VIABLE HIGHER DERIVATIVE THEORIES

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ŞENER ÖZÖNDER

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Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Sinan Bilikmen
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Bayram Tekin
Supervisor

Examining Committee Members

Prof. Dr. Ayşe Kalkanlı Karasu (METU,PHYS)

Assoc. Prof. Dr. Bayram Tekin (METU,PHYS)

Prof. Dr. Atalay Karasu (METU,PHYS)

Assoc. Prof. Dr. Bahtiyar Özgür Sarıoğlu (METU,PHYS)

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Name, Last Name: Şener Özönder

Signature :

ABSTRACT

VIABLE HIGHER DERIVATIVE THEORIES

Özönder, Şener

M.S., Department of Physics

Supervisor : Assoc. Prof. Dr. Bayram Tekin

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In this thesis, higher derivative theories are investigated. Ostrogradskian instability of higher derivative theories is examined both at the classical and quantum levels. It is shown that avoiding the instability in nondegenerate higher derivative theories is impossible. Moreover, the degenerate model of relativistic particle with a curvature term is studied as a viable higher derivative theory.

Most of the work we present here is not original. We give a review of the literature and compile various detached works that already exist.

Keywords: Higher Derivative, Ostrogradski, Instability, Nonlocality

ÖZ

ELVERİŞLİ YÜKSEK TÜREVLİ KURAMLAR

Özönder, Şener

Yüksek Lisans, Fizik Bölümü

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Bu tezde, yüksek türevli kuramlar araştırıldı. Yüksek türevli teorilerdeki Ostrogradski kararsızlığı klasik ve kuantum düzeyinde incelendi. Dejenere olmayan yüksek türevli kuramlarda kararsızlığın kaçınılmaz olduğu gösterildi. Ayrıca dejenere olan eğrilik terimli relativistik parçacık modeli elverişli yüksek türevli kuram olarak çalışıldı.

Burada sunduğumuz çalışmanın büyük bir kısmı özgün değildir. Literatürün bir incelemesini yapmakta ve dağınık durumda olan halihazırdaki çeşitli çalışmaları derlemekteyiz.

Anahtar Kelimeler: Yüksek Türevli, Ostrogradski, Kararsızlık, Yerelsizlik

To my family

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CHAPTER 1

INTRODUCTION

Since Newton, all fundamental theories in physics have been based on the equations of motion which do not include terms which are more than second time derivatives of the dynamical variables. In the Lagrangian formalism, this means that the Lagrangian can only be a function of the dynamical variable and its first time derivative, and accordingly, the phase space of the theory has $2N$ dimensions per canonical coordinate for N particles. The Lagrangian surely may include higher derivative terms if they can be written as a total time derivative of some function, which does not modify the equations of motion.

On the other hand, higher derivative terms have long been introduced in an attempt to modify the fundamental theories to render them free of some theoretical complications or make them compatible with phenomenology.

A very old higher derivative equation is the Abraham-Lorentz equation of motion for a point-like electron which is losing energy through radiation during acceleration. Later, higher derivative theories appeared in the context of generalized classical electrodynamics due to Fokker, Bopp, Podolsky, Feynman and also as a generalized meson field theory due to Green [1]. Euler-Heisenberg [2] and Born-Infeld [3] also worked on higher curvature electrodynamics where Maxwell's theory is modified with various powers of the field strength. These theories give finite self-energy for point particles.

Even in gravity, higher curvature theories were proposed immediately after Einstein's field equations by Weyl and Eddington [4].

One of the motivations for generalized field equations was the regularization in order to tame divergences arising from the assumption of taking the electron as a point-like particle and this motivation endured also in the pre-renormalization era of the quantum field theory. Effective quantum field theories endowed with higher derivative terms were employed to obtain finite results. However, after the successful results of propagator cutoffs (Pauli-Villars) and renormalization techniques in quantum field theory such as quantum electrodynamics, the interest in these effective theories decreased.

In fact, adding higher derivative terms to the Lagrangian and adding regulator cutoffs to propagators are essentially the same thing. This is easily seen when the equation of motion is written in the convolution form [5]. Higher differentiation comes out as a convolution of the field with a "low pass filter" which integrates out high energy oscillations. In quantum field theory, this procedure cures the ultraviolet divergence problem caused by interactions in very short distances.

The field of "higher order variational problems" is occupied not only by physicists, but also by mathematicians and engineers [1]. The sort of above-mentioned smoothing functions are also used in a different context: In image processing, introducing higher derivative terms to the "energy functional" brings about noise removal and also blurring effect on the sharp sides of the patterns (see [6]). This is basically what happens in regularized quantum field theory; that is, point-like particles are no longer that much "sharp", but smeared or averaged out over spacetime to the extend determined by the cutoff term.

Since late 90's, there has also been an increasing interest in higher derivative dynamical models without paying attention to the underlying canonical structure [7]. Having been coined as "generalized dynamics" or "jerky mechanics", these models attract attention in realistic areas of physics as diverse as fluid mechanics and acoustics.

In this thesis, we investigate higher derivative theories and in particular search for the viable ones that are free of instabilities. In Chapter 2, the canonical formalism of higher derivative theories is constructed. In Chapter 3, the problems of higher derivative theories are explicitly shown and the suggestions to tackle these problems are examined. In Chapter 4, the viability conditions for higher derivative theories are provided and the model of relativistic particle with a curvature term is presented as a viable higher derivative theory. It is also shown that some higher derivative nonlocal Lagrangians may be free of problems.

[1] *Isoperimetrical* systems is the name of the systems of differential equations which occur in higher order derivative variational problems.

CHAPTER 2

THE FORMALISM

The canonical formalism for Lagrangians which depend on more than one time derivative was first constructed by Ostrogradski [8]. The construction of the Hamiltonian for a usual Lagrangian will be reviewed first; Ostrogradski's construction of the Hamiltonian for a higher derivative Lagrangian and its field theoretical generalization will be presented later.

2.1 Canonical Formulation of Lower Derivative Theories

For the sake of simplicity, let us consider a one dimensional time-independent system whose action is given by

$$S[q] = \int dt L(q, \dot{q}) \tag{2.1}$$

whose variation yields the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \; . \tag{2.2}$$

Unless otherwise stated, we shall assume that the Lagrangian is *nondegenerate*. This means that the expression $p = \frac{\partial L}{\partial \dot{q}}$ depends on $\dot{q}$ so that we can invert it to solve $\dot{q}$ in terms of q and the conjugate momentum p. The generalization of this condition reads: if the determinant of the matrix $M_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}$, called the "Hessian", is nonzero (*i.e.* the matrix M is nonsingular), we say that the theory is *nondegenerate* (or *nonsingular*) [9]. Constructing the Hamiltonization for of the singular and nonsingular theories is different.

Since we need two pieces of initial value data, the phase space is 2-dimensional and consequently there should be two canonical coordinates, Q and P. These can simply be defined as

$$Q \equiv q \qquad \text{and} \qquad P \equiv \frac{\partial L}{\partial \dot{q}} \; . \tag{2.3}$$

Due to nondegeneracy, the phase space transformation (2.3) can be inverted to solve $\dot{q}$ in terms of Q and P. That is, there exists a function $v(Q,P)$ such that

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{\substack{q=Q \\ \dot{q}=v}} = P \; . \tag{2.4}$$

The canonical Hamiltonian is obtained by a Legendre transformation

$$\begin{aligned} H(Q,P) &\equiv P\dot{q} - L \; , && (2.5) \\ &= Pv(Q,P) - L\Big(Q, v(Q,P)\Big) \; . && (2.6) \end{aligned}$$

The canonical evolution equations follow:

$$\begin{aligned} \dot{Q} &\equiv \frac{\partial H}{\partial P} = v + P\frac{\partial v}{\partial P} - \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = v \; , && (2.7) \\ \dot{P} &\equiv -\frac{\partial H}{\partial Q} = -P\frac{\partial v}{\partial Q} + \frac{\partial L}{\partial Q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial Q} = \frac{\partial L}{\partial Q} \; . && (2.8) \end{aligned}$$

It is clear that (2.7) makes the inverse phase space transformation (2.4), and (2.8) gives the Euler-Lagrange equation (2.2). This proves that the Hamiltonian generates time evolution. If the Lagrangian is independent of time explicitly, this means that H is the conserved quantity, *i.e.*, the energy.

