ON FORWARD INTEREST RATE MODELS: VIA RANDOM FIELDS AND MARKOV JUMP PROCESSES

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Abstract

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The essence of the interest rate modeling by using Heath-Jarrow-Morton framework is to find the drift condition of the instantaneous forward rate dynamics so that the entire term structure is arbitrage free. In this study, instantaneous forward interest rates are modeled using random fields and Markov Jump processes and the drift conditions of the forward rate dynamics are given. Moreover, the methodology presented in this study is extended to certain financial settings and instruments such as multi-country interest rate models, term structure of defaultable bond prices and forward measures. Also a general framework for bond prices via nuclear space valued semi-martingales is introduced.

Keywords: Term Structure of Interest Rates, Forward Interest Rates, HJM Framework, Nuclear Space.

ÖZ

İLERİ TARİHLİ FAİZ ORANLARI ÜZERİNE:RASSAL ALANLAR VE MARKOV SIÇRAMA SÜREÇLERİ

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Heath-Jarrow-Morton çerçevesi içinde yapılan faiz oranı modellerinin en önemli tarafı ileri tarihli faiz oranlarının dinamiğinin sürüklenme koşulunun verim eğrisininde arbitrajı engelleyecek şekilde bulunmasıdır.Bu çalışmada, anlık ileri tarihli faiz oranları rassal alanlar ve Markov sıçrama süreçleri ile modellenmiş ve anlık ileri tarihli faiz oranlarının dinamikleri ile ilgili şartlar gösterilmiştir. Ayrıca, burada öne sürülen methodoloji, çoklu ülke faiz modeli, temerrüde düşebilen bono fiyatlaması ve ileri tarih ölçüleri gibi finansal durumlara ve enstrümanlara da uygulanmıştır.Son olarak, bono fiyatlarının nükleer uzay değerli yarımartingaleler ile modellenmesini gösteren genel bir çerçeve sunulmuştur.

Anahtar Kelimeler:Faiz Oranları,İleri tarihli faiz oranları,HJM modeli,Nükleer uzaylar.

To everyone in my life.

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Chapter 1

Introduction

Modeling the term structure of interest rates has always been an intriguing activity for both scholars and practitioners. The quest for understanding the variations in interest rates should not be a surprise to us if we consider the economic and financial environment that we live in. Every economic activity more or less has some sort of relation with interest rates. Hence the understanding of the behavior of this phenomena is important for gaining insights about the mainstream economic activities. These economic activities may vary from macroeconomically oriented ones such as the conduct of monetary policy, the financing of public debt or forming expectations about real economic activities and inflation, to more microeconomically oriented ones such as risk management and pricing of interest rates. In this work, our motivation can be relatively placed on the microeconomics side of the continuum. If it is needed to have a further classification, this work can be placed on the shelves of the mathematical finance part of interest rate modeling, which most of the time deals with the understanding of the stochastic behavior of interest rates. Having positioned our work in the interest rate literature, it is advisable to address our specific concerns that we try to deal with in this work.

In this work, the term structure of interest rates are investigated in a disparate fashion yet an unified construction is tried to be given at the end. The disparity of this work mainly stems from the abundance of the previous works in term structure modeling. Hence in order to capture this fact, we try to give a complete picture of interest rate modeling in a concise manner and also try to give a modest contribution to the existing literature. Our contribution is within the track of identifying the conditions that preclude arbitrage opportunities in our proposed settings. Our settings primarily fall into the category of *infinite dimensional* and *jump-augmented* models of the term structure of interest rates. While doing this, we extensively use the framework proposed by Heath, Jarrow and Morton (HJM) [26], in which the instantaneous forward rates and the volatility structures of them are utilized. The main result of HJM [26] is that in order to preclude the arbitrage opportunity the drift term of the instantaneous forward rate dynamics should be related to the volatility term of the forward rate dynamics. This relation is cited as HJM drift condition in the literature. In this work, firstly, we give drift conditions for the forward interest rate dynamics generated by a jump-diffusion process, in which the jumps are generated by a Markov Jump Process. The second contribution of this work is to give extensions of the certain financial settings and instruments such as multi-country interest rate models, term structure of defaultable bond prices and forward measures, by positing the forward rates as a two-parameter random field proposed by Korezlioglu [38]. The last but not least contribution is to give a general framework for pricing of zero coupon bonds by modeling the instantaneous forward rates as a nuclear space valued semi-martingales.

The organization of this study is as follows. In Chapter 2, the basic definitions and concepts related to the interest rate modeling and risk-neutral pricing are given. Then the existing interest rate models are investigated in Chapter 3. In Chapter 4, the HJM framework that is frequently employed in this paper is reviewed in detail with its necessary extensions such as Musiela parameterization. In Chapter 5, we introduce the Markov Jump augmented interest rate model and characterize the drift condition associated with the model. Beginning with Chapter 6, we turn to the term structure of interest rates modeled by random fields by presenting an extensive survey on random field models and giving the details of our modeling approach that we use in Chapter 7. In Chapter 7, we present the applications of random field models that we proposed as extensions to the existing models. And finally, in Chapter 9, as a unifying framework for the cases that we discussed in previous chapters, pricing of zero-coupon bonds in a nuclear space framework is postulated.

Chapter 2

Preliminaries

2.1 Basics of Interest Rate Modelling

In this section, we review some basic concepts of interest rate modeling. The primary objects of the investigation are different notions of interest rates such as spot or forward rates and concepts pertaining to the joint evolution of interest rates, namely fundamental interest rate curves such as yield curve or term structure of interest rates.

The first definition we consider is the definition of a bank account, or money-market account. A bank account characterizes a locally riskless investments, in which profit accrued continuously at the risk-free rate prevailing in the market at every instant.

Definition 2.1.1 (Bank Account)**.** We denote the value of a bank account at time $t \geq 0$ as $B(t)$. It is assumed that $B(0) = 1$ and that the bank account evolves according to the following differential equation

$$dB(t) = r(t)B(t)dt, \qquad B(0) = 1. \tag{2.1}$$

where $r(t)$ is the instantaneous rate at which the bank account accrues. It should be noted that $r(t)$ is a positive function of time in this setting. As a consequence

$$B(t) = \exp\left(\int_0^t r(s)ds\right). \tag{2.2}$$

Remark 2.1.1. The instantaneous rate, $r(t)$, is usually referred to as *instantaneous spot rate* or as *short rate*.

It is clear from the above definition that the value of one unit of currency payable at time T, as of time t, is $\frac{B(t)}{B(T)}$. Hence we have another fundamental definition.

Definition 2.1.2 (Discount factor)**.** The discount factor $D(t,T)$ between two time instants t and T, $t \leq T$, is the amount at time t equivalent to one unit of currency payable at time T, and is given by

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(s)ds\right). \tag{2.3}$$

Remark 2.1.2. Since our primary concern is to model the variability of interest rates themselves, apart from classical Black-Scholes modeling of contingent claims in which interest rate is assumed to be deterministic, the evolution of r is a stochastic process, consequently making the bank account and the discount factors stochastic processes.

Next we turn to the basic instrument in interest rate markets, zero coupon bond.

Definition 2.1.3 (Zero-coupon bond)**.** A T maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at $t < T$ is denoted by $P(t,T)$. From the setting, $P(T,T) = 1$ for all T.

Zero-coupon bond prices are the basic quantities in interest rate theory, and all interest rates can be defined in terms of any given family of interest rates. However, we should bear in mind that although interest rates are always quoted in financial markets, zero-coupon bonds are relatively theoretical instruments. Most of the time zero-coupon bond prices are stripped out from more complex interest rate instruments such as coupon bearing bonds and callable bonds.

In the following, we give definitions of spot rates, which are necessary for computing zero-coupon bond prices.

Definition 2.1.4 (Continuously compounded spot interest rate)**.** (Yield) The continuously compounded spot interest rate at time t for the maturity T is denoted by $Y(t,T)$ and is the constant rate at which an investment of $P(t,T)$ units of currency at time t accrues continuously yield one unit of currency at maturity T. Mathematically,

$$Y(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{2.4}$$

the price of the zero-coupon bond with maturity T can be given as,

$$P(t,T) = e^{-Y(t,T)(T-t)}. \tag{2.5}$$

Another compounding type that should be mentioned is the simple compounding, in which London Interbank Offer Rates (LIBOR) are quoted.

Definition 2.1.5 (Simply compounded spot interest rate)**.** The simply compounded spot interest rate prevailing at t for the maturity T is denoted by $L(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when accruing occurs proportionally to the investment time. In mathematical terms,

$$L(t,T) = \frac{1 - P(t,T)}{(T-t)P(t,T)}. \tag{2.6}$$

The bond price can be expressed in terms of LIBOR or simply compounded spot interest rate as

$$P(t,T) = \frac{1}{1 + L(t,T)(T-t)}. \tag{2.7}$$

A further compounding method is n-times per year compounded spot interest rate. n-times per year compounding is obtained as follows. If we invest today one unit of currency at the simply compounded rate R, in one period we will obtain the amount $(1+\frac{R}{n})$. After this period, we invest such an amount for one more period and we have $(1+\frac{R}{n})^2$. If we keep on reinvesting for n period, final amount is $(1+\frac{R}{n})^n$. Based on this, we have

Definition 2.1.6 (n-times per year compounded spot interest rate)**.** The n-times per year compounded interest rate at time t for the maturity T is denoted by $R^n(t,T)$ and is the constant rate at which an investment has to be made to produce of an amount one unit currency at maturity, starting from $P(t,T)$ units of currency at time t, when reinvesting the obtained amounts n times a year. That is,

$$R^n(t,T) = \frac{n}{P(t,T)^{\frac{1}{n(T-t)}}} - n. \tag{2.8}$$

For $n = 1$, R is referred to an annually-compounded spot interest rate.

Remark 2.1.3. If number n of compounding times goes to infinity, $Y(t,T)$ is obtained.

$$\lim_{n\to\infty} \frac{n}{P(t,T)^{\frac{1}{n(T-t)}}} - n = -\frac{\ln(P(t,T))}{(T-t)} = Y(t,T) \tag{2.9}$$

Moreover, the instantaneous spot rate or short rate is equivalent to above definitions in infinitesimal time intervals. In other words, the short rate can be obtained as a limit of all the different spot rates.

$$\begin{aligned} r(t) &= \lim_{T\to t} Y(t,T), \\ &= \lim_{T\to t} L(t,T), \\ &= \lim_{T\to t} R(t,T). \end{aligned} \tag{2.10}$$

The next fundamental concept we consider is the forward interest rate, which is characterized by three time instants, namely time t at which the rate is considered, its expiry time T and its time of maturity S, with $t \leq T \leq S$. In order to define forward rate, we need to describe a plain vanilla forward rate agreement (FRA) since the notion of forward rate is closely tied to the value of this instrument.

A FRA is a financial contract that involves three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period between T and S. According to this contract, a fixed payment based on a fixed rate K is exchanged against a floating based on the LIBOR rate, $L(T,S)$, at the time of maturity. Formally, at time S, one receives $(S-T)KN$ units of currency and pays the amount $(S-T)L(T,S)N$ where N denotes the notional amount of the contract. The value of the contract in S is by assuming $N = 1$ is

$$(S-T)(K - L(T,S)). \tag{2.11}$$

By using the definition of the LIBOR rate, (2.11) can be rewritten as

$$(S-T)K - \frac{1}{P(T,S)} + 1. \tag{2.12}$$

The value of the term $\frac{1}{P(T,S)}$ at T is equal to 1, which in turn equals $P(t,T)$ at time t. Similarly, the remaining terms are equal to $P(t,S)(S-T) + P(t,S)$ at time t. Therefore the total value of the FRA is

$$FRA = P(t,S)(S-T)K - P(t,T) + P(t,S). \tag{2.13}$$

In order to preclude arbitrage, this contract's value has to be zero at time t. By equating this value to zero and solving for K, the simply compounded forward rate is defined.

Definition 2.1.7. The simply-compounded forward interest rate prevailing at time t for expiry $T > t$ and maturity $S > T$ is denoted by $F(t,T,S)$ and is defined by

$$F(t,T,S) := \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{2.14}$$

In this work, we most of the time deal with the instantaneous counterpart of the forward rate, which is intuitively a forward interest rate at time t whose maturity is close to its expiry T. In other words, if the maturity of the forward rate collapses towards its expiration date we have the instantaneous forward rate.

$$\begin{aligned} \lim_{S\to T} F(t,T,S) &= -\lim_{S\to T} \frac{1}{P(t,S)} \frac{P(t,S)-P(t,T)}{S-T}, \\ &= -\frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T}. \\ &= -\frac{\partial \ln P(t,T)}{\partial T}. \end{aligned} \tag{2.15}$$

Definition 2.1.8 (Instantaneous forward interest rate)**.** The instantaneous forward interest rate prevailing at time t for the maturity $T > t$ is denoted by $f(t,T)$ and is defined as

$$f(t,T) := \lim_{S\to T} F(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T}, \tag{2.16}$$

and therefore price of the zero coupon bond is given by

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right). \tag{2.17}$$

The most important concept in interest rate modeling is the relationship between interest rate and the time to maturity (time of maturity). A simple, yet strong tool that shows this relationship is the yield curve (also referred to term structure of interest rates or zero-coupon curve). Although, there is a slight ambiguity in the literature about which interest rate is used for depicting the relationship between interest rates and time to maturity, market convention is to plot simply-compounded interest rates for all maturities T up to one year, and annually compounded rates for maturities T larger than one year. Therefore, we have

Definition 2.1.9 (Term Structure of Interest Rates)**.** The term structure of interest rates (also referred to "yield curve" or "zero-coupon curve") at time t

is the graph of the function

$$T \mapsto \begin{cases} L(t,T) & \text{if } t < T \leq t+1 \text{ years} \\ R(t,T) & \text{if } T > t+1 \text{ years.} \end{cases}$$

Yield curves can display a wide variety of shapes. Typically, a yield curve will slope upwards, with longer term rates being higher, though certain economic conditions imply inverted yield curves where longer rates are less than shorter ones. Therefore, a variety of shapes and changes need to be described by a robust interest rate model.

2.2 Basics of Risk-Neutral Pricing

-Ex nilhilo nihil fit-[1]

The main assumption in the seminal paper by Black and Scholes [9] is the absence of the arbitrage in the financial markets. Intuitively, this assumption is equivalent to the impossibility of investing zero today and receiving a nonnegative amount that is positive with positive probability. The absence of arbitrage hence requires that the two portfolios having the same payoff at a given future date have the same price today. The morale of the Black-Scholes theory concludes that a portfolio constructed suitably should have the same instantaneous return as that of risk free investment. The first work to develop a model for the evolution of the term structure of interest rates is done by Vasicek [58] using Black-Scholes's arguments. Before reviewing the models of Vasicek and others, in this section we give a summary of risk-neutral pricing by following Harrison and Kreps [23], Harrison and Pliska [24, 25], Lamberton and Lapeyre [42], and Brigo and Mercurio [12].

Note 2.2.1. *This construction is generally valid for models that we considered in this work. However, deviations from this context, if any, will be stated where it is needed in the following chapters.*

Let us define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a right continuous filtration $\mathfrak{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$. $N + 1$ non-dividend paying securities are traded continuously from time 0 until time T in the considered financial market. Their prices are modeled by $N + 1$ dimensional adapted semi-martingale $S = \{S_t :$

[1] Latin proverb meaning "nothing comes out nothing"

$0 \leq t \leq T\}$, whose components are positive. The asset indexed by 0 is a bank account. Its price hence evolves according to

$$dS_t^0 = r(t)S_t^0 dt$$

with $S_0^0 = 1$ and $r(t)$ is the instantaneous short rate. $S_t^0 = B(t)$ and $1/S_t^0 = D(0,t)$ by using the previous section's notation.

Definition 2.2.1. A trading strategy is ($N+1$ dimensional) process $\phi = \{\phi_t : 0 \leq t \leq T\}$, whose components ϕ^0, ϕ^1, ..., ϕ^N are locally bounded and predictable. The value process associated with a strategy ϕ is defined by

$$V_t(\phi) = \phi_t S_t = \sum_{n=0}^{N} \phi_t^n S_t^n, \tag{2.18}$$

and the gains process associated with a strategy ϕ by

$$G_t(\phi) = \int_0^t \phi_u dS_u = \sum_{n=0}^{N} \int_0^t \phi_u^n dS_u^n, \qquad 0 \leq t \leq T. \tag{2.19}$$

Definition 2.2.2. A trading strategy is self-financing if $V(\phi) \geq 0$ and

$$V_t(\phi) = V_0(\phi) + G_t(\phi), \qquad 0 \leq t \leq T.$$

Roughly speaking, a strategy is self-financing if its value changes only due to changes in asset prices. In other words no additional cash inflows or outflows occur after the initial value. Similarly it can easily seen that the above relation holds for discounted processes [24, 25].

Proposition 2.2.1. *Let ϕ be a trading strategy. Then ϕ is a self-financing strategy if and only if*

$$\widetilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_u d\widetilde{S}_u$$

where $\widetilde{V}_t(\phi) = D(0,t)V_t(\phi)$ and $\widetilde{S}_t = D(0,t)S_u$.

From these definitions and proposition, the arbitrage opportunity and the equivalent martingale measure (risk-neutral measure) can be defined as follows [23, 24].

Definition 2.2.3. An arbitrage opportunity is a self-financing strategy ϕ such that $V_0(\phi) = 0$ but $\mathbb{P}(V_T(\phi) > 0) > 0$.

Definition 2.2.4. An equivalent martingale measure Q is a probability measure on the space $(\Omega, \mathcal{F})$ such that

1. $\mathbb{P}$ and Q are equivalent measures, that is $\mathbb{P}(A) = 0$ if and only if $Q(A) = 0$, for every $A \in \mathcal{F}$.
2. the Radon-Nikodym derivative $\frac{dQ}{d\mathbb{P}}$ belongs to $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
3. the discounted asset price process $\widetilde{S}$ is an $(\mathfrak{F}, Q)$ martingale, that is $E^Q(\widetilde{S}_t^n)|\mathcal{F}_u) = \widetilde{S}_u^n$, for all $n = 1, 2, ..., N$ and all $0 \leq u \leq t \leq T$.

Harrison and Pliska [25] proved the fundamental result that the existence of an equivalent martingale measure implies the absence of arbitrage opportunities and provides the mathematical characterization of the unique no-arbitrage price of any attainable contingent claim.

Definition 2.2.5. A contingent claim is square-integrable and positive random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. A contingent claim H is attainable if there exists some self-financing ϕ such that $V_T(\phi) = H$. Such a ϕ is said to generate H, and $\Pi_t = V_t(\phi)$ is the price at time t associated with H.

Proposition 2.2.2. *Assume there exists an equivalent martingale measure Q and let H be an attainable contingent claim. Then for each time t, $0 \leq t \leq T$, there exists a unique price Π_t associated with H.*

$$\Pi_t = E^Q(D(t,T)H \mid \mathcal{F}_t). \tag{2.20}$$

If the set of all equivalent martingale measures is nonempty, then it is possible to derive the unique no-arbitrage price associated with any attainable contingent claim. And such a price is given by the expected value of the discounted payoff under the measure Q. Harrison and Pliska [24, 25] also proved that a financial market is arbitrage free and complete if and only if there exists a unique equivalent martingale measure.

Definition 2.2.6. A financial market is complete if and only if every contingent claim is attainable.

The risk neutral pricing and no-arbitrage conditions can be summarized as follows:

- The market is free of arbitrage if and only if there exists a martingale measure.
- The market is complete if and only if the martingale measure is unique.
- In an arbitrage free market, not necessarily complete, the price of any attainable claim uniquely given, either by the value of the associated replicating strategy, or by the risk neutral expectation of the discounted claim payoff under any of the equivalent risk-neutral martingale measure.

Chapter 3

Term Structure Models

3.1 Overview

The development of term-structure models has been marked by certain milestones. Even before Black-Scholes, studies are concentrated on the stochastic nature of the interest rates and researchers modeled the evolution of interest rates as a random walk. After Black-Scholes seminal paper, Vasicek [58] introduced a general no-arbitrage framework in which the short rate has a mean-reverting property. Cox, Ingersoll, and Ross (CIR) [29] then extended the previous works by modeling the term structure in the context of a well-defined economic environment and constructed a model that does not allow negative interest rates. In term structure literature the Vasicek and the CIR models are coined as *equilibrium models* since they explicitly specify the market prices of risk and describe the variations of the term structure based on economic fundamentals. Several extensions to equilibrium models are presented by incorporating the stochastic volatility or multiple factors to the existing ones. However, due to their dependence on economic fundamentals, equilibrium models are considered not desirable for pricing interest rate derivatives. In practical applications, matching the model's bond prices to the current term structure is more important than understanding the relationship between the shape of the term structure and its forecast for future economic conditions. This can be justified by the fact that the trading of derivatives involves simultaneous hedging of the the risk exposure by using the underlying security. This necessitates that the derivative price should be based on the market price of the underlying security. Pioneered by Ho and Lee [27] and then followed by

Heath, Jarrow and Morton (HJM) [26], *no-arbitrage* models have the feature of matching the term structure and hence enable pricing of interest rate derivatives by considering the entire yield curve. Although no-arbitrage models give impetus to interest rate models that use the framework of HJM in different contexts such as jump-augmented diffusion, multi-factor and multi-state, they are not free from complications. One complication is the unobservable nature of the instantaneous forward rates. Additionally, continuous compounding of the instantaneous forward rate rules out the popular specification of interest rates as a log-normal process. This led to to the development of the *market* or *LIBOR models* [10, 32, 47] that study the directly observable interest rates such as LIBOR and swap rates. As a completely different approach, *random field* or *string models* describe the dynamics of the forward curve by infinite dimensional shocks to it. In other words, each point on the forward curve has its own shock parameterized by its time of maturity or time to maturity. The first works by using infinite dimensional framework are developed by Kennedy [34, 35] for Gaussian case. After then Goldstein [20] and Santa-Clara and Sornette [52] extended Kennedy's work to more general processes.

