THE EFFECT OF ROTATION, UP TO SECOND ORDER, ON THE OSCILLATION FREQUENCIES OF SOME DELTA-SCUTI STARS

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ABSTRACT

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DELTA-SCUTI STARS

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In this work, the effect of rotation on the oscillation frequencies of some radially and nonradially oscillating Delta-Scuti stars have been explored. Rotation has been considered as a perturbation and treated up to the second order. Series of evolutionary models have been calculated for the oscillating stars in question and compared with the observational parameters. Three stars are considered: V350 Peg with no rotation, CC And with a rotational velocity $v\sin i$=20 km/s, and BS Tuc with $v\sin i$=130 km/s.

We find that splitting in the oscillation frequencies are conspicuous especially in fast rotating stars, with a considerable contribution from the related terms due to second order effect.

Keywords: stellar oscillations, oscillation frequencies, rotating stars, pulsating stars, Delta-Scuti stars

ÖZ

DELTA-SCUTI YILDIZLARINDA,
EKSENEL DÖNMENİN İKİNCİ DERECE ETKİSİ DAHİL EDİLEREK,
TİTREŞİM FREKANSLARININ İNCELENMESİ

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Bu çalışmada, çapsal ve çapsal olmayan titreşimler gösteren Delta-Scuti yıldızlarında, eksenel dönmenin titreşim frekansları üzerindeki etkisi incelenmiştir. Dönme pertürbasyon olarak ele alınmış ve dönmenin ikinci dereceye kadar etkisi hesaplamalara dahil edilmiştir. Söz konusu yıldızlar icin evrimsel model dizileri hesaplanmış ve sonuçlar gözlemsel parametrelerle karşılaştırılmıştır. Bu amaçla üç yıldızın verileri kullanılmıştır. Bunlar dönmeyen bir değişen yıldız olan V350 Peg, dönme hızı $v\sin i$=20 km/s olan CC And ve $v\sin i$=130 km/s hızla dönen BS Tuc'tur.

Sonuç olarak, titreşim frekanslarındaki saçılmanın özellikle hızlı dönen yıldızlarda dikkat çekici olduğu ve dönmenin ikinci derece etkisinden gelen te- rimlerin, saçılımın büyüklüğüne önemli bir katkıda bulunduğu görülmüştür.

Anahtar Kelimeler: yıldız titreşimleri, titreşim frekansları, dönen yıldızlar, zonklayan yıldızlar, Delta-Scuti yıldızları

To my parents

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

The subject of stellar oscillations is of great interest as it is a leading area in stellar astrophysics to explore the deep interior of stars. Studying the seismic waves within the star helps us to understand the processes taking place inside and gives us some clue about the stellar structure.

Stars are self-gravitating gaseous spheres radiating a great amount of energy to their surrounding. Energy radiated from the surface of a star is generated in the deep interior by thermonuclear reactions. Once a star is born out of an interstellar cloud, it survives by burning the hydrogen in its core and spends most of its time in the hydrogen burning stage, which is referred as main sequence stage. As evolution proceeds, the star consumes the nuclear fuel in its core causing its internal structure to change (Unno et al. 1989).

Stars are considered to be very active objects, some blow out stellar winds from their surfaces with speeds up to a few thousand kilometers per hour, while some others are pulsating variables (Unno et al. 1989).

Pulsating stars are stars in which large scale dynamical motions, usually including the entire star are present. The motion in question is generally rhythmic and resembles the breathing of a human body. The simplest kind of such motion is a purely radial oscillation (or radial pulsation), in which the star always maintains its spherical shape, but changes its volume by expanding and contracting. Nonradial oscillation (or nonradial pulsation) is a more general type in which a star oscillates distorting its spherical shape (Cox 1980, Unno et al. 1989).

The study of pulsating stars is a relatively small but highly important and promising area of modern stellar astrophysics. Pulsating stars constitute a subset of the class of intrinsic variable stars. These are stars whose variability is caused by the activity within themselves, not by geometric effects such as eclipses in binary stars or external agencies such as interaction with the interstellar medium. Variable stars can be most generally defined as stars, physical properties of which change in time. But since physical properties of all stars change during their lifetimes, for a star to be classified as variable star, the time of change should be in a scale which can be detectable by astronomers, i.e, ranging between fractions of seconds and a few years or decades (Cox, 1980).

The most obvious and most easily detectable distinguishing feature of a variable star is its apparent brightness: most such stars, in fact, are detected by their light variations. Other observable properties, such as spectral type or color, and radial velocity, usually also vary during the light variations (Cox 1980).

The theory of stellar pulsation was originally developed in order to explain the pulsations of classical variable stars such as the Cepheids and RR Lyrae stars. These variables are thought to be radial pulsators. However, pulsations and oscillation-related phenomena have been discovered in many stars that were regarded as non-pulsating stars before. They include our sun itself, white dwarfs, Ap stars , and early-type O and B stars with slow and rapid rotation. The most important characteristics of oscillations in these stars are that they are thought to be nonradial and are usually multiperiodic with several modes of oscillations involved(Unno et al. 1989).

The variability of stars was first explained in terms of stellar pulsations by the German physicist August Ritter. In 1879 he suggested that radial pulsations, accompanied by variations in surface temperature, might be responsible for the variability of some stars showing periodic variations of light. Pulsation hypothesis was presented in a quite definite form by Shapley in 1914. This hypothesis was analyzed in detail by Eddington (1926). His work laid the basis for the theory of adiabatic natural radial oscillations of gaseous spheres. Stellar oscillations were then analyzed by Rosseland (1949), Ledoux (1958,1965,1974), Ledoux and Walraven (1958), Ledoux and Whitney (1961), Cox (1967,1980). Further details about historical and theoretical background of the subject may be found in many texts (Cox and Giuli 1968, Cox 1980, Unno et al. 1989, Zhevakin 1975, Rosseland 1964, Ledoux 1974, Gautschy and Saio 1995&1996, Christensen-Dalsgaard and Dziembowski 2000). Also Christensen-Dalsgaard et al. (1999) present helioseismic studies of the solar interior.

In this study, we are interested in δ-Scuti type stars, which belong to the class of pulsating stars mentioned above. δ-Scuti stars are located in the lower part of the instability strip in H-R diagram. Their pulsation period is up to a few hours. They are of spectral type A-F and they are either main sequence (burning hydrogen in their cores) or early post MS objects (just finished hydrogen in their cores).

They show small or large amplitude variations in their light curves, pulsation amplitude ranging from milimagnitude to one magnitude, and these variations may be multiperiodic. From observations and model calculations it is understood that δ-Scuti stars show radial and nonradial pulsations simultaneously.

Generally, axial rotation is observed in these stars. They have equatorial rotational velocities such as $\upsilon \sin i$=50 km/s, where the inclination angle, i is the angle between the line of sight and rotation axis (for example; if i=30^o, υ = 100 km/s). There are some cases where i is such that υ = 250 km/s. Kırbıyık et al. (2004) studied FG Vir, providing an analysis of the inclination angle.

From theoretical and observational studies it is known that δ-Scuti stars have masses around 2 $M_\odot$ and their surface temperature, log T_{eff}, ranges from 3.6 to 4.0.

δ-Scuti stars were studied first by Chevalier (1971). Detailed information about these variable stars was given by Frolov (1975). Breger (1979) has given a review about these stars. Kurtz (1988) published an observational review on δ-Scuti pulsators and Ostermann et al. (1991) presented multiside campaign on a specific δ-Scuti star. Dziembowski (1982), and Dziembowski & Krolikowska (1985) explored nonlinear mode coupling in oscillating stars. Theoretical aspects of mode identification has also been given by Pamyatnykh (2003). The problems related to modeling of δ-Scuti stars have been discussed by Christensen-Dalsgaard (2000), and Kjeldsen (2001). Pulsational instability domain of δ-Scuti stars has been studied by Pamyathnykh (2000), while detailed introductory information and a seismological analysis of δ-Scuti stars in the Pleiades cluster is given by Fox Machado et al. (2006).

However, there are some points about pulsation properties of δ-Scuti stars, which are still unclear. Actually, it is known that there are many factors that could effect stellar pulsation such as rotation, magnetic field, tidal effect and relativistic effects (Matalgah 2004). Extensive studies have been done on the nature of oscillations in rotating stars by Smeyers and Denis (1971), Hansen and Van Horn (1979), Cox (1980), Saio (1981), Carroll and Hansen (1982), Unno et al (1989), Dziembowski and Goode (1992), Strohmayer and Lee (1996), Christensen-Dalsgaard and Dziembowski (2000), Pamyathnykh (2003), Reese et al. (2006).

The aim of this work is to study the effect of rotation on the oscillation frequencies. Rotation is treated as a perturbation and is assumed to be uniform. Effects of magnetic field and viscosity are neglected. Stellar mass elements are considered to be spherically symmetric. The star is assumed to be in hydrostatic equilibrium and adiabatic condition is used. We calculated the oscillations up to the second order in the rotation frequency Ω . We modified and used the oscillation program written in Fortran by Al-Murad and Kırbıyık in 1993, to calculate oscillation frequencies. Evolutionary models are constructed by N. Kızıloğlu using Ezer's stellar evolution code (Ezer and Cameron 1967) that was modified by Yıldız and Kızıloğlu (1997). We computed oscillation frequencies of several models for low spherical harmonic degrees (l up to 3). We tried to see the effect of rotation on different models with slow and fast rotation.

Our work consists of five chapters along with two appendices. We summarize a theoretical background and fundamental properties of nonradial oscillations of stars in the second chapter, in the third chapter we analyze the effects of rotation on the oscillation frequencies and in the fourth chapter some model calculations and corresponding results are presented. Discussion and conclusion is given in the last chapter. Finally, Appendix A includes the oscillation program that we used, while Appendix B includes the MATHEMATICA results for some angular integrals which are numerically calculated.

CHAPTER 2

SOME THEORETICAL CONSIDERATIONS RELATED TO STELLAR OSCILLATIONS

To study stellar oscillations we should be familiar with some basic mathematical and physical concepts. In this chapter, an introduction will be made by discussing some of these concepts.

2.1 Eulerian and Lagrangian Descriptions

To define a hydrodynamical system, the physical quantities are represented as functions of position $\overrightarrow{r}$ and time t. Some of these quantities are the local density $\rho(\overrightarrow{r}, t)$, local pressure P($\overrightarrow{r}$, t), local temperature T($\overrightarrow{r}$, t), and the local instantaneous velocity $\overrightarrow{v}$($\overrightarrow{r}$, t), where $\overrightarrow{r}$ is the position vector to a given point in space, i.e; point of observation. Here $\overrightarrow{r}$ and t are independent variables. This description corresponds to what is seen by a stationary observer. It is known as the Eulerian description and denoted by ∂/∂t (local time derivative). On the other hand, it is often convenient to use the Lagrangian description, which is also referred as the material (or Stokes) time derivative and denoted by d/dt. It corresponds to what is seen by the observer moving with the given fluid element (Cox 1980, Christensen-Dalsgaard & Dziembowski 2000). The local velocity is determined by the rate of change of position $\overrightarrow{r}$:

$$\overrightarrow{v}(\overrightarrow{r}, t) = \frac{d\overrightarrow{r}}{dt}. \tag{2.1}$$

And the relation between two time derivatives is given as (Cox 1980, Christensen-Dalsgaard & Dziembowski 2000)

$$\frac{d}{dt} = (\frac{\partial}{\partial t})_r + \overrightarrow{\nabla}.\frac{d\overrightarrow{r}}{dt} = \frac{\partial}{\partial t} + \overrightarrow{v}.\overrightarrow{\nabla}. \tag{2.2}$$

2.2 Basic Hydrodynamical Equations

Since stars can be treated as gaseous fluid, the equations of fluid dynamics, hence the hydrodynamical equations apply to the stars. In this section; basic hydrodynamical equations, which originate from the conservation of mass, momentum and energy, will be shortly summarized. Derivation of the equations may be found in many textbooks related to fluid mechanics (for details, see Landau and Lifshitz 1987).

Equation of Continuity (Conservation of Mass)

Conservation of mass is usually expressed by the equation of continuity and it states that the amount of mass that enters a volume element is the same with the amount leaving the volume.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{v}) = 0, \tag{2.3}$$

where ρ is density. By using the relation (2.2), equation (2.3) can also be written as

$$\frac{d\rho}{dt} + \rho \vec{\nabla}.\vec{v} = 0. \tag{2.4}$$

Equation of Motion (Conservation of Momentum)

For stars, we may ignore internal friction (viscosity) in the gas. Therefore, forces on a volume of gas consist of surface forces (the pressure on the surface of the volume) and the body forces. Thus the equation of motion, per unit volume, can be written as

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P + \rho \vec{f}, \tag{2.5}$$

where P is the pressure and $\vec{f}$ is the body force per unit mass. Combining equations (2.2) and (2.5), we get the equation of motion (Euler equations) as

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}.\vec{\nabla})\vec{v} = -\vec{\nabla} P + \rho \vec{f}. \tag{2.6}$$

Gravity will be considered as the only body force, since the effect of the magnetic field is neglected. Gravitational force per unit mass is the gravitational acceleration $\vec{g}$, and can be written as

$$\vec{g} = -\vec{\nabla}\Phi, \tag{2.7}$$

where Φ is the gravitational potential and satisfies Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho, \tag{2.8}$$

where G is the gravitational constant.

Thus, equation of motion can be written as

$$\rho\frac{\partial\overrightarrow{v}}{\partial t} + \rho(\overrightarrow{v}.\overrightarrow{\nabla})\overrightarrow{v} = -\overrightarrow{\nabla}P - \rho\overrightarrow{\nabla}\Phi. \tag{2.9}$$

And knowing that velocity is the rate of change of position, for a displacement $\delta\overrightarrow{r}$ of a mass element, we have

$$\overrightarrow{v} = \frac{\partial}{\partial t}\delta\overrightarrow{r}. \tag{2.10}$$

Energy Equation

Energy equation gives us the relation between P and ρ. In general it can be written as

$$\frac{dQ}{dt} = \frac{dE}{dt} + P\frac{d}{dt}(\frac{1}{\rho}) = \frac{dE}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} = \frac{dE}{dt} + \frac{P}{\rho}\overrightarrow{\nabla}.\overrightarrow{v}, \tag{2.11}$$

where dQ/dt gives the rate of heat gain or loss per unit mass, and E is the internal energy per unit mass (Christensen-Dalsgaard & Dziembowski 2000).

Using the adiabatic approximation (i.e. no net heat gain or loss by the oscillating mass element), this equation turns out to be

$$\frac{dP}{dt} = \frac{\Gamma_1 P}{\rho}\frac{d\rho}{dt}, \tag{2.12}$$

where

$$\Gamma_1 = (\frac{\partial\ln P}{\partial\ln\rho})_{ad} \tag{2.13}$$

is the adiabatic exponent.

2.2.1 Linearized Equations

Next step is to linearize the equations. Since the amplitude of the pulsations are small, they may be considered as small perturbations around the equilibrium and hence we may describe each quantity by adding this small perturbation to the quantity in the equilibrium state. Thus ;

$$\rho = \rho_0 + \rho' \tag{2.14}$$

$$P = P_0 + P' \tag{2.15}$$

$$\overrightarrow{v} = \overrightarrow{v_0} + \overrightarrow{v}' \tag{2.16}$$

$$\Phi = \Phi_0 + \Phi' \tag{2.17}$$

If we replace these in the nonlinear equations and use the relation between Eulerian and Lagrangian time derivatives (eqn. (2.2)) we end up with the linearized set of equations as follows

$$\frac{\partial \rho'}{\partial t} + \overrightarrow{\nabla}.(\rho_0 \overrightarrow{v}') = 0, \tag{2.18}$$

$$\frac{\partial \overrightarrow{v}'}{\partial t} = \frac{\rho'}{\rho_0^2} \overrightarrow{\nabla} P - \frac{1}{\rho_0} \overrightarrow{\nabla} P' - \overrightarrow{\nabla} \Phi', \tag{2.19}$$

$$\frac{P'}{P} = \Gamma_1 \frac{\rho'}{\rho_0} + \delta \overrightarrow{r}.(\frac{\Gamma_1}{\rho_0} \overrightarrow{\nabla} \rho - \frac{1}{P} \overrightarrow{\nabla} P), \tag{2.20}$$

$$\nabla^2 \Phi' = 4\pi G \rho'. \tag{2.21}$$

In our calculations we will use Cowling approximation, Φ'=0 , (Cowling 1941), so that we will treat the problem without taking into account of gravitational potential perturbation.

2.3 Fundamental Properties of Nonradial Oscillations

In this section we present some basic concepts about the nonradial oscillations of stars. As described before when a star is oscillating nonradially, some regions of the stellar surface expand while others contract.

A spherically symmetric star in time independent equilibrium is considered as an unperturbed state and we think of the oscillations as small perturbations. Hence we take the perturbations of the physical variables as being proportional to $Y_l^m(\theta, \phi) e^{i\sigma t}$, where Y_l^m represents the spherical harmonics, θ the colatitude and ϕ the azimuth angle in the spherical polar coordinates, σ the angular frequency -which is a function of n and l-, and t the time. The spherical harmonic degree l represents the number of border lines by which the stellar surface is divided to oscillate in the opposite phase(Unno et al. 1989). The azimuthal order m is the number of nodal lines at the longitude, while the quantum number n determines the nodal surfaces in the radial direction(Matalgah 2004). The eigenfrequencies (σ) of a normal mode show (2l+1)-fold degeneracy depending on the value of m, which takes the values from -l to l.

We shall here note that radial oscillation is a special case of nonradial oscillations with $l = 0$.

Two characteristic frequencies are used in order to describe the local vibrational property. One of them is the Lamb frequency and denoted by L_l

$$\Rightarrow L_l^2 = \frac{l(l+1)c_s^2}{r^2}. \tag{2.22}$$

The other characteristic frequency is the Brunt-Vaisala frequency, denoted by N

$$\Rightarrow N^2 = g(\frac{d\ln P}{\Gamma_1 dr} - \frac{d\ln\rho}{dr}), \tag{2.23}$$

where c_s is the speed of sound and given as

$$c_s^2 = \frac{\Gamma_1 P}{\rho}. \tag{2.24}$$

For the oscillations with high frequency ($\sigma^2 > L^2\ and\ N^2$), the relative Eulerian pressure perturbation dominates the relative radial displacement, leaving the excess pressure to be the main restoring force. In this case, oscillation shows locally the characteristics of the acoustic wave and the mode that appears is referred as p-mode. For low frequency oscillations ($\sigma^2 < L^2\ and\ N^2$), pressure perturbation is less than radial displacement, and in this case the restoring force is due mainly to buoyancy (Unno et al. 1989). This type of oscillation which shows the characteristics of the gravity wave leads to a mode referred as g-mode to appear. There is also a third mode referred as fundamental mode (f-mode), which occurs between the p and g mode regions.

We may represent the oscillation frequency σ in terms of a dimensionless frequency ω using the relation

$$\omega^2 = \frac{\sigma^2 R^3}{GM}. \tag{2.25}$$

If we plot the so-called propagation diagram, we see the characteristics of the mentioned frequencies from the center to the surface of the star (e.g., see Figure 4.10). General properties of the oscillations are characterized by N^2 and L_l^2 as mentioned before. As seen in the figure high frequency region is occupied by p-modes, while g modes lie in the low frequency region which is the interior of the star.

While the radial oscillation has only the spectrum of the pressure mode (p-mode) or the acoustic (wave) mode (see Figure 4.1), the nonradial oscillation shows the spectrum of the gravity (wave) mode (g-mode) as well (see Figures 4.10 and 4.16).

CHAPTER 3

EFFECT OF ROTATION ON THE OSCILLATION FREQUENCIES

Two additional forces act on the oscillating mass elements in a star which is pulsating as well as rotating. These are centrifugal force and Coriolis force. Complication is caused by the nonsphericity, which is a result of the former. Centrifugal force is proportional to the square of the angular rotation velocity, whereas the Coriolis force is proportional only to the first power of the velocity. Thus, the Coriolis force can not be neglected even in a star that is rotating sufficiently slow to neglect centrifugal force -hence nonsphericity (Cox 1980).

The effect of rotation has been studied theoretically by many authors, some of which are Simon (1969), Smeyers and Denis (1971), Chlebowski (1978), Saio (1981), Strohmayer and Lee (1996), and Christensen-Dalsgaard & Dziembowski (2000).

As explained before, the degeneracy in m arises from the rotational symmetry of the equilibrium structure around an arbitrary axis. Therefore, if a slow rotation or a weak magnetic field is introduced, the degeneracy is resolved. The degeneracy is lifted by slow rotation, giving (2l+1) separate eigenfrequencies with equal spacing (Unno et al. 1989).