2.2 Canonical Formulation of Higher Derivative Theories

Now consider a system whose action is given by [10, 11, 12]

$$S\left[q\right] = \int dt L(q, \dot{q}, \ldots, q^{(N)}). \tag{2.9}$$

where $q^{(n)}$ means $\frac{d^n q}{dt^n}$. The variation of the action $(q(t) \rightarrow q(t) + \delta(t))$ yields

$$\delta S\left[q\right] = \int dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} + \cdots + \frac{\partial L}{\partial q^{(N)}}\delta q^{(N)}\right] \; . \tag{2.10}$$

Using integrations by parts, we convert the terms of the form

$$\frac{\partial L}{\partial q^{(n)}}\delta q^{(n)} \tag{2.11}$$

into terms proportional to δq and eliminating the surface terms (using the assumption that δq vanishes there), we find the generalized Euler-Lagrange equation

$$\sum_{j=0}^{N}\left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial q^{(j)}} = 0 \; , \tag{2.12}$$

which contains the term $q^{(2N)}$. Hence the canonical phase space must contain N coordinates and N conjugate momenta. These are given by Ostrogradski as

$$Q_i \equiv \Big(\frac{d}{dt}\Big)^{i-1} q \,, \tag{2.13}$$

$$P_i \equiv \sum_{j=i}^{N} \Big(-\frac{d}{dt}\Big)^{j-i} \frac{\partial L}{\partial q^{(j)}} \,, \qquad i = 1, 2, \ldots, N \,. \tag{2.14}$$

If the nondegeneracy condition is satisfied, the action's dependence on $q^{(N)}$ cannot be eliminated by partial integrations. Due to nondegeneracy, we can solve $q^{(N)}$ in terms of P_N, q and the first $N-1$ derivatives of q. That is, there exists a function $\mathcal{A}(Q_1, \ldots, Q_N, P_N)$ such that

$$\frac{\partial L}{\partial q^{(N)}}\bigg|_{\substack{q^{(i-1)}=Q_i \\ q^{(N)}=\mathcal{A}}} = P_N \,. \tag{2.15}$$

Accordingly, Ostrogradski's Hamiltonian takes the form

$$H \equiv \sum_{i=1}^{N} P_i q^{(i)} - L \,, \tag{2.16}$$

$$= P_1 Q_2 + P_2 Q_3 + \cdots + P_{N-1} Q_N + P_N \mathcal{A} - L\Big(Q_1, \ldots, Q_N, \mathcal{A}\Big) \,. \tag{2.17}$$

As the previous case, the evolution equations

$$\dot{Q}_i \equiv \frac{\partial H}{\partial P_i} \qquad \text{and} \qquad \dot{P}_i \equiv -\frac{\partial H}{\partial Q_i} \,, \qquad i = 1, 2, \ldots, N \tag{2.18}$$

again generate the canonical transformations and give the Euler-Lagrange equation: $\dot{Q}_i$ gives the canonical definition (2.13) for Q_{i+1}; $\dot{P}_{i+1}$ gives the canonical definition (2.14) for P_i; and $\dot{P}_1$ gives the equation of motion (2.12) (similar to (2.8)). So there is no doubt that Ostrogradski's Hamiltonian generates time evolution. When the Lagrangian is independent of time explicitly, this implies that H is the conserved quantity, *i.e.* the energy.

It is seen that the Hamiltonian (2.17) is not always positive valued. This follows from the fact that H is linear in $P_1, P_2, \ldots P_{N-1}$ and it might be bounded only for P_N (see (2.51)). So Ostrogradski's result is that higher derivative theories, for which higher derivative terms (more than first time derivative in the action) cannot be eliminated by partial integration, are unstable due to the linear dependence of the Hamiltonian to the conjugate momenta. That is to say, one can increase or decrease the energy without any bound by moving in different directions in the $2N$-dimensional phase space.

In the literature, this result is widely expressed as "the Hamiltonian is not bounded from below". However to be more exact, the phrase "the Hamiltonian is not bounded from below and above" must be used [13] because a Hamiltonian not bounded from below but from above would be as valuable as the positive valued Hamiltonians. In a theory where the energy is bounded from above, we can change the sign of the L and H and get the energy bounded from below. This is possible because the sign of H is not preferred, in contrast to classical mechanics where the kinetic energy should be positive valued and the sign of H is determined. If both positive and negative energy states exist in the theory, changing the sign of the L and H does not change this situation. The fact that energy can take both positive and negative values without any bound is the source of the difficulty in higher derivative theories.

Since no special form for the Lagrangian is assumed in getting (2.17), the generality of this result should be emphasized: The energy is not positive valued for nondegenerate higher derivative theories. The result cannot be changed by any kind of interaction terms or by adjusting the parameters. (The degenerate case will be discussed in Chapter 4).

The same Hamiltonian can also be obtained by means of the Lagrange multiplier method which was developed by Dirac in the early 1950's [14] and was applied to the higher derivative systems in the 1980's [15]. Hamiltonization of the Pais-Uhlenbeck oscillator

$$L = \frac{1}{2}\left\{\ddot{q}^2 - (\omega_1^2 + \omega_2^2)\dot{q}^2 + \omega_1^2\omega_2^2 q^2\right\} \tag{2.19}$$

with Dirac constraints was carried out [16] and it was shown that the result exactly matches the Ostrogradskian Hamiltonian.

Generalization of the above formalism to higher spatial dimensions is trivial. In this case, we have a copy of above formulas for each spatial dimension. The spatial higher dimensional action is given by

$$S = \int dt L(x^\alpha, \dot{x}^\alpha, \ldots, x^{\alpha\,(N)}) \tag{2.20}$$

where $x^{\alpha\,(n)}$ means $\frac{d^n x^\alpha}{dt^n}$ and the index α is to count the spatial dimensions, *i.e.*, $\alpha = 1, 2, \ldots, m$ for m spatial dimensions. The Euler-Lagrangian equation in m dimensions is

$$\sum_{j=0}^{N}\left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial x^{\alpha\,(j)}} = 0\,, \qquad \alpha = 1, 2, \ldots, m \tag{2.21}$$

The Ostrogradskian canonical coordinates and conjugate momenta are

$$
\begin{aligned}
Q_i^\alpha &\equiv \Big(\frac{d}{dt}\Big)^{i-1} x^\alpha \ , && (2.22)\\
P_i^\alpha &\equiv \sum_{j=i}^{N} \Big(-\frac{d}{dt}\Big)^{j-i} \frac{\partial L}{\partial x^{\alpha\,(j)}} \ , \quad i=1,2,\ldots,N \quad , \alpha = 1,2,\ldots,m. && (2.23)
\end{aligned}
$$

If the nondegeneracy condition is satisfied, there should be a function $\mathcal{A}^\alpha(Q_1^\alpha, \ldots, Q_N^\alpha, P_N^\alpha)$ such that

$$
\frac{\partial L}{\partial x^{\alpha\,(N)}}\bigg|_{\substack{x^{\alpha\,(i-1)}=Q_i^\alpha \\ x^{\alpha\,(N)}=\mathcal{A}^\alpha}} = P_N^\alpha \ . \tag{2.24}
$$

Thus, higher spatial dimensional Ostrogradskian Hamiltonian follows

$$
\begin{aligned}
H &\equiv \sum_{\alpha=1}^{m}\sum_{i=1}^{N} P_i^\alpha x^{\alpha\,(i)} - L \ , && (2.25)\\
&= \sum_{\alpha=1}^{m} (P_1^\alpha Q_2^\alpha + \cdots + P_{N-1}^\alpha Q_N^\alpha + P_N^\alpha \mathcal{A}^\alpha) - L\Big(Q_1^\alpha, \ldots, Q_N^\alpha, \mathcal{A}^\alpha\Big) \ . && (2.26)
\end{aligned}
$$

where the evolution equations are

$$
\dot{Q}_i^\alpha \equiv \frac{\partial H}{\partial P_i^\alpha} \qquad \text{and} \qquad \dot{P}_i^\alpha \equiv -\frac{\partial H}{\partial Q_i^\alpha} \ , \quad i=1,2,\ldots,N \quad , \alpha=1,2,\ldots,m \tag{2.27}
$$

Now let us construct the canonical formalism of the higher derivative field theories (see for example [17]). Consider the action in D dimensions

$$
S = \int d^D x \ \mathcal{L}(\phi, \partial_\mu \phi, \partial_{\mu\nu}\phi, \partial_{\mu\nu\alpha}\phi, \ldots) \ , \tag{2.28}
$$

where ϕ is a real scalar field. Variation with respect to ϕ leads to terms of the form

$$
\frac{\partial \mathcal{L}}{\partial(\partial_{\mu\nu\ldots\beta}\phi)} \partial_{\mu\nu\ldots\beta}\delta\phi \ . \tag{2.29}
$$

Using integrations by parts, we convert these into terms proportional to $\delta\phi$ and eliminate the surface terms. As a result, the generalized Euler-Lagrange equations for a real scalar field follows

$$
\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \partial_{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu\nu}\phi)} - \partial_{\mu\nu\alpha} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu\nu\alpha}\phi)} + \cdots = 0 \ . \tag{2.30}
$$

The conserved energy-momentum tensor $T^{\alpha\mu}$ can be derived by means of the Noether's theorem.

$$
\begin{aligned}
\delta\mathcal{L} &= \partial_\alpha \Bigg\{ \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} - \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)} + \cdots \right) \delta\phi \\
&\qquad + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)} - \partial_\gamma \frac{\partial \mathcal{L}}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} + \cdots \right) \delta\partial_\beta\phi \\
&\qquad\quad + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} - \cdots \right) \delta\partial_\beta\partial_\gamma\phi + \cdots \Bigg\} \ . && (2.31)
\end{aligned}
$$

For a constant infinitesimal translation $\delta x^\mu = \epsilon^\mu$,

$$\delta\mathcal{L} = \epsilon^\mu\,\partial_\mu\mathcal{L} \quad , \quad \delta\phi = \epsilon^\mu\,\partial_\mu\phi \quad , \quad \delta\partial_\alpha\phi = \epsilon^\mu\,\partial_\mu\partial_\alpha\phi \quad , \quad \ldots \tag{2.32}$$

are substituted into (2.31). Hence the result is

$$\partial_\alpha T^{\alpha\mu} = 0 \tag{2.33}$$

where

$$\begin{aligned} T^{\alpha\mu} \;=\; & \left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} - \partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)} + \partial_\beta\partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} - \cdots\right)\partial^\mu\phi \\ & + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)} - \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} + \cdots\right)\partial_\beta\partial^\mu\phi \\ & + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} - \cdots\right)\partial_\beta\partial_\gamma\partial^\mu\phi + \cdots - \eta^{\alpha\mu}\mathcal{L}\,. \end{aligned} \tag{2.34}$$