In this section, we give a brief review of *equilibrium models* and *no-arbitrage models* by mentioning the most renowned models such as Vasicek, CIR, Ho-Lee, Hull-White and etc. Among the no-arbitrage models, the HJM framework, which embodies most of the interest rate models deserves a special attention since our work frequently uses the methodology employed by HJM [26]. Therefore, we thoroughly examine its properties in Chapter 4. Also we devote an entire chapter about infinite dimensional models so that we prefer to discuss their properties at that part.

3.2 Equilibrium Models

The equilibrium models starts with certain assumptions about the economic environment in which the modeling occurs. Moreover, these assumptions led certain specifications about the state variables that describe the state of the economy. The foremost aspect of those specifications is the explicit definition of the market price of the risk. Although equilibrium models can be investigated by a more general framework called *affine term structure models* postulated by Duffie and Kan [16] and further developed by Dai and Singleton [15], we prefer to investigate them by two main section, one-factor models and multi-factor models.

3.2.1 One-Factor Models

The motivation behind modeling the term structure as a one-factor model is attributed not only to its simplicity but also to the fact that most of the variation in changes of the yield curve is due to the variation in the level of interest rates. Therefore, it is both desirable and pragmatic to take $r(t)$ as the single factor related to the yield curve. That is why one-factor models are generally cited as short rate models. The dynamics of the short rate are given by the following stochastic differential equation

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{3.1}$$

meaning that the change in the short rate can be separated into a drift term μdt and a random shock term represented by an increment of a Brownian motion $dW(t)$, with an instantaneous volatility of $\sigma(r)$. Similarly, the bond dynamics can be represented as

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt + \sigma_P(t,T)dW(t). \tag{3.2}$$

Of course, the drift of the bond price dynamics, $\mu_P(t,T)$, is directly related to the drift of the short rate dynamics, $\mu(r)$. The volatility terms are also related to each each other. The no arbitrage condition implies the following relation

$$\lambda(r(t)) := \frac{\mu_P - r(t)}{\sigma_P(t,T)}, \tag{3.3}$$

where $\lambda(r)$ is the market price of risk, which can be described as the required compensation in the form of expected excess return over the risk-free rate per unit of risk measured by the volatility of the return. It can be shown that the market price of risk is independent of the maturity date and hence same for all bonds in the market. As mentioned before, the specification of the market price of risk makes implicit assumptions about investor preferences, production technologies and endowment processes in the considered economy. Once the market price of risk has been determined, it is easy to determine the short rate process under risk-neutral probability measure Q. That is,

$$dr(t) = [\mu(t, r(t)) - \lambda(r(t))\sigma(t, r(t))]dt + \sigma(t, r(t))dW^Q(t), \tag{3.4}$$

and the bond price dynamics under Q becomes

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sigma_p(t,T)dW^Q(t). \tag{3.5}$$

All one-factor models are constructed by specifying the drift and volatility terms as well as determining the λ, market price of risk. Once these figures are determined bond price can be found by risk-neutral pricing formula

$$P(t,T) = E^Q\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right]. \tag{3.6}$$

We presented here two famous one-factor model examples, namely Vasicek and CIR, without dealing with their derivations. More detailed treatment of one-factor models can be found in [59, 30].

The Vasicek model

In the Vasicek Model [58], short rate follows an Ornstein-Uhlenbeck process given by

$$dr(t) = \alpha[\beta - r(t)]dt + \sigma dW(t), \tag{3.7}$$

where α measures the speed of mean reversion, β is the long-run mean to which the short rate is reverting, and σ is the volatility of the short rate. Here all the parameters are assumed to be constant. Therefore the market price of risk is constant. In a risk-neutral world, the dynamics of the short rate becomes

$$dr(t) = \alpha[\widetilde{\beta} - r(t)]dt + \sigma dW^Q(t), \tag{3.8}$$

where $\widetilde{\beta} = \frac{\beta - \lambda\sigma}{\alpha}$. The price of the zero-coupon bond then can be given by

$$P(t,T) = \exp[A(t,T) - B(t,T)r(t)], \tag{3.9}$$

in which $A(t,T)$ and $B(t,T)$ are deterministic functions and found by solving ordinary differential equations. This not only allows closed form solutions for bond prices and interest derivatives but also implies that spot rates for all maturities are linear in short rate. Although bringing easiness in applications for pricing interest rate instruments, Vasicek model is subject to generating negative interest rates, an undesirable property for pricing interest rate derivatives.

The Cox-Ingersoll-Ross model

Cox, Ingersoll and Ross [29] developed the term structure model that uses a square-root diffusion process in a general equilibrium framework for the considered economy. The dynamics of the short rates under observed measure is given by

$$dr(t) = \alpha[\beta - r(t)]dt + \sigma\sqrt{r(t)}dW(t). \tag{3.10}$$

The main feature of this model is that the above process has a reflecting boundary at $r(t) = 0$ under the proper choice of α and β and therefore does not allow negative interest rates. The market price of risk becomes $\lambda(r(t)) = \frac{\gamma\sqrt{r(t)}}{\sigma}$. Hence, under the risk-neutral measure Q, the short rate evolves according to

$$dr(t) = \widetilde{\alpha}[\widetilde{\beta} - r(t)]dt + \sigma\sqrt{r(t)}dW^Q(t), \tag{3.11}$$

in which short rate has a mean-reverting speed of $\widetilde{\alpha} = \alpha + \gamma$ and a long-run mean of $\widetilde{\beta} = \frac{\gamma\beta}{\alpha+\gamma}$. The bond price is exponentially affine in short rate as in the Vasicek model.

Both Vasicek and CIR models are parts of the family of affine models, where bond price is an exponentially affine function of factors. In the following, we review some models of multi-factor models along with the affine term structure models.

3.2.2 Multi-factor Models

Multi-factor models assumes that the dynamics of the term structure of interest rates driven by several factors, making the yield a function of these factors. Several representations of factors related to macroeconomic shocks and or level, slope or curvature of the yield curve can be postulated in such modeling. There are lots of empirical evidence supporting the multi factor explanation of the variation in interest rates [43]. The first model that we consider here is the model presented by Brennan and Schwartz [11].

The Brennan-Schwartz model

Developed by Brennan and Schwartz [11], the model uses the dynamics of the two yields, namely short rate and long rate. The rationale to model the yield curve by short and long rate is that difference between short rate and long rate approximates the slope of the yield curve. So this model accounts both level and slope effects of the term structure. The model dynamics is given by

$$\begin{aligned} dr(t) &= \alpha_1(r(t), \ell(t), t)dt + \beta_1(r(t), \ell(t), t)dW_1(t), \\ d\ell(t) &= \alpha_2(r(t), \ell(t), t)dt + \beta_2(r(t), \ell(t), t)dW_2(t). \end{aligned} \tag{3.12}$$

where $W_1(t)$ and $W_2(t)$ are correlated Brownian motions. Once the market price of risk associated with each risk factor is determined, bond prices can be determined as in the case of one-factor models. The Brennan-Schwartz model can be exploited with any two yields of finite maturities instead of short and long rates, which are extracted from consol yields.

The Longstaff-Schwartz model

Another popular two-factor model that incorporates the dynamics of the short rate and its variance was developed by Longstaff and Schwartz [44]. They developed the model within the CIR framework by using the general equilibrium setting. In that setting, two state variables, X and Y, represent the state of the economy; each follows a square-root process:

$$dX(t) = (\alpha - \beta X(t)) + \sigma\sqrt{X}dW_1(t), \tag{3.13}$$

$$dY(t) = (\mu - \nu X(t)) + \phi\sqrt{X}dW_2(t), \tag{3.14}$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions. As in the CIR case, the short rate and its variance is linear in its state variables. The price of a zero-coupon bond is shown to be exponentially linear in r and its variance v. That is,

$$P(t,T) = \exp[A(t,T)r(t) + B(t,T)v(t)],$$

where A and B are functions obtained analytically and relate the bond price to the state variables.

Affine Models

As it is illustrated in the models presented so far, if it is modeled properly, certain models can give an tractable way of computing bond prices and interest rate options. That is, if yields are linear of factor(s) that explain variations of interest rate dynamics, the complexity of interest rate modeling decreases substantially. In this section, very concise review of the affine models, or more precisely, multi-factor affine models are presented. In a multi-factor model, there are N factors, $X_1, X_2, ..., X_N$. Let $X(t)$ be the vector $(X_1, X_2, ..., X_N)'$ that evolves according to a multidimensional diffusion process

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \tag{3.15}$$

where $\mu(X(t))$ is an N-dimensional vector, $\sigma(X(t))$ is an $N \times N$ matrix, and $W(t)$ is a vector of N independent Brownian motions. As Duffie and Kan [16] postulated, in an affine model the instantaneous short rate is a linear combination of the factors:

$$r(t) = a_0 + \sum_{i=1}^{N} a_i X_i(t), \tag{3.16}$$

where a_i are constant coefficients. The drift and volatility matrix for the factors are also affine functions of the factors $X(t)$. The elements of $X(t)$ in this setup can be any macroeconomic variable or state variable that is pertaining to variations in the interest rate evolutions. Duffie and Kan [16] showed that, the zero coupon bond prices in these models are given by,

$$P(t,T) = \exp[A(t,T) - B(t,T)'X(t)] \tag{3.17}$$

where $A(t,T)$ and $B(t,T)$ satisfy a set of ordinary differential equations with proper initial conditions.

Remark 3.2.1. Although we placed the affine type models in the equilibrium models section since most of the affine models in the literature are analyzed in the equilibrium framework, certain no-arbitrage models that we will discuss in the next section can be analyzed in an general affine framework.

3.3 No-Arbitrage Models

The equilibrium models discussed so far are analyzed in a specified economy hence leaving the interest rate models prone to arbitrage opportunities, since these models are generally estimated to explain the observed historical patterns in the dynamics of the term structure. This approach may not be practical for the pricing and hedging of interest rate derivatives, which need matching of the current yield curve so that there is no arbitrage opportunity. In order to make a factor model match the current yield curve, most cited way is make the coefficients in a factor model vary deterministically with time. This type of models take the bond prices prevailing in the market as given and price interest rates derivatives accordingly. Therefore, the main objective of this modeling framework is finding a fair value of derivatives as in the Black-Scholes case, rather than spotting mispricing in the underlying bond themselves. In this section, we review the Ho-Lee model [27], the Hull-White model [28] and the Black-Derman-Toy model [7], which are still popular models in industry practice.

The Ho-Lee model

Ho and Lee [27] developed an interest rate model that allows the drift of the short rate process to be time varying by postulating that the short rate follows a random walk. In that model, short rate dynamics is given by

$$dr(t) = \theta(t)dt + \sigma dW(t). \tag{3.18}$$

By specifying the volatility term σ as a constant, drift term $\theta(t)$ can be completely characterized to match the current market conditions. That is, the drift θ can be found as

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t, \tag{3.19}$$

where $f(0,t)$ is the instantaneous forward rate for time t seen at time zero (current yield curve). We will turn to the Ho-Lee model back when discussing HJM framework.

The Hull-White model

Despite its simplicity, the Ho-Lee model does not account for the mean-reversion in the short rate dynamics. To encounter this problem and to match the current yield curve, Hull and White [28] proposed time varying counterparts of the Vasicek and CIR models. As an extended Vasicek model, the short rate dynamics under risk neutral measure is given by

$$dr(t) = [\theta(t) - \beta r(t)]dt + \sigma dW^Q(t). \tag{3.20}$$

Matching the current term structure requires calibrating $\theta(t)$ to be

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \beta f(0,t) + \frac{\sigma^2}{2\beta}(1 - e^{-2\beta}). \tag{3.21}$$

The practical implementation of Hull-White model entails that the zero-coupon bond prices are computed as in the affine class of term structure and that the option pricing can be obtained by a trinomial tree efficiently.

Remark 3.3.1. Bond price formula both in the Ho-Lee and Hull-White models are exponentially affine in short-rate. That is, $P(t,T) = \exp[A(t,T) + B(t,T)r(t)]$.

The Black-Derman-Toy

In an attempt to model the interest rates as a lognormally distributed process, Black, Derman and Toy [7] proposed to build a binomial tree equivalent to the following continuous time process

$$d \ln r(t) = \left[\theta(t) + \frac{\partial\sigma(t)/\partial t}{\sigma(t)} \ln r(t)\right] + \sigma(t)dW(t). \tag{3.22}$$

In that setting, short rate $r(t)$ has a log-normal distribution, thus avoiding negative interest rates, and also has time-varying parameters $\theta(t)$ and $\sigma(t)$ for matching the current yield curve. As an extension of Black-Derman-Toy model, Black and Karasinski [8] proposed a model that accounts for the mean-reversion property. This model has the form

$$d \ln r(t) = [\theta(t) - \beta(t) \ln r(t)]dt + \sigma(t)dW(t). \tag{3.23}$$

Both Black-Derman-Toy and Black-Karasinki model can be implemented through a trinomial tree procedure.

3.4 Market Models

The models, most of the time cited as *market models* or *LIBOR models*, deal with directly observable market rates, such as LIBOR or swap rates quoted in the market. This class of term structure models has a widespread practice in the financial industry not only because they use directly observable interest rates but also they are consistent with the well-established market formulas, Black's cap and swaption formula [6] for two basic interest rate derivative products. Before market models, actually there were no modeling approach that validates Black's formulas, which use the same underlying hypotheses as Black-Scholes's stock option pricing formula. Therefore, the advent of market models provides a legitimate and rigorous treatment of Black's formulas.

The first market models are developed by Brace, Gaterek, and Musiela (BGM) [10], Jamshidian [32], and Miltersen, Sandmann, and Sondermann [47]. Although their results are similar, all of them have different assumptions and methodologies to obtain a Black type formulas. For example, BGM derive the processes of market quoted rates in the HJM framework and deduce the necessary restrictions in the HJM model to ensure that market rates are log-normal. On the other hand, Jamshidian's approach is to find a numeraire and a measure in which the market quoted rates are martingales. Both approaches, however, result in Black like formulas for interest rate cap and swaps.

Chapter 4

Heath-Jarrow-Morton Framework

Interest rate models that we review in previous chapter, models using instantaneous short rate as the state variable, have certain advantages. For example, specifying $r(t)$ as the solution of an stochastic differential equation allows us to work within the partial differential framework and also makes possible to obtain tractable formulas for bond and derivative prices. However, short-rate models have also clear drawbacks. Some of them can be summarized as follows. First of all, it is unreasonable to assume that the entire bond market is governed by one or few explanatory variables. Moreover, it is very difficult to obtain a realistic volatility structure for the forward rates without introducing a complex short rate model. Also, matching the current yield curve becomes difficult as the short rate model becomes realistic [3].

Realizing these facts, Heath, Jarrow and Morton (HJM) [26] developed a continuous time general framework for modeling the entire yield curve. The stepping stone of their approach is to choose instantaneous forward rates as fundamental quantities to construct an arbitrage free term structure. In that framework, instantaneous volatility structures are also used to build the forward rate dynamics. We now give a motivating example [12] to facilitate the understanding of HJM framework.

Let us take the following dynamics for the short rate under the risk-neutral measure. This is a very simple case of Ho-Lee model.

$$dr(t) = \theta dt + \sigma dW(t). \tag{4.1}$$

Since this model is an affine model, the price of the zero-coupon bond is given by

$$P(t,T) = \exp\left[\frac{\sigma^2}{6}(T-t)^3 - \frac{\theta}{2}(T-t)^2 - (T-t)r(t)\right]. \tag{4.2}$$

From the definition of instantaneous forward rate

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\sigma^2}{2}(T-t)^2 + \theta(T-t) + r(t). \tag{4.3}$$

Differentiating this and substituting the short rate dynamics we obtain

$$df(t,T) = \sigma^2(T-t)dt + \sigma dW(t). \tag{4.4}$$

As it is seen from the last equation, the drift $\sigma^2(T-t)$ is determined via volatility σ. That is, drift term is determined by a certain transformation of the volatility term. This is not a mere coincidence for our example, instead it is a general fact proved by Heath, Jarrow and Morton [26]. In the sequel, we give a detailed analysis of the HJM framework both under objective and risk-neutral probability measures and a different approach to the HJM framework where the forward rates are parameterized by time to maturity rather than time of maturity.

4.1 Dynamics of Forward Rates

Assume that $f(0,T)$, $0 \leq T \leq \tau$, where τ is the longest maturity in the market, is known at time 0. $f(0,T)$ is called the initial forward curve. The dynamics of instantaneous forward rates under actual measure $\mathbb{P}$ are given by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s). \tag{4.5}$$

In a differential form, it is written as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t). \tag{4.6}$$

The process $\alpha(t,T)$ and $\sigma(t,T)$ may be random. For each fixed T, they are adapted processes in t variable. In this section we assume that forward dynamics are governed by a single Brownian motion, yet the results can be easily generalized to multiple Brownian motions.

First of all, we know that bond prices are given by

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

Note that the differential of $-\int_t^T f(t,u)du$ is given by

$$d\left(-\int_t^T f(t,u)du\right) = f(t,t)dt - \int_t^T df(t,u)du. \tag{4.7}$$

Therefore, by using $f(t,t) = r(t)$

$$d\left(-\int_t^T f(t,u)du\right) = r(t)dt - \int_t^T [\alpha(t,u)dt + \sigma(t,u)dW(t)]du \tag{4.8}$$

Let us define

$$\alpha^*(t,T) := \int_t^T \alpha(t,u)du \tag{4.9}$$

$$\sigma^*(t,T) := \int_t^T \sigma(t,u)du \tag{4.10}$$

By using the above definitions and changing the order of integration by Fubini theorem we have

$$d\left(-\int_t^T f(t,u)du\right) = r(t)dt - \alpha^*(t,T)dt - \sigma^*(t,T)dW(t). \tag{4.11}$$

Let $g(x) = e^x$. Then the price of the zero-coupon bond price is given by

$$P(t,T) = g\left(-\int_t^T f(t,u)du\right). \tag{4.12}$$

By using the Ito's formula, we have

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) - \alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2\right]dt - \sigma^*(t,T)dW(t). \tag{4.13}$$

According to the risk-neutral pricing principles, in order to guarantee that there is no arbitrage opportunity, we should seek a probability measure Q under which each discounted bond price, $\widetilde{P}(t,T)$, is a martingale. The discounted bond price is defined

$$\widetilde{P}(t,T) = \frac{P(t,T)}{\exp\left(\int_0^t r(u)du\right)}. \tag{4.14}$$

By using Ito's integration by parts formula, we reach

$$\frac{d\widetilde{P}(t,T)}{\widetilde{P}(t,T)} = \left(-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2\right)dt - \sigma^*(t,T)dW(t). \tag{4.15}$$

In order to use the Girsanov theorem, right hand side of the above equation should be written as $-\sigma^*(t,T)[\lambda(t)dt + dW(t)]$. Then using the Girsanov theorem to change the probability measure Q (risk-neutral measure) under which

$$W^Q(t) = \int_0^t \lambda(u)du + W(t), \tag{4.16}$$

is a Brownian motion, the dynamics of the discounted bond prices can be written as

$$d\widetilde{P}(t,T) = -\widetilde{P}(t,T)\sigma^*(t,T)dW^Q(t). \tag{4.17}$$

Then, it would follow that the discounted bond price process is a martingale. However, in order to say that we must solve the following equation for $\lambda(t)$.

$$\left(-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2\right)dt - \sigma^*(t,T)dW(t) = -\sigma^*(t,T)[\lambda(t)dt + dW(t)]. \tag{4.18}$$

That is, we should find a process $\lambda(t)$, denoted as market price of risk in the literature, satisfying

$$-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2 = -\sigma^*(t,T)\lambda(t) \tag{4.19}$$

These are the market price of risk equations, one for each maturity. However, in our case we have only one process λ since source of uncertainty is due to a single Brownian motion. By differentiating the above equation and using the definitions of $\alpha^*(t,T)$ and $\sigma^*(t,T)$ we have the HJM drift condition under actual measure.

$$\alpha(t,T) = \sigma(t,T)[\sigma^*(t,T) + \lambda(t)] \tag{4.20}$$

Hence the following theorem summarizes the above discussion.

Theorem 4.1.1. *A term structure model for a zero-coupon bond prices of all maturities, $0 \leq T \leq T^*$, is arbitrage free if there exists a process $\lambda(t)$ such that*

$$\alpha(t,T) = \sigma(t,T)[\sigma^*(t,T) + \lambda(t)] \tag{4.21}$$

holds for all $0 \leq t \leq T \leq \tau$.

4.2 Heath Jarrow Morton Framework under Risk Neutral Measure

In order to reach a drift condition under risk-neutral measure, we may apply equation (4.21) with $\lambda(t) = 0$. Thus, if we start modeling directly under the martingale measure the drift condition becomes

$$\begin{aligned}\alpha(t,T) =& \sigma(t,T)\sigma^*(t,T) \\ =& \sigma(t,T)\int_t^T \sigma(t,u)du\end{aligned} \tag{4.22}$$

And the forward rate dynamics are written as

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)dW^Q(t). \tag{4.23}$$

We know from our previous discussion that the discounted bond price process follows the dynamics

$$d\widetilde{P}(t,T) = -\widetilde{P}(t,T)\sigma^*(t,T)dW^Q(t).$$

By applying Ito's formula to $d(B(t)d\widetilde{P}(t,T))$ where $B(t)$ is the bank account process defined before, we can reach the dynamics of the bond prices

$$dP(t,T) = r(t)P(t,T)dt - \sigma^*(t,T)P(t,T)dW^Q(t). \tag{4.24}$$

As it is postulated in risk-neutral pricing framework, the underlying assets, zero-coupon bonds, have an risk free return under the martingale measure Q.

