j^*(t, \boldsymbol{x}, f_j) = \begin{cases} 1 & \textit{if } f_j \geq \sum_{i \in A_j} \mu_i(t, \boldsymbol{x}) \textit{ and } \boldsymbol{A}_j \leq \boldsymbol{x}, \\ 0 & \textit{otherwise.} \end{cases}$$

In this definition, μ_i is the opportunity cost of a seat on a flight leg i, and the opportunity cost of an itinerary is approximated by the sum of the costs of each flight leg that is on the route of that itinerary. Throughout the reservation period, as the state of the network changes, the opportunity costs change. Therefore, $\mu_i(t, \mathbf{x})$ is defined which shows the opportunity cost of the flight leg i for each level of capacity $\mathbf{x}$ and for each time period t. These opportunity costs are called bid prices. An arriving request for an ODF is accepted if and only if there is available capacity and the corresponding fare exceeds the sum of the bid prices of the flight leg that the ODF uses. Namely, the opportunity cost of an itinerary is approximated by the sum of the bid prices of the flight legs that it crosses. Bid prices are optimal only when the opportunity cost of an itinerary $V_{t+1}(\mathbf{x}) - V_{t+1}(\mathbf{x} - A_j)$ is equal to the sum of the opportunity costs of selling each flight leg separately. However, most of the time, a change in the capacity of a flight leg does not affect the revenue linearly. Talluri and van Ryzin (1998) provide examples, where they show that bid prices are suboptimal. However, they show that bid prices are asymptotically optimal and under certain "large number" scalings of the network revenue management problems. Talluri and van Ryzin (1998) state that the performance of the bid price control policy depends on the mathematical models that are used for determining bid prices and if the variance of demand is properly included in the models, more revenue can be generated.

Bid price control policy is commonly used in real life. The major factor leading to this popularity is that bid price control policy is easy to implement. It requires much less data to be stored compared to nested booking limit policy. In nested booking limit control policy, the booking limits are updated according the realized demand and available capacity. Hence, the booking limits of each ODF should be stored at each point in time. Also, at any point in time the net contribution of the ODFs change as it is explained in the nesting heuristic presented in Section 3.5.2. Therefore, additional data like the dual prices of the capacities or the reduced costs of the variables is required to be stored in order to rank the ODFs. Working with bid price control policy, on the other hand, the only information that should be stored is the bid price of each flight leg. Moreover, the accept/reject decision requires only the comparison of the fare of the arriving request with the sum of the bid prices.

The LP models presented in this chapter, namely DLP, SLP, and use of EMVLP and CVLP together, give approximate bid prices to be used throughout the booking process. Note that EMVLP model does not provide reasonable bid prices, but it is used together with CVLP. The bid prices are determined using dual prices of the capacity constraints of each flight leg in the LP models. The main difference between nested booking limit control and bid price control policies results from the use of the model results. Booking limit control approach uses directly the optimal allocations obtained by the mathematical model whereas the bid price control approach makes accept/reject decisions working with the bid prices obtained using the dual prices of the capacity constraints of the flight legs.

A major drawback of the bid price control policy is that once the bookings are opened to an ODF, there is no limit on the number of seats that will be sold to that ODF. However, as the departure time approaches, it may be more profitable to close bookings to that ODF and sell the seats to ODFs having higher fares. Hence, if the bid prices are set at the beginning of the reservation period and the same set of bid prices are used throughout the whole period, the aircraft may be filled with customers of ODFs with lower fares compared to optimal control policy. Therefore, the bid prices should be revised according to the realized bookings throughout the reservation horizon. Otherwise, the bid price control approach would not be effective for making accept/reject decisions.

CHAPTER 4

NUMERICAL STUDY

This chapter is devoted to a numerical comparison of the performances of different mathematical models and booking policies. The network structure and the parameters for the numerical experiments are the same as the ones used by de Boer *et al.* (2002). The network representation of the problem de Boer *et al.* (2002) work with is given in Figure 4.1. The flight network has 6 different itineraries *(AB, AC, AD, BC, BD, CD)* operating on 3 flight legs, each having a capacity of 200 seats. There are 3 different fare classes offered for each itinerary. Hence, we consider 18 origin-destination and fare combinations in total. The length of the reservation period is 150 days.

We analyze three different problem scenarios: base problem, the problem with increased low fare demand variance and the problem with decreased fare variance. In the base problem, the variance of demand for fare classes 2 and 3 are small as compared to fare class 1 and fare of class 1 is relatively expensive. In the second scenario, the variance of fare class 2 and 3 are increased. In the third scenario, demand variances are the same as the variances in the base case, but the fare of fare class 1 is decreased. Changing demand distribution parameters and fare structures allows us to compare the performance of the models for different cases. The information related to the fares and demand distributions of the ODFs are given in Appendix A. Tables A.1 and A.3 represents the fare and demand data used for the first scenario and Tables A.2 and A.4 show the data used in the other scenarios.

We determine the optimal allocations by using the mathematical models presented in Chapter 3 including the proposed model EMVLP, then, simulate the

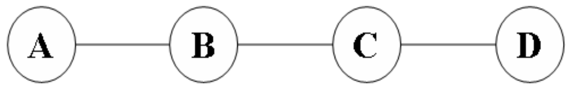

Figure 4.1: The multi-leg flight network.

booking process according to these allocations. In simulation model, we use the three control policies presented in Section 3.5, namely partitioned booking limit control policy, nested booking limit control policy and bid price control policy. When working with bid price control policy, we use EMVLP and CVLP models together because as it is explained in Chapter 3, EMVLP model alone does not give bid prices comparable with fares. For the three control policies, the model performances are compared in terms of expected revenue, standard deviation and coefficient of variation of the revenue, and the risk of poor performance.

Throughout the reservation period, we update the booking control policies at certain points in time. The demand update mechanism is as follows: as new information about the demand becomes available, the demand estimations for the remaining periods are revised by conditioning on the previous realizations. Then the allocations for the ODFs are updated throughout the reservation period by taking the available seat capacity into account. For updating the demand, we use Bayesian demand updating mechanism presented in Section 4.3. For partitioned booking limit control policy and nested booking control policy, we present the performances when the allocations are determined only once at the beginning of the reservation period. For bid price control policy, on the other hand, we present the performances attained with the updated bid prices and the performances when the bid prices are determined only once at the beginning of the reservation period. We do not update partitioned booking limit control policy since it is not an effective policy. Furthermore, it is shown by de Boer *et al.*

(2002) that bid price control policy and nested booking limit control policy give very similar results when they are used for the same models. Therefore, we choose to update only bid price control policy which is easier to implement.

In Section 4.1, a preliminary analytical study is performed in order to observe the distribution of revenue under a given allocation $\mathbf{x}$. In Section 4.2, the model used for the customer arrival process and the assumptions of demand distributions are presented. Section 4.3 is on the Bayesian demand updates. Lastly, Section 4.4 is devoted to the evaluation of the numerical results.

4.1 Numerical Analysis of the Revenue Distribution

In this section, we analyze the revenue distribution in our base problem for seat allocations obtained by solving DLP and SLP models. As it is explained in Section 3.4, we solve the DLP and SLP models and obtain the optimal seat allocations x_j. Then, for each ODF j, we calculate the $Z_j$values and their probabilities. The possible revenue values for each ODF j are found by multiplying these Z_js with the associated fares. The possible values of network revenue are found by adding these revenues obtained from each ODF j and their probabilities are calculated by multiplying the occurrence probabilities of each possible value of the revenue for ODF j. As an example, the possible revenues of the whole network and their probabilities are given in Appendix D for the DLP model. Evaluating the probabilities and the revenue values allows us to fit the probability distribution and determine its type. When the type of the probability distribution of the revenue is known, it is possible to compute any measure related to the network revenue without using simulation. Then, at any point in time, the decision maker can know the probability distribution of the revenue that would be earned for any feasible allocation.

As it is seen in Figures B.1 and B.2 in Appendix B, for our base problem, the probability distributions of the network revenue obtained by using the seat allocations of DLP and SLP models fit well to the Normal Distribution. The resulting revenue is the revenue that would be obtained by a partitioned booking limit control policy, and it provides a lower bound for the revenue that would be obtained by a nested control policy. In Section 4.4.1, it will be seen that this revenue distribution is very similar to the one obtained by simulating partitioned booking control policy. Note that the expected revenue of the SLP model is higher as compared to DLP model. We expect to observe this when simulating partitioned allocation policy with the seat allocations obtained by DLP and SLP models.

4.2 Simulation Model

The solutions obtained by the mathematical models are used in the simulation of the partitioned booking limit policy, nested booking limit policy and bid price policy. In the simulation of partitioned booking limit control policy, we directly use the allocations obtained by the mathematical models. For simulating the nested booking limit control policy, the nesting heuristic proposed by de Boer *et al.* (2002) and presented in Section 3.5.2 is used. In the simulation of bid price control policy, at the beginning of the reservation period the mathematical model is solved using initial capacities. Then, the dual prices of the capacity constraints *(bid prices)* that result from the mathematical models are given as an input to the simulation model. The booking policy is such that a request is accepted if and only if its fare exceeds the sum of bid prices that the ODF traverses. The output of the simulation model gives the remaining available seat capacity at any point in time. Throughout the reservation period, the bid prices are updated using the available seat capacity and the updated demand estimations for the remaining part of the reservation period.

The major issue while simulating the booking process is how to determine the total demand of each ODF to come over the whole reservation period and how to model arrivals of the reservation requests for different ODFs in time. The number of booking requests to arrive until the departure is not known and an assumption should be made about the demand distribution. Knowing the total demand distribution is not sufficient to simulate the booking process. The distribution of the arrivals in time is also important in determining the revenue. The characteristic of the airline booking process is that the arrival rate of the customers is not constant throughout the reservation period. Moreover, different fare classes have different arrival patterns. Lower fare class customers arrive usually at the early stages of the reservation period whereas higher fare classes usually arrive close to the end of the reservation period. In order to model these features, a Non-Homogeneous Poisson Process (NHPP) is used. With a NHPP, both the non-stationary arrival rates and different arrival patterns over the reservation period can be considered in the simulation model. The non-stationary demand for each ODF j at time t is assumed to be Poisson distributed with rate $\lambda_j(t)$. McGill and van Ryzin (1999) note that Poisson distribution is appropriate for modeling the arrival of airline reservation requests. However, it is stated in their study that there is the following deficiency arising while working with Poisson arrivals for airline demand: if the arrivals in each period are Poisson distributed, then the aggregate demand distribution over the whole reservation period will be Poisson with equal mean and variance. For perishable items and also for airline demand, such an assumption is not reasonable, because variance of the airline demand distribution is usually much higher compared to the mean. The simulation model is given in Appendix E.

This problem is handled by modeling the time dependent Poisson arrival rate of an ODF j as follows:

$$\lambda_j(t) = Y_j \beta_j(t). \tag{4.1}$$

Here Y_j is the expected total number of demand for ODF j over the whole reservation period and $\beta_j(t)$ represents the Beta distribution used for spreading the total demand over the whole reservation period. In other words, Beta distribution models the arrival pattern of the requests of different fare classes such that $\int_0^T \beta_j(t)dt = 1$ for each ODF j.

Y_j is assumed to have a Gamma distribution with parameters p_j and δ_j. That is, $Y_j \sim$ Gamma (p_j, δ_j). Beckmann and Bobkowski (1958) show that Gamma distribution is reasonable for modeling the aggregated airline demand. The probability density function of Gamma distributed Y_j with parameters p_j and δ_j is:

$$f(y_j) = \frac{\delta^{p_j} y_j^{(p_j-1)} e^{(-\delta y_j)}}{\Gamma(p_j)} \qquad for \quad p_j \geq 0, \ \delta_j \geq 0, \tag{4.2}$$

where Γ is the following Gamma function:

$$\Gamma(p_j) = \int_0^\infty y_j^{p_j-1} e^{-y_j} dy_j. \tag{4.3}$$

If p is integer, $\Gamma(p_j) = (p_j - 1)!$. The related reference is due to Hines and Montgomery (1980).

With the formulation given in (4.1), the resulting arrival process is a conditional Poisson process rather than a pure Poisson process. Hence, the aggregate distribution of demand over the whole reservation period is not Poisson.

Remark 4. *Suppose the demand in a period is Poisson distributed with rate* λ*, and* λ *is Gamma distributed with parameters* p *and* δ*. Then, the unconditional distribution of demand* D_t *over* $[0,t]$ *is Negative Binomial with parameters* p *and* $\delta/(\delta+t)$*.*

The proof given by Popovic (1987) is presented below.

Proof. Suppose that the random rate of the demand in a time unit is λ. Then, the conditional probability of observing k units of demand in a the time interval [0,t] is

$$Pr(D_t = k|\lambda) = \frac{(\lambda t)^k}{k!} e^{(-\lambda t)}, k = 0, 1, ... \tag{4.4}$$

Since the distribution of the arrival rate λ is Gamma(p, δ), its probability density function is

$$f(\lambda) = \frac{\delta^p \lambda^{(p-1)} e^{(-\delta\lambda)}}{\Gamma(p)}. \tag{4.5}$$

Then, by using the equations (4.4) and (4.5), the unconditional demand distribution is the following Negative Binomial Distribution with parameters p and $\delta/(\delta+t)$.

$$\begin{aligned} Pr(D_t = k) &= \int_0^\infty Pr(D_t = k|\lambda) f(\lambda) d\lambda \\ = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{(-\lambda t)} f(\lambda) d\lambda &= \begin{pmatrix} p+k-1 \\ k \end{pmatrix} (\frac{\delta}{\delta+t})^p (\frac{t}{\delta+t})^k \\ & for\ k = 0, 1, ... \end{aligned} \tag{4.6}$$

□

In our problem, the arrival rate λ_j of each ODF j for the whole reservation period is Gamma distributed with parameters p_j and δ_j. Then, the equation

(4.6) implies that the distribution of aggregate demand throughout the reservation period is Negative Binomial with parameters p_j and $\delta_j/(\delta_j+1)$ for each ODF j, where $t=1$ corresponds to the whole reservation period. Expectation and variance of Negative Binomial distribution are $\frac{p_j}{\delta_j}$ and $\frac{p_j}{\delta_j}(1+\frac{1}{\delta_j})$, respectively. The parameters of the aggregate demand distributions, expectation and standard deviation of demands used for the numerical experiments are given in Appendix A for each ODF.

In order to model the passenger arrivals throughout the reservation period, Beta distribution is used as it is given in equation (4.1). $\beta(t)$ actually shows the percentage of arrivals to come in period t. Figure 4.2 illustrates behaviour of the Beta distribution with different parameters.

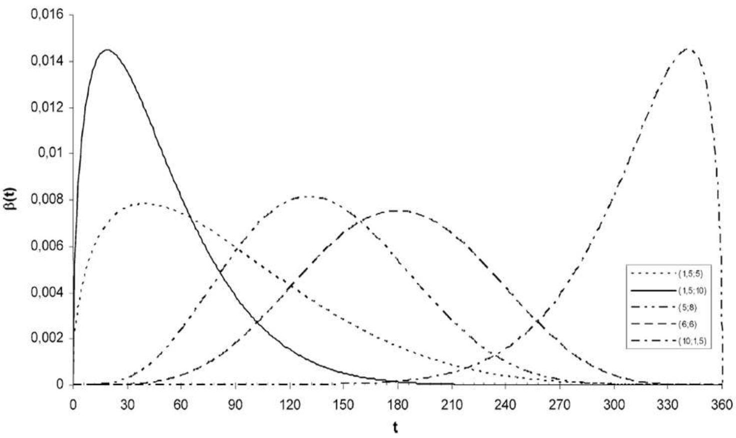

Figure 4.2: Beta density functions with different parameters.
(Kimms and Mller-Bungart, 2007)

In our problem, different parameters are used for modeling the Beta arrival patterns of different fare classes denoted by α_j and γ_j. For fare class 3, a Beta distribution with parameters $(5,6)$ is used. For fare class 2, the distribution is Beta $(2,5)$; and it is Beta $(2,13)$ for fare class 1. As it is seen in Figure 4.2, the different parameters of the beta distribution allow us to model different arrival patterns of the fare classes such that lower fare classes arrive densely at the beginning of the reservation period whereas most of the higher fare class customers arrive close to the end of the reservation period.

The probability density function of Beta distribution with parameters α and γ is:

$$\frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\gamma)}x^{\alpha-1}(1-x)^{\gamma-1} \qquad \alpha \geq 0, \quad \gamma \geq 0. \tag{4.7}$$

Then,

$$\beta(t) = (\frac{1}{T})^{\alpha-1}(1-\frac{t}{T})^{\gamma-1}\frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\gamma)}$$

as shown by de Boer (1999). Let $t = T$ denote the beginning of the reservation period and $t = 0$ be the time of the departure of the flight. In other words, the periods are numbered such that period t shows the remaining number of periods until departure. In our setting, periods correspond to days. The distribution of the arrivals of each ODF j is given as, when there are t periods (days) remaining until the departure,

$$\int_0^t (\frac{\lambda_j}{t})(\frac{x}{t})^{\alpha_j-1}(1-\frac{x}{t})^{\gamma-1}\frac{\Gamma(\alpha_j+\gamma_j)}{\Gamma(\alpha_j)\Gamma(\gamma_j)}dx. \tag{4.8}$$

Equation (4.8) is evaluated in Remark 5 below for integers α_j and γ_j. These observations are due to de Boer (1999).

Remark 5. *If α_j and γ_j are both integers, the arrival rate for remaining t periods will be,*

$$\int_0^t (\frac{\lambda_j}{t})(\frac{x}{t})^{\alpha_j-1}(1-\frac{x}{t})^{\gamma_j-1}\frac{\Gamma(\alpha_j+\gamma_j)}{\Gamma(\alpha_j)\Gamma(\gamma_j)}dx$$

$$= \lambda_j \frac{\Gamma(\alpha_j+\gamma_j)}{\Gamma(\alpha_j)\Gamma(\gamma_j)} \sum_{l=0}^{\gamma_j-1} \begin{pmatrix} \gamma_j - 1 \\ l \end{pmatrix} (-1)^l \frac{1}{l+\alpha_j}(\frac{t}{T})^{\alpha_j+l}. \tag{4.9}$$

Different than the approach used by de Boer *et al.* (2002), we make an approximation and take the arrival rates constant in each period, but different across time periods. Recall that, days are numbered such that the beginning of the reservation period is T, and t shows the number of remaining days until departure. Then for an ODFj, let $\beta_j(t;0)$ shows the percentage of the arrivals to come after day t, namely in the remaining t days of time and $\beta_j(T;t)$ is the percentage of the arrivals before day t. We divide the reservation period into N time periods of equal length such that a period corresponds to T/N days. The period compasses the time interval [t,t-T/N). Then the percentage of the arrivals in that time interval is found by $\beta_j(t;0) - \beta_j(t - T/N;0)$, which are found by the equation (4.9).

However, with this approach the variance of total demand will change. We make an approximation and adjust the distribution parameters such that the expectation and variance of the demand throughout the whole reservation period satisfy our initial demand assumptions. Suppose $\lambda_j(i)$ denote the arrival rate in period i and $\beta_j(i)$ is the constant beta rate for period i, where $\beta_j(i)$ actually gives the ratio of the number of arrivals in period i to the total number of arrivals.

Let $D_j(i)$ denote the demand for a ODF j in period i and assume that demand in different periods are independent. Then, the expectation of the demand will be,

$$E(\sum_{i=1}^{N} D_i(j)) = \frac{\sum_{i=1}^{N} E(D_i(j))}{N} = \frac{\sum_{i=1}^{N} \frac{p_j^{'}}{\delta_j^{'}/\beta_j(i)}}{N} = \frac{p_j^{'}}{\delta_j^{'}} \sum_{i=1}^{N} \beta_j(i)$$

By using the fact that $\sum_{i=1}^{N} \beta_j(i) = 1$, we obtain the following equation for the expectation.

$$E(\sum_{i=1}^{N} D_i(j)) = \frac{p_j^{'}}{\delta_j^{'}}.$$

$$\begin{aligned}
Var(\sum_{i=1}^{N} D_i(j)) = \sum_{i=1}^{N} Var(D_i(j)) &= \sum_{i=1}^{N} \frac{p_j{'}}{\delta_j^{'}/\beta_j(i)}(1 + \frac{1}{\delta_j^{'}/\beta_j(i)}) \\
&= \frac{p_j^{'}}{\delta_j^{'}} \sum_{i=1}^{N} \beta_j(i) + \frac{p_j^{'}}{\delta_j^{'2}} \sum_{i=1}^{N} \beta_j(i)^2 \\
&= \frac{p_j^{'}}{\delta_j^{'}} + \frac{p_j^{'}}{\delta_j^{'2}} \sum_{i=1}^{N} \beta_j(i)^2 \\
&= \frac{p_j^{'}}{\delta_j^{'}}(1 + \frac{\sum_{i=1}^{N} \beta_j(i)^2}{\delta_j^{'}}) = \frac{p_j}{\delta_j}(1 + \frac{1}{\delta_j})
\end{aligned}$$

Hence, from the equations above, the parameters of Gamma distribution $p^{'}$ and $\delta^{'}$ should satisfy the following equations:

$$\frac{p_j^{'}}{\delta_j^{'}} = \frac{p_j}{\delta_j} \tag{4.10}$$

$$\frac{p_j^{'}}{\delta_j^{'}}(1 + \frac{\sum_{i=1}^{N} \beta_j(i)^2}{\delta_j^{'}}) = \frac{p_j}{\delta_j}(1 + \frac{1}{\delta_j}) \tag{4.11}$$

Then, by the equations (4.10) and (4.11) we obtain,

$$
\begin{aligned}
1 + \frac{\sum_{i=1}^{N} \beta_j(i)^2}{\delta_j'} &= 1 + \frac{1}{\delta_j} \\
\delta_j' &= \delta_j \sum_{i=1}^{N} \beta_j(i)^2
\end{aligned}
$$

In each period, the arrival rate of ODF j is assumed to be Gamma distributed with parameters $p_j^{'}$ and $delta_j^{'}$. Note that the assumption of independent demand throughout the horizon is not correct, but we only make an approximation. Also, we do not obtain the same probability distribution, but make a two moment approximation by equating the expectation and variance.