We assume uniform rotation in our calculations, in which the equation of motion is modified as follows (Matalgah 2004);

$$\frac{d\overrightarrow{v}}{dt} + 2(\overrightarrow{\Omega} \times \overrightarrow{v}) + \overrightarrow{\Omega} \times (\overrightarrow{\Omega} \times \overrightarrow{r}) = -\overrightarrow{\nabla}\Phi' + \frac{\rho'}{\rho_0^2}\overrightarrow{\nabla}P_0 - \frac{1}{\rho_0}\overrightarrow{\nabla}P'. \tag{3.1}$$

3.1 First Order Rotational Effect

To study the first order rotational effect, which is the aim of this section, we use the equation (3.1) with the first order term in Ω only. Thus;

$$\frac{d\overrightarrow{v}}{dt} + 2(\overrightarrow{\Omega} \times \overrightarrow{v}) = -\overrightarrow{\nabla}\Phi' + \frac{\rho'}{\rho_0^2}\overrightarrow{\nabla}P_0 - \frac{1}{\rho_0}\overrightarrow{\nabla}P'. \tag{3.2}$$

Generalizing the expansion of the perturbed quantities to rotation as;

$$f' = f'(r)Y_l^m(\theta, \phi)e^{i\sigma t}, \tag{3.3}$$

and letting the displacement be along the radius vector in any direction as;

$$\overrightarrow{\xi}(r, \theta, \phi; t) = \delta\overrightarrow{r}(r, \theta, \phi; t) = \overrightarrow{\xi}(r)Y_l^m(\theta, \phi)e^{i\sigma t}, \tag{3.4}$$

where σ is the oscillation frequency in the rotating frame, we have

$$\overrightarrow{v} = \frac{\partial}{\partial t}\delta\overrightarrow{r} = \frac{\partial}{\partial t}\overrightarrow{\xi} = (i\sigma)\overrightarrow{\xi}(r, \theta, \phi; t) \tag{3.5}$$

and

$$\frac{d\overrightarrow{v}}{dt} = -\sigma^2\overrightarrow{\xi}(r, \theta, \phi; t). \tag{3.6}$$

Substituting equations (3.5) and (3.6) in equation (3.2), we get

$$-\sigma^2\overrightarrow{\xi} + 2i\sigma(\overrightarrow{\Omega} \times \overrightarrow{\xi}) = -\overrightarrow{\nabla}\Phi' + \frac{\rho'}{\rho_0^2}\overrightarrow{\nabla}P_0 - \frac{1}{\rho_0}\overrightarrow{\nabla}P'. \tag{3.7}$$

Now let oscillation frequency $\sigma = \sigma_0 + \sigma_1$ in rotating frame, where '0' denotes the oscillation frequency with no rotation and '1' denotes the oscillation frequency with 1^{st} order rotation. We have

$$\sigma^2 = (\sigma_0 + \sigma_1)^2 = \sigma_0^2 + 2\sigma_0\sigma_1 + \sigma_1^2 \tag{3.8}$$

and

$$\overrightarrow{\xi} = \overrightarrow{\xi}_0 + \overrightarrow{\xi}_1. \tag{3.9}$$

Then, let us expand equation (3.7) up to the first order in rotation as;

$$-(\sigma_0^2 + 2\sigma_0\sigma_1 + \sigma_1^2)(\overrightarrow{\xi}_0 + \overrightarrow{\xi}_1) + 2i(\sigma_0 + \sigma_1)(\overrightarrow{\Omega} \times \overrightarrow{\xi}_0) = -\overrightarrow{\nabla}\Phi' + \frac{\rho'}{\rho_0^2}\overrightarrow{\nabla}P_0 - \frac{1}{\rho_0}\overrightarrow{\nabla}P'. \quad (3.10)$$

Equating the 1^{st} order perturbed terms in left and right hand sides;

$$-\sigma_0^2\overrightarrow{\xi}_1 - 2\sigma_0\sigma_1\overrightarrow{\xi}_0 + 2i\sigma_0(\overrightarrow{\Omega} \times \overrightarrow{\xi}_0) = -\overrightarrow{\nabla}\Phi_1' + \frac{\rho_1'}{\rho_0^2}\overrightarrow{\nabla}P_0 - \frac{1}{\rho_0}\overrightarrow{\nabla}P_1'. \quad (3.11)$$

To solve this, we write the first order quantities in terms of zeroth order quantities as;

$$\overrightarrow{\xi_1} = \sum a_n\overrightarrow{\xi_{0n}}. \quad (3.12)$$

Applying the similar representation to P_1, Φ_1 and ρ_1, substituting in the equation (3.11), multiplying both sides with $\overrightarrow{\xi_0}^*$ and integrating over the mass, we get σ_1, after some manipulations, as

$$\sigma_1 = \frac{i\int(\overrightarrow{\Omega} \times \overrightarrow{\xi_0}).\overrightarrow{\xi_0}^*\rho_0 r^2 dr}{\int|\overrightarrow{\xi_0}|^2\rho_0 r^2 dr} \quad (3.13)$$

where

$$\overrightarrow{\xi_0} = \{\xi_{0r}, \xi_{0h}\frac{\partial}{\partial\theta}, \xi_{0h}\frac{1}{sin\theta}\frac{\partial}{\partial\phi}\}Y_l^m(\theta,\phi)e^{i\sigma t}. \quad (3.14)$$

After proper integrations equation (3.13) becomes

$$\sigma_1 = m\Omega\frac{\int[2\xi_{0r}\xi_{0h} + \xi_{0h}^2]\rho_0 r^2 dr}{\int[\xi_{0r}^2 + l(l+1)\xi_{0h}^2]\rho_0 r^2 dr}. \quad (3.15)$$

Call

$$\frac{\int[2\xi_{0r}\xi_{0h} + \xi_{0h}^2]\rho_0 r^2 dr}{\int[\xi_{0r}^2 + l(l+1)\xi_{0h}^2]\rho_0 r^2 dr} = C. \quad (3.16)$$

Thus,

$$\sigma_1 = m\Omega C, \quad (3.17)$$

and

$$\sigma = \sigma_0 + m\Omega C, \quad (3.18)$$

in rotating frame, while the oscillation frequency in inertial reference frame turns out to be (Al-Murad 1993)

$$\sigma = \sigma_0 + m\Omega[1 - C]. \quad (3.19)$$

To solve these, we define new dimensionless variables following Dziembowski (1971) as

$$Y_1 = \frac{\delta r}{r} = \frac{\xi_r}{r}, \tag{3.20}$$

$$Y_2 = \frac{1}{rg}(\frac{P'}{\rho_0} + \Phi'), \tag{3.21}$$

$$Y_3 = \frac{1}{rg}\Phi', \tag{3.22}$$

and

$$Y_4 = \frac{1}{g}\frac{d\Phi'}{dr}. \tag{3.23}$$

Using these new variables and combining the four equations to be solved, we have

$$r\frac{dY_1}{dr} = (\frac{V}{\Gamma_1} - 3)Y_1 + [\frac{l(l+1)}{C_1\omega_0^2} - \frac{V}{\Gamma_1}]Y_2 + \frac{V}{\Gamma_1}Y_3 + \frac{2m\Omega}{\sigma_0}\{Y_{0,1} + [\frac{1}{C_1\omega_0^2} - \frac{\sigma_1}{m\Omega}\frac{l(l+1)}{C_1\omega_0^2}]Y_{0,2}\}, \tag{3.24}$$

$$r\frac{dY_2}{dr} = (C_1\omega_0^2 + rA)Y_1 + (1 - U - rA)Y_2 + rY_3 + \frac{2m\Omega}{\sigma_0}[\frac{\sigma_1}{m\Omega}C_1\omega_0^2 Y_{0,1} - Y_{0,2}], \tag{3.25}$$

$$r\frac{dY_3}{dr} = (1 - U)Y_3 + Y_4, \tag{3.26}$$

$$r\frac{dY_4}{dr} = -UrAY_1 + \frac{UV}{\Gamma_1}Y_2 + [l(l+1) - \frac{UV}{\Gamma_1}]Y_3 - UY_4, \tag{3.27}$$

where

$$V = -\frac{d\ln P}{d\ln r} = \frac{gr}{c_s}, \tag{3.28}$$

$$U = \frac{d\ln M}{d\ln r} = \frac{4\pi\rho r^3}{M(r)}, \tag{3.29}$$

$$C_1 = (\frac{r}{R})^3\frac{M_{total}}{M(r)}, \tag{3.30}$$

$$\omega_0^2 = \frac{\sigma^2 R^3}{GM_{total}}, \tag{3.31}$$

$$A = \frac{d\ln g}{dr} - \frac{1}{\Gamma_1}\frac{d\ln P}{dr}, \tag{3.32}$$

$$c_s = (\frac{\Gamma_1 P}{\rho})^{1/2}, \tag{3.33}$$

where c_s is the speed of sound and Γ_1 is referred as the adiabatic exponent and given as

$$\Gamma_1 = (\frac{d\ln P}{d\ln \rho})_{ad.}. \tag{3.34}$$

In the equations (3.24)-(3.27), $Y_{0,i}$ (i=1,2,3,4) are variables in nonrotating star, hence have known solutions, whereas Y_1, Y_2, Y_3, Y_4, and σ_1 are the unknowns. Hence we have 4 equations and 5 unknowns.

We shall now introduce the boundary conditions. At the center, i.e. $r = 0$, $(\delta r/r)$ must be finite, while at the surface, i.e. $r = R$, $(\delta P/P)$ must be so. Thus we have the following conditions.

At the center;

$$(i)\ C_1\omega_0^2 Y_1 - lY_2 + \frac{2m\Omega}{\sigma_0}C_1\omega_0^2[\frac{\sigma_1}{m\Omega} - \frac{1}{l}]Y_{0,1} = 0 \tag{3.35}$$

and

$$(ii)\ lY_3 - Y_4 = 0. \tag{3.36}$$

At the surface;

$$\begin{aligned}(i)\ &(1 - \frac{4 + C_1\omega_0^2}{V})Y_1 + \{\frac{l(l+1)}{C_1\omega_0^2 V} - 1\}Y_2 + (1 - \frac{l+1}{V})Y_3 \\ &+ \frac{2m\Omega}{\sigma_0 V}\{(1 - \frac{\sigma_1}{m\Omega}C_1\omega_0^2)Y_{0,1} + [1 - \frac{\sigma_1}{m\Omega}l(l+1) + C_1\omega_0^2]\frac{Y_{0,2}}{C_1\omega_0^2}\} = 0\end{aligned} \tag{3.37}$$

and

$$(ii)\ UY_1 + (1+l)Y_3 + Y_4 = 0. \tag{3.38}$$

At this point, to simplify the problem, we use Cowling approximation (Cowling 1941), which states $\Phi' = 0$. The effect of gravitational potential perturbation ($\Phi' \neq 0$) has been studied for both rotating and nonrotating star models by Özel (2003), and for a specific source V2109 Cyg by Özel et al. (2005).

When this approximation is employed, our equations to be solved reduce to 2 equations since the variables Y_3 and Y_4 vanish together with the boundary conditions involving them.

So we are left with 2 equations and 2 boundary conditions which are equations (3.24)&(3.25) and (3.35)&(3.37), respectively. The unknowns to be found are Y_1, Y_2, and σ_1.

In our program the solution is achieved by first assigning a value to ω which is related to σ, using this value in order to find Y_1 and Y_2, and then checking whether the boundary conditions are satisfied. If they are not satisfied, small increments are added to the initial value step by step until the equations of boundary conditions hold. In this way we obtain the value of σ up to a certain accuracy by defining a tolerance value at the beginning to determine up to which digit we want our value to be precise. In our calculations we usually used a tolerance value of 10^{-3}. This tells the program when to stop iteration, i.e. when the numerical values are substituted in the equations and boundary conditions hold up to a certain precision. In the case of a tolerance value of 10^{-3} it means that it is enough to find the frequency value precisely up to the third digit after the decimal point. A version of our oscillation program for an arbitrary case is given in Appendix A.

3.2 Second Order Rotational Effect

To proceed in second order rotational calculations let us once again state our basic equations.

Energy equation:

$$\frac{P'}{P_0} = \Gamma_1 \frac{\rho'}{\rho_0} + \vec{\xi}.(\frac{\Gamma_1}{\rho_0}\vec{\nabla}\rho_0 - \frac{1}{P_0}\vec{\nabla}P_0). \tag{3.39}$$

Poisson's equation:

$$\nabla^2\Phi' = 4\pi G\rho' \tag{3.40}$$

Continuity equation:

$$\rho' + \vec{\nabla}.(\rho_0\vec{\xi}) = 0 \tag{3.41}$$

And equation of motion (Matalgah 2004):

$$-\sigma^2\overrightarrow{\xi}+2i\sigma(\overrightarrow{\Omega}\times\overrightarrow{\xi})+\overrightarrow{\Omega}\times(\overrightarrow{\Omega}\times\overrightarrow{r})=-\frac{1}{\rho_0}\overrightarrow{\nabla}P'-\frac{\rho'}{\rho_0^2}\overrightarrow{\nabla}P_0-\overrightarrow{\nabla}\Phi'. \tag{3.42}$$

Now let us take equation (3.42) and rewrite it as

$$-\sigma^2\overrightarrow{\xi}+2i\sigma(\overrightarrow{\Omega}\times\overrightarrow{\xi})+\overrightarrow{\Omega}\times(\overrightarrow{\Omega}\times\overrightarrow{r})=-\overrightarrow{\nabla}(\Phi'+\frac{P'}{\rho_0})-\frac{P'}{\rho_0^2}\overrightarrow{\nabla}\rho_0+\frac{\rho'}{\rho_0^2}\overrightarrow{\nabla}P_0. \tag{3.43}$$

Call

$$\chi=(\Phi'+\frac{P'}{\rho_0}). \tag{3.44}$$

The sum in the paranthesis on the right hand side of equation (3.39) is equal to $\Gamma_1\overrightarrow{A}$, since $\overrightarrow{A}$ was defined as the convection criterion as

$$\overrightarrow{A}=\frac{1}{\rho_0}\overrightarrow{\nabla}\rho_0-\frac{1}{\Gamma_1 P_0}\overrightarrow{\nabla}P_0. \tag{3.45}$$

After some manipulations, our equations turn out to be as follows;

$$\frac{P'}{P_0}=\Gamma_1\frac{\rho'}{\rho_0}+\Gamma_1\overrightarrow{\xi}.\overrightarrow{A}, \tag{3.46}$$

$$\nabla^2\Phi'=4\pi G\rho', \tag{3.47}$$

$$\frac{\rho'}{\rho_0}+\frac{\overrightarrow{\xi}.\overrightarrow{\nabla}\rho_0}{\rho_0}=-\overrightarrow{\nabla}.\overrightarrow{\xi}, \tag{3.48}$$

and

$$-\sigma^2\overrightarrow{\xi}+2i\sigma(\overrightarrow{\Omega}\times\overrightarrow{\xi})+\overrightarrow{\Omega}\times(\overrightarrow{\Omega}\times\overrightarrow{r})=-\overrightarrow{\nabla}\chi+\frac{\Gamma_1 P_0}{\rho_0}(\overrightarrow{\nabla}.\overrightarrow{\xi})\overrightarrow{A}, \tag{3.49}$$

where

$$\sigma=\sigma_0+\sigma_1+\sigma_2, \tag{3.50}$$

$$P'=P_0+P_1+P_2, \tag{3.51}$$

$$\rho'=\rho_0+\rho_1+\rho_2, \tag{3.52}$$

$$\Phi' = \Phi_0 + \Phi_1 + \Phi_2, \tag{3.53}$$

$$\chi' = \chi_0 + \chi_1 + \chi_2, \tag{3.54}$$

$$\overrightarrow{\xi} = \overrightarrow{\xi_0} + \overrightarrow{\xi_1} + \overrightarrow{\xi_2}. \tag{3.55}$$

To find $\overrightarrow{\xi_1}$, which we will need in our calculations, we use equation (3.49) for the first order case as

$$-2\sigma_0\sigma_1\overrightarrow{\xi_0} - \sigma_0^2\overrightarrow{\xi_1} = 2i\sigma_0(\overrightarrow{\Omega} \times \overrightarrow{\xi_0}) - \overrightarrow{\nabla}\chi_1 + \frac{\Gamma_1 P_0}{\rho_0}(\overrightarrow{\nabla}.\overrightarrow{\xi_1})\overrightarrow{A}. \tag{3.56}$$

For a spherically symmetric case, $\overrightarrow{A}$ has only r-component; $\overrightarrow{A} = A\hat{e}_r$.So,

$$\overrightarrow{\xi_1} = -\frac{2\sigma_1}{\sigma_0}\overrightarrow{\xi_0} + \frac{2i}{\sigma_0}(\overrightarrow{\Omega} \times \overrightarrow{\xi_0}) + \frac{1}{\sigma_0^2}\overrightarrow{\nabla}\chi_1 - \frac{\Gamma_1 P_0}{\sigma_0^2\rho_0}(\overrightarrow{\nabla}.\overrightarrow{\xi_1})A\hat{e_r}, \tag{3.57}$$

where

$$\overrightarrow{\xi_0} = (\xi_{0r}, \xi_{0h}\frac{\partial}{\partial\theta}, \frac{\xi_{0h}}{sin\theta}\frac{\partial}{\partial\phi})Y_l^m \tag{3.58}$$

$$\overrightarrow{\Omega} = (\Omega\cos\theta, -\Omega\sin\theta, 0) \tag{3.59}$$

After some manipulations, we get the components of $\overrightarrow{\xi_1}$ as

$$\xi_{1r} = \xi_{1r}(r)Y_l^m, \tag{3.60}$$

$$\xi_{1\theta} = (b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\frac{\partial}{\partial\theta}Y_l^m + \frac{2m\Omega}{\sigma_0}\xi_{0h}\cot\theta Y_l^m, \tag{3.61}$$

and

$$\xi_{1\phi} = (b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\frac{im}{sin\theta}Y_l^m + \frac{2i\Omega}{\sigma_0}[\xi_{0r}\sin\theta Y_l^m + \xi_{0h}\cos\theta\frac{\partial Y_l^m}{\partial\theta}], \tag{3.62}$$

where

$$b_1 = \frac{\chi_1}{\sigma_0^2\ r\ Y_l^m} = \frac{\chi(r)}{\sigma_0^2\ r}. \tag{3.63}$$

Now let us rewrite equation (3.49) keeping only the second order terms;

$$
\begin{aligned}
&-\sigma_1^2\overrightarrow{\xi_0} - 2\sigma_0\sigma_2\overrightarrow{\xi_0} - 2\sigma_0\sigma_1\overrightarrow{\xi_1} - \sigma_0^2\overrightarrow{\xi_2} \\
&+ 2i\sigma_0(\overrightarrow{\Omega} \times \overrightarrow{\xi_1}) + 2i\sigma_1(\overrightarrow{\Omega} \times \overrightarrow{\xi_0}) + \overrightarrow{\Omega}(\overrightarrow{\Omega}.\overrightarrow{r}) - \Omega^2\overrightarrow{r} = 0.
\end{aligned} \tag{3.64}
$$

Taking the dot product of equation (3.65) with $(-\overrightarrow{\xi_0}^*)$, integrating over mass and solving for σ_2, we have

$$
\sigma_2 = \frac{\sigma_1^2}{2\sigma_0} + \frac{1}{J}\int(\overrightarrow{\Omega} \times \overrightarrow{\xi_1}).\overrightarrow{\xi_0}^* dm_r + \frac{1}{2\sigma_0 J}\int(\overrightarrow{\Omega}.\overrightarrow{r})(\overrightarrow{\Omega}.\overrightarrow{\xi_0}^*)dm_r - \frac{\Omega^2}{2\sigma_0 J}\int\overrightarrow{r}.\overrightarrow{\xi_0}^* dm_r, \tag{3.65}
$$

where

$$
J = \int\overrightarrow{\xi_0}.\overrightarrow{\xi_0}^* dm_r \tag{3.66}
$$

and

$$
dm_r = \rho r^2 dr sin\theta d\theta d\phi \tag{3.67}
$$

$$
\sigma_2 = \sigma_{21} + \sigma_{22} + \sigma_{23} + \sigma_{24} \tag{3.68}
$$

$$
\sigma_{21} = \frac{\sigma_1^2}{2\sigma_0} = \frac{[m\Omega(1-C)]^2}{2\sigma_0} \tag{3.69}
$$

$$
\sigma_{22} = \frac{i}{J}\int(\overrightarrow{\Omega} \times \overrightarrow{\xi_1}).\overrightarrow{\xi_0}^* dm_r \tag{3.70}
$$

$$
\sigma_{23} = \frac{1}{2\sigma_0 J}\int(\overrightarrow{\Omega}.\overrightarrow{r})(\overrightarrow{\Omega}.\overrightarrow{\xi_0}^*)dm_r \tag{3.71}
$$

$$
\sigma_{24} = -\frac{\Omega^2}{2\sigma_0 J}\int(\overrightarrow{r}.\overrightarrow{\xi_0}^*)dm_r \tag{3.72}
$$

Let us remember the coordinates of $\overrightarrow{\Omega}, \overrightarrow{r}$ and $\overrightarrow{\xi_0}^*$;

$$
\overrightarrow{\Omega} = [\Omega(r,\theta)cos\theta, -\Omega(r,\theta)sin\theta, 0] \tag{3.73}
$$

$$
\overrightarrow{r} = (r, 0, 0) \tag{3.74}
$$

$$
\overrightarrow{\xi_0}^* = (\xi_{0r}Y_l^{m*}, \xi_{0h}\frac{\partial}{\partial\theta}Y_l^{m*}, \frac{\xi_{0h}}{sin\theta}\frac{\partial}{\partial\phi}Y_l^{m*}) \tag{3.75}
$$

Using (3.8)-(3.10) we have σ_{23} and σ_{24} as

$$\sigma_{23} = \frac{\Omega^2}{2\sigma_0 J}\{\int rcos^2\theta\xi_{0r}Y_l^{m*}dm_r - \int rcos\theta sin\theta\xi_{0h}\frac{\partial}{\partial\theta}Y_l^{m*}dm_r\} \tag{3.76}$$

$$\sigma_{24} = -\frac{\Omega^2}{2\sigma_0 J}\int r\xi_{0r}Y_l^{m*}dm_r \tag{3.77}$$

$$\begin{aligned}\sigma_{23}+\sigma_{24} = -&\frac{\Omega^2}{2\sigma_0 J}[\int r\xi_{0r}\rho r^2 dr\int(1-cos^2\theta)Y_l^{m*}sin\theta d\theta d\phi \\ &+\int r\xi_{0h}\rho r^2 dr\int cos\theta sin\theta\frac{\partial}{\partial\theta}Y_l^{m*}sin\theta d\theta d\phi]\end{aligned} \tag{3.78}$$

$$= -\frac{\Omega^2}{2\sigma_0 J}\{\int r^3\xi_{0r}\rho dr\int(sin\theta - sin\theta cos^2\theta)Y_l^{m*}d\theta d\phi + \int r^3\xi_{0h}\rho dr\int sin^2\theta cos\theta\frac{\partial}{\partial\theta}Y_l^{m*}d\theta d\phi\} \tag{3.79}$$

The angular integrals above and all the others hereafter are calculated numerically using MATHEMATICA and the results are given in the Appendix B.