4.3 How to Use HJM Methodology ?

As it is evident from the above discussions, in order to use HJM framework for asset pricing, the only needed parameter is the volatility term $\sigma(t,T)$ of the instantaneous forward rates. Once the volatility term is specified under actual measure, thanks to Girsanov theorem, it does not change under risk-neutral measure. And then it is very easy to compute the bond price of any maturity. Here is the methodology,

- Specify the volatility structure, $\sigma(t,T)$.
- Determine the drift parameters according to the HJM condition $\alpha(t,T) = \sigma(t,T)\sigma^*(t,T)$.
- Extract the current forward rate structure $f(0,T)$ from the market data.
- Find forward rates by integrating

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s).$$

- Compute the bond prices by

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

4.4 Musiela Parameterization

In most of the practical applications, it is convenient to use *time to maturity* rather than *time of maturity.* That is, the forward rates are modeled with a fixed time to maturity rather than a fixed maturity date. Mathematically speaking,

Definition 4.4.1. The forward rates $g(t,x)$ are defined for all $x \geq 0$ by the relation

$$g(t,x) = f(t, t+x). \tag{4.25}$$

Also the dynamics of the forward rates under risk-neutral measure are given as

$$g(t,x) = g(0,x) + \int_0^t \alpha(s,x)ds + \int_0^t \sigma(s,x)dW(s). \tag{4.26}$$

And from the knowledge of the instantaneous forward rates for all times to maturity between 0 and $T-t$, the price at time t of a zero-coupon bond with maturity T can be obtained by

$$P(t,T) = \exp\left(-\int_0^{T-t} g(t,x)dx\right). \tag{4.27}$$

The differential of $-\int_0^{T-t} g(t,x)dx$ can be given by

$$d\left(-\int_0^{T-t} g(t,x)dx\right) = g(t,T-t)dt - \int_0^{T-t} dg(t,x)dx. \tag{4.28}$$

Let $h(x) = e^x$. Then the price of the zero-coupon bond price is given by

$$P(t,T) = h\left(-\int_0^{T-t} g(t,x)dx\right). \tag{4.29}$$

By using the Ito's formula, we have

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} = &\left[g(t,T-t) - \int_0^{T-t} \alpha(t,y)dy + \frac{1}{2}\left(\int_0^{T-t} \sigma(t,y)dy\right)^2\right] dt \\ &- \left(\int_0^{T-t} \sigma(t,y)dy\right) dW(t) \end{aligned} \tag{4.30}$$

Hence the discounted bond price dynamics is governed by

$$\begin{aligned} \frac{d\widetilde{P}(t,T)}{\widetilde{P}(t,T)} = &\left[-r(t) + g(t,T-t) - \int_0^{T-t} \alpha(t,y)dy + \frac{1}{2}\left(\int_0^{T-t} \sigma(t,y)dy\right)^2\right] dt \\ &- \left(\int_0^{T-t} \sigma(t,y)dy\right) dW(t). \end{aligned} \tag{4.31}$$

The no-arbitrage condition requires that the dynamics of the discounted bond prices be martingale. This is equivalent to imposing that the drift of $d\widetilde{P}(t,T)$ be zero. Hence we have

$$-r(t) + g(t, T-t) - \int_0^{T-t} \alpha(t,y)dy + \frac{1}{2}\left(\int_0^{T-t} \sigma(t,y)dy\right)^2 = 0. \qquad (4.32)$$

If we denote $x = T - t$ we can rewrite above equation as

$$g(t,x) = r(t) + \int_0^x \alpha(t,y)dy - \frac{1}{2}\left(\int_0^x \sigma(t,y)dy\right)^2 \qquad (4.33)$$

If we differentiate this modified condition with respect to x we obtain

$$\alpha(t,x) = \frac{\partial g(t,x)}{\partial x} + \sigma(t,x)\left(\int_0^x \sigma(t,y)dy\right) \qquad (4.34)$$

This condition is similar to the drift condition in the classical HJM framework, where forward rates are parameterized by time of maturity, except the first term that involves the slope of the forward rate at time t for time to maturity x. The parameterization that we discuss here is first proposed by Musiela [48].

Chapter 5

Applications of HJM Frameworks in a Jump-Diffusion Setting

The existence of jumps in interest rate markets is evident by both central market interventions and reactions to the unexpected news in the financial markets. Monetary authorities, mostly central banks, often use interest rates as their toolkit for implementing their monetary policies. And this will eventually alters the term structure of interest rates prevailing in the market. Several interest rate models are proposed to account this fact, namely the jump behavior of interest rates. The basic types of models include firstly the ones that exploits the HJM framework such as Jarrow and Madan [33], Bjork et. al. [4, 5] and Shirakawa [55]. The other models generalizes the ordinary factor models that we discussed before and augments them with jump processes. Another approaches try to explicitly model the interest rate setting behavior of the monetary authorities. A detailed literature review on jump models can be found in the book of Jessica and Weber [30].

In this section we briefly mention the general interest rate setting proposed by Bjork et. al [4, 5] and give a specialized example, in which the jump part is modeled by a general Markov jump process. In our case, we identify the related drift condition in order to preclude the arbitrage.

5.1 General Formulation of the Term Structure with a Marked Point Process

The most general framework that encompasses most of the term structure models, including jump augmented ones, was developed by Bjork, Masi, Kabanov, and Rungaldier [5] and Bjork, Kabanov and Rungaldier [4]. The key feature of their modeling approach is that one needs a measure over a continuum of possible hedging instruments instead of a vector of quantities of a finite set of hedging instruments [30]. In that setting, suppose that the bond price process under actual measure is

$$\frac{dP(t,T)}{P(t-,T)} = \alpha(t,T)dt + \sigma(t,T)dW(t) + \int_E \theta(t,T,x)\mu(dt,dx) \tag{5.1}$$

where $W(t)$ is a Brownian motion under $\mathbb{P}$, $\alpha(t,T)$ and $\sigma(t,T)$ are the ordinary drift and volatility functions, and μ is a marked point process on a mark space E. E can be regarded as $\mathbb{R}^n$, $\mathbb{N}$ or any finite set. μ can also be seen as a jump process taking values in E, and hence $\theta(t,T,x)$ can be thought as the volatility function for the jumps. In this setting, μ has a compensator $\nu(dt,x)$ of the form $\nu(dt,dx) = \lambda(t,dx)dt$ so that the λ is the intensity of μ and

$$\mu(dt,x) - \lambda(t,dx)dt$$

is a martingale.

Bjork et al. [4] showed that the a martingale measure exists if

$$\alpha(t,T) + \Gamma(t)\sigma(t,T) + \int_E \Phi(t,x)\theta(t,T,x)\lambda(t,dx) = r(t). \tag{5.2}$$

This is the extension of the HJM forward rate drift restriction that we previously discussed. Here $\Gamma(t)$ is the market price of risk for the Brownian motion and $\Phi(t,x)$ corresponds to a market price of risk for the marked point process.

For the forward rate process under risk-neutral measure, Bjork et. al [4, 5] showed that if the dynamics of the forward rate is given by

$$df(t,T) = a(t,T)dt + b(t,T)dW^Q(t) + \int_E \delta(t,T,x)\mu(dt,dx), \tag{5.3}$$

the drift condition is specified as

$$a(t,T) = b(t,T)\int_t^T b(t,s)ds - \int_E \delta(t,T,x)e^{\gamma(t,T,x)}\lambda^Q(t,dx), \tag{5.4}$$

and the bond price process under Q is

$$\frac{dP(t,T)}{P(t-,T)} = r(t)dt + S(t,T)dW^Q(t) + \int_E (e^{\gamma(t,T,x)}\widetilde{\mu}(dt,dx), \tag{5.5}$$

where $\widetilde{\mu}$ is the Q-compensated process $\widetilde{\mu}(dt,dx) = \mu(t,dx) - \lambda^Q(t,dx)dt$, and

$$S(t,T) = -\int_t^T \sigma(t,s)ds,$$

$$\gamma(t,T,x) = -\int_t^T \delta(t,T,x)ds,$$

$$\lambda^Q(t,dx) = \Phi(t,x)\lambda(t,x).$$

5.2 Term Structure Modeling with Markov Jump-Diffusion Processes

In this framework, we model the forward rate dynamics as a Jump-Diffusion Process, where the jump part is characterized by a *Markov Jump Process.* Before going into mathematical foundations of Markov Jump Processes, they can be described as follows. If the process is in state x then it stays there for an exponential period of time with mean λ^{-1}, that is, time between jumps is exponentially distributed with parameter $\lambda(x)$, after which it jumps from x to a new state $x + \zeta(x)$, where the distribution of the jump sizes is given by $P(\zeta(x) \leq y) = R(x,y)$.

5.2.1 Markov Jump Process

Definition 5.2.1 (Markov Jump Process)**.** For all $t \geq 0$.

$$X(t) = X(0) + \sum_{n=1}^{\infty} U_n(I(T_n \leq t) \tag{5.6}$$

where T_n and U_n have the following conditional distributions. Given that $X(T_n) = x$, $T_{n+1} - T_n$ is exponentially distributed with mean $\lambda^{-1}(x)$, and independent of the past; the jump $U_{n+1} = X(T_{n+1}) - X(T_n)$ is independent of the past and has a distribution that depends only on x.

$$F_n(t) = \mathbb{P}(T_{n+1} - T_n \leq t | \mathcal{F}_{T_n}) = 1 - e^{-\lambda(X(T_n))t} \tag{5.7}$$

and for some family of distribution function $C(x,.)$

$$\mathbb{P}(X(T_{n+1}) - X(T_n) \leq y|\mathcal{F}_{T_n}) = C(X(T_n), y)$$

$$\mathbb{E}(X(T_{n+1}) - X(T_n)|\mathcal{F}_{T_n}) = a(X(T_n))$$

In the sequel, we assume that $\lambda(x)$ is always non-negative since if for some x, $\lambda(x) = 0$ then once the process gets into x it stays in it forever, which we called such a state, x, absorbing. Also we assume that $\lambda(x)$ is finite on finite intervals, so that there are no states that process leaves x instantaneously.

In order to use the Markov jump processes in our settings, we need to identify the compensator of X.

Theorem 5.2.1. *Let X be a Markov jump process such that for all x, the holding time parameter is positive, $\lambda(x) > 0$, and the size of the jump from x is integrable with mean a(x).The compensator of X is given by*

$$S(t) = \int_0^t \lambda(X(s))a(X(s))ds \tag{5.8}$$

Proof. See Klebaner Theorem 9.15 [37] □

We can conclude from the above theorem that a Markov Jump Process X has a semimartingale representation.

$$X(t) = X(0) + S(t) + D(t) = X(0) + \int_0^t \lambda(X(s))a(X(s))ds + D(t) \tag{5.9}$$

where $D(t)$ denotes the purely discontinuous martingale. In differential form, it can be written as,

$$dX(t) = \lambda(X(t))a(X(t))dt + dD(t). \tag{5.10}$$

Now we are ready to begin with the term structure modeling.

5.2.2 Bond Price Dynamics

Price of a zero coupon bond, $P(t,T)$, at time t with time of maturity T is given by,

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right) \tag{5.11}$$

where forward rate, $f(t,s)$, satisfies the equation,

$$\begin{aligned} f(t,T) = f(0,t) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) \\ + \int_0^t \Phi(s,T)[dX(s) - \lambda(X(s))a(X(s))ds] \end{aligned} \tag{5.12}$$

Then $P(t,T) = \exp(L(t))$. where $L(t) = -\int_t^T f(t,u)du$. We can find the differential of $L(t)$ by,

$$dL(t) = d\left(-\int_t^T f(t,u)du\right) = f(t,t)dt - \int_t^T df(t,u) \tag{5.13}$$

Hence $dL(t)$ is given by using $f(t,t) = r(t)$ where $r(t)$ denotes short rate.

$$\begin{aligned} dL(t) = r(t)dt - \int_t^T \alpha(t,u)dtdu - \int_t^T \sigma(t,u)dW(t)du \\ - \int_t^T \Phi(t,u)dX(t)du + \int_t^T \Phi(t,u)\lambda(X(t))a(X(t))dtdu \end{aligned} \tag{5.14}$$

Defining

$$\alpha^*(t,T) := \int_t^T \alpha(t,u)du$$

$$\sigma^*(t,T) := \int_t^T \sigma(t,u)du$$

$$\Phi^*(t,T) := \int_t^T \Phi(t,u)du$$

and using the Stochastic Fubini Theorem for changing the order of integration, we can represent $dL(t)$ as follows,

$$\begin{aligned} dL(t) = r(t)dt - \alpha^*(t,T)dt - \sigma^*(t,T)dW(t) - \Phi^*(t,T)dX(t) \\ + \Phi^*(t,T)\lambda(X(t))a(X(t))dt \end{aligned} \tag{5.15}$$

In an integral form, we read

$$\begin{aligned} L(t) = L(0) + \int_0^t [r(s) - \alpha^*(s,T)]ds - \int_0^t \Phi^*(s,T)[dX(s) - \lambda(X(s))a(X(s))ds] \\ - \int_0^t \sigma^*(s,T)dW(s) \end{aligned} \tag{5.16}$$

Under the proper choice of functions $\alpha(t,T)$, $\sigma(t,T)$ and $\Phi(t,T)$, $L(t)$ can be characterized as a regular right continuous with left limits process (cádlag) and it can be represented as a sum of two processes, a local martingale(sum of the second and the third term) and a process of finite variation. Therefore we can conclude that $L(t)$ is a semi-martingale.

Proposition 5.2.1. *Itó formula for semi-martingales.*
Let $X(t)$ be a semi-martingale. For any C^2 function $f : [0,T] \times R \longmapsto R$

$$\begin{aligned} f(X(t)) - f(X(0)) = &\int_0^t f'(X(s-))dX(s)) + \frac{1}{2}\int_0^t f''(X(s-))d\langle X^{cm}, X^{cm}\rangle(s) \\ &+ \sum_{s\leq t}(f(X(s)) - f(X(s-)) - f'(X(s-))\Delta X(s)) \end{aligned} \tag{5.17}$$

where X^{cm} refers to the continuous martingale part of X.

Proof. See R. Cont and Tankov [14] Proposition 8.19 □

By applying Ito's formula for $f(x) = e^x$, we have the expression for the zero coupon bond.

$$\begin{aligned} P(t,T) - P(0,T) = &\int_0^t P(s-,T)\left(r(s)ds - \alpha^*(s,T)ds + \frac{1}{2}\,\sigma^*(s,T)^2 ds\right) \\ &- \int_0^t P(s-,T)\Phi^*(s,T)[dX(s) - \lambda(X(s))a(X(s))ds] \\ &- \int_0^t P(s-,T)\sigma^*(s,T)dW(s) \\ &+ \sum_{s\leq t}(P(s,T) - P(s-,T) - P(s-,T)\Delta L(s)) \end{aligned} \tag{5.18}$$

Since $P(s,T) = f(L(s))$ we obtain

$$\frac{P(s,T)}{P(s-,T)} = \exp(\Phi^*(s,T)\Delta X(s))$$

and therefore

$$\begin{aligned} \Delta P(s,T) &= P(s-,T)\left[\frac{P(s,T}{P(s-,T} - 1\right] \\ &= P(s-,T)\,(\exp(\Phi^*(s,T)\Delta X(s)) \end{aligned} \tag{5.19}$$

This leads to the last term of $P(t,T)$ to be,

$$\begin{aligned} &\sum_{s\leq t}(P(s,T) - P(s-,T) - P(s-,T)\Delta L(s)) \\ &\quad = \sum_{s\leq t} P(s-,T)\,[\exp(\Phi^*(s,T)\Delta X(s)) - 1 - \Phi^*(s,T)\Delta X(s)] \end{aligned} \tag{5.20}$$

This term can be seen as a jump process in $X(s)$ and therefore its compensator can be computed as follows heuristically.

$$A(t) = \sum_{s \le t} P(s-,T) \sum_{k \in \Lambda} \{\exp(\Phi^*(s,T)[k - X(s)]) - 1 - \Phi^*(s,T)[k - X(s)]\} \\ \times \mathbb{P}(X(s),k)\lambda(X(s))ds \tag{5.21}$$

where $\mathbb{P}(X(s),.)$ is the transition probability of the Markov jump process $X(s)$ and Λ is the state space of $X(s)$. This term can also be expressed as

$$A(t) = \int_0^t \int_\Lambda P(s-,T)[exp(\Phi^*(s,T)y) - 1 - \Phi^*(s,T)y)]C(dy,x)\lambda(x)ds \tag{5.22}$$

Therefore, after discounting the bond price process, $P(t,T)$, by $\beta(t) = \int_0^t r(s)ds$, we obtain discounted bond price process.

$$\begin{aligned}\widetilde{P}(t,T) = \widetilde{P}(0,T) &+ \int_0^t \widetilde{P}(s-,T)\left(-\alpha^*(s,T)ds + \frac{1}{2}\,\sigma^*(s,T)^2 ds\right) \\ &+ \int_0^t \widetilde{P}(s-,T)\Phi^*(s,T)\lambda(X(s))a(X(s))ds \\ &- \int_0^t \widetilde{P}(s-,T)\sigma^*(s,T)dW(s) - \int_0^t P^d(s-,T)\Phi^*(s,T)dX(s)) \\ &+ \int_0^t \int_\Lambda \widetilde{P}(s-,T)[exp(\Phi^*(s,T)y) - 1 - \Phi^*(s,T)y)](\widetilde{\mu}^X(ds,dy)) \\ &+ \int_0^t \int_\Lambda \widetilde{P}(s-,T)[exp(\Phi^*(s,T)y) - 1 - \Phi^*(s,T)y)]C(dy,x)\lambda(x)ds\end{aligned} \tag{5.23}$$

where $\widetilde{\mu}^X(ds,dy)$ is the compensated process such that

$$\widetilde{\mu}^X(ds,dy) = \mu^X(ds,dy) - \nu^X(ds,dy),$$

in which $\mu^X((0,t] \times A) = \sum_{s \le t} I_A(\Delta X(s))$ is defined to be the random measure of jumps and ν^X its compensator. We require discounted bond price process $\widetilde{P}(t,T)$ be a martingale, therefore, we can characterize the no-arbitrage condition as follows.

$$\alpha^*(t,T) = \frac{1}{2}\,\sigma^*(t,T)^2 + \int_\Lambda [exp(\Phi^*(s,T)y) - 1 - \Phi^*(s,T)y)]C(dy,x)\lambda(x) \tag{5.24}$$

Chapter 6

Random Field Models

6.1 Overview and Motivation

One of the most important problems in modeling the term structure and pricing the interest rate derivatives is to adequately account for the correlation structure between rates of different maturities. This affects directly the hedging of derivatives and hence results in continuous calibration of the term structure modeling. The multi-factor models that previously discussed in Chapter 3 could not handle this situation since the correlation structure is specified by a limited number of factors, resulting a inadequate representation of the natural relationship between bonds. Although the no-arbitrage models, generalized by HJM framework, are supposed to be consistent with the current term structure, they are not guaranteed to be consistent with the future innovations of the term structure. This inconsistency result in continuous recalibration of the model parameters, which are assumed to be deterministic [20]. Therefore, both multi-factor and no-arbitrage models have inherent inconsistencies, which stems from the complications between models assumptions and practical applications.

These problems are addressed by many researchers through the last decade and increasing number of studies are conducted in order to properly capture the correlation among bonds with different maturities. One of the pioneered and advanced techniques employed for modeling term structure dynamics in order to address these problems is using the infinite dimensional framework, referred most of the time as *random field* or *string models*, popularized by Kennedy [34, 35], Goldstein [20], and Santa-Clara and Sornette [52]. They showed that the correlation structure can be modeled in a parsimonious manner without need to continuous recalibration. The basic idea of their approach is to allow each instantaneous forward rate driven by its own shock, parameterized by time of maturity. Each of these shocks is imperfectly correlated with shocks to other instantaneous forward rates of different maturities. This approach, in addition to handle inconsistent practice of recalibration, offers a more parsimonious description of the term structure dynamics. Indeed, only one parameter needs to be estimated to measure how the correlation of innovations between two forward rates drop as a function of difference in time to maturity[20]. As another advantage, random field framework entails that the best hedging instrument for a bond is another one with similar maturity. This fact, shown by [20], is clearly contrary to the predictions of multi-factor models, which asserts that a 30-year bond can be perfectly hedged by an appropriate position in shorter maturity bonds such as 1-month or 1-year.

Although the theoretical framework of random field models is much more complex than their previous counterparts, the pricing of bond option is surprisingly straightforward. If the model is Gaussian, the volatility and correlation structures are deterministic and therefore bond prices are shown to be lognormally distributed. Since the bond prices are lognormally distributed, it is possible to obtain Black type option prices[34]. If the volatility and correlation structures are stochastic, as Goldstein [20] showed, the closed form solutions to the option prices are not attainable. However, by using the characteristics functions it is possible to represent option prices in a way that facilitate numerical computation.

6.2 Random Fields

Definition 6.2.1 (Random Field)**.** A real, scalar random field $X(\mathbf{t})$ is a collection of random variables at points with coordinates $\mathbf{t} = (t_1, ..., t_n)$ in an n-dimensional parameter space.

Definition 6.2.2 (Gaussian Random Field)**.** A Gaussian random field is a random field where all finite dimensional distributions (fidis) are multivariate normal. As a consequence, a Gaussian random field is completely determined by specifying the mean and the covariance structure.

Random fields are defined by their covariance structure. A simple example is given by the Brownian sheet. $W(t,s)$ is a Brownian sheet if

$$\mathbb{E}[W(t_1,s_1) - W(t_2,s_2)] = 0 \tag{6.1}$$

$$\mathbb{E}[W(t_1,s_1)W(t_2,s_2)] = (t_1 \wedge t_2)(s_1 \wedge s_2) \tag{6.2}$$

for all t_1, t_2, s_1, s_2. For example, if $W_1(t)$ and $W_2(t)$ are two independent Brownian motions, then $W(t,s) = W_1(t)W_2(s)$ is a random field having (6.1) and (6.2).

6.3 Pioneers of the Random Field Models

In this section we give a thorough summary of the first studies that employed random field approach in term structure modeling. Our primary concerns are the works done by Kennedy [34, 35], Goldstein [20], and Santa-Clara and Sornette [52].