4.3 Bayesian Demand Updating Model

In order to come up with a dynamic solution, at some predetermined points in time, we update the demand for the remaining reservation period taking the realized demands into account. Incorporating Beta distribution in the arrival pattern induces correlation between the arrivals. Therefore, bookings on hand provide information about bookings to arrive in the remaining part of the reservation period. In finding the distribution of the demand for the remaining periods, we use the derivations given by de Boer *et al.* (1999).

Suppose we are at period t. That is, there are t periods remaining until departure. Demand in the time interval $[t, T]$ is realized and we want to update the demand distribution for the remaining number of periods. Let $N_j(t; 0)$ denote the number of arrivals for ODF j for the remaining t periods and $N_j(T; t)$ denote the realized demand before time t. According to realized demand $N_j(T; t)$, we should find the distribution of the remaining demand $N_j(t; 0)$. Using Bayes' rule, the distribution of $N_j(t;0)$ conditional on $N_j(T;t)$ is obtained as follows:

$$\begin{aligned}
&Pr(N_j(t;0) = k | N_j(T;t) = n_j(T;t)) = \\
&= \int_0^\infty Pr(N_j(t;0) = k | N_j(T;t) = n_j(T;t); \lambda_j)\ f(\lambda_j | N_j(T;t) = n_j(T;t)) d\lambda_j \\
&= \int_0^\infty Pr(N_j(t;0) = k | N_j(T;t) = n_j(T;t); \lambda_j) \frac{Pr(N_j(T;t) = n_j(T;t) | \lambda_j) f(\lambda_j)}{Pr(N_j(T;t) = n_j(T;t)} d\lambda_j.
\end{aligned}$$

In order to solve the above equation, we must first derive the distribution of the number of arrivals $N_j(T;t)$. Using the equation (4.6) for the time interval $[t,T]$, the distribution of the number of arrivals $N_j(T;t)$ will be,

$$Pr(N_j(T;t) = k) = (\frac{\delta_j}{\delta_j + \beta_j(T;t)})^{p_j} (\frac{\beta_j(T;t)}{\delta_j + \beta_j(T;t)})^k \frac{\Gamma(k + p_j)}{\Gamma(p_j) k!}.$$

where $\beta_j(t;0)$ shows the percentage of the arrivals to come after period t, namely in the remaining t periods of time. Similarly, $\beta_j(T;t)$ is the percentage of the arrivals to come before the period t. Recalling the equation (4.9)

$$\beta_j(t;0) = \lambda_j \frac{\Gamma(\alpha_j + \gamma_j)}{\Gamma(\alpha_j)\Gamma(\gamma_j)} \sum_{l=0}^{\beta_j - 1} \begin{pmatrix} \gamma_j - 1 \\ l \end{pmatrix} (-1)^l \frac{1}{l + \alpha_j} (\frac{t}{T})^{\alpha_j + l}.$$

and $\beta_j(T;t) = 1 - \beta_j(t;0)$.

Then, the conditional distribution of demand for the remaining reservation period $N_j(t;0)$ on the realized demand $N_j(T;t)$ is given by de Boer (1999) as follows:

$$
\begin{aligned}
&Pr(N_j(t;0) = k | N_j(T;t) = n_j(T;t)) = \\
&= \frac{(\delta_j + \beta_j(T;t))^{p_j + n_j(T;t)}}{\Gamma(p_j + n_j(T;t))} \frac{\beta_j(t;0)^k}{k!} \int_0^\infty \lambda_j^{k+n_j(T;t)+p_j-1} e^{-(\delta_j+1)\lambda_j} d\lambda_j \\
&= (\frac{\beta_j(t;0)}{\delta_j+1})^k (1 - \frac{\beta_j(t;0)}{\delta_j+1})^{n_j(T;t)+p} \frac{\Gamma(n_j(T;t)+p_j+k)}{\Gamma(n_j(T;t)+p_j)k!} \qquad (4.12)
\end{aligned}
$$

which is Negative Binomial Distribution with parameters $n_j(T;t) + p_j$ and $\frac{\delta_j+\beta_j(T;t)}{\delta_j+1}$. In the probabilistic models, we explicitly use the probability distribution of the Negative Binomial distribution given in equation (4.12). However, for deterministic models, it is sufficient to know the expected demand for the remaining horizon which is denoted by $\mathrm{E}(N_j(0;t))$ and given by the following equation.

$$
E(N_j(0;t)) = \frac{n_j(T;t) + p}{\delta_j + \beta_j(T;t)} \beta_j(t;0). \qquad (4.13)
$$

4.4 Evaluation of the Numerical Results

For the three problem scenarios (base problem, problem with increased low fare demand variance, problem with decreased fare variance), we solve the DLP, SLP and the EMVLP model with different values of θ. The optimal number of seats allocated to each itinerary and fare class combination are given in Table 4.1 for the base problem. The results indicate that as θ increases, the number of seats allocated to the highest fare class decreases. Recall that SLP is equivalent to EMVLP with a variance penalty factor of zero. Among the probabilistic models, SLP assigns the largest number of seats to fare class 1 as compared to the others. The optimal seat allocations for the other two scenarios are presented in Tables C.1 and C.2 in Appendix C. Similar solutions are obtained also for the other two problem scenarios investigated. However, the penalty factor for variance of the revenue is mostly effective in the problem where the difference between the fares is small. Since class 1 is not so much profitable as in the other problem scenarios, the model assigns even less number of seats to that class. The main reason for the decrease in the number of seats allocated to higher fare classes is that the variance of demand for higher fare classes is high compared to lower fare classes. Therefore, the EMVLP model has a tendency for reserving the seats for less variable lower classes.

Table 4.1: Optimal allocations of the mathematical models for the base problem

ODF				EMVLP					
itinerary	*class*	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
	3	41	42	44	44	43	42	42	45
AB	2	40	40	41	40	39	38	36	35
	1	30	40	39	35	31	25	18	13
	3	0	0	1	6	11	18	28	31
AC	2	25	18	20	19	19	18	18	19
	1	20	22	19	16	13	10	7	5
	3	0	0	0	8	15	23	28	31
AD	2	24	21	21	20	19	18	18	17
	1	20	17	15	12	10	8	5	4
	3	30	23	25	25	25	24	24	25
BC	2	20	19	19	19	18	18	16	15
	1	20	27	24	21	18	13	8	6
	3	1	15	20	20	21	21	22	23
BD	2	20	16	16	16	15	15	15	15
	1	20	22	20	18	16	14	11	9
	3	45	38	39	39	39	40	43	47
CD	2	40	36	36	36	36	35	36	36
	1	30	35	33	31	29	26	22	18

The graphs in Figures 4.3, 4.4 and 4.5 compare the optimal seat allocations of the models for the itineraries AB, AC and AD with respect to fare classes, for the base problem. As θ increases, the number of seats allocated to fare class 1 decrease whereas the number of seats allocated to fare class 3 shows increase. The comparisons for other itineraries are given in Figures C.1, C.2 and C.3 in Appendix C.

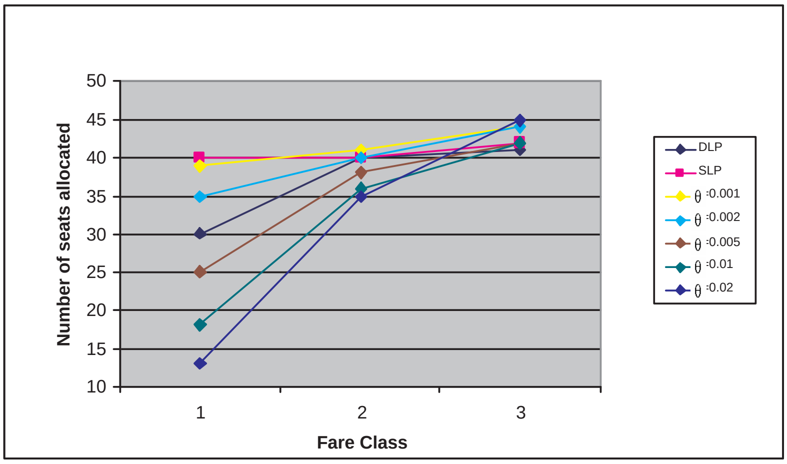

Figure 4.3: Optimal seat allocations for itinerary AB.

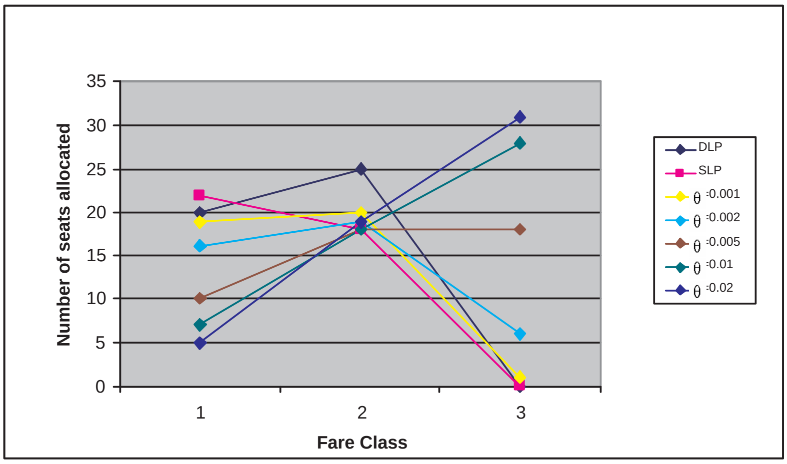

Figure 4.4: Optimal seat allocations for itinerary AC.

4.4.1 Results of Partitioned Booking Control Policy

After obtaining the optimal seat allocations by solving the mathematical models, we simulate the allocations with the booking control policies given in Section 3.5. Tables 4.2, 4.3 and 4.4 show the simulation results for the three problem scenarios with partitioned booking control policy. In simulation of the partitioned booking control policy we assume that low fare class customers usually arrive before high

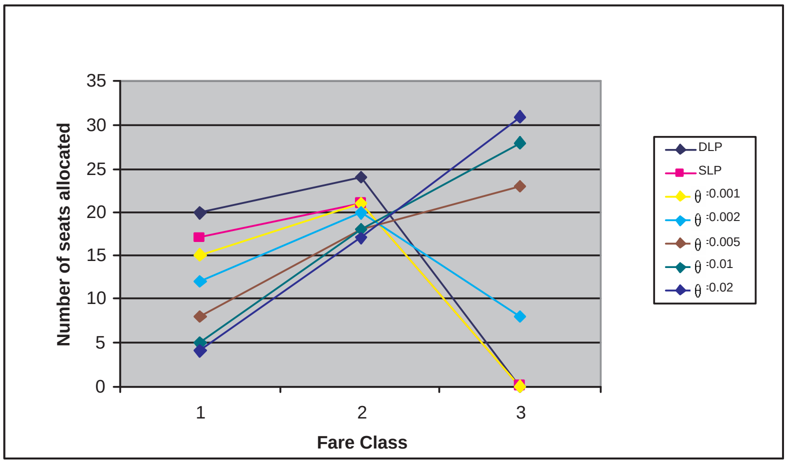

Figure 4.5: Optimal seat allocations for itinerary AD.

fare class customers which is called as low-before-high arrival pattern. However, it is not necessary that all of the lower fare class customers book before all of the higher fare class customers. The models are compared in terms of expected revenue, standard deviation and coefficient of variation of revenue and load factor of the network. Load factor of a flight is determined as the ratio of the accepted number of requests to the capacity of the flight. For calculating the load factor of the total network, we simply take the average of load factors of the three flight legs. It is a reasonable approximation because the load factors of the individual flights are very close to each other.

Table 4.2: Simulation results for partitioned booking limit control, base problem (without update).

with low before high arrival pattern								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	70615	71735	71630	70899	69744	67776	63726	60055
Std. Dev	5390	6070	5440	4590	3770	2820	1800	1440
Coef. of Var.	0.08	0.08	0.08	0.06	0.05	0.04	0.03	0.02
Load Factor	0.86	0.87	0.88	0.91	0.93	0.95	0.97	0.96

It is observed that SLP gives the highest expected revenue with a partitioned booking limit control policy. Moreover, expected revenue decreases as variance penalty θ increases. Note that the load factor of the flights are higher for the proposed model EMVLP as compared to DLP and SLP. EMVLP penalizes the variance and assigns more seats to lower fare classes which causes the aircraft to be filled with more passengers. The other models, on the other hand, fill the aircraft with less number of passengers with higher fares. However, the impact of increased load factor is smaller on the expected revenue than the impact of the loss due to accepting less number of high fare class passengers. As a result, the expected revenue decreases as variance penalty θ increases. With a partitioned booking control policy, expected revenue strictly decreases in θ, but with nested booking limit control and bid price control approaches we do not expect the same behaviour. When the seat allocations are used with nested policies, penalizing the variance would improve the expected revenue by reducing the effect of overprotection.

Table 4.3: Simulation results for partitioned booking limit control, problem with increased low fare demand variance (without update).

with low before high arrival pattern								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	69178	70614	70392	69670	68694	66900	63512	60034
Std. Dev	5500	6320	5440	4630	3930	3090	2270	1910
Coef. of Var.	0.08	0.09	0.08	0.07	0.06	0.05	0.04	0.03
Load Factor	0.84	0.85	0.88	0.90	0.92	0.94	0.94	0.94

Another point to be emphasized is that as θ increases, standard deviation and coefficient of variation of the revenue decrease. Since partitioned booking control policy is seldom used in real life, we do not make any further analysis. However,

Table 4.4: Simulation results for partitioned booking limit control, problem with decreased fare variance (without update).

with low before high arrival pattern								
			EMVLP					
Exp. Rev	59759	60692	60591	60277	59901	59131	57987	56485
Std. Dev	4070	3690	3160	2700	2400	1950	1540	1300
Coef. of Var.	0.07	0.06	0.05	0.04	0.04	0.03	0.03	0.02
Load Factor	0.86	0.89	0.92	0.93	0.94	0.96	0.97	0.97

for nested booking limit control and bid price control policies, we analyze the impact of decreasing standard deviation on decreasing the probability of poor performance in Section 4.4.2 under the subtitle of "Probability of poor performance".

4.4.2 Results of Nested Booking Control Policy

The simulation results of the optimal allocations with nested booking control policy for the base problem, for the problem with increased low fare demand variance and for the problem with decreased fare variance are presented in Tables 4.5, 4.6 and 4.7, respectively. Note that these results are obtained by solving the mathematical models once at the beginning of the reservation period. In other words, they are not updated. In the aforementioned tables, the results with and without low-before-high arrival pattern assumptions are presented. Since bid prices are effective only if they are updated, we do not present the results obtained by bid price control policy here.

The first observation is that the revenues under nested booking control policy significantly increase as compared to partitioned booking limit control policy. Furthermore, the expected revenues are higher when there is no assumption about the arrival order of different fare classes.

Table 4.5: Simulation results for nested booking limit control, base problem (without update).

low before high arrival pattern								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	75973	74880	75902	75818	75151	74663	71233	69443
Std. Dev	7540	7520	6950	5870	5350	5030	3440	3210
Coef. of Var.	0.10	0.10	0.09	0.08	0.07	0.07	0.05	0.05
Load Factor	0.88	0.87	0.89	0.92	0.94	0.97	0.97	0.98
without any assumption about the arrival patterns								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	76772	75650	76692	77317	77306	76763	75935	75552
Std.Dev.	8290	8100	7650	6880	6130	5400	5080	4910
Coef. of Var.	0.11	0.11	0.10	0.09	0.08	0.07	0.07	0.06
Load Factor	0.87	0.87	0.89	0.92	0.94	0.96	0.97	0.98

Expected Revenue

It is observed that for some values of θ, the variance of revenue decreases and, at the same time, expectation of the revenue increases. Figures 4.6, 4.7 and 4.8 compare the expected revenues resulting from different models for the three problem scenarios. In all of the problems, for θ being less than 0.003, the expected revenue obtained with our proposed models are greater than the ones obtained by SLP. Furthermore, in the base problem, the expected revenue obtained by the proposed model when $\theta = 0.001$ is very close to the expected revenue of DLP. For the decreased fare problem, penalizing the variance causes the expected revenue to increase.

When Table 4.8 is analyzed, it is observed that the expected revenues are greater than the DLP results for $\theta \leq 0.005$.

Table 4.6: Simulation results for nested booking limit control, problem with increased low fare demand variance (without update).

low before high arrival pattern								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	75367	74509	74946	74852	74358	73130	71696	70399
Std. Dev	7870	7980	6920	5890	5020	4210	3710	3550
Coef. of Var.	0.09	0.08	0.07	0.06	0.06	0.06	99.89	99.99
Load Factor	0.86	0.85	0.89	0.92	0.94	0.96	0.97	0.98
without any assumption about the arrival patterns								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	76262	75380	76036	76452	76302	75809	75764	75606
Std.Dev.	8620	8880	7670	6870	6140	5530	5270	5160
Coef. of Var.	0.11	0.12	0.10	0.09	0.08	0.07	0.07	0.07
Load Factor	0.87	0.85	0.89	0.91	0.93	0.95	0.97	0.97

Table 4.7: Simulation results for nested booking limit control, problem with decreased fare variance(without update).

low before high arrival pattern								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	62705	63316	63597	63574	63792	62879	61856	61235
Std. Dev	5700	4780	4020	3440	3050	2550	2090	1900
Coef. of Var.	0.09	0.08	0.06	0.05	0.05	0.04	0.03	0.03
Load Factor	0.88	0.90	0.93	0.94	0.95	0.97	0.98	0.98
without any assumption about the arrival patterns								
			EMVLP					
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	63058	64046	64092	64973	65063	64544	64188	64038
Std.Dev.	6080	5560	4720	4540	4250	3460	3180	2800
Coef. of Var.	0.10	0.09	0.07	0.07	0.07	0.05	0.05	0.04
Load Factor	0.87	0.90	0.92	0.94	0.96	0.97	0.97	0.97

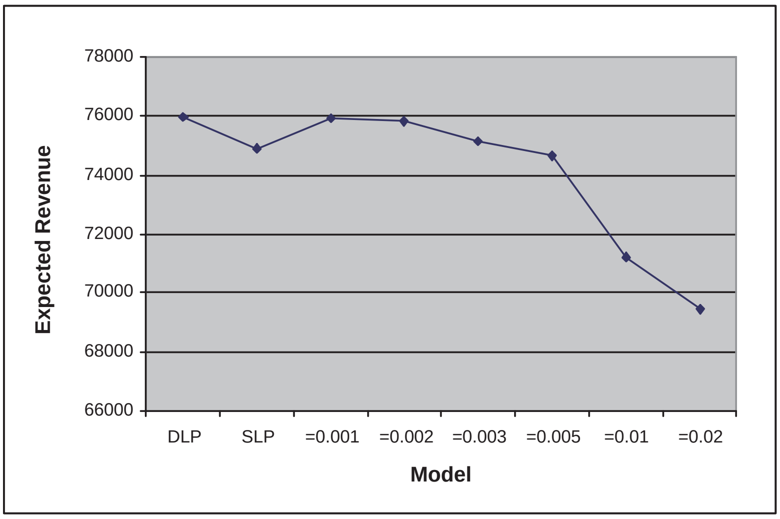

Figure 4.6: Comparison of expected revenues for base problem.