The result of $\sigma_{23}+\sigma_{24}$ is equal to zero for all (l,m) in consideration ($l \leq 3$) except for $(l,m) = (0,0)$ and (2,0). For (0,0)

$$\sigma_{23}+\sigma_{24} = -\frac{2\Omega^2\sqrt{\pi}}{3\sigma_0 J}\int r^3\xi_{0r}\rho dr, \tag{3.80}$$

while for (2,0)

$$\sigma_{23}+\sigma_{24} = -\frac{2\Omega^2\sqrt{\pi/5}}{\sigma_0 J}\int(\xi_{0h} - \frac{1}{3}\xi_{0r})\rho r^3 dr. \tag{3.81}$$

Now we have σ_{21}, σ_{23} and σ_{24}. We should next find σ_{22}, which is a little bit more complicated.

$$\sigma_{22} = \frac{i}{J}\int(\overrightarrow{\Omega}\times\overrightarrow{\xi_1}).\overrightarrow{\xi_0}^* dm_r = \frac{i}{J}\int S dm_r, \tag{3.82}$$

where

$$\xi_1 = (\xi_{1r}, \xi_{1\theta}, \xi_{1\phi}). \tag{3.83}$$

$$\overrightarrow{\Omega}\times\overrightarrow{\xi_1} = (-\xi_{1\phi}\Omega sin\theta)\hat{e}_r - (\xi_{1\phi}\Omega cos\theta)\hat{e}_\theta + (\xi_{1\theta}\Omega cos\theta + \xi_{1r}\Omega sin\theta)\hat{e}_\phi \tag{3.84}$$

Then,

$$S = -\xi_{1\phi}\Omega sin\theta\xi_{0r}Y_l^{m*} - \xi_{1\phi}\Omega cos\theta\xi_{0h}\frac{\partial}{\partial\theta}Y_l^{m*} + \xi_{1\theta}\Omega cos\theta\frac{\xi_{0h}}{sin\theta}\frac{\partial}{\partial\phi}Y_l^{m*} + \xi_{1r}\Omega\xi_{0h}\frac{\partial}{\partial\phi}Y_l^{m*} \tag{3.85}$$

$$S = \xi_{1r}\Omega\xi_{0h}\frac{\partial}{\partial\phi}Y_l^{m*} + \xi_{1\theta}\Omega cos\theta\frac{\xi_{0h}}{sin\theta}\frac{\partial}{\partial\phi}Y_l^{m*} - \xi_{1\phi}\Omega sin\theta\xi_{0r}Y_l^{m*} - \xi_{1\phi}\Omega cos\theta\xi_{0h}\frac{\partial}{\partial\theta}Y_l^{m*} \tag{3.86}$$

Now,

$$\sigma_{22} = \frac{i}{J}\int Sdm_r = \frac{i}{J}\int(S_1 + S_2 + S_3 + S_4)dm_r = \sigma_{22,S_1} + \sigma_{22,S_2} + \sigma_{22,S_3} + \sigma_{22,S_4}. \tag{3.87}$$

Let us remember the expression of spherical harmonics;

$$Y_l^m = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(cos\theta). \tag{3.88}$$

So we have

$$S_1 = -im\xi_{1r}\Omega\xi_{0h}Y_l^{m*}, \tag{3.89}$$

where

$$\xi_{1r} = \xi_{1r}(r)Y_l^m. \tag{3.90}$$

Then,

$$\sigma_{22,S_1} = \frac{m\Omega}{J}\int\xi_{1r}(r)\xi_{0h}\rho r^2dr, \tag{3.91}$$

where the orthogonality relation of spherical harmonics (Arfken and Weber 1995)

$$\int Y_l^mY_l^{m*}sin\theta d\theta d\phi = 1 \tag{3.92}$$

is used.

$$S_2 = -im\Omega\xi_{0h}\xi_{1\theta}cot\theta Y_l^{m*} \tag{3.93}$$

Remember

$$\xi_{1\theta} = (b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\frac{\partial}{\partial\theta}Y_l^m + \frac{2m\Omega}{\sigma_0}\xi_{0h}cot\theta Y_l^m. \tag{3.94}$$

Since

$$\sigma_{22,S_2} = \frac{i}{J}\int S_2dm_r, \tag{3.95}$$

we have

$$\begin{aligned}\sigma_{22,S_2} = &\frac{m\Omega}{J}\int\xi_{0h}(b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\rho r^2dr\int\frac{\partial}{\partial\theta}(Y_l^mY_l^{m*})cos\theta d\theta d\phi\\ &+\frac{2(m\Omega)^2}{\sigma_0J}\int\xi_{0h}^2\rho r^2dr\int cot^2\theta|Y_l^m|^2sin\theta d\theta d\phi,\end{aligned} \tag{3.96}$$

where the first term will be referred as $\sigma_{22,S_{2a}}$, while the second as $\sigma_{22,S_{2b}}$.

$$S_3 = -\Omega \xi_{0r} \xi_{1\phi} sin\theta Y_l^{m*}, \tag{3.97}$$

and

$$\sigma_{22,S_3} = \frac{i}{J} \int S_3 dm_r. \tag{3.98}$$

Remember

$$\xi_{1\phi} = (b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\frac{im}{sin\theta}Y_l^m + \frac{2i\Omega}{\sigma_0}[\xi_{0r} sin\theta Y_l^m + \xi_{0h} cos\theta \frac{\partial}{\partial\theta} Y_l^m]. \tag{3.99}$$

We have

$$\begin{aligned}
\sigma_{22,S_3} = & \frac{m\Omega}{J}\int (b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\xi_{0r}\rho r^2 dr \\
& + \frac{2\Omega^2}{\sigma_0 J}\int \xi_{0r}^2 \rho r^2 dr \int sin^3\theta Y_l^m Y_l^{m*} d\theta d\phi \\
& + \frac{2\Omega^2}{\sigma_0 J}\int \xi_{0r}\xi_{0h}\rho r^2 dr \int cos\theta sin^2\theta \frac{\partial}{\partial\theta} Y_l^m Y_l^{m*} d\theta d\phi,
\end{aligned} \tag{3.100}$$

where the first, second and third terms will be referred as $\sigma_{22,S_{3a}}$, $\sigma_{22,S_{3b}}$ and $\sigma_{22,S_{3c}}$, respectively.

$$S_4 = -\Omega \xi_{0h} \xi_{1\phi} cos\theta \frac{\partial}{\partial\theta} Y_l^{m*} \; and \; \sigma_{22,S_4} = \frac{i}{J}\int S_4 dm_r \tag{3.101}$$

So,

$$\begin{aligned}
\sigma_{22,S_4} = & \frac{m\Omega}{J}\int (b_1 - \frac{2\sigma_1}{\sigma_0}\xi_{0h})\xi_{0h}\rho r^2 dr \int cos\theta Y_l^m \frac{\partial}{\partial\theta} Y_l^{m*} d\theta d\phi \\
& + \frac{2\Omega^2}{\sigma_0 J}\int \xi_{0r}\xi_{0h}\rho r^2 dr \int sin^2\theta cos\theta Y_l^m \frac{\partial}{\partial\theta} Y_l^{m*} d\theta d\phi \\
& + \frac{2\Omega^2}{\sigma_0 J}\int \xi_{0h}^2 \rho r^2 dr \int cos^2\theta sin\theta \frac{\partial}{\partial\theta} Y_l^m \frac{\partial}{\partial\theta} Y_l^{m*} d\theta d\phi,
\end{aligned} \tag{3.102}$$

where the first, second and third terms will be referred as $\sigma_{22,S_{4a}}$, $\sigma_{22,S_{4b}}$ and $\sigma_{22,S_{4c}}$, respectively.

To have σ_{22} in a better form, we will gather some terms, which have common variables as;

$$\sigma_{22,S_{2a}+S_{3a}+S_{4a}} = -\frac{2(m\Omega)^2(1-C)}{\sigma_0 J}\int (\xi_{0h}^2 + \xi_{0h}\xi_{0r})\rho r^2 dr + \frac{m\Omega}{J}\int b_1(\xi_{0h} + \xi_{0r})\rho r^2 dr, \tag{3.103}$$

and

$$\sigma_{22,S_{3c}+S_{4b}} = \frac{2\Omega^2}{\sigma_0 J} \int \xi_{0r}\xi_{0h}\rho r^2 dr \int sin^2\theta cos\theta \frac{\partial}{\partial\theta}(Y_l^m Y_l^{m*}) d\theta d\phi. \quad (3.104)$$

Hence, we finally have σ_2 as composed of following terms:

$$\sigma_{21} = \frac{\sigma_1^2}{2\sigma_0} = \frac{[m\Omega(1-C)]^2}{2\sigma_0}, \quad (3.105)$$

$$\sigma_{22,S_1} = \frac{m\Omega}{J} \int \xi_{1r}(r)\xi_{0h}\rho r^2 dr, \quad (3.106)$$

$$\sigma_{22,S_{2a}+S_{3a}+S_{4a}} = -\frac{2(m\Omega)^2(1-C)}{\sigma_0 J} \int (\xi_{0h}^2 + \xi_{0h}\xi_{0r})\rho r^2 dr + \frac{m\Omega}{J} \int b_1(\xi_{0h}+\xi_{0r})\rho r^2 dr, \quad (3.107)$$

$$\sigma_{22,S_{3c}+S_{4b}} = \frac{2\Omega^2}{\sigma_0 J} \int \xi_{0r}\xi_{0h}\rho r^2 dr \int sin^2\theta cos\theta \frac{\partial}{\partial\theta}(Y_l^m Y_l^{m*}) d\theta d\phi, \quad (3.108)$$

$$\sigma_{22,S_{3b}} = \frac{2\Omega^2}{\sigma_0 J} \int \xi_{0r}^2 \rho r^2 dr \int sin^3\theta Y_l^m Y_l^{m*} d\theta d\phi, \quad (3.109)$$

$$\sigma_{22,S_{2b}} = \frac{2(m\Omega)^2}{\sigma_0 J} \int \xi_{0h}^2 \rho r^2 dr \int cot^2\theta |Y_l^m|^2 sin\theta d\theta d\phi, \quad (3.110)$$

$$\sigma_{22,S_{4c}} = \frac{2\Omega^2}{\sigma_0 J} \int \xi_{0h}^2 \rho r^2 dr \int cos^2\theta sin\theta \frac{\partial}{\partial\theta} Y_l^m \frac{\partial}{\partial\theta} Y_l^{m*} d\theta d\phi, \quad (3.111)$$

$$\sigma_{23} + \sigma_{24} = \{-\frac{2\Omega^2\sqrt{\pi}}{3\sigma_0 J} \int r^3 \xi_{0r}\rho dr, \ for(l,m) = (0,0) \quad (3.112)$$

$$= \{\frac{2\Omega^2\sqrt{\pi/5}}{\sigma_0 J} \int (\xi_{0h} + \frac{1}{3}\xi_{0r})\rho r^3 dr, \ for(l,m) = (2,0). \quad (3.113)$$

Once calculating the angular parts of the integrals numerically for all the (l,m) values of interest (in our case; up to $l = 3$), we write all the integral terms that constitute σ_2 as multiplication of the angular integral result -if there is an angular part- with the radial integral part. Each integral term for a given case of different (l,m) combination that we want to calculate is introduced in our computer program. The program for an arbitrary case is given in APPENDIX A. This is the last version of the program including our modifications, while the original version may be found in M.S. thesis of Al-Murad (1993). The program was also used and modified by Özel (2003), and Matalgah (2004).

CHAPTER 4

SOME MODEL CALCULATIONS AND RESULTS

In this chapter, results of calculations done by using our oscillation program and star models are presented. To choose suitable models for the observed stars, it was first attempted to match the observed fundamental frequency of each star to the frequencies calculated using our models. After having various models giving the frequency in question, the best model to represent the observed star is determined by trying to match the parameters (like temperature, luminosity, mass and radius) given in literature, up to a certain precision.

The models that are used for rotating stars have been evolved including rotation and considering global angular momentum conservation. Our evolutionary models are constructed by N. Kızıloğlu using the modified version (Yıldız and Kızıloğlu 1997) of Ezer's stellar evolution code (Ezer and Cameron 1967). The modification was made according to the MHD equation of state(Mihalas et al. 1990). OPAL opacity tables (Iglesias et al. 1992) were applied to the stellar models. The initial chemical composition used in our models is such that $X = 0.698$, and $Z = 0.019$. Other details about the stellar evolutionary code are described by Yıldız and Kızıloğlu (1997).

To start with, oscillation program is tested using a model without rotation. For this model star with no rotation, radial fundamental frequency was calculated and matched to the corresponding observed one. After that a star with slow rotation and then one with fast rotation were chosen. Results will be presented in this order.

4.1 V350 Peg

V350 Peg (=HIP 115563, HD 220564) has been classified as a δ-Scuti type pulsating star since its discovery during the Hipparcos mission (ESA 1997). It has a pulsation period of 0.2012 days and a total pulsation amplitude of 0.05 mag (Vidal-Sainz et al.

2002, Ekmekçi and Topal 2007) with a minimum visual magnitude of 7.32 (SIMBAD Astronomical Database). No information about its rotation has been recorded. Some parameters of this star are given in Table 4.1.

Table 4.1: Parameters of V350 Peg given in literature

V350 Peg in literature	Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	T_{eff} (K)	Freq. (c/d) $(l=0)$
Ekmekçi and Topal (2007)	1.7-2	4.3	23.988	6137.6 / 6760.8	6.67332 (F_1) 4.66181 (F_2)
Vidal-Sainz et al.(2002)				6800 (100)	5.668 (F_2) 5.840 (F_1)

The given pulsation period (0.2012 days) leads to a frequency of 4.97 c/d. None of the authors quoted above had confirmed the main period given in the Hipparcos Catalogue. Instead, Vidal-Sainz et al. identified two frequencies -of almost similar amplitude-,the ratio of which indicates nonradial pulsation for at least one of the corresponding modes (2002). They achieved the frequencies using the observational data of 31 nights between July 1997 and January 1998, and 4 nights in Novenber and December 2001. Ekmekçi and Topal had also identified two frequencies (by using different temperatures), which are not in agreement with that of the former authors, as possible fundamental frequencies (2007). They used the observational data obtained between August 2005-December 2005.

It has been attempted to obtain each frequency given by the authors quoted above. It has not been concluded on one certain model but the possible models that might represent the star, i.e. the models with the closest values to the given parameters and frequency range are presented in Table 4.2 along with the corresponding parameters.

As can be seen in Table 4.2, the proposed eligible models are in a mass range 1.9-2 $M_\odot$. It was not possible to match both the frequency and the parameters for models with smaller masses. And all the models that are proposed belong to the region towards the end of the Main Sequence because of their quite low core hydrogen abundance. These results are in agreement with the results of Ekmekçi and Topal. They proposed that V350 Peg is located between the evolutionary tracks of 1.7 and

Table 4.2: Frequency results of the suitable models, calculated by our oscillation program with no rotation

Model Number	Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	T_{eff} (K)	ρ_c/ρ_{mean} §1	X_c §2	Freq. (c/d) $(l=0,f)$
652	1.9	3.400792	24.45154	6965.300	26658.0	≈ 0	6.67
716	1.9	3.693506	24.20325	6666.568	50903.3	≈ 0	5.851
717	1.9	3.701962	24.19243	6658.206	52046.9	≈ 0	5.829
724	1.9	3.762146	24.09741	6598.239	58779.9	≈ 0	5.68
725	1.9	3.770911	24.08478	6589.702	59829.3	≈ 0	5.659
771	1.9	4.146194	22.87361	6203.862	125134.7	≈ 0	4.66
965	2.0	3.769423	29.63301	6941.603	139058.3	≈ 0	5.848
979	2.0	3.840460	29.49271	6868.950	53110.6	≈ 0	5.667
1030	2.0	4.324073	28.07006	6393.926	119606.7	≈ 0	4.674
1031	2.0	4.333165	28.02848	6384.848	121313.3	≈ 0	4.655

§1 Ratio of the core density to the mean density of the star

§2 Central Hydrogen abundance

2 $M_\odot$, but nearer to that of 2 $M_\odot$ with the point close to the upper end of the main sequence or may be at the beginning of the subgiant branch to be classified as F2 IV-V (Ekmekçi and Topal 2007).

The propagation diagram of one of the selected models can be found in Figure 4.1. Lamb frequency does not exist in the diagram since l=0.

While trying to find the appropriate models in the given frequency range, several models have been calculated at different evolutionary stages of 1.9 and 2 $M_\odot$ stars without rotation. The frequency trend is shown in Figure 4.2. It is obvious from Figure 4.2 that as stellar mass increases, oscillation frequency corresponding to the same temperature gets higher. The lines corresponding to fundamental frequency, first and second overtones are not equidistant as the temperature varies. This is a precursor of the difficulty in the mode identification as the temperature decreases, which will be encountered again in the following models.

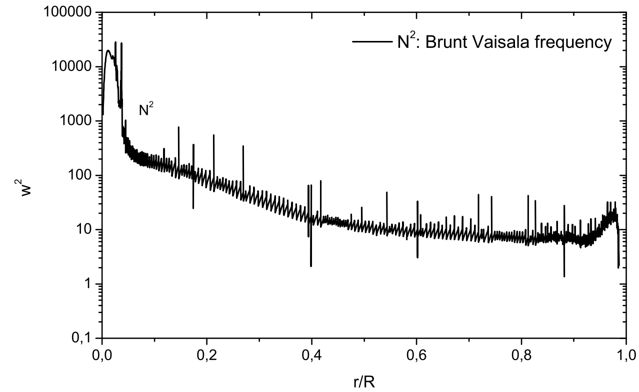


Figure 4.1: The Propagation Diagram for model 724 (1.9 $M_\odot$ star, $X_c \approx 0$), plotted for l=0 in the absence of rotation

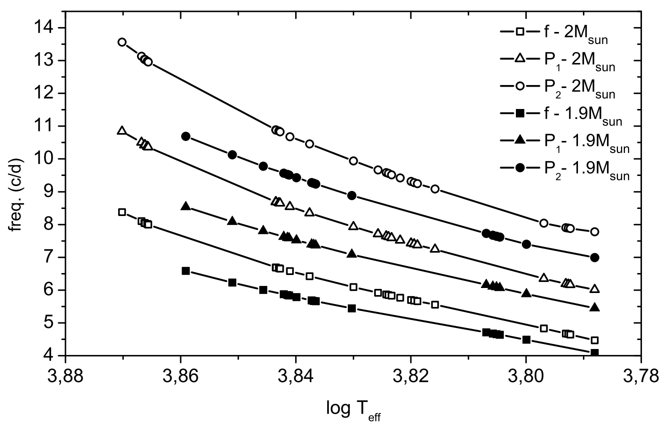


Figure 4.2: Frequency change along some part of the evolutionary track of 1.9 and 2 $M_\odot$ stars in the absence of rotation

Luminosity versus effective temperature belonging to the series of models that are calculated for 1.9 $M_{\odot}$ star is plotted in Figure 4.3.

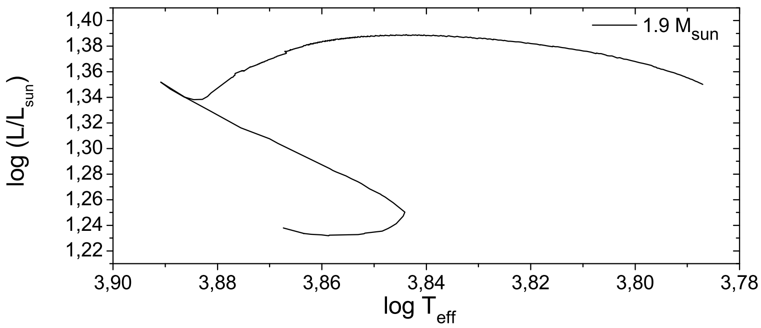


Figure 4.3: Change in luminosity and effective temperature with evolution (1.9 $M_{\odot}$) Only part of the track, that includes the models we used, is presented.

In order to see where our models are located in the Hertzsprung-Russell (HR) diagram, the track of 1.9 $M_{\odot}$ star, obtained by using our models, is placed onto the HR diagram presented by Maeder and Meynet (1988) (Figure 4.4).

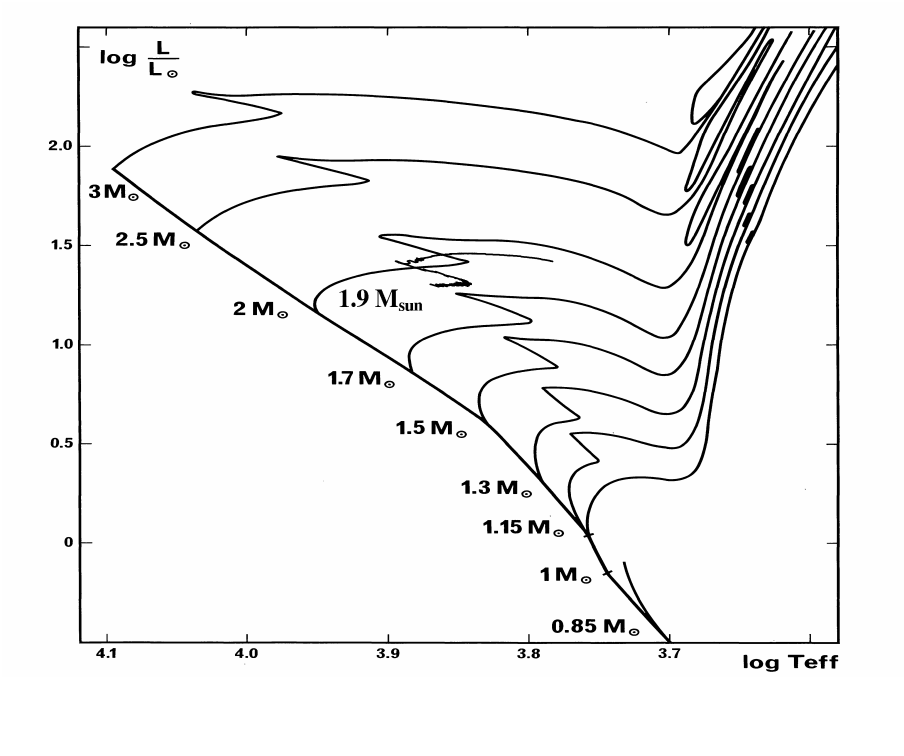


Figure 4.4: Evolutionary tracks of low mass stars in the HR diagram as presented by Maeder and Meynet (1988) and 1.9 $M_{\odot}$ track added using the calculated models

It seems that the track corresponding to the 1.9 $M_{\odot}$ exactly corresponds to the expected region. By combining with the result presented in Figure 4.2, it can be concluded that as the star evolves, the pulsation frequency decreases.

Since all the results have been in accordance with our expectations, calculations are carried on by selecting another star (CC And) with slow rotation and analyzing the rotational effect on the oscillation frequencies.

4.2 CC And

CC And (=HIP3432) is a δ Scuti type variable star with a spectral type F3IV-V. It was discovered in 1952 due to its light variation by Lindblad and Eggen (Lindblad and Eggen 1953). It has a rotational speed of $v\sin i$=20 km/s (Lopez de Coca et al. 1990) which categorizes this star as a slow rotator. Quite a few observed oscillation frequencies have been reported in literature. All frequencies have been identified as nonradial oscillations by Fu and Jiang (1995), however some of the frequencies were recently identified as radial oscillations by Ekmekçi and Topal (2007). Fu and Jiang used the observational data obtained between 25 September and 20 November 1984, while Ekmekçi and Topal used tha data obtained in the period August-December 2005. Visual magnitude V of this source is 9.39 with ΔV=0.240 (Solano and Fernley 1997), and the pulsation period was given as 0.1249 days in many papers. Those mentioned and some other parameters of this star are given in Table 4.3.