6.3.1 Kennedy Model

Kennedy [34] postulated his term structure model by specifying the instantaneous forward rates as

$$f(t,T) = \mu(t,T) + X(t,T), \tag{6.3}$$

where $X(t,T)$ is a mean zero random field and $\mu(t,T)$ is a drift term to determined by no-arbitrage. Also, it is assumed that the covariance structure is

$$cov(X(t_1,T_1), X(t_2,T_2)) = c(t_1 \wedge t_2, T_1, T_2) \tag{6.4}$$

for some function c with $c(s,T_1,T_2) = c(s,T_2,T_1)$. When $X(t,T)$ is Gaussian, c is a function of $t_1 \wedge t_2$ if $X(t,T)$ has independent increments in the t-direction. In order to find closed form solutions for option prices, Kennedy imposed certain assumptions in a Gaussian setup and is able to reduce the random field model to a tree parameter family interest rate model. The main result of Kennedy is given

Theorem 6.3.1 (Kennedy). *If $X(t,T)$ is Gaussian, the discounted bond prices are martingales under risk-neutral measure Q if*

$$\mu^Q(t,T) = \mu^Q(0,T) + \int_0^t [c(t \wedge u, u, T) - c(0,u,T)]du, \tag{6.5}$$

for all $t < T$. Moreover, if $X(t,T)$ is Gaussian and discounted bond prices are martingales under Q, then the covariances must be of the form

$$cov(X(t_1,T_1), X(t_2,T_2)) = c(t_1 \wedge t_2, T_1, T_2) \tag{6.6}$$

The conditions imposed are mostly related to the generalizations of the usual Markov property in a random field setting. These properties place very strong constraints on the underlying process.

Definition 6.3.1. $X(t,T)$ is Markov if

1. For $0 \le t_1 \le t_2 \le t_3$, $t_1 \le T_1$, $t_3 \le T_2$, given $X(t_2,T_2)$ then $X(t_1,T_1)$ and $X(t_3,T_2)$ are independent.
2. For $0 \le t_1 \le t_2 \le T_1 \wedge T_2$, given $X(t_2,T_1)$ then $X(t_1,T_1)$ and $X(t_2,T_2)$ are independent

The first property implies that $X(t,T)$ is Markov in t-direction for $t \le T$, with T fixed. Additionally, if $X(t,T)$ has independent increments in the t-direction, then $X(t,T)$ has the first property.

Theorem 6.3.2 (Kennedy)**.** *If $f(t,T)$ is Markov and has independent increments then the covariance function satisfies*

$$c(t,T_1,T_2) = \nu(t)g(T_1,T_2) \tag{6.7}$$

for some function ν and g where ν is non-decreasing and g is symmetric and non-negative definite.

Definition 6.3.2. A process $X(t,T)$ is stationary if for all $T > 0$ the joint distributions of $X(t,T)$ are the same as those of $X(t+u,T+u)$, for every $u > 0$.

The stationary assumption makes the covariance structure more specific.

Theorem 6.3.3 (Kennedy)**.** *If $f(t,T)$ is Markov, stationary, and has independent increments, then*

$$c(t,T_1,T_2) = e^{\lambda(t-T_1\wedge T_2)}h(|T_1-T_2|) \tag{6.8}$$

where $\lambda > 0$ and $|h(x)| \le h(0)e^{-(1/2)\lambda x}$.

Definition 6.3.3 (Markov in T-direction)**.** $X(t,T)$ is Markov in the T-direction if for $t \leq T_1 \leq T_2 \leq T_3$, $X(t,T_1)$ and $X(t,T_3)$ are independent, given $X(t,T_2)$.

Definition 6.3.4 (Strict Markov Property)**.** If $X(t,T)$ is Markov and Markov in the T-direction then it is strictly Markov.

As stated in [35], short rate process is Markov if and only if $f(t,T)$ is strictly Markov. Also strict Markov property defines a three parameter family of models with a restricted functional form of covariance structure. This result is given in the following theorem.

Theorem 6.3.4 (Kennedy)**.** *If $f(t,T)$ is strictly Markov, stationary and has independent increments, then*

$$c(t,T_1,T_2) = \sigma^2 e^{\lambda t + (2\mu - \lambda)(T_1 \wedge T_2) - \mu(T_1 + T_2)} \tag{6.9}$$

for constants $\sigma, \lambda > 0$, $\mu \geq \frac{1}{2}\lambda$.

6.3.2 Goldstein Model

Goldstein [20] generalized Kennedy's results [34, 35] to non-Gaussian processes and proposes a methodology to price the interest rate options using the Heston's approach, which uses characteristic functions. Goldstein postulated that the forward rates are governed under risk-neutral measure by

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dZ_T(t) \tag{6.10}$$

with the correlation structure

$$corr[dZ_{T_1}(t), dZ_{T_2}(t)] = c(t,T_1,T_2)dt \tag{6.11}$$

Goldstein claims that the drift condition is given by the following theorem.

Theorem 6.3.5 (Goldstein)**.** *The discounted bond process*

$$\widetilde{P}(t,T) = e^{-\int_0^t r(u)du} P(t,T)$$

is a martingale under the martingale measure if

$$\mu(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)c(t,T,u)du. \tag{6.12}$$

Proof. By the definition of $P(t,T)$

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

Using Ito's Lemma the differential of the $dP(t,T)$ is obtained as

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \int_t^T df(t,u)du + \frac{1}{2}\left[\int_t^T df(t,u)du\right]^2 \tag{6.13}$$

Since

$$\frac{d\widetilde{P}(t,T)}{\widetilde{P}(t,T)} = \frac{dP(t,T)}{P(t,T)} - r(t)dt \tag{6.14}$$

and in order $X(t,T)$ be a martingale, the drift of the instantaneous forward rate must satisfy

$$\int_t^T \mu(t,u)du = \frac{1}{2}\left[\int_t^T df(t,u)du\right]^2 \tag{6.15}$$

Taking derivative with respect to T

$$\begin{aligned}\mu(t,T) = &\left[\int_t^T df(t,u)du\right] df(t,T) = \int_t^T \sigma(t,u)dZ_u(t)\sigma(t,T)dZ_T(t)du \\ =&\sigma(t,T)\int_t^T \sigma(t,u)c(t,T,u)du\end{aligned} \tag{6.16}$$

Remark 6.3.1. When we take $c(.)$ as unity, the above model recovers the original HJM formulation.

□

6.3.3 String Model

Santa-Clara and Sornette [52] offered a new class of models for the term structure of interest rates by allowing each instantaneous forward rate to be driven by a stochastic shock while constraining the shocks so that the forward rate curve is always continuous. They postulated the shocks to the forward curve as stochastic string shocks and constructed them as solutions of stochastic partial differential equation (SPDE). Santa-Clara and Sornette modeled the dynamics of the forward rate under Q by using Musiela notation, that is, they parameterized the forward rates by time to maturity instead of time of maturity.

$$df(t,x) = \alpha(t,x)dt + \sigma(t,x)dZ(t,x) \tag{6.17}$$

They also imposed several requirements on Z to qualify as a string shock to the forward rate curve

1. $Z(t, x)$ is continuous in x at all times t.

2. $Z(t, x)$ is continuous in t for all x.

3. The string is martingale in time t, $\mathbb{E}[d_t Z(t, x)] = 0$, for all x.

4. The variance of the increments is equal to the time change, $var[d_t Z(t, x)] = dt$, for all x.

5. The correlation of the increments does not depend on t.

They showed that the first two conditions are satisfied by taking Z to be the solution of a stochastic partial differential equation with at least one partial derivative in x and t. Condition 3, 4, and 5 make strings Markovian. Moreover, they showed that all strings shocks produced as solutions to SPDEs have Gaussian distributions. The drift (no-arbitrage) condition is stated as

$$\alpha(t, x) = \frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \left(\int_0^x c(x, y)\sigma(t, y)dy \right) \tag{6.18}$$

where $c(x, y) = corr(d_t Z(t, x), d_t Z(t, y))$.

6.4 Other Related Models

Besides the pioneering works that we discussed in previous section, there are numerous works that extend the basic models using random field approach. In this section we try to give very brief summary of these works. These works incorporates stochastic volatility, modeling of the defaultable term structures, estimation and simulations of bond prices by taking random field models as their starting point.

Kimmel [36] modeled the term structure of interest rates by using a random field with a conditional volatility. Kimmel developed a class of random field models in which the volatility of bond yields and forward rates depend on a set of latent variables. These latent variables themselves follow a diffusion process. By following this approach, Kimmel found a way that easily characterizes necessary and sufficient conditions for existence and uniqueness of a forward rate process for the absence of arbitrage.

There are relatively few studies about the estimation and calibration of random field models. One of them is Pang's [51] work that show how a Gaussian random field interest rate term structure model can be calibrated to the market prices. In that work, Pang also showed that calibrating a Gaussian random field model is easier than calibrating a multi-factor Gaussian HJM type interest rate model. Another study that is related to the estimation of random field models is due to Bester[2] in which he compared the performance of affine type interest rate models to that of random field models by using Monte-Carlo Markov Chain estimation technique. The striking result is that the random field models are much better able to fit the patterns of volatility and correlation in existing bond price data. Also there is a study by McDonald and Beard [45] that investigates the simulation procedures for random field models.

As an extension to defaultable term structure modeling we can mention two works, one by Furrer [18] and the other by Ozkan and Schmidt [50]. Furrer studied applications of random field modeling to default intensities of a class of obligors and to the modeling of firm values in structural credit risk framework. On the other hand, Ozkan and Schmidt [50] studied the modeling of the defaultable term structure by using Levy random fields hence extended the existing framework to a more general class of processes that incorporate discontinuities.

Lastly, we would like to mention the work done by Hamza et. al [22]. In that work, rather than using a random field approach, researchers try to characterize a mathematical framework, which generalizes every interest rate model, including random field models and jump models, by using semi-martingale theory.

6.5 Korezlioglu's Approach

Korezlioglu [41] studied the representation of zero coupon bond prices in terms of the random fields generated by a two-parameter Brownian sheet. In his work, the main objective is to present the adequate representation theorem for the Radon-Nikodym density of the risk neutral probability following previous works [41] and [40] and to look for the drift conditions under the risk-neutral probability as well as the working probability.

In this section, we give the necessary mathematical setting for the two-parameter processes and stochastic calculus related to them. Moreover, in order to form the basis for the applications that we will present in the next chapter, we review [38] and [39] in detail.

6.5.1 Model Settings

The probability space is $(\Omega, \mathcal{A}, \mathbb{P})$ on which the Brownian sheet is defined.

Definition 6.5.1. A Brownian sheet $W = \{W(t,s), (t,s) \in \mathbb{R}\}$ is a process having the following properties:

- Almost all trajectories of W are continuous,
- $\forall (t,s)$,$W(t,0) = W(0,s) = 0$ a.s.
- W has independent increments. That is, if $t_1 \leq t_2$, $t_1' \leq t_2'$, $s_1 \leq s_2$, $s_1' \leq s_2'$ are such that the rectangles $(t_1,t_2) \times (s_1,s_2)$ and $(t_1',t_2') \times (s_1',s_2')$ are disjoint then the variations of W over the these rectangles are independent, i.e., $W(t_2,s_2) - W(t_2,s_1) - W(t_1,s_2) + W(t_1,s_1)$ and $W(t_2',s_2') - W(t_2',s_1') - W(t_1',s_2') + W(t_1',s_1')$ are independent.
- $W(t_2,s_2) - W(t_2,s_1) - W(t_1,s_2) + W(t_1,s_1)$ has the normal distribution with mean zero and variance $(t_2 - t_1) \times (s_2 - s_1)$.

Given a filtration $\mathfrak{F} = \{\mathcal{F}_t \in \mathbb{R}_+\}$ a two-parameter process W is called a $\mathfrak{F}$-Brownian sheet, if it is a Brownian sheet such that $\forall (t,s)$, $W(t,s)$ is $\mathcal{F}_t$-measurable and for $t_1 < t_2$ and $s_1 < s_2$ the increment $W(t_2,s_2) - W(t_2,s_1) - W(t_1,s_2) + W(t_1,s_1)$ is independent of $\mathcal{F}_{t_1}$

Definition 6.5.2. A process $X(t,s)$ is said to be $\mathcal{F}$-progressive if $\forall t$ the map $(u,s,\omega) \mapsto X(u,s,\omega)$ from $[0,t] \times \mathbb{R}_+ \times \Omega$ is $\mathcal{B}_{[0,t]} \otimes \mathcal{B}_+ \otimes \mathcal{A}$ measurable, where $\mathcal{B}$ indicates the Borel field.

The parameter domain is restricted to $\{(t,s) : 0 \leq t \leq \tau, 0 \leq s \ \leq \tau\}$, where τ is the highest maturity time on market. Let $\mathcal{F}^W_{t,s}$ be the σ-algebra generated by $W(u,v), 0 \leq y \leq t, 0 \leq s \leq \tau\}$ and all $\mathcal{A}$-negligible sets of Ω and define $\mathcal{F}_{t,s} = \cap_{\epsilon>0,\lambda>0}\mathcal{F}^W_{t+\epsilon,s+\lambda}$. $\{\mathcal{F}_{t,s}\}$ is called the natural filtration of W and it is taken that $\mathcal{A} = \mathcal{F}_{\tau,\tau}$.

A two-parameter Brownian martingale used in [38] and our work is defined by

$$M(t,s) = \int_0^t \int_0^s F(u,v)W(du,dv), \tag{6.19}$$

where F is a bounded two-parameter process progressively measurable with respect to the filtration $\mathfrak{F}$ and satisfying

$$\mathbb{E}\int_0^\tau \int_0^\tau F^2(u,v)dudv < \infty. \tag{6.20}$$

The dynamics of instantaneous forward interest rates under risk-neutral measure is represented by

$$f(dt,s) = \mu^Q(t,s)dt + \sigma^Q(t,s)M^Q(dt,s) \tag{6.21}$$

As in the case of one-parameter processes, in order to change the working probability measure to the risk neutral probability, the Martingale Representation and Girsanov theorems are needed in a two-parameter setting. The given theorems are based on [41, 40], where the Brownian sheet is represented as a distribution-valued Brownian motion.

Theorem 6.5.1. *Let M be a local $\mathfrak{F}$-martingale. Then there is a $\mathfrak{F}$-progressive process $\{H(t,s),(t,s)\in[0,\tau]\times[0,\tau]\}$ such that*

$$\int_0^\tau \int_0^\tau H^2(t,s)dtds < \infty \tag{6.22}$$

and

$$M(t) = M(0) + \int_0^t \int_0^\tau H(u,v)W(du,dv). \tag{6.23}$$

Proof. The proof of this theorem is based on the Martingale Representation Theorem given in [41, 40] for distribution valued square-integrable martingales, which can be extended to local martingales by the usual localization method. □

From this, it can be stated

Theorem 6.5.2. *Let Q be a probability measure on $\mathcal{A}$, equivalent to $\mathbb{P}$. Then the martingale*

$$L(t) = \mathbb{E}\left(\frac{dQ}{d\mathbb{P}}|\mathcal{F}_t\right) \tag{6.24}$$

has the following representation

$$L(t) = \exp\left\{-\int_0^t \int_0^\tau \lambda(u,v)W(du,dv) - \frac{1}{2}\int_0^t \int_0^\tau \lambda^2(u,v)dudv\right\} \tag{6.25}$$

where λ is $\mathfrak{F}$-progressive process such that $\int_0^\tau \int_0^\tau \lambda^2(u,v)dudv < \infty$.

Proof. According to the martingale representation theorem $L(t)$ has the following representation

$$L(t) = 1 + \int_0^t \int_0^\tau H(u,v)W(du,dv). \tag{6.26}$$

If the Ito's formula is applied to $lnL(t)$

$$d(lnL(t)) = -\frac{1}{L(t)} \int_0^\tau H(t,v)W(dt,dv) - \frac{1}{2}\frac{1}{L^2(t)} \left(\int_0^\tau H^2(t,v)dv \right) dt. \tag{6.27}$$

Notice $L(t) > 0$ for each t. Hence,

$$\lambda(t,s) := \frac{H(t,s)}{L(t)} \tag{6.28}$$

then λ is progressively measurable and

$$\int_0^\tau \int_0^\tau \lambda^2(t,s)dtds < \infty \tag{6.29}$$

Thus $d(ln(L(t))$ can be written as

$$d(ln(L(t)) = -\int_0^\tau \lambda(t,v)W(dt,dv) - \frac{1}{2}\int_0^\tau \lambda^2(t,v)dvdt. \tag{6.30}$$

And the representation of $L(t)$ follows from above equation. □

The Girsanov Theorem can be stated then,

Theorem 6.5.3. *Under the equivalent martingale probability measure Q, the process*

$$W^Q(t,s) = \int_0^t \int_0^s \lambda(u,v)dudv + W(t,s) \tag{6.31}$$

is a $\mathfrak{F}$-Brownian sheet.

Proof. The proof is given in [41]. Another proof without using nuclear space valued martingales is given in [38]. □

Korezlioglu showed that the drift conditions of HJM type and Musiela type models both under risk-neutral and working probabilities. Additionally, a new model, "mixed HJM-Musiela", is proposed. The drift conditions found are similar to the ones that studied before [20], [34, 35] and [52].

6.5.2 HJM-type Model

In that model $P(t,T)$ is represented as follows; $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$ where f is the instantaneous forward rate that satisfies the equation under Q.

$$f(t,s) = k(s) + \int_0^t \mu^Q(u,s)du + \int_0^t \sigma(u,s)M^Q(du,s) \tag{6.32}$$

where $k(s)$ ia a non-random Lebesgue-integrable function, μ^Q is Lebesgue-integrable and σ is continuous both being $\mathfrak{F}$-progressive processes. To find the drift condition, Korezlioglu [39] followed an approach similar to that of we reviewed in the chapter related to HJM framework. A longer and more involving approach that gave the same results is given in [38]. The differential of $-\int_t^T f(t,s)ds$ is given by

$$d\left(-\int_t^T f(t,s)ds\right) = f(t,t) - \left(\int_t^T \mu^Q(t,s)ds\right)dt - \int_t^T \sigma(t,s)M^Q(dt,s)dt \tag{6.33}$$

The last integral above can be rewritten by using the Stochastic Fubini Theorem and Lemma 4.1 of [38]

$$\begin{aligned}\int_t^T \sigma(t,s)M^Q(dt,s)dt = &\int_0^t \left(\int_t^T \sigma(t,s)ds\right)M^Q(dt,dv)\\ &+\int_t^T \left(\int_v^T \sigma(t,s)ds\right)M^Q(dt,dv)\end{aligned} \tag{6.34}$$

Hence the dynamics of the discounted bond prices, $\widetilde{P}(t,T)$, can be written as

$$\begin{aligned}\frac{d\widetilde{P}(t,T)}{\widetilde{P}(t,T)} = &\left(\int_t^T \mu^Q(t,s)ds\right)dt - \int_0^t \left(\int_t^T \sigma(t,s)ds\right)M^Q(dt,dv)\\ &-\int_t^T \left(\int_v^T \sigma(t,s)ds\right)M^Q(dt,dv)\\ &+\frac{1}{2}\left[\int_0^t \left(\int_t^T \sigma(t,s)ds\right)F^2(t,v)dv\right]dt\\ &+\frac{1}{2}\left[\int_t^T \left(\int_v^T \sigma(t,s)ds\right)F^2(t,v)dv\right]dt\end{aligned} \tag{6.35}$$

Drift condition under risk-neutral probability measure

By the fundamental theorem of asset pricing, discounted asset prices should be martingale in order to preclude arbitrage. So, technically this corresponds to that of the drift term should be zero. Then

$$\int_t^T \mu^Q(t,s)ds = \frac{1}{2}\left[\int_0^t \left(\int_t^T \sigma(t,s)ds\right) F^2(t,v)dv\right] + \frac{1}{2}\left[\int_t^T \left(\int_v^T \sigma(t,s)ds\right) F^2(t,v)dv\right] \tag{6.36}$$

Differentiating this with respect to T, and collecting the integrals, the drift condition is obtained as

$$\mu^Q(t,T) = \sigma(t,T)\left[\int_t^T \left(\sigma(t,s)\int_0^s F^2(t,v)dv\right) ds\right] \tag{6.37}$$

The model can be extended to the case where the process $F(t,s)$ depends on T. Let $F_T(t,s)$ can be a progressive process process in t such that

$$\mathbb{E}\int_0^{\tau}\int_0^{\tau}\left(\int_0^s F_s^2(u,v)dv\right) duds < \infty \tag{6.38}$$

And the martingale M can be replaced by

$$M(t,s) = \int_0^t\int_0^s F_s^2(u,v)W(du,dv). \tag{6.39}$$

Then the drift condition becomes

$$\mu^Q(t,T) = \sigma(t,T)\left[\int_t^T \sigma(t,s)\left(\int_0^s F_T(t,v)F_s(t,v)dv\right) ds\right] \tag{6.40}$$

By using the fact that the covariation of the two martingales $M^Q(.,s)$ and $M^Q(.,s')$ is

$$\langle M^Q(.,s), M^Q(.,s')\rangle = \int_0^t \int_0^{s\wedge s'} F_s(u,v)F_s'(u,v)dudv \tag{6.41}$$

and using the definition of the covariance in previous works [34, 35, 20].

$$d\langle M^Q(.,s), M^Q(.,s')\rangle = c(t,s,s') \tag{6.42}$$

the drift condition (6.40) can be written as

$$\mu^Q(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)c(t,T,s)ds \tag{6.43}$$

Drift condition under working probability

In order to derive the drift condition, the only thing to do is replace W^Q by its expression in terms of W in the equation of $f(t,T)$. Hence,

$$\begin{aligned} f(dt,s) &= \mu^Q(t,s)dt + \sigma(t,s)\left(M(dt,s) + \left(\int_0^s F(t,v)\lambda(t,v)dv\right)dt\right) \\ &= \left(\mu^Q(t,s) + \sigma(t,s)\int_0^s F(t,v)\lambda(t,v)dv\right)dt + \sigma(t,s)M(dt,s) \\ &= \mu(t,s)dt + \sigma(t,s)M(dt,s). \end{aligned} \tag{6.44}$$

Then the drift condition under working probability is shown to be

$$\mu(t,T) = \sigma(t,T)\left[\int_t^T \sigma(t,s)\left(\int_0^s F^2(t,v)dv + \int_0^s F(t,v)\lambda(t,v)dv\right)ds\right]. \tag{6.45}$$

6.5.3 Musiela-type Model

In that model $P(t,T)$ is represented as follows; $P(t,T) = \exp\left(-\int_0^{T-t} g(t,x)dx\right)$ where g is the instantaneous forward rate that satisfies the equation under Q.

$$g(t,x) = h(s) + \int_0^t \mu^Q(u,x)du + \int_0^t \sigma(u,x)M^Q(du,x) \tag{6.46}$$

where $h(s)$ ia a non-random Lebesgue-integrable function, μ^Q is Lebesgue-integrable and σ is continuous both being $\mathfrak{F}$-progressive processes. By using the same methodology, the drift condition under risk-neutral measure Q is specified as,

$$\mu^Q(t,x) = \frac{\partial g(t,x)}{\partial x} + \sigma(t,x)\int_0^x \left(\int_v^x \sigma(t,y)dy\right)F^2(t,v)dv \tag{6.47}$$

6.5.4 Mixed-Musiela-type Model

Korezlioglu [39] proposed a new type of interest rate model that eliminates the computation of the slope of the yield curve in drift condition yet has ability to parameterize the forward interest rate with time to maturity. This type of modeling will be used in our modeling of the term structure in a nuclear space framework. In this model, $P(t,T)$ is represented by

$$P(t,T) = \exp\left(-\int_t^T h(t,s)ds\right) \tag{6.48}$$

where h is the instantaneous forward rate that satisfies the equation under Q.

$$h(t,s) = m(s) + \int_0^t \mu^Q(u,s-u)du + \int_0^t \sigma(u,s-u)M^Q(du,s-u) \tag{6.49}$$

where $m(s)$ ia a non-random Lebesgue-integrable function, μ^Q is Lebesgue-integrable and σ is continuous both being $\mathfrak{F}$-progressive processes. The drift condition under risk-neutral measure Q is given by

$$\mu^Q(t,x) = \sigma(t,x)\int_0^x \left(\int_n^x \sigma(t,v)dv\right)F^2(t,n)dn. \tag{6.50}$$

Chapter 7

Applications of the Random Field Model

In that section, we give three applications of the random field model by using Korezlioglu's setting that we discussed in previous chapter. The applications considered here are the identification of the forward measure, the term structure modeling in a multi-country setting and the defaultable term structure modeling respectively. In all of these applications, the two-parameter setting is utilized. We try to keep all of them self contained, meaning that the necessary literature review and the definitions pertaining to each subject are mentioned briefly.