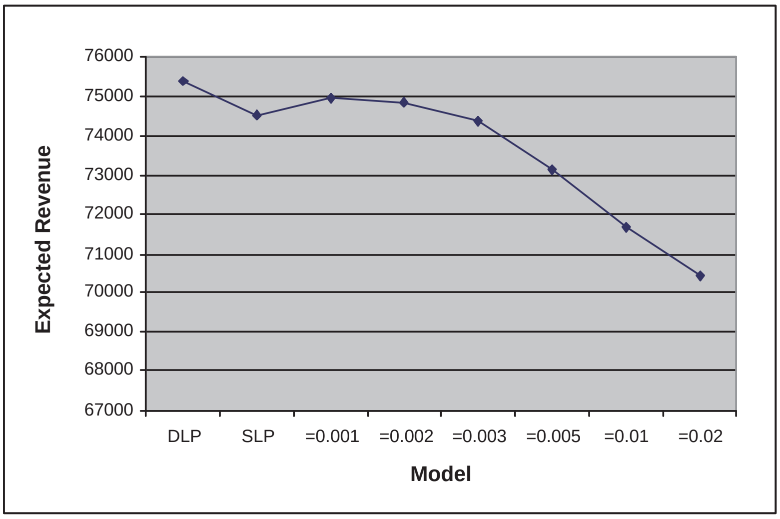

Figure 4.7: Comparison of expected revenues for problem with increased low fare demand variance.

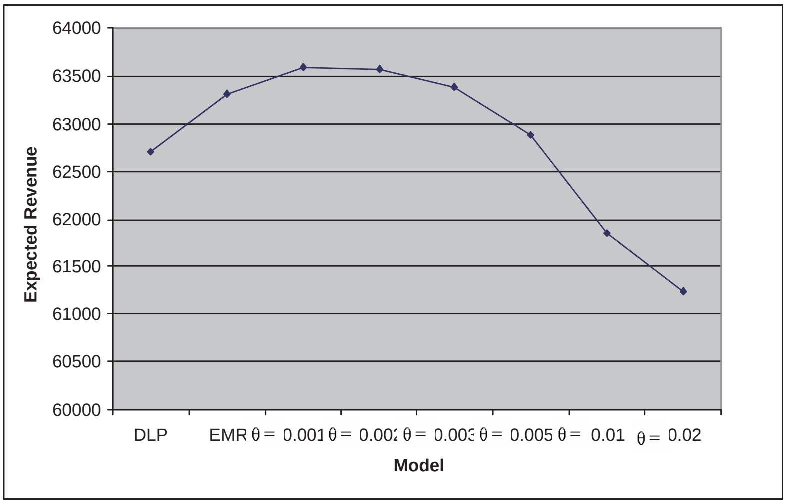

Figure 4.8: Comparison of expected revenues for problem with decreased fare variance.

Standard Deviation and Coefficient of Variation

Variability of revenue is measured with standard deviation and coefficient of variation. As it is illustrated in the Tables 4.5, 4.6 and 4.7, for the first two problem scenarios, the highest variance in revenue is induced by the SLP model. When the difference between the fares is small, DLP gives the highest variance. As it is expected, standard deviation of revenue decreases as θ increases. The coefficient of variation of revenue turns out to be similar for the three problem except that it is slightly smaller in problem with decreased fare variance and a bit higher in the problem with increased low fare demand variance. These solutions indicate that with a reasonable penalty factor, more stable policies can be attained without sacrificing from the expected revenue. To see the impact of decreasing variance, we analyze the probability of poor performance resulting from using the model solutions with nested booking control policy in the later parts of this section.

Probability of Poor Performance

Probability of poor performance is defined as the probability that revenue R is less than a predetermined threshold level κ. We denote it by $Pr(R \leq \kappa)$. In order to calculate this probability, we need to know the distribution of revenue. The simulation results show that revenues obtained by nested booking limit control policy without any update throughout the horizon fit well to Normal Distribution. Then, assuming Normal Distribution, we calculate probability of poor performance. As threshold levels, we take arbitrary revenue values that are mostly within the interval of $(\mu, \mu - 2\sigma)$, where μ and σ are Normal Distribution parameters. For Normal Distribution the interval $[\mu - 2\sigma), \mu + 2\sigma]$ cover 95% of all possible values. We believe that $\mu - 2\sigma$ is sufficient for representing poor revenue values. Table 4.8 shows the probabilities that revenue is less than the threshold levels for the base problem. It is observed that SLP model gives the highest probabilities.

Table 4.8: Comparison of probability of poor performance for the base problem

	DLP	SLP	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	75973	74880	75902	75818	75151	74663	71233	69443
Thr. Val.(κ)	Pr $(R \leq \kappa)$							
75000	0.45	0.51	0.45	0.44	0.49	0.53	0.84	0.96
72000	0.30	0.35	0.29	0.29	0.28	0.3	0.59	0.79
70000	0.21	0.26	0.20	0.15	0.17	0.18	0.36	0.57
67000	0.12	0.15	0.10	0.07	0.07	0.07	0.11	0.23
65000	0.07	0.10	0.06	0.03	0.03	0.03	0.04	0.08

The proposed EMVLP models with $\theta \leq 0.003$ are preferable to SLP because they give higher revenues and lower variances. Although DLP model gives the best expected revenue, it is comparable with the proposed model with $\theta = 0.002$.

DLP model gives an expected revenue of 75973 and the expected revenue of EMVLP model with $\theta = 0.002$ is 75818. However, probability that revenue being less than 70000 is 0.21 in the DLP whereas it is 0.15 in the proposed model. The risk is decreased by 5% at the expense of a decrease of 0.2% in expected revenue. For large flight networks, 5% decrease in risk would be critical and the decision maker may choose to work with the proposed model instead of DLP. Also, for threshold values of 67000 and 65000, the risk in EMVLP Model with $\theta = 0.002$ is %5 less than the risk contained in DLP model. For θ being 0.01 and 0.02, the expected revenues decrease so much that the probabilities that revenue falls below the specified threshold values become higher than the probabilities for DLP and SLP.

Table 4.9 shows the probability of poor performance for the problem with increased low fare demand variance. Similar to the base problem, SLP is more risky.

Table 4.9: Comparison of probability of poor performance for the problem with increased low fare demand variance

	DLP	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	75367	74509	74946	74852	74358	73130	71696	70399
Thr. Val.(κ)	Pr $(R \leq \kappa)$							
75000	0.48	0.52	0.50	0.51	0.55	0.67	0.81	0.99
72000	0.29	0.38	0.33	0.32	0.32	0.39	0.53	0.99
70000	0.21	0.28	0.24	0.21	0.20	0.23	0.32	0.54
67000	0.14	0.17	0.12	0.09	0.07	0.07	0.10	0.17
65000	0.07	0.14	0.08	0.05	0.06	0.03	0.03	0.03

The impact of decreasing variance is not as strong as it is in the base problem. However, the probability of revenue being less than 67000 is 14% in DLP model.

It is 9% and 7% in EMVLP model with θ being 0.001 and 0.002, respectively. Expected revenue is the highest in DLP model. However, using EMVLP with $\theta = 0.002$ decreases the risk 5% at the expense of decreasing expected revenue by 0.06%. Also, with $\theta = 0.003$, the risk probability decreases 7% and the expected revenue decreases 0.13%. Analogous to the observations for the base problem, for some values of θ, allocation policies obtained by EMVLP may be preferable to SLP and DLP.

The most striking impact of variance penalization is seen in the third problem scenario where the difference between fares is small. The results are given in Table 4.10. For $\theta \leq 0.005$, the expected revenues are higher than DLP model and probability of poor performance significantly decreases. In EMVLP model with $\theta = 0.001$, expected revenue is at the highest level and risk probability falls to 0.19 (for threshold value 60000) from 0.32 in DLP model. The similar behaviour is seen for other threshold values. When the difference between the fares is small and the demand variance of high fare class is large compared to other classes, penalizing variability is an effective method for both increasing expected revenue and decreasing variance.

Table 4.10: Comparison of probability of poor performance for the problem with decreased fare variance

	DLP	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
Exp. Rev	62705	63316	63597	63574	63792	62879	61856	61235
Thr. Val.(κ)	Pr $(R \leq \kappa)$							
62000	0.45	0.39	0.34	0.32	0.33	0.37	0.53	0.66
60000	0.32	0.25	0.19	0.15	0.14	0.13	0.33	0.48
58000	0.21	0.16	0.09	0.06	0.04	0.03	0.04	0.05
57000	0.16	0.10	0.05	0.03	0.02	0.01	0.01	0.01
55000	0.09	0.04	0.02	0.01	0.01	0	0	0

Bid Price Control Policy

To see performances of the models with an updated booking policy, we use bid price control policy. We update bid prices 10 times during the reservation period by taking the realized bookings into account. We update DLP, SLP and EMVLP solutions with the value of θ that performs best in the solutions without update. The simulation results for the problem with decreased fare variance is given in Table 4.11.

Table 4.11: Simulation results for updated bid price control, problem with decreased fare variance.

	The Model		
Performance Measure	*DLP*	*SLP*	$\theta = 0.001$
Exp. Rev	66274	65139	65776
Std. Dev.	3910	3430	2900
Coef. of Var.	0.06	0.05	0.04
Load Factor	0.95	0.96	0.97

As compared to booking control policy without update, variance of revenue obtained with updating booking control policy turns out to be lower. Furthermore, using proposed EMVLP model with $\theta = 0.001$ causes further decrease in standard deviation as compared to DLP and SLP models. Although the expected revenue obtained by the EMVLP model is less than the one obtained by DLP model, it may be a preferable option to DLP model because of the decreased variance. Note that the load factor and the expected revenue obtained by EMVLP model is greater than the SLP model. This indicates that EMVLP accepts more passengers with higher revenues as compared to SLP.

CHAPTER 5

CONCLUSION

In this study, airline network seat inventory control problem without cancellations and no-shows is analyzed. The study is carried out with the aim of proposing seat allocation policies that considers other risk measures in addition to expected revenue. This approach differentiates the study from the previous studies existing in the literature.

We proposed two linear programming models that incorporate variability of the revenue induced by seat allocation policies. In the first model, variance of the total marginal revenue is penalized by a factor θ in the objective function, and the objective function maximizes expected revenue minus this penalized variance. Although the variance of total marginal revenue is not equal to the variance of the revenue, our simulation studies show that it affects the variability of revenue significantly. In the second model, the objective function maximizes expected revenue, but we add a constraint which sets an upper limit for the ratio of the expected revenue to the variance of the total marginal revenue. This ratio is determined from the solution of the first model. The motivation for developing a second model is to use our model with bid price control policies. In applications, where there is information about the coefficient of variation of the revenue, the use of this model will be appropriate.

We simulate the arrival booking process with booking limit and bid price control policies. For different problem scenarios, we compare the proposed models with other models existing in the literature. It is observed that use of the proposed model decreases the risk of poor performance by decreasing variability of the revenue. Previous studies show that deterministic model (DLP) results in better expected revenues compared to probabilistic models. However, the proposed model becomes a preferable option to DLP since it decreases the risk of poor performance without causing an important decrease in expected revenue. Furthermore, as long as variance penalty factor θ is within some interval, also the expected revenue increases compared to other probabilistic models. Especially when the difference between the fares of different fare classes is small or when the variance of demand for high fare classes is high, the proposed model results in expected revenues higher than DLP. Also for other cases, the expected revenue obtained by the proposed model is higher than SLP and it is very close to DLP.

As an extension of this study, including correlation between the demands of different fare classes for an origin-destination pair can be considered. In this study, we assume that demands for each ODF are independent. However, including correlation between demands in the experiments may be a more realistic representation of the problem.

We do not have the opportunity to collaborate and work with an airline company. The study can further be improved by working with real airline data and testing the performance of proposed models for real operations.

Furthermore, the development of more sophisticated models still constitutes a gap in the literature. It is certain that integrating nesting into the models will significantly improve the network revenue. Also, development of easily to implementable dynamic control policies is an important research direction.

Bibliography

Beckman, M.J. and Bobkoski, F., 1958. Airline demand: An analysis of some frequency distributions. *Naval Research Logistics Quarterly* 5, 43-51.

Belobaba, P.P., 1987. Air travel demand and airline seat inventory management. Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA.

Bertsimas, D. and Tsitsiklis, J.N., 1997. *Introduction to Linear Optimization.* Belmont, MA: Athena Scientific.

Brumelle, S.L. and McGill, J.I., 1993. Airline seat allocation with multiple nested fare classes. *Operations Research* 41, 127-137.

Curry, R.E., 1990. Optimal airline seat allocation with fare classes nested by origins and destinations. *Transportation Sci.* 24, 193-204.

de Boer, S.V., 1999 Mathematical Programming in Airline Seat Inventory Control. Master's Thesis, Vrije Universiteit Amsterdam, The Netherlands

de Boer, S.V., Freling, R. and Piersma, N., 2002. Mathematical programming for network revenue management revisited. *European Journal of Operations Research* 137, 72-92.

Filar, J.A., Kallenberg, L.C.M., Lee, H., 1989. Variance Penalized Markov Decision Processes *Mathematics of Operations Research* 14, 147-161.

Glover, F., Glover, R., Lorenzo, J. and McMillan, C., 1982. The passenger mix problem in the scheduled airlines. *Interfaces* 12, 73-79.

Hahn, G.J., Shapiro, S.S., 1967. *Statistical Models in Engineering.* New York: Wiley

Higle, J.L., 2007. Bid-price control with origin-destination demand: A stochastic programming approach. *Journal of Revenue and Pricing Management* 5, 291-304

Hines, W.W. and Montgomery, D.C., 1980. *Probability and Statistics in Engineering and Management Science.* New York: Wiley.

Kimms, A. and Müller-Bungart, M., 2007. Simulation of stochastic demand data streams for network revenue management problems. *OR Spectrum* 29, 5-20

Lautenbacher, C.J. and Stidham, S.J., 1999. The underlying Markov decision process in the single-leg airline yield management problem. *Transportation Science* 33, 136-146.

Lee, T.C. and Hersh, M., 1993. A model for dynamic airline seat allocation control with multiple seat bookings. *Transportation Science* 27, 252-265.

Littlewood, K., 1972. Forecasting and control of passenger bookings. *AGIFORS Sympos. Proc.* 12, 95-117.

McGill, J.I. and van Ryzin, G.J., 1999. Revenue management: Research Overview and prospects. *Transportation Science* 33, 233-256.

Pak, K. and Piersma, N., 2002. Overview of OR techniques for airline revenue management. *Statistica Neerlandica.* 56, 480-496.

Popovic, J.B., 1987. Decision making on stock levels in cases of uncertain demand. *European Journal of Operations Research* 32, 276-290.

Simpson, R.W., 1989. Using network flow techniques to find shadow prices for market and seat inventory control, MIT Flight Transportation Laboratory Memorandum M 89-1, Cambridge, Massachusetts.

Subramanian, J., Stidham, S.J. and Lautenbacher C.J., 1999 Airline yield management with overbooking, cancellations and no-shows. *Transportation Science* 33, 147-167.

Talluri, K.T. and van Ryzin, G.J., 1998. An analysis of bid-price controls for network revenue management. *Management Science* 44, 1577-1593.

Talluri, K.T. and van Ryzin, G.J., 1999. A randomized linear programming method for computing network bid prices. *Transportation Science* 33, 207-216

Talluri, K.T. and van Ryzin, G.J., 2005. *The theory and practice of revenue management.* New York: Springer.

Williamson, E.L., 1992. Airline network seat control: Methodologies and revenue impacts. PhD. Thesis, Massachusetts Institute of Technology, Cambridge, MA.

Wollmer, R.D., 1986. A hub-spoke seat management model. Unpublished Internal Report, Mc Donnell Douglas Corporation, Long Beach, CA.

Wollmer, R.D., 1992. An airline seat management model for a single leg route when lower fare classes book first. *Operations Research* 40, 26-37

APPENDIX A

Table A.1: Fares of the itineraries with respect to fare class

Itinerary	*Fare Class* 3	*Fare Class* 2	*Fare Class* 1
AB	75	125	250
AC	130	170	400
AD	200	320	460
BC	100	150	330
BD	160	200	420
CD	80	110	235

Table A.2: Adjusted fare settings of the itineraries with respect to fare class

Itinerary	*Fare Class* 3	*Fare Class* 2	*Fare Class* 1
AB	75	125	150
AC	130	170	220
AD	200	320	440
BC	100	150	210
BD	160	200	250
CD	80	110	160

Table A.3: The demand distribution parameters of the itineraries with respect to fare class

	Fare Class 3				Fare Class 2				Fare Class 1			
Itinerary	p_j	δ_j	E_j	SD_j	p_j	δ_j	E_j	SD_j	p_j	δ_j	E_j	SD_j
AB	80	1.6	50	9.01	80	2	40	7.75	3	0.1	30	18.17
AC	80	2	40	7.75	50	2	25	6.12	2	0.1	20	14.83
AD	60	2	30	6.71	72	3	24	5.66	2	0.1	20	14.83
BC	60	2	30	6.71	40	2	20	5.48	2	0.1	20	14.83
BD	60	2	30	6.71	60	3	20	5.16	6	0.3	20	9.31
CD	80	1.6	50	9.01	80	2	40	7.75	6	0.2	30	13.42

Table A.4: Adjusted demand distribution parameters of the itineraries with respect to fare class

	Fare Class 3				Fare Class 2				Fare Class 1			
Itinerary	p	δ	E	SD	p	δ	E	SD	p	δ	E	SD
AB	20	0.4	50	13.23	20	0.5	40	10.95	3	0.1	30	18.17
AC	20	0.5	40	10.95	5	0.2	25	12.25	2	0.1	20	14.83
AD	15	0.5	30	9.49	18	0.75	24	7.48	2	0.1	20	14.83
BC	15	0.5	30	9.49	10	0.5	20	7.75	2	0.1	20	14.83
BD	15	0.5	30	9.49	15	0.75	20	6.83	6	0.3	20	9.31
CD	20	0.4	50	13.23	20	0.5	40	10.95	6	0.2	30	13.42

APPENDIX B

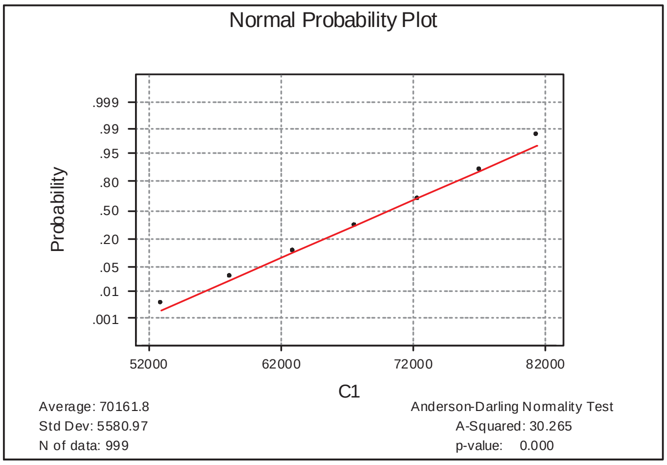

Figure B.1: Normal probability plot of the revenue with the optimal DLP allocations for the base problem.

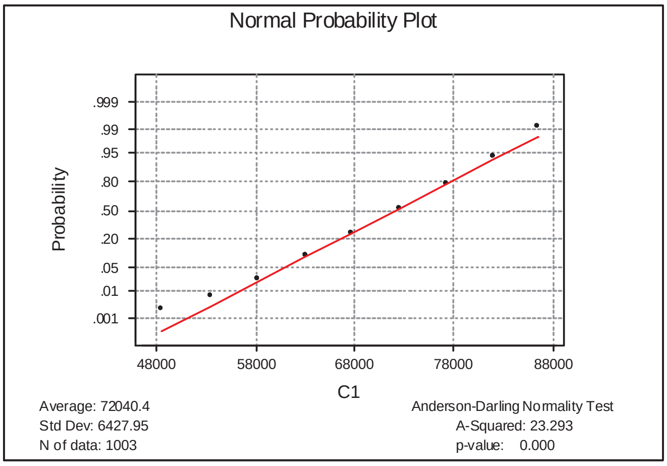

Figure B.2: Normal probability plot of the revenue with the optimal SLP allocations for the base problem.

APPENDIX C

Table C.1: Optimal allocations for the problem with increased low fare demand variance.