Table 4.3: Parameters of CC And given in literature

CC And in literature	Rot. Vel. ($v\sin i$) (km/s)	Spectral type	Period (days)	Mass ($M_{\odot}$)	Radius ($R_{\odot}$)	Luminosity ($L_{\odot}$)	Q §1 (days)	T_{eff} (K)
Breger (1979)		F3 IV	0.125					
Halprin and Moon (1983)		F3 IV	0.125					
Lopez de Coca et al. (1990)	20	F3 IV	0.1249					
Claret et al. (1990)			0.1249	1.98	3.04	25.1189 (M_{bol}=1.25)		
Rodriguez et al. (1994)	20	F3 IV-V	0.1249					
Fu and Jiang (1995)	20	F3 IV-V	0.1249	1.98	3.04	25.1189 (M_{bol}=1.25)	0.0331	7400
Solano and Fernley (1997)		F3 IV-V	0.1249					
Ekmekçi and Topal (2007)				~1.7	3.04	25.1189 (M_{bol}=1.25)	0.033	7413
				~1.7	3.04	25.1189 (M_{bol}=1.25)	0.031	6918.3

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_{\odot}>}}$, where P is the period in days.

In Tables 4.4 and 4.5, observed frequencies presented by the authors quoted above are given.

The pulsation frequency of the star (8.006 c/d, calculated by taking the reciprocal of the given pulsation period) is considered as the radial fundamental frequency. So, our attempt was to obtain this value, in the case of l=0. Trying to match both the parameters given in Table 4.3 and the fundamental frequency up to a certain precision, the model which represents the star best is chosen.

Parameters of the chosen model can be seen in Table 4.6 and the results corresponding to l=0 are shown in Table 4.7.

Table 4.4: Observed frequencies given by Fu and Jiang (1995)

	Frequency (cycle/day)	Q §1 (days)	Identification
F_1	8.005890	0.0331	*l=3,f,m=0*
F_2	7.814795	0.0339	*l=3,f,m=2*
F_3	8.101026	0.0327	*l=3,f,m=-1*
F_4	13.34628	0.0198	$l=3,p_2$ or radial *2H*
F_5	7.902449	0.0335	*l=3,f,m=1*
F_6	16.01199	0.0165	$l=3,p_3,m=0$
F_7	15.82091	0.0167	$l=3,p_3,m=2$

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

Table 4.5: Observed frequencies given by Ekmekçi and Topal (2007)

	Frequency (cycle/day)	Q §1 (days)	Identification
F_1	8.00576	0.031	Fundamental
F_2	8.81625	0.028	
F_3	6.84994	0.036	
F_4	6.56455	0.038	
F_5	16.88150	0.015	

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

Table 4.6: Parameters of the selected model for CC And

Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	T_{eff} (K)	ρ_c/ρ_{mean} §1	X_c §2	Rot. Vel. §3 (km/s)
1.93	3.040220	25.80512	7466.674	9249.00	≈ 0	21.3341994

§1 Ratio of the core density to the mean density of the star

§2 Central Hydrogen abundance

§3 Equatorial rotational velocity, v=$v$sin$i$, with the inclination angle i=90^o

Table 4.7: Frequency results for l=0 calculated using the selected model for CC And

Mode	Frequency (cycle/day)	Q §1 (days)
f	8.005	0.03274
p_1	10.36	0.02530
p_2	12.96	0.02023

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

After achieving the fundamental frequency (up to a certain precision) by simultaneously matching the parameters (letting an error of 1% to 8% -depending on the variety of the given values- for the effective temperature and around 3% for the luminosity) we ended up with a series of models with a mass of $\geq$ 1.9 $M_\odot$. We then calculated the other frequencies belonging to the low degree spherical harmonics ($l \leq$3). Our main purpose was to see how much effect slow rotation has on oscillation frequencies. We also tried to see whether we could obtain the observed frequencies. We finally selected the model with 1.93 $M_\odot$ since we could match many of the observed frequencies. Some results of our calculations for l=1 can be seen in Table 4.8.

In Figure 4.5 the splitting in the frequency of the selected model for l=1 can be seen. Splitting for l=1 of individual modes is shown in Figure 4.6 in large scale.

It is seen that for a rotational velocity around 20 km/s, contribution of the second order rotational effect to the change in frequency is not large enough to be taken into account. Frequency splitting between consecutive m values is around 0.15 c/d, while only around 15% of this value is the contribution of the second order rotational effect.

Table 4.8: Several frequency results of the selected model for CC And (l=1)

m	Freq. (c/d) (without rotation)	Freq.1 (c/d) (1^{st} order rot. effect added)	Freq.2 (c/d) (1^{st} & 2^{nd} order rot. effect added)	Freq. ratio (Freq.2/Freq.1)	Q §1 (days)	N_p §2	N_g §3	estimated mode
-1	20.1185	20.0096	20.0051	0.99978	0.0131	5	6	g_1
0	20.1185	20.1185	20.1191	1.00003	0.01303	5	6	g_1
1	20.1185	20.2274	20.2367	1.00046	0.01295	5	6	g_1
-1	21.5046	21.4185	21.3825	0.99832	0.01226	5	5	f
0	21.5046	21.5046	21.505	1.00002	0.01219	5	5	f
1	21.5046	21.5907	21.6289	1.00177	0.01212	5	5	f
-1	22.6896	22.5745	22.5713	0.99986	0.01161	6	5	p_1
0	22.6896	22.6896	22.6901	1.00002	0.01155	6	5	p_1
1	22.6896	22.8047	22.8123	1.00033	0.01149	6	5	p_1
-1	25.9699	25.8782	25.849	0.99887	0.01014	7	4	p_3
0	25.9699	25.9699	25.9703	1.00002	0.01009	7	4	p_3
1	25.9699	26.0616	26.0932	1.00121	0.01004	7	4	p_3
-1	28.1602	28.0345	28.0323	0.99992	0.00935	8	4	p_4
0	28.1602	28.1602	28.1607	1.00002	0.00931	8	4	p_4
1	28.1602	28.2859	28.2922	1.00022	0.00926	8	4	p_4
-1	31.4993	31.3957	31.3738	0.9993	0.00835	9	3	p_6
0	31.4993	31.4993	31.4997	1.00001	0.00832	9	3	p_6
1	31.4993	31.6029	31.6274	1.00078	0.00829	9	3	p_6
-1	34.0738	33.944	33.9428	0.99996	0.00772	10	3	p_7
0	34.0738	34.0738	34.0743	1.00001	0.00769	10	3	p_7
1	34.0738	34.2037	34.2085	1.00014	0.00766	10	3	p_7

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

§2 Number of p nodes from the center to the surface of the star

§3 Number of g nodes from the center to the surface of the star

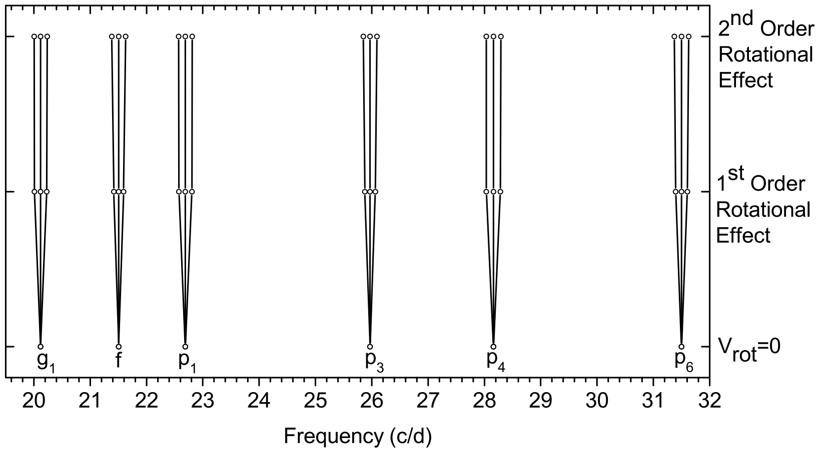


Figure 4.5: Frequency splitting for l=1, belonging to the selected model for CC And

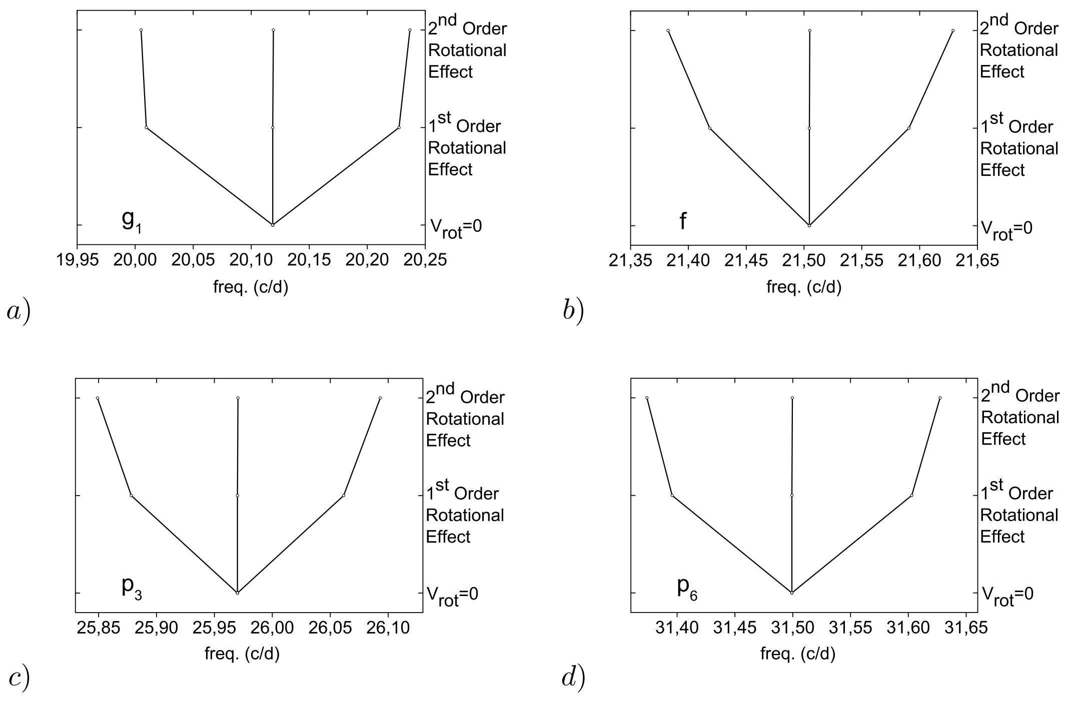


Figure 4.6: Frequency splitting for several selected modes of l=1

Calculated frequencies for the case of l=2 are presented in Table 4.9, while splitting for several modes of l=2 is shown in Figure 4.7, and in a larger scale in Figure 4.8.

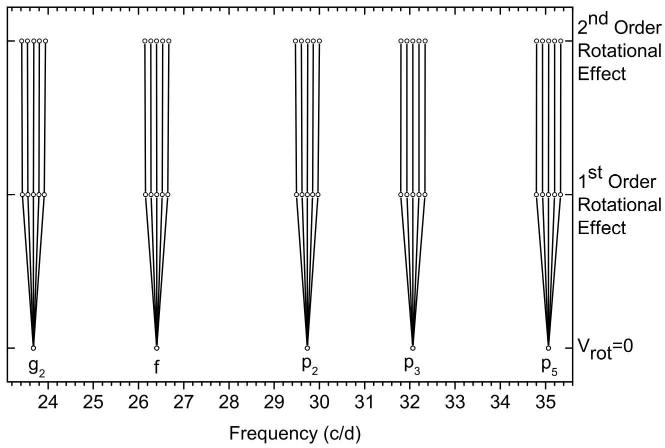


Figure 4.7: Frequency splitting for l=2, belonging to the selected model for CC And

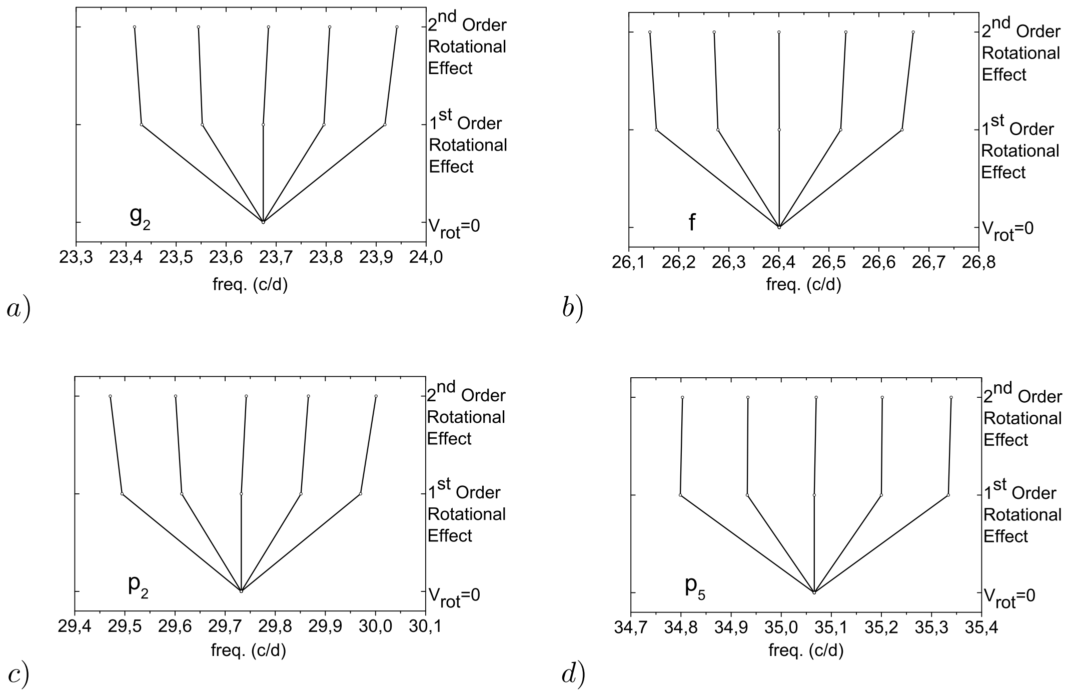


Figure 4.8: Frequency splitting of several modes of the selected model for CC And (l=2)

Table 4.9: Several frequency results of the selected model for CC And (l=2)

m	Freq. (c/d) (without rotation)	Freq.1 (c/d) (1^{st} order rot. effect added)	Freq.2 (c/d) (1^{st} & 2^{nd} order rot. effect added)	Freq. ratio (Freq.2/Freq.1)	Q [§1] (days)	N_p [§2]	N_g [§3]	estimated mode
-2	23.6741	23.4307	23.4167	0.9994	0.01119	6	8	g_2
-1	23.6741	23.5524	23.5447	0.99967	0.01113	6	8	g_2
0	23.6741	23.6741	23.6846	1.00044	0.01107	6	8	g_2
1	23.6741	23.7957	23.8071	1.00048	0.01101	6	8	g_2
2	23.6741	23.9174	23.9415	1.00101	0.01095	6	8	g_2
-2	26.4007	26.155	26.1421	0.99951	0.01003	7	7	f
-1	26.4007	26.2779	26.2707	0.99973	0.00998	7	7	f
0	26.4007	26.4007	26.4001	0.99998	0.00993	7	7	f
1	26.4007	26.5236	26.534	1.00039	0.00988	7	7	f
2	26.4007	26.6464	26.6687	1.00084	0.00983	7	7	f
-2	29.7323	29.4944	29.4709	0.9992	0.00889	8	6	p_2
-1	29.7323	29.6133	29.6011	0.99959	0.00885	8	6	p_2
0	29.7323	29.7323	29.7422	1.00033	0.00881	8	6	p_2
1	29.7323	29.8513	29.8662	1.0005	0.00878	8	6	p_2
2	29.7323	29.9702	30.001	1.00103	0.00874	8	6	p_2
-2	32.0668	31.8002	31.8053	1.00016	0.00824	9	6	p_3
-1	32.0668	31.9335	31.9352	1.00005	0.00821	9	6	p_3
0	32.0668	32.0668	32.0655	0.99996	0.00817	9	6	p_3
1	32.0668	32.2001	32.2020	1.00006	0.00814	9	6	p_3
2	32.0668	32.3334	32.3389	1.00017	0.0081	9	6	p_3
-2	35.0661	34.7986	34.8029	1.00012	0.00753	10	5	p_5
-1	35.0661	34.9323	34.9337	1.00004	0.0075	10	5	p_5
0	35.0661	35.0661	35.0697	1.0001	0.00747	10	5	p_5
1	35.0661	35.1999	35.2017	1.00005	0.00745	10	5	p_5
2	35.0661	35.3336	35.3391	1.00016	0.00742	10	5	p_5

[§1] $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

[§2] Number of p nodes from the center to the surface of the star

[§3] Number of g nodes from the center to the surface of the star

In this case it is seen again that the splitting between the consecutive m values is around 0.15 c/d, with a second order rotational contribution of about 15%. A similar conclusion is reached from the analysis of l=3 (Table 4.10, Figure 4.9).

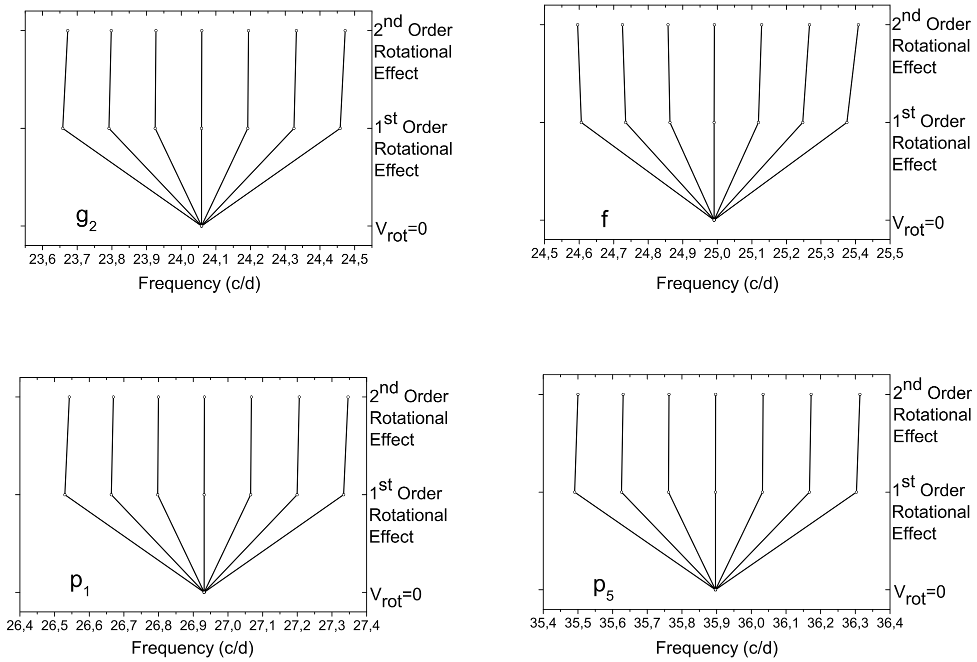


Figure 4.9: Frequency splitting of several modes of the selected model for CC And ($l = 3$)

In Figure 4.10 the propagation diagram is plotted for l=1, 2 and 3.

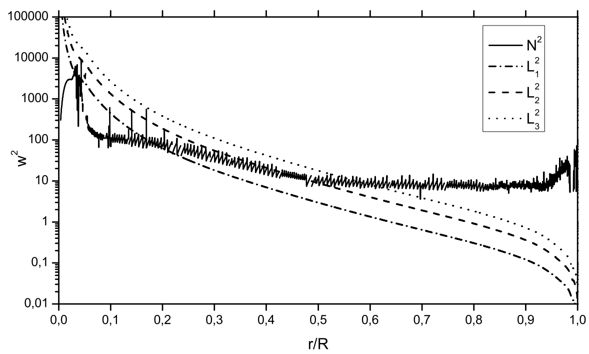


Figure 4.10: The Propagation Diagram for 1.93 $M_{\odot}$ star, $X_c \approx 0$, $v\sin i \simeq 21.3$ km/s

Table 4.10: Several frequency results of the selected model for CC And (l=3)

m	Freq. (c/d) (without rotation)	Freq.1 (c/d) (1^{st} order rot. effect added)	Freq.2 (c/d) (1^{st} & 2^{nd} order rot. effect added)	Freq. ratio (Freq.2/Freq.1)	Q §1 (days)	N_p §2	N_g §3	estimated mode
-3	24.0584	23.6581	23.6725	1.00061	0.01107	6	8	g_2
-2	24.0584	23.7915	23.7983	1.00029	0.01101	6	8	g_2
-1	24.0584	23.9249	23.9272	1.0001	0.01095	6	8	g_2
0	24.0584	24.0584	24.0591	1.00003	0.01089	6	8	g_2
1	24.0584	24.1918	24.1941	1.0001	0.01083	6	8	g_2
2	24.0584	24.3252	24.332	1.00028	0.01077	6	8	g_2
3	24.0584	24.4586	24.4729	1.00058	0.01071	6	8	g_2
-3	24.9908	24.6065	24.5957	0.99956	0.01066	7	7	f
-2	24.9908	24.7346	24.7253	0.99962	0.0106	7	7	f
-1	24.9908	24.8627	24.8573	0.99978	0.01054	7	7	f
0	24.9908	24.9908	24.9915	1.00003	0.01049	7	7	f
1	24.9908	25.1189	25.1281	1.00037	0.01043	7	7	f
2	24.9908	25.2471	25.267	1.00079	0.01037	7	7	f
3	24.9908	25.3752	25.4083	1.0013	0.01032	7	7	f
-3	26.9315	26.5296	26.5425	1.00049	0.00987	7	6	p_1
-2	26.9315	26.6636	26.6697	1.00023	0.00983	7	6	p_1
-1	26.9315	26.7975	26.7996	1.00008	0.00978	7	6	p_1
0	26.9315	26.9315	26.9322	1.00003	0.00973	7	6	p_1
1	26.9315	27.0655	27.0676	1.00008	0.00968	7	6	p_1
2	26.9315	27.1995	27.2056	1.00022	0.00963	7	6	p_1
3	26.9315	27.3335	27.3464	1.00047	0.00958	7	6	p_1
-3	35.8967	35.4904	35.4999	1.00027	0.00738	10	5	p_5
-2	35.8967	35.6258	35.6303	1.00013	0.00736	10	5	p_5
-1	35.8967	35.7613	35.7627	1.00004	0.00733	10	5	p_5
0	35.8967	35.8967	35.8972	1.00001	0.0073	10	5	p_5
1	35.8967	36.0322	36.0339	1.00005	0.00727	10	5	p_5
2	35.8967	36.1676	36.1726	1.00014	0.00725	10	5	p_5
3	35.8967	36.3031	36.3134	1.00028	0.00722	10	5	p_5

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

§2 Number of p nodes from the center to the surface of the star

§3 Number of g nodes from the center to the surface of the star

In Figure 4.11 we show the correspondence of our results which are in the range of observed frequencies.

