GENERALIZED FINITE DIFFERENCES FOR THE SOLUTION OF ONE DIMENSIONAL ELASTIC PLASTIC PROBLEMS OF NONHOMOGENEOUS MATERIALS

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ABSTRACT

GENERALIZED FINITE DIFFERENCES FOR THE SOLUTION OF
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NONHOMOGENEOUS MATERIALS

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In this thesis, the Generalized Finite Difference (GFD) method is applied to analyze the elastoplastic deformation behavior of a long functionally graded (FGM) tube subjected to internal pressure.

First, the method is explained in detail by considering the elastic response of a rotating FGM tube. Then, the pressurized tube problem is treated. A long FGM tube with fixed ends (axially constrained ends) is taken into consideration. The two cases in which the modulus of elasticity only and both the modulus of elasticity and the yield limit are graded properties are analyzed. The plastic model here is based on incremental theory of plasticity, Tresca's yield criterion and its associated flow rule. The numerical results are compared to those of analytical ones. Furthermore, the elastic response of an FGM tube with free ends is studied considering graded modulus of elasticity and Poisson's ratio. The results of these computations are compared to those of Shooting solutions.

In the light of analyses and comparisons stated above, the applicability of the GFD method to the solution of similar problems is discussed. It is observed that, in purely elastic deformations the accuracy of the method is sufficient. However, in case of elastic-plastic deformations, the discrepancies between numerical and analytical results may increase in determining plastic displacements. It is also noteworthy that the predictions for tubes with two graded properties, i. e. the modulus of elasticity and the yield limit, turn out to be better than those with one graded property in this regard.

Keywords: Generalized Finite Difference, Functionally Graded Material, Elastoplasticity, Tresca's Criterion

ÖZ

HOMOJEN OLMAYAN MALZEMELERİN
TEK BOYUTLU ELASTİK PLASTİK PROBLEMLERİ İÇİN
GENELLEŞTİRİLMİŞ SONLU FARKLAR METODU

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Bu tezde, Genelleştirilmiş Sonlu Farklar (GFD) metodu uygulanarak, fonksiyonel derecelendirilmiş malzemeden (FGM) yapılmış uzun bir tüpün iç basınç etkisi altında elastoplastik gerilme analizi yapılmıştır.

Metot, dönen uzun bir FGM tüpün elastik davranışı probleminin örnek çözümüyle detaylı bir şekilde açıklanmıştır. Bu açıklamadan sonra, bu tezde ele alınan elastik-plastik basınç tüpü problemi çözülmüştür. Uçları sabit uzun bir FGM tüpün öncelikle elastisite modülünün daha sonra hem elastisite modülünün hem de akma gerilmesinin radyal olarak değiştiği kabul edilmiştir. Tüpün plastik davranışı Tresca akma kriteri ve ilgili akma kuralı yardımıyla formüle edilmiştir. Elde edilen sonuçlar analitik çalışmaların sonuçlarıyla karşılaştırılmıştır. Buna ilaveten, uçları serbest uzun bir FGM tüpün elastik davranışı, elastisite modülü ve Poisson oranı değişimi göz önüne alınarak incelenmiştir. Burada elde edilen sonuçlar ise Shooting (atış) yöntemi kullanılarak bulunan sayısal sonuçlarla karşılaştırılmıştır.

Yukarıda yapılan analiz ve karşılaştırmaların ışığı altında GFD yönteminin benzer problemlere uygulanabilirliği incelenmiştir. Elastik gerilme durumunda metodun doğruluğu tatmin edicidir. Fakat elastik-plastik deformasyonun incelendiği durumlarda, nümerik ve analitik yöntemler kullanılarak hesaplanan plastik deplasmanlarda görülen farklar artmaktadır. Bu bağlamda, dereceli elastisite modülü ve dereceli akma gerilmesi dikkate alınarak yapılan çözümlerin sadece dereceli elatisite modülü göz önüne alınarak yapılan çözümlere göre analitik çalışmalarla daha uyumlu sonuçlar vermesi kayda değerdir.

Anahtar Kelimeler: Genelleştirilmiş Sonlu Farklar, Fonksiyonel Derecelendirilmiş Malzeme, Elastoplastiklik, Tresca Kriteri

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TABLE OF CONTENTS

ABSTRACT iv

ÖZ vi

ACKNOWLEDGEMENTS viii

TABLE OF CONTENTS ix

LIST OF TABLES xi

LIST OF FIGURES xiii

NOMENCLATURE xvi

CHAPTER

1. INTRODUCTION 1
    - 1.1 General 1
    - 1.2 Literature Survey Related to the Numerical Method 1
2. DESCRIPTION OF THE NUMERICAL METHOD 4
    - 2.1 General 4
    - 2.2 Theory of the Generalized Finite Difference Method 4
    - 2.3 Example Problem: Elastic Response of a Rotating Functionally Graded Tube with Fixed Ends 6
3. ELASTOPLASTIC RESPONSE OF A LONG FUNCTIONALLY GRADED TUBE SUBJECTED TO INTERNAL PRESSURE 22
    - 3.1 General 22
    - 3.2 Elastoplastic Response of a Pressurized Functionally Graded Tube with Fixed Ends 22
        - 3.2.1 Solution for Graded Modulus of Elasticity 22
            - 3.2.1.1 Elastic Stress State 23
            - 3.2.1.2 Elastoplastic Stress State 26

3.2.2 Solution for Graded Modulus of Elasticity and Yield Stress 33

3.2.2.1 Elastic Stress State... 33

3.2.2.2 Elastoplastic Stress State 41

3.3 Elastic Response of a Pressurized Functionally Graded Tube with Free Ends 54

3.3.1 Solution for Graded Modulus of Elasticity and Poisson's Ratio 54

3.3.1.1 Elastic Stress State... 54

4. CONCLUSIONS 63

4.1 General 63

4.2 Recommendations for Future Work 64

REFERENCES... 65

APPENDICES

A. SUMMARY OF COMPUTER PROGRAMS 66

A.1 Subroutine GFD 66

A.2 Subroutine ELASTIC_ROT 66

A.3 Subroutine EPLSTC.. 66

A.4 Subroutine ELASTIC_FREE 68

B. THE CODES OF COMPUTER PROGRAMS 69

B.1 Subroutine GFD 69

B.2 Subroutine ELASTIC_ROT 73

B.3 Subroutine EPLSTC.. 77

B.4 Subroutine ELASTIC_FREE 89

B.5 Subroutine GRIDS... 94

B.6 Subroutine DAVINT 95

B.7 Subroutine IDNT... 98

LIST OF TABLES

TABLES

Table 1 Stress and displacement values in a rotating FGM tube for $n = n_{cr}$ $= 1.38260$ at $\Omega = \Omega_e = 1.28780$ for $N = 10$, $w = 1/d$ and $dm = 2h + \delta$ in GFD formulae 13