ODF				EMVLP					
itinerary	*class*	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
	3	41	41	41	40	40	41	45	50
AB	2	40	41	40	39	38	38	37	35
	1	30	41	38	35	31	26	20	14
	3	0	0	3	12	20	26	31	35
AC	2	25	15	15	14	14	14	16	17
	1	20	23	19	16	13	10	7	5
	3	0	0	9	13	15	18	21	23
AD	2	24	21	20	19	18	18	17	16
	1	20	18	15	12	11	9	6	5
	3	30	22	23	23	23	24	26	28
BC	2	20	19	18	18	18	17	17	16
	1	20	28	25	21	18	14	9	6
	3	1	17	18	19	19	21	23	24
BD	2	20	15	15	15	15	15	15	15
	1	20	22	20	18	16	14	12	10
	3	45	35	35	37	39	41	46	51
CD	2	40	36	35	35	36	36	37	37
	1	30	36	33	32	31	28	23	19

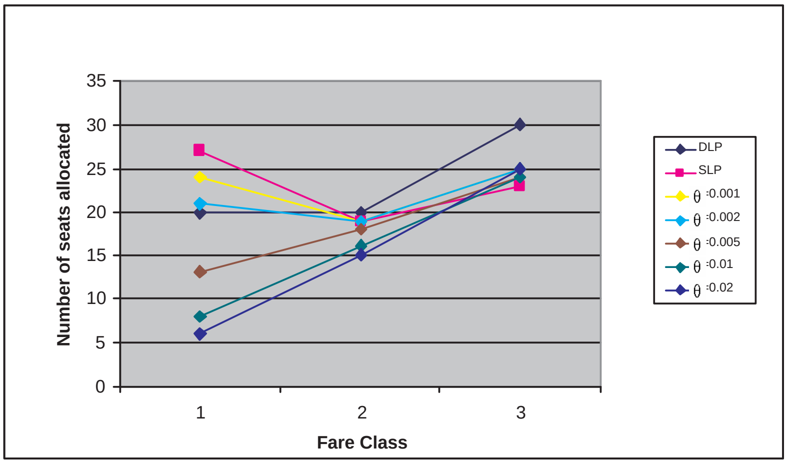

Figure C.1: Optimal seat allocations for itinerary BC.

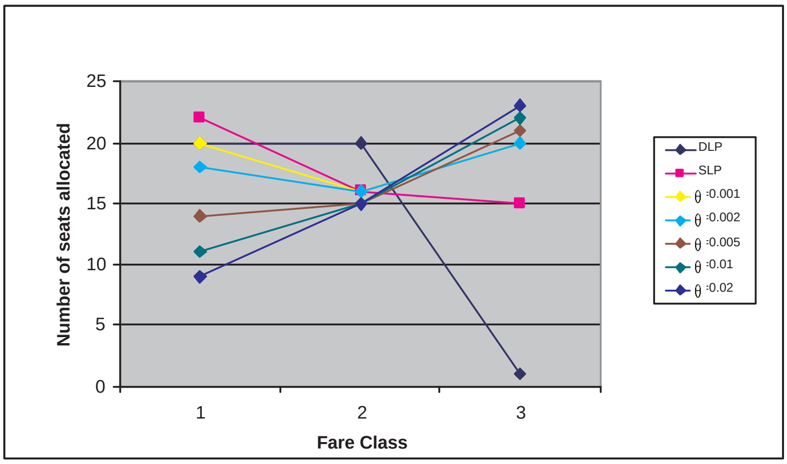

Figure C.2: Optimal seat allocations for itinerary BD.

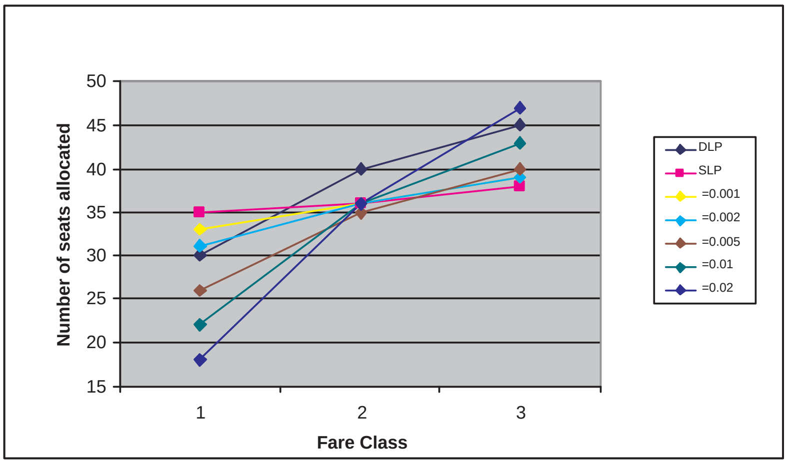

Figure C.3: Optimal seat allocations for itinerary CD.

Table C.2: Optimal allocations for the problem with decreased fare variance.

ODF				EMVLP					
itinerary	*class*	*DLP*	*SLP*	$\theta = 0.001$	$\theta = 0.002$	$\theta = 0.003$	$\theta = 0.005$	$\theta = 0.01$	$\theta = 0.02$
	3	41	45	44	43	42	42	44	46
AB	2	40	41	40	40	39	38	37	36
	1	30	36	33	31	29	25	20	16
	3	0	4	9	11	15	23	29	33
AC	2	25	20	20	19	19	19	18	19
	1	20	14	12	11	10	8	7	6
	3	0	0	6	13	16	18	21	23
AD	2	24	22	21	20	20	19	18	17
	1	20	18	15	12	10	8	6	4
	3	30	25	25	25	25	24	24	25
BC	2	20	20	19	19	18	18	17	16
	1	20	22	20	18	16	13	10	7
	3	1	21	21	21	21	22	23	24
BD	2	20	17	16	16	16	15	15	15
	1	20	17	16	15	14	13	12	11
	3	45	38	39	39	40	42	44	48
CD	2	40	37	37	36	36	37	37	37
	1	30	30	29	28	27	26	24	21

APPENDIX D

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
34660	1.65E-13	36540	5.99E-12	37360	6.06E-11	38180	2.06E-10	39000	8.24E-10
34900	4.51E-13	36560	1.13E-10	37380	8.55E-11	38200	3.26E-10	39020	6.66E-09
35140	8.20E-13	36580	1.82E-11	37400	1.02E-11	38220	2.27E-10	39040	3.98E-10
35360	1.24E-12	36600	1.87E-11	37420	2.20E-11	38240	7.02E-11	39060	9.58E-10
35600	1.69E-12	36620	6.41E-13	37440	1.09E-11	38260	1.35E-09	39080	4.31E-10
35620	9.73E-12	36640	8.01E-12	37460	9.02E-11	38280	8.28E-11	39100	4.43E-10
35840	2.16E-12	36660	1.44E-11	37480	7.37E-11	38300	2.40E-10	39120	5.49E-10
35860	2.65E-11	36680	4.58E-11	37500	7.63E-11	38320	6.52E-11	39140	8.88E-10
35880	5.73E-14	36700	1.43E-12	37520	5.23E-11	38340	9.00E-11	39160	4.47E-10
35900	1.76E-14	36720	1.93E-12	37540	1.03E-09	38360	5.94E-10	39180	5.07E-10
35920	1.76E-14	36740	1.02E-12	37560	3.58E-10	38380	1.21E-10	39200	8.29E-10
35940	3.10E-14	36760	8.79E-12	37580	7.95E-11	38400	2.75E-10	39220	1.66E-09
35960	6.98E-14	36780	1.56E-11	37600	8.70E-11	38420	1.76E-10	39240	1.01E-09
35980	2.91E-12	36800	1.78E-10	37620	4.82E-11	38440	3.84E-10	39260	6.35E-10
36000	2.25E-14	36820	2.09E-12	37640	2.85E-11	38460	2.41E-10	39280	1.51E-09
36020	7.62E-15	36840	2.51E-12	37660	2.73E-11	38480	6.90E-10	39300	4.50E-10
36040	8.36E-14	36860	8.82E-12	37680	6.15E-11	38500	1.83E-10	39320	7.93E-10
36060	7.24E-14	36880	1.88E-11	37700	1.02E-10	38520	2.08E-10	39340	4.97E-10
36080	4.83E-11	36900	5.71E-11	37720	1.07E-10	38540	3.16E-10	39360	9.06E-10
36100	3.53E-12	36920	4.05E-11	37740	4.64E-11	38560	1.86E-10	39380	7.83E-10
36120	2.55E-13	36940	5.43E-12	37760	7.10E-11	38580	8.53E-10	39400	7.57E-10
36140	9.01E-14	36960	3.87E-12	37780	4.84E-10	38600	1.38E-10	39420	1.42E-09
36160	2.64E-13	36980	1.03E-11	37800	7.91E-11	38620	5.79E-10	39440	1.10E-09
36180	8.75E-14	37000	1.99E-11	37820	1.27E-10	38640	3.69E-10	39460	5.57E-10
36200	4.25E-12	37020	6.33E-11	37840	3.99E-11	38660	3.05E-10	39480	1.15E-09
36220	7.98E-12	37040	4.31E-11	37860	6.97E-11	38680	5.14E-10	39500	1.16E-09
36240	1.83E-13	37060	5.24E-12	37880	4.74E-11	38700	7.72E-10	39520	1.98E-09
36260	2.45E-14	37080	6.44E-11	37900	4.86E-11	38720	2.24E-10	39540	1.57E-09
36280	4.91E-13	37100	2.24E-11	37920	3.98E-10	38740	3.91E-10	39560	9.83E-10
36300	2.58E-13	37120	7.19E-11	37940	1.69E-10	38760	2.94E-10	39580	6.84E-10
36320	7.78E-11	37140	4.68E-11	37960	6.53E-11	38780	4.93E-10	39600	1.53E-09
36340	9.66E-12	37160	5.74E-11	37980	1.58E-10	38800	9.16E-10	39620	9.50E-10
36360	7.77E-13	37180	1.27E-11	38000	4.26E-11	38820	5.07E-10	39640	2.96E-09
36380	2.73E-13	37200	6.87E-12	38020	6.66E-10	38840	3.59E-10	39660	2.72E-09
36400	8.83E-13	37220	2.54E-11	38040	4.89E-10	38860	2.93E-10	39680	1.28E-09
36420	5.18E-12	37240	8.01E-11	38060	9.28E-11	38880	6.36E-10	39700	7.39E-10
36440	1.16E-11	37260	5.40E-11	38080	6.97E-11	38900	7.90E-10	39720	1.17E-09
36460	1.47E-11	37280	7.10E-11	38100	6.99E-11	38920	9.44E-10	39740	1.39E-09

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
36480	7.22E-13	37300	9.73E-12	38120	5.76E-11	38940	2.28E-10	39760	3.70E-09
36500	1.10E-13	37320	1.71E-10	38140	4.91E-10	38960	5.19E-10	39780	7.72E-10
36520	1.38E-12	37340	8.55E-11	38160	1.56E-10	38980	6.58E-10	39800	1.33E-09
39820	1.02E-09	40640	3.43E-09	41460	9.48E-09	42180	4.34E-08	43000	3.75E-08
39840	2.65E-09	40660	4.59E-09	41480	6.65E-09	42200	4.12E-08	43020	4.21E-08
39860	2.60E-09	40680	2.91E-09	41500	1.78E-08	42220	1.09E-08	43040	3.88E-08
39880	1.26E-09	40700	2.69E-09	41520	1.39E-08	42240	1.49E-08	43060	6.18E-08
39900	3.75E-09	40720	4.45E-09	41540	4.95E-08	42260	4.08E-08	43080	4.03E-08
39920	1.33E-09	40740	4.26E-09	41560	1.10E-08	42280	2.53E-08	43100	4.44E-08
39940	1.49E-09	40760	3.12E-09	41580	5.67E-09	42300	1.63E-08	43120	4.83E-08
39960	2.15E-09	40780	1.06E-08	41600	1.64E-08	42320	1.77E-08	43140	6.18E-08
39980	2.54E-09	40800	6.94E-09	41620	1.52E-08	42340	2.53E-08	43160	4.18E-08
40000	2.41E-09	40820	3.77E-09	41640	1.41E-08	42360	1.48E-08	43180	8.08E-08
40020	1.05E-09	40840	6.22E-09	41660	6.84E-09	42380	1.50E-08	43200	4.75E-08
40040	1.60E-09	40860	5.22E-09	41680	7.06E-09	42400	4.52E-08	43220	3.90E-08
40060	1.73E-09	40880	7.91E-09	41700	8.72E-09	42420	3.79E-08	43240	3.93E-08
40080	4.77E-09	40900	3.76E-09	41720	1.15E-08	42440	1.58E-08	43260	5.85E-08
40100	1.54E-09	40920	2.86E-09	41740	2.96E-08	42460	1.67E-08	43280	1.00E-07
40120	1.91E-09	40940	4.42E-09	41760	8.27E-09	42480	1.93E-08	43300	4.28E-08
40140	3.47E-09	40960	9.14E-09	41780	2.54E-08	42500	1.54E-07	43320	5.92E-08
40160	2.34E-09	40980	4.27E-09	41800	1.55E-08	42520	3.78E-08	43340	5.78E-08
40180	1.09E-09	41000	7.20E-09	41820	9.58E-09	42540	1.89E-08	43360	6.00E-08
40200	4.52E-09	41020	4.02E-09	41840	1.97E-08	42560	2.03E-08	43380	8.16E-08
40220	1.62E-09	41040	9.79E-09	41860	1.79E-08	42580	2.14E-08	43400	7.69E-08
40240	5.83E-09	41060	5.44E-09	41880	1.36E-08	42600	2.08E-08	43420	7.59E-08
40260	1.52E-09	41080	5.22E-09	41900	6.82E-09	42620	4.09E-08	43440	4.71E-08
40280	3.29E-09	41100	7.58E-09	41920	7.00E-09	42640	2.27E-08	43460	4.91E-08
40300	5.83E-09	41120	6.76E-09	41940	1.57E-08	42660	4.04E-08	43480	5.34E-08
40320	4.12E-09	41140	4.56E-09	41960	2.89E-08	42680	2.14E-08	43500	3.34E-07
40340	1.98E-09	41160	9.03E-09	41980	1.29E-08	42700	2.76E-08	43520	5.47E-08
40360	2.57E-09	41180	4.48E-09	42000	8.84E-09	42720	2.77E-08	43540	6.81E-08
40380	4.34E-09	41200	1.92E-08	42020	1.60E-08	42740	8.23E-08	43560	6.13E-08
40400	2.68E-09	41220	9.60E-09	42040	2.54E-08	42760	2.61E-08	43580	5.47E-08
40420	2.60E-09	41240	6.30E-09	42060	2.05E-08	42780	2.89E-08	43600	6.44E-08
40440	5.65E-09	41260	1.19E-08	42080	1.60E-08	42800	2.56E-08	43620	9.49E-08
40460	2.30E-09	41280	6.57E-09	42100	2.17E-08	42820	2.64E-08	43640	7.09E-08
40480	2.13E-09	41300	7.72E-09	42120	1.18E-08	42840	5.75E-08	43660	6.65E-08
40500	3.67E-09	41320	9.36E-09	42140	9.63E-09	42860	2.99E-08	43680	6.11E-08
40520	1.49E-08	41340	5.88E-09	42160	1.32E-08	42880	2.69E-08	43700	7.03E-08
40540	6.04E-09	41360	6.18E-09	42080	1.60E-08	42900	4.00E-08	43720	6.99E-08
40560	5.95E-09	41380	9.13E-09	42100	2.17E-08	42920	2.83E-08	43740	7.35E-08
40580	2.29E-09	41400	1.51E-08	42120	1.18E-08	42940	3.53E-08	43760	8.29E-08

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
40600	8.04E-09	41420	6.70E-09	42140	9.63E-09	42960	6.38E-08	43780	7.86E-08
40620	3.16E-09	41440	1.29E-08	42160	1.32E-08	42980	3.35E-08	43800	6.40E-08
43820	8.41E-08	44640	1.78E-07	45460	3.00E-07	41720	1.15E-08	43720	6.99E-08
43840	7.79E-08	44660	1.60E-07	45480	3.28E-07	41740	2.96E-08	43740	7.35E-08
43860	1.21E-07	44680	1.61E-07	45500	3.46E-07	41760	8.27E-09	43760	8.29E-08
43880	7.83E-08	44700	1.68E-07	45520	3.08E-07	41780	2.54E-08	43780	7.86E-08
43900	8.33E-08	44720	1.98E-07	45540	3.45E-07	41800	1.55E-08	43800	6.40E-08
43920	8.70E-08	44740	1.60E-07	45560	3.28E-07	41820	9.58E-09	43820	8.41E-08
43940	8.71E-08	44760	1.94E-07	45580	3.23E-07	41840	1.97E-08	43840	7.79E-08
43960	1.13E-07	44780	1.79E-07	45600	3.81E-07	41860	1.79E-08	43860	1.21E-07
43980	1.17E-07	44800	1.68E-07	45620	3.51E-07	41880	1.36E-08	43880	7.83E-08
44000	1.10E-07	44820	1.94E-07	45640	4.17E-07	41900	6.82E-09	43900	8.33E-08
44020	9.56E-08	44840	2.16E-07	45660	3.37E-07	41920	7.00E-09	43920	8.70E-08
44040	1.02E-07	44860	2.81E-07	45680	3.82E-07	41940	1.57E-08	43940	8.71E-08
44060	1.11E-07	44880	2.05E-07	45700	3.76E-07	41960	2.89E-08	43960	1.13E-07
44080	1.44E-07	44900	1.88E-07	45460	3.00E-07	41980	1.29E-08	43980	1.17E-07
44100	1.06E-07	44920	2.03E-07	45480	3.28E-07	42000	8.84E-09	44000	1.10E-07
44120	9.19E-08	44940	2.31E-07	45500	3.46E-07	42020	1.60E-08	44020	9.56E-08
44140	2.14E-07	44960	2.54E-07	45520	3.08E-07	42040	2.54E-08	44040	1.02E-07
44160	1.20E-07	44980	2.17E-07	45540	3.45E-07	42060	2.05E-08	44060	1.11E-07
44180	1.10E-07	45000	2.14E-07	45560	3.28E-07	42080	1.60E-08	44080	1.44E-07
44200	1.23E-07	45020	2.12E-07	45580	3.23E-07	42100	2.17E-08	44100	1.06E-07
44220	1.29E-07	45040	2.37E-07	45600	3.81E-07	42120	1.18E-08	44120	9.19E-08
44240	1.08E-07	45060	2.28E-07	45620	3.51E-07	42140	9.63E-09	44140	2.14E-07
44260	1.16E-07	45080	3.15E-07	45640	4.17E-07	42160	1.32E-08	44160	1.20E-07
44280	1.32E-07	45100	2.39E-07	45660	3.37E-07	42180	4.34E-08	44180	1.10E-07
44300	2.26E-07	45120	2.28E-07	45680	3.82E-07	42200	4.12E-08	44200	1.23E-07
44320	1.59E-07	45140	2.36E-07	45700	3.76E-07	42220	1.09E-08	44220	1.29E-07
44340	1.19E-07	45160	3.37E-07	45460	3.00E-07	42240	1.49E-08	44240	1.08E-07
44360	1.32E-07	45180	2.47E-07	45480	3.28E-07	42260	4.08E-08	44260	1.16E-07
44380	2.19E-07	45200	2.40E-07	45500	3.46E-07	42280	2.53E-08	44280	1.32E-07
44400	1.37E-07	45220	2.47E-07	45520	3.08E-07	42300	1.63E-08	44300	2.26E-07
44420	1.54E-07	45240	2.65E-07	45540	3.45E-07	42320	1.77E-08	44320	1.59E-07
44440	1.36E-07	45260	5.13E-07	45560	3.28E-07	42340	2.53E-08	44340	1.19E-07
44460	1.36E-07	45280	2.94E-07	45580	3.23E-07	42360	1.48E-08	44360	1.32E-07
44480	1.31E-07	45300	2.73E-07	45600	3.81E-07	42380	1.50E-08	44380	2.19E-07
44500	1.52E-07	45320	3.58E-07	45620	3.51E-07	42400	4.52E-08	44400	1.37E-07
44520	1.44E-07	45340	2.57E-07	45640	4.17E-07	42420	3.79E-08	44420	1.54E-07
44540	1.62E-07	45360	2.94E-07	45660	3.37E-07	42440	1.58E-08	44440	1.36E-07
44560	1.41E-07	45380	3.31E-07	45680	3.82E-07	42460	1.67E-08	44460	1.36E-07
44580	1.35E-07	45400	2.88E-07	45700	3.76E-07	42480	1.93E-08	44480	1.31E-07