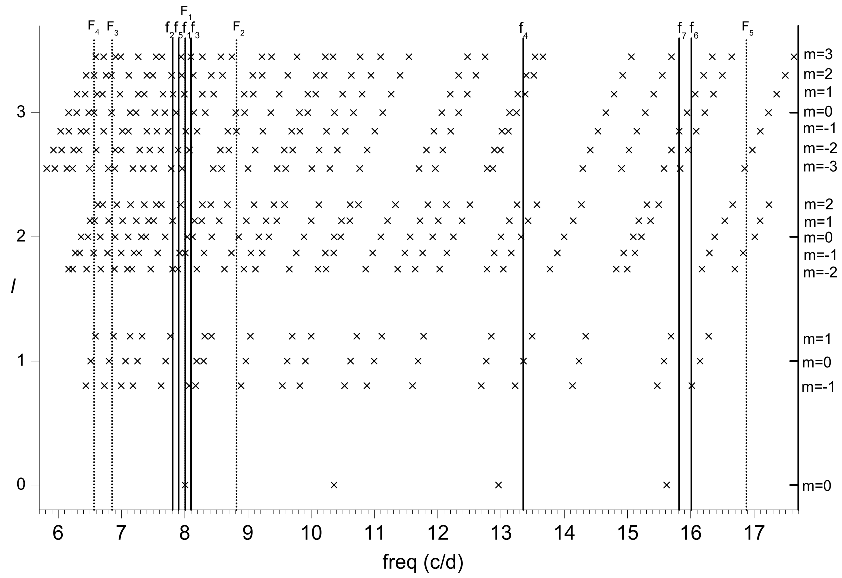


Figure 4.11: Calculated frequencies of our model for CC And, corresponding to the range of observed frequencies

(The results are obtained by using the model with a mass of 1.93 $M_\odot$ and a rotation velocity of 21.3km/s. The results are given for *l*=0,1,2 and 3. The splitted frequencies according to the azimuthal order m is also shown. The solid lines f_1-f_7 correspond to the observational frequencies given by Fu and Jiang (1995) while the dotted lines F_1-F_5 correspond to the observational frequencies given by Ekmekci and Topal (2007), which were presented in Tables 4.4 and 4.5.)

It is clear that the first frequency of each group, which are very close to each other, agrees with our calculated fundamental frequency. Other frequencies may also be considered to match several calculated frequencies. As the spherical harmonic degree *l* gets higher, there is an increase in the number of calculated frequencies in the same frequency range. An important remark is that mode identification may not be completely reliable and that is why the expression of "estimated modes" is used. Our program determines the modes using the number of p and g nodes. One is referred as p-mode if there is a larger number of p nodes than g nodes and as g-mode in the opposite case. For that matter, number of p and g nodes, which are reliable are also given. Frequency results which are in the observed frequency range, are mostly estimated as g modes by the program, but in this case, since these frequencies are observed, another phenomenon like "avoided crossing" (Aizenman et al., 1977) must have occurred so that the nature of p and g modes may change and one mode may show the properties of the other mode. Another reason for the g modes to be observed may be that; each estimated mode is not pure g or pure p mode because of having nodes both in g and in p region.

At this point, sticking to our main purpose, which is to investigate the rotational effect on the oscillation frequencies, we propose that it may be sufficient to include the effect of slow rotation up to the first order only, since the second order rotational effect is not very effective on the oscillation frequencies. In the next section, second order rotational effect will be analyzed for the fast rotating variable star BS Tuc.

4.3 Bs Tuc

Bs Tuc (=HD6870) is listed as a δ-Scuti type variable star by Michel Breger (1979). It was then classified as λ Bootis star (one of the δ Scuti subgroups) by Rodriguez and Breger (2001). Its rotational speed is given as $v\sin i$=130 km/s (Lopez de Coca et al. 1990). The pulsation period is reported as 0.065 days which turns out to be 15.38 cycles/day. Visual magnitude V of this variable is 7.49 with ΔV=0.02 (Rodriguez and Breger 2001). The parameters of this star is listed in Table 4.11.

To start with; a model is selected, which has the closest frequency value to the fundamental frequency and at the same time it was attempted to match the other parameters. The most appropriate model is presented in Table 4.12.

The frequency results of this model calculated for l=0 can be seen in Table 4.13, while calculated frequencies for l=1 and 2 are listed in Tables 4.14 and 4.15, respectively.

In Figure 4.12, the rotational effect on the oscillation frequencies for low spherical harmonic degrees l=1 and l=2 are plotted.

Table 4.11: Parameters of BS Tuc given in literature.

Bs Tuc in literature	Rot. Vel. (v sin i) (km/s)	Spectral type	Period (days)	T$_{eff}$ (K)	M$_{bol}$ §1	Q §2 (days)
Breger (1979)		A5 III	0.065			
Halprin and Moon (1983)		A5 III	0.065			
Tsvetkov (1985)			0.065	7744.62	1.98 (lum.=12.82L$_{\odot}$)	0.0327
Lopez de Coca et al. (1990)	130	A5 III	0.065			
Rodriguez et al. (1994)	130	A5 III	0.065			

§1 Bolometric magnitude

§2 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

Table 4.12: Parameters of the selected model for BS Tuc

Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	T$_{eff}$ (K)	ρ_c/ρ_{mean} §1	X_c §2	Rot. Vel. §3 (km/s)
1.8	1.93	12.185	7766.174	216.18	0.4501	132.763712

§1 Ratio of the core density to the mean density of the star

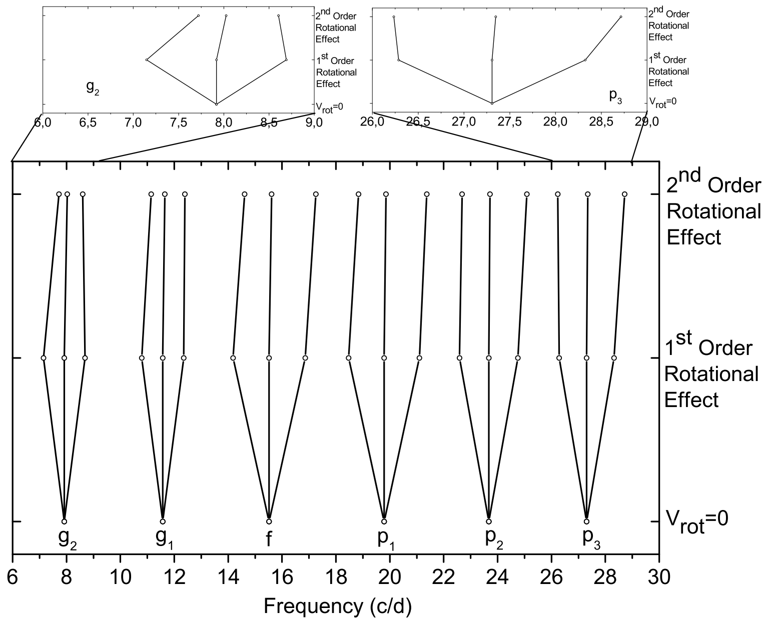
§2 Central Hydrogen abundance

§3 Equatorial rotational velocity, v=$v$sin$i$, with the inclination angle i=90^o

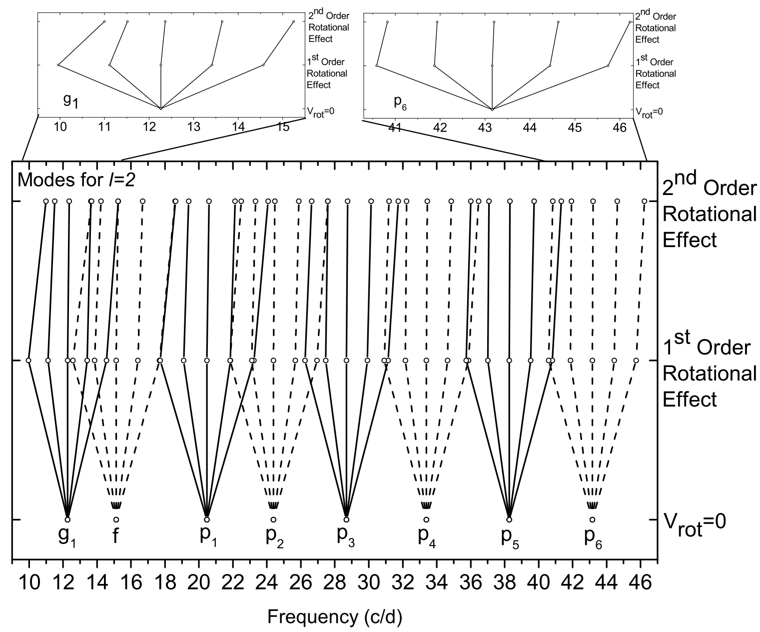
Table 4.13: Frequency results for l=0 calculated using the selected model for BS Tuc

Mode	Frequency (cycle/days)	Q §1 (days)
f	15.34	0.03294
p_1	19.52	0.02561
p_2	24.02	0.02082

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.



a)



b)

Figure 4.12: Rotational splitting for various modes

Results belong to 1.8 $M_{\odot}$ star, with $T_{eff} \simeq 7766$K and rotational velocity $\upsilon \sin i \simeq 132.8$ km/s. Figure 4.12.a corresponds to the results for l=1, while Figure 4.12.b corresponds to the results for l=2. Frequencies calculated in the absence of rotation for the same model are shown closest to the lower frequency axis, with the first and second order rotational effect added respectively upwards.

In both Figures 4.12.a and 4.12.b the rather symmetric splitting of the frequencies is seen when the rotational effect is considered up to first order in angular frequency Ω, (effect of Coriolis force). When the Centrifugal force effect (contribution from Ω^2) is also taken into account, the asymmetry in the splitting is conspicious for both harmonic degrees.

As can be seen from the Figure 4.12.a , splitting in the acoustic modes are larger than that of gravity modes, for low spherical harmonic degree l=1. This is because of the relatively large Ledoux constant, C for gravity modes, however, for acoustic modes, C is small (Pamyatnykh 2003). C gets smaller for acoustic modes, because amplitude of the radial displacement vector ξ_{0r} increases towards the surface (See equation 3.16).

As the spherical harmonic degree increases, the frequencies of different modes get closer to each other, hence it gets harder to identify the mode. In Figures 4.14 and 4.15, larger scaled graphs can be seen, which show the splitting in each mode for l=2 separately. Frequency results of several modes for l=3 are listed in Table 4.16, and plotted in Figure 4.16.

And in Figure 4.13, the propagation diagram is plotted for l=1, 2 and 3.

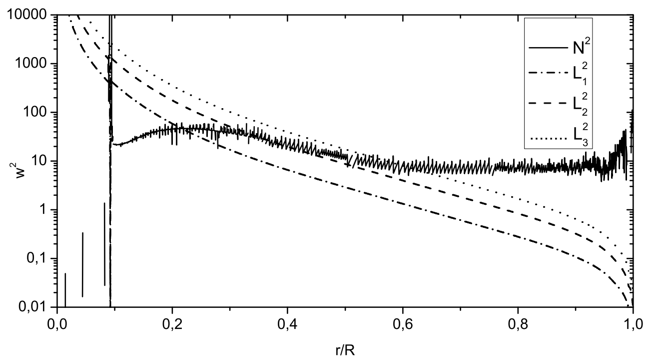


Figure 4.13: The Propagation Diagram for 1.8 $M_\odot$ star, X_c=0.45, l=1, 2, 3 and rotational velocity≈132.8 km/s

Table 4.14: Calculated frequencies of the selected model for BS Tuc (l=1)

m	Freq. (c/d) (without rotation)	Freq.1 (c/d) (1^{st} order rot. effect added)	Freq.2 (c/d) (1^{st} & 2^{nd} order rot. effect added)	Freq. ratio (Freq.2/Freq.1)	Q [§1] (days)	N_p [§2]	N_g [§3]	estimated mode
-1	7.91967	7.14765	7.72015	1.0801	0.06476	0	2	g_2
0	7.91967	7.91967	8.02565	1.01338	0.06229	0	2	g_2
1	7.91967	8.69168	8.60507	0.99004	0.0581	0	2	g_2
-1	11.5731	10.7926	11.1451	1.03266	0.04486	0	1	g_1
0	11.5731	11.5731	11.6464	1.00633	0.04293	0	1	g_1
1	11.5731	12.3536	12.3898	1.00293	0.04035	0	1	g_1
-1	15.5212	14.185	14.6096	1.02993	0.03422	1	1	f
0	15.5212	15.5212	15.6148	1.00603	0.03202	1	1	f
1	15.5212	16.8574	17.2551	1.02359	0.02897	1	1	f
-1	19.7878	18.4749	18.8407	1.0198	0.02653	1	0	p_1
0	19.7878	19.7878	19.8599	1.00364	0.02517	1	0	p_1
1	19.7878	21.1006	21.3705	1.01279	0.02339	1	0	p_1
-1	23.6715	22.5902	22.6790	1.00393	0.02204	2	0	p_2
0	23.6715	23.6715	23.7212	1.0021	0.02108	2	0	p_2
1	23.6715	24.7529	25.0836	1.01336	0.01993	2	0	p_2
-1	27.3063	26.2865	26.2342	0.99801	0.01906	3	0	p_3
0	27.3063	27.3063	27.3469	1.00149	0.01828	3	0	p_3
1	27.3063	28.3261	28.7152	1.01374	0.01741	3	0	p_3
-1	31.7617	30.5965	30.6220	1.00083	0.01633	4	0	p_4
0	31.7617	31.7617	31.8016	1.00126	0.01572	4	0	p_4
1	31.7617	32.9269	33.2368	1.00941	0.01504	4	0	p_4
-1	36.6334	35.3913	35.4589	1.00191	0.0141	5	0	p_5
0	36.6334	36.6334	36.6703	1.00101	0.01363	5	0	p_5
1	36.6334	37.8755	38.1237	1.00655	0.01311	5	0	p_5
-1	41.54151	40.2630	40.3419	1.00196	0.01239	6	0	p_6
0	41.5415	41.5415	41.575	1.00081	0.01202	6	0	p_6
1	41.5415	42.8200	43.0309	1.00493	0.01162	6	0	p_6

[§1] $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

[§2] Number of p nodes from the center to the surface of the star

[§3] Number of g nodes from the center to the surface of the star

Table 4.15: Calculated frequencies of the selected model for BS Tuc (l=2)

m	Freq. (c/d) (without rotation)	Freq.1 (c/d) (1^{st} order rot. effect added)	Freq.2 (c/d) (1^{st} & 2^{nd} order rot. effect added)	Freq. ratio (Freq.2/Freq.1)	Q §1 (days)	N_p §2	N_g §3	estimated mode
-2	12.2626	9.9605	10.9980	1.10416	0.04546	1	2	g_1
-1	12.2626	11.1115	11.5130	1.03613	0.04342	1	2	g_1
0	12.2626	12.2626	12.3641	1.00828	0.04043	1	2	g_1
1	12.2626	13.4136	13.6376	1.0167	0.03666	1	2	g_1
2	12.2626	14.5647	15.2471	1.04685	0.03279	1	2	g_1
-2	15.1275	12.5729	13.6732	1.08751	0.03656	1	1	f
-1	15.1275	13.8502	14.2328	1.02762	0.03513	1	1	f
0	15.1275	15.1275	15.2342	1.00705	0.03282	1	1	f
1	15.1275	16.4048	16.6935	1.0176	0.02995	1	1	f
2	15.1275	17.6821	18.5946	1.05161	0.02686	1	1	f
-2	20.4736	17.7188	18.6309	1.05148	0.02683	1	0	p_1
-1	20.4736	19.0962	19.3977	1.01579	0.02577	1	0	p_1
0	20.4736	20.4736	20.5874	1.00556	0.02428	1	0	p_1
1	20.4736	21.8510	22.1185	1.01224	0.0226	1	0	p_1
2	20.4736	23.2284	24.0724	1.03633	0.02077	1	0	p_1
-2	24.3923	21.8392	22.4884	1.02973	0.02223	2	0	p_2
-1	24.3923	23.1158	23.3316	1.00934	0.02143	2	0	p_2
0	24.3923	24.3923	24.4698	1.00318	0.02043	2	0	p_2
1	24.3923	25.6689	25.8789	1.00818	0.01932	2	0	p_2
2	24.3923	26.9454	27.5830	1.02366	0.01812	2	0	p_2
-2	28.6904	26.2442	26.6414	1.01513	0.01877	3	0	p_3
-1	28.6904	27.4673	27.5843	1.00426	0.01812	3	0	p_3
0	28.6904	28.6904	28.7491	1.00205	0.01739	3	0	p_3
1	28.6904	29.9136	30.1327	1.00732	0.01659	3	0	p_3
2	28.6904	31.1367	31.7381	1.01931	0.01575	3	0	p_3
-2	33.3899	30.9105	31.1936	1.00916	0.01603	4	0	p_4
-1	33.3899	32.1502	32.2195	1.00216	0.01552	4	0	p_4
0	33.3899	33.3899	33.439	1.00147	0.01495	4	0	p_4
1	33.3899	34.6297	34.8547	1.0065	0.01434	4	0	p_4
2	33.3899	35.8694	36.464	1.01658	0.01371	4	0	p_4

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

§2 Number of p nodes from the center to the surface of the star

§3 Number of g nodes from the center to the surface of the star

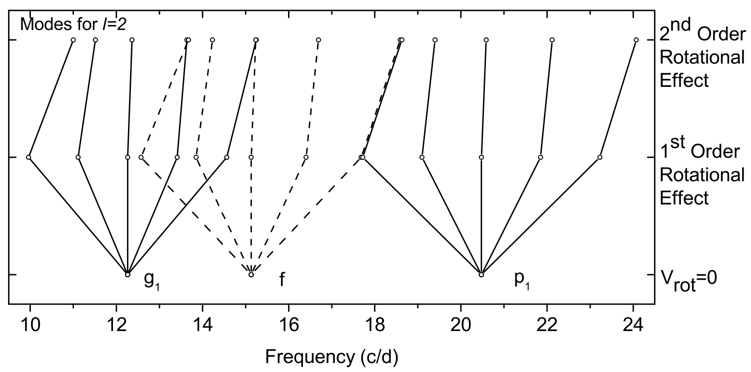


Figure 4.14: Large scale illustration for some of the modes shown in Figure 4.12.b

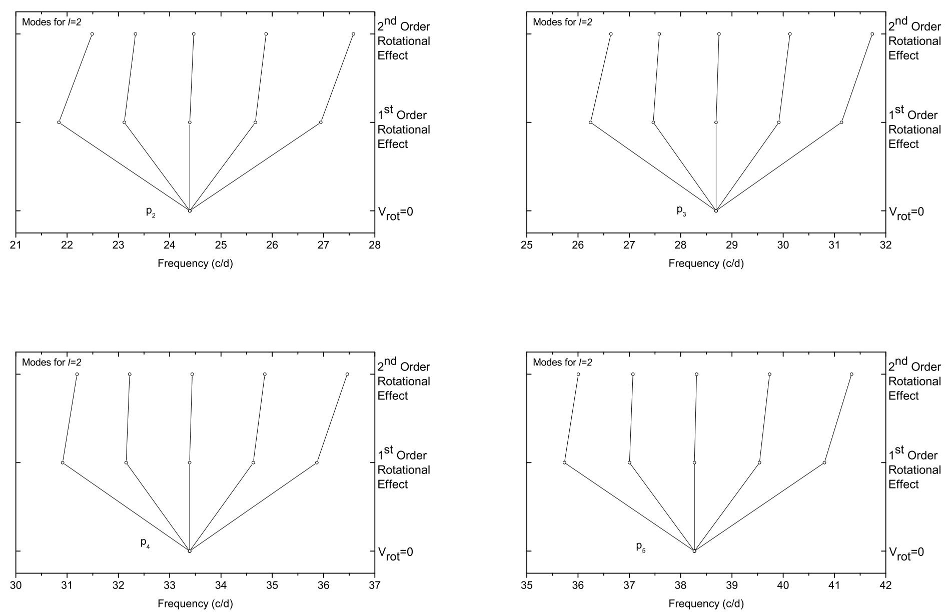


Figure 4.15: Rotational splitting of some acoustic modes for l=2 calculated by using the same model presented in Figure 4.12.b

Table 4.16: Several calculated frequencies of the selected model for BS Tuc (l=3)

m	Freq. (c/d) (without rotation)	Freq.1 (c/d) (1^{st} order rot. effect added)	Freq.2 (c/d) (1^{st} & 2^{nd} order rot. effect added)	Freq. ratio (Freq.2/Freq.1)	Q §1 (days)	N_p §2	N_g §3	estimated mode
-3	12.5579	8.78328	11.1699	1.27172	0.04476	1	2	g_1
-2	12.5579	10.0415	11.2147	1.11684	0.04458	1	2	g_1
-1	12.5579	11.2997	11.7233	1.03749	0.04264	1	2	g_1
0	12.5579	12.5579	12.6958	1.01098	0.03938	1	2	g_1
1	12.5579	13.816	14.1323	1.02289	0.03538	1	2	g_1
2	12.5579	15.0742	16.0326	1.06358	0.03118	1	2	g_1
3	12.5579	16.3324	18.3967	1.12639	0.02718	1	2	g_1
-3	14.6198	10.9408	13.032	1.19114	0.03836	1	1	f
-2	14.6198	12.1672	13.2025	1.08509	0.03787	1	1	f
-1	14.6198	13.3935	13.7707	1.02816	0.0363	1	1	f
0	14.6198	14.6198	14.7368	1.008	0.03392	1	1	f
1	14.6198	15.8461	16.1006	1.01606	0.03105	1	1	f
2	14.6198	17.0725	17.8621	1.04625	0.02799	1	1	f
3	14.6198	18.2988	20.0215	1.09414	0.02497	1	1	f
-3	24.8449	20.9322	22.3129	1.06596	0.02241	2	0	p_2
-2	24.8449	22.2364	22.9008	1.02988	0.02183	2	0	p_2
-1	24.8449	23.5407	23.7685	1.00968	0.02103	2	0	p_2
0	24.8449	24.8449	24.916	1.00286	0.02006	2	0	p_2
1	24.8449	26.1491	26.3432	1.00742	0.01898	2	0	p_2
2	24.8449	27.4533	28.0502	1.02174	0.01782	2	0	p_2
3	24.8449	28.7576	30.037	1.04449	0.01664	2	0	p_2
-3	39,4877	35,6	36,3335	1,0206	0,01376	5	0	p_5
-2	39,4877	36,8959	37,2272	1,00898	0,01343	5	0	p_5
-1	39,4877	38,1918	38,2934	1,00266	0,01306	5	0	p_5
0	39,4877	39,4877	39,5323	1,00113	0,01265	5	0	p_5
1	39,4877	40,7837	40,9438	1,00393	0,01221	5	0	p_5
2	39,4877	42,0796	42,5278	1,01065	0,01176	5	0	p_5
3	39,4877	43,3755	44,2844	1,02095	0,01129	5	0	p_5

§1 $Q = P\sqrt{\frac{<\rho_*>}{<\rho_\odot>}}$, where P is the period in days.