Therefore the energy-momentum vector and the Hamiltonian are defined as

$$P^\mu \;=\; \int d^{D-1}x\; T^{0\mu}\,, \tag{2.35}$$

$$H \;=\; \int d^{D-1}x\; T^{00}\,, \tag{2.36}$$

where both integrals are spatial. If we have a complex field and the Lagrangian is phase invariant, by using $\delta\phi = i\epsilon\phi$ and $\delta\phi^* = -i\epsilon\phi^*$, we find from (2.31)

$$\begin{aligned} j^\alpha \;=\; i\epsilon & \left\{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} - \partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)} + \cdots\right)\phi \right. \\ & \left. + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi)} - \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\partial_\gamma\phi)} + \cdots\right)\partial_\beta\phi + \cdots - c.c.\right\}. \end{aligned} \tag{2.37}$$

As an example, let us consider the Lagrangian

$$L = \frac{1}{2}(\Box\phi)(\Box\phi) - \frac{1}{2}\mu^2\phi^2\,, \tag{2.38}$$

where $\Box = \partial_\mu\partial^\mu$. The equation of motion reads

$$(\Box^2 - \mu^2)\phi = 0\,. \tag{2.39}$$

The energy-momentum tensor and the Hamiltonian density are given by

$$T^{\mu\nu} \;=\; (\Box\phi)\partial^\mu\partial^\nu\phi - \partial^\mu(\Box\phi)\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}\,, \tag{2.40}$$

$$T^{00} \;\equiv\; \ddot{\phi}\Box\phi - \dot{\phi}\Box\dot{\phi} + \frac{1}{2}(\Box\phi)\Box\phi + \frac{1}{2}\mu^2\phi^2\,. \tag{2.41}$$

2.3 An Example: Higher Derivative Harmonic Oscillator

Second derivative generalization of the harmonic oscillator [11] is considered here as an example of higher derivative theories which was examined by Pais and Uhlenbeck in detail long time ago [18].

$$L = -\frac{gm}{2\omega^2}\ddot{q}^2 + \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \ . \tag{2.42}$$

Here m is the particle mass, ω is the frequency and g is a small positive real number that can be thought of as a coupling constant. The Euler-Lagrange equation for second order Lagrangians

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0 \tag{2.43}$$

gives the equation of motion

$$m\Big(\frac{g}{\omega^2}\, q^{(4)} + \ddot{q} + \omega^2 q\Big) = 0 \ , \tag{2.44}$$

for which the general solution is

$$q(t) = A_+ \cos(k_+ t) + B_+ \sin(k_+ t) + A_- \cos(k_- t) + B_- \sin(k_- t) \ . \tag{2.45}$$

Here the two frequencies are

$$k_\pm \equiv \omega\sqrt{\frac{1 \mp \sqrt{1-4g}}{2g}} \ , \tag{2.46}$$

where $0 < g < 1/4$. In the limit $g \to 0$, we obtain $k_+ = \omega$ (*i.e.* usual harmonic oscillator) while k_- diverges. The solution of (2.44) is no more pure oscillations if g is equal or greater than $1/4$. For example, the case $g = 1/4$ corresponds to the equal frequency case ($\omega_1 = \omega_2 = \omega$) of the Pais-Uhlenbeck oscillator (2.19) for which the solution includes terms like $\sin(\omega t)$, $\cos(\omega t)$, $t\sin(\omega t)$ and $t\cos(\omega t)$. It is obvious that this kind of solution is not stable since the last two terms have a runaway character.

Here k_+ and k_- modes denote positive and negative energy excitations accordingly. For $q_0^{(n)} = \frac{d^n}{dt^n}q(t)\Big|_{t=0}$, the constants in (2.45) are given by

$$A_+ = \frac{k_-^2 q_0 + \ddot{q}_0}{k_-^2 - k_+^2} \ , \qquad B_+ = \frac{k_-^2 \dot{q}_0 + q_0^{(3)}}{k_+(k_-^2 - k_+^2)} \ , \tag{2.47}$$

$$A_- = \frac{k_+^2 q_0 + \ddot{q}_0}{k_+^2 - k_-^2} \ , \qquad B_- = \frac{k_+^2 \dot{q}_0 + q_0^{(3)}}{k_-(k_+^2 - k_-^2)} \ . \tag{2.48}$$

The conjugate momenta are

$$P_1 = m\dot{q} + \frac{gm}{\omega^2}q^{(3)} \qquad \Leftrightarrow \qquad q^{(3)} = \frac{\omega^2 P_1 - m\omega^2 Q_2}{gm}\ , \tag{2.49}$$

$$P_2 = -\frac{gm}{\omega^2}\ddot{q} \qquad \Leftrightarrow \qquad \ddot{q} = -\frac{\omega^2 P_2}{gm}\ . \tag{2.50}$$

where $Q_2 = \dot{q}$ from (2.13).

The Hamiltonian can be written as

$$H = P_1 Q_2 - \frac{\omega^2}{2gm}P_2^2 - \frac{m}{2}Q_2^2 + \frac{m\omega^2}{2}Q_1^2\ , \tag{2.51}$$

$$= \frac{gm}{\omega^2}\dot{q}q^{(3)} - \frac{gm}{2\omega^2}\ddot{q}^2 + \frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2}q^2\ , \tag{2.52}$$

$$= \frac{m}{2}\sqrt{1-4g}\ k_+^2(A_+^2 + B_+^2) - \frac{m}{2}\sqrt{1-4g}\ k_-^2(A_-^2 + B_-^2)\ . \tag{2.53}$$

By using the Noether's theorem, we can also check that H is really the conserved quantity corresponding to the energy. Since there is a time translation symmetry in the theory (see 2.42), *i.e.*, the action is invariant under the transformation $t \to t' = t + \delta t$, there should be an associated conserved quantity which is the energy. The differentiation of $L = L(q, \dot{q}, \ddot{q})$ gives

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}q^{(3)} + \frac{\partial L}{\partial t} \tag{2.54}$$

where the last term vanishes since the Lagrangian (2.42) does not depend on time explicitly. Substituting the equation of motion (2.43) for the $\frac{\partial L}{\partial q}$ term in (2.54) yields

$$\frac{dL}{dt} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\dot{q} - \left(\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}q^{(3)}\ . \tag{2.55}$$

This can be rewritten as a total derivative

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} - L\right) = 0\ . \tag{2.56}$$

Using $\frac{\partial L}{\partial \dot{q}} = m\dot{q}$ and $\frac{\partial L}{\partial \ddot{q}} = -\frac{gm}{\omega^2}\ddot{q}$, it is seen that the expression in the parentheses in (2.56) is equal to H defined in (2.52). So the energy due to Noether's theorem exactly matches the energy derived by means of Ostrogradski's method.

From (2.53), it is seen that the "+" modes carry positive energy whereas the "−" modes carry negative energy. Moreover, it is seen from (2.53) that the associated mass for positive and negative energy modes are different since k_- and k_+ are different[1]. This model was examined as the "different frequency case" of the higher derivative harmonic oscillator by Pais and Uhlenbeck [18].

[1] This becomes more transparent when the Lagrangian is written as a "difference" of two usual harmonic oscillators (see (3.7))

2.4 Quantization of Higher Derivative Harmonic Oscillator

Let us denote the "empty" state wavefunction ("vacuum" in quantum field theory) by $\Omega(Q_1, Q_2)$, which is the minimum excitation for both negative and positive energy states. We define the positive energy lowering operator as a, positive energy raising operator as $a^\dagger$, negative energy lowering operator as b and negative energy raising operator as $b^\dagger$.

In order to find the empty state function, one should solve

$$a\,|\Omega\rangle \;=\; 0\,, \tag{2.57}$$

$$b\,|\Omega\rangle \;=\; 0\,. \tag{2.58}$$

The solution (2.45) can be expressed in terms of complex exponentials so that the raising and lowering operators can easily be extracted for quantization

$$\begin{aligned} q(t) = \frac{1}{2}(A_+ + iB_+)e^{-ik_+t} + \frac{1}{2}(A_+ - iB_+)e^{ik_+t} \\ + \frac{1}{2}(A_- + iB_-)e^{-ik_-t} + \frac{1}{2}(A_- - iB_-)e^{ik_-t}\,. \end{aligned} \tag{2.59}$$

Now the ladder operators will be constructed explicitly by means of A_+, A_-, B_+, B_-, which were just constants in the classical analysis. Since the k_+ mode carries positive energy, the lowering operator for positive energy excitations must be proportional to the e^{-ik_+t} term and one can easily conclude from (2.59) that

$$a \;\propto\; A_+ + iB_+\,, \tag{2.60}$$

$$\;\propto\; \frac{mk_+}{2}\left(1+\sqrt{1-4g}\right)Q_1 + iP_1 - k_+P_2 - \frac{im}{2}\left(1-\sqrt{1-4g}\right)Q_2\,, \tag{2.61}$$

Here first (2.47) is substituted for A_+ and B_+, and then $q = Q_1$, $\dot{q} = Q_2$, (2.49) and (2.50) are used.