Table 2 Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega = \Omega_e = 1.28780$ 14

Table 3 Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.1$ at $\Omega = \Omega_e = 1.23340$ 15

Table 4 Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.5$ at $\Omega = \Omega_e = 1.26734$ 17

Table 5 Error in GFD method for n_{cr} and Ω_e in a rotating FGM tube in the case of yielding at both surfaces simultaneously 21

Table 6 Error in GFD method for Ω_e in a rotating FGM tube for $n = 1.1$ and $n = 1.5$ 21

Table 7 Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in homogeneous and nonhomogeneous tubes in elastic stress state subjected to internal pressure $\overline{p}_{in} = 0.255$ 24

Table 8 Error in GFD method ($N = 100$ and $N = 200$) for elastic limit pressures in FGM tubes for different values of parameter k 26

Table 9 Error in GFD method ($N = 100$ and $N = 200$) for $\overline{r}_{ep}$ in partially plastic FGM tubes subjected to internal pressure for values of parameter n 30

Table 10 Error in GFD method ($N = 200$) for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$ 30

Table 11 Error in GFD method ($N = 200$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$ 31

Table 12 Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.269497$ 34

Table 13 Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.264938$ 36

Table 14 Error in GFD method $(N = 100)$ for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in an FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.273478$ 37

Table 15 Error in GFD method $(N = 100)$ for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in an FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.336428$ 39

Table 16 Error in GFD method for $\overline{p}_e$ in FGM tubes versus increasing number of nodes in four cases considered 41

Table 17 Error in GFD method ($N = 100$ and $N = 200$) for $\overline{r}_{ep}$ in partially plastic FGM tubes subjected to internal pressure in four cases considered 44

Table 18 Error in GFD method $(N = 200)$ for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$ 49

Table 19 Error in GFD method $(N = 200)$ for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$ 50

Table 20 Error in GFD method $(N = 200)$ for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$ 50

Table 21 Error in GFD method $(N = 200)$ for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$ 53

Table 22 Error in GFD method $(N = 100)$ for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in FGM tubes with free ends under internal pressure $\overline{p}_{in} = 0.24$ in various cases for variation of E and ν 62

Table 23 Nondimensional axial strains in FGM tubes under internal pressure $\overline{p}_{in} = 0.24$ calculated by shooting method (Ref.[12]) and GFD method in various cases for variation of E and ν 62

LIST OF FIGURES

FIGURES

Figure 1 Cross section of the tube 6

Figure 2 Nodes included in the computations at $P = 6$ for $dm = 2h+\delta$ 8

Figures 3 & 4 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = n_{cr}$ $= 1.38260$ at $\Omega = \Omega_e = 1.28780$ versus number of nodes 14

Figure 5 Elastic response of a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega = \Omega_e = 1.28780$ 16

Figures 6 & 7 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.1$ at $\Omega = \Omega_e = 1.23340$ versus number of nodes 17

Figure 8 Elastic response of a rotating FGM tube for $n = 1.1$ at $\Omega = \Omega_e$ $= 1.23340$ 18

Figures 9 & 10 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.5$ at $\Omega = \Omega_e = 1.26734$ versus number of nodes 19

Figure 11 Elastic response of a rotating FGM tube for $n = 1.5$ at $\Omega = \Omega_e$ $= 1.26734$ 20

Figures 12 & 13 Error in GFD method for u and $\frac{du}{dr}$ in homogeneous ($n = 0$) and FGM ($n = -0.4$, $k = 0.6$) tubes in elastic stress state under internal pressure $\overline{p}_{in} = 0.255$ versus number of nodes 24

Figure 14 Comparison of stresses and displacement in an FGM tube ($n = -0.4$, $k = 0.6$) to those in a homogeneous tube ($n = 0$) under internal pressure $\overline{p}_{in} = 0.255$ 25

Figure 15 Variation of elastic limit pressure $\overline{p}_e$ in pressurized FGM tubes with material parameters n and k 27

Figure 16 Propagation of elastic-plastic border radius in partially plastic FGM tubes ($k = 0.6$) with increasing internal pressures using n as a parameter 29

Figure 17 Error in GFD method for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$ versus number of nodes 31

Figure 18 Stresses, displacement and plastic strains in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$ 32

Figures 19 & 20 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.269497$ versus number of nodes 34

Figure 21 Elastic response of an FGM tube ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.269497$ 35

Figures 22 & 23 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.264938$ versus number of nodes 36

Figure 24 Elastic response of an FGM tube ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.264938$ 38

Figures 25 & 26 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.273478$ versus number of nodes 39

Figure 27 Elastic response of an FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.273478$ 40

Figures 28 & 29 Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.336428$ versus number of nodes 41

Figure 30 Elastic response of an FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.336428$ 42

Figure 31 Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) with increasing internal pressures 45

Figure 32 Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) with increasing internal pressures 46

Figure 33 Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) with increasing internal pressures 47

Figure 34 Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) with increasing internal pressures 48

Figure 35 Error in GFD method for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$ versus number of nodes 49

Figure 36 Stresses and displacement in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$ 51

Figure 37 Plastic strains in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7, m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$ 52

Figure 38 Error in GFD method for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$ versus number of nodes 53

Figure 39 Stresses and displacement in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$ 55

Figure 40 Plastic strains in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$ 56

Figure 41 Elastic responses of FGM tubes ($n = 0.4$, $k = 0.6$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of ν as a parameter 58

Figure 42 Elastic responses of FGM tubes ($n = 0$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of ν as a parameter 59

Figure 43 Elastic responses of FGM tubes ($n = -0.4$, $k = 0.6$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of ν as a parameter 60

Figure 44 Elastic responses of FGM tubes ($\nu_1 = \nu_2 = 0$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of E as a parameter 61

NOMENCLATURE

B	:	A norm that is to be minimized
CON	:	Connectivity array
d	:	Distance between two nodes
dm	:	Distance of influence
Df	:	A vector containing derivatives at a node
D_r	:	A vector containing first derivatives at all nodes
D_{rr}	:	A vector containing second derivatives at all nodes
E	:	Modulus of elasticity
E_0	:	Reference value of modulus of elasticity
h	:	Grid distance
k	:	Material parameter
K	:	Coefficient matrix
K_r	:	Coefficient matrix for first derivatives
K_{rr}	:	Coefficient matrix for second derivatives
m	:	Material parameter
n	:	Material parameter
N	:	Number of nodes in a domain
r	:	Radial coordinate
r_{ep}	:	Elastic-plastic border radius
r_{in}	:	Inner radius of the tube
r_{out}	:	Outer radius of the tube
p_e	:	Elastic limit pressure
p_{in}	:	Internal pressure
s	:	Material parameter
u	:	Radial component of displacement vector
w	:	Weighting function
ε_j	:	Normal strain component in j-direction

ε_j^p	:	Normal plastic strain component in j-direction
ν	:	Poisson's ratio
ρ	:	Mass density
σ_j	:	Normal stress component in j-direction
σ_j^p	:	Normal plastic stress component in j-direction
σ_0	:	Reference value of yield limit
σ_Y	:	Yield limit
ω	:	Angular speed
ϕ_Y	:	A stress variable
Ω	:	Nondimensional angular speed
Ω_e	:	Nondimensional elastic limit angular speed

CHAPTER 1

INTRODUCTION

1.1 General

The Generalized Finite Difference (GFD) method is regarded as an evolution of the Finite Difference method since it can be easily applied to the problems with irregular domains and free moving boundaries. GFD method is a meshless method. Therefore, it does not require a mesh discretization of the domain of the problem. Approximation scheme is constructed entirely from a set of nodes. The method is based on the use of a weighted least-square approximation procedure together with a Taylor series expansion of the unknown function.