§2 Number of p nodes from the center to the surface of the star

§3 Number of g nodes from the center to the surface of the star

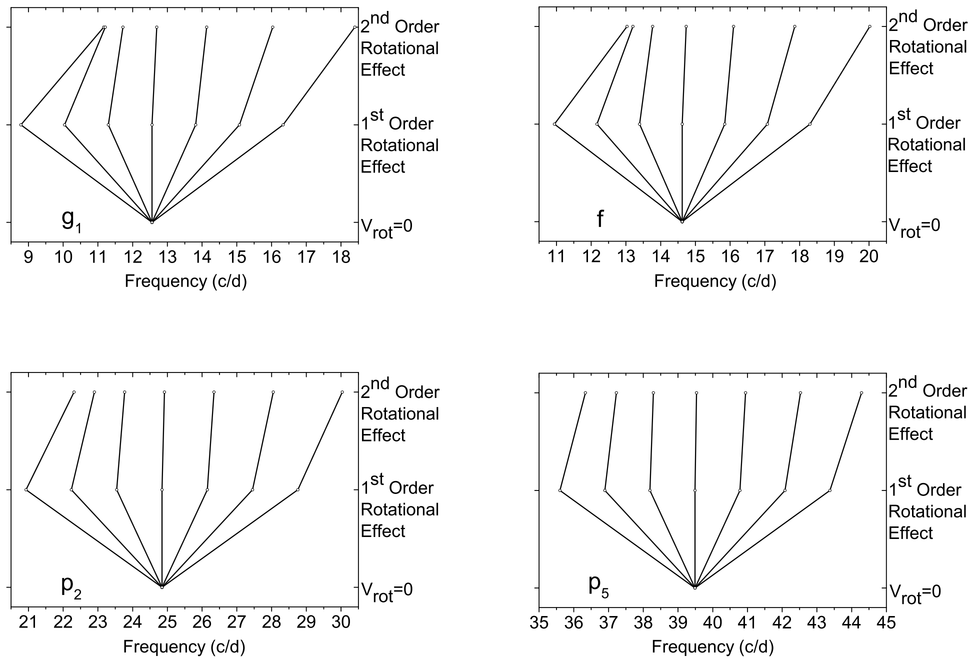


Figure 4.16: Frequency splitting of several modes for the selected model for BS Tuc ($l = 3$)

Another important point is the change in frequency when the azimuthal degree, m, is equal to zero (if not clearly seen in the figures, obvious in the tables), for the second order rotational effect, while there is no change when only the first order effect is taken into account. The reason is that the contribution from the first order rotational effect is as σ_1=m Ω (1-C), while there are some terms -in the calculation of σ_2- in which m does not appear, as can be seen in the equations (3.78), (3.108), (3.109), (3.111) in the previous chapter.

After analyzing the rotational effect in the case of fast rotation, it is seen that the frequency difference between each consecutive azimuthal order m, ranges from 0.5 to 2.5 c/d, but mostly around 1.5 c/d, while the contribution of the second order effect on this difference ranges from 30% to 75%. This effect is large enough, much larger than that in the case of slow rotation, to take into account. Therefore we may conclude that rotational effect should be considered including at least up to second order to achieve accurate results in the case of fast rotation.

There are still some effects which we did not take into consideration in order to simplify the problem, such as magnetic field effects and rotational mode coupling. The latter may be significant when the frequency separation between the modes is around the same order with rotational frequency (Pamyatnykh 2003). There are also other factors to be considered when comparing some cases, such as the evolutionary stage of the star, different chemical composition and stellar parameters. Of course, these leave us a great deal to explore more in our future work.

CHAPTER 5

DISCUSSION AND CONCLUSION

When a star is rotating as well as pulsating we have more complicated problems to solve. Axial rotation of a star is an important agent which has effect both on evolutionary structure of a star and also observed quantities such as oscillation frequencies.

We tried to investigate the effect of rotation on the oscillation frequencies of some Delta Scuti type pulsating stars. We included this effect up to the second order in rotation velocity Ω. Calculations are made for low order spherical harmonic degrees ($l \leq 3$).

We modified an oscillation program that was written by Al-Murad and Kırbıyık in 1993. Our modifications are related to the inclusion of the second order rotational effect in calculating the oscillation frequencies. All the radial integral terms -presented in Chapter 3- constituting the frequencies are calculated numerically by our program and the frequency results are presented in Chapter 4, while the angular integrals are calculated by the program MATHEMATICA and the corresponding results are listed in Appendix B.

According to our results, second order rotational effect has a significant contribution on the total rotational effect for fast rotating stars. Therefore it should not be neglected for that case, while the effect in question may be neglected for the case of slow rotation.

Analyzing the tables 4.8 - 4.10 and 4.14 - 4.16 presented in the previous chapter, we see that the ratio of the second order rotation to first order rotation increases as the spherical harmonic degree l increases. But the difficulty in determining the modes increases with the degree l, as well. As seen in figure 4.12, the modes gets very close to each other, and sometimes the frequency values of different modes can be almost same, so that we need better methods for mode identification.

The propagation diagrams shown in figures 4.1, 4.10 and 4.16 give us ideas about the frequency range that may be observed. As clearly seen in figures 4.10 and 4.16, the formation region of the pressure waves gets closer to the surface of the star when the spherical harmonic degree l increases, while for smaller l we can get information via pressure (acoustic) waves from the deep regions closer to the center.

Nonradial oscillations are still not clear in all aspects and there are many specific cases to consider, and many more effects to take into account. The nonadiabatic case could be a possible topic for future work; while rotational mode coupling, the analysis of the inclination angle, gravitation potential perturbation, third order rotational effect, and inclusion of higher degrees of spherical harmonics into the calculations still remain as appropriate future study goals.

In the slow rotation case, calculated frequencies match most of the observed ones. For CC And, observed frequencies given by Fu and Jiang (1995) are 8.005890 c/d, 7.814795 c/d, 8.101026 c/d, 13.34628 c/d, 7.902449 c/d, 16.01199 c/d, and 15.82091 c/d. Corresponding calculated frequencies that are closest to these values are 8.005 c/d (l=0, f), 7.81346 c/d (l=3, m=1), 8.13371 c/d (l=3, m=2), 13.3542 c/d (l=1, m=0), 7.91887 c/d (l=2, m=-1), 16.0174 c/d (l=1, m=-1), and 15.8166 c/d (l=3, m=-1), respectively. Except the first one which is referred as the radial fundamental frequency, all the frequencies that are calculated are estimated as g-modes as mentioned before. The other observed frequencies given by Ekmekçi and Topal (2007) are 8.00576 c/d, 8.81625 c/d, 6.84994 c/d, 6.56455 c/d, and 16.88150 c/d, while the corresponding calculated ones with closest values are 8.005 c/d (l=0, f), 8.81911 c/d (l=3, m=-1), 6.84242 c/d (l=3, m=0), 6.56602 c/d (l=3, m=2), and 16.8514 c/d (l=3, m=-3). The mode estimation is again as explained for the other group of frequencies. But it is appropriate to remark again that the crossing of the modes is expected for this star model, as the temperature of the model corresponds to a region where the modes get unstable (See Figure 1 in Pamyatnykh 2003). Along the evolution of the star, as the temperature decreases, mixed character of the modes is expected.

In the case of fast rotation, there are not observational data available for the star (BS Tuc) we consider. However, the pulsation frequency of the star is reported as 15.38 c/d, while calculated radial fundamental frequency is 15.34 c/d. It may be interesting to make a further study in the near future on a similar fast rotating star having reported observational frequencies.

REFERENCES

[1] Aizenman, M., Smeyers, P., and Weigert, A., 1977, Astron. Astrophys., 58, 41-46

[2] Al-Murad, M. A., Non-radial Oscillations of Polytropes and of a Standard Solar Model (MS. thesis), 1993, METU

[3] Al-Murad, M. A., Kırbıyık, H., 1995, in: Stellar Evolution: What should be done, ed. A. Noels, Proceedings of 32nd Liege International Astrophysics Colloquium, p.389

[4] Arfken, G. B., and Weber, H. J., 1995, *Mathematical Methods For Physicists*, Academic Press

[5] Breger, M., 1979, Publications of the Astronomical Society of the Pacific, 91, 5-26

[6] Carroll, B. W., Hansen, C. J., 1982, ApJ, 263, 352-365

[7] Chevalier, C., 1971, A & A, 14,24

[8] Chlebowski, T., 1978, Acta Astronomica, vol. 28, no. 4, p. 441-463

[9] Christensen-Dalsgaard, J., 2000, in: Delta Scuti and Related Stars, eds. M. Breger, and H. Montgomery, ASP Conf. Ser., 210, 187

[10] Christensen-Dalsgaard, J., Dappen, W., Dziembowski, W. A., and Guzik, J. A., 1999,in: Variable Stars as Essential Astrophysical Tools, ed. C. İbanoğlu , (NATO ASI Ser. C, 544; Dordrecht: Kluwer), Chapter II

[11] Christensen-Dalsgaard, J., and Dziembowski, W. A., 2000, in: Variable Stars as Essential Astrophysical Tools, ed. C. İbanoğlu , (NATO ASI Ser. C, 544; Dordrecht: Kluwer), 1-57

[12] Claret, A., Rodriguez, E., Rolland, A., and Lopez de Coca, P., 1990, ASPC, 11, 481-482

[13] Cowling, T. G., 1941, MNRAS, 101, 367-375

[14] Cox, J. P., 1967, in: Aerodynamic Phenomena in Stellar Atmospheres, IAU Symposium No. 28, ed. R. N. Thomas, London: Academic Press, p.3

[15] Cox, J. P., 1980, *Theory of Stellar Pulsation*, Princeton University Press

[16] Cox, J. P., and Giuli, R. T., 1968, *Principles of Stellar Structure*, Gordon and Breach, New York

[17] Dziembowski, W., 1971, Acta. Astron., 21, 289

[18] Dziembowski, W., 1982, Acta. Astron. ,32. 3-4, 147-171

[19] Dziembowski, W. , and Krolikowska, M., 1985, Acta. Astron., 35. 1-2, 5-28

[20] Dziembowski, W. A., and Goode, P. R., 1992, ApJ 394, 670 -687

[21] Eddington, A. S., 1926, *The Internal Constitution of the Stars*, Cambridge: Cambridge University Press

[22] Ekmekçi, F., and Topal, S., In Press, Astron. Nachr.

[23] Ekmekçi, F., and Topal, S., 2007, New Astronomy, 13, 118-123

[24] ESA 1997, The Hipparcos and Tycho Catalogues, vols. 1-17, ESA SP-1200

[25] Ezer, D., and Cameron, A. G. W., 1967, Canadian J. Phys., 45, 3461

[26] Fox Machado, L., Perez Hernandez, F., Suarez, J. C., Michel, E., and Lebreton, Y., 2006, A&A, 446, 611-620

[27] Frolov, M. S., 1975, in: Pulsating Stars, ed. B. V. Kukarkin, Keter Publishing House, Jerusalem Ltd.

[28] Fu, J., and Jiang, S., 1995, Astron. Astrophys. Supp. Ser., 110, 303-311

[29] Gautschy, and A., Saio, H., 1995, ARA&A, 33, 75-114

[30] Gautschy, and A., Saio, H., 1996, ARA&A, 34, 551-606

[31] Halprin, L., and Moon, T. T., 1983, Astrophysics and Space Science, 91, 43-51

[32] Hansen, C. J., and Van Horn, H. M., 1979, Astrophys. J., 233, 253

[33] Iglesisas, C. A., Rogers, F. J., and Wilson, B. G., 1992, ApJ 397, 717

[34] Kırbıyık, H., Civelek, R., Kızıloğlu, N., 2004, Astrophysics and Space Science, 295, 473-484

[35] Kjeldsen, H., 2001, in: Proceedings of SOHO 10/GONG 2000 Workshop Helio- and Asteroseismology at the Dawn of the Millenium, ESA SP-464, 361

[36] Kurtz, D. W., 1988, in: Multimode Stellar Pulsations, eds. G. Kovacs, L. Szabados, and B. Szeidl, (Konkoly Observatory, Budapest), p.95

[37] Landau, L. D., and Lifshitz, E. M., 2^{nd} ed., 1987, Fluid Mechanics, Pergamon Press

[38] Ledoux, P., 1958, Stellar Variability, in: Handbuch der Physik, ed. S. Flügge, Berlin: Springer-Verlag, 51, 605-688

[39] Ledoux, P., 1965, in: Stellar Structure - Stars and Stellar Systems: Compendium of Astronomy and Astrophysics, eds. L. H. Aller and D. B. McLaughlin, Vol. VIII, p.499

[40] Ledoux, P., 1974, in: Stellar Instability and Evolution, IAU Symp., eds. Ledoux et al., 59, 135-173

[41] Ledoux, P., and Walraven, Th., 1958, in: Handbuch der Physik, ed. S. Flügge, Berlin: Springer-Verlag, 51, 353-604

[42] Ledoux, P., and Whitney, C.A., 1961, in: Aerodynamic Phenomena in Stellar Atmospheres, Nuovo Cim., Suppl., 22, 1

[43] Lindblad, P. O., and Eggen, O. J., 1953, Publications of the Astronomical Society of the Pacific, Vol. 65, No. 387, p.291

[44] Lopez de Coca, P., Rolland, A., Rodriguez, E., and Garrido, R., 1990, Astron. Astrophys. Suppl. Ser., 83, 51-69

[45] Maeder, A., and Meynet, G., 1988, Astron. Astrophys. Suppl. Ser. 1988, 76, 411-425

[46] Matalgah, Z., Second Order Rotational Effect on Nonradial Oscillations in δ-Scuti Stars (MS. thesis), 2004, METU

[47] Mihalas, D., Hummer, D. G. , Mihalas, B. W., and Dappen, W., 1990, ApJ 350, 300

[48] Ostermann, W. M., Breger, M., Garrido, R., Martin, B., Paparo, M., Jiang, S., Scheck, M., Stich, J., and Li, Z., 1991, A&A, 245, 543-547

[49] Özel, N., Radial and Nonradial Oscillations in Stars with Gravitational Potential Perturbation (MS. thesis), 2003, METU

[50] Özel, N., Kırbıyık, H., Civelek, R., Kızıloğlu, N., 2005, Astronomische Nachrichten, Vol.326, Issue 2, p.118-121

[51] Pamyatnykh, A. A., 2000, ASPC, 210, 215P

[52] Pamyatnykh, A. A., Astrophysics and Space Science, 2003, 284, 97-107

[53] Reese, D., Lignieres, F., Rieutord, M., 2006, Comm. in Asteroseismology, Vol. 147

[54] Ritter, A., 1879, Wiedemanns Ann., 8, 179

[55] Rodriguez, E., Lopez de Coca, P., Rolland, A., Garrido, R., and Costa, V., 1994, Astron. Astrophys. Suppl. Ser., 106, 21-28

[56] Rodriguez, E., and Breger, M., 2001, A&A 366, 178-196

[57] Rosseland S., 1964, *The Pulsation Theory of Variable Stars*, DOVER PUBLICATIONS (First published in 1949, Oxford: Clarendon Press)

[58] Saio, H., 1981, The Astrophysical Journal, 244, 299-315

[59] Shapley, H., 1914, Ap. J., 40, 448

[60] Simon, R., 1969, A&A, 2, 390

[61] Smeyers, P., and Denis, J., 1971, Astron. Astrophys., 14, 311

[62] Solano, E., and Fernley, J., 1997, Astron. Astrophys. Suppl. Ser., 122, 131-147

[63] Strohmayer, T. E., and Lee, U., 1996, A & A, 311, 155-171

[64] Tsvetkov, TS. G., 1985, Astrophysics and Space Science, 117, 137-149

[65] Unno, W., Osaki, Y., Ando, H., Saio, H., and Shibahashi H., 1989, *Nonradial Oscillations of Stars*, Universtiy of Tokyo Press

[66] Vidal-Sainz, J., Wils, P., Lampens, P., and Garcia-Melendo, E., 2002, A&A, 394, 585-588

[67] Yıldız, M., and Kızıloğlu, N. 1997, A&A, 326, 187-194

[68] Zhevakin, S. A., 1975, in: Pulsating Stars, ed. B. V. Kukarkin, Keter Publishing House, Jerusalem Ltd.