7.1 Forward Measure

The method of a forward risk adjustment was first pioneered by Jamshidian [31] under the name of a forward risk-adjusted process. Then the formal definition of a forward probability measure was introduced by Geman et. al. [19] under the name of forward neutral probability. The main idea in Geman's study is that the forward price process of any financial asset follows a (local) martingale under the forward neutral probability. In that section we try to identify the forward measure related to the interest rate models driven by a two-parameter processes. For a complete discussion and the approach taken in this section, we referred to the book of Musiela and Rutkowski [49].

In this section, we identify the forward measure in a two-parameter setting in which the dynamics of the instantaneous forward rates are given by HJM-type and Musiela-type equations discussed previously.

7.1.1 Forward Measure under HJM-type model

Price of a zero coupon bond, $P(t,T)$, at time t with time of maturity T is given by,

$$P(t,T) = exp\left(-\int_t^T f(t,s)ds\right) \tag{7.1}$$

where forward rate, $f(t,s)$, satisfies the equation,

$$f(t,s) = k_1(s) + \int_0^t \mu^Q(u,s)du + \int_0^t \varphi(u,s)M^Q(du,s) \tag{7.2}$$

Let us also first define an auxiliary forward process $F_P(t)$ as,

$$F_P(t) := F_P(t,T,T^*) := \frac{P(t,T)}{P(t,T^*)} = exp\left(\int_T^{T^*} f(t,s)ds\right) \tag{7.3}$$

for all $T \leq T^*$. In order to identify the forward measure associated with time T^* we need to find an equivalent martingale measure under which $F_P(t)$ is a martingale [49]. $F_P(t)$ can be written as,

$$F_P(t) = \exp(L(t)), \tag{7.4}$$

where $L(t)$ is defined as below

$$\begin{aligned} L(t) = \int_T^{T^*} k_1(s)ds + \int_T^{T^*}\left(\int_0^t \mu^Q(u,s)du\right)ds \\ + \int_T^{T^*}\left(\int_0^t \varphi(u,s)M^Q(du,s)\right)ds. \end{aligned} \tag{7.5}$$

Hence,

$$dL(t) = \left(\int_T^{T^*} \mu^Q(t,s)ds\right)dt + \int_T^{T^*} \varphi(t,s)M^Q(dt,s)ds \tag{7.6}$$

The second term in the above equation can be written as,

$$\begin{aligned} \int_T^{T^*} \varphi(t,s)M^Q(dt,s)ds = \int_T^{T^*}\left(\int_{s=v}^{T^*} \varphi(t,s)ds\right)M^Q(dt,dv) \\ + \int_0^T\left(\int_{s=T}^{T^*} \varphi(t,s)ds\right)M^Q(dt,dv). \end{aligned} \tag{7.7}$$

By applying Ito's formula,

$$
\begin{aligned}
\frac{d_t F_P(t)}{F_P(t)} &= \left(\int_T^{T^*} \mu^Q(t,s)ds\right) dt + \int_T^{T^*} \left(\int_{s=v}^{T^*} \varphi(t,s)ds\right) M^Q(dt,dv) \\
&+ \int_0^T \left(\int_{s=T}^{T^*} \varphi(t,s)ds\right) M^Q(dt,dv) \\
&+ \left[\frac{1}{2}\int_{v=T}^{T^*} \left(\int_{s=v}^{T^*} \varphi(t,s)ds\right)^2 F^2(t,v)dv\right] dt \\
&+ \left[\frac{1}{2}\int_{v=0}^{T} \left(\int_{s=T}^{T^*} \varphi(t,s)ds\right)^2 F^2(t,v)dv\right] dt
\end{aligned} \tag{7.8}
$$

Now, in order to find an equivalent probability measure P^*, we have to translate W^Q in such a way that $F_P(t)$ becomes a martingale under P^*. To do that, we use the Martingale Representation and Girsanov Theorems for two parameter processes. That is, if

$$
\mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{Q}}|\mathcal{F}_t\right] = exp\left\{-\int_0^t \int_0^{T^*} \lambda(u,v)W^Q(du,dv) - \frac{1}{2}\int_0^t \int_0^{T^*} \lambda(u,v)^2 dudv\right\} \tag{7.9}
$$

for some predictable process λ for which the right hand of side of the above equation (7.9) has $\mathbb{Q}$ expectation 1, then by the Girsanov theorem

$$
W^*(t,s) = W^Q(t,s) + \int_0^t \int_0^s \lambda(u,v)dvdu \tag{7.10}
$$

is a $\mathbb{P}^*$ Brownian sheet. Let us put,

$$
M^*(t,s) = \int_0^t \int_0^s F(u,v)W^*(du,dv) \tag{7.11}
$$

and $M^*(t,s)$ becomes

$$
M^*(t,s) = M^Q(t,s) + \int_0^t \int_0^s F(u,v)\lambda(u,v)dvdu \tag{7.12}
$$

Replacing M^Q with M^*, we can write equation (7.8) as

$$
\begin{aligned}
\frac{d_t F_P(t)}{F_P(t)} &= \left[\ldots\ldots\ldots\ldots\right] dt + \int_T^{T^*} \left(\int_{s=v}^{T^*} \varphi(t,s)ds\right) M^*(dt,dv) \\
&+ \int_0^T \left(\int_{s=T}^{T^*} \varphi(t,s)ds\right) M^*(dt,dv)
\end{aligned} \tag{7.13}
$$

The martingale condition is that the expression which is the coefficient of dt must be equal to zero. That is,

$$
\begin{aligned}
0 = &\int_T^{T^*} \mu^Q(t,s)ds + \frac{1}{2}\int_{v=T}^{T^*}\left(\int_{s=v}^{T^*}\varphi(t,s)ds\right)^2 F^2(t,v)dv \\
&+ \frac{1}{2}\int_{v=0}^{T}\left(\int_{s=T}^{T^*}\varphi(t,s)ds\right)^2 F^2(t,v)dv \\
&- \int_T^{T^*}\left(\int_{s=v}^{T^*}\varphi(t,s)ds\right)F(t,v)\lambda(t,v)dv \\
&- \int_0^{T}\left(\int_{s=T}^{T^*}\varphi(t,s)ds\right)F(t,v)\lambda(t,v)dv
\end{aligned}
\tag{7.14}
$$

If we differentiate equation (7.14) with respect to T^*, we obtain,

$$
\begin{aligned}
0 = &\,\mu^Q(t,T^*) + \varphi(t,T^*)\int_{v=T}^{T^*}\left(\int_{s=v}^{T^*}\varphi(t,s)ds\right)F^2(t,v)dv \\
&+ \varphi(t,T^*)\int_{v=0}^{T}\left(\int_{s=T}^{T^*}\varphi(t,s)ds\right)F^2(t,v)dv - \varphi(t,T^*)\int_{v=T}^{T^*}F(t,v)\lambda(t,v)dv \\
&- \varphi(t,T^*)\int_{v=0}^{T}F(t,v)\lambda(t,v)dv
\end{aligned}
\tag{7.15}
$$

After rearranging the above equation,we can find the condition that ensures $F_P(t)$ be a martingale.

$$
\begin{aligned}
\mu^Q(t,T^*) = &\,\varphi(t,T^*)\int_{v=0}^{T^*}F(t,v)\lambda(t,v)dv \\
&- \varphi(t,T^*)\int_{s=T}^{T^*}\left(\int_{v=0}^{s}\varphi(t,s)F^2(t,v)dv\right)ds
\end{aligned}
\tag{7.16}
$$

Under the hypothesis that $\varphi(t,T^*) \neq 0$ for every t, we can write,

$$
\frac{\mu^Q(t,T^*)}{\varphi(t,T^*)} = \int_{v=0}^{T^*}F(t,v)\lambda(t,v)dv - \int_{s=T}^{T^*}\left(\int_{v=0}^{s}\varphi(t,s)F^2(t,v)dv\right)ds \tag{7.17}
$$

Furthermore suppose that

$$
\frac{\mu(t,T^*)}{\varphi(t,T^*)}
$$

is differentiable in T^*, then we can write

$$
\frac{\partial}{\partial T^*}\left[\frac{\mu^Q(t,T^*)}{\varphi(t,T^*)}\right] = F(t,T^*)\lambda(t,T^*) - \int_{v=0}^{T^*}\varphi(t,T^*)F^2(t,v)dv \tag{7.18}
$$

If moreover $F(t,T^*) \neq 0$ for every t then we get the market price of risk associated with time of maturity T^*,

$$\lambda(t,T^*) = \frac{1}{F(t,T^*)}\left[\frac{\partial}{\partial T^*}\left[\frac{\mu^Q(t,T^*)}{\varphi(t,T^*)}\right] + \int_{v=0}^{T^*} \varphi(t,T^*)F^2(t,v)dv\right] \tag{7.19}$$

Therefore, we can characterize the forward martingale measure associated with the date T^* by specifying $\lambda(t,T^*)$ as above.

7.1.2 Forward Measure under Mixed Musiela-HJM type model

Price of a zero coupon bond, $P(t,T)$, at time t with time of maturity T is given by,

$$P(t,T) = exp\left(-\int_t^T h(t,s)ds\right) \tag{7.20}$$

where forward rate, $h(t,s)$, satisfies the equation,

$$h(t,s) = k_3(s) + \int_0^t \beta^Q(u,s-u)du + \int_0^t \delta(u,s-u)M^Q(du,s-u) \tag{7.21}$$

Let us also define an auxiliary forward process $F_P(t)$ as we did previously,

$$F_P(t) := F_P(t,T,T^*) := \frac{P(t,T)}{P(t,T^*)} = exp\left(\int_T^{T^*} h(t,s)ds\right) \tag{7.22}$$

for all $T \leq T^*$. We want to find an equivalent martingale measure under which $F_P(t)$ is a martingale. $F_P(t)$ can be written as

$$F_P(t) = \exp(L(t)) \tag{7.23}$$

where $L(t)$ is defined as below

$$\begin{aligned} L(t) = &\int_T^{T^*} k_3(s)ds + \int_T^{T^*}\left(\int_0^t \beta^Q(u,s-u)du\right)ds \\ &+ \int_T^{T^*}\left(\int_0^t \delta(u,s-u)M^Q(du,s-u)\right)ds \end{aligned} \tag{7.24}$$

Hence,

$$\begin{aligned} d_tL(t) &= \left(\int_T^{T^*} \beta^Q(t,s-t)ds\right)dt + \int_T^{T^*} \delta(t,s-t)\left(\int_0^{s-t} M^Q(dt,dn)\right)ds \\ &= \left(\int_{T-t}^{T^*-t} \beta^Q(t,v)dv\right)dt + \int_{T-t}^{T^*-t} \delta(t,v)\left(\int_{n=0}^{v} M^Q(dt,dn)\right)dv \end{aligned} \tag{7.25}$$

The second term in equation (7.25) can be written as,

$$\int_{T-t}^{T^*-t} \delta(t,v)\left(\int_{n=0}^{v} M^Q(dt,dn)\right)dv = \int_{n=0}^{T-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)M^Q(dt,dn) + \int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)M^Q(dt,dn) \tag{7.26}$$

By applying Ito's formula,

$$\begin{aligned}\frac{d_tF_P(t)}{F_P(t)} &= \left(\int_{T-t}^{T^*-t}\beta^Q(t,v)dv\right)dt + \int_{n=0}^{T-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)M^Q(dt,dn)\\ &+ \int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)M^Q(dt,dn)\\ &+ \left[\frac{1}{2}\int_{n=0}^{T-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)^2 F^2(t,n)dn\right]dt\\ &+ \left[\frac{1}{2}\int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)^2 F^2(t,n)dn\right]dt\end{aligned} \tag{7.27}$$

Now, in order to find an equivalent probability measure P^*,we have to translate W^Q in such a way that $F_P(t)$ becomes a martingale under P^*. To do this, we use the Martingale Representation and Girsanov Theorems for two parameter processes.Then,

$$\mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{Q}}|\mathcal{F}_t\right] = exp\left\{-\int_0^t\int_0^{T^*}\lambda(u,v)W^Q(du,dv) - \frac{1}{2}\int_0^t\int_0^{T^*}\lambda(u,v)^2dudv\right\} \tag{7.28}$$

for some predictable process λ for which the right hand of equation(9) has $\mathbb{Q}$ expectation 1. By the Girsanov theorem

$$W^*(t,s) = W^Q(t,s) + \int_0^t\int_0^s\lambda(u,v)dvdu \tag{7.29}$$

is a $\mathbb{P}^*$ Brownian sheet. Let us put,

$$M^*(t,s) = \int_0^t\int_0^s F(u,v)W^*(du,dv) \tag{7.30}$$

and $M^*(t,s)$ becomes

$$M^*(t,s) = M^Q(t,s) + \int_0^t\int_0^s F(u,v)\lambda(u,v)dvdu \tag{7.31}$$

Replacing M^Q with M^*,we can write equation (7.27) as

$$\frac{d_tF_P(t)}{F_P(t)} = \Big[\ldots\ldots\ldots\ldots\Big]dt + \int_{n=0}^{T^*-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)M^Q(dt,dn) + \int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)M^Q(dt,dn) \tag{7.32}$$

The martingale condition is that the expression which is the coefficient of dt must equal to zero.That is,

$$\begin{aligned}0 &= \int_{T-t}^{T^*-t}\beta^Q(t,v)dv + \frac{1}{2}\int_{n=0}^{T-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)^2 F^2(t,n)dn \\ &+ \frac{1}{2}\int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)^2 F^2(t,n)dn \\ &- \int_{n=0}^{T-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)F(t,n)\lambda(t,n)dn \\ &- \int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)F(t,n)\lambda(t,n)dn\end{aligned} \tag{7.33}$$

If we differentiate equation (7.33) with respect to T^*, we obtain,

$$\begin{aligned}0 &= \beta^Q(t,T^*-t) + \delta(t,T^*-t)\int_{n=0}^{T-t}\left(\int_{v=T-t}^{T^*-t}\delta(t,v)dv\right)F^2(t,n)dn \\ &+ \delta(t,T^*-t)\int_{n=T-t}^{T^*-t}\left(\int_{v=n}^{T^*-t}\delta(t,v)dv\right)F^2(t,n)dn \\ &- \delta(t,T^*-t)\int_{n=0}^{T-t}F(t,n)\lambda(t,n)dn \\ &- \delta(t,T^*-t)\int_{n=T-t}^{T^*-t}F(t,n)\lambda(t,n)dn\end{aligned} \tag{7.34}$$

After rearranging the above equation,we can find the condition that ensures $F_P(t)$ is a martingale.

$$\begin{aligned}\beta^Q(t,T^*-t) &= \delta(t,T^*-t)\int_{n=0}^{T^*-t}F(t,n)\lambda(t,n)dn \\ &- \delta(t,T^*-t)\int_{v=T-t}^{T^*-t}\left(\int_{n=0}^{v}\delta(t,v)F^2(t,n)dn\right)dv\end{aligned} \tag{7.35}$$

Under the hypothesis that $\delta(t,T^*-t) \neq 0$ for every t,we can write,

$$\frac{\beta^Q(t,T^*-t)}{\delta(t,T^*-t)} = \int_{n=0}^{T^*-t}F(t,n)\lambda(t,n)dn - \int_{v=T-t}^{T^*-t}\left(\int_{n=0}^{v}\delta(t,v)F^2(t,n)dn\right)dv \tag{7.36}$$

Furthermore suppose that

$$\frac{\beta^Q(t, T^* - t)}{\delta(t, T^* - t)}$$

is differentiable in T^*, then we can write by denoting $T^* - t = x$

$$\frac{\partial}{\partial x}\left[\frac{\beta^Q(t,x)}{\delta(t,x)}\right] = F(t,x)\lambda(t,x) - \delta(t,x)\int_{n=0}^{x} F^2(t,n)dn \tag{7.37}$$

If moreover $F(t,x) \neq 0$ for every t then we get the market price of risk associated with time to maturity x,

$$\lambda(t,x) = \frac{1}{F(t,x)}\left[\frac{\partial}{\partial T^*}\left[\frac{\beta^Q(t,x)}{\delta(t,x)}\right] + \varphi(t,x)\int_{n=0}^{x} F^2(t,n)dn\right] \tag{7.38}$$

Therefore, we can characterize the forward martingale measure associated with the date T^* by specifying $\lambda(t, T^*)$ as above.

7.2 Term Structure Modeling in a Multi-Country Setting

7.2.1 Overview

In this section, an arbitrage-free model of the term structure of interest rates on multi country setting is extended by assuming that the processes driving the foreign and domestic instantaneous forward rates are two parameter processes, namely random fields parameterized by current time and time to maturity. In most of the applications related to derivative securities with two or more economies, exchange rates are assumed to be random, mostly modeled as a geometric brownian motion, whereas interest rates are assumed to be constant. The first models that incorporate stochastic interest rates in the framework of Black-Scholes [9] are due to Feiger and Jacquillant [17] and Grabbe [21]. However, as mentioned by Amin and Jarrow [1], this type of modeling could not integrate the complete characteristics of the term structure into the valuation because of its incapabilities in valuing American-type options. As the shortcomings of BS approach to interest rate modeling was overcomed by the seminal paper of Heath, Jarrow and Morton [26] (see Chapter 4), the first application of HJM methodology in foreign currency options are given by Amin and Jarrow [1]. In their work, they characterize the drift conditions of the foreign and domestic forward rates and exchange rates by using the same technique, basing on martingale measure approach, developed by HJM for stochastic interest rates.

The modeling of stochastic interest rates in a cross-currency setting has been investigated by many researchers, for a complete discussion and lists of publications reader is referred to [49].

Like in the setting of one country term structure models driven by random fields that models the forward rate dynamics by incorporating the second parameter as representing the random shocks arising from time to maturity, we use a two parameter process for both domestic and foreign forward rates. Moreover, for the modeling of term structures in multi-country setting, it is usual to take into account the fact that the fluctuations of interest rates and exchange rates are correlated. Therefore, we use the same two parameter process in modeling both interest rates and exchange rates, although for exchange rate it reduces to a one parameter process. Yet, by appropriate choice of the driving process this approach is satisfactory enough to model the correlation both between exchange rate and interest rates and among different maturities of instantaneous forward rates.

In the sequel, apart from deriving the drift condition for exchange rate dynamics, our modeling is based on the domestic martingale measure, Q. It is easy to represent the same conditions and price processes in the objective probability measure P by using the discussions of Korezlioglu [38, 39] and Chapter 6.

7.2.2 Model

Assume that there are two bond markets, one domestic and one foreign. We take as given a standard HJM model for the domestic forward rates $f_d(t,T)$ dynamics of the form

$$df_d(t,s) = \alpha_d(t,s)dt + \sigma_d(t,s)M(dt,s) \tag{7.39}$$

where $M(t,s)$ is a two parameter martingale under the domestic martingale measure Q defined by

$$M(t,s) = \int_0^t \int_0^s F(u,v)W(du,dv) \tag{7.40}$$

The foreign forward rates are denoted by $f_f(t,T)$, and their dynamics, still under the domestic martingale measure Q, are assumed to be given by

$$df_f(t,s) = \alpha_f(t,s)dt + \sigma_f(t,s)M(dt,s) \tag{7.41}$$

Note 7.2.1. *The same two parameter process defined by (7.40) is driving both the domestic and the foreign bond market.*

The exchange rate $X(t)$ (denoted in units of domestic currency per unit of foreign currency) has the Q dynamics.