In this thesis, GFD method is applied to analyze the elastoplastic behavior of a long functionally graded (FGM) tube subjected to internal pressure. Applicability of the method to one dimensional problems with regular grids is evaluated.

After a brief summary of the thesis and literature survey related to the numerical method are told in this chapter, theory of the method is given, and then elastic response of a rotating FGM tube is studied as a sample problem in Chapter 2. Formulae and steps leading to coefficient matrices of GFD method are presented in detail. The numerical results are compared to those of analytical solution obtained from Eraslan and Akış [9].

In Chapter 3, the pressurized tube problem is treated. Firstly, a long FGM tube with fixed ends (axially constrained ends) in elastoplastic stress state is considered. In this case, FGM tubes with graded modulus of elasticity only and with graded modulus of elasticity and graded yield stress together are examined. Results obtained by GFD method are compared to those of analytical studies in Eraslan and Akış [10] and [11]. Afterwards, elastic response of an FGM tube with free ends (axially unconstrained ends) is considered. Graded material properties for this case are modulus of elasticity and Poisson's ratio. The results of the calculations are compared to those of Shooting solution in Eraslan [12].

Finally, conclusions of this study are given in Chapter 4.

1.2 Literature Survey Related to the Numerical Method

Jensen [3] published the basis of GFD method. He emphasized the difficulty of dealing non-rectangular problem domains with regular grids, and considered arbitrary grid using in order to drop the requirement that the grid be rectangular. In the paper, a six point scheme was formed by the selection of five closest points to the central node (distance criterion). Taylor series expansion around the central node was utilized to obtain difference coefficients. In order to avoid singularity or ill conditioning of the star which refers to a group of established nodes related to the central node, he used an extra node to reclassify node distribution if necessary. The main drawbacks of this method were frequent singularity or ill-conditioning of the stars despite the reclassification of nodes and irregular density of nodes due to distance criterion.

Parrone and Kao [4] continued investigation of irregular finite difference techniques and suggested a scheme which can not only avoid singularity in the derivative coefficient matrix but also improve the accuracy of derivatives obtained (Ref.[4]). The stars were formed by dividing the area around central node into eight zones with 45° central angle (eight segment criterion). They used six point stars but applied an averaging process to obtain good derivative approximations. In the paper two problems were solved: Poisson's equation and large deflection response of a flat membrane. Results were satisfactory compared with the previous attempts.

Liszka and Orkizs [5] made important contributions to the development of GFD method. They underlined the importance of number and position of the nodes in each star for the method. Four quadrants were formed by the help of Cartesian axes around the central node, and two nearest nodes per quadrant were included in the star (four quadrant criterion). They also used moving least square interpolation which leads to employ more nodes in the stars and improve accuracy in the approximation of derivatives.

Benito et al. [6] examined influences of several factors in GFD method such as number of nodes in the star, arrangement of the star, the weight function and stability parameter in time dependent problems. According to the results, as the number of nodes in the star was increased, the precision of the results increased. But after a certain number of nodes, this improvement seemed to be relatively small compared with the increasing calculation effort. The error in the method also decreased when the number of nodes was increased in the domain. After the comparison of distance criterion and four quadrant criterion for selection of nodes in the star, it was shown that four quadrant criterion gave more accurate results than the other. With regard to weighting function, smooth functions were proved to give best results. On the other hand, the results obtained in the paper showed the suitability of the method for time dependent problems.

Benito et al. [7] described an h-adaptive method in GFD. In the method, an error indicator based on third and fourth order terms in Taylor series expansion was proposed. Mean value of the error indicator or a multiple was reduced by adding selective nodes to the domain. Selection criteria of the additional nodes also depended on the minimum distance between each node to avoid ill-conditioning of the stars. Decreasing the limit of error indicator and minimum distance step by step was advised in order to obtain more balanced clouds of nodes. Results showed that even with a few nodes added in each cloud of nodes by h-adaptive algorithm, solution error decreased significantly. In the paper, the reduction of solution error by adding third order terms of Taylor series expansion to GFD formulae which utilizes derivative expressions up to the second order was also examined. Results showed that the effect of the additional terms did not justify the increase in calculation time required.

Gavete et al.[8] introduced the criterion of radius of influence to reduce errors and avoid ill-conditioned stars. The weighting functions were also based on this criterion that the resulting value was zero if the distance of any node to the central node was greater than the radius of influence. Four quadrant criterion was used for selection of nodes in the star. Two or more nodes closest to the central node in each quadrant were chosen. Laplace equation was solved to evaluate accuracy of the method. Fixed and variable radii of influence were employed in the solutions, and the latter gave better results. In the paper, the method was compared with another meshless method, Element Free Galerkin method. For the same problem, GFD method gave more accurate results than the other method.

CHAPTER 2

DESCRIPTION OF THE NUMERICAL METHOD

2.1 General

In this chapter GFD method is described, and then elastic response of a long rotating FGM tube with fixed ends is analyzed by the method. Each step in the calculations are presented. Finally, results are compared to the analytical solution given in Ref.[9].

2.2 Theory of the Generalized Finite Difference Method

GFD method is based on Taylor series expansion and least squares approximation briefly. The steps which lead to derivative coefficient matrices of the method are explained below.

For any sufficiently differentiable function $f(x)$, Taylor series expansion around a point $p(x_0)$ up to the second order derivatives is expressed in the form

$$f = f_0 + h\frac{df_0}{dx} + \frac{h^2}{2}\frac{d^2 f_0}{dx^2} + o(h^3) \tag{1}$$

where $f = f(x)$, $f_0 = f(x_0)$, $h = x - x_0$.

Norm B is considered.

$$B = \sum_{i=1}^{N}\left[\left[f_0 - f_i + h_i\frac{df_0}{dx} + \frac{h_i^2}{2}\frac{d^2 f_0}{dx^2}\right] w_i\right]^2 \tag{2}$$

where $f_i = f(x_i)$, $h_i = x_i - x_0$, N the number of nodes in the domain, and w_i the weighting function.