APPENDIX A

OSCILLATION PROGRAM

```
      IMPLICIT REAL*8(A-H,O-Z)
c     GAMA1 her shell de degisiyor.
c     Yildizin kutlesi ve yaricapi her model icin ayri ayri yazilacak
c     L degeri 0,1,2,3...hangisi icin yapilacaksa ona gore her run da
c     degisecek. Kac shell oldugu belirtilecek.
      DIMENSION X(10000),Y(10000),Z(10000),YY(10000),
     * ZZ(10000),RR(10000),DTERM(2000,2000),FQV(2000,2000),
     * RM(10000),PR(10000),DNS(10000),RMI(10000),PRI(10000),
     * DLTH(10000),FIRSTI(2000,2000),SECONDI(2000,2000),
     * SECONDII(2000,2000),
     * DNSI(10000),SD(10000),BV(10000),FL(10000),FRATIO(2000,2000),
     * COMEGA(2000,2000),THIRDI(2000,2000),FORTHI(2000,2000),
     * DNSD(10000),DLTR(10000),DLPP(10000),FIFTHI(2000,2000),
     * SIGMA(2000,2000),QV(2000,2000),SIXTHI(2000,2000),
     * CNUM(10000),CDEN(10000),XPHASE(10000),SQV(2000,2000),
     * YPHASE(10000),ERPR(10000),THIRDC(10000),SRATIO(2000,2000),
     * A1(10000),A2(10000),B1(10000),B2(10000),FORTHC(10000),
     * tl(10000),alrl(10000),el(10000),bwl(10000),wl(10000),
     * psi(10000),beta(10000),il(10000),rrr(10000),TRATIO(2000,2000),
     * rmm(10000),rdns(10000),FIFTHC(10000),SEVENTHI(2000,2000),
     * SIXTHC(10000),SEVENTHC(10000),EIGHTHC(10000),FSIGMA(2000,2000),
     * rpr(10000),gama1(10000),gamval(10000),rgamval(10000),
     * SSIGMA(2000,2000),SIGMAT(2000,2000),SCOMEGA(2000,2000),
     * FCOMEGA(2000,2000),EIGHTHI(2000,2000),NINTHC(10000)
```

```
      CHARACTER*4 bdchar
      character*4 aster/' * '/
      CHARACTER MODE
c     data aster/'* '/
      G=6.6732D-8
      PI=3.141593d0
      J=1
      NA=0
      NG=0
      np=0
      IORD=0
      bc1=0.d0
      L=0
      j2=1
c     do 109 L1=1,3
c     l=l1-1
c     GAMA1=5./3.
      vsini=132.7637122d5
      amin=1.0d0
c     amax=50.0d0
      amax=200.0d0
      tol0=1.d-3
      tol1=1.d-3
      del0=(amax-amin)/50000.d0
c     del0=1.d-1
c     TE=3.840d0
      H=3.0d-4
      tol=tol0
      DELTA =DEL0
      OMEGA=AMIN
c     PRINT*,'============================
c    * ====================='
c     PRINT*,' THE FOLLOWING MODES HAS BEEN APPROACHED'
```

```
c        PRINT*,'============================
c    *   ======================'
c        PRINT*,' '
         TR=1.3433d11
         TM=3.5730d33
         open(7,file='123.dta',status='old')
         OPEN (5,FILE='freq.dta',STATUS='UNKNOWN')
         OPEN (3,FILE='p.dta',STATUS='UNKNOWN')
c         if(j2.gt.1)go to 108
         DO 15 M=1,358
c10      READ (7,*) RM(M), RR(M), DNS(M), PR(M), gamval(i)
c        ********el(m)=delrr, wl(m)=deladi,gamval(m)=gama1,beta(m)=gam2(m)
         read(7,119) bdchar,il(m),rm(m),rr(m),dns(m),tl(m),pr(m),alrl(m),
     *   el(m),bwl(m),wl(m),gamval(m),beta(m)
 119     format(A4,I3,8e12.5,e12.4,1x,f6.4,e12.5)
c        PRINT*,'TEST',m,RM(M),RR(M),DNS(M),PR(M)
c         write(5,1020) bdchar,m,rm(m),rr(m),dns(m),pr(m),gamval(m)
 1020    format(A4,I4,5(e11.4))
c        if(m.eq.357) go to 15
         if(m.gt.100) go to 15
c        write(5,107)aster
c        print*,aster
c107     format(A4)
         if(bdchar. eq. aster) m11=m
         m11=2
c        x11=rr(m)/tr
c        x11=x11+0.0009d0
c        x11=x11+0.0005d0
c        x11=x11+0.025d0
c        endif
c        write(5,1035) x11
c1035    format(e11.4)
 15      CONTINUE
```

```
      x11=rr(m11)/tr
      m=m-1
      DNSM=TM/((4.D0/3.D0)*PI*TR**3.D0)
c     j2=2
c108  continue
 1000 FORMAT(E11.3,4(E11.3),2I4,2(E11.3))
c     **************************************************
 187  continue
      I=1
c     OPEN (5,FILE='pulsout.dta',STATUS='UNKNOWN')
      X(1)=0.D0
      i1=0
      tp=pr(1)
      td=dns(1)
      do 3 j=1,m
      rrr(j)=rr(j)/tr
c     if(rrr(j).eq.rrr(j-1))print *,j
      rmm(j)=rm(j)/tm
      rdns(j)=dns(j)/td
      rpr(j)=pr(j)/tp
      rgamval(j)=gamval(j)
 3    continue
 20   X1=X(I)
c     if(rrr(i).lt.0.1d0) h=1.0d-6
      CALL INTRP (X1,RM1,RRR,RMM,m)
      CALL INTRP (X1,DNS1,RRR,rDNS,m)
      CALL INTRP (X1,PR1,RRR,rPR,m)
      call intrp (x1,gm1,rrr,rgamval,m)
      gama1(i)=gm1
      RMI(I)=RM1
      dns1=dns1*td
      pr1=pr1*tp
      DNSI(I)=DNS1
```

```
      PRI(I)=PR1
c     WRITE(5,1000) X(I),PRI(I),RMI(I),DNSI(I),i,m
      if(y1.gt.1.0d8) go to 33
c     WRITE(5,1000) X1,PR1,RM1,DNS1,gm1,i,m
 33   continue
      I=I+1
      x(i)=x(i-1)+H
c     if(x(i).lt.0.084d0)go to 33
      if(x(i).lt.x11)go to 33
      if(i1.eq.2)go to 34
      x(2)=x(i)
      x11=x(2)
      i=2
      i1=2
 34   continue
c     **************************************m=1 ? m=385?
      if(x(i).gt.1.0d0) go to 25
      GOTO 20
 25   RMI(I)=1.0D0
      DNSI(I)=0.0D0
      PRI(I)=0.0D0
      CALL DNSDIF (DNSI,DNSD,TR,H,I)
c     OPEN (5,FILE='output1.dat',STATUS='NEW')
c     WRITE(5,101)
c     WRITE(5,104)
     WRITE(5,101)
      DO 35 N=2,i-1
      SD(N)=DNSD(N)/DNSI(N)+DNSI(N)*G*TM*RMI(N)/
     * (GAMA1(N)*PRI(N)*TR*TR*X(N)*X(N))
      BV(N)=-SD(N)*G*TM*RMI(N)/(TR*TR*X(N)*X(N))
      BV(N)=BV(N)*TR**3/(G*TM)
c     if(bv(n).lt.0.0d0) bv(n)=bv(n-1)
      tr2=tr*tr
```

```
      FL(N)=DFLOAT(L*(L+1))*GAMA1(N)*PRI(N)/(DNSI(N)*(TR2*X(N)*X(N)))
      FL(N)=FL(N)*TR**3/(G*TM)
      if(y1.gt.1.0d8) go to 36
      WRITE(5,2000) X(N),BV(N),FL(N)
 2000 FORMAT(F10.4,2(E20.4))
 35   CONTINUE
 36   continue
 104  FORMAT(7X,'X',11X,'BRUNT V.FRQ.',9X,'LAMB FRQ.')
c     CLOSE(5)
c     ******************************************************c
 40   CONTINUE
      Y(2)=1.D0
      IF (L.EQ.0) THEN
      Z(2)=1.d0-3.D0*GAMA1(2)*PRI(2)*TR*X(2)/(G*TM*RMI(2)*DNSI(2))
c     print*, z(2),x(2),rmi(2),dnsi(2),gama1(2),pri(2)
      GOTO 5
      ENDIF
      Z(2)=OMEGA*X(2)**3/(RMI(2)*DFLOAT(L))
c     ***********************************************
c5     DO 50 N=3,300
 5    do 50 n=2,l-1
      X0=X(N)
      Y0=Y(N)
      Z0=Z(N)
      RMI0=RMI(N)
      DNSI0=DNSI(N)
      PRI0=PRI(N)
      DNSD0=DNSD(N)
      gama0=gama1(N)
      CONST1=0.D0
      CONST2=0.D0
c     print*, gama0,rmi0,dnsi0,pri0,dnsd0,y0,z0
      CALL RUNGE(X0,Y0,Z0,Y1,Z1,G,TM,RMI0,DNSI0,PRI0,TR,GAMA0,L,
```

```
     *  OMEGA,PI,DNSD0,H,CONST1,CONST2)
        if(y1.gt.1.0d8) then
        x11=x11+h
        go to 187
        endif
c       print*, x0,y0,z0,y1,z1
        Y(N+1)=Y1
        Z(N+1)=Z1
 50     CONTINUE
c       ***************************************************************
        BC=Y(I-1)-Z(I-1)
c       bc1=Y(i-2)-z(i-2)
        SIGM=DSQRT(G*TM*OMEGA/(TR**3))
c       fr=1.d6*sigm/(2.d0*pi)
        fr=86400.D0*SIGM/(2.D0*PI)
        p=2.D0*PI/SIGM/86400.D0
        rhomean=3.d0*TM/(4.d0*pi*(TR*TR*TR))
        q=p*(dsqrt(rhomean/1.408d0))
        cd1=1.d0/p
c       if(bc.gt.0.d0.and.bc1.lt.0.d0)print*,bc,bc1,omega,fr
c       if(bc.lt.0.d0.and.bc1.gt.0.d0)print*,bc,bc1,omega,fr
c       if(bc.gt.0.d0.and.bc1.lt.0.d0)write(5,37)I,bc,bc1,omega,fr
c       if(bc.lt.0.d0.and.bc1.gt.0.d0)write(5,37)I,bc,bc1,omega,fr
c       if(bc.gt.0.d0.and.bc1.lt.0.d0)write(5,37)I,omega,fr,p,cd1,q
c       if(bc.lt.0.d0.and.bc1.gt.0.d0)write(5,37)I,omega,fr,p,cd1,q
        if(bc.gt.0.d0.and.bc1.lt.0.d0) tol=1.0d-4
        if(bc.lt.0.d0.and.bc1.gt.0.d0) tol=1.0d-4
 37     format(i3,e13.5,f12.7,3e13.5)
        bc1=bc
c       print*, omega,fr,bc
c       if(dabs(bc).lt.0.2) print *,omega,bc,fr
        IF (OMEGA.GE.AMAX) GOTO 999
        IF (tol.GE.TOL1) GOTO 888
```

```
      tol=1.0d-3
c     SIGM=DSQRT(G*TM*OMEGA/(TR**3))
c     DO 55 N=4,300
c     print*,sigm,qva
      do 55 n=2,l-1
      ERPR(N)=G*TM*RMI(N)*DNSI(N)*Z(N)/(TR*X(N))
      DLPP(N)=(ERPR(N)-y(n)*G*TM*RMI(N)*DNSI(N)/(TR*X(N)))/PRI(N)
      DLTR(N)=y(n)*X(N)*TR
      DLTH(N)=(ERPR(N)/DNSI(N))/ (X(N)*TR*SIGM**2.D0)
      IF (Y(N)*Y(N+1).LT.0.D0) THEN
c     IF (OMEGA.LT.BV(N).AND.OMEGA.LT.FL(N)) NG=NG+1
c     IF (OMEGA.GT.BV(N).AND.OMEGA.GT.FL(N)) NP=NP+1
      IF (OMEGA.LT.BV(N).AND.OMEGA.LT.FL(N)) then
      ng=ng+1
      if(ng.eq.1) xg1=x(n)
      if(ng.eq.2) xg2=x(n)
      if(ng.eq.3) xg3=x(n)
      if(ng.eq.4) xg4=x(n)
      if(ng.eq.5) xg5=x(n)
      if(ng.eq.6) xg6=x(n)
      if(ng.eq.7) xg7=x(n)
      if(ng.eq.8) xg8=x(n)
      if(ng.eq.9) xg9=x(n)
      if(ng.eq.10) xg10=x(n)
      endif
      IF (OMEGA.GT.BV(N).AND.OMEGA.GT.FL(N)) then
      np=np+1
      if(np.eq.1) xp1=x(n)
      if(np.eq.2) xp2=x(n)
      if(np.eq.3) xp3=x(n)
      if(np.eq.4) xp4=x(n)
      if(np.eq.5) xp5=x(n)
      if(np.eq.6) xp6=x(n)
```

```
      if(np.eq.7) xp7=x(n)
      if(np.eq.8) xp8=x(n)
      if(np.eq.9) xp9=x(n)
      if(np.eq.10) xp10=x(n)
      endif
      ENDIF
      CNUM(N)=(2.D0*Y(N)*X(N)*TR*(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0))+(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0))*(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0)))*DNSI(N)*(X(N)*TR)**2.D0
      CDEN(N)=(Y(N)*X(N)*TR*Y(N)*X(N)*TR+L*(L+1)*(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0))*(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0)))*DNSI(N)*(X(N)*TR)**2.D0
      THIRDC(N)=(Y(N)*X(N)*TR*X(N)*TR)**2.D0*DNSI(N)
      FORTHC(N)=(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0))**2.D0*(X(N)*TR)**2.D0*DNSI(N)
      FIFTHC(N)=Y(N)*X(N)*TR*(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0))*(X(N)*TR)**2.D0*DNSI(N)
      SIXTHC(N)=((Y(N)*X(N)*TR)+((G*TM*RMI(N)*Z(N))/
     * (SIGM**2.D0*(X(N)*TR)**2.D0)))*G*TM*RMI(N)*DNSI(N)*ZZ(N)
      SEVENTHC(N)=(G*TM*RMI(N)*Z(N)/(SIGM**2.D0*(X(N)*TR)**2.D0))*
     * YY(N)*X(N)*TR*(X(N)*TR)**2.D0*DNSI(N)
      EIGHTHC(N)=(Y(N)*X(N)*TR*(1.D0/3.D0)+(G*TM*RMI(N)*Z(N)/
     * (SIGM**2.D0*(X(N)*TR)**2.D0)))*DNSI(N)*(X(N)*TR)**3.D0
      NINTHC(N)=Y(N)*X(N)*TR*DNSI(N)*(X(N)*TR)**3.D0
 55   CONTINUE
c     ************************************************************
      IORD=NP-NG
      IF (IORD.LT.0.D0) MODE='G'
      IF (IORD.GT.0.D0) MODE='P'
      IF (IORD.EQ.0.D0) MODE='F'
      WRITE(5,105)J,OMEGA,MODE,IORD
 105  FORMAT(I10,10X,'OMEGA 2=',F10.4,A14,I3)
```

```
      IF (L.EQ.0) THEN
      OPEN (9,FILE='zeroth.dat',STATUS='NEW')
c     WRITE(5,101)
c     WRITE(5,102)
c     WRITE(5,101)
      WRITE(9,100)K,OMEGA,SIGM,q,FR,MODE,NP,NG,IORD
c     WRITE(5,101)
c     WRITE(5,103)
c     WRITE(5,101)
      DO 71 N= 2,I-1
      XPHASE(N)=DLOG10(1.D0+DABS(DLTR(N)/TR))
      YPHASE(N)=DLOG10(1.D0+DABS(Z(N)*RMI(N)/X(N)))
      IF (DLTR(N).LT.0.D0) XPHASE(N)= -XPHASE(N)
      IF(ZZ(N).LT.0.D0) YPHASE(N)= -YPHASE(N)
c     WRITE(5,3000)X(N),y(n),DLPP(N),XPHASE(N),YPHASE(N)
 71   CONTINUE
c     GOTO 70
      ENDIF
c     *******************************************************
      CALL ROTAT(CNUM,CDEN,CINTGRL,DEN,I,H)
      CALL SECONDFRQ(THIRDC,FORTHC,FIFTHC,SIXTHC,SEVENTHC,
     * EIGHTHC,NINTHC,THIRD,FORTH,FIFTH,SIXTH,SEVENTH,
     * EIGHTH,NINTH,I,H)
c     ROTFRQ=VSINI/(2.D0*PI*TR)
      ROTFRQ=VSINI/TR
c     *******************************************************
      OPEN (7,FILE='1stORD.dat',STATUS='NEW')
c
c     DO 65 K= -L,L,1
      SIGMA(J,K) =SIGM+DFLOAT(K)*ROTFRQ*(1.D0-CINTGRL)
      FR1=86400*SIGMA(J,K)/(2.D0*PI)
      QV(J,K) =(2.D0*PI/SIGMA(J,K)/86400.D0)*(DSQRT(DNSM/1.408d0))
c     DNSMsun=1.408d0 g/cm**3
```

```
      COMEGA(J,K) =SIGMA(J,K)**2.D0*TR**3.D0/(G*TM)
c
      OPEN (11,FILE='2ndORD.dat',STATUS='NEW')
c     OPEN (12,FILE='1stORDcont.dat',STATUS='NEW')
      OPEN (14,FILE='freqs.dat',STATUS='NEW')
      OPEN (15,FILE='integrals.dat',STATUS='NEW')
c
      FSIGMA(J,K)=DFLOAT(K)*ROTFRQ*(1.D0-CINTGRL)
      DTERM(J,K)=(DFLOAT(K)*ROTFRQ*(1.D0-CINTGRL))**2.d0/SIGM*2.d0
      FIRSTI(J,K)=DFLOAT(K)*ROTFRQ/DEN*SEVENTH
      SECONDI(J,K)=(2.D0*(DFLOAT(K)*ROTFRQ)**2.D0*(CINTGRL-1.D0))/
     * (SIGM*DEN)*(FIFTH+FORTH)
      SECONDII(J,K)=(DFLOAT(K)*ROTFRQ)/
     * (SIGM*SIGM*DEN)*SIXTH
      THIRDI(J,K)=((2.D0*(ROTFRQ)**2.D0)/
     * (SIGM*DEN))*FIFTH
      FORTHI(J,K)=((2.D0*(ROTFRQ)**2.D0)/
     * (SIGM*DEN))*THIRD
      FIFTHI(J,K)=(2.D0*(DFLOAT(K)*ROTFRQ)**2.D0)/
     * (SIGM*DEN)*FORTH
      SIXTHI(J,K)=(2.D0*(ROTFRQ)**2.D0)/
     * (SIGM*DEN)*FORTH
      SEVENTHI(J,K)=(2.D0*(DSQRT(PI/5.D0))*(ROTFRQ)**2.D0)/
     * (SIGM*DEN)*EIGHTH
      EIGHTHI(J,K)=-(2.D0*(DSQRT(PI))*(ROTFRQ)**2.D0)/
     * (3.D0*SIGM*DEN)*NINTH
      IF(L.EQ.3 .AND. K.EQ.3) THEN
      THIRDI(J,K)=THIRDI(J,K)*(2.D0/3.D0)
      FORTHI(J,K)=FORTHI(J,K)*(8.D0/9.D0)
      FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/6.D0)
      SIXTHI(J,K)=SIXTHI(J,K)*(1.D0/2.D0)
      ELSE IF(L.EQ.3 .AND. K.EQ.2) THEN
      THIRDI(J,K)=0.D0
```

```
FORTHI(J,K)=FORTHI(J,K)*(2.D0/3.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(3.D0/4.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(1.D0)
ELSE IF(L.EQ.3 .AND. K.EQ.1) THEN
THIRDI(J,K)=THIRDI(J,K)*(-2.D0/5.D0)
FORTHI(J,K)=FORTHI(J,K)*(8.D0/15.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(5.D0/2.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(27.D0/10.D0)
ELSE IF(L.EQ.3 .AND. K.EQ.0) THEN
THIRDI(J,K)=THIRDI(J,K)*(-8.D0/15.D0)
FORTHI(J,K)=FORTHI(J,K)*(22.D0/45.D0)
FIFTHI(J,K)=0.D0
SIXTHI(J,K)=SIXTHI(J,K)*(28.D0/5.D0)
ELSE IF(L.EQ.3 .AND. K.EQ.-1) THEN
THIRDI(J,K)=THIRDI(J,K)*(-2.D0/5.D0)
FORTHI(J,K)=FORTHI(J,K)*(8.D0/15.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(5.D0/2.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(27.D0/10.D0)
ELSE IF(L.EQ.3 .AND. K.EQ.-2) THEN
THIRDI(J,K)=0.D0
FORTHI(J,K)=FORTHI(J,K)*(2.D0/3.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(3.D0/4.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(1.D0)
ELSE IF(L.EQ.3 .AND. K.EQ.-3) THEN
THIRDI(J,K)=THIRDI(J,K)*(2.D0/3.D0)
FORTHI(J,K)=FORTHI(J,K)*(8.D0/9.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/6.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(1.D0/2.D0)
ELSE IF(L.EQ.2 .AND. K.EQ.2) THEN
THIRDI(J,K)=THIRDI(J,K)*(4.D0/7.D0)
FORTHI(J,K)=FORTHI(J,K)*(6.D0/7.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/4.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(3.D0/7.D0)
```

```
ELSE IF(L.EQ.2 .AND. K.EQ.1) THEN
THIRDI(J,K)=THIRDI(J,K)*(-2.D0/7.D0)
FORTHI(J,K)=FORTHI(J,K)*(4.D0/7.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(3.D0/2.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(11.D0/14.D0)
ELSE IF(L.EQ.2 .AND. K.EQ.0) THEN
THIRDI(J,K)=THIRDI(J,K)*(-4.D0/7.D0)
FORTHI(J,K)=FORTHI(J,K)*(10.D0/21.D0)
FIFTHI(J,K)=0.D0
SIXTHI(J,K)=SIXTHI(J,K)*(18.D0/7.D0)
ELSE IF(L.EQ.2 .AND. K.EQ.-1) THEN
THIRDI(J,K)=THIRDI(J,K)*(-2.D0/7.D0)
FORTHI(J,K)=FORTHI(J,K)*(4.D0/7.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(3.D0/2.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(11.D0/14.D0)
ELSE IF(L.EQ.2 .AND. K.EQ.-2) THEN
THIRDI(J,K)=THIRDI(J,K)*(4.D0/7.D0)
FORTHI(J,K)=FORTHI(J,K)*(6.D0/7.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/4.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(3.D0/7.D0)
ELSE IF(L.EQ.1 .AND. K.EQ.1) THEN
THIRDI(J,K)=THIRDI(J,K)*(2.D0/5.D0)
FORTHI(J,K)=FORTHI(J,K)*(4.D0/5.D0)
FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/2.D0)
SIXTHI(J,K)=SIXTHI(J,K)*(3.D0/10.D0)
ELSE IF(L.EQ.1 .AND. K.EQ.0) THEN
THIRDI(J,K)=THIRDI(J,K)*(-4.D0/5.D0)
FORTHI(J,K)=FORTHI(J,K)*(2.D0/5.D0)
FIFTHI(J,K)=0.D0
SIXTHI(J,K)=SIXTHI(J,K)*(2.D0/5.D0)
ELSE IF(L.EQ.1 .AND. K.EQ.-1) THEN
THIRDI(J,K)=THIRDI(J,K)*(2.D0/5.D0)
FORTHI(J,K)=FORTHI(J,K)*(4.D0/5.D0)
```

```
      FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/2.D0)
      SIXTHI(J,K)=SIXTHI(J,K)*(3.D0/10.D0)
      ELSE IF(L.EQ.0 .AND. K.EQ.0) THEN
      THIRDI(J,K)=0.D0
      FORTHI(J,K)=FORTHI(J,K)*(2.D0/3.D0)
      FIFTHI(J,K)=0.D0
      SIXTHI(J,K)=0.D0
      ENDIF
      SSIGMA(J,K)=FIRSTI(J,K)+SECONDI(J,K)+SECONDII(J,K)+THIRDI(J,K)+
     * FORTHI(J,K)+FIFTHI(J,K)+DTERM(J,K)+SIXTHI(J,K)
      IF(L.EQ.2 .AND. K.EQ.0) THEN
      SSIGMA(J,K)=FIRSTI(J,K)+SECONDI(J,K)+SECONDII(J,K)+THIRDI(J,K)+
     * FORTHI(J,K)+FIFTHI(J,K)+DTERM(J,K)+SIXTHI(J,K)+SEVENTHI(J,K)
      ELSE IF(L.EQ.0 .AND. K.EQ.0) THEN
      SSIGMA(J,K)=FIRSTI(J,K)+SECONDI(J,K)+SECONDII(J,K)+THIRDI(J,K)+
     * FORTHI(J,K)+FIFTHI(J,K)+DTERM(J,K)+SIXTHI(J,K)+EIGHTHI(J,K)
      ENDIF
      SIGMAT(J,K)=SIGMA(J,K)+SSIGMA(J,K)
      FRATIO(J,K)=FSIGMA(J,K)/SIGM
      SRATIO(J,K)=SSIGMA(J,K)/SIGM
      TRATIO(J,K)=SSIGMA(J,K)/FSIGMA(J,K)
      IFR=SIGM*1.D+6/(2.D0*PI)
      SFR=SIGMAT(J,K)*1.D+6/(2.D0*PI)
      FFR=SIGMA(J,K)*1.D+6/(2.D0*PI)
      FQV(J,K)=(2.D0*PI/SIGMA(J,K)/86400.D0)*(DSQRT(DNSM/1.408d0))
      IQV=(2.D0*PI/SIGM/86400.D0)*(DSQRT(DNSM/1.408d0))
      SQV(J,K) =(2.D0*PI/SIGMAT(J,K)/86400.D0)*(DSQRT(DNSM/1.408d0))

      rqva=rhom*qv(j,k)
      FRFR=SIGMA(J,K)*86400.D0/(2.D0*PI)
      IRFR=(SIGM*86400.D0)/(2.D0*PI)
      SRFR=SIGMAT(J,K)*86400.D0/(2.D0*PI)
      ICOMEGA=(SIGM**2.D0*TR**3.D0)/(G*TM)
```

```
      FCOMEGA(J,K)=SIGMA(J,K)**2.D0*TR**3.D0/(G*TM)
      SCOMEGA(J,K)=SIGMAT(J,K)**2.D0*TR**3.D0/(G*TM)
      com=comega(j,k)
c
c     WRITE(5,101)
c     WRITE(5,102)
      WRITE(7,101)
c     if(k .ne. 0) go to 111
      WRITE(7,100)K,COMEGA(J,K),SIGMA(J,K),QV(J,K),FR1,MODE,np,ng,IORD
      WRITE(11,402)K,SIGM,FSIGMA(J,K),SSIGMA(J,K)
c     WRITE(11,400)K,SIGMA(J,K),SIGMAT(J,K)
c     WRITE(12,413)K,SIGMA(J,K),FCOMEGA(J,K),FQV(J,K),FFR,FRFR,
     * MODE,IORD
      WRITE(11,409)SIGMA(J,K),SIGMAT(J,K),SCOMEGA(J,K),SQV(J,K),SRFR,
     * MODE,IORD
c     WRITE(11,414)K,SIGM,ICOMEGA,IQV,IFR,IRFR
c     WRITE(11,410)K,SCOMEGA(J,K),SIGMAT(J,K),SQV(J,K),SRFR,MODE,IORD
      WRITE(14,412)K,FRFR,SRFR,MODE,np,ng,IORD
      WRITE(15,415)K,DTERM(J,K),FIRSTI(J,K),SECONDI(J,K),SECONDII(J,K)
     * ,THIRDI(J,K),FORTHI(J,K),FIFTHI(J,K),SIXTHI(J,K)
     * ,SEVENTHI(J,K),EIGHTHI(J,K)
c     WRITE(3,151)L,TE,qv(j,k),comega(j,k),iord,np,ng,xg1,xg2,xg3,xg4
c    * ,xg5,xg6
c     write(3,151)L,TE,qv(j,k),comega(j,k),iord,np,ng,xp1,xp2,xp3,xp4
c    * ,xp5,xp6
      WRITE(3,151)L,comega(j,k),iord,np,ng,xg1,xg2,xg3,xg4,xg5,xg6,xg7,
     * xg8,xg9,xg10
      write(3,151)L,comega(j,k),iord,np,ng,xp1,xp2,xp3,xp4,xp5,xp6,xp7,
     * xp8,xp9,xp10
c151  FORMAT(I2,F6.3,2F6.2,3I3,7f7.3)
 151  FORMAT(I2,F6.2,3I3,10f6.3)
 111  continue
c     WRITE(5,101)
```

```
c       WRITE(5,103)
c       WRITE(5,101)
        YY(2)=1.D0
        ZZ(2)=((X(2)**3/RMI(2))*OMEGA*(1.D0+2.D0*K*ROTFRQ/
c    *  SIGM*(CINTGRL+1.D0/L)*Y(2)))/L
     *  SIGM*(-CINTGRL-1.D0/L)*Y(2)))/L
        DO 75 N=2,I-1
        X0=X(N)
        Y0=YY(N)
        Z0=ZZ(N)
        RMI0=RMI(N)
        DNSI0=DNSI(N)
        PRI0=PRI(N)
        DNSD0=DNSD(N)
        gama0=gama1(N)
        CONST1=2.D0*K*ROTFRQ/SIGM*(Y(N)+(RMI(N)/(X(N)**3*
     *  OMEGA)+CINTGRL*L*(L+1)*RMI(N)/(X(N)**3*OMEGA)))*Z(N)
        CONST2=2.D0*K*ROTFRQ/SIGM*(-CINTGRL*OMEGA*X(N)**3*
     *  Y(N)/RMI(N)-Z(N))
        CALL RUNGE(X0,Y0,Z0,Y1,Z1,G,TM,RMI0,DNSI0,PRI0,TR,GAMA0,L,OMEGA
     *  ,PI,DNSD0,H,CONST1,CONST2)
        YY(N+1)=Y1
        ZZ(N+1)=Z1
        ERPR(N)=G*TM*RMI(N)*DNSI(N)*ZZ(N)/(TR*X(N))
c       print *,yy(1),zz(1),yy(2),zz(2),yy(3),zz(3)
        DLPP(N)= (ERPR(N)-YY(N)*G*TM*RMI(N)*DNSI(N)/(TR*X(N)))/PRI(N)
        DLTR(N)= YY(N)*X(N)*TR
        DLTH(N)=(ERPR(N)/DNSI(N))/(X(N)*TR*SIGM**2.D0)
        XPHASE(N)=DLOG10(1.D0+DABS(DLTR(N)/TR))
        YPHASE(N)=DLOG10(1.D0+DABS(ZZ(N)*RMI(N)/X(N)))
        IF (DLTR(N).LT.0.D0) XPHASE(N)= -XPHASE(N)
        IF (ZZ(N).LT.0.D0) YPHASE(N)=-YPHASE(N)
c       WRITE(5,3000)X(N),DLTR(N),DLPP(N),XPHASE(N),YPHASE(N)
```

```
c       WRITE(5,3000)X(N),yy(N),DLPP(N)
 75     CONTINUE
 65     CONTINUE
c       CLOSE(5)
 100    FORMAT(I5,F13.4,E14.4,2E12.4,A6,3I3)
 101    FORMAT(60('='))
 102    FORMAT(4X,'M',7X,' OMEGA',7X,'SIGMA',6X,'c/d(lne0)',6X,'MODE')
 103    FORMAT(4X,'X',8X,'DR/R',8X,'DP/P',6X,'XPHASE',6X,'YPHASE')
 400    FORMAT(I5,2e14.6)
 409    FORMAT(5e14.6,A6,I3)
 402    FORMAT(I5,3e14.6)
 410    FORMAT(I5,4e14.6,A6,I3)
 412    FORMAT(I5,2e14.6,A6,3I3)
 413    FORMAT(I5,5e14.6,A6,I3)
 414    FORMAT(I5,5e14.6)
 415    FORMAT(I5,9e14.6)
 3000   FORMAT(F7.3,4(E12.3))
 70     CONTINUE
c       ****************************************************
        J=J+1
        TOL=TOL0
        NA=0
        NG=0
        np=0
        IORD=0
        OMEGA=OMEGA+DELTA
        GOTO 40
c       ****************************************************
 888    IF (DABS(BC).LT.0.002d0) THEN
c       print*, omega,bc,fr
c       write(5,58) omega,bc,fr
c58     format(3e12.3)
        endif
```

```
c       PRINT*,'THE',J,'APPROACH','TOL=',TOL,OMEGA
c       TOL=TOL/10.D0
c       DELTA=DELTA/10.D0
c       OMEGA=OMEGA+DELTA
c       GOTO 40
c       ENDIF
c       ************************************************************
 80     OMEGA=OMEGA+DELTA
        GOTO 40
 999    CONTINUE
c109    continue
c       PRINT*,'==========================================
c    *  THE CHARECTERISTICS OF THESE MODES'
c       PRINT*,' ARE IN /NONRAD OUT A1'
c       PRINT*,'==========================================
c       PRINT*,'END'
        STOP
        END
c       *************************************************************************
c       INTERPOLATION. REF: S.CONTE&C.de.BOORE,1981,numerical analysis
c       *************************************************************************
        SUBROUTINE INTRP(XU,YU,X,F,NP)
        IMPLICIT REAL*8 (A-H,O-Z)
c       DIMENSION F(NP),X(NP),A(20),XK(20)
        DIMENSION F(10000),X(10000),A(20),XK(20)
c       do 113 j=1,385
c       write(5,1002) j,f(j),x(j)
c113    continue
c1002   format(I4, 2(e11.4))
c       TOL11=1.0D-2
        TOL11=1.0D-3
        IF (XU.GT.X(1).AND.XU.LE.X(NP)) THEN
        DO 11 NEXT=2,NP
```

```
      IF (XU.LE.X(NEXT)) GOTO 12
   11 CONTINUE
      ENDIF
      IF (XU.LE.X(1)) THEN
      YU=F(1)
      ELSE
      YU=F(NP)
      ENDIF
      IFLAG=3
      RETURN
12    XK(1)=X(NEXT)
      NEXTL=NEXT-1
      NEXTR=NEXT+1
      A(1)=F(NEXT)
      YU=A(1)
      PSIK=1.D0
      KP1MAX=MIN(20,NP)
      DO 21 KP1=2,KP1MAX
      IF (NEXTL.EQ.0.D0) THEN
      NEXT=NEXTR
      NEXTR=NEXTR+1
      ELSEIF (NEXTR.GT.NP) THEN
      NEXT=NEXTL
      NEXTL=NEXTL-1
      ELSEIF (XU-X(NEXTL).GT.X(NEXTR)-XU) THEN
      NEXT=NEXTR
      NEXTR=NEXTR+1
      ELSE
      NEXT=NEXTL
      NEXTL=NEXTL-1
c     print*, xu,kp1
      ENDIF
      XK(KP1)=X(NEXT)
```

```
      A(KP1)=F(NEXT)
      DO 13 J=KP1-1,1,-1
c     if(xk(kp1).eq.xk(j)) go to 133
      A(J)=(A(J+1)-A(J))/(XK(KP1)-XK(J))
c     go to 13
c133   a(j)=a(j-1)
c     print *,j,xu
 13   CONTINUE
      PSIK=PSIK*(XU-XK(KP1-1))
      ERROR=A(1)*PSIK
      YU=YU+ERROR
c     print*,yu,psik,a(1),a(2),a(3)
      IF (DABS(ERROR).LE.TOL11) THEN
      IFLAG=1
      RETURN
      ENDIF
 21   CONTINUE
      IFLAG=2
      RETURN
      END
c     ******************************************************
c     NUMERICAL DIFFERENTIATION using 3 points formula
c     ******************************************************
      SUBROUTINE DNSDIF(DNSI,DNSD,TR,H,I)
      IMPLICIT REAL*8 (A-H,O-Z)
      save
      DIMENSION DNSI(10000),DNSD(10000)
      DO 30 N=1,i-3
      DNSD(N)=(-3.D0*DNSI(N)+4.D0*DNSI(N+1)-DNSI(N+2))/(2.D0*H*TR)
c     write(5,55)n,dnsd(n),dnsi(n),dnsi(n+1),dnsi(n+2)
 55   format(i5,3x,4e13.6)
 30   CONTINUE
      DNSD(I-2)=(-DNSI(I-3)+DNSI(I-1))/(2.D0*H*TR)
```

```
      DNSD(I-1)=(DNSI(I-3)-4.D0*DNSI(I-2)+3.D0*DNSI(I-1))/(2.D0*H*TR)
      RETURN
      END
c     *****************************************************
c     NUMERICAL INTEGRATION
c     *****************************************************
      SUBROUTINE ROTAT(CNUM,CDEN,CINTGRL,DEN,I,H)
      IMPLICIT REAL*8 (A-H,O-Z)
      save
      DIMENSION CNUM(10000),CDEN(10000)
      SNUM1=0.D0
      SNUM2=0.D0
      SDEN1=0.D0
      SDEN2=0.D0
      SNUM0=CNUM(2)+CNUM(I+1)
      SDEN0=CDEN(2)+CDEN(I+1)
      DO 60 N=2,I-4,2
      SNUM1=SNUM1+CNUM(N+1)
      SNUM2=SNUM2+CNUM(N+2)
      SDEN1=SDEN1+CDEN(N+1)
      SDEN2=SDEN2+CDEN(N+2)
 60   CONTINUE
      CNUMI=H/3.D0*(SNUM0+4.D0*SNUM1+2.D0*SNUM2)
      CDENI=H/3.D0*(SDEN0+4.D0*SDEN1+2.D0*SDEN2)
      DEN=CDENI
      CINTGRL=CNUMI/CDENI
      RETURN
      END
c     *****************************************************
c     RUNGE-KUTTA METHOD
c     *****************************************************
      SUBROUTINE RUNGE(X,Y0,Z0,Y,Z,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,
     * OMEGA,PI,DNSD,H,CONST1,CONST2)
```

```fortran
      IMPLICIT REAL*8 (A-H,O-Z)
      A=H*F1(X,Y0,Z0,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1
      AA=H*F2(X,Y0,Z0,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+
     * H*CONST2
      X=X+0.5D0*H
      Y1=Y0+0.5D0*A
      Z1=Z0+0.5D0*AA
      B=H*F1(X,Y1,Z1,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1
      BB=H*F2(X,Y1,Z1,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+
     * H*CONST2
      Y1=Y0+0.5D0*B
      Z1=Z0+0.5D0*BB
      C=H*F1(X,Y1,Z1,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1
      CC=H*F2(X,Y1,Z1,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+
     * (H*CONST2)
      X=X+0.5D0*H
      Y1=Y0+C
      Z1=Z0+CC
      D=H*F1(X,Y1,Z1,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1
      DD=H*F2(X,Y1,Z1,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+
     * H*CONST2
      Y=Y0+(A+2.D0*B+2.D0*C+D)/6.D0
      Z=Z0+(AA+2.D0*BB+2.D0*CC+DD)/6.D0
      RETURN
      END
c     ************************************************************
      FUNCTION F1(X,Y0,Z0,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)
      IMPLICIT REAL*8 (A-H,O-Z)
      F1 = ((G*TM*RMI*DNSI/(PRI*TR*X*GAMA1)-3.D0)*Y0+(L*(L+1)/
     * (OMEGA*X**3/RMI)-G*TM*RMI*DNSI/(PRI*TR*X*GAMA1))*Z0)/X
      RETURN
      END
c     ************************************************************
```

```
      FUNCTION F2(X,Y0,Z0,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)
      IMPLICIT REAL*8 (A-H,O-Z)
      F2 = ((1.D0-4.D0*PI*DNSI*(TR*X)**3/(TM*RMI)-TR*X*(DNSD/DNSI+
     *  DNSI*G*TM*RMI/(GAMA1*PRI*TR*TR*X*X)))*Z0+(OMEGA*X**3/RMI+
     * X*TR*(DNSD/DNSI+DNSI*G*TM*RMI/
     * (GAMA1*PRI*TR*TR*X*X)))*Y0)/X
      IF (L.EQ.0) THEN
      F2=F2+((4.D0*PI*DNSI*(TR*X)**3/(TM*RMI))*Y0)/X
      ENDIF
      RETURN
      END
c     ***********************************************
c     FINDING THE SECOND ORDER EIGEN FREQUENCY
c     ***********************************************
      SUBROUTINE SECONDFRQ(THIRDC,FORTHC,FIFTHC,SIXTHC,
     * SEVENTHC,EIGHTHC,NINTHC,THIRD,FORTH,FIFTH,SIXTH,
     * SEVENTH,EIGHTH,NINTH,I,H)
      IMPLICIT REAL*8 (A-H,O-Z)
      SAVE
      DIMENSION THIRDC(5000),FIFTHC(5000),FORTHC(5000),SIXTHC(5000),
     * SEVENTHC(5000),EIGHTHC(5000),NINTHC(5000)
      STHIRDC1=0.D0
      SFORTHC1=0.D0
      SFIFTHC1=0.D0
      SSIXTHC1=0.D0
      SSEVENTHC1=0.D0
      SEIGHTHC1=0.D0
      SNINTHC1=0.D0
      STHIRDC2=0.D0
      SFORTHC2=0.D0
      SFIFTHC2=0.D0
      SSIXTHC2=0.D0
      SSEVENTHC2=0.D0
```

```
      SEIGHTHC2=0.D0
      SNINTHC2=0.D0
      STHIRDC0=THIRDC(2)+THIRDC(I+1)
      SFORTHC0=FORTHC(2)+FORTHC(I+1)
      SFIFTHC0=FIFTHC(2)+FIFTHC(I+1)
      SSIXTHC0=SIXTHC(2)+SIXTHC(I+1)
      SSEVENTHC0=SEVENTHC(2)+SEVENTHC(I+1)
      SEIGHTHC0=EIGHTHC(2)+EIGHTHC(I+1)
      SNINTHC0=NINTHC(2)+NINTHC(I+1)
      DO 60 N=2,I-4,2
      STHIRDC1=STHIRDC1+THIRDC(N+1)
      SFORTHC1=SFORTHC1+FORTHC(N+1)
      SFIFTHC1=SFIFTHC1+FIFTHC(N+1)
      SSIXTHC1=SSIXTHC1+SIXTHC(N+1)
      SSEVENTHC1=SSEVENTHC1+SEVENTHC(N+1)
      SEIGHTHC1=SEIGHTHC1+EIGHTHC(N+1)
      SNINTHC1=SNINTHC1+NINTHC(N+1)
      STHIRDC2=STHIRDC2+THIRDC(N+2)
      SFORTHC2=SFORTHC2+FORTHC(N+2)
      SFIFTHC2=SFIFTHC2+FIFTHC(N+2)
      SSIXTHC2=SSIXTHC2+SIXTHC(N+2)
      SSEVENTHC2=SSEVENTHC2+SEVENTHC(N+2)
      SEIGHTHC2=SEIGHTHC2+EIGHTHC(N+2)
      SNINTHC2=SNINTHC2+NINTHC(N+2)
60    CONTINUE
      THIRDCI=H/3.D0*(STHIRDC0+4.D0*STHIRDC1+2.D0*STHIRDC2)
      FORTHCI=H/3.D0*(SFORTHC0+4.D0*SFORTHC1+2.D0*SFORTHC2)
      FIFTHCI=H/3.D0*(SFIFTHC0+4.D0*SFIFTHC1+2.D0*SFIFTHC2)
      SIXTHCI=H/3.D0*(SSIXTHC0+4.D0*SSIXTHC1+2.D0*SSIXTHC2)
      SEVENTHCI=H/3.D0*(SSEVENTHC0+4.D0*SSEVENTHC1+
     * 2.D0*SSEVENTHC2)
      EIGHTHCI=H/3.D0*(SEIGHTHC0+4.D0*SEIGHTHC1+2.D0*SEIGHTHC2)
      NINTHCI=H/3.D0*(SNINTHC0+4.D0*SNINTHC1+2.D0*SNINTHC2)
```

```
THIRD=THIRDCI
FORTH=FORTHCI
FIFTH=FIFTHCI
SIXTH=SIXTHCI
SEVENTH=SEVENTHCI
EIGHTH=EIGHTHCI
NINTH=NINTHCI
RETURN
END
```