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
44600	1.68E-07	45420	2.83E-07	45460	3.00E-07	42500	1.54E-07	44500	1.52E-07
44620	2.45E-07	45440	2.85E-07	45480	3.28E-07	42520	3.78E-08	44520	1.44E-07
34660	1.65E-13	37720	1.07E-10	45520	3.08E-07	45880	4.08E-07	46700	7.32E-07
34900	4.51E-13	37740	4.64E-11	45540	3.45E-07	45900	4.30E-07	46720	8.15E-07
35140	8.20E-13	37760	7.10E-11	45560	3.28E-07	45920	4.14E-07	46740	8.37E-07
35360	1.24E-12	37780	4.84E-10	45580	3.23E-07	45940	5.62E-07	46760	8.22E-07
35600	1.69E-12	37800	7.91E-11	45600	3.81E-07	45960	4.45E-07	46780	7.82E-07
35620	9.73E-12	37820	1.27E-10	45620	3.51E-07	45980	4.37E-07	46800	7.98E-07
35840	2.16E-12	37840	3.99E-11	45640	4.17E-07	46000	4.65E-07	46820	8.14E-07
35860	2.65E-11	37860	6.97E-11	45660	3.37E-07	46020	4.73E-07	46840	9.24E-07
35880	5.73E-14	37880	4.74E-11	45680	3.82E-07	46040	5.24E-07	46860	8.69E-07
35900	1.76E-14	37900	4.86E-11	45700	3.76E-07	46060	4.55E-07	46880	8.29E-07
35920	1.76E-14	37920	3.98E-10	45720	4.50E-07	46080	5.01E-07	46900	8.74E-07
35940	3.10E-14	37940	1.69E-10	45740	3.72E-07	46100	5.14E-07	46920	8.67E-07
35960	6.98E-14	37960	6.53E-11	45760	3.67E-07	46120	5.01E-07	46940	1.01E-06
35980	2.91E-12	37980	1.58E-10	45780	4.19E-07	46140	5.10E-07	46960	9.22E-07
36000	2.25E-14	38000	4.26E-11	45800	3.87E-07	46160	5.48E-07	46980	9.50E-07
36020	7.62E-15	38020	6.66E-10	45820	4.39E-07	46180	5.78E-07	47000	9.05E-07
36040	8.36E-14	38040	4.89E-10	45840	4.09E-07	46200	5.46E-07	47020	9.33E-07
36060	7.24E-14	38060	9.28E-11	45860	4.27E-07	46220	5.36E-07	47040	9.32E-07
36080	4.83E-11	38080	6.97E-11	45880	4.08E-07	46240	5.23E-07	47060	9.64E-07
36100	3.53E-12	38100	6.99E-11	45900	4.30E-07	46260	1.17E-06	47080	1.05E-06
36120	2.55E-13	38120	5.76E-11	45920	4.14E-07	46280	5.59E-07	47100	9.59E-07
36140	9.01E-14	38140	4.91E-10	45940	5.62E-07	46300	5.49E-07	47120	9.72E-07
36160	2.64E-13	38160	1.56E-10	45960	4.45E-07	46320	6.27E-07	47140	1.04E-06
36180	8.75E-14	38180	2.06E-10	45980	4.37E-07	46340	6.08E-07	47160	1.03E-06
36200	4.25E-12	38200	3.26E-10	46000	4.65E-07	46360	5.69E-07	47180	1.11E-06
36220	7.98E-12	38220	2.27E-10	46020	4.73E-07	46380	5.85E-07	47200	1.12E-06
36240	1.83E-13	38240	7.02E-11	46040	5.24E-07	46400	5.95E-07	47220	1.07E-06
36260	2.45E-14	38260	1.35E-09	46060	4.55E-07	46420	6.30E-07	47240	1.05E-06
36280	4.91E-13	38280	8.28E-11	46080	5.01E-07	46440	6.64E-07	47260	1.09E-06
36300	2.58E-13	38300	2.40E-10	46100	5.14E-07	46460	6.07E-07	47280	1.08E-06
36320	7.78E-11	38320	6.52E-11	46120	5.01E-07	46480	6.14E-07	47300	1.18E-06
36340	9.66E-12	38340	9.00E-11	46140	5.10E-07	46500	6.48E-07	47320	1.28E-06
36360	7.77E-13	38360	5.94E-10	46160	5.48E-07	46520	6.95E-07	47340	1.12E-06
36380	2.73E-13	38380	1.21E-10	45720	4.50E-07	46540	6.70E-07	47360	1.15E-06
36400	8.83E-13	38400	2.75E-10	45740	3.72E-07	46560	6.85E-07	47380	1.23E-06
36420	5.18E-12	38420	1.76E-10	45760	3.67E-07	46580	6.85E-07	47400	1.24E-06
36440	1.16E-11	38440	3.84E-10	45780	4.19E-07	46600	6.79E-07	47420	1.22E-06
36460	1.47E-11	38460	2.41E-10	45800	3.87E-07	46620	7.22E-07	47440	1.28E-06
36480	7.22E-13	38480	6.90E-10	45820	4.39E-07	46640	7.24E-07	47460	1.25E-06
36500	1.10E-13	38500	1.83E-10	45840	4.09E-07	46660	9.59E-07	47480	1.25E-06

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
36520	1.38E-12	38520	2.08E-10	45860	4.27E-07	46680	7.45E-07	47500	1.27E-06
47520	1.41E-06	48040	1.75E-06	48860	2.94E-06	49680	4.73E-06	50500	7.43E-06
47540	1.30E-06	48060	1.88E-06	48880	2.96E-06	49700	5.11E-06	50520	7.49E-06
47560	1.28E-06	48080	1.95E-06	48900	3.06E-06	49720	4.84E-06	50540	7.57E-06
47580	1.37E-06	48100	1.82E-06	48920	3.00E-06	49740	5.00E-06	50560	7.65E-06
47600	1.40E-06	48120	1.85E-06	48940	3.10E-06	49760	4.98E-06	50580	7.82E-06
47620	1.99E-06	48140	2.00E-06	48960	3.10E-06	49780	5.02E-06	50600	7.79E-06
47640	1.46E-06	48160	1.89E-06	48980	3.17E-06	49800	5.09E-06	50620	7.92E-06
47660	1.41E-06	48180	2.07E-06	49000	3.18E-06	49820	5.10E-06	50640	8.01E-06
47680	1.47E-06	48200	2.01E-06	49020	3.22E-06	49840	5.21E-06	50660	8.09E-06
47700	1.43E-06	48220	2.07E-06	49040	3.27E-06	49860	5.22E-06	50680	8.12E-06
47720	1.46E-06	48240	2.00E-06	49060	3.36E-06	49880	5.33E-06	50700	8.22E-06
47740	1.60E-06	48260	2.00E-06	49080	3.39E-06	49900	5.34E-06	50720	8.35E-06
47760	1.48E-06	48280	2.04E-06	49100	3.47E-06	49920	5.40E-06	50740	8.41E-06
47780	1.48E-06	48300	2.37E-06	49120	3.53E-06	49940	5.45E-06	50760	8.55E-06
47800	1.52E-06	48320	2.09E-06	49140	3.51E-06	49960	5.69E-06	50780	8.68E-06
47820	1.65E-06	48340	2.11E-06	49160	3.58E-06	49980	5.68E-06	50800	8.79E-06
47840	1.55E-06	48360	2.15E-06	49180	3.60E-06	50000	5.67E-06	50820	8.76E-06
47860	1.73E-06	48380	2.32E-06	49200	3.61E-06	50020	5.70E-06	50840	8.88E-06
47880	1.58E-06	48400	2.40E-06	49220	3.66E-06	50040	5.80E-06	50860	9.00E-06
47900	1.72E-06	48420	2.29E-06	49240	3.69E-06	50060	5.86E-06	50880	9.05E-06
47920	1.72E-06	48440	2.33E-06	49260	3.82E-06	50080	5.97E-06	50900	9.13E-06
47940	1.64E-06	48460	2.37E-06	49280	3.81E-06	50100	6.00E-06	50920	9.20E-06
47960	1.85E-06	48480	2.29E-06	49300	3.90E-06	50120	6.04E-06	50940	9.36E-06
47980	1.73E-06	48500	2.37E-06	49320	3.87E-06	50140	6.09E-06	50960	9.49E-06
48000	1.72E-06	48520	2.58E-06	49340	3.95E-06	50160	6.19E-06	50980	9.63E-06
48020	1.81E-06	48540	2.50E-06	49360	3.95E-06	50180	6.31E-06	51000	9.57E-06
48040	1.75E-06	48560	2.42E-06	49380	4.12E-06	50200	6.40E-06	51020	9.73E-06
48060	1.88E-06	48580	2.45E-06	49400	4.06E-06	50220	6.36E-06	51040	9.77E-06
48080	1.95E-06	48600	2.49E-06	49420	4.09E-06	50240	6.44E-06	51060	1.02E-05
48100	1.82E-06	48620	4.38E-06	49440	4.17E-06	50260	6.50E-06	51080	1.00E-05
48120	1.85E-06	48640	2.54E-06	49460	4.18E-06	50280	6.84E-06	51100	1.01E-05
48140	2.00E-06	48660	2.63E-06	49480	4.29E-06	50300	6.64E-06	51120	1.02E-05
48160	1.89E-06	48680	2.62E-06	49500	4.43E-06	50320	6.72E-06	51140	1.03E-05
48180	2.07E-06	48700	2.72E-06	49520	4.35E-06	50340	6.79E-06	51160	1.05E-05
48200	2.01E-06	48720	2.69E-06	49540	4.51E-06	50360	6.88E-06	51180	1.07E-05
48220	2.07E-06	48740	2.72E-06	49560	4.46E-06	50380	7.13E-06	51200	1.06E-05
48240	2.00E-06	48760	2.77E-06	49580	4.48E-06	50400	7.05E-06	51220	1.08E-05
48260	2.00E-06	48780	2.78E-06	49600	4.75E-06	50420	7.11E-06	51240	1.09E-05
47980	1.73E-06	48800	2.81E-06	49620	4.60E-06	50440	7.24E-06	51260	1.10E-05
48000	1.72E-06	48820	2.87E-06	49640	4.66E-06	50460	7.23E-06	51280	1.11E-05
48020	1.81E-06	48840	2.90E-06	49660	4.69E-06	50480	7.33E-06	51300	1.13E-05

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
51320	1.13E-05	54140	4.11E-05	54960	5.73E-05	55780	7.85E-05	56600	1.06E-04
51340	1.14E-05	54160	4.14E-05	54980	5.78E-05	55800	7.91E-05	56620	1.06E-04
51360	1.15E-05	54180	4.18E-05	55000	5.82E-05	55820	7.97E-05	56640	1.07E-04
51380	1.23E-05	54200	4.21E-05	55020	5.87E-05	55840	8.02E-05	56660	1.08E-04
51400	1.18E-05	54220	4.25E-05	55040	5.92E-05	55860	8.08E-05	56680	1.09E-04
51420	1.19E-05	54240	4.28E-05	55060	5.96E-05	55880	8.14E-05	56700	1.09E-04
51440	1.20E-05	54260	4.32E-05	55080	6.01E-05	55900	8.20E-05	56720	1.10E-04
51460	1.22E-05	54280	4.36E-05	55100	6.06E-05	55920	8.26E-05	56740	1.11E-04
51480	1.22E-05	54300	4.39E-05	55120	6.10E-05	55940	8.33E-05	56760	1.12E-04
51500	1.23E-05	54320	4.43E-05	55140	6.15E-05	55960	8.39E-05	56780	1.12E-04
51520	1.26E-05	54340	4.46E-05	55160	6.20E-05	55980	8.45E-05	56800	1.13E-04
51540	1.26E-05	54360	4.50E-05	55180	6.25E-05	56000	8.51E-05	56820	1.14E-04
51560	1.27E-05	54380	4.54E-05	55200	6.30E-05	56020	8.57E-05	56840	1.15E-04
51580	1.29E-05	54400	4.58E-05	55220	6.35E-05	56040	8.64E-05	56860	1.15E-04
51600	1.29E-05	54420	4.61E-05	55240	6.40E-05	56060	8.70E-05	56880	1.16E-04
51620	1.31E-05	54440	4.65E-05	55260	6.44E-05	56080	8.76E-05	56900	1.17E-04
51640	1.33E-05	54460	4.69E-05	55280	6.49E-05	56100	8.83E-05	56920	1.18E-04
51660	1.35E-05	54480	4.73E-05	55300	6.54E-05	56120	8.89E-05	56940	1.19E-04
51680	1.35E-05	54500	4.77E-05	55320	6.60E-05	56140	8.96E-05	56960	1.20E-04
51700	1.36E-05	54520	4.81E-05	55340	6.65E-05	56160	9.02E-05	56980	1.20E-04
53720	3.44E-05	54540	4.85E-05	55360	6.70E-05	56180	9.09E-05	57000	1.21E-04
53740	3.64E-05	54560	4.89E-05	55380	6.75E-05	56200	9.15E-05	57020	1.22E-04
53760	3.50E-05	54580	4.93E-05	55400	6.80E-05	56220	9.22E-05	57040	1.23E-04
53780	3.53E-05	54600	4.97E-05	55420	6.85E-05	56240	9.29E-05	57060	1.24E-04
53800	3.56E-05	54620	5.01E-05	55440	6.91E-05	56260	9.35E-05	57080	1.25E-04
53820	3.59E-05	54640	5.05E-05	55460	6.96E-05	56280	9.42E-05	57100	1.25E-04
53840	3.62E-05	54660	5.09E-05	55480	7.01E-05	56300	9.49E-05	57120	1.26E-04
53860	3.65E-05	54680	5.13E-05	55500	7.06E-05	56320	9.56E-05	57140	1.27E-04
53880	3.68E-05	54700	5.17E-05	55520	7.12E-05	56340	9.63E-05	57160	1.28E-04
53900	3.71E-05	54720	5.21E-05	55540	7.17E-05	56360	9.69E-05	57180	1.29E-04
53920	3.74E-05	54740	5.25E-05	55560	7.23E-05	56380	9.76E-05	57200	1.30E-04
53940	3.78E-05	54760	5.30E-05	55580	7.28E-05	56400	9.83E-05	57220	1.31E-04
53960	3.81E-05	54780	5.34E-05	55600	7.34E-05	56420	9.90E-05	57240	1.31E-04
53980	3.84E-05	54800	5.38E-05	55620	7.39E-05	56440	9.97E-05	57260	1.32E-04
54000	3.87E-05	54820	5.42E-05	55640	7.45E-05	56460	1.00E-04	57280	1.33E-04
54020	3.91E-05	54840	5.47E-05	55660	7.50E-05	56480	1.01E-04	57300	1.34E-04
54040	3.94E-05	54860	5.51E-05	55680	7.56E-05	56500	1.02E-04	57320	1.35E-04
54060	3.97E-05	54880	5.56E-05	55700	7.62E-05	56520	1.03E-04	57340	1.36E-04
54080	4.01E-05	54900	5.60E-05	55720	7.67E-05	56540	1.03E-04	57360	1.37E-04
54100	4.04E-05	54920	5.64E-05	55740	7.73E-05	56560	1.04E-04	57380	1.38E-04
54120	4.07E-05	54940	5.69E-05	55760	7.79E-05	56580	1.05E-04	57400	1.39E-04

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
57420	1.39E-04	57640	1.50E-04	62800	5.79E-04	63620	6.78E-04	66440	1.03E-03
57440	1.40E-04	57660	1.51E-04	62820	5.82E-04	63640	6.80E-04	66460	1.04E-03
57460	1.41E-04	57680	1.52E-04	62840	5.84E-04	63660	6.83E-04	66480	1.04E-03
57480	1.42E-04	57700	1.53E-04	62860	5.86E-04	63680	6.85E-04	66500	1.04E-03
57500	1.43E-04	62060	4.96E-04	62880	5.89E-04	63700	6.88E-04	66520	1.04E-03
57520	1.44E-04	62080	4.98E-04	62900	5.91E-04	65720	9.44E-04	66540	1.05E-03
57540	1.45E-04	62100	5.00E-04	62920	5.93E-04	65740	9.47E-04	66560	1.05E-03
57560	1.46E-04	62120	5.02E-04	62940	5.96E-04	65760	9.50E-04	66580	1.05E-03
57580	1.47E-04	62140	5.05E-04	62960	5.98E-04	65780	9.52E-04	66600	1.05E-03
57600	1.48E-04	62160	5.07E-04	62980	6.00E-04	65800	9.55E-04	66620	1.05E-03
57620	1.49E-04	62180	5.09E-04	63000	6.03E-04	65820	9.57E-04	66640	1.06E-03
57640	1.50E-04	62200	5.11E-04	63020	6.05E-04	65840	9.60E-04	66660	1.06E-03
57660	1.51E-04	62220	5.13E-04	63040	6.08E-04	65860	9.62E-04	66680	1.06E-03
57680	1.52E-04	62240	5.16E-04	63060	6.10E-04	65880	9.65E-04	66700	1.06E-03
57700	1.53E-04	62260	5.18E-04	63080	6.12E-04	65900	9.67E-04	66720	1.07E-03
57420	1.39E-04	62280	5.20E-04	63100	6.15E-04	65920	9.70E-04	66740	1.07E-03
57440	1.40E-04	62300	5.22E-04	63120	6.17E-04	65940	9.72E-04	66760	1.07E-03
57460	1.41E-04	62320	5.24E-04	63140	6.19E-04	65960	9.75E-04	66780	1.07E-03
57480	1.42E-04	62340	5.27E-04	63160	6.22E-04	65980	9.77E-04	66800	1.08E-03
57500	1.43E-04	62360	5.29E-04	63180	6.24E-04	66000	9.80E-04	66820	1.08E-03
57520	1.44E-04	62380	5.31E-04	63200	6.27E-04	66020	9.82E-04	66840	1.08E-03
57540	1.45E-04	62400	5.33E-04	63220	6.29E-04	66040	9.84E-04	66860	1.08E-03
57560	1.46E-04	62420	5.36E-04	63240	6.31E-04	66060	9.87E-04	66880	1.09E-03
57580	1.47E-04	62440	5.38E-04	63260	6.34E-04	66080	9.89E-04	66900	1.09E-03
57600	1.48E-04	62460	5.40E-04	63280	6.36E-04	66100	9.92E-04	66920	1.09E-03
57620	1.49E-04	62480	5.42E-04	63300	6.39E-04	66120	9.94E-04	66940	1.09E-03
57640	1.50E-04	62500	5.45E-04	63320	6.41E-04	66140	9.97E-04	66960	1.09E-03
57660	1.51E-04	62520	5.47E-04	63340	6.44E-04	66160	9.99E-04	66980	1.10E-03
57680	1.52E-04	62540	5.49E-04	63360	6.46E-04	66180	1.00E-03	67000	1.10E-03
57700	1.53E-04	62560	5.52E-04	63380	6.48E-04	66200	1.00E-03	67020	1.10E-03
57420	1.39E-04	62580	5.54E-04	63400	6.51E-04	66220	1.01E-03	67040	1.10E-03
57440	1.40E-04	62600	5.56E-04	63420	6.53E-04	66240	1.01E-03	67060	1.11E-03
57460	1.41E-04	62620	5.58E-04	63440	6.56E-04	66260	1.01E-03	67080	1.11E-03
57480	1.42E-04	62640	5.61E-04	63460	6.58E-04	66280	1.01E-03	67100	1.11E-03
57500	1.43E-04	62660	5.63E-04	63480	6.61E-04	66300	1.02E-03	67120	1.11E-03
57520	1.44E-04	62680	5.65E-04	63500	6.63E-04	66320	1.02E-03	67140	1.11E-03
57540	1.45E-04	62700	5.68E-04	63520	6.66E-04	66340	1.02E-03	67160	1.12E-03
57560	1.46E-04	62720	5.70E-04	63540	6.68E-04	66360	1.02E-03	67180	1.12E-03
57580	1.47E-04	62740	5.72E-04	63560	6.71E-04	66380	1.03E-03	67200	1.12E-03
57600	1.48E-04	62760	5.75E-04	63580	6.73E-04	66400	1.03E-03	67220	1.12E-03
57620	1.49E-04	62780	5.77E-04	63600	6.75E-04	66420	1.03E-03	67240	1.13E-03