Since the k_- mode carries negative energy, its lowering operator must be proportional to the e^{+ik_-t} term

$$b \;\propto\; A_- - iB_-\,, \tag{2.62}$$

$$\;\propto\; \frac{mk_-}{2}\left(1-\sqrt{1-4g}\right)Q_1 - iP_1 - k_-P_2 + \frac{im}{2}\left(1+\sqrt{1-4g}\right)Q_2\,. \tag{2.63}$$

Substituting $P_i = -i\frac{\partial}{\partial Q_i}$, the two coupled equations follow from (2.57) and (2.58)

$$\left[\frac{mk_+}{2}\left(1+\sqrt{1-4g}\right)Q_1 + \frac{\partial}{\partial Q_1} + ik_+\frac{\partial}{\partial Q_2} - \frac{im}{2}\left(1-\sqrt{1-4g}\right)Q_2\right]|\Omega\rangle \;=\; 0, \tag{2.64}$$

$$\left[\frac{mk_-}{2}\left(1-\sqrt{1-4g}\right)Q_1 - \frac{\partial}{\partial Q_1} + ik_-\frac{\partial}{\partial Q_2} + \frac{im}{2}\left(1+\sqrt{1-4g}\right)Q_2\right]|\Omega\rangle \;=\; 0, \tag{2.65}$$

for which the unique solution is

$$\Omega(Q_1, Q_2) = N \exp\left[-\frac{m\sqrt{1-4g}}{2(k_+ + k_-)}\left(k_+ k_- Q_1^2 + Q_2^2\right) - i\sqrt{g} m Q_1 Q_2\right] . \tag{2.66}$$

To have a sensible quantum theory, one must have normalizable wave functions

$$\langle\Omega|\Omega\rangle < \infty , \tag{2.67}$$

which is satisfied in this case since the non-oscillating part of $\Omega(Q_1, Q_2)$ is decaying.

Consequently, any normalized state can be built from the empty state wave function $\Omega(Q_1, Q_2)$ by acting with the desired number of $a^\dagger$ and $b^\dagger$ operators

$$|N_+, N_-\rangle \equiv \frac{(a^\dagger)^{N_+}}{\sqrt{N_+!}} \frac{(b^\dagger)^{N_-}}{\sqrt{N_-!}} |\Omega\rangle , \tag{2.68}$$

where N_+ and N_- denote the positive and negative energy states, respectively. The commutation relations are

$$[a, a^\dagger] = 1 = [b, b^\dagger] , \tag{2.69}$$

and the Hamiltonian is given by

$$H|N_+, N_-\rangle = (N_+\, k_+ - N_-\, k_-)\, |N_+, N_-\rangle . \tag{2.70}$$

The spectrum of the field theoretical version of this model involves negative energy particle and anti-particle pairs in addition to the positive energy particle and anti-particle pairs of the "usual" second order Klein-Gordon equation. The doubling of the spectrum is a consequence of the fact that we started with a fourth derivative order equation of motion (2.44).

Since N_+ and N_- can take arbitrary values, namely, arbitrary number of positive and negative energy excitations can take place, the Hamiltonian of quantized *nondegenerate* higher derivative model is unbounded. In the sense of the unboundedness of energy, the similarity of this result to the classical analysis is indeed natural since the underlying canonical structure for both classical and quantum theory is the same. The *degenerate* case will be examined in Chapter 4.

CHAPTER 3

PROBLEMS

In the previous chapter, we showed that the Hamiltonian of a higher derivative theory is not positive valued. This makes interacting higher derivative theories necessarily unstable. The analysis of the higher derivative harmonic oscillator in the previous chapter was classical. People sometimes think that quantization might cure the problem of instability driven by an unbounded Hamiltonian. However, unlike the case of the Hydrogen atom, instability stays alive after quantization.

Ostrogradskian instability appears in interacting higher derivative quantum field theories such that the vacuum abruptly decays into some collection of positive and negative energy particles "for free", *viz.*, without violating the conservation of energy. Because of this, it is believed that *nondegenerate* higher derivative theories cannot be realistic models of the nature [10, 11].

Now let us recall the positive and negative "frequency" parts in the free field expansion of the solution of the Klein-Gordon equation. In that case, "positive frequency part" (which corresponds to the creation of negative energy particles) can be interpreted as the annihilation of positive energy particles to get rid of the negative energy problem. However, in the case of higher derivative harmonic oscillator, the Hamiltonian appears as the difference of energies of two harmonic oscillators (2.70), and therefore, according to the number of positive and negative excitations, it may take any positive or negative value. The result is irrelevant to the previous antiparticle interpretation this time, because here we consider not positive and negative frequencies, but truly positive and negative energies.

3.1 Nature of the Instability

During the earlier times of higher derivative theories, the issue of instability was not given much attention and the efforts were mainly concentrated on some stratagems to make energy positive valued, however all these efforts yielded null results nevertheless.

It is essential to make a stability analysis from the very first as the instability of a theory makes it usually, if not always[1], useless. As a matter of fact, for a "free" (no interaction) higher derivative theory, where the terms in the Lagrangian are at most quadratic (see (2.42)), there is no consideration of instability if the solutions are oscillations and if there is no external driving force, friction, *etc.* Recalling the example of the higher derivative oscillator, the system oscillates with two different frequencies, k_+ and k_-, with two oscillating solutions for each (2.59). Positive and negative energy "modes" or "particles" will live on in their own land since it's a linear theory without self or external interactions. However, we do not count non-interacting higher derivative quantum field theories as "viable" because there is no use of them in modeling our universe.

Unfortunately, the phrase "Hamiltonian is not bounded from below" led most of the people in the field to mix it up with the phrase "potential is not bounded below" and consequently incorrect conclusions claiming even free higher derivative theories to be unstable ensued. This is also the reason why we intentionally avoided pronouncing the phrase "Hamiltonian is not bounded from below" as usual in the literature, but we preferred "Hamiltonian is not positive valued" instead throughout the previous parts. Incorrect claims originating from this misunderstanding were debunked by Elizer and Woodard [10, 11] several times and this issue deserves further attention to get a better understanding of Ostrogradskian instability which higher derivative theories suffer from.

Consider first the *wrong harmonic oscillator* Lagrangian [20],

$$L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 \, , \tag{3.1}$$

where the *potential is not bounded from below* (Fig. 3.1) and the Hamiltonian is constant. The solution to this system is,

$$q(t) = Ae^{-\omega t} + Be^{\omega t} \, , \qquad \omega = \sqrt{\frac{k}{m}} \, . \tag{3.2}$$

[1] Unstable higher derivative theories are sometimes employed as a *mechanism*, for example, to explain the D-brane decay in gravity coupled p-adic string theory. (see [5, 19])

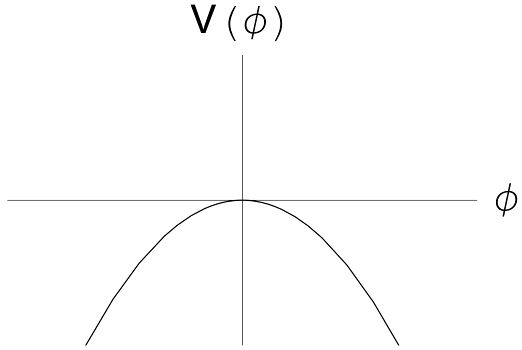


Figure 3.1: *Instability due to an unbounded potential.* Runaway solutions here cause instability and the instability is associated with the value of the dynamical variable, but not the energy. However, the Ostrogradskian instability is directly related to the lack of a positive valued Hamiltonian.

Obviously the solution has a *runaway* character due to the exponentials and the instability is associated with the value of the dynamical variable which gets higher and higher (or decays) by time. This is also the reason why the lowest order self-interacting term of scalar field theory is ϕ^4, but not ϕ^3.

However, the instability due to unbounded potentials is not the same as the Ostrogradskian instability which appears in higher derivative theories. Recall that our higher derivative oscillator (2.59) does not suffer from runaways; *i.e.*, the solution itself is stable. In higher derivative models the instability is due to the interaction which causes the excitation of both positive and negative energy degrees of freedom in the same time while conserving the total energy. So the problem in *interacting* higher derivative theories is that the dynamical variable becomes arbitrarily highly excited.[2] This means that the positive energy solution can be excited arbitrarily by exciting the negative energy solution. This is allowed because the total energy is conserved.

To fix the ideas associated with the Ostrogradskian instability in quantum field theory, it is useful to examine the case of the Hydrogen atom. Consider a Hydrogen atom in which electron-photon interactions are turned off for a moment (we do not bother how the Hydrogen atom could form in that case). Then the Schrödinger equation for the Coulomb potential will give the stationary states of Hydrogen atom. If there is no interaction between electron and photon fields, any excited state will

[2] Smilga [21] claimed that it is possible to attain a stable higher derivative theory by adjusting the parameters in front of the higher derivative terms in the interaction part of the Lagrangian. However, according to our numerical calculations done in `Mathematica`, it seems that the solutions sooner or later hit the singularity.

keep staying excited as all the states are stationary solutions and there is no leakage of wave function to other states. However, when the interactions are turned on, the photon field will perturb those states and therefore the previous solutions given by the Schrödinger equation will not be stationary states anymore. Putting aside the misleading idea “the system wants to decrease its energy”, the interaction will mix the stationary states and generate transitions between them [22]. As coined *spontaneous emission*, the wave function of the excited electron will leak to lower energy states and finally the electron will go down to the ground state by emitting photons. Once the photon is emitted, there is no way for the electron to go back to an excited state, because the energy will have already been delivered to the photon field (excitations by absorption of photons from the environment is irrelevant to what is mentioned here).

Now if we consider the *interacting* nondegenerate higher derivative quantum field theory, similar to the case of the Hydrogen atom, there is always some finite possibility that the vacuum decays into a collection of positive and negative energy particles while conserving the energy. The vacuum decay diagrams are determined by the interaction terms. The diagrams depicting scatterings in the lower derivative theory mostly will turn to vacuum decay diagrams with four external outgoing legs in the higher derivative generalization of the lower derivative theory.

Here it’s natural to ask whether the decay of vacuum might entail much longer time, for example, longer than the age of the universe. However, the vacuum can decay to the particles with arbitrarily high energies provided that the total energy of the particles sum to zero. Owing to this freedom it can be concluded that the vacuum evaporates into positive and negative energy particles *instantaneously* [11]. Further, the vacuum decay does not occur once in contrast to the particle decays.