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \mu_x(t)dt + \sigma_x(t)M(dt,t) \\ &= \mu_x(t)dt + \int_0^t \sigma_x(t)M(dt,dn) \end{aligned} \tag{7.42}$$

Provided that domestic and foreign savings accounts are defined by

$$B_d(t) = e^{\int_0^t r_d(u)du} \tag{7.43}$$

$$B_f(t) = e^{\int_0^t r_f(u)du} \tag{7.44}$$

where $r_d(t)$ and $r_f(t)$ are domestic and foreign short interest rates respectively, we can show that under domestic martingale measure $\mu_x(t)$ is equal to the spread between domestic short rate and foreign short rate.

Suppose that the dynamics of $X(t)$ under the objective probability measure P are given by

$$dX(t) = X(t)\alpha_x(t)dt + X(t)\sigma_x(t)M^P(dt,t) \tag{7.45}$$

One can invest by buying foreign currency and depositing it in a foreign currency bank account, however he can also achieve the same investment strategy by investing in a domestic asset with the price process, $B_e(t)$ where,

$$B_e(t) = B_f(t)X(t)$$

The dynamics of $B_e(t)$ can be found by applying Ito's Lemma

$$dB_e(t) = B_e(t)(\alpha_x(t) + r_f(t))dt + B_e(t)\sigma_x(t)M^P(dt,t) \tag{7.46}$$

Using the general result that every domestic asset has the short rate as its local rate of return under martingale measure Q in order to preclude arbitrage , it can be shown that the dynamics of B_e under Q are given by,

$$dB_e(t) = B_e(t)r_d(t) + B_e(t)\sigma_x(t)M(dt,t) \tag{7.47}$$

Applying the Ito's formula to $\frac{B_e(t)}{B_d(t)}$, which equals $X(t)$ by definition,we obtain the dynamics of foreign exchange, $X(t)$ as

$$\frac{dX(t)}{X(t)} = (r_d(t) - r_f(t))dt + \sigma_x(t)M(dt,t) \tag{7.48}$$

Under a foreign martingale measure, the coefficient processes for the foreign forward rates will definitely satisfy HJM drift condition, however what we want to accomplish in this paper is to establish a foreign market drift condition under the domestic martingale measure Q.

Let $U(t,s)$ denote the time t value of the foreign zero-coupon bond maturing at time s in units of the domestic currency. We then have the following,

$$\frac{U(t,s)}{B_d(t)} = X(t)e^{Y(t)} := V(t) \tag{7.49}$$

where $Y(t) = -\int_t^s f_f(t,u)du - \int_0^t r_d(u)du$.

In order to identify the drift condition $V(t)$ should be martingale,i.e., the expression which is the coefficient of dt term in $dV(t)$ must be equal to zero. Therefore, next we find $dV(t)$ by applying Ito's Formula

$$dV(t) = X(t)d(e^{Y(t)}) + e^{Y(t)}dX(t) + d\langle X(t), e^{Y(t)}\rangle_t \tag{7.50}$$

Then

$$\begin{aligned} d(Y(t)) = d_t\left(-\int_t^s f_f(t,u)du - \int_0^t r_d(u)du\right) &= r_f(t)dt - r_d(t)dt \\ &\quad - \int_t^s \alpha_f(t,u)dudt \\ &\quad - \int_t^s \sigma_f(t,u)M(dt,u)du \end{aligned} \tag{7.51}$$

The last integral can be written as by changing the order of integration using stochastic Fubini Theorem,

$$\begin{aligned} \int_t^s \sigma_f(t,u)M(dt,u)du \quad &= \quad \int_0^t \left(\int_t^s \sigma_f(t,u)du\right) M(dt,dn) \\ &\quad + \int_t^s \left(\int_n^s \sigma_f(t,u)du\right) M(dt,dn) \end{aligned} \tag{7.52}$$

Therefore $d(e^{Y(t)})$ is equal to

$$\begin{aligned} d(e^{Y(t)}) = e^{Y(t)}[(r_f(t)dt - r_d(t)dt - \int_t^s \alpha_f(t,u)dudt \\ + \frac{1}{2}\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dndt \\ + \frac{1}{2}\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dndt \\ - \int_0^t (\int_t^s \sigma_f(t,u)du) M(dt,dn) \\ - \int_t^s (\int_n^s \sigma_f(t,u)du) M(dt,dn)] \end{aligned} \tag{7.53}$$

Now, in order to find $dV(t)$ we have to find $d\langle X(t), e^{Y(t)}\rangle_t$. As it is evident from the domain of integration of the martingales $M(dt,dn)$ in $dX(t)$ and $d(e^{Y(t)})$, $d\langle X(t), e^{Y(t)}\rangle_t$ can be given as,

$$d\langle X(t), e^{Y(t)}\rangle_t = -V(t)\int_0^t \sigma_x(t)\left(\int_t^s \sigma_f(t,u)du\right)F^2(t,n)dndt \tag{7.54}$$

Finally we find the dynamics of the $V(t)$

$$\frac{dV(t)}{V(t)} = \underbrace{\Big[\ldots\ldots\ldots\ldots\Big]}_{A(t)} dt - \int_t^s \sigma_f(t,u)M(dt,u)du \tag{7.55}$$

where $A(t)$ is given by,

$$\begin{aligned} A(t) = \frac{1}{2}\left[\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dn\right] + \frac{1}{2}\left[\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dn\right] \\ - \int_t^s \alpha_f(t,u)du - \int_0^t \sigma_x(t)\left(\int_t^s \sigma_f(t,u)du\right)F^2(t,n)dn \end{aligned} \tag{7.56}$$

The martingale condition for $V(t)$ is that $A(t)$, which is the coefficient of dt, must equal to zero. That is,

$$\begin{aligned} \int_t^s \alpha_f(t,u)du &= \frac{1}{2}\left[\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dn\right] \\ &+ \frac{1}{2}\left[\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dn\right] \\ &- \int_0^t \sigma_x(t)\left(\int_t^s \sigma_f(t,u)du\right)F^2(t,n)dn \end{aligned} \tag{7.57}$$

If we differentiate this expression with respect to s, we find the modified drift condition of foreign bond dynamics under domestic martingale measure.

$$\alpha_f(t,s) = \sigma_f(t,s)\left[\int_{u=t}^{s}\left(\int_{n=0}^{u}\sigma_f(t,u)F^2(t,n)dn\right)du - \int_0^t \sigma_x(t)F^2(t,n)dn\right] \tag{7.58}$$

7.2.3 Modeling of the Yield Spread

What we want to do next is to define the process $g(t,s)$ known as yield spread by the following equation

$$g(t,s) = f_f(t,s) - f_d(t,s)$$

and governed by the dynamics under domestic martingale measure.

$$dg(t,s) = \alpha_g(t,s) + \sigma_g(t,s)M(dt,s)$$

Our aim now is to derive the appropriate drift condition for the coefficient process α_g in terms of σ_g, σ_d. By the definition of the yield spread, it is obvious that we have the relationships

$$\alpha_f(t,s) = \alpha_d(t,s) + \alpha_g(t,s) \tag{7.59}$$

$$\sigma_f(t,s) = \sigma_d(t,s) + \sigma_g(t,s) \tag{7.60}$$

Using the fact that the domestic and foreign forward rates satisfy the HJM drift condition and the foreign market yield spread equation (7.59) and (7.60), we can conclude that

$$\alpha_g(t,s) = \sigma_d(t,s)\left[\int_{u=t}^{s}\left(\int_{n=0}^{u}\sigma_g(t,u)F^2(t,n)dn\right)du - \int_0^t \sigma_x(t)F^2(t,n)dn\right] \tag{7.61}$$

Summing up all the findings up to now, we now provide two theorems that characterize the necessary and sufficient conditions needed on certain dynamics to ensure that the multi-country model of interest rates is arbitrage free.

Theorem 7.2.1. *The arbitrage free dynamics of the foreign and domestic forward interest rates.*

(i)The dynamics of the domestic forward interest rates under domestic martingale measure are given by

$$df_d(t,s) = \sigma_n(t,s)\left[\int_{u=t}^{s}\left(\int_{n=0}^{u}\sigma_d(t,u)F^2(t,n)dn\right)du\right] + \sigma_d(t,s)M(dt,s) \tag{7.62}$$

(ii)The dynamics of the foreign forward interest rates under domestic martingale measure are given by

$$\begin{aligned} df_f(t,s) = \sigma_f(t,s)&\left[\int_{u=t}^{s}\left(\int_{n=0}^{u}\sigma_f(t,u)F^2(t,n)dn\right)du - \int_0^t \sigma_x(t)F^2(t,n)dn\right] \\ &+ \sigma_f(t,s)M(dt,s) \end{aligned} \tag{7.63}$$

(iii)The dynamics of the exchange rate under domestic martingale measure are given by

$$\frac{dX(t)}{X(t)} = (r_d(t) - r_f(t))dt + \sigma_x(t)M(dt,t) \tag{7.64}$$

These expressions for the domestic and foreign interest rates led to the following.

Theorem 7.2.2. *Given the dynamics of the foreign and domestic forward rates (7.62),(7.63) and the exchange rate (7.64),*

(i)The dynamics of the foreign zero coupon bond prices under domestic martingale measure are given by

$$\begin{aligned} \frac{dP_f(t,s)}{P_f(t,s)} =& [r_f(t) + \int_0^t \sigma_x(t)\left(\int_t^s \sigma_f(t,u)du\right)F^2(t,n)dn]dt \\ &- \int_t^s \sigma_f(t,u)M(dt,u)du \end{aligned} \tag{7.65}$$

(ii)The dynamics of the domestic zero coupon bond prices under domestic martingale measure are given by

$$\frac{dP_d(t,s)}{P_d(t,s)} = r_d(t)dt - \int_t^s \sigma_d(t,u)M(dt,u)du \tag{7.66}$$

(iii)The dynamics of the foreign zero coupon bond prices in terms of domestic currency, $P_{f^d}(t,s)$, under domestic martingale measure are given by

$$\frac{dP_{f^d}(t,s)}{P_{f^d}(t,s)} = r_d(t)dt + \sigma_x(t)M(dt,t) - \int_t^s \sigma_f(t,u)M(dt,u)du \tag{7.67}$$

where $\int_t^s \sigma_k(t,u)M(dt,u)du$ *for* $k \in \{d, f\}$ *can be decomposed as*

$$\begin{aligned}\int_t^s \sigma_k(t,u)M(dt,u)du &= \int_0^t \left(\int_t^s \sigma_k(t,u)du\right) M(dt,dn) \\ &\quad + \int_t^s \left(\int_n^s \sigma_k(t,u)du\right) M(dt,dn)\end{aligned} \tag{7.68}$$

Corollary 7.2.3. *The arbitrage free dynamics of the yield spreads under the domestic martingale measure are given by*

$$\begin{aligned}dg(t,s) = \sigma_d(t,s)&\left[\int_{u=t}^s \left(\int_{n=0}^u \sigma_g(t,u)F^2(t,n)dn\right) du - \int_0^t \sigma_x(t)F^2(t,n)dn\right] \\ &+ \sigma_g(t,s)M(dt,s)\end{aligned} \tag{7.69}$$

Now, we can extend our model as [38, 39] do via defining the $F(t,.)$ function explicitly depending on the time to maturity in $M(t,s)$. That is we can express $M(dt,s)$ as

$$M(dt,s) = \int_0^s F_s(t,v)M(dt,dv) \tag{7.70}$$

If we do so, we can write the equation (19) as

$$\begin{aligned}\int_t^s \alpha_f(t,u)du &= \frac{1}{2}\left[\int_0^t \left(\int_t^s \sigma_f(t,u)F_u(t,n)du\right)^2 dn\right] \\ &\quad + \frac{1}{2}\left[\int_t^s \left(\int_n^s \sigma_f(t,u)F_u(t,n)du\right)^2 dn\right] \\ &\quad - \int_0^t \sigma_x(t)\left(\int_t^s \sigma_f(t,u)F_u(t,n)du\right) F_t(t,n)dn\end{aligned} \tag{7.71}$$

Hence differentiating this expression with respect to s our drift condition becomes,

$$\begin{aligned}\alpha_f(t,s) = \sigma_f(t,s)[&\int_t^s \left(\int_0^u \sigma_f(t,u)F_s(t,n)F_u(t,n)dn\right) du \\ &- \int_0^t \sigma_x(t)F_s(t,n)F_t(t,n)dn]\end{aligned} \tag{7.72}$$

By using the definition of covariance structure $c(t,s_1,s_2)$ of the random field $M(t,s)$ as we did in Chapter 6.

$$c(t,s_1,s_2)dt = d\langle M(.,s_1), M(.,s_2)\rangle = \int_0^{s_1 \wedge s_2} F_{s_1}(t,n)F_{s_2}(t,n)dndt, \tag{7.73}$$

we reached the drift condition for foreign forward rates under domestic martingale measure as,

$$\alpha_f(t,s) = \sigma_f(t,s)\left[\int_t^s \sigma_f(t,u)c(t,s,u)du + \sigma_x(t)c(t,t,s)\right] \tag{7.74}$$

Similarly, the drift condition for the yield spread dynamics under domestic martingale measure is given by,

$$\alpha_g(t,s) = \sigma_d(t,s)\left[\int_t^s \sigma_g(t,u)c(t,s,u)du + \sigma_x(t)c(t,t,s)\right] \tag{7.75}$$

7.3 Defaultable Term Structure Modeling

In this chapter, we extend the HJM approach for two parameter processes to the defaultable term structure modeling. In the first part, modeling is done by a classical HJM approach, finding the drift restrictions, applied to any defaultable zero coupon bond price with zero recovery as in Schonbucher's study [54].

7.3.1 Model Settings

In credit risk modeling, the main ingredient is the modeling of the default event, which can be defined as any random event whose occurrence affects the ability of the counterparty in a financial contract to fulfill a contractual commitment to meet his or her obligations stated in the contract. Most of the mathematical literature on modeling the default event is devoted to the modeling of the random time when the default event occurs,that is, default time.

Let us denote the default time by τ. We assume that default time, τ is a stopping time, which means that τ is a random variable $\tau : \Omega \mapsto \mathbb{R}_+ \bigcup\{\infty\}$ such that $\{\tau \leq t\} \in F_t$, for every $t \geq 0$. We denote $N(t) := \mathbb{I}_{\{\tau\leq t\}}$ as the default indicator function and $A(t)$ as the predictable compensator of $N(t)$. In our context $A(t)$ has an intensity $h(t)$, which is a non-negative and progressively measurable process called default intensity. Therefore, we can define the martingale $K(t)$ as the compensated process of $N(t)$ such that

$$K(t) = N(t) - A(t) \tag{7.76}$$

where

$$A(t) = \int_0^t h(u)du$$

In this context, we enlarge the filtration $\mathfrak{F}$ as by including the information on default process, $N(t)$. Hence in this part, filtration is generated by both the two parameter process and default indicator process. For using in the sequel, we now give certain definitions related to the defaultable zero-coupon bond pricing.

Definition 7.3.1. Setup and Basics

1. The risk free instantaneous forward rates $f(t,s)$ have Q dynamics

$$df(t,s) = \alpha(t,s)dt + \sigma(t,s)M(dt,s) \tag{7.77}$$

2. The defaultable instantaneous forward rates $f_c(t,s)$ have Q dynamics

$$df_c(t,s) = \alpha_c(t,s)dt + \sigma_c(t,s)M(dt,s) \tag{7.78}$$

3. The instantaneous risk-free short rates $r(t)$ and the instantaneous defaultable short rates $r_c(t)$ are defined as

$$r(t) = f(t,t)$$

and

$$r_c(t) = f_c(t,t)$$

4. The corresponding bank account processes for risk-free and defaultable bank accounts are given by

$$B(t) = \exp\left(\int_0^t r(u)du\right)$$

and

$$B_c(t) = \mathbb{I}_{\{\tau>t\}} \exp\left(\int_0^t r_c(u)du\right)$$

Definition 7.3.2. The time t price of the bond with maturity s is denoted by $P_c(t,s)$. The payoff at maturity time s of this bond can be given as $\mathbb{I}_{\{\tau>s\}} = 1 - N(t)$. The price of the risky zero coupon bond price can be given by

$$P_c(t,s) = (1-N(t))exp\left(-\int_t^s f_c(t,u)du\right) \tag{7.79}$$

7.3.2 Defaultable Bond Price Dynamics

Now, we are ready to develop an arbitrage free term structure model. Firstly, by the Ito's formula, we can find the dynamics of the defaultable zero-coupon bond. Writing $X(t) = -\int_t^s f_c(t,u)du$, we can find for $\tau > t$.

$$\begin{aligned} dP_c(t,s) =&(1-N(t))d(e^{X(t)}) + d(1-N(t))e^{X(t)} \\ =&(1-N(t))e^{X(t)}[dX(t) + \frac{1}{2}d\langle X(t), X(t)\rangle] - dN(t)e^X(t) \end{aligned} \tag{7.80}$$

By using the facts that $1-N(t)$ is equal to one for $\tau > t$ and $X(t)$ is continuous.

$$\frac{dP_c(t,s)}{P_c(t-,s)} = dX(t) + \frac{1}{2}d\langle X(t), X(t)\rangle - dN(t) \tag{7.81}$$

where

$$\begin{aligned} d(X(t)) = d_t\left(-\int_t^s f_c(t,u)du\right) &= r_c(t)dt - \int_t^s \alpha_c(t,u)dudt \\ &\quad - \int_t^s \sigma_c(t,u)M(dt,u)du \end{aligned} \tag{7.82}$$

The last integral can be written as by changing the order of integration using stochastic Fubini Theorem,

$$\begin{aligned} \int_t^s \sigma_c(t,u)M(dt,u)du &= \int_0^t \left(\int_t^s \sigma_c(t,u)du\right) M(dt,dn) \\ &\quad + \int_t^s \left(\int_n^s \sigma_c(t,u)du\right) M(dt,dn) \end{aligned} \tag{7.83}$$

Then, by using (7.82), (7.83) and the definition of compensated process $K(t)$.

$$\begin{aligned} \frac{dP_c(t,s)}{P_c(t-,s)} = &\left[r_c(t) - h(t) - \int_t^s \alpha_c(t,u)du + \frac{1}{2}\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dn \right. \\ &\left. + \frac{1}{2}\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dn\right] dt \\ &- \int_0^t \left(\int_t^s \sigma_c(t,u)du\right) M(dt,dn) \\ &- \int_t^s \left(\int_n^s \sigma_c(t,u)du\right) M(dt,dn) - dK(t) \end{aligned} \tag{7.84}$$

The dynamics of the discounted bond prices $\widetilde{P}_c(t,s)$ can be obtained as follows,

$$\begin{aligned} \frac{d\widetilde{P}_c(t,s)}{\widetilde{P}_c(t-,s)} = &\left[r_c(t) - r(t) - h(t) - \int_t^s \alpha_c(t,u)du \right. \\ &+ \frac{1}{2}\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dn \\ &\left. + \frac{1}{2}\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dn\right] dt \\ &- \int_0^t \left(\int_t^s \sigma_c(t,u)du\right) M(dt,dn) \\ &- \int_t^s \left(\int_n^s \sigma_c(t,u)du\right) M(dt,dn) - dK(t) \end{aligned} \tag{7.85}$$

From the main hypothesis that the discounted asset prices are martingale, it is required that the drift term of the above equation be zero. Therefore, the drift condition becomes

$$
\begin{aligned}
r(t) =& r_c(t) - h(t) - \int_t^s \alpha_c(t,u)du + \frac{1}{2}\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dn \\
&+ \frac{1}{2}\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dn
\end{aligned}
\tag{7.86}
$$

7.3.3 Default Intensity

The compensated process $K(t)$ is a discountinuous martingale, which have a finite variation by the definition of $N(t)$ and $A(t)$. Therefore the stochastic exponential of $-K(t)$ can be written as

$$
\varepsilon(-K(t)) = e^{-K^c(t)} \prod_{u \le t}(1 - \Delta K(u)) = \exp\{\int_0^t h(u)du\} \prod_{u \le t}(1 - \Delta K(u)) \tag{7.87}
$$

Since $K(t)$ has only one jump at $t = \tau$, the stochastic exponential can be expressed as

$$
\varepsilon(-K(t)) = \mathbb{I}_{\{\tau > t\}} \exp\{\int_0^t h(u)du\} \tag{7.88}
$$

Now, considering the discounted defaultable bank account process $\widetilde{B}_c(t)$, which can be stated as,

$$
\widetilde{B}_c(t) = \mathbb{I}_{\{\tau > t\}} exp\{\int_0^t (r_c(u) - r(u))du\} \tag{7.89}
$$

By the main hypothesis that every discounted(with risk-free) asset is a martingale under risk-neutral measure Q, we assume that this expression is a martingale. However, it is seen that (7.89) is the stochastic exponential of the process

$$
K^*(t) = -N(t) + \int_0^t (r_c(u) - r(u))du \tag{7.90}
$$

This can also be verified by taking the logarithm of the (7.89). Since the compensator of the process $N(t)$ is unique, we reached the following conclusion similar to [54].

$$
K(t) - K^*(t) = \int_0^t (h(u) - r_c(u) - r(u))du \tag{7.91}
$$

Hence, the default intensity is equal to the credit spread defined by $\gamma(t) = r_c(t) - r(t)$

$$
h(t) = r_c(t) - r(t) \tag{7.92}
$$

By inserting (7.92) into the drift condition we found in (7.86), drift condition becomes,

$$\int_t^s \alpha_c(t,u)du = \frac{1}{2}\int_0^t (\int_t^s \sigma_f(t,u)du)^2 F^2(t,n)dn + \frac{1}{2}\int_t^s (\int_n^s \sigma_f(t,u)du)^2 F^2(t,n)dn \tag{7.93}$$

Differentiating this expression with respect to s we obtain the drift condition for the defaultable instantaneous forward rates as,

$$\alpha_c(t,s) = \sigma_c(t,s)\left[\int_{u=t}^s \left(\int_{n=0}^u \sigma_c(t,u)F^2(t,n)dn\right) du\right] \tag{7.94}$$

By using the definition of the covariance structure $c(t,s_1,s_2)$ of the random field $M(t,s)$ as we did in Chapter 6.

$$c(t,s_1,s_2)dt = d\langle M(.,s_1), M(.,s_2)\rangle = \int_0^{s_1 \wedge s_2} F_{s_1}(t,n)F_{s_2}(t,n)dndt \tag{7.95}$$

We reached the drift condition for defaultable forward rates under martingale measure as,

$$\alpha_c(t,s) = \sigma_c(t,s)\left[\int_t^s \sigma_c(t,u)c(t,s,u)du\right] \tag{7.96}$$

Up to now, the modeling is under the risk-neutral martingale measure Q. Now we give the corresponding change of measure theorem that is used for transferring from objective probability measure to risk-neutral probability measure.