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
67260	1.13E-03	68080	1.21E-03	68900	1.28E-03	69720	1.33E-03	70540	1.36E-03
67280	1.13E-03	68100	1.21E-03	68920	1.28E-03	69740	1.33E-03	70560	1.36E-03
67300	1.13E-03	68120	1.22E-03	68940	1.28E-03	69760	1.33E-03	70580	1.36E-03
67320	1.13E-03	68140	1.22E-03	68960	1.28E-03	69780	1.33E-03	70600	1.36E-03
67340	1.14E-03	68160	1.22E-03	68980	1.29E-03	69800	1.33E-03	70620	1.36E-03
67360	1.14E-03	68180	1.22E-03	69000	1.29E-03	69820	1.33E-03	70640	1.36E-03
67380	1.14E-03	68200	1.22E-03	69020	1.29E-03	69840	1.34E-03	70660	1.36E-03
67400	1.14E-03	68220	1.22E-03	69040	1.29E-03	69860	1.34E-03	70680	1.36E-03
67420	1.15E-03	68240	1.23E-03	69060	1.29E-03	69880	1.34E-03	70700	1.36E-03
67440	1.15E-03	68260	1.23E-03	69080	1.29E-03	69900	1.34E-03	70720	1.36E-03
67460	1.15E-03	68280	1.23E-03	69100	1.29E-03	69920	1.34E-03	70740	1.36E-03
67480	1.15E-03	68300	1.23E-03	69120	1.30E-03	69940	1.34E-03	70760	1.36E-03
67500	1.15E-03	68320	1.23E-03	69140	1.30E-03	69960	1.34E-03	70780	1.36E-03
67520	1.16E-03	68340	1.24E-03	69160	1.30E-03	69980	1.34E-03	70800	1.36E-03
67540	1.16E-03	68360	1.24E-03	69180	1.30E-03	70000	1.34E-03	70820	1.36E-03
67560	1.16E-03	68380	1.24E-03	69200	1.30E-03	70020	1.34E-03	70840	1.36E-03
67580	1.16E-03	68400	1.24E-03	69220	1.30E-03	70040	1.34E-03	70860	1.36E-03
67600	1.16E-03	68420	1.24E-03	69240	1.30E-03	70060	1.34E-03	70880	1.36E-03
67620	1.17E-03	68440	1.24E-03	69260	1.30E-03	70080	1.35E-03	70900	1.36E-03
67640	1.17E-03	68460	1.25E-03	69280	1.31E-03	70100	1.35E-03	70920	1.36E-03
67660	1.17E-03	68480	1.25E-03	69300	1.31E-03	70120	1.35E-03	70940	1.36E-03
67680	1.17E-03	68500	1.25E-03	69320	1.31E-03	70140	1.35E-03	70960	1.36E-03
67700	1.17E-03	68520	1.25E-03	69340	1.31E-03	70160	1.35E-03	70980	1.36E-03
67720	1.18E-03	68540	1.25E-03	69360	1.31E-03	70180	1.35E-03	71000	1.36E-03
67740	1.18E-03	68560	1.25E-03	69380	1.31E-03	70200	1.35E-03	71020	1.36E-03
67760	1.18E-03	68580	1.26E-03	69400	1.31E-03	70220	1.35E-03	71040	1.36E-03
67780	1.18E-03	68600	1.26E-03	69420	1.31E-03	70240	1.35E-03	71060	1.36E-03
67800	1.18E-03	68620	1.26E-03	69440	1.32E-03	70260	1.35E-03	71080	1.36E-03
67820	1.19E-03	68640	1.26E-03	69460	1.32E-03	70280	1.35E-03	71100	1.36E-03
67840	1.19E-03	68660	1.26E-03	69480	1.32E-03	70300	1.35E-03	71120	1.36E-03
67860	1.19E-03	68680	1.26E-03	69500	1.32E-03	70320	1.35E-03	71140	1.36E-03
67880	1.19E-03	68700	1.26E-03	69520	1.32E-03	70340	1.35E-03	71160	1.36E-03
67900	1.19E-03	68720	1.27E-03	69540	1.32E-03	70360	1.35E-03	71180	1.36E-03
67920	1.20E-03	68740	1.27E-03	69560	1.32E-03	70380	1.35E-03	71200	1.36E-03
67940	1.20E-03	68760	1.27E-03	69580	1.32E-03	70400	1.35E-03	71220	1.36E-03
67960	1.20E-03	68780	1.27E-03	69600	1.32E-03	70420	1.36E-03	71240	1.36E-03
67980	1.20E-03	68800	1.27E-03	69620	1.33E-03	70440	1.36E-03	71260	1.36E-03
68000	1.20E-03	68820	1.27E-03	69640	1.33E-03	70460	1.36E-03	71280	1.36E-03
68020	1.21E-03	68840	1.28E-03	69660	1.33E-03	70480	1.36E-03	71300	1.36E-03
68040	1.21E-03	68860	1.28E-03	69680	1.33E-03	70500	1.36E-03	71320	1.36E-03
68060	1.21E-03	68880	1.28E-03	69700	1.33E-03	70520	1.36E-03	71340	1.36E-03

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
71360	1.36E-03	72060	1.34E-03	72880	1.30E-03	73700	1.23E-03	74520	1.15E-03
71380	1.36E-03	72080	1.34E-03	72900	1.30E-03	73720	1.23E-03	74540	1.14E-03
71400	1.36E-03	72100	1.34E-03	72920	1.30E-03	73740	1.23E-03	74560	1.14E-03
71420	1.36E-03	72120	1.34E-03	72940	1.30E-03	73760	1.23E-03	74580	1.14E-03
71440	1.36E-03	72140	1.34E-03	72960	1.30E-03	73780	1.23E-03	74600	1.14E-03
71460	1.36E-03	72160	1.34E-03	72980	1.29E-03	73800	1.23E-03	74620	1.14E-03
71360	1.36E-03	72180	1.34E-03	73000	1.29E-03	73820	1.22E-03	74640	1.13E-03
71380	1.36E-03	72200	1.34E-03	73020	1.29E-03	73840	1.22E-03	74660	1.13E-03
71400	1.36E-03	72220	1.34E-03	73040	1.29E-03	73860	1.22E-03	74680	1.13E-03
71420	1.36E-03	72240	1.34E-03	73060	1.29E-03	73880	1.22E-03	74700	1.12E-03
71440	1.36E-03	72260	1.34E-03	73080	1.29E-03	73900	1.21E-03	74720	1.12E-03
71460	1.36E-03	72280	1.34E-03	73100	1.29E-03	73920	1.21E-03	74740	1.12E-03
71480	1.36E-03	72300	1.33E-03	73120	1.28E-03	73940	1.21E-03	74760	1.12E-03
71500	1.36E-03	72320	1.33E-03	73140	1.28E-03	73960	1.21E-03	74780	1.12E-03
71520	1.36E-03	72340	1.33E-03	73160	1.28E-03	73980	1.21E-03	74800	1.11E-03
71540	1.36E-03	72360	1.33E-03	73180	1.28E-03	74000	1.20E-03	74820	1.11E-03
71560	1.36E-03	72380	1.33E-03	73200	1.28E-03	74020	1.20E-03	74840	1.11E-03
71580	1.36E-03	72400	1.33E-03	73220	1.28E-03	74040	1.20E-03	74860	1.10E-03
71600	1.36E-03	72420	1.33E-03	73240	1.28E-03	74060	1.20E-03	74880	1.11E-03
71620	1.36E-03	72440	1.33E-03	73260	1.27E-03	74080	1.20E-03	74900	1.10E-03
71640	1.36E-03	72460	1.33E-03	73280	1.27E-03	74100	1.19E-03	74920	1.10E-03
71660	1.36E-03	72480	1.33E-03	73300	1.27E-03	74120	1.19E-03	74940	1.09E-03
71680	1.36E-03	72500	1.32E-03	73320	1.27E-03	74140	1.19E-03	74960	1.09E-03
71700	1.36E-03	72520	1.32E-03	73340	1.27E-03	74160	1.19E-03	74980	1.09E-03
71720	1.36E-03	72540	1.32E-03	73360	1.27E-03	74180	1.19E-03	75000	1.09E-03
71740	1.35E-03	72560	1.32E-03	73380	1.26E-03	74200	1.19E-03	75020	1.08E-03
71760	1.35E-03	72580	1.32E-03	73400	1.26E-03	74220	1.18E-03	75040	1.08E-03
71780	1.35E-03	72600	1.32E-03	73420	1.26E-03	74240	1.18E-03	75060	1.08E-03
71800	1.35E-03	72620	1.32E-03	73440	1.26E-03	74260	1.18E-03	75080	1.08E-03
71820	1.35E-03	72640	1.32E-03	73460	1.26E-03	74280	1.18E-03	75100	1.07E-03
71840	1.35E-03	72660	1.31E-03	73480	1.26E-03	74300	1.17E-03	75120	1.07E-03
71860	1.35E-03	72680	1.32E-03	73500	1.25E-03	74320	1.17E-03	75140	1.07E-03
71880	1.35E-03	72700	1.31E-03	73520	1.25E-03	74340	1.17E-03	75160	1.07E-03
71900	1.35E-03	72720	1.31E-03	73540	1.25E-03	74360	1.17E-03	75180	1.06E-03
71920	1.35E-03	72740	1.31E-03	73560	1.25E-03	74380	1.16E-03	75200	1.06E-03
71940	1.35E-03	72760	1.31E-03	73580	1.25E-03	74400	1.16E-03	75220	1.06E-03
71960	1.35E-03	72780	1.31E-03	73600	1.24E-03	74420	1.16E-03	75240	1.06E-03
71980	1.35E-03	72800	1.31E-03	73620	1.24E-03	74440	1.16E-03	75260	1.05E-03
72000	1.35E-03	72820	1.31E-03	73640	1.24E-03	74460	1.15E-03	75280	1.05E-03
72020	1.35E-03	72840	1.30E-03	73660	1.24E-03	74480	1.15E-03	75300	1.05E-03
72040	1.35E-03	72860	1.30E-03	73680	1.24E-03	74500	1.15E-03	75320	1.05E-03

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
75340	1.04E-03	76160	9.22E-04	76980	7.95E-04	77800	6.66E-04	78620	5.36E-04
75360	1.04E-03	76180	9.21E-04	77000	7.93E-04	77820	6.61E-04	78640	5.35E-04
75380	1.04E-03	76200	9.18E-04	77020	7.88E-04	77840	6.55E-04	78660	5.27E-04
75400	1.03E-03	76220	9.12E-04	77040	7.86E-04	77860	6.50E-04	78680	5.33E-04
75420	1.03E-03	76240	9.10E-04	77060	7.82E-04	77880	6.58E-04	78700	5.22E-04
75440	1.03E-03	76260	9.06E-04	77080	7.82E-04	77900	6.44E-04	78720	5.23E-04
75460	1.02E-03	76280	9.07E-04	77100	7.74E-04	77920	6.46E-04	78740	5.17E-04
75480	1.02E-03	76300	9.00E-04	77120	7.76E-04	77940	6.42E-04	78760	5.15E-04
75500	1.02E-03	76320	9.01E-04	77140	7.68E-04	77960	6.42E-04	78780	5.12E-04
75520	1.02E-03	76340	8.94E-04	77160	7.66E-04	77980	6.36E-04	78800	5.10E-04
75540	1.01E-03	76360	8.93E-04	77180	7.64E-04	78000	6.32E-04	78820	5.07E-04
75560	1.01E-03	76380	8.90E-04	77200	7.59E-04	78020	6.30E-04	78840	5.03E-04
75580	1.01E-03	76400	8.86E-04	77220	7.56E-04	78040	6.28E-04	78860	4.95E-04
75600	1.01E-03	76420	8.83E-04	77240	7.55E-04	78060	6.21E-04	78880	5.01E-04
75620	1.00E-03	76440	8.82E-04	77260	7.48E-04	78080	6.22E-04	78900	4.91E-04
75640	1.00E-03	76460	8.75E-04	77280	7.51E-04	78100	6.13E-04	78920	4.96E-04
75660	9.96E-04	76480	8.75E-04	77300	7.42E-04	78120	6.20E-04	78940	4.88E-04
75680	9.95E-04	76500	8.70E-04	77320	7.47E-04	78140	6.09E-04	78960	4.87E-04
75700	9.90E-04	76520	8.70E-04	77340	7.34E-04	78160	6.06E-04	78980	4.83E-04
75720	9.90E-04	76540	8.62E-04	77360	7.34E-04	78180	6.05E-04	79000	4.83E-04
75740	9.84E-04	76560	8.63E-04	77380	7.33E-04	78200	6.03E-04	79020	4.76E-04
75760	9.81E-04	76580	8.59E-04	77400	7.29E-04	78220	5.95E-04	79040	4.74E-04
75780	9.80E-04	76600	8.56E-04	77420	7.24E-04	78240	5.95E-04	79060	4.69E-04
75800	9.78E-04	76620	8.52E-04	77440	7.20E-04	78260	5.88E-04	79080	4.73E-04
75820	9.74E-04	76640	8.51E-04	77460	7.15E-04	78280	5.93E-04	79100	4.62E-04
75840	9.70E-04	76660	8.43E-04	77480	7.20E-04	78300	5.84E-04	79120	4.64E-04
75860	9.66E-04	76680	8.46E-04	77500	7.11E-04	78320	5.85E-04	79140	4.58E-04
75880	9.69E-04	76700	8.37E-04	77520	7.11E-04	78340	5.76E-04	79160	4.56E-04
75900	9.60E-04	76720	8.38E-04	77540	7.02E-04	78360	5.77E-04	79180	4.56E-04
75920	9.59E-04	76740	8.32E-04	77560	7.06E-04	78380	5.75E-04	79200	4.50E-04
75940	9.55E-04	76760	8.31E-04	77580	7.00E-04	78400	5.69E-04	79220	4.46E-04
75960	9.54E-04	76780	8.27E-04	77600	6.93E-04	78420	5.67E-04	79240	4.49E-04
75980	9.50E-04	76800	8.24E-04	77620	6.94E-04	78440	5.66E-04	79260	4.41E-04
76000	9.46E-04	76820	8.22E-04	77640	6.93E-04	78460	5.57E-04	79280	4.42E-04
76020	9.44E-04	76840	8.17E-04	77660	6.82E-04	78480	5.61E-04	79300	4.34E-04
76040	9.43E-04	76860	8.12E-04	77680	6.86E-04	78500	5.53E-04	79320	4.44E-04
76060	9.37E-04	76880	8.14E-04	77700	6.79E-04	78520	5.54E-04	79340	4.28E-04
76080	9.36E-04	76900	8.05E-04	77720	6.82E-04	78540	5.44E-04	79360	4.28E-04
76100	9.30E-04	76920	8.08E-04	77740	6.72E-04	78560	5.46E-04	79380	4.28E-04
76120	9.33E-04	76940	7.99E-04	77760	6.69E-04	78580	5.42E-04	79400	4.26E-04
76140	9.25E-04	76960	8.00E-04	77780	6.68E-04	78600	5.41E-04	79420	4.20E-04

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
79440	4.16E-04	80260	3.11E-04	81080	2.27E-04	81900	1.47E-04	82720	8.61E-05
79460	4.12E-04	80280	3.18E-04	81100	2.17E-04	81920	1.53E-04	82740	8.72E-05
79480	4.18E-04	80300	3.07E-04	81120	2.20E-04	81940	1.45E-04	82760	8.77E-05
79500	4.07E-04	80320	3.13E-04	81140	2.17E-04	81960	1.50E-04	82780	8.65E-05
79520	4.06E-04	80340	3.00E-04	81160	2.18E-04	81980	1.44E-04	82800	8.31E-05
79540	4.00E-04	80360	3.01E-04	81180	2.16E-04	82000	1.43E-04	82820	8.01E-05
79560	4.06E-04	80380	3.00E-04	81200	2.09E-04	82020	1.40E-04	82840	8.29E-05
79580	3.99E-04	80400	2.95E-04	81220	2.09E-04	82040	1.40E-04	82860	7.66E-05
79600	3.93E-04	80420	2.96E-04	81240	2.12E-04	82060	1.36E-04	82880	8.11E-05
79620	3.95E-04	80440	2.95E-04	81260	2.06E-04	82080	1.36E-04	82900	7.41E-05
79640	3.96E-04	80460	2.86E-04	81280	2.05E-04	82100	1.30E-04	82920	8.62E-05
79660	3.83E-04	80480	2.90E-04	81300	2.02E-04	82120	1.39E-04	82940	7.61E-05
79680	3.91E-04	80500	2.83E-04	81320	2.10E-04	82140	1.27E-04	82960	7.04E-05
79700	3.81E-04	80520	2.87E-04	81340	1.94E-04	82160	1.28E-04	82980	7.19E-05
79720	3.84E-04	80540	2.77E-04	81360	1.97E-04	82180	1.30E-04	83000	8.33E-05
79740	3.75E-04	80560	2.81E-04	81380	1.99E-04	82200	1.26E-04	83020	7.02E-05
79760	3.74E-04	80580	2.77E-04	81400	1.93E-04	82220	1.21E-04	83040	6.82E-05
79780	3.72E-04	80600	2.73E-04	81420	1.92E-04	82240	1.27E-04	83060	6.44E-05
79800	3.75E-04	80620	2.71E-04	81440	1.88E-04	82260	1.20E-04	83080	7.55E-05
79820	3.68E-04	80640	2.75E-04	81460	1.85E-04	82280	1.25E-04	83100	6.38E-05
79840	3.61E-04	80660	2.64E-04	81480	1.88E-04	82300	1.14E-04	83120	6.57E-05
79860	3.59E-04	80680	2.73E-04	81500	1.83E-04	82320	1.25E-04	83140	6.09E-05
79880	3.66E-04	80700	2.61E-04	81520	1.83E-04	82340	1.14E-04	83160	6.10E-05
79900	3.52E-04	80720	2.67E-04	81540	1.77E-04	82360	1.14E-04	83180	6.56E-05
79920	3.58E-04	80740	2.56E-04	81560	1.83E-04	82380	1.12E-04	83200	5.67E-05
79940	3.51E-04	80760	2.57E-04	81580	1.77E-04	82400	1.11E-04	83220	5.90E-05
79960	3.49E-04	80780	2.56E-04	81600	1.75E-04	82420	1.11E-04	83240	7.29E-05
79980	3.50E-04	80800	2.49E-04	81620	1.77E-04	82440	1.11E-04	83260	5.47E-05
80000	3.45E-04	80820	2.52E-04	81640	1.74E-04	82460	1.02E-04	83280	6.02E-05
80020	3.42E-04	80840	2.46E-04	81660	1.68E-04	82480	1.06E-04	83300	5.33E-05
80040	3.41E-04	80860	2.43E-04	81680	1.73E-04	82500	1.03E-04	83320	6.19E-05
80060	3.36E-04	80880	2.50E-04	81700	1.65E-04	82520	1.04E-04	83340	4.83E-05
80080	3.38E-04	80900	2.37E-04	81720	1.71E-04	82540	9.77E-05	83360	5.11E-05
80100	3.27E-04	80920	2.50E-04	81740	1.62E-04	82560	1.12E-04	83380	5.25E-05
80120	3.35E-04	80940	2.37E-04	81760	1.61E-04	82580	9.86E-05	83400	4.93E-05
80140	3.25E-04	80960	2.39E-04	81780	1.58E-04	82600	9.73E-05	83420	5.27E-05
80160	3.22E-04	80980	2.32E-04	81800	1.58E-04	82620	9.84E-05	83440	4.54E-05
80180	3.25E-04	81000	2.35E-04	81820	1.59E-04	82640	9.57E-05	83460	4.43E-05
80200	3.20E-04	81020	2.30E-04	81840	1.51E-04	82660	8.66E-05	83480	4.98E-05
80220	3.16E-04	81040	2.24E-04	81860	1.48E-04	82680	1.01E-04	83500	4.50E-05
80240	3.15E-04	81060	2.26E-04	81880	1.56E-04	82700	8.85E-05	83520	4.66E-05

R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)	R	P (R = r)
83540	4.12E-05	83820	3.39E-05	84100	1.77E-05	84380	1.42E-05	84660	1.47E-06
83560	4.38E-05	83840	2.74E-05	84120	2.75E-05	84400	6.39E-06	84680	2.67E-05
83580	4.35E-05	83860	2.79E-05	84140	1.53E-05	84420	1.80E-05	84700	5.30E-06
83600	4.00E-05	83880	3.39E-05	84160	1.84E-05	84440	1.22E-05	84720	5.99E-06
83620	4.30E-05	83900	2.43E-05	84180	2.67E-05	84460	1.07E-05	84740	1.97E-07
83640	3.98E-05	83920	3.59E-05	84200	1.68E-05	84480	1.56E-05	84760	6.63E-06
83660	3.56E-05	83940	2.82E-05	84220	1.67E-05	84500	7.05E-06	84780	6.31E-06
83680	4.40E-05	83960	2.90E-05	84240	1.80E-05	84520	1.46E-05	84800	3.73E-06
83700	3.36E-05	83980	2.53E-05	84260	1.81E-05	84540	5.92E-06	84820	3.73E-06
83720	4.22E-05	84000	2.51E-05	84280	2.16E-05	84560	8.46E-06	84840	1.94E-06
83740	3.35E-05	84020	2.82E-05	84300	1.24E-05	84580	8.76E-06	84860	1.11E-06
83760	3.34E-05	84040	2.30E-05	84320	2.22E-05	84600	1.50E-05	84920	3.66E-05
83780	3.42E-05	84060	2.06E-05	84340	8.93E-06	84620	8.34E-06		
83800	3.24E-05	84080	2.61E-05	84360	1.63E-05	84640	1.22E-06		