As an example, consider a *higher derivative* “Yukawa” theory where the Lagrangian is given by,

$$L = L_{Dirac} + L_{Double\ Klein-Gordon} - g\bar{\psi}\psi\phi \ , \tag{3.3}$$

where

$$L_{Double\ Klein-Gordon} = -\frac{1}{2}\phi\left(\Box - m_1^2\right)\left(\Box - m_2^2\right)\phi \ . \tag{3.4}$$

From the previous discussions, it’s obvious that the ϕ field carries both negative and positive energy particles. Hence, the electron-positron pair creation diagram turn to the vacuum decay diagram in this model (Fig. 3.2). Note also that the Dirac part

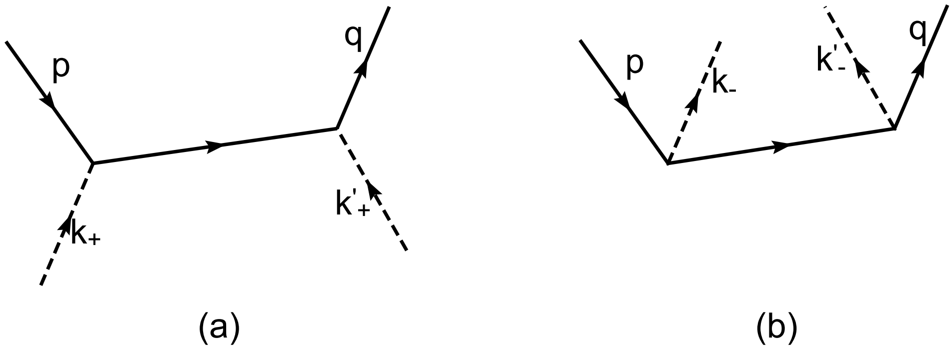


Figure 3.2: *Vacuum decay in the higher derivative Yukawa theory.* Scattering and pair annihilation/creation processes in usual theories turn out to be a vacuum decay process in higher derivative theories. In the usual Yukawa theory, two scalar particles are annihilated and two fermions are created (a). In the higher derivative generalization of the Yukawa theory, positive and negative energy scalar particles exist in the spectrum. The second diagram depicts the decay of the vacuum to two fermions and two negative energy scalar particles while conserving the total energy (b).

was not modified and therefore the fermions still carry only positive energy (3.3).

It is instructive to look at the classical analog of the vacuum decay in higher derivative ϕ^4 theory [23],

$$L = -\frac{1}{2}\phi\left(\Box - m_1^2\right)\left(\Box - m_2^2\right)\phi - \lambda\phi^4 \tag{3.5}$$

where $m_2 > m_1$. With the definitions

$$\psi_1 \equiv \frac{(\Box - m_2^2)\,\phi}{[2(m_2^2 - m_1^2)]^{1/2}}\ , \qquad \psi_2 \equiv \frac{(\Box - m_1^2)\,\phi}{[2(m_2^2 - m_1^2)]^{1/2}}\ , \tag{3.6}$$

the Lagrangian can be rewritten as

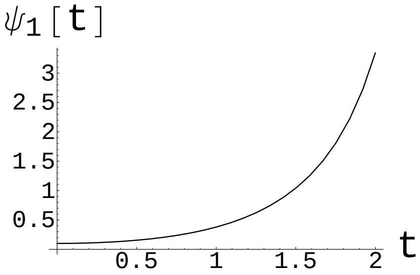
$$L = \frac{1}{2}\psi_1\left(\Box - m_1^2\right)\psi_1 - \frac{1}{2}\psi_2\left(\Box - m_2^2\right)\psi_2 - \frac{4\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^4 \tag{3.7}$$

The energy of the field ψ_2 is negative as it appears with the wrong sign in the action. The coupled equations of motion read

$$\left(\Box - m_1^2\right)\psi_1 - \frac{16\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^3 \ = \ 0\ , \tag{3.8}$$

$$\left(\Box - m_2^2\right)\psi_2 - \frac{16\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^3 \ = \ 0\ . \tag{3.9}$$

If there hadn't been any interactions (*i.e.* $\lambda = 0$), the negative energy field would not have caused a problem. However, the interaction term will couple ψ_1 and ψ_2 together and there will be a flow of negative energy from positive energy field ψ_1 to negative



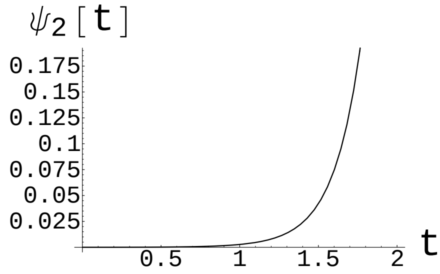


Figure 3.3: *Numerical solutions of the higher derivative ϕ^4 theory.* Space homogeneous (*i.e.* $\nabla^2\psi_1 = \nabla^2\psi_2 = 0$) numerical solutions of the equations (3.8) and (3.9) with the set of initial conditions $\psi_1(0) = 0.1, \dot{\psi}_1(0) = 0, \psi_2(0) = 0, \dot{\psi}_2(0) = 0$ show that both of the positive energy field ψ_1 and negative energy field ψ_2 increase exponentially without bound. The small difference in the values of the two graphs is due to the mass difference (*i.e.* $m_{\psi_1} \neq m_{\psi_2}$) of the two fields. These two graphs show how Ostrogradskian instability takes place at the classical level.

energy field ψ_2 by conserving the total energy. So the arbitrary excitation of the fields means that the theory is not stable. This can be easily seen through numerical solutions of equations (3.8) and (3.9) (Fig. 3.3).

A similar analysis of instability in interacting higher derivative systems was given by Nesterenko. In [24] a damping term $\gamma\frac{dq}{dt}$ is added to the equation of motion of the Pais-Uhlenbeck oscillator (see 2.42). Then it is shown that this damping term contributes a decaying part to the positive mode oscillations whereas it contributes an exponentially growing part to the negative mode oscillations. In another words, the damping term behaves like a sink for the positive mode solutions while it behaves like a source for the negative mode solutions. This result stems from the fact that Hamiltonian is unbounded.

This negative energy problem is manifest in quantum theory through ghost states. We can calculate the free field propagator for the Lagrangian (3.5),

$$G(p) = \frac{1}{(m_2^2 - m_1^2)}\left(\frac{1}{(p^2 + m_1^2)} - \frac{1}{(p^2 + m_2^2)}\right) . \tag{3.10}$$

Due to the minus sign in (3.10), we immediately recognize that one of the particles in the spectrum, m_1 or m_2, is a ghost. An important note is in order: This ghost is not the same kind of the Faddeev-Popov ghosts which cancel with unphysical components of the original fields; on the contrary, the ghost here exists on its own and it cannot be cancelled or excluded from the theory in any way [25].

Another point is that in the equal mass (or equal frequency) limit, (3.10) diverges and the Hamiltonian is no more diagonalizable, but "Jordan diagonalizable", which is similar to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{3.11}$$

This Jordan matrix has only one eigenvector $\propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Even though there are some $n \times n$ Jordon diagonalizable matrices which have n eigenvectors, for a higher derivative theory which employes the Jordon matrix (3.11), an immediate criticism follows stating that it is not possible to make a measurement in this theory where the Hamiltonian is Jordan diagonalized because there is not enough number of base vector to span the whole Hilbert space.

3.2 Indefinite Metric Stratagem

In the previous chapter, we showed that half of the degrees of freedom contributes negative energy to the Hamiltonian and quantization does not stabilize higher derivative theories (2.70). On the other hand, some people try to quantize the theory in a different way.

It's thought that one can treat the negative energy lowering operator as a positive energy raising operator. So in contrast to the formalism given in (2.57, 2.58), they [16] use

$$a\,|\overline{\Omega}\rangle = 0 \;, \tag{3.12}$$

$$b^\dagger\;|\overline{\Omega}\rangle = 0 \;, \tag{3.13}$$

with the usual commutation relations

$$[a, a^\dagger] = 1 = [b, b^\dagger] \;. \tag{3.14}$$

Hence positive N_+ and negative N_- energy states can be produced from vacuum by

$$|\overline{N_+}\rangle \;\; = \;\; a^\dagger\;|\overline{\Omega}\rangle \;, \tag{3.15}$$

$$|\overline{N_-}\rangle \;\; = \;\; b\,|\overline{\Omega}\rangle \;. \tag{3.16}$$

Using the commutation relations (3.14), we find

$$\left\langle \overline{N_+} | \overline{N_+} \right\rangle \;\; = \;\; \left\langle \overline{\Omega} | \overline{\Omega} \right\rangle \;, \tag{3.17}$$

$$\left\langle \overline{N_-} | \overline{N_-} \right\rangle \;\; = \;\; -\left\langle \overline{\Omega} | \overline{\Omega} \right\rangle \;. \tag{3.18}$$

Here the norm of the negative energy particles is negative. Furthermore, the reason why we didn't put 1 instead of $\left\langle \overline{\Omega} | \overline{\Omega} \right\rangle$ will be seen below. It's claimed that this formalism is advantageous as it makes the energy positive valued. To examine this claim, we can look at the vacuum expectation value of H from (2.70)

$$\begin{aligned}
\left\langle \overline{\Omega} | H | \overline{\Omega} \right\rangle &= k_+ \left\langle \overline{\Omega} | a^\dagger a | \overline{\Omega} \right\rangle - k_- \left\langle \overline{\Omega} | b^\dagger b | \overline{\Omega} \right\rangle \ , && (3.19)\\
&= k_+ \left\langle \overline{\Omega} | (a a^\dagger - 1) | \overline{\Omega} \right\rangle - k_- \left\langle \overline{\Omega} | b^\dagger b | \overline{\Omega} \right\rangle \ , && (3.20)\\
&= k_+ \left\langle \overline{N_+} | \overline{N_+} \right\rangle - k_+ \left\langle \overline{\Omega} | \overline{\Omega} \right\rangle - k_- \left\langle \overline{N_-} | \overline{N_-} \right\rangle \ , && (3.21)\\
&= k_+ \left\langle \overline{\Omega} | \overline{\Omega} \right\rangle - k_+ \left\langle \overline{\Omega} | \overline{\Omega} \right\rangle + k_- \left\langle \overline{\Omega} | \overline{\Omega} \right\rangle \ . && (3.22)
\end{aligned}$$

Here the energy seems to be positive valued. So this formalism renders positive norm negative energy states into negative norm positive energy states. To preserve unitarity of the S-matrix and to be able to make probabilistic interpretation, negative norm states should be excluded from the physical sector. For this, a new definition of norm in Hilbert space was proposed by means of an indefinite metric, but to date no one was able to solve the problem in a consistent way.