The formulae are obtained minimizing norm B.

$$\frac{\partial B}{\partial\,(Df)} = 0 \tag{3}$$

where

$$Df = \begin{bmatrix} \dfrac{df_0}{dx} & \dfrac{d^2 f_0}{dx^2} \end{bmatrix}^T$$

A set of two equations with two unknowns is formed by Eq. (3). The equations are as follows:

$$f_0 \sum_{i=1}^{N} w_i^2 h_i - \sum_{i=1}^{N} f_i w_i^2 h_i + \frac{df_0}{dx} \sum_{i=1}^{N} w_i^2 h_i^2 + \frac{d^2 f_0}{dx^2} \sum_{i=1}^{N} w_i^2 \frac{h_i^3}{2} = 0 \tag{4}$$

$$f_0 \sum_{i=1}^{N} w_i^2 \frac{h_i^2}{2} - \sum_{i=1}^{N} f_i w_i^2 \frac{h_i^2}{2} + \frac{df_0}{dx} \sum_{i=1}^{N} w_i^2 \frac{h_i^3}{2} + \frac{d^2 f_0}{dx^2} \sum_{i=1}^{N} w_i^2 \frac{h_i^4}{4} = 0 \tag{5}$$

Eqs. (4) and (5) are expressed by the following system of equations.

$$\begin{bmatrix} \sum w_i^2 h_i^2 & \sum w_i^2 \frac{h_i^3}{2} \\ \sum w_i^2 \frac{h_i^3}{2} & \sum w_i^2 \frac{h_i^4}{4} \end{bmatrix} \begin{bmatrix} \frac{df_0}{dx} \\ \frac{d^2 f_0}{dx^2} \end{bmatrix} = \begin{bmatrix} -f_0 \sum w_i^2 h_i + \sum f_i w_i^2 h_i \\ -f_0 \sum w_i^2 \frac{h_i^2}{2} + \sum f_i w_i^2 \frac{h_i^2}{2} \end{bmatrix} \tag{6}$$

Eq. (6) in resumed notation is given by

$$A_P D f_P = b_P \tag{7}$$

Right hand side of Eq. (7) is rearranged. For instance, for $P = 3$ and consequently $x_0 = x_3$, right hand side of the equation is as follows:

$$b_3 = \begin{bmatrix} w_1^2 h_1 & w_2^2 h_2 & -\sum w_i^2 h_i & w_4^2 h_4 & ... & w_N^2 h_N \\ w_1^2 \frac{h_1^2}{2} & w_2^2 \frac{h_2^2}{2} & -\sum w_i^2 \frac{h_i^2}{2} & w_4^2 \frac{h_4^2}{2} & ... & w_N^2 \frac{h_N^2}{2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix} \tag{8}$$

Eq. (8) in resumed notation is given by

$$b_P = B_P f \tag{9}$$

Eq. (9) is inserted in Eq. (7), and resulting equation is modified to give derivative values as a result.

$$D f_P = C_P f \tag{10}$$

where

$$C_P = A_P^{-1} B_P \tag{11}$$

Eq. (10) is based on Taylor series expansion around point x_0, central node. The equation is constructed for each node in the domain, being central node in turn. Matrices C are combined to form coefficient matrices of first and second derivatives, formulation of which are explained in detail in the next section. Any ordinary differential equation can be solved utilizing these coefficient matrices.

2.3 Example Problem: Elastic Response of a Rotating Functionally Graded Tube with Fixed Ends

Response of a rotating FGM tube with fixed ends in elastic stress state was analyzed analytically in Eraslan and Akış [9]. In this section, the problem is solved by GFD method in detail. Results obtained are compared to the results of analytical solution. Cross section of the rotating tube can be seen in Figure 1.

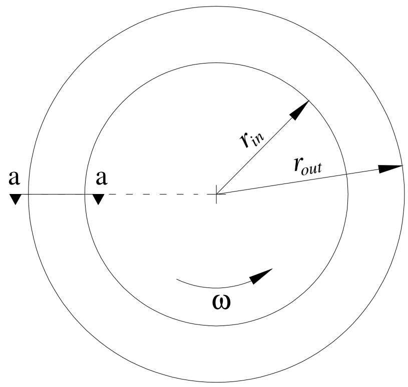


Figure 1. Cross section of the tube

Elastic modulus of the FGM tube varies radially with following relation:

$$E(r) = E_0 \left(\frac{r}{r_{out}}\right)^n \tag{12}$$

where E_0 is the reference value of E, r the radial coordinate, and n is a material parameter.

In the problem, cylindrical polar coordinates (r,θ,z) are considered, and the notation of Timoshenko and Goodier [1] is used. A state of plane strain ($\varepsilon_z = 0$), and infinitesimal deformations are assumed. The equation of equilibrium in the radial direction (Ref.[1]) is as follows:

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = -\rho\omega^2 r^2 \tag{13}$$

where σ_j designates a normal stress component, ρ the mass density, and ω the constant angular speed.

Stress displacement relations for plane strain problems are expressed in the following forms:

$$\sigma_r = \frac{E(r)}{(1+\nu)(1-2\nu)}\left[\nu\frac{u}{r} + (1-\nu)\frac{du}{dr}\right] \tag{14}$$

$$\sigma_\theta = \frac{E(r)}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{u}{r} + \nu\frac{du}{dr}\right] \tag{15}$$

$$\sigma_z = \frac{E(r)}{(1+\nu)(1-2\nu)}\left[\nu\frac{u}{r} + \nu\frac{du}{dr}\right] \tag{16}$$

where ν is Poisson's ratio, and u the radial component of displacement vector.

Substitution of Eq. (12) into Eqs. (14) and (15), and then those into Eq. (13) result in the governing differential equation for radial displacement u.

$$r^2\frac{d^2u}{dr^2} + (1+n)\,r\frac{du}{dr} - \frac{1-(1+n)\,\nu}{1-\nu}u = -\frac{r_{out}^n r^{3-n}(1+\nu)(1-2\nu)\rho\omega^2}{(1-\nu)E_0} \tag{17}$$

Boundary conditions of the problem are $\sigma_r(r_{in}) = \sigma_r(r_{out}) = 0$.

Following non-dimensional and normalized values are used in the presentation of the results.

$$\overline{r} = \frac{r}{r_{out}}\ ,\ \ \overline{\sigma}_j = \frac{\sigma_j}{\sigma_0}\ ,\ \overline{u} = \frac{uE_0}{\sigma_0 r_{out}}\ ,\ \ \Omega = \omega r_{out}(\frac{\rho}{\sigma_0})^{1/2} \tag{18}$$

where σ_0 is the uniaxial yield limit and Ω the nondimensional angular speed.