APPENDIX E

```
51$       CREATE,     1,NSexpo(arrsch3):NSexpo(arrsch3):NEXT(35$);

35$       ASSIGN:     fare3=250:
                x3=40:
                b3=MX(x3-accept3,0):
                limit3_1=b6+b9;
92$       ASSIGN:     cmin3=rc1-limit3_1;
74$       COUNT:      arr3,1;
73$       BRANCH:      Else,dispose3,Yes:
                If,cmin3,capacity assign3,Yes;
dispose3    DISPOSE:     No;

capacity assign3 ASSIGN:    revenue=revenue+fare3:
                rc1=rc1-1:
                accept3=accept3+1;
32$       COUNT:      sold3,1:NEXT(dispose3);


52$       CREATE,     1,NSEXPO(arrsch4):NSEXPO(arrsch4):NEXT(36$);

36$       ASSIGN:     fare4=130:
                x4=0:
                b4=MX(x4-accept4,0):
                limit4_1=b1+b2+b3+b5+b6+b8+b9:
                limit4_2=b5+b6+b8+b9+b10+b11+b12+b13+b14+b15;
93$       ASSIGN:     cmin4=MN(rc1-limit4_1,rc2-limit4_2);
75$       COUNT:      arr4,1;
0$       BRANCH:      Else,dispose4,Yes:
                If,cmin4>0,capacity assign4,Yes;
dispose4    DISPOSE:     No;

capacity assign4 ASSIGN:    revenue=revenue+fare4:
                rc1=rc1-1:
                rc2=rc2-1:
                accept4=accept4+1;
17$       COUNT:      sold4,1:NEXT(dispose4);


53$       CREATE,     1,NSEXPO(arrsch5):NSEXPO(arrsch5):NEXT(37$);

37$       ASSIGN:     fare5=170:
                x5=18:
                b5=MX(x5-accept5,0):
                limit5_1=b2+b3+b6+b8+b9:
                limit5_2=b6+b8+b9+b11+b12+b14+b15;
94$       ASSIGN:     cmin5=MN(rc1-limit5_1,rc2-limit5_2);
76$       COUNT:      arr5,1;
1$       BRANCH:      Else,dispose5,Yes:
                If,cmin5>0,capacity assign5,Yes;
dispose5    DISPOSE:     No;

capacity assign5 ASSIGN:    revenue=revenue+fare5:
                rc2=rc2-1:
                accept5=accept5+1:
                rc1=rc1-1;
18$       COUNT:      sold5,1:NEXT(dispose5);
```

```
54$         CREATE,       1,NSEXPO(arrsch6):NSEXPO(arrsch6):NEXT(38$);

38$         ASSIGN:       fare6=400:
                    x6=22:
                    b6=MX(x6-accept6,0):
                    limit6_1=0:
                    limit6_2=b15;
95$         ASSIGN:       cmin6=MN(rc1-limit6_1,rc2-limit6_2);
77$         COUNT:        arr6,1;
2$         BRANCH:       Else,dispose6,Yes:
                    If,cmin6>0,capacity assign6,Yes;
dispose6     DISPOSE:      No;

capacity assign6 ASSIGN:     accept6=accept6+1:
                    revenue=revenue+fare6:
                    rc2=rc2-1:
                    rc1=rc1-1;
19$         COUNT:        sold6,1:NEXT(dispose6);


55$         CREATE,       1,NSEXPO(arrsch7):NSEXPO(arrsch7):NEXT(39$);

39$         ASSIGN:       fare7=200:
                    x7=0:
                    b7=MX(x7-accept7,0):
                    limit7_1=b1+b2+b3+b4+b5+b6+b8+b9:
                    limit7_2=b4+b5+b6+b8+b9+b10+b11+b12+b13+b14+b15:
                    limit7_3=b8+b9+b13+b14+b15+b16+b17+b18;
96$         ASSIGN:       cmin7=MN(rc1-limit7_1,rc2-limit7_2,rc3-limit7_3);
78$         COUNT:        arr7,1;
3$         BRANCH:       Else,dispose7,Yes:
                    If,cmin7>0,capacity assign7,Yes;
dispose7     DISPOSE:      No;

capacity assign7 ASSIGN:     revenue=revenue+fare7:
                    rc1=rc1-1:
                    rc2=rc2-1:
                    rc3=rc3-1:
                    accept7=accept7+1;
20$         COUNT:        sold7,1:NEXT(dispose7);


56$         CREATE,       1,NSEXPO(arrsch8):NSEXPO(arrsch8):NEXT(40$);

40$         ASSIGN:       fare8=320:
                    x8=21:
                    b8=MX(x8-accept8,0):
                    limit8_1=b3+b6+b9:
                    limit8_2=b6+b9+b12+b15:
                    limit8_3=b9+b18+b15;
97$         ASSIGN:       cmin8=MN(rc1-limit8_1,rc2-limit8_2,rc3-limit8_3);
79$         COUNT:        arr8,1;
4$         BRANCH:       Else,dispose8,Yes:
                    If,cmin8>0,capacity assign8,Yes;
dispose8     DISPOSE:      No;

capacity assign8 ASSIGN:     revenue=revenue+fare8:
                    rc1=rc1-1:
                    rc2=rc2-1:
                    rc3=rc3-1:
                    accept8=accept8+1;
```

```
21$         COUNT:        sold8,1:NEXT(dispose8);


57$         CREATE,       1,NSEXPO(arrsch9):NSEXPO(arrsch9):NEXT(41$);

41$         ASSIGN:       fare9=460:
                    x9=17:
                    b9=MX(x9-accept9,0):
                    limit9_1=b6:
                    limit9_2=b6+b12+b15:
                    limit9_3=15;
98$         ASSIGN:       cmin9=MN(rc1-limit9_1,rc2-limit9_2,rc3-limit9_3);
80$         COUNT:        arr9,1;
5$          BRANCH:       Else,dispose9,Yes:
                    If,cmin9>0,capacity assign9,Yes;
dispose9    DISPOSE:      No;

capacity assign9 ASSIGN:     revenue=revenue+fare9:
                    rc1=rc1-1:
                    rc2=rc2-1:
                    rc3=rc3-1:
                    accept9=accept9+1;
22$         COUNT:        sold9,1:NEXT(dispose9);


58$         CREATE,       1,NSEXPO(arrsch10):NSEXPO(arrsch10):NEXT(42$);

42$         ASSIGN:       fare10=100:
                    x10=23:
                    b10=MX(x10-accept10,0):
                    limit10_2=b5+b6+b8+b9+b11+b12+b14+b15;
99$         ASSIGN:       cmin10=rc2-limit10_2;
81$         COUNT:        arr10,1;
6$          BRANCH:       Else,dispose10,Yes:
                    If,cmin10>0,capacity assign10,Yes;
dispose10   DISPOSE:      No;

capacity assign10 ASSIGN:    revenue=revenue+fare10:
                    rc2=rc2-1:
                    accept10=accept10+1;
23$         COUNT:        sold10,1:NEXT(dispose10);


59$         CREATE,       1,NSEXPO(arrsch11):NSEXPO(arrsch11):NEXT(43$);

43$         ASSIGN:       fare11=150:
                    x11=19:
                    b11=MX(x11-accept11,0):
                    limit11_2=b6+b8+b9+b12+b15;
100$        ASSIGN:       cmin11=rc2-limit11_2;
82$         COUNT:        arr11,1;
7$          BRANCH:       Else,dispose11,Yes:
                    If,cmin11>0,capacity assign11,Yes;
dispose11   DISPOSE:      No;

capacity assign11 ASSIGN:    revenue=revenue+fare11:
                    accept11=accept11+1:
                    rc2=rc2-1;
24$         COUNT:        sold11,1:NEXT(dispose11);


60$         CREATE,       1,NSEXPO(arrsch12):NSEXPO(arrsch12):NEXT(44$);
```

```
44$          ASSIGN:       fare12=330:
                     x12=27:
                     b12=MX(x12-accept12,0):
                     limit12_2=b6+b15;
101$          ASSIGN:       cmin12=rc2-limit12_2;
83$          COUNT:        arr12,1;
8$          BRANCH:        Else,dispose12,Yes:
                     If,cmin12>0,capacity assign12,Yes;
dispose12     DISPOSE:      No;

capacity assign12 ASSIGN:    revenue=revenue+fare12:
                     rc2=rc2-1:
                     accept12=accept12+1;
31$          COUNT:        sold12,1:NEXT(dispose12);


61$          CREATE,       1,NSEXPO(arrsch13):NSEXPO(arrsch13):NEXT(45$);

45$          ASSIGN:       fare13=160:
                     x13=15:
                     b13=MX(x13-accept13,0):
                     limit13_2=b5+b6+b8+b9+b10+b11+b12+b14+b15:
                     limit13_3=b8+b9+b14+b15+b16+b17+b18;
102$          ASSIGN:       cmin13=MN(rc2-limit13_2,rc3-limit13_3);
84$          COUNT:        arr13,1;
9$          BRANCH:        Else,dispose13,Yes:
                     If,cmin13>0,capacity assign13,No;
dispose13     DISPOSE:      No;

capacity assign13 ASSIGN:    revenue=revenue+fare13:
                     rc2=rc2-1:
                     rc3=rc3-1:
                     accept13=accept13+1;
25$          COUNT:        sold13,1:NEXT(dispose13);


62$          CREATE,       1,NSEXPO(arrsch14):NSEXPO(arrsch14):NEXT(46$);

46$          ASSIGN:       fare14=200:
                     x14=16:
                     b14=MX(x14-accept14,0):
                     limit14_2=b6+b8+b9+b11+b12+b15:
                     limit14_3=b8+b9+b15+b18;
103$          ASSIGN:       cmin14=MN(rc2-limit14_2,rc3-limit14_3);
85$          COUNT:        arr14,1;
10$          BRANCH:        Else,dispose14,Yes:
                     If,cmin14>0,capacity assign14,Yes;
dispose14     DISPOSE:      No;

capacity assign14 ASSIGN:    revenue=revenue+fare14:
                     rc2=rc2-1:
                     rc3=rc3-1:
                     accept14=accept14+1;
26$          COUNT:        sold14,1:NEXT(dispose14);


63$          CREATE,       1,NSEXPO(arrsch15):NSEXPO(arrsch15):NEXT(47$);

47$          ASSIGN:       fare15=420:
                     x15=22:
                     b15=MX(x15-accept15,0):
```

```
                    limit15_2=0:
                    limit15_3=0;
104$       ASSIGN:      cmin15=MN(rc2-limit15_2,rc3-limit15_3);
86$        COUNT:       arr15,1;
11$        BRANCH:      Else,dispose15,Yes:
                    If,cmin15>0,capacity assign15,Yes;
dispose15    DISPOSE:     No;

capacity assign15 ASSIGN:    revenue=revenue+fare15:
                    rc2=rc2-1:
                    rc3=rc3-1:
                    accept15=accept15+1;
27$        COUNT:       sold15,1:NEXT(dispose15);


64$        CREATE,      1,NSEXPO(arrsch16):NSEXPO(arrsch16):NEXT(48$);

48$        ASSIGN:      fare16=80:
                    x16=38:
                    b16=MX(x16-accept16,0):
                    limit16_3=b8+b9+b14+b15+b17+b18;
105$       ASSIGN:      cmin16=rc3-limit16_3;
87$        COUNT:       arr16,1;
12$        BRANCH:      Else,dispose16,Yes:
                    If,cmin16>0,capacity assign16,Yes;
dispose16    DISPOSE:     No;

capacity assign16 ASSIGN:    revenue=revenue+fare16:
                    accept16=accept16+1:
                    rc3=rc3-1;
28$        COUNT:       sold16,1:NEXT(dispose16);


65$        CREATE,      1,NSEXPO(arrsch17):NSEXPO(arrsch17):NEXT(49$);

49$        ASSIGN:      fare17=110:
                    x17=36:
                    b17=MX(x17-accept17,0):
                    limit17_3=b8+b9+b14+b15+b18;
106$       ASSIGN:      cmin17=rc3-limit17_3;
88$        COUNT:       arr17,1;
13$        BRANCH:      Else,dispose17,Yes:
                    If,cmin17>0,capacity assign17,Yes;
dispose17    DISPOSE:     No;

capacity assign17 ASSIGN:    revenue=revenue+fare17:
                    accept17=accept17+1:
                    rc3=rc3-1;
29$        COUNT:       sold17,1:NEXT(dispose17);


66$        CREATE,      1,NSEXPO(arrsch18):NSEXPO(arrsch18):NEXT(50$);

50$        ASSIGN:      fare18=235:
                    x18=35:
                    b18=MX(x18-accept18,0):
                    limit18_3=b9+b15;
107$       ASSIGN:      cmin18=rc3-limit18_3;
89$        COUNT:       arr18,1;
14$        BRANCH:      Else,dispose18,Yes:
                    If,cmin18>0,capacity assign18,Yes;
dispose18    DISPOSE:     No;
```

```
capacity assign18 ASSIGN:    revenue=revenue+fare18:
                        accept18=accept18+1:
                        rc3=rc3-1;
30$          COUNT:        sold18,1:NEXT(dispose18);


67$          CREATE,        1,NSexpo(arrsch1):NSEXPO(arrsch1):NEXT(33$);

33$          ASSIGN:        fare1=75:
                        x1=42:
                        b1=MX(x1-accept1,0):
                        limit1_1=b2+b3+b5+b6+b8+b9;
90$          ASSIGN:        cmin1=rc1-limit1_1;
70$          COUNT:        arr1,1;
69$          BRANCH:        Else,dispose1,Yes:
                        If,cmin1>0,capacity assign1,Yes;
dispose1     DISPOSE:       No;

capacity assign1 ASSIGN:    rc1=rc1-1:
                        revenue=revenue+fare1:
                        accept1=accept1+1;
15$          COUNT:        sold1,1:NEXT(dispose1);


68$          CREATE,        1,NSexpo(arrsch2):NSexpo(arrsch2):NEXT(34$);

34$          ASSIGN:        fare2=125:
                        x2=40:
                        b2=MX(x2-accept2,0):
                        limit2_1=b3+b6+b8+b9;
91$          ASSIGN:        cmin2=rc1-limit2_1;
72$          COUNT:        arr2,1;
71$          BRANCH:        Else,dispose2,Yes:
                        If,cmin2,capacity assign2,Yes;
dispose2     DISPOSE:       No;

capacity assign2 ASSIGN:    revenue=revenue+fare2:
                        accept2=accept2+1:
                        rc1=rc1-1;
16$          COUNT:        sold2,1:NEXT(dispose2);
```

```
PROJECT,     "Unnamed Project","Department of Industrial Engineering",,,No,Yes,Yes,Yes,No,No,No,No,No;

ATTRIBUTES:  x10:
             x11:
             x12:
             x13:
             x14:
             x15:
             x16:
             x17:
             x18:
             fare1,:
             fare2,:
             fare3,:
             fare4,:
             fare5,:
             fare6,:
             fare7,:
             fare8,:
             fare9,:
             x1:
             x2:
             x3:
             x4:
             x5:
             x6:
             x7:
             x8:
             x9:
             fare10,:
             fare11:
             fare12:
             fare13:
             fare14:
             fare15:
             fare16:
             fare17:
             fare18;

SCHEDULES:
arrsch10,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.000536635,index10_y)*
0.002777778,15),

DATA(GAMM(0.02253866,index10_y)*0.002777778,15),DATA(GAMM(0.133085422,index10_y)*0.002777778,15)
,DATA(GAMM(0.322517495,index10_y)*0.002777778,15),

DATA(GAMM(0.567759584,index10_y)*0.002777778,15),DATA(GAMM(0.690648946,index10_y)*0.002777778,1
5),DATA(GAMM(0.572589297,index10_y)*0.002777778,15),

DATA(GAMM(0.293539218,index10_y)*0.002777778,15),DATA(GAMM(0.077275407,index10_y)*0.002777778,1
5),DATA(GAMM(0.002683174,index10_y)*0.002777778,15):

arrsch11,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.001773,index11_y)*0.0
02777778,15),DATA(GAMM(0.011822,index11_y)*0.002777778,15),

DATA(GAMM(0.044922,index11_y)*0.002777778,15),DATA(GAMM(0.122353,index11_y)*0.002777778,15),DAT
A(GAMM(0.213971,index11_y)*0.002777778,15),

DATA(GAMM(0.384201,index11_y)*0.002777778,15),DATA(GAMM(0.564481,index11_y)*0.002777778,15),DAT
A(GAMM(0.641912,index11_y)*0.002777778,15),

DATA(GAMM(0.666737,index11_y)*0.002777778,15),DATA(GAMM(0.303224,index11_y)*0.002777778,15):
```

arrsch12,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.009592,index12_y)*0.002777778,15),DATA(GAMM(0.124690,index12_y)*0.002777778,15),

DATA(GAMM(0.287746,index12_y)*0.002777778,15),DATA(GAMM(0.728956,index12_y)*0.002777778,15),DATA(GAMM(1.659334,index12_y)*0.002777778,15),

DATA(GAMM(3.702330,index12_y)*0.002777778,15),DATA(GAMM(6.627746,index12_y)*0.002777778,15),DATA(GAMM(11.020665,index12_y)*0.002777778,15),

DATA(GAMM(14.540756,index12_y)*0.002777778,15),DATA(GAMM(9.255824,index12_y)*0.002777778,15):

arrsch13,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.000536635,index13_y)*0.002777778,15),

DATA(GAMM(0.02253866,index13_y)*0.002777778,15),DATA(GAMM(0.133085422,index13_y)*0.002777778,15),DATA(GAMM(0.322517495,index13_y)*0.002777778,15),

DATA(GAMM(0.567759584,index13_y)*0.002777778,15),DATA(GAMM(0.690648946,index13_y)*0.002777778,15),DATA(GAMM(0.572589297,index13_y)*0.002777778,15),

DATA(GAMM(0.293539218,index13_y)*0.002777778,15),DATA(GAMM(0.077275407,index13_y)*0.002777778,15),DATA(GAMM(0.002683174,index13_y)*0.002777778,15):

arrsch14,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.001182,index14_y)*0.002777778,15),DATA(GAMM(0.007881,index14_y)*0.002777778,15),

DATA(GAMM(0.029948,index14_y)*0.002777778,15),DATA(GAMM(0.081569,index14_y)*0.002777778,15),DATA(GAMM(0.142647,index14_y)*0.002777778,15),

DATA(GAMM(0.256134,index14_y)*0.002777778,15),DATA(GAMM(0.376320,index14_y)*0.002777778,15),DATA(GAMM(0.427941,index14_y)*0.002777778,15),

DATA(GAMM(0.444492,index14_y)*0.002777778,15),DATA(GAMM(0.202149,index14_y)*0.002777778,15):

arrsch15,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.003197,index15_y)*0.002777778,15),DATA(GAMM(0.041563,index15_y)*0.002777778,15),

DATA(GAMM(0.095915,index15_y)*0.002777778,15),DATA(GAMM(0.242985,index15_y)*0.002777778,15),DATA(GAMM(0.553111,index15_y)*0.002777778,15),

DATA(GAMM(1.234110,index15_y)*0.002777778,15),DATA(GAMM(2.209249,index15_y)*0.002777778,15),DATA(GAMM(3.673555,index15_y)*0.002777778,15),

DATA(GAMM(4.846919,index15_y)*0.002777778,15),DATA(GAMM(3.085275,index15_y)*0.002777778,15):

arrsch16,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.000670793,index16_y)*0.002777778,15),

DATA(GAMM(0.028173325,index16_y)*0.002777778,15),DATA(GAMM(0.166356778,index16_y)*0.002777778,15),DATA(GAMM(0.403146869,index16_y)*0.002777778,15),

DATA(GAMM(0.70969948,index16_y)*0.002777778,15),DATA(GAMM(0.863311182,index16_y)*0.002777778,15),DATA(GAMM(0.715736621,index16_y)*0.002777778,15),

DATA(GAMM(0.366924022,index16_y)*0.002777778,15),DATA(GAMM(0.096594258,index16_y)*0.002777778,15),DATA(GAMM(0.003353967,index16_y)*0.002777778,15):

arrsch17,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.001773,index17_y)*0.002777778,15),DATA(GAMM(0.011822,index17_y)*0.002777778,15), DATA(GAMM(0.044922,index17_y)*0.002777778,15),DATA(GAMM(0.122353,index17_y)*0.002777778,15),DATA(GAMM(0.213971,index17_y)*0.002777778,15),

DATA(GAMM(0.384201,index17_y)*0.002777778,15),DATA(GAMM(0.564481,index17_y)*0.002777778,15),DATA(GAMM(0.641912,index17_y)*0.002777778,15),

DATA(GAMM(0.666737,index17_y)*0.002777778,15),DATA(GAMM(0.303224,index17_y)*0.002777778,15):

arrsch18,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.004796,index18_y)*0.002777778,15),DATA(GAMM(0.062345,index18_y)*0.002777778,15),