Besides, there is another problem with this formalism which is at least as important as the unitarity. If we solve (3.12) and (3.13) for the empty state wave function, we get

$$\overline{\Omega}(Q_1, Q_2) = N \exp\left[-\frac{m\sqrt{1-4g}}{2(k_- - k_+)} \left(k_+ k_- Q_1^2 - Q_2^2 \right) + i\sqrt{g} m Q_1 Q_2 \right] . \qquad (3.23)$$

Unfortunately the wave function does not fulfill the normalizability condition (2.67) this time. This means that we're no longer doing quantum mechanics and we are not able to obtain finite results from this theory. So it seems that there is no use of this formalism.

3.3 Perturbative Approach

Higher derivative terms increase the dimension of the phase space and the number of solutions irrespective of how they appear in the Lagrangian and this result is not altered when higher derivatives are present only in the interaction part with small coupling constants.

If higher derivative terms are gathered in the interaction part, then one can solve the equations perturbatively by the ansatz [10, 26],

$$q_{pert}(t) = \sum_{n=0}^{\infty} g^n q_n(t) \ . \tag{3.24}$$

As an example, let us consider the Lagrangian [10]

$$L = -\frac{mg(1-g)}{2\omega^2}\ddot{q}^2 + \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \tag{3.25}$$

for $0 < g < 1$. Note that here the constant term in front of $\ddot{q}^2$ is different from the one in (2.42) for further convenience. The Euler-Lagrange equation for (3.25) reads

$$\frac{g(1-g)}{\omega^2}q^{(4)} + \ddot{q} + \omega^2 q = 0 \ . \tag{3.26}$$

The solution is the same with (2.45), but the frequencies here are found as

$$k_+ = \frac{\omega}{\sqrt{1-g}} \qquad \text{and} \qquad k_- = \frac{\omega}{\sqrt{g}} \tag{3.27}$$

If we substitute the ansatz (3.24) into (3.26) and separate the terms with respect to powers of g, we get the following system of equations

$$\ddot{q}_0(t) + \omega^2 q_0(t) = 0 \ , \tag{3.28}$$

$$\ddot{q}_1(t) + \omega^2 q_1(t) = -\frac{1}{\omega^2} q_0^{(4)}(t) \ , \tag{3.29}$$

$$\vdots \tag{3.30}$$

$$\ddot{q}_n(t) + \omega^2 q_n(t) = -\frac{1}{\omega^2}\left(q^{(4)}_{(n-1)} - q^{(4)}_{(n-2)}\right) \ . \tag{3.31}$$

This system of equations can be written in a more compact form

$$\ddot{q}_k(t) = -\omega^2 \sum_{l=0}^{k} q_l(t) \ . \tag{3.32}$$

This is achieved by differentiating each equation and substituting it into the next equation so that the fourth derivative terms on the right-hand side of the equations are removed. If we multiply both sides of (3.32) by g^k and sum over k, we obtain

$$\sum_{k=0}^{\infty} g^k \ddot{q}_k(t) = -\omega^2 \sum_{k=0}^{\infty} g^k \sum_{l=0}^{k} q_l(t) \ . \tag{3.33}$$

The left-hand side of (3.33) is $\ddot{q}_{pert}(t)$ from (3.24). Interchanging the order of summations on the right-hand side yields

$$\ddot{q}_{pert}(t) = -\omega^2 \sum_{l=0}^{\infty} q_l(t) \sum_{k=l}^{\infty} g^k \tag{3.34}$$

$$= -\omega^2 \sum_{l=0}^{\infty} q_l(t) \frac{g^l}{1-g} \tag{3.35}$$

$$= -\frac{\omega^2}{1-g}\, q_{pert}(t) \ . \tag{3.36}$$

Hence we obtained $q_{pert}(t)$ by means of the perturbative ansatz and the Euler-Lagrange equation without any reference to initial conditions.

Now notice that the term $\frac{\omega^2}{1-g}$ in (3.36) is nothing but k_+^2. So we can conclude that the perturbation theory does not follow from the original theory since it respects the solutions with positive energy but discards the ones with negative energy. However, there is a *nonperturbative* amplitude for the system to decay to the higher excitations (*i.e.* Ostrogradskian instability). This eventually makes perturbation theory useless.

CHAPTER 4

VIABLE HIGHER DERIVATIVE THEORIES

In the previous chapter, it was shown that all *nondegenerate* higher derivative theories suffer from Ostrogradskian instability irrespective of the form of the Lagrangian. On the other hand, if the assumption of nondegeneracy, which is central for Ostrogradski's canonical formulation, is relinquished, then it may be possible to get a stable higher derivative theory.

If the system is *degenerate*, this means that one cannot solve $q^{(N)}$ in terms of P_N, q and the first $N-1$ derivatives of q (see 2.4 and 2.15) and therefore the Hamiltonian cannot be obtained by simply Legendre transforming the Lagrangian. Degeneracy implies the presence of a continuous symmetry (or symmetries). This continuous symmetry can be used to eliminate a dynamical variable by gauge fixing and this yields *constraints*.

A simple rule to check the stability of a *degenerate* higher derivative theory was given by Woodard [11]:

> *"If the number of gauge constraints is less than the number of unstable directions in the canonical phase space then there is no chance for avoiding the problem [of Ostrogradskian instability]."*

The number of unstable directions increases with higher derivative terms, but any continuous symmetry results fixed number of constraints. So constraints may stabilize only the theories which contain some fixed number of higher derivative terms. For example, the model of relativistic particle with a second derivative curvature term is stable whereas it is not so if a third derivative torsion term is included.

4.1 Relativistic Particle with "Curvature" and "Torsion"

Relativistic particle with a higher derivative curvature term which was worked out by Plyushchay [27] is the only hitherto known theory free of Ostrogradskian instability. In order to see that the third order additional torsion term produces tachyonic (negative mass) states, here we will make the calculations for the action of relativistic particle with both curvature and torsion terms and then we will show that tachyonic states in the mass spectrum disappear provided that the torsion term is excluded.

The second derivative curvature term in the Lagrangian is sometimes called "rigidity" and also studied in the context of string theory. Polyakov [28] introduced a second derivative term to the original Nambu-Goto action in order to give strings some kind of rigidity so as to prevent the "creasing" (or "crumpling") effect. In addition, rigidity is introduced with higher derivative terms both in the theory of cosmic strings and the theory of flexural vibrations of beams and rods in mechanical engineering [29].

The action of the relativistic particle of mass m with additional curvature and torsion terms is given by [30]

$$S = -m\int ds - \alpha \int k(s)\ ds - \beta \int \kappa(s)\ ds\ , \tag{4.1}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$; $k(s) = \left[\left(\frac{d^2x^\mu}{ds^2}\right)^2\right]^{1/2}$ and $\kappa(s) = \left[\left(\frac{d^3x^\mu}{ds^3}\right)^2\right]^{1/2}$ are curvature and torsion of the world curve of the particle, respectively, m is a constant in mass dimensions, and α and β are dimensionless constants.

For the parametrization $x^\mu = x^\mu(\tau)$, we use

$$\frac{d}{ds} \longrightarrow \frac{d\tau}{ds}\frac{d}{d\tau} \quad \text{and} \quad ds^2 = \left(\frac{dx^\mu}{d\tau}\right)^2 d\tau^2 = \dot{x}^2 d\tau^2\ , \tag{4.2}$$

to rewrite the action (4.1) in the form

$$S = -m\int d\tau\sqrt{\dot{x}^2} - \alpha\int d\tau \frac{((\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2)^{1/2}}{\dot{x}^2} - \beta\int d\tau \frac{\sqrt{\dot{x}^2}\ d}{(\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2}\ , \tag{4.3}$$

where

$$\dot{x} \equiv \frac{dx}{d\tau}, \quad d = \det(d_{\rho\sigma}), \quad d_{\rho\sigma} = x^{(\rho)\ \mu}x^{(\sigma)}_\mu, \quad x^{(\rho)} \equiv \frac{d^\rho x}{d\tau^\rho}, \quad \rho,\sigma = 1,2,3. \tag{4.4}$$

As a digression, we may look at the nonrelativistic limit of the model of relativistic particle with a curvature term. Here we will include the constant of speed of light c in the action in contrast to (4.3) so as to make the limiting procedure more evident.

$$S = -mc\int d\tau\sqrt{\dot{x}^2} - \alpha c\int d\tau \frac{((\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2)^{1/2}}{\dot{x}^2}\ , \tag{4.5}$$

Without loss of generality, we can take $x^0(\tau) = c\tau$, $\dot{x}^0(\tau) = c$, $\ddot{x}^0(\tau) = 0$ where the "overdot" denotes the differentiation with respect to τ as defined in (4.2). The nonrelativistic limit of the first part of the action (4.5) is well known. The nonrelativistic limit occurs when $\frac{\dot{\vec{x}}^2}{c^2} \ll 1$.

$$
\begin{aligned}
-mc\sqrt{\dot{x}^2} &= -mc\sqrt{\dot{x}^0\dot{x}^0 - \dot{\vec{x}}\dot{\vec{x}}} = -mc\sqrt{c^2 - \dot{\vec{x}}^2} && (4.6)\\
&= -mc^2\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}} = -mc^2(1 - \frac{1}{2}\frac{\dot{\vec{x}}^2}{c^2}) = -mc^2 + \frac{1}{2}m\dot{\vec{x}}^2 \ . && (4.7)
\end{aligned}
$$

Obviously the first term has no effect on the equation of motion as it is just a constant and the second term is the kinetic energy of a point particle.