Theorem 7.3.1 (Change of Measure)**.** *Assume that the default process has an intensity $h(t)$. Let $\theta(t,s)$ and $\phi(t)$ a strictly positive F_t progressive processes such that*

$$\int_0^{T^*}\int_0^{T^*} \theta(u,v)dudv < \infty$$

and

$$\int_0^{T^*} |\phi(u)-1|h(u)du < \infty$$

Defining the process

$$\frac{dL(t)}{L(t-)} = -\int_0^s \theta(t,u)W(dt,du) + [\phi(t)-1]K(dt) \tag{7.97}$$

where L(0)=1. Then there is a probability measure Q equivalent to P such that

$$W^Q(dt,s) = \int_0^s \theta(t,v)dtdv + W(dt,s) \tag{7.98}$$

gives the W^Q Brownian sheet and

$$h_Q(t) = \phi(t)h(t) \tag{7.99}$$

is the intensity of the default indicator process under Q.

Proof. It follows directly by replacing $W(t)$ in [54] with our $M(t, s)$. $\square$

Chapter 8

Zero Coupon Bond Prices Via Nuclear Space Valued Semi-Martingales

8.1 Introduction

One of the most recent and advanced techniques employed for modeling term structure dynamics is using the infinite dimensional framework, referred most of the time as random field or string models. The basic motivation behind using infinite dimensional settings is their ability to capture the correlation structure of zero-coupon bonds with different maturities in a parsimonious and accurate manner. Using finite dimensional models leads to certain inconsistencies not only in practical implementations such as hedging contingent claims and calibration of term structure but also in statistical descriptions of the bond prices [13]. Although infinite dimensional models are not a panacea for all the complications inherited with finite dimensional counterparts, they become popular in term structure modeling especially with the introduction of random field models by Kennedy [34, 35] and Goldstein [20]. For an extensive review of these model, reader is referred to the Chapter 6 of this thesis. In this section, we extend the infinite-dimensional framework by placing the stochastic components, both continuous and discontinuous, of the forward rate on a pair of nuclear spaces in duality and finding the martingale condition of discounted zero-coupon bond prices to preclude the arbitrage.

In this part, we introduce a noise process taking values in the dual F' of a nuclear space F, which is also supposed to be nuclear. Such examples of nuclear dual pairs are well known items of mathematical analysis. For example, any finite dimensional vector space is nuclear, since any operator on a finite dimensional vector space is nuclear. Additionally and more relevant to our interest, the space of smooth functions on any compact manifold, the Schwartz space of smooth functions on $\mathbb{R}^n$, for which the derivatives of all orders are rapidly decreasing and the space of infinitely differentiable functions with compact support can be given as examples of nuclear spaces. This would stress the importance of our approach. The stochastic integration with respect to F'-valued square integrable martingales have almost all trajectories in a Hilbertian subspace of F'. Hence, this observation reduces the stochastic integration with respect to F' valued square integrable martingales to the stochastic integration on a Hilbert space. This idea is fully used in [41] where the method of Metivier and Pistone [46] was revised. This same approach has been used here as follows. The forward interest rate is represented as

$$f(t,s) = f(0,s) + \int_0^t \mu(u, s-u)du + \int_0^t \sigma(u, s-u)dM(u)$$

This type of model was also considered by Ozkan and Schmidt [50] for the case of square integrable Hilbert space valued Levy Processes where σ was considered as a uniformly bounded functional. In our approach, the stochastic integral with respect to F'-valued square integrable martingales has an extended version to not necessarily continuous functional valued processes. Instead of giving a result concerning the global representation of $f(t,T)$, given above, we wanted to separate the continuous and discontinuous parts of M. However, as a remark, we can not use the same σ for both the continuous and discontinuous parts of M because they do not guarantee the same space of integrable processes.

In the following, we give first the necessary mathematical preliminaries regarding the nuclear space valued martingales and their stochastic integral. After giving foundations, the next section is devoted to the case of a general square integrable martingale and the corresponding extended HJM condition. The second model that we consider concerns the case where the discontinuous part of M is generated by a F' valued Markov Jump Process. Finally we consider the case of a square integrable Levy Process with values in F'. Our result is very similar to that of [50]. The difference lies only in our extended definition of stochastic integrals.

8.2 Preliminaries and Construction of Stochastic Integral

8.2.1 Nuclear Spaces

The topological vector spaces considered here are over the field $\mathbb{R}$. Given two locally convex vector spaces in duality (E, E'), where E' denotes the dual of E, $e'(e)$ or (e', e) or, if more precision is needed, $(e', e)_{E'E}$ will represent the value of $e' \in E'$ at $e \in E$. For any absolutely convex set $A \subset E$, p_A will denote its gauge. For two locally convex spaces E and F, the space of continuous linear mappings of E into F is denoted by $L(E, F)$. We refer to Schaefer's book [53] for the general properties of topological vector spaces and definitions used in this work.

Definition 8.2.1 (Nuclear Space)**.** A nuclear space is a locally convex topological vector space V such that for any seminorm p we can find a larger seminorm q so that the natural map from V_q, Banach space given by completing V using the seminorm q, to V_p is a nuclear operator.

Let E be a complete nuclear space. If U is an absolutely convex neighborhood of zero in E, $E(U)$ is the completion of the normed space $(E/p_U^{-1}(0), p_U)$ and $k(U)$ the canonical mapping of E into $E(U)$. For two absolutely continuous convex neighborhoods of 0, U and V in E such that $U \subset V$, the canonical mapping of $E(U)$ into $E(V)$ is denoted by $k(V, U)$ and satisfies the relation: $k(V, U) \circ K(U) = k(V)$. Since E is nuclear there exists a neighborhood base $\mathcal{U}_h(E)$ such that $\forall\ U \in \mathcal{U}_h(E)$, $E(U)$ is a separable Hilbert space and for all $U, V \in \mathcal{U}_h(E)$ such that $U \subset V$ the canonical mappings $k(U)$ and $k(V, U)$ are nuclear operators.

If B is any non-empty closed, bounded and absolutely convex subset of E, then $E[B]$ denotes the Banach subspace of E generated by B and equipped with the norm p_B. The canonical injection of $E[B]$ into E is denoted by $i(B)$. For two bounded and absolutely convex closed subsets A and B of E such that $A \subset B$, the canonical injection of $E[A]$ into $E[B]$ is denoted by $i(B, A)$.

8.2.2 Construction of the Stochastic Integral

The stochastic integral on nuclear spaces was introduced by Ustunel [56] for semi-martingales, by Korezlioglu and Martias [41] for square integrable martingales and Brownian motion.

In this work F represents a nuclear space which is separable and complete. Its strong topological dual F' is also supposed to be complete and nuclear.The fact that F and F' are complete nuclear spaces implies their reflexivity.

For $U \in \mathcal{U}_h(F)$, U^0 denotes its polar and $F'[U^0]$ is shown to be isometric to $F(U)'$, the topological dual of $F(U)$.

All random variables and processes considered here are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfying the usual conditions.

We put $\Omega' = \mathbb{R}_+ \times \Omega$. A mapping $X : \Omega' \to F'$ is called a weakly measurable process, if for all $\phi \in F'$ and all $t \in \mathbb{R}_+$, $X_t(\phi)$ is a real random variable. We refer to Ustunel [56] for an introduction to nuclear space valued semi-martingales. A real-valued semi-martingale is a right-continuous adapted process $\{X_t, t \in \mathbb{R}_t\}$ having the following decomposition.

$$x_t = m_t + a_t, \qquad a_0 = 0$$

where m is a local martingale and a is a finite(bounded) variation process. The space S denotes the Banach space of real-semimartingales having finite norms

$$\|x\|_1 = inf\left[\mathbb{E}\left([x,x]_\infty^{\frac{1}{2}} + \int_0^\infty |da_s|\right)\right]$$

where the infimum is taken over all decompositions $x = m+a$ and where $[x,x]_\infty$ denotes the total quadratic variation of x. If x is a special semi-martingale, i.e., if a is predictable, then the decomposition $x = m + a$ is unique. Now we give the definition of a F' valued semi-martingale.

Definition 8.2.2. Let X be a weakly measurable process on $(\mathbb{R}_t\times\Omega, \mathcal{B}(\mathbb{R}_t) \bigotimes \mathcal{A})$ with values in F'. Then X is called a (F'-valued) semi-martingale if for any $\phi \in F$ the stochastic process $(t,\omega) \mapsto (\phi, X(t,\omega))_{F,F'}$ has a modification $(t,\omega) \mapsto \tilde{X}_t(\phi)(\omega)$ which is a semi-martingale in S.

Ustunel [56] defined a F'-valued semi-martingale as a projective system of semi-martingales and gave the above class of semi-martingales as a particular class.

We consider here only this type of semi-martingales because of their interesting property expressed in the following.

Proposition 8.2.1. *There is a neighborhood $G \in \mathcal{U}_h(F)$ such that almost all trajectories of X are in $F'[G^0]$.*

Definition 8.2.3. A weakly measurable F'-valued process X is called a square integrable martingale if for all $\phi \in F$, $X(\phi) := ((X_t(\omega), \phi)_{F',F}; (t,\omega) \in \Omega)$, has a modification in $\mathfrak{M}^2(\mathbb{R})$, the space of real valued square integrable martingales. Similarly, M is said to be a square integrable martingale if for all $\phi \in F$, $M(\phi)$ has a modification in $\mathfrak{M}^2_c(\mathbb{R})$, the space of continuous real valued square integrable martingales.

Remark 8.2.1. We rather adopted here the above definition for possible future applications. In this paper we only deal with F'-valued square integrable martingales. Therefore we could give the characterization of square integrable F'-valued martingales by replacing the space S of semi-martingales by the space S of real-valued square-integrable martingales as in [41, 40].The property we are interested in is that there is a neighborhood $G \in \mathcal{U}_h(F)$ such that almost all trajectories of a F'-valued square integrable martingale are in $F'[G^0]$.

In what follows $\mathfrak{M}^2(F, F')$ will represent the space of F'-valued square integrable martingales and M a particular element of this space. We denote by $\mathcal{U}_h(F, M)$ the set of all neighborhoods $U \in \mathcal{F}_h(F)$ such that M is the injection of an $F'[U^0]$-valued square integrable martingales according to the above proposition.

Now we fix a neighborhood $G \in \mathcal{U}_h(F, M)$. We identify $F(G)$ with $F'[G^0]$ and we denote both of them by H. This is a separable Hilbert space and M is a H-valued square-integrable martingale. At this point we need some notations concerning nuclear and Hilbert-Schmidt operators on H.

- $L(H, H)$ is the space of all bounded operators on H into H with the uniform norm $\| \,.\, \|$.
- $L^1(H, H)$ is the space of nuclear operators on H into H with the norm $\| \,.\, \|_1$.
- $L^2(H, H)$ is the space of Hilbert-Schmidt operators on H into H with the Hilbert Schmidt norm $\| \,.\, \|_2$.

- $H\widehat{\bigotimes}_1 H$ (resp. $H\widehat{\bigotimes}_2 H$) is the projective(resp. Hilbertian) tensor products of H with H.

For notational convenience, $H\widehat{\bigotimes}_1 H$ (resp. $H\widehat{\bigotimes}_2 H$) are identified with $L^1(H, H)$ (resp.$L^2(H, H)$ under the isometry which puts $h \bigotimes k$ into a one-to-one correspondence with $(., h)_H$. Here and in what follows $(. , .)_H$ denotes the scalar product on H. H-valued martingales are always taken with their cadlag versions.We denote by $\mathfrak{M}^2(H)$, the space of H(separable)-valued square-integrable martingales.

Definition 8.2.4. Given two martingales $M, N \in \mathfrak{M}^2(H)$, the space of square integrable H-valued martingale, there is a unique $H\widehat{\bigotimes}_1 H$-valued cadlag predictable process with integrable variation, denoted by $\langle M, N\rangle$ and called the "oblique" bracket of (M, N), such that $M \bigotimes N - \langle M, N\rangle$ is a $H\widehat{\bigotimes}_1 H$-valued martingale vanishing at $t = 0$. The bracket process $\langle M, M\rangle$ that we denote by $\langle M\rangle$ is called the increasing process of M. We put $\beta_t := \|\langle M\rangle_t\|_1$. This process is the unique predictable increasing process with integrable variation for which $\|M\|^2 - \beta$ is a martingale vanishing at $t = 0$.

From now on M will represent a given martingale in $\mathfrak{M}^2(H)$ and λ will denote the measure $d\mathbb{P}d\beta$. All the operations that we carry out here on stochastic processes and operators are only valued $\lambda - a.e.$ and in order to simplify the notations, we will not always mention it. There exists a predictable process Q with values in the cone of symmetric and non-negative elements of $L^1(H, H)$, unique up to a λ-equivalence, such that $\|Q\|_1 = 1$, $\lambda - a.e.$ and $\langle M\rangle_t = \int_{0-}^t Q_s d\beta_s$.

We consider the following factorization of $Q(t, \omega)$; for $\lambda - a.e.$ there is a predictable operator $D(t, w) \in L^2(H)$ such that $Q = DD^*$, where D^* denotes the adjoint of D.

We define a new scalar product on H by

$$\forall f, g \in H \qquad (f, g)_{\widetilde{H}(t,w)} = (D^*(t, \omega)f, D^*(t, \omega)g)_H \tag{8.1}$$

We complete H with respect to this scalar product and get a Hilbert space that we denote by $\widetilde{H}(t, \omega)$. Obviously, $f \mapsto D^*(t, \omega)f$ is extended to an isometry from $\widetilde{H}(t, \omega)$ into H.

We can construct an orthonormal basis $\{\tilde{e}_n(t,\omega), n \in \mathbb{N}\}$ of $\widetilde{H}(t,\omega)$ of predictable processes such that $\{\tilde{e}_n(t,\omega)\}$ is also an element of H [41]. Let X be a predictable process such that $X(t,w) \in \widetilde{H}(t,\omega)$ and

$$\int_{0-}^{\infty} \|X(t,\omega)\|^2_{\widetilde{H}(t,\omega)} d\lambda(t,\omega) < \infty \tag{8.2}$$

Such a process can be written as follows

$$X(t,\omega) = \sum_{n=0}^{\infty} a_n(t,\omega)\tilde{e}_n(t,\omega) \tag{8.3}$$

where a_n is a predictable real process such that

$$\sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n^2(s,\omega) d\lambda(s,w) < \infty. \tag{8.4}$$

The vector space Λ^2 of all predictable $\widetilde{H}$-valued processes X satisfying (8.2) or equivalently (8.4) is a Hilbert Space, denoted as $\Lambda^2(D,H)$ with the scalar product,

$$(X,Y)_{\Lambda^2} = \int_{0-}^{\infty} (X(s,\omega), Y(s,\omega))_{\widetilde{H}(s,\omega)} d\lambda(s,\omega) \tag{8.5}$$

or equivalently

$$(X,Y)_{\Lambda^2} = \sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n(s,\omega) b(s,\omega) d\lambda(s,\omega) \tag{8.6}$$

with

$$Y(t,\omega) = \sum_{n=0}^{\infty} b_n(t,\omega)\tilde{e}_n(t,\omega) \tag{8.7}$$

We refer to [46, 41] for the definitions and properties of a stochastic integral on Hilbert spaces. Let $X.M$ be the square integrable martingale defined by

$$(X.M)(t,\omega) = \int_{0-}^{t} X(s,\omega) dM(s,\omega) \tag{8.8}$$

We have

$$\langle X.M, Y.M\rangle(t,\omega) = \int_{0-}^{t} (X(s,\omega), Y(s,\omega))_{\widetilde{H}(s,\omega)} d\beta(s,\omega) \tag{8.9}$$

Let us put

$$M_n(t,\omega) = \int_{0-}^{t} \tilde{e}_n(s,\omega) dM(s,\omega) \tag{8.10}$$

Finally, using (8.9) we get

$$\begin{aligned} \langle M_n, M_m\rangle(t,\omega) &= \int_{0-}^{t} (\tilde{e}_n(s,\omega), \tilde{e}_m(s,\omega))_{\widetilde{H}(s,\omega)} d\beta(s,\omega) \\ &= \beta(t,\omega), \qquad if \quad n = m \\ &= 0, \qquad\qquad if \quad n \neq m \end{aligned} \tag{8.11}$$

$\{M_n, n \in \mathbb{N}\}$ is thus a sequence of mutually orthogonal square-integrable real martingales, with a common increasing process $\beta(t, \omega)$. If X is represented by (8.3) the stochastic integral (8.8) is then given by

$$(X.M)(t, \omega) = \sum_{n=0}^{\infty} \int_{0-}^{t} a_n(s, \omega) dM_n(s, \omega) \tag{8.12}$$

with

$$\langle X.M \rangle(t, \omega) = \sum_{n=0}^{\infty} \int_{0-}^{t} a_n^2(s, \omega) d\beta(s, \omega) \tag{8.13}$$

and

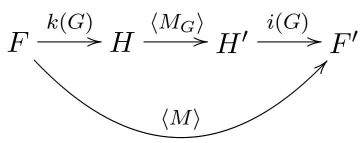
$$\mathbb{E}[(X.M)_t]^2 = \sum_{n=0}^{\infty} \int_{0-}^{t} a_n^2(s, \omega d\lambda(s, \omega) \tag{8.14}$$

In conclusion, the stochastic integral (with respect to M) of any process X in $\Lambda^2(D, H)$ provides a real square integrable martingale such that $\|X\|_{\Lambda^2}^2$ is equal to the left hand side of (8.2). Such a process X has the representation (8.3), with $\|X\|_{\Lambda^2}^2$ given by the left hand side of (8.4). Each element of Λ^2 has the representation (8.3) with the stochastic integral given by (8.12) and having the properties (8.13) and (8.14).

Now, we go back to our martingale $M \in \mathfrak{M}^2(F, F')$ considered as a $H(\cong F(G) \cong F'[G^0])$-valued squared martingale. We know from [41] that there is a predictable process $\langle M \rangle$, unique up to an evanescent set (with respect to λ), with values in the set of symmetric, nonnegative nuclear operators $L(F, F')$ such that, $\forall f, g \in F$

$$(\langle M \rangle f, g)_{F', F} = \langle M(f), M(g) \rangle \tag{8.15}$$

with $M(f)(t, \omega) = (M(t, \omega), f)_{F', F}$.We call $\langle M \rangle$ the increasing process of M. Let $\langle M_G \rangle$ be the increasing process of M considered as a H-valued martingale. We then have the following diagram.



Consider the following representation of $\langle M_G \rangle$:

$$\langle M_G \rangle(t, \omega) = \int_{0-}^{t} Q_G(s, \omega) d\beta_G(s, \omega) \tag{8.16}$$

We use again the factorization $Q_G = D_G \circ D_G^*$ where D_G is a predictable process with values in the Hilbert-Schmidt operators on H. Let us put $D = i(G) \circ D_G$. Then $D^* = D_G^* \circ k(G)$. For $f, g \in F$ we introduce a scalar product

$$(f, g)_{\widetilde{H}(t,\omega)} = (D^* f, D^* g)_H \tag{8.17}$$

We complete F with respect to this scalar product. As in the Hilbertian case we can construct a Hilbert space of $\widetilde{H}(t, \omega)$-valued predictable process X such that

$$\int_{0-}^{\infty} \|X(t, \omega\|^2_{\widetilde{H}(t,\omega)} d\lambda(s, \omega) < \infty. \tag{8.18}$$

Remark 8.2.2. Remember that this space is constructed over the equivalence classes with equivalence relation $f_1 \sim f_2$ if and only if $p_G(f_1 - f_2) = 0$. We can also identify this space with space Λ^2 generated by $H(\cong F(G) \cong F'[G^0])$.

Here $\Lambda^2(D, F, F')$ will denote the space of $\widetilde{H}(t, \omega)$-valued predictable processes satisfying the above diagram. We can repeat here verbatim what we have said for $\Lambda^2(D, F)$. We are interested in the representation (8.3) of elements of $\Lambda(D, F, F')$ with the series representation (8.10) of the elements of $\mathfrak{M}^2(F, F')$ and the definition of the stochastic integral by (8.12), followed by (8.13) and (8.14).

Remark 8.2.3. The construction of the stochastic integral we developed here entirely depends on the factorization of $Q = DD^*$. One can prove that $\Lambda^2(D, H)$ does not depend on this factorization. If $Q = BB^*$ is another factorization, then $\Lambda^2(D, H) = \Lambda^2(B, H)$. Moreover, each element of $\Lambda^2(B, H)$ is an isometric image of an element of $\Lambda^2(D, H)$ [41]. One can prove a better statement. $X.M$ can be defined independently on the factorization of Q. The proof of this fact will be given in a forthcoming paper.

Next we will give the modeling approach of bond prices by using the above construction.

8.3 Application to Bond Prices

In this section we consider zero-coupon bond prices where forward interest rates are expressed in terms of semi-martingales. More precisely, if $P(t, T)$ represents the zero-coupon bond price at t with maturity T, then

$$P(t,T) = exp\left(-\int_t^T f(t,s)ds\right) \tag{8.19}$$

where the forward interest rate $f(t,s)$ is

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s-u)du + \int_0^t \sigma_c(u,s-u)dM_c + \int_0^t \sigma_d(u,s-u)dM_d \tag{8.20}$$

with M_c and M_d being square integrable continuous and discontinuous martingales, respectively. This type of forward interest rate formula was already considered by [39] under the name of mixed HJM-Musiela type model.