APPENDIX B

MATHEMATICA RESULTS OF SOME ANGULAR INTEGRALS

$\int cos\theta \ \frac{\partial}{\partial\theta}(Y_l^m Y_l^{m*}) \ d\theta d\phi=$

0 , for (l,m)=(0,0)

-2 , for (l,m)=(1,0)

-4 , for (l,m)=(2,0)

-6 , for (l,m)=(3,0)

1, for other cases of (l,m) ($l \leq 3$)

This integral is in equation (3.96), and its value is taken as 1 afterwards, since for m=0 the term including this integral totally vanishes.

$\int cot^2\theta \ |Y_l^m|^2 \ sin\theta \ d\theta d\phi$=

diverges , for (l,m)=(0,0)

$\frac{1}{2}$, for (l,m)=(1,-1)

diverges , for (l,m)=(1,0)

$\frac{1}{2}$, for (l,m)=(1,1)

$\frac{1}{4}$, for (l,m)=(2,-2)

$\frac{3}{2}$, for (l,m)=(2,-1)

diverges , for (l,m)=(2,0)

$\frac{3}{2}$, for (l,m)=(2,1)

$\frac{1}{4}$, for (l,m)=(2,2)

$\frac{1}{6}$, for (l,m)=(3,-3)

$\frac{3}{4}$, for (l,mm)=(3,-2)

$\frac{5}{2}$, for (l,m)=(3,-1)

diverges , for (l,m)=(3,0)

$\frac{5}{2}$, for (l,m)=(3,1)

$\frac{3}{4}$, for (l,m)=(3,2)

$\frac{1}{6}$, for (l,m)=(3,3)

As seen in equation (3.110), when m=0, $\sigma_{22,S_{2b}}$ totally vanishes.

$\int sin^3\theta \ Y_l^m Y_l^{m*} \ d\theta d\phi$=

$\frac{2}{3}$, for (l,m)=(0,0)

$\frac{4}{5}$, for (l,m)=(1,-1)

$\frac{2}{5}$, for (l,m)=(1,0)

$\frac{4}{5}$, for (l,m)=(1,1)

$\frac{6}{7}$, for (l,m)=(2,-2)

$\frac{4}{7}$, for (l,m)=(2,-1)

$\frac{10}{21}$, for (l,m)=(2,0)

$\frac{4}{7}$, for (l,m)=(2,1)

$\frac{6}{7}$, for (l,m)=(2,2)

$\frac{8}{9}$, for (l,m)=(3,-3)

$\frac{2}{3}$, for (l,m)=(3,-2)

$\frac{8}{15}$, for (l,m)=(3,-1)

$\frac{22}{45}$, for (l,m)=(3,0)

$\frac{8}{15}$, for (l,m)=(3,1)

$\frac{2}{3}$, for (l,m)=(3,2)

$\frac{8}{9}$, for (l,m)=(3,3)

(See equation (3.109).)

$\int sin^2\theta \; cos\theta \; \frac{\partial}{\partial\theta}(Y_l^m Y_l^{m*}) \; d\theta d\phi$=

0 , for (l,m)=(0,0)

$\frac{2}{5}$, for (l,m)=(1,-1)

-$\frac{4}{5}$, for (l,m)=(1,0)

$\frac{2}{5}$, for (l,m)=(1,1)

$\frac{4}{7}$, for (l,m)=(2,-2)

-$\frac{2}{7}$, for (l,m)=(2,-1)

-$\frac{4}{7}$, for (l,m)=(2,0)

-$\frac{2}{7}$, for (l,m)=(2,1)

$\frac{4}{7}$, for (l,m)=(2,2)

$\frac{2}{3}$, for (l,m)=(3,-3)

0 , for (l,m)=(3,-2)

-$\frac{2}{5}$, for (l,m)=(3,-1)

-$\frac{8}{15}$, for (l,m)=(3,0)

-$\frac{2}{5}$, for (l,m)=(3,1)

0 , for (l,m)=(3,2)

$\frac{2}{3}$, for (l,m)=(3,3)

(See equation (3.108).)

$\int cos^2\theta \ sin\theta \ \frac{\partial}{\partial\theta}Y_l^m \frac{\partial}{\partial\theta}Y_l^{m*} \ d\theta d\phi=$

0 , for (l,m)=(0,0)

$\frac{3}{10}$, for (l,m)=(1,-1)

$\frac{2}{5}$, for (l,m)=(1,0)

$\frac{3}{10}$, for (l,m)=(1,1)

$\frac{3}{7}$, for (l,m)=(2,-2)

$\frac{11}{14}$, for (l,m)=(2,-1)

$\frac{18}{7}$, for (l,m)=(2,0)

$\frac{11}{14}$, for (l,m)=(2,1)

$\frac{3}{7}$, for (l,m)=(2,2)

$\frac{1}{2}$, for (l,m)=(3,-3)

1 , for (l,m)=(3,-2)

$\frac{27}{10}$, for (l,m)=(3,-1)

$\frac{28}{5}$, for (l,m)=(3,0)

$\frac{27}{10}$, for (l,m)=(3,1)

1 , for (l,m)=(3,2)

$\frac{1}{2}$, for (l,m)=(3,3)

(See equation (3.111).)

$\int sin\theta\ Y_l^{m*}\ d\theta d\phi=$

$2\sqrt{\pi}$, for $(l,m)=(0,0)$
0 , for $(l,m)\neq(0,0)$ and $l\leq 3$

(See equation (3.79).)

$\int sin\theta\ cos^2\theta\ Y_l^{m*}\ d\theta d\phi=$

$\frac{2\sqrt{\pi}}{3}$, for $(l,m)=(0,0)$
$\frac{4}{3}\ \sqrt{\frac{\pi}{5}}$, for $(l,m)=(2,0)$
0, for other cases of (l,m) $(l\leq 3)$

(See equation (3.79).)

$\int sin^2\theta\ cos\theta\ \frac{\partial}{\partial\theta} Y_l^{m*}\ d\theta d\phi=$

-4 $\sqrt{\frac{\pi}{5}}$, for $(l,m)=(2,0)$
0, for other cases of (l,m) $(l\leq 3)$
(See equation (3.79).)