As for the curvature term in (4.5), the nonrelativistic limit of it is as follows:

$$
\begin{aligned}
-\alpha c\frac{((\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2)^{1/2}}{\dot{x}^2} &= -\alpha c\frac{((\dot{x}^0\ddot{x}^0 - \dot{\vec{x}}\ddot{\vec{x}})^2 - (\dot{x}^0\dot{x}^0 - \dot{\vec{x}}\dot{\vec{x}})(\ddot{x}^0\ddot{x}^0 - \ddot{\vec{x}}\ddot{\vec{x}}))^{1/2}}{\dot{x}^0\dot{x}^0 - \dot{\vec{x}}\dot{\vec{x}}} && (4.8)\\
&= -\alpha c\frac{((\dot{\vec{x}}\ddot{\vec{x}})^2 - (c^2 - \dot{\vec{x}}\dot{\vec{x}})(-\ddot{\vec{x}}\ddot{\vec{x}}))^{1/2}}{c^2 - \dot{\vec{x}}\dot{\vec{x}}} && (4.9)\\
&= -\alpha c\left(\frac{(\dot{\vec{x}}\ddot{\vec{x}})^2}{(c^2 - \dot{\vec{x}}^2)^2} + \frac{\ddot{\vec{x}}^2}{(c^2 - \dot{\vec{x}}^2)}\right)^{1/2} && (4.10)\\
&= -\alpha c\left((\dot{\vec{x}}\ddot{\vec{x}})^2\frac{1}{c^4}\frac{1}{(1 - \frac{\dot{\vec{x}}^2}{c^2})^2} + \ddot{\vec{x}}^2\frac{1}{c^2}\frac{1}{(1 - \frac{\dot{\vec{x}}^2}{c^2})}\right)^{1/2} && (4.11)\\
&= -\alpha\left((\dot{\vec{x}}\ddot{\vec{x}})^2\frac{1}{c^2}\Big(1 + 2\frac{\dot{\vec{x}}^2}{c^2}\Big) + \ddot{\vec{x}}^2\Big(1 + \frac{\dot{\vec{x}}^2}{c^2}\Big)\right)^{1/2} && (4.12)\\
&= -\alpha\left(\ddot{\vec{x}}^2\Big(\frac{(\dot{\vec{x}}\ddot{\vec{x}})^2}{\ddot{\vec{x}}^2 c^2} + 1\Big)\right)^{1/2} = -\alpha\,|\ddot{\vec{x}}| \ . && (4.13)
\end{aligned}
$$

In the step (4.12) we used the assumption $\Big(1 + \frac{\dot{\vec{x}}^2}{c^2}\Big) \approx \Big(1 + 2\frac{\dot{\vec{x}}^2}{c^2}\Big) \approx 1$ and in the last step we adopted $\frac{(\dot{\vec{x}}\ddot{\vec{x}})^2}{\ddot{\vec{x}}^2 c^2} \ll 1$. Thus the nonrelativistic limit of the curvature term gives the magnitude of the acceleration of a point-like particle. Here notice that the result in (4.13) is not a total derivative and it contributes higher derivative terms to the equation of motion.

Going back to the main discussion of the relativistic model (4.3), the canonical variables are defined according to (2.14)

$$
q_1 = x, \;\; q_2 = \dot{x}, \;\; q_3 = \ddot{x} \ , \tag{4.14}
$$

$$
\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{x}} + \frac{d}{d\tau}\frac{\partial L}{\partial \ddot{x}} + \frac{d^2}{d\tau^2}\frac{\partial L}{\partial x^{(3)}} \ , && (4.15)\\
p_2 &= \frac{\partial L}{\partial \ddot{x}} - \frac{d}{d\tau}\frac{\partial L}{\partial x^{(3)}} \ , && (4.16)
\end{aligned}
$$

$$p_3 = \frac{\partial L}{\partial x^{(3)}} \,, \tag{4.17}$$

where p_3 is given explicitely by

$$p_3^\mu = \beta \, \frac{\sqrt{\dot{x}^2}}{(\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2} \sqrt{d} \, \sum_{\rho=1}^{3} d^{3\rho} x^{(\rho)\mu} \,. \tag{4.18}$$

Here $d^{\rho\sigma}$ is the matrix inverse to $d_{\rho\sigma}$ so that $d_{\rho\sigma}\, d^{\sigma\gamma} = \delta_\rho^\gamma$.

Notice that the power of $x^{(3)}$ is one in the term $\sqrt{d}$, whereas it is minus one in the summation part. So the right-hand side of (4.18) is homogeneous of degree zero and one cannot solve for $x^{(3)}$. This means that our model is singular and therefore some constraints should be used for the Hamiltonization. The associated continuous symmetry is invariance under reparametrizations

$$\tau \longrightarrow f(\tau) \quad \text{and} \quad x^\mu(\tau) \longrightarrow x'^\mu(\tau) \equiv x^\mu(f^{-1}(\tau)) \tag{4.19}$$

It is easy to check that the action (4.3) is invariant under this reparametrization.

From (4.18) we can deduce three primary constraints:

$$\overset{(1)}{\phi}_1 = p_3^2 + \beta^2(q_2^2/g) = 0 \,, \tag{4.20}$$

$$\overset{(1)}{\phi}_2 = p_3 q_2 = 0 \,, \tag{4.21}$$

$$\overset{(1)}{\phi}_3 = p_3 q_3 = 0 \,, \tag{4.22}$$

where $g = (q_2 q_3)^2 - q_2^2 q_3^2$ and the overscript "(1)" denotes that the constraint is "primary". According to Dirac's classification, primary constraints are the ones derived directly from the conjugate momentum.

En route to quantization, we first examine the invariant Casimir operators in the model. The angular momentum is given by

$$M_{\mu\nu} = \sum_{\alpha=1}^{3} (q_{\alpha\mu} p_{\alpha\nu} - q_{\alpha\nu} p_{\alpha\mu}) \,. \tag{4.23}$$

Since our theory is invariant under the Poincaré group of transformations, we consider the following Casimir operators of the Poincaré group:

$$C_1 = p_1^2 = p_{1\mu} p_1^\mu \,, \tag{4.24}$$

$$C_2 = W = -w_\mu w^\mu = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} p_1^2 - (M_{\mu\sigma} p_1^\mu)^2 \,, \tag{4.25}$$

where $w_\mu = (1/2)\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p_1^\sigma$ is the Pauli-Lubanski pseudovector. Here the mass corresponds to the Casimir operator (4.24) whereas the spin corresponds to the Casimir operator (4.25).

If we go to the rest frame where $p_1^\mu = (p_1^0 = M, \vec{p}_1 = 0)$, then we have,

$$W = (p_1^0)^2 M_{12} M^{12} = M^2 \, \frac{C(SO(2))}{2} \, , \tag{4.26}$$

where $C(SO(2))$ is the squared Casimir operator of the SO(2) group which is equal to $2j^2$ [31] [1]. The representations of the group of rotations on the plane $SO(2)$ are integers, half-odd-integers, and continuous values for spin j. The eigenvalues of the Casimir operator W are

$$W = M^2 j^2, \qquad M^2 > 0, \quad j \geq 0 \, . \tag{4.27}$$

The state vectors $|\psi\rangle$ are eigenvectors of the operators (4.24) and (4.25)

$$\begin{aligned} p_1^2 \; |\psi\rangle \;\; &= \;\; M^2 \, |\psi\rangle \;\; , && (4.28) \\ W \; |\psi\rangle \;\; &= \;\; M^2 j^2 \, |\psi\rangle \;\; . && (4.29) \end{aligned}$$

Here the constraint $\overset{(1)}{\phi}_1$ (4.20) will give the wave equation. But we first fix the gauge (recall the reparametrization invariance (4.19)) by $q_2 q_3 = 0$ and subsequently the constraint (4.20) becomes $\overset{(1)}{\phi}_1 = (p_3^2 q_3^2 - \beta^2)$. Note that this is indeed the proper time gauge, $x^0(\tau) = \tau$, considering (4.14). So the wave function is given by

$$\overset{(1)}{\phi}_1 \; |\psi\rangle = (p_3^2 q_3^2 - \beta^2) \, |\psi\rangle \;\; . \tag{4.30}$$

The left-hand side of (4.30) can be expressed in terms of the Casimir operators p_1^2 and W. According to the calculation done in the appendix of [30], we have

$$1 - \chi = \frac{W - (\alpha\sqrt{m^2 - p_1^2} + |\beta| \, m\sqrt{\chi})^2}{\beta^2 p_1^2} \, \frac{\chi}{1 + \chi} \, , \tag{4.31}$$

where $\chi = \beta^2/(p_3^2 q_3^2)$. So (4.30) becomes

$$(1 - \chi) \;\; |\psi\rangle = 0 \; , \tag{4.32}$$

which is equivalent to

$$\left[W - (\alpha\sqrt{m^2 - p_1^2} + |\beta| \, m)^2 \right] \;\; |\psi\rangle = 0 \; . \tag{4.33}$$

[1] We concluded this result in the rest frame but in fact the result is frame independent, because the subgroup of the Poincaré group that leaves p^μ invariant has the same structure in all frames. This subgroup is called the *little group* and in this case it's a rotation group. So the representations of the rotation group give the representations of the Lorentz group for a timelike state [32].