Note 8.3.1. *We use here the notation of the preceding section by putting an index c or d to the element, corresponding to M_c and M_d, respectively. For β_i, λ_i, Q_i, D_i, $H_i(t,\omega)$, $e_{i,n}(t,\omega)$, Λ_i^2 with $i = c$ or d, respectively.*

In what follows, M_d can be seen as the compensated process of X_d, a pure jump process which may have the semimartingale representation

$$X_d(t) = M_d(t) + K(t)$$

where $K(t)$ is the compensator of the process $X_d(t)$. We can define the random measure of the jumps of the process X_d as

$$\delta((0,t] \times B) = \sum_{u \leq t} \mathbb{I}_B(\Delta X_d)$$

for a Borel set of H. Additionally, the compensator of δ^d, ν^d can be defined such that

$$M_d = \int_0^t \int_H m(\delta^d - \nu^d)(du, dm).$$

We suppose that the probability measure $\mathbb{P}$ we have been considering here is a risk neutral probability measure, implying that the discounted bond prices $P(t,T)$ are $\mathbb{P}$ martingales. We have the following assumptions for forward rate dynamics.

Assumption 8.3.1. *For each (u,s), such that $u \leq t \leq s \leq T$, $\sigma_i(u,s-u)$ is supposed to belong to Λ_i^2. Moreover it is assumed that*

$$\int_0^T \int_0^T |\mu(u,s)|duds < \infty \tag{8.21}$$

We define

$$\mu^*(u,T) = \int_0^{T-u} \mu(u,s)ds \tag{8.22}$$

and

$$\sigma_i^*(u,T) = \int_0^{T-u} \sigma_i(u,s)ds \tag{8.23}$$

It can be seen that $\sigma_i^*(.,T)$ is also in Λ_i^2. Notice that the short term interest rate is $f(t,t)$.

Under the above assumptions, we can prove the following theorem.

Theorem 8.3.1. *Let $\widetilde{P}(t,T)$ be the discounted bond price,that is,*

$$\widetilde{P}(t,T) = \left[exp(-\int_0^t f(u,u)du)\right] P(t,T).$$

Then it has the following expression as a semi-martingale.

$$\begin{aligned}\widetilde{P}(t,T) = \widetilde{P}(0,T) &- \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) \\ &- \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u) + \frac{1}{2}\int_0^t \sum_{n=0}^{\infty} \widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} \\ &+ \int_{0-}^t \int_H \widetilde{P}(u-,T)\left\{exp\left[\sigma_d^*(u-,T)m\right] - 1 - \sigma^*(u-,T)m\right\}\delta^d(du,dm)\end{aligned} \tag{8.24}$$

Remark 8.3.1. Let us consider the series expansion

$$\sigma_c^*(t,T) = \sum_{n=0}^{\infty} \int_0^{\infty} \sigma_{c,n}^*(u,T)e_{c,n} \tag{8.25}$$

and

$$\int_0^t \sigma_c^*(u,T)dM_c(u) = \sum_{n=0}^{\infty} \int_0^t \sigma_{c,n}^*(u,T)dM_{c,n}(u) \tag{8.26}$$

where

$$\sigma_{c,n}^*(u,T) = \int_0^{T-u} \sigma_{c,n}(u,s)ds$$

We could then write

$$\int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) = \int_0^t \sum_{n=0}^{\infty} \widetilde{P}(u-,T)\sigma_{c,n}^*(u,T)dM_{c,n}(u) \tag{8.27}$$

and

$$\int_0^t \widetilde{P}(u-,T)d\langle\sigma_c^*(.,T).M\rangle_u = \int_0^t \sum_{n=0}^{\infty} \widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} \tag{8.28}$$

Proof. The proof of the theorem is an immediate consequence of the Ito formula applied to $\widetilde{P}(t,T)$. Let us first give its expression. Put

$$Y(t) = \int_0^T f(t,s)ds$$

Differentiating $Y(t)$ and using the Fubini Theorem we find,

$$\begin{aligned} Y(t) = &- \int_0^t f(u,u)du + \int_0^t \left(\int_t^T \mu(u,s-u)ds \right) du \\ &+ \int_0^t \left(\int_t^T \sigma_c(u,s-u)ds \right) dM_c(u) + \int_0^t \left(\int_t^T \sigma_d(u,s-u)ds \right) dM_d(u) \\ = &- \int_0^t f(u,u)du + \int_0^t \mu^*(u,T)du \\ &+ \int_0^t \sigma_c^*(u,T)dM_c(u) + \int_0^t \sigma_d^*(u,T)dM_d(u) \end{aligned} \tag{8.29}$$

According to the above assumptions, the last two stochastic integrals are well defined and give square-integrable martingales. The Ito formula applied to

$$\widetilde{P}(t,T) = exp\left[\left(- \int_0^t f(u,u)du - Y(t) \right) \right]$$

gives

$$\begin{aligned} \widetilde{P}(t,T) = &\widetilde{P}(0,T) - \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) \\ &- \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u) + \frac{1}{2}\int_0^t \widetilde{P}(u-,T)d\langle \sigma_c^*(.,T).M\rangle_u \\ &+ \sum_{u\le t} \left[\widetilde{P}(u,t) - \widetilde{P}(u-,T) - \widetilde{P}(u-,T)\Delta Y(u) \right] \end{aligned} \tag{8.30}$$

We need to express the last sum in terms of ΔM_d. We have

$$\frac{\widetilde{P}(u,T)}{\widetilde{P}(u-,T)} = exp\left[\sigma_d^*(u,T)\Delta M_d(u)\right] \tag{8.31}$$

and

$$\Delta\widetilde{P}(u,T) = \widetilde{P}(u-,T)\left[\frac{\widetilde{P}(u,T)}{\widetilde{P}(u-,T)} - 1 \right] \tag{8.32}$$

Therefore,

$$\begin{aligned} &\sum_{u\le t} \left[\widetilde{P}(u,t) - \widetilde{P}(u-,T) - \widetilde{P}(u-,T)\Delta Y(u) \right] \\ =&\sum_{u\le t} \widetilde{P}(u-,T)\left\{ exp[\sigma_d^*(u,T)\Delta M_d(u)] - 1 - \sigma_d^*(u,T)\Delta M_d(u) \right\} \end{aligned} \tag{8.33}$$

The last term can also be written as

$$\int_{0-}^{t}\int_{H}\widetilde{P}(u-,T)\left\{exp\left[\sigma_d^*(u-,T)m\right]-1-\sigma^*(u-,T)m\right\}\delta^d(du,dm)$$

Then by using remark (8.3.1), we find the bond price formula. □

From this we deduce the following proposition.

Proposition 8.3.1 (Extended Heath-Jarrow-Morton drift condition)**.** *Since $\widetilde{P}(t,T)$ should be a martingale, we have*

$$\int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du = \frac{1}{2}\int_0^t\sum_{n=0}^{\infty}\widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n} + \int_{0-}^{t}\int_{H}\widetilde{P}(u-,T)\rho_m(u,T)\nu^d(du,dm) \tag{8.34}$$

where

$$\rho_m(u,T) = exp\left[\sigma_d^*(u-,T).m\right]-1-\sigma^*(u-,T).m$$

Proof. In order for $\widetilde{P}(t,T)$ to be a martingale we need to cancel all the terms of (8.24) which are not stochastic integrals. By rearranging the equation (8.24) as

$$\begin{aligned}\widetilde{P}(t,T) = \widetilde{P}(0,T) &- \int_0^t \widetilde{P}(u-,T)\mu^*(u,T)du - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u)\\ &- \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u) + \frac{1}{2}\int_0^t\sum_{n=0}^{\infty}\widetilde{P}(u-,T)[\sigma_{c,n}^*(u,T)]^2 d\beta_{c,n}\\ &+ \int_{0-}^{t}\int_{H}\widetilde{P}(u-,T)\rho_m(u,T)(\delta^d(du,dm)-\nu^d(du,dm))\\ &+ \int_{0-}^{t}\int_{H}\widetilde{P}(u-,T)\rho_m(u,T)\nu^d(du,dm)\end{aligned} \tag{8.35}$$

the martingale condition implies

$$\begin{aligned}\widetilde{P}(t,T) =& \widetilde{P}(0,T) - \int_0^t \widetilde{P}(u-,T)\sigma_c^*(u,T)dM_c(u) - \int_0^t \widetilde{P}(u-,T)\sigma_d^*(u,T)dM_d(u)\\ &+ \int_{0-}^{t}\int_{H}\widetilde{P}(u-,T)\rho_m(u,T)(\delta^d(du,dm)-\nu^d(du,dm))\end{aligned} \tag{8.36}$$

Then condition (8.34) is automatically satisfied. □

Corollary 8.3.2. *If the measures β and ν are absolutely continuous with respect to the Lebesque measure, then the drift condition becomes,*

$$\mu^*(t,T) = \frac{1}{2}\sum_{n=0}^{\infty}[\sigma^*_{c,n}(u,T)]^2\frac{d\beta_c(t)}{dt} + \frac{d\chi(t)}{dt} \tag{8.37}$$

where

$$\chi(t) = \int_0^t \int_H \{exp\,[\sigma^*_d(u-,T).m] - 1 - \sigma^*_d(u-,T).m\}\,\nu(du,dm)$$

We will see various cases where the above absolute continuity conditions are satisfied. As the first example, the absolute continuity of ν is obtained in the case where M_d is derived from a Markov Jump Process.

8.3.1 Bond Prices with Markov Jump Process

Let $\{X(t),\, t \in [0,T]\}$ be a Markov Jump Process with values $\{f_n,\, n \in \mathbb{N}\} \subset F'$.In order to guarantee the right continuity we suppose that X takes its values at the jump points. The sojourn time at the value $X(t)$ is an exponential random variable with parameter $\lambda(X(t)) \in (0,+\infty)$. We suppose that

$$sup_n\,\lambda(f_n) < \infty$$

This ensures the regularity, i.e., the process does not explode. Let $p(f_j, f_i)$ be the transition probability of X from state f_j to state f_k. If $X(t)$ is the state just before the jump to f_j, the jump size is represented by

$$\xi(X(t)) = f_j - X(t)$$

Suppose that

$$\mathbb{E}[|\xi(X(t))|] = \sum_{j\in\mathbb{N}} p(X(t), f_j)|f_j - X(t)| < \infty. \tag{8.38}$$

we then put

$$\mathbb{E}[\xi(X(t))] = \sum_{j\in\mathbb{N}} p(X(t), f_j)(f_j - X(t)) \tag{8.39}$$

This is the conditional expected size of jumps. We define

$$K(t) = \int_0^t \lambda(X(s))\mathbb{E}[\xi(X(s))]ds \tag{8.40}$$

We claim that this is the compensator of X. In fact, the probability that a jump occurs in the elementary interval $(t, t+dt]$ when X is in state $X(t)$ is $\lambda(X(t))dt$

and the conditional expected jump size is $\mathbb{E}[\xi(X(t))]$. Therefore $dX(t) - dK(t)$ is the increment of a martingale.

In view of Definiton 2.3, we would like to look for conditions under which

$$N(t) = X(t) - K(t)$$

is a square integrable martingale. Let us choose $\phi \in F$ and define,

$$N_\phi(t) := (N(t), \phi)_{F,F'}$$

$$X_\phi(t) := (X(t), \phi)_{F,F'}$$

$$K_\phi(t) := (K(t), \phi)_{F,F'}$$

Consider first the real valued Markov Jump Process $X_\phi(t)$. It has a cadlag version because of the right continuity and regularity of X. Let us put

$$\begin{aligned}\mathbb{E}[\xi_\phi(X_\phi(t))] &= \sum_{j=0}^{\infty} p\left(X(t), f(j)\right)[(f_j, \phi)_{F,F'} - X_\phi(t)] \\ &= \sum_{j=0}^{\infty} p\left(X(t), f(j)\right)(f_j - X(t), \phi)_{F,F'} \\ &= (\mathbb{E}[\xi(X(t))], \phi)_{F'F}.\end{aligned} \tag{8.41}$$

It is also seen that

$$K_\phi(t) = (K(t), \phi)_{F,F'} = \int_0^t \lambda(X(s))(\mathbb{E}[\xi(X(s))]ds \tag{8.42}$$

is the compensator of X_ϕ. Here again $\lambda(X(t))(\mathbb{E}[\xi(X(t))]dt$ represents the conditional expectation of the jump size of X_ϕ given that that there is a jump of $X(t)$ in the interval $(t, t+dt]$. Now, we need to give conditions under which N_ϕ is a square integrable martingale.

Proposition 8.3.2. *For $k = 1\, and\, 2$, define*

$$\mathbb{E}[|\xi_\phi(X_\phi(t))|^k] = \sum_{j=0}^{\infty} p\left(X(t), f(j)\right)|(f_j, \phi)_{F,F'} - X_\phi(t)|^k \tag{8.43}$$

If there is a positive constant C such that

$$\mathbb{E}[|\xi_\phi(X_\phi(t))|^k] \leq C(1 + |X_\phi(t)|^k) \tag{8.44}$$

then N_ϕ is a square integrable martingale.

Proof. If (8.44) holds we can write

$$\lambda(X(t))\mathbb{E}[|\xi_\phi(X(t))|^k] \leq sup_n \, \lambda(f_n)C(1 + |X_\phi(t)|^k)$$

Therefore, according to Klebaner[37], N_ϕ is a square-integrable martingale. □

As a consequence, according to Definition 8.2.3 we see that the compensated Markov Jump Process $N(t)$ is a F'-valued square-integrable martingale. This will represent our M^d in the general setting of the previous section.

Proposition 8.3.3. *If the dynamics of the forward interest rates are given by the following,*

$$f(t,s) = f(0,s) + \int_0^t \mu(u, s-u)du + \int_0^t \sigma(u, s-u)dX(u) \tag{8.45}$$

where X is a Markov Jump Process, then the drift condition becomes

$$\mu^*(t,T) = \int_H \{exp\,[\sigma^*(u-,T).m] - 1 - \sigma^*(u-,T).m\}\, \lambda(X(t))\mathbb{E}(\xi(X(t)) \tag{8.46}$$

Proof. By using the definition of the compensator of $X(t)$ and the Corollary (8.3.2) □

8.3.2 Bond Prices with Levy Processes

Here we consider a centered Levy process

$$L(t) = W(t) + M_d(t)$$

where $M_d(t)$ is the compensated jump part of L. In order to follow our approach, we suppose that L is a square integrable martingale. We use again the Hilbertian space $H(\cong F(G) \cong F'[G])$ where all trajectories of L are concentrated. It is known [57] that the Hilbert space valued Levy Process has the increasing process

$$\langle L \rangle_t = \int_0^t Qdt = \int_0^t (Q/TrQ)(TrQ)dt.$$

With our preceding notations, the covariance operator $Q_t = Q/TrQ$ and $\beta_t^L = tTrQ$. Instead of factorizing Q_t we construct $\widetilde{H}(t,\omega)$ in such a way that the measure $\lambda(dt, d\omega)$ is replaced by $dt\mathbb{P}(d\omega)$. We consider the factorization $D \circ D^*$

of Q with $D \in L^2(H, H)$. Let us define a scalar product $(f, g)_{\widetilde{H}} = (D^* f, D^* g)_H$. The completion of H under this scalar product $(f, g)_{\widetilde{H}}$ is denoted by $\widetilde{H}$. If $\{\tilde{e}_n, n \in \mathbb{N}\}$ is an orthogonal basis in $\widetilde{H}$, then the space Λ^2 is the space of predictable processes X such that

$$X(t, \omega) = \sum_{n=0}^{\infty} a_n(t, \omega)\tilde{e}_n(t, \omega) \tag{8.47}$$

where a_n is a predictable real process such that

$$\sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n^2(s, \omega) ds \mathbb{P}(d\omega) < \infty. \tag{8.48}$$

The vector space Λ^2 of all predictable $\widetilde{H}$-valued processes X satisfying (8.2) or equivalently (8.4) is a Hilbert Space, denoted as $\Lambda^2(D, H)$ with the scalar product,

$$(X, Y)_{\Lambda^2} = \int_{0-}^{\infty} (X(s, \omega), Y(s, \omega))_{\widetilde{H}(s,\omega)} ds \mathbb{P}(d\omega) \tag{8.49}$$

or equivalently

$$(X, Y)_{\Lambda^2} = \sum_{n=0}^{\infty} \int_{0-}^{\infty} a_n(s, \omega) b(s, \omega) ds \mathbb{P}(d\omega) \tag{8.50}$$

with

$$Y(t, \omega) = \sum_{n=0}^{\infty} b_n(t, \omega)\tilde{e}_n(t, \omega) \tag{8.51}$$

With this setting, we have the following HJM condition.

Proposition 8.3.4. *If the dynamics of the forward interest rates are given by the following,*

$$f(t, s) = f(0, s) + \int_0^t \mu(u, s-u) du + \int_0^t \sigma(u, s-u) dL(u) \tag{8.52}$$

where

$$L(t) = W(t) + M_d(t),$$

then the drift condition becomes

$$\mu^*(t, T) = \frac{1}{2} \sum_{n=0}^{\infty} [\sigma^*(u, T)]^2 + \int_H \{exp\,[\sigma^*(u-, T).x] - 1 - \sigma^*(u-, T).x\}\, F(dx) \tag{8.53}$$

where $F(dx)$ denotes the Levy measure of the jumps.

Proof. By taking $M_c(t) = W(t)$ and using corollary 8.3.2 with $\nu(dt, dx) = dtF(dx)$. □

Remark 8.3.2. The above condition is similar to the one found in [50] for default free case where the Levy random field drives the forward interest rates. The condition in Proposition (8.3.4) differs from the one in [50] that the eigenvalues of the covariance operator do not appear in (8.53). This is due to the construction of the stochastic integral described in section 2.

8.4 Relation between Nuclear-Spaced Martingales and Two Parameter Processes

In this part, we try to find the connection between the nuclear-spaced martingales that we discuss in this section and the two-parameter processes we investigated in chapter 7. Let $M(t,s) = \int_0^t \int_0^s F(u,v)M(du,dv)$ be a two-parameter continuous square integrable martingale. $M(t,.)$ can then be considered as martingale with values in $L^2[0,T]$. Let us consider the dual pair $(\mathcal{D}, \mathcal{D}')$. We can inject $L^2[0,T]$ into $(D)'$ and take $H' = L^2[0,T]$. Let $\phi, \psi \in \mathcal{D}$. The quadratic covariance of the integrals $\phi(s)M(.,s)$ and $\psi(s)M(.,s)$ is given by

$$X(t) = \langle \int_0^T M(.,s)\phi(s)ds, \int_0^T M(.,s)\psi(s)ds\rangle_t \tag{8.54}$$

This is equivalent to

$$\begin{aligned} X(t) &= \langle \int_0^T \phi(s) \left(\int_0^s M(.,v)dv \right) ds, \int_0^T \psi(s) \left(\int_0^s M(.,v)dv \right) ds\rangle_t \\ &= \langle \int_0^T \left(\int_v^T \phi(s)ds \right) M(.,v)dv, \int_0^T \left(\int_v^T \psi(s)ds \right) M(.,v)dv\rangle_t \\ &= \int_0^T \left(\int_v^T \phi(s)ds \right) F(t,v)dv \left(\int_v^T \psi(s)ds \right) F(t,v)dv \end{aligned} \tag{8.55}$$

The last term can also be expressed as

$$\int_0^T \int_0^s \left(\int_v^T \phi(v')dv' \right) F^2(t,v)dv\psi(s)ds. \tag{8.56}$$

Additionally, let Q denote the covariance operator of $M(t,.)$. We can deduce that

$$(D_t^*\phi)(s) = \int_0^T \phi(s)dvF(t,s) \tag{8.57}$$

Hence we can conclude that by $Q_t = D_t D_t^*$

$$
\begin{aligned}
(D_t D_t^*)(s) &= \int_0^s \left(\int_v^T \phi(v')dv' \right) F^2(t,v)dv \\
Q_t(s) &= \int_0^s \left(\int_v^T \phi(v')dv' \right) F^2(t,v)dv
\end{aligned}
\tag{8.58}
$$

Therefore, the relation that we are looking for can be given as

$$
X(t) = \int_0^T Q_t \phi(s)\psi(s)ds \tag{8.59}
$$

Chapter 9

Conclusion

In this work, besides giving a concise review of term structure models, we tried to focus on the two class of interest rate models. One is jump-augmented and the other is the random field models. By using the HJM framework, we characterized the drift conditions of the models that we proposed. Our first proposed term structure model is the one where the instantaneous forward interest rates are governed by both a Brownian motion and a Markov-Jump Process. The other models that we investigated are related to random field extensions of certain financial settings. In this respect, we proposed a multi-country term structure model in which term structures of both countries and the exchange rate between them is modeled in a two-parameter process setting. Our aim here is to capture the covariance between markets. Therefore, we characterize drift conditions necessary to prevent arbitrage opportunities by taking this into account. Moreover, a defaultable term structure model is investigated in the same manner. Lastly forward measure approach for the two-parameter process is clarified.

As a unified framework, we gave a model that encompasses most of the infinite dimensional interest rate models. To do this, we used the nuclear space valued semi-martingales. In order to show how our proposed methodology work, we gave an example, where the instantaneous forward interest rate dynamics is governed by a Levy Random Field Process.

Although countless further research topics are identified, in our opinion the most important progress can be made by investigating how a real data match with our models. For instance, performance of the Markov-Jump Process in estimation of interest rate model parameters or the benefits of random field models in a multi-country or in a credit risk environment can be investigated empirically to justify the existence of our models.

A more general and realistic models can be tackled by utilizing the infinite dimensional tools, since they allow for any dynamical system use of patterns (functions of an independent variable) as parameters. Thus histories, spatial distributions, probability distributions can be used as parameters. This is a distinguishing advantage of infinite dimensional techniques.

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