In the solution, following material properties are considered: $E_0 = 200$ GPa, $\sigma_0 = 430$ MPa, $\nu = 0.3$, $\rho = 7800$ kg/m^3. Inner and outer radii of the tube are 0.55 m and 1.0 m, respectively.

It was shown in Ref.[9] that for $n = 1.38260$ at nondimensional elastic limit angular speed $\Omega_e = 1.28780$, plastic deformation commences at inner and outer surfaces simultaneously. Therefore, the value of n is denoted as n_{cr}.

Stresses and displacement in the tube for the above mentioned case are calculated by the numerical method step by step. Ten regular nodes are considered in the domain of the problem. For each node Eq. (10) is constructed, and considered node is denoted as central node.

The weighting function used in the calculations is as follows:

$$w_i = \begin{cases} \frac{1}{d} \text{ for } d \le dm \\ 0 \text{ for } d > dm \end{cases} \tag{19}$$

where d is the distance between ith node and central node, and dm is the distance of influence.

For each node, nodes within the distance of influence are employed to construct GFD formulae instead of employing all nodes. In this problem, dm is chosen as $2h + \delta$, where h is the grid distance, and $\delta = 1.00E - 10$. δ is used in order to ensure that required points are included in the computations. Figure 2 shows nodes included in the computations at Section a-a in Figure 1 when $P = 6$ and consequently $r_6 = r_0$.

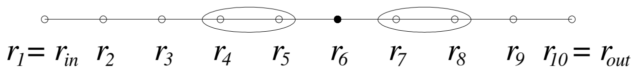


Figure 2. Nodes included in the computations at $P = 6$ for $dm = 2h + \delta$.

For $N = 10$, nodal coordinates are

$$r^T = \begin{bmatrix} 0.55 & 0.6 & 0.65 & 0.7 & 0.75 & 0.8 & 0.85 & 0.9 & 0.95 & 1 \end{bmatrix}$$

Matrices A_P, B_P and C_P in Eqs. (7), (9) and (10) are to be calculated. For each node neighboring nodes within the distance of influence are kept in a connectivity array CON_P in order to calculate the matrices using these nodes, and finally construct global derivative coefficient matrices. Since $w = 0$ for $d > dm$, nodes which are out of the range of dm do not contribute to the computations. These nodes are omitted in the calculations. Results for each node are given below.

$P = 1$

$$CON_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 0.075 \\ 0.075 & 0.003125 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -30 & 20 & 10 \\ -1 & 0.5 & 0.5 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -30 & 40 & -10 \\ 400 & -800 & 400 \end{bmatrix}$$

Eq. (10) is further examined here. For $P = 1$ Eq. (10) takes the form

$$Df_1 = C_1 u_{1A}$$

where $Df_1^T = \begin{bmatrix} \frac{du_1}{dr} & \frac{d^2u_1}{dr^2} \end{bmatrix}$ and $u_{1A}^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$

$P = 2$

$$CON_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & 0.05 \\ 0.05 & 0.00375 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -20 & -10 & 20 & 10 \\ 0.5 & -1.5 & 0.5 & 0.5 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -11.4286 & 4.28571 & 5.71429 & 1.42857 \\ 285.714 & -457.143 & 57.1429 & 114.286 \end{bmatrix}$$

$P = 3$

$$CON_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 0 \\ 0 & 0.00625 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -10 & -20 & 0 & 20 & 10 \\ 0.5 & 0.5 & -2 & 0.5 & 0.5 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -2.5 & -5 & 0 & 5 & 2.5 \\ 80 & 80 & -320 & 80 & 80 \end{bmatrix}$$

For P = 3, 4, ... , 8 distribution of nodes related to the central node is the same. Two nodes from both sides are included in the computations. Since distances between the central node and neighboring nodes within the distance of influence are taken into consideration, and these nodes are regularly scattered, matrices A_P, B_P and C_P for mentioned nodes are the same. The matrices are seen above at $P = 3$. However, connectivity arrays at these nodes are different. General form of array CON is given.

$P = 4, 5, ..., 8$

$$CON_P = \begin{bmatrix} r_{P-2} & r_{P-1} & r_P & r_{P+1} & r_{P+2} \end{bmatrix}$$

$P = 9$

$$CON_9 = \begin{bmatrix} 7 & 8 & 9 & 10 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 3 & -0.05 \\ -0.05 & 0.00375 \end{bmatrix}, \quad B_9 = \begin{bmatrix} -10 & -20 & 10 & 20 \\ 0.5 & 0.5 & -1.5 & 0.5 \end{bmatrix}$$

$$C_9 = \begin{bmatrix} -1.42857 & -5.71429 & -4.28571 & 11.4286 \\ 114.286 & 57.1429 & -457.143 & 285.714 \end{bmatrix}$$

$P = 10$

$$CON_{10} = \begin{bmatrix} 8 & 9 & 10 \end{bmatrix}$$

$$A_{10} = \begin{bmatrix} 2 & \text{-0.075} \\ \text{-0.075} & 0.003125 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} \text{-10} & \text{-20} & 30 \\ 0.5 & 0.5 & \text{-1} \end{bmatrix}$$

$$C_{10} = \begin{bmatrix} 10 & \text{-40} & 30 \\ 400 & \text{-800} & 400 \end{bmatrix}$$

Matrices C_P are utilized to construct two global coefficient matrices in order to relate function values to the first and second derivatives.

$$D_r = K_r.u \tag{20}$$

$$D_{rr} = K_{rr}.u \tag{21}$$

where

$$D_r^T = \begin{bmatrix} \frac{du_1}{dr} & \frac{du_2}{dr} & \frac{du_3}{dr} & \frac{du_4}{dr} & \frac{du_5}{dr} & \frac{du_6}{dr} & \frac{du_7}{dr} & \frac{du_8}{dr} & \frac{du_9}{dr} & \frac{du_{10}}{dr} \end{bmatrix}$$

$$D_{rr}^T = \begin{bmatrix} \frac{d^2u_1}{dr^2} & \frac{d^2u_2}{dr^2} & \frac{d^2u_3}{dr^2} & \frac{d^2u_4}{dr^2} & \frac{d^2u_5}{dr^2} & \frac{d^2u_6}{dr^2} & \frac{d^2u_7}{dr^2} & \frac{d^2u_8}{dr^2} & \frac{d^2u_9}{dr^2} & \frac{d^2u_{10}}{dr^2} \end{bmatrix}$$

and

$$u^T = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} \end{bmatrix}$$

K_r and K_{rr} are matrices of length 10 x 10 in which row number refers to node number. For example, in order to construct first row of K_r, elements of first row of C_1 are distributed into the columns of K_r, the numbers of which are taken from array CON_1. For K_{rr} second row of C_1 is employed. This procedure is repeated for remaining nodes.

In order to construct Eq. (17), terms $\frac{du}{dr}$ and $\frac{d^2u}{dr^2}$ are replaced with K_r and K_{rr}. For radial displacement u, a 10 by 10 identity matrix is used. These matrices are multiplied with corresponding terms in the equation, and summed up in the name of coefficient matrix K. Finally, the equation in resumed notation takes the form

$$K.u = F \tag{22}$$

Next, boundary conditions $\sigma_r(r_{in}) = \sigma_r(r_{out}) = 0$ are to be imposed in Eq. (22). First and last rows of K, and F are modified according to the boundary conditions. K_r, K_{rr} and final form of K are displayed in the following pages.

$$K_r = \begin{bmatrix} -30 & 40 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -11.4286 & 4.28571 & 5.71429 & 1.42857 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & -5 & 0 & 5 & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.5 & -5 & 0 & 5 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & -5 & 0 & 5 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.5 & -5 & 0 & 5 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & -5 & 0 & 5 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.5 & -5 & 0 & 5 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.42857 & -5.71429 & -4.28571 & 11.4286 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -40 & 30 \end{bmatrix}$$

$$K_{rr} = \begin{bmatrix} 400 & -800 & 400 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 285.714 & -457.143 & 57.1429 & 114.286 & 0 & 0 & 0 & 0 & 0 & 0 \\ 80 & 80 & -320 & 80 & 80 & 0 & 0 & 0 & 0 & 0 \\ 0 & 80 & 80 & -320 & 80 & 80 & 0 & 0 & 0 & 0 \\ 0 & 0 & 80 & 80 & -320 & 80 & 80 & 0 & 0 & 0 \\ 0 & 0 & 0 & 80 & 80 & -320 & 80 & 80 & 0 & 0 \\ 0 & 0 & 0 & 0 & 80 & 80 & -320 & 80 & 80 & 0 \\ 0 & 0 & 0 & 0 & 0 & 80 & 80 & -320 & 80 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 114.286 & 57.1429 & -457.143 & 285.714 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & -800 & 400 \end{bmatrix}$$

$$K=\begin{bmatrix} -3.44226E12 & 4.71207E12 & -1.17802E12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 86.5193 & -158.852 & 28.7403 & 43.1851 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29.9283 & 26.0566 & -135.607 & 41.5434 & 37.6717 & 0 & 0 & 0 & 0 & 0 \\ 0 & 35.0305 & 30.8609 & -157.207 & 47.5391 & 43.3695 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40.5326 & 36.0653 & -180.407 & 53.9347 & 49.4674 & 0 & 0 & 0 \\ 0 & 0 & 0 & 46.4348 & 41.6696 & -205.207 & 60.7304 & 55.9652 & 0 & 0 \\ 0 & 0 & 0 & 0 & 52.7370 & 47.6740 & -231.608 & 67.9260 & 62.8630 & 0 \\ 0 & 0 & 0 & 0 & 0 & 59.4392 & 54.0783 & -259.607 & 75.5217 & 70.1608 \\ 0 & 0 & 0 & 0 & 0 & 0 & 99.9093 & 38.6373 & -422.679 & 283.725 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.69231E12 & -1.07692E13 & 8.19231E12 \end{bmatrix}$$

The modified form of Eq. (22) is solved, and displacement values are obtained. Using displacement values, first and second derivatives are computed by Eqs. (20) and (21). Once u and du/dr values are known, then stress values are calculated from Eqs. (14), (15) and (16). Results in nondimensional forms are shown in Table 1.

Table 1. Stress and displacement values in a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega = \Omega_e = 1.28780$ for $N = 10$, $w = 1/d$ and $dm = 2h + \delta$ in GFD formulae

$\overline{r}$	$\overline{u}$	$\frac{d\overline{u}}{d\overline{r}}$	$\frac{d^2\overline{u}}{d\overline{r}^2}$	$\overline{\sigma}_r$	$\overline{\sigma}_\theta$	$\overline{\sigma}_z$
0.55	1.14328	-0.890868	3.10946	0	0.999483	0.299845
0.6	1.10262	-0.740624	2.69109	0.0311985	1.00994	0.312342
0.65	1.06974	-0.627521	1.87913	0.0577328	1.02166	0.323818
0.7	1.04097	-0.546474	1.31354	0.0746923	1.03002	0.331412
0.75	1.01558	-0.490595	0.918846	0.0811556	1.03448	0.334692
0.8	0.992361	-0.451443	0.634514	0.0792809	1.03525	0.334358
0.85	0.970724	-0.424758	0.427044	0.0695503	1.03223	0.330534
0.9	0.950102	-0.407191	0.273243	0.0526426	1.02538	0.323407
0.95	0.930192	-0.396183	0.157185	0.0294094	1.01493	0.313301
1	0.910547	-0.390235	0.106226	0	1.00060	0.300180

Different weighting functions can be used in the solution. Weighting functions considered throughout the thesis besides $w = 1/d$ are

$$\text{Exponential weighting function} = \begin{cases} \exp(-10d) \text{ for } d \leq dm \\ 0 \text{ for } d > dm \end{cases} \tag{23}$$

$$\text{Cubic Spline weighting function} = \begin{cases} \frac{2}{3} - 4\left(\frac{d}{dm}\right)^2 - 4\left(\frac{d}{dm}\right)^3 \text{ for } d \leq \frac{dm}{2} \\ \frac{4}{3} - 4\left(\frac{d}{dm}\right) + 4\left(\frac{d}{dm}\right)^2 - \frac{4}{3}\left(\frac{d}{dm}\right)^3 \text{ for } \frac{dm}{2} < d \leq dm \\ 0 \text{ for } d > dm \end{cases} \tag{24}$$

Results obtained with different GFD parameters are compared to those of analytical solution in Ref.[9]. Relative percent error is used for presentation of errors throughout the thesis.

$$\%error(f) = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{fe_i - fc_i}{fe_i}\right| \text{x100} \tag{25}$$

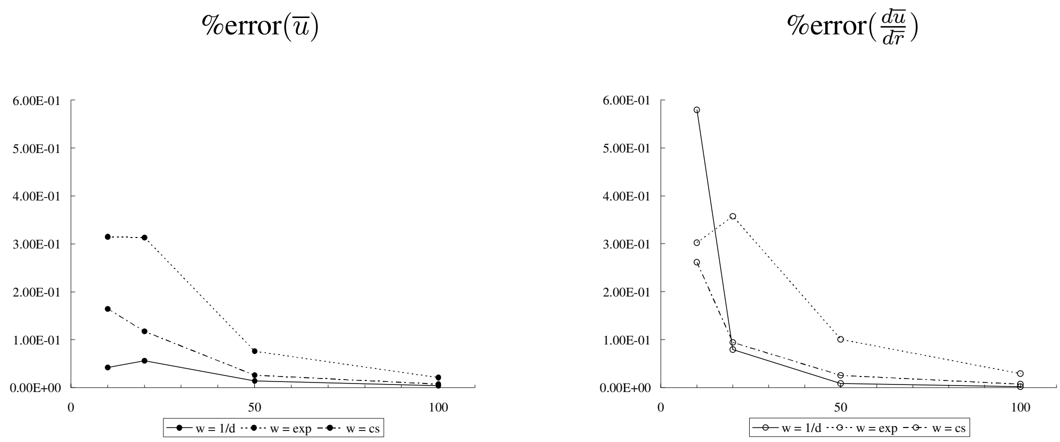
where fe and fc are exact and calculated function values, respectively.

Error in the method for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega_e = 1.28780$ with $N = 100$ are shown in Table 2. In the solution with cubic spline weighting function, when $d = dm$, $w \approx 0$, and corresponding nodes have nearly zero effect in the calculations. Therefore, in the presentation of errors with cubic spline weighting function, dm is taken h more than the ones in the other weighting functions.

Table 2. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega = \Omega_e = 1.28780$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{dr}$)
	2h + δ	4.04E-03	2.11E-03
$1/d$	3h + δ	2.52E-02	3.80E-02
	4h + δ	6.02E-02	1.02E-01
	2h + δ	2.13E-02	2.91E-02
exponential	3h + δ	6.52E-02	1.04E-01
	4h + δ	1.60E-01	2.78E-01
	3h + δ	1.16E-02	1.26E-02
cubic spline	4h + δ	3.92E-02	6.14E-02
	5h + δ	7.12E-03	7.38E-03

Variation of errors in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ with number of nodes is seen in Figures 3 and 4. In the calculations $dm = 2h + \delta$ for $w = 1/d$ and exponential weighting function. dm is $5h + \delta$ for cubic spline weighting function.



Figures 3 & 4. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega = \Omega_e = 1.28780$ versus number of nodes.

Matching stresses and displacement are plotted in Figure 5. Analytical solution from Ref.[9] and solution by GFD method considering 10 and 100 nodes in the domain for $w = 1/d$ and $dm = 2h + \delta$ are considered. Dashed and continuous lines show numerical results with $N = 10$ and $N = 100$, respectively. Dots show analytical results.

Plastic deformation commences at inner surface of the tube for $n < n_{cr}$. The problem is solved for material parameter $n = 1.1 < n_{cr}$ at nondimensional elastic limit angular velocity $\Omega_e = 1.23340$ calculated in Ref.[9]. Arrays K_r and K_{rr} are the same as the previous case because they are dependent on grid formulation. Eq. (17) is constructed by considered n and Ω_e values. Error in GFD method with $N = 100$ for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ are shown in Table 3.

Table 3. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.1$ at $\Omega = \Omega_e = 1.23340$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	2h + δ	3.79E-03	1.99E-03
$1/d$	3h + δ	2.69E-02	4.10E-02
	4h + δ	6.54E-02	1.12E-01
	2h + δ	2.25E-02	3.11E-02
exponential	3h + δ	7.03E-02	1.14E-01
	4h + δ	1.75E-01	3.06E-01
	3h + δ	1.19E-02	1.29E-02
cubic spline	4h + δ	4.22E-02	6.67E-02
	5h + δ	7.34E-03	7.58E-03

Figures 6 & 7 show variation of errors in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ with number of nodes employing $dm = 2h + \delta$ for $w = 1/d$ and exponential weighting function in the calculations. $dm = 5h + \delta$ for cubic spline weighting function.

For $n = 1.1$, stresses and displacement in the tube calculated by GFD method considering 10 and 100 nodes in the domain with $w = 1/d$ and $dm = 2h + \delta$ are presented in Figure 8. Dashed and continuous lines show numerical results with $N = 10$ and $N = 100$, respectively. Dots show analytical results from Ref.[9].

For values of $n > n_{cr}$, yielding begins at outer surface. In order to represent this behavior, $n = 1.5 > n_{cr}$ and $\Omega_e = 1.26734$ obtained from Ref.[9] are used in the calculations. Table 4 shows %error($\overline{u}$) and %error($\frac{d\overline{u}}{d\overline{r}}$) in GFD method with $N = 100$.

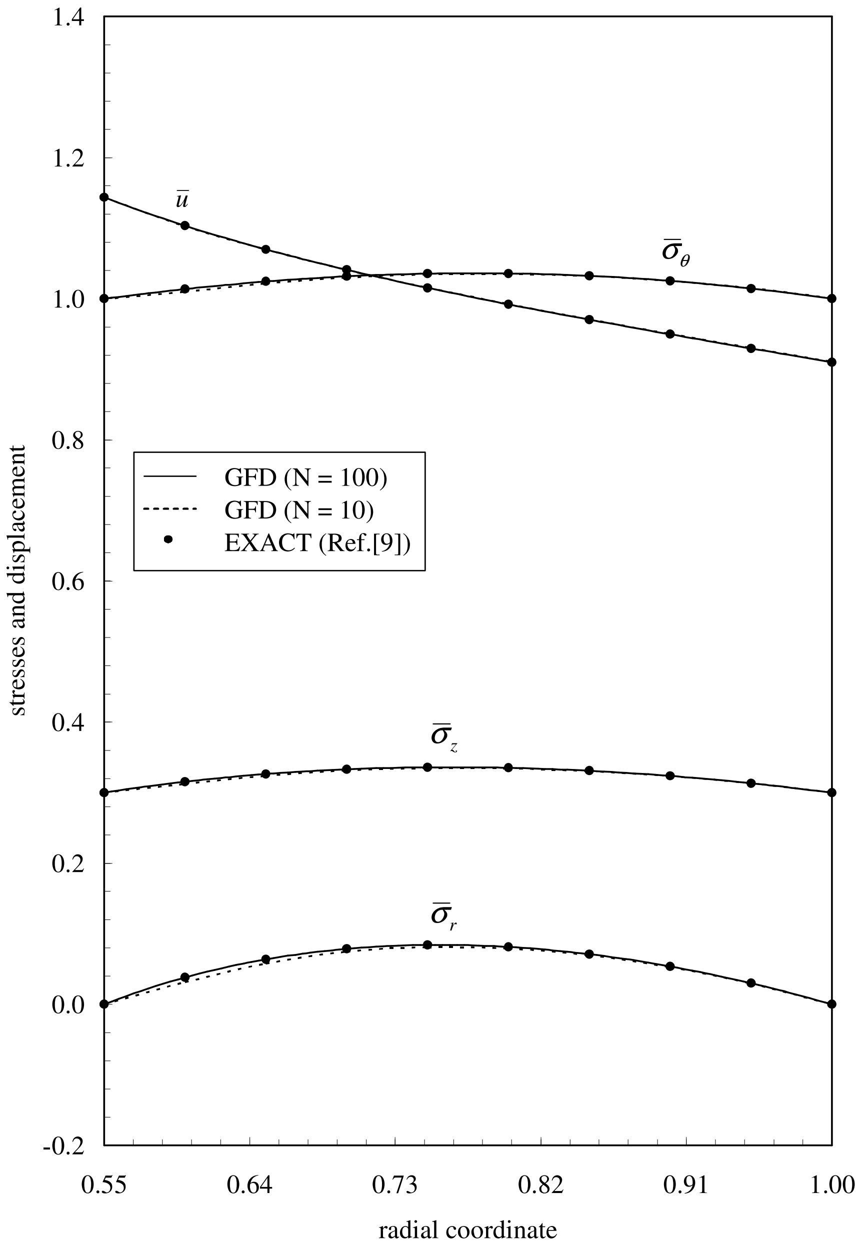
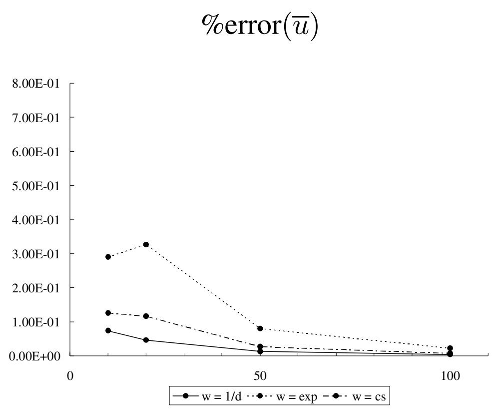
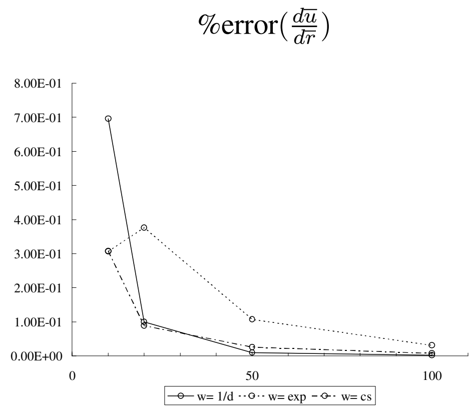


Figure 5: Elastic response of a rotating FGM tube for $n = n_{cr} = 1.38260$ at $\Omega = \Omega_e = 1.28780$.





Figures 6 & 7. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.1$ at $\Omega = \Omega_e = 1.23340$ versus number of nodes.

Table 4. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.5$ at $\Omega = \Omega_e = 1.26734$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	2h + δ	4.13E-03	2.18E-03
$1/d$	3h + δ	2.44E-02	3.66E-02
	4h + δ	5.79E-02	9.80E-02
	2h + δ	2.07E-02	2.82E-02
exponential	3h + δ	6.29E-02	1.00E-01
	4h + δ	1.53E-01	2.67E-01
	3h + δ	1.14E-02	1.24E-02
cubic spline	4h + δ	3.79E-02	5.91E-02
	5h + δ	7.00E-03	7.27E-03

Variation of errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ with increasing number of nodes are shown in Figures 9 & 10. In the calculations, for $w = 1/d$ and exponential weighting function $dm = 2h + \delta$, and for cubic spline weighting function $dm = 5h + \delta$.

Stresses and displacement calculated for $w = 1/d$ and $dm = 2h + \delta$ in GFD formulae are shown in Figure 11. Dashed and continuous lines show numerical results with $N = 10$ and $N = 100$, respectively. Dots show analytic results in Ref.[9].

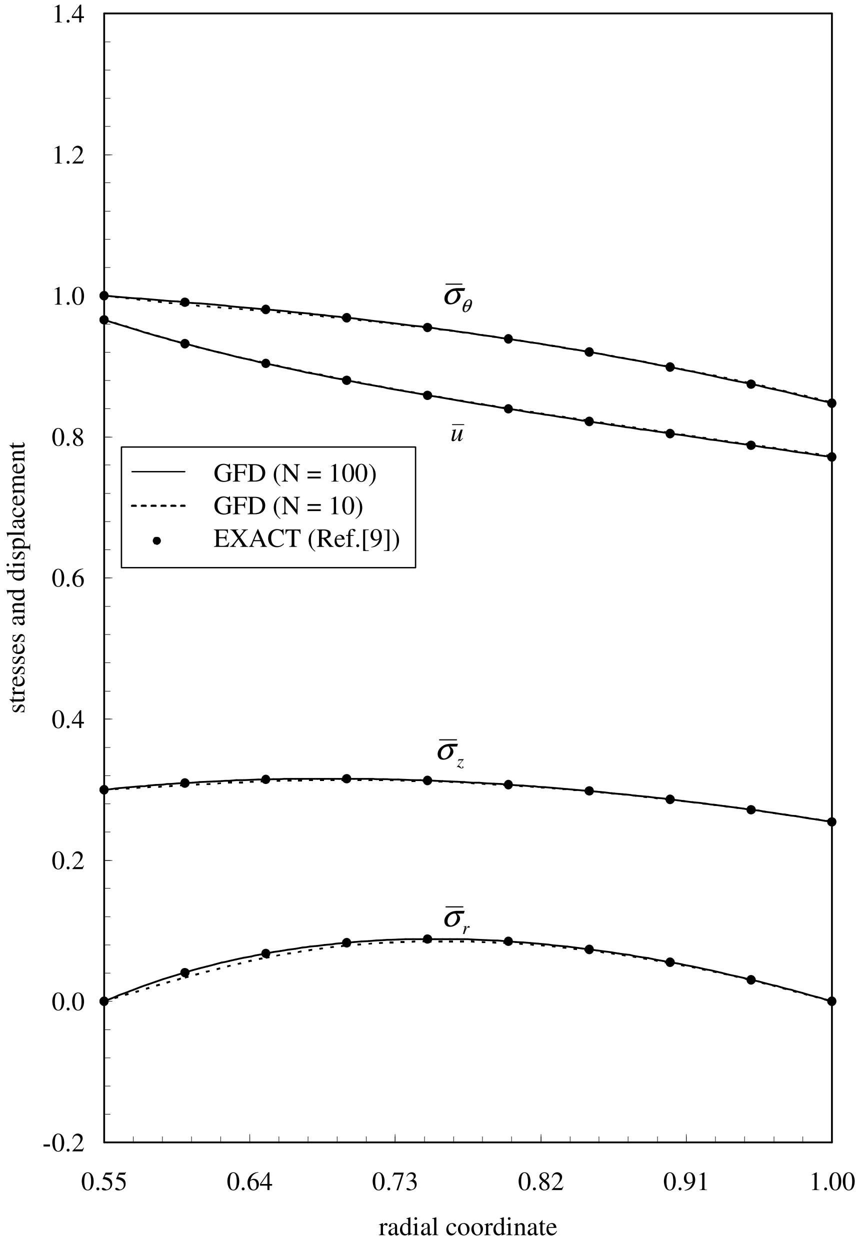