Substituting the operators (4.28) and (4.29) into (4.33) yields

$$M^2 j^2 = (\alpha\sqrt{m^2 - M^2} + |\beta|\, m)^2 \ . \tag{4.34}$$

To obtain the mass spectrum, we solve (4.34) for the ratio M^2/m^2

$$\frac{M^2}{m^2} = \frac{\alpha^2(j^2+\alpha^2) + \beta^2(j^2-\alpha^2) \pm 2\,|\beta|\,\sqrt{j^2\alpha^2(j^2+\alpha^2-\beta^2)}}{(j^2+\alpha^2)^2} \ . \tag{4.35}$$

Obviously tachyonic solutions are allowed in the spectrum. For example, if $j = 0$ then (4.35) becomes

$$\frac{M^2}{m^2} = 1 - \frac{\beta^2}{\alpha^2} \ , \tag{4.36}$$

where tachyonic states appear if $|\beta| > |\alpha|$.

However, if we set $\beta = 0$ in (4.34); *i.e.*, kill the torsion term in the action (4.1), we end up with a positive valued mass spectrum of a relativistic particle with second order curvature term

$$\frac{M^2}{m^2} = \frac{1}{1 + j^2/\alpha^2} \ . \tag{4.37}$$

It's also shown in [33] that the torsion term leads to tachyonic solutions. Thus the only hitherto known higher derivative theory free of tachyons is the relativistic particle with a curvature term.

4.2 Nonlocal Theories

Nonlocality is the case where objects or fields in different points of the spacetime interact *directly* with each other without any mediator between them. Nonlocality appears in diverse areas of physics such as quantum field theory, string theory, noncommutative geometry, regularization techniques and even cosmology (see references in [11]). The point is that if a nonlocal theory can be represented as the limit of a sequence of higher derivative theories, then the Ostrogradskian instability is unavoidable. String theory for which such a representation is possible and nonlocal actions in which terms such as $q(t)$ and $q(t+\tau)$ appear [2] hold this instability [10, 12]. Higher derivative limit of a nonlocal theory can be reached if the action is analytic in derivatives.

As an illustration, consider the analytic higher derivative Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q \exp\left(-\frac{g}{\omega^2}\frac{d^2}{dt^2}\right) q \ , \tag{4.38}$$

[2] These kind of equations are studied also by mathematicians under the name of *delay differential equations*.

where g is real and (4.38) becomes the usual harmonic oscillator for $g = 0$. The associated Euler-Lagrange equation is,

$$\ddot{q} = -\omega^2 \exp\left(-\frac{g}{\omega^2}\frac{d^2}{dt^2}\right) q \ . \tag{4.39}$$

After imposing a solution of the form

$$q(t) = e^{i\kappa\omega t} \ , \tag{4.40}$$

we obtain the equation

$$\kappa^2 = e^{\,g\kappa^2} \ . \tag{4.41}$$

Setting $\kappa^2 = a + ib$ yields

$$a + ib = e^{ga}e^{igb} \ . \tag{4.42}$$

Taking the square of both sides of (4.42) and solving for b yields

$$b = \pm(e^{2ga} - a^2)^{1/2} \ . \tag{4.43}$$

Substituting (4.43) into (4.42), for positive values of b we obtain

$$a = e^{ga}\cos\left[g\left(e^{2ga} - a^2\right)^{1/2}\right] \ . \tag{4.44}$$

For a large and positive, (4.43) and (4.44) become

$$b \;\approx\; e^{ga} \ , \tag{4.45}$$

$$a \;\approx\; e^{ga}\cos\left(ge^{ga}\right) \ . \tag{4.46}$$

The left-hand side of (4.46) is order of a whereas e^{ga} term exists in the right-hand side. Hence the value of the cosine function should be very close to zero for (4.46) to be valid. Since zeros of the cosine function are at $\frac{(2N+1)}{2}\pi$ for N integer, we conclude

$$ge^{ga} \;\sim\; \frac{(2N+1)}{2}\pi \ , \tag{4.47}$$

$$b \;\sim\; \frac{1}{g}\frac{(2N+1)}{2}\pi \ . \tag{4.48}$$

Consequently we end up with

$$\kappa^2 \sim \frac{1}{g}\ln\left[\frac{1}{g}\frac{1}{2}(2N+1)\pi\right] \pm i\,\frac{1}{g}\frac{1}{2}(2N+1)\pi \ . \tag{4.49}$$

There are infinitely many solutions due to the exponential term in the Lagrangian (4.38), which is actually nothing but an infinite sequence of higher derivatives in disguise. The energy will obviously be unbounded if we consider the Hamiltonian which depends on infinite number of canonical momenta linearly.

There are runaway solutions in addition to the oscillations as the square root of κ^2 has real and imaginary parts; nevertheless, these runaway solutions are not necessarily associated with the Ostrogradskian instability [3].

On the other hand, if there is a nonanalytic dependence upon the derivative operator, the theory may survive [10]. One example to it is

$$L[q] = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \frac{1}{2}g^2\omega^2 q\left(\frac{\omega^2}{d^2/dt^2 + \omega^2}\right)q \; , \tag{4.50}$$

which the Lagrangian depends upon inverse differential operators.

Owing to the poles at $d/dt = \pm i\omega$, higher derivative limit is not applicable. This kind of nonlocality is obtained by integrating out one or more dynamical variables and it does not change the stability or instability of the original theory. This sort of nonlocality was named *derived* nonlocality [10]. The Lagrangian (4.50) is actually derived from

$$L'[q,x] = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - g\omega^2 xq \; . \tag{4.51}$$

To derive (4.50) from (4.51) one first obtains the equation of motion for x

$$\left[\frac{d^2}{dt^2} + \omega^2\right]x = -g\omega^2 q \implies x = -g\omega^2\frac{1}{d^2/dt^2 + \omega^2}q \; . \tag{4.52}$$

In (4.51), we can make the substitution $\dot{x}^2 \rightarrow -x\frac{d^2x}{dt^2} + \frac{d}{dt}(x\frac{dx}{dt})$ and can ignore the $\frac{d}{dt}(x\frac{dx}{dt})$ term as it is a total derivative and does not contribute to the equataion of motion. Hence, (4.51) can be written in the form

$$L'[q,x] = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 - \frac{1}{2}x\left[\frac{d^2}{dt^2} + \omega^2\right]x - g\omega^2 xq \; . \tag{4.53}$$

So it can be readily checked that substitution of (4.52) into (4.53) gives (4.50). Obviously there is neither negative energy solutions nor runaways since the original theory is nothing but two coupled harmonic oscillators.

Therefore, if a nonanalytic dependence of the Lagrangian upon derivative operators is present, we may have a viable higher derivative theory.

[3] Recall that in section 3.1 we distinguished the Ostrogradskian instability and the instability caused by unbounded potentials. The Ostrogradskian instability is associated with the runaway solutions which take place if interactions are allowed. However, runaways due to unbounded potentials may appear in usual lower derivative theories even without any interaction.

CHAPTER 5

CONCLUSION

In this work, it is shown that all *nondegenerate* higher derivative theories suffer from Ostrogradskian instability. There is no way to circumvent this problem since the Hamiltonian of higher derivative theories is not positive valued and this result is independent of the form of the Lagrangian.

In contrast to the case of the Hydrogen atom, quantization here does not remove the instability. This is indeed expected because the emergence of Ostrogradskian instability is due to the interaction between positive and negative energy modes (or particles). The Ostrogradskian instability is manifest both at the classical and quantum levels and it causes the solutions to become arbitrarily highly excited.

To date no one has been able to find a way to get rid of the negative energy problem. It should be emphasized that we mean "truly" negative energy, not negative frequency. It was also seen that quantizing a higher derivative theory in a somewhat different way by reinterpreting the negative energy ladder operators seems to produce a positive valued energy, however, negative norm states spring up this time. Also the wave functions are not normalizable in this "wrongly" quantized theory.

When a higher derivative field theory is considered, we encounter the problem of ghost particles with negative energies. Thus the decay of the vacuum into some collection of positive and negative energy particles is allowed as the total energy is conserved. We can conclude that scattering and pair annihilation/creation processes in usual lower derivative theories turn to the vacuum decay process in higher derivative theories. Hence, an unstable vacuum makes these theories useless.

The spectrum of higher derivative models include both positive and negative energy modes (or particles). However, the perturbative approach discards the negative energy modes and therefore it becomes useless since there is always a nonperturbative amplitude for the system to decay.

As for the *degenerate* case, the model of the relativistic particle with a higher derivative curvature term is free of the Ostrogradskian instability. A simple rule applies to check the stability of degenerate theories: The number of gauge constraints should be more than the number of unstable directions in the canonical phase space.

Since the number of constraints due to a continuous symmetry is fixed, adding more higher derivative terms increase the dimension of the phase space and also the instable directions on it. We showed that third order derivate torsion term spoils the stability of the model of the relativistic particle with a curvature term and in the presence of a torsion term, we find negative energy tachyons in the spectrum.

Besides, an example is given for the *derived* higher derivative theories which are obtained by integrating out one or more dynamical variables from the Lagrangian.

We have not touched the higher curvature gravity theories in this thesis, however, studies in this area are still going on. For example, it is proposed that higher curvature terms may mimic dark energy. The question of the stability of higher derivative theories on anti-de Sitter spacetime seems to be an open problem.

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