DATA(GAMM(0.143873,index18_y)*0.002777778,15),DATA(GAMM(0.364478,index18_y)*0.002777778,15),DATA(GAMM(0.829667,index18_y)*0.002777778,15),

DATA(GAMM(1.851165,index18_y)*0.002777778,15),DATA(GAMM(3.313873,index18_y)*0.002777778,15),DATA(GAMM(5.510333,index18_y)*0.002777778,15),

DATA(GAMM(7.270378,index18_y)*0.002777778,15),DATA(GAMM(4.627912,index18_y)*0.002777778,15):

arrsch1,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.000670793,index1_y)*0.002777778,15),

DATA(GAMM(0.028173325,index1_y)*0.002777778,15),DATA(GAMM(0.166356778,index1_y)*0.002777778,15),DATA(GAMM(0.403146869,index1_y)*0.002777778,15),

DATA(GAMM(0.70969948,index1_y)*0.002777778,15),DATA(GAMM(0.863311182,index1_y)*0.002777778,15),DATA(GAMM(0.715736621,index1_y)*0.002777778,15),

DATA(GAMM(0.366924022,index1_y)*0.002777778,15),DATA(GAMM(0.096594258,index1_y)*0.002777778,15),DATA(GAMM(0.003353967,index1_y)*0.002777778,15):

arrsch2,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.001773,index2_y)*0.002777778,15),DATA(GAMM(0.011822,index2_y)*0.002777778,15),

DATA(GAMM(0.044922,index2_y)*0.002777778,15),DATA(GAMM(0.122353,index2_y)*0.002777778,15),DATA(GAMM(0.213971,index2_y)*0.002777778,15),

DATA(GAMM(0.384201,index2_y)*0.002777778,15),DATA(GAMM(0.564481,index2_y)*0.002777778,15),DATA(GAMM(0.641912,index2_y)*0.002777778,15),

DATA(GAMM(0.666737,index2_y)*0.002777778,15),DATA(GAMM(0.303224,index2_y)*0.002777778,15):

arrsch3,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.009592,index3_y)*0.002777778,15),DATA(GAMM(0.124690,index3_y)*0.002777778,15),

DATA(GAMM(0.287746,index3_y)*0.002777778,15),DATA(GAMM(0.728956,index3_y)*0.002777778,15),DATA(GAMM(1.659334,index3_y)*0.002777778,15),

DATA(GAMM(3.702330,index3_y)*0.002777778,15),DATA(GAMM(6.627746,index3_y)*0.002777778,15),DATA(GAMM(11.020665,index3_y)*0.002777778,15),

DATA(GAMM(14.540756,index3_y)*0.002777778,15),DATA(GAMM(9.255824,index3_y)*0.002777778,15):

arrsch4,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.000536635,index4_y)*0.00277777777777778,15),

DATA(GAMM(0.02253866,index4_y)*0.00277777777777778,15),DATA(GAMM(0.133085422,index4_y)*0.00277777777777778,15),

DATA(GAMM(0.322517495,index4_y)*0.00277777777777778,15),DATA(GAMM(0.567759584,index4_y)*0.00277777777777778,15), DATA(GAMM(0.690648946,index4_y)*0.00277777777777778,15),DATA(GAMM(0.572589297,index4_y)*0.00277777777777778,15),

DATA(GAMM(0.293539218,index4_y)*0.00277777777777778,15),DATA(GAMM(0.077275407,index4_y)*0.00277777777777778,15),
DATA(GAMM(0.002683174,index4_y)*0.00277777777777778,15):

arrsch5,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.001773,index5_y)*0.002777778,15),DATA(GAMM(0.011822,index5_y)*0.002777778,15),

DATA(GAMM(0.044922,index5_y)*0.002777778,15),DATA(GAMM(0.122353,index5_y)*0.002777778,15),DATA(GAMM(0.213971,index5_y)*0.002777778,15),

DATA(GAMM(0.384201,index5_y)*0.002777778,15),DATA(GAMM(0.564481,index5_y)*0.002777778,15),DATA(GAMM(0.641912,index5_y)*0.002777778,15),

DATA(GAMM(0.666737,index5_y)*0.002777778,15),DATA(GAMM(0.303224,index5_y)*0.002777778,15):

arrsch6,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.009592,index6_y)*0.002777778,15),DATA(GAMM(0.124690,index6_y)*0.002777778,15),

DATA(GAMM(0.287746,index6_y)*0.002777778,15),DATA(GAMM(0.728956,index6_y)*0.002777778,15),DATA(GAMM(1.659334,index6_y)*0.002777778,15),

DATA(GAMM(3.702330,index6_y)*0.002777778,15),DATA(GAMM(6.627746,index6_y)*0.002777778,15),DATA(GAMM(11.020665,index6_y)*0.002777778,15),

DATA(GAMM(14.540756,index6_y)*0.002777778,15),DATA(GAMM(9.255824,index6_y)*0.002777778,15):

arrsch7,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.000536635,index7_y)*0.00277777777777778,15),

DATA(GAMM(0.02253866,index7_y)*0.00277777777777778,15),DATA(GAMM(0.133085422,index7_y)*0.00277777777777778,15),

DATA(GAMM(0.322517495,index7_y)*0.00277777777777778,15),DATA(GAMM(0.567759584,index7_y)*0.00277777777777778,15),

DATA(GAMM(0.690648946,index7_y)*0.00277777777777778,15),DATA(GAMM(0.572589297,index7_y)*0.00277777777777778,15),

DATA(GAMM(0.293539218,index7_y)*0.00277777777777778,15),DATA(GAMM(0.077275407,index7_y)*0.00277777777777778,15),
DATA(GAMM(0.002683174,index7_y)*0.00277777777777778,15):

arrsch8,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.001182,index8_y)*0.002777778,15),DATA(GAMM(0.007881,index8_y)*0.002777778,15),

DATA(GAMM(0.029948,index8_y)*0.002777778,15),DATA(GAMM(0.081569,index8_y)*0.002777778,15),DATA(GAMM(0.142647,index8_y)*0.002777778,15),

DATA(GAMM(0.256134,index8_y)*0.002777778,15),DATA(GAMM(0.376320,index8_y)*0.002777778,15),DATA(GAMM(0.427941,index8_y)*0.002777778,15),

DATA(GAMM(0.444492,index8_y)*0.002777778,15),DATA(GAMM(0.202149,index8_y)*0.002777778,15):

arrsch9,TYPE(Arrival),FORMAT(Duration),FACTOR(1.0),UNITS(Days),DATA(GAMM(0.009592,index9_y)*0.002777778,15),DATA(GAMM(0.124690,index9_y)*0.002777778,15),

DATA(GAMM(0.287746,index9_y)*0.002777778,15),DATA(GAMM(0.728956,index9_y)*0.002777778,15),DATA(GAMM(1.659334,index9_y)*0.002777778,15),

```
DATA(GAMM(3.702330,index9_y)*0.002777778,15),DATA(GAMM(6.627746,index9_y)*0.002777778,15),DATA(
GAMM(11.020665,index9_y)*0.002777778,15),

DATA(GAMM(14.540756,index9_y)*0.002777778,15),DATA(GAMM(9.255824,index9_y)*0.002777778,15);

VARIABLES:    limit6_1,CLEAR(System),CATEGORY("None-None"):
        limit6_2,CLEAR(System),CATEGORY("None-None"):
        index8_y,CLEAR(System),CATEGORY("None-None"),12.18:
        limit14_2,CLEAR(System),CATEGORY("None-None"):
        limit14_3,CLEAR(System),CATEGORY("None-None"):
        limit1_1,CLEAR(System),CATEGORY("None-None"):
        index15_y,CLEAR(System),CATEGORY("None-None"),1.25:
        index3_y,CLEAR(System),CATEGORY("None-None"),0.63:
        index10_y,CLEAR(System),CATEGORY("None-None"),11.18:
        limit7_1,CLEAR(System),CATEGORY("None-None"):
        limit7_2,CLEAR(System),CATEGORY("None-None"):
        limit7_3,CLEAR(System),CATEGORY("None-None"):
        index9_y,CLEAR(System),CATEGORY("None-None"),0.42:
        limit15_2,CLEAR(System),CATEGORY("None-None"):
        limit15_3,CLEAR(System),CATEGORY("None-None"):
        limit2_1,CLEAR(System),CATEGORY("None-None"):
        index16_y,CLEAR(System),CATEGORY("None-None"),14.91:
        index4_y,CLEAR(System),CATEGORY("None-None"),14.91:
        b10,CLEAR(System),CATEGORY("None-None"),23:
        b11,CLEAR(System),CATEGORY("None-None"),19:
        b12,CLEAR(System),CATEGORY("None-None"),27:
        b13,CLEAR(System),CATEGORY("None-None"),15:
        b14,CLEAR(System),CATEGORY("None-None"),16:
        b15,CLEAR(System),CATEGORY("None-None"),22:
        b16,CLEAR(System),CATEGORY("None-None"),38:
        b17,CLEAR(System),CATEGORY("None-None"),36:
        b18,CLEAR(System),CATEGORY("None-None"),35:
        limit10_2,CLEAR(System),CATEGORY("None-None"):
        sold10,CLEAR(System),CATEGORY("None-None"),0:
        sold11,CLEAR(System),CATEGORY("None-None"),0:
        sold12,CLEAR(System),CATEGORY("None-None"),0:
        sold13,CLEAR(System),CATEGORY("None-None"),0:
        sold14,CLEAR(System),CATEGORY("None-None"),0:
        sold15,CLEAR(System),CATEGORY("None-None"),0:
        sold16,CLEAR(System),CATEGORY("None-None"),0:
        sold17,CLEAR(System),CATEGORY("None-None"),0:
        sold18,CLEAR(System),CATEGORY("None-None"),0:
        index11_y,CLEAR(System),CATEGORY("None-None"),6.77:
        limit8_1,CLEAR(System),CATEGORY("None-None"):
        limit8_2,CLEAR(System),CATEGORY("None-None"):
        limit8_3,CLEAR(System),CATEGORY("None-None"):
        b1,CLEAR(System),CATEGORY("None-None"),42:
        b2,CLEAR(System),CATEGORY("None-None"),40:
        b3,CLEAR(System),CATEGORY("None-None"),40:
        b4,CLEAR(System),CATEGORY("None-None"),0:
        b5,CLEAR(System),CATEGORY("None-None"),18:
        b6,CLEAR(System),CATEGORY("None-None"),22:
        b7,CLEAR(System),CATEGORY("None-None"),0:
        b8,CLEAR(System),CATEGORY("None-None"),21:
        b9,CLEAR(System),CATEGORY("None-None"),17:
        rc1,CLEAR(System),CATEGORY("None-None"),200:
        rc2,CLEAR(System),CATEGORY("None-None"),200:
        rc3,CLEAR(System),CATEGORY("None-None"),200:
        limit16_3,CLEAR(System),CATEGORY("None-None"):
        limit3_1,CLEAR(System),CATEGORY("None-None"):
        index17_y,CLEAR(System),CATEGORY("None-None"),13.53:
```

```
        index5_y,CLEAR(System),CATEGORY("None-None"),8.46:
        limit11_2,CLEAR(System),CATEGORY("None-None"):
        sold1,CLEAR(System),CATEGORY("None-None"),0:
        sold2,CLEAR(System),CATEGORY("None-None"),0:
        sold3,CLEAR(System),CATEGORY("None-None"),0:
        sold4,CLEAR(System),CATEGORY("None-None"),0:
        sold5,CLEAR(System),CATEGORY("None-None"),0:
        index12_y,CLEAR(System),CATEGORY("None-None"),0.42:
        sold6,CLEAR(System),CATEGORY("None-None"),0:
        limit9_1,CLEAR(System),CATEGORY("None-None"):
        sold7,CLEAR(System),CATEGORY("None-None"),0:
        limit9_2,CLEAR(System),CATEGORY("None-None"):
        sold8,CLEAR(System),CATEGORY("None-None"),0:
        limit9_3,CLEAR(System),CATEGORY("None-None"):
        sold9,CLEAR(System),CATEGORY("None-None"),0:
        cmin10,CLEAR(System),CATEGORY("None-None"):
        limit17_3,CLEAR(System),CATEGORY("None-None"):
        cmin11,CLEAR(System),CATEGORY("None-None"):
        cmin12,CLEAR(System),CATEGORY("None-None"):
        cmin13,CLEAR(System),CATEGORY("None-None"):
        cmin14,CLEAR(System),CATEGORY("None-None"):
        cmin15,CLEAR(System),CATEGORY("None-None"):
        cmin16,CLEAR(System),CATEGORY("None-None"):
        cmin17,CLEAR(System),CATEGORY("None-None"):
        cmin18,CLEAR(System),CATEGORY("None-None"):
        limit4_1,CLEAR(System),CATEGORY("None-None"):
        limit4_2,CLEAR(System),CATEGORY("None-None"):
        index18_y,CLEAR(System),CATEGORY("None-None"),1.25:
        index6_y,CLEAR(System),CATEGORY("None-None"),0.42:
        revenue,CLEAR(System),CATEGORY("None-None"):
        limit12_2,CLEAR(System),CATEGORY("None-None"):
        index13_y,CLEAR(System),CATEGORY("None-None"),11.18:
        index1_y,CLEAR(System),CATEGORY("None-None"),14.91:
        cmin1,CLEAR(System),CATEGORY("None-None"):
        cmin2,CLEAR(System),CATEGORY("None-None"):
        cmin3,CLEAR(System),CATEGORY("None-None"):
        cmin4,CLEAR(System),CATEGORY("None-None"):
        cmin5,CLEAR(System),CATEGORY("None-None"):
        cmin6,CLEAR(System),CATEGORY("None-None"):
        cmin7,CLEAR(System),CATEGORY("None-None"):
        cmin8,CLEAR(System),CATEGORY("None-None"):
        cmin9,CLEAR(System),CATEGORY("None-None"):
        limit18_3,CLEAR(System),CATEGORY("None-None"):
        limit5_1,CLEAR(System),CATEGORY("None-None"):
        limit5_2,CLEAR(System),CATEGORY("None-None"):
        index7_y,CLEAR(System),CATEGORY("None-None"),11.18:
        limit13_2,CLEAR(System),CATEGORY("None-None"):
        limit13_3,CLEAR(System),CATEGORY("None-None"):
        index14_y,CLEAR(System),CATEGORY("None-None"),10.15:
        index2_y,CLEAR(System),CATEGORY("None-None"),13.53;

COUNTERS:     sold1,,Replicate,,DATABASE(,,"User Specified","sold requests1"):
        sold2,,Replicate,,DATABASE(,,"user Specified","sold requests2"):
        sold3,,Replicate,,DATABASE(,,"User Specified","sold requests3"):
        sold4,,Replicate,,DATABASE(,,"User Specified","sold requests4"):
        sold5,,Replicate,,DATABASE(,,"User Specified","sold requests5"):
        sold6,,Replicate,,DATABASE(,,"User Specified","sold requests6"):
        sold7,,Replicate,,DATABASE(,,"User Specified","sold requests7"):
        sold8,,Replicate,,DATABASE(,,"User Specified","sold requests8"):
        sold9,,Replicate,,DATABASE(,,"User Specified","sold requests9"):
        arr10,,Replicate,,DATABASE(,,"User Specified","arrivals10"):
        arr11,,Replicate,,DATABASE(,,"User Specified","arrivals11"):
```

```
          arr12,,Replicate,,DATABASE(,,"User Specified","arrivals12"):
          arr13,,Replicate,,DATABASE(,,"User Specified","arrivals13"):
          arr14,,Replicate,,DATABASE(,,"User Specified","arrivals14"):
          arr15,,Replicate,,DATABASE(,,"User Specified","arrivals15"):
          arr16,,Replicate,,DATABASE(,,"User Specified","arrivals16"):
          arr17,,Replicate,,DATABASE(,,"User Specified","arrivals17"):
          arr18,,Replicate,,DATABASE(,,"User Specified","arrivals18"):
          sold10,,Replicate,,DATABASE(,,"User Specified","sold requests10"):
          sold11,,Replicate,,DATABASE(,,"User Specified","sold requests11"):
          sold12,,Replicate,,DATABASE(,,"User Specified","sold requests12"):
          sold13,,Replicate,,DATABASE(,,"User Specified","sold requests13"):
          sold14,,Replicate,,DATABASE(,,"User Specified","sold requests14"):
          sold15,,Replicate,,DATABASE(,,"User Specified","sold requests15"):
          sold16,,Replicate,,DATABASE(,,"User Specified","sold requests16"):
          sold17,,Replicate,,DATABASE(,,"User Specified","sold requests17"):
          sold18,,Replicate,,DATABASE(,,"User Specified","sold requests18"):
          arr1,,Replicate,,DATABASE(,,"User Specified","arrivals1"):
          arr2,,Replicate,,DATABASE(,,"User Specified","arrivals2"):
          arr3,,Replicate,,DATABASE(,,"User Specified","arrivals3"):
          arr4,,Replicate,,DATABASE(,,"User Specified","arrivals4"):
          arr5,,Replicate,,DATABASE(,,"User Specified","arrivals5"):
          arr6,,Replicate,,DATABASE(,,"User Specified","arrivals6"):
          arr7,,Replicate,,DATABASE(,,"User Specified","arrivals7"):
          arr8,,Replicate,,DATABASE(,,"User Specified","arrivals8"):
          arr9,,Replicate,,DATABASE(,,"User Specified","arrivals9");

OUTPUTS:  1,NC(arr1),"arr1.dat",,DATABASE(,,"User Specified","arr1"):
          2,NC(arr2),"arr2.dat",,DATABASE(,,"User Specified","arr2"):
          3,NC(arr3),"arr3.dat",,DATABASE(,,"User Specified","arr3"):
          4,NC(arr4),"arr4.dat",,DATABASE(,,"User Specified","arr4"):
          5,NC(arr5),"arr5.dat",,DATABASE(,,"User Specified","arr5"):
          6,NC(arr6),"arr6.dat",,DATABASE(,,"User Specified","arr6"):
          7,NC(arr7),"arr7.dat",,DATABASE(,,"User Specified","arr7"):
          8,NC(arr8),"arr8.dat",,DATABASE(,,"User Specified","arr8"):
          9,NC(arr9),"arr9.dat",,DATABASE(,,"User Specified","arr9"):
          10,NC(arr10),"arr10.dat",,DATABASE(,,"User Specified","arr10"):
          11,NC(arr11),"arr11.dat",,DATABASE(,,"User Specified","arr11"):
          12,NC(arr12),"arr12.dat",,DATABASE(,,"User Specified","arr12"):
          13,NC(arr13),"arr13.dat",,DATABASE(,,"User Specified","arr13"):
          14,NC(arr14),"arr14.dat",,DATABASE(,,"User Specified","arr14"):
          15,NC(arr15),"arr15.dat",,DATABASE(,,"User Specified","arr15"):
          16,NC(arr16),"arr16.dat",,DATABASE(,,"User Specified","arr16"):
          17,NC(arr17),"arr17.dat",,DATABASE(,,"User Specified","arr17"):
          18,NC(arr18),"arr18.dat",,DATABASE(,,"User Specified","arr18"):
          NC(sold15),,,DATABASE(,,,"sold15"):
          NC(sold1),,,DATABASE(,,,"sold1"):
          leg2 capacity,"leg2 capacity.dat",,DATABASE(,,"User Specified","leg2 capacity.dat"):
          NC(sold16),,,DATABASE(,,,"sold16"):
          NC(sold2),,,DATABASE(,,,"sold2"):
          NC(sold17),,,DATABASE(,,,"sold17"):
          NC(sold3),,,DATABASE(,,,"sold3"):
          NC(sold18),,,DATABASE(,,,"sold18"):
          NC(sold4),,,DATABASE(,,,"sold4"):
          NC(sold5),,,DATABASE(,,,"sold5"):
          NC(sold6),,,DATABASE(,,,"sold6"):
          NC(sold7),,,DATABASE(,,,"sold7"):
          NC(sold8),,,DATABASE(,,,"sold8"):
          leg1 capacity,"leg1 capacity.dat",,DATABASE(,,"User Specified","leg1 capacity"):
          leg3 capacity,"leg3 capacity.dat",,DATABASE(,,,"leg3 capacity"):
          NC(sold9),,,DATABASE(,,,"sold9"):
          NC(sold10),,,DATABASE(,,,"sold10"):
          revenue,"revenue.dat",,DATABASE(,,"User Specified","revenue"):
```

```
    NC(sold11),,,DATABASE(,,,"sold11"):
    NC(sold12),,,DATABASE(,,,"sold12"):
    NC(sold13),,,DATABASE(,,,"sold13"):
    NC(sold14),,,DATABASE(,,,"sold14");

REPLICATE,    2500,0.0,150,Yes,Yes,0.0,,,24.0,Days,No,No,,DATETIME("Feb 19, 2007 19:33:52"),Yes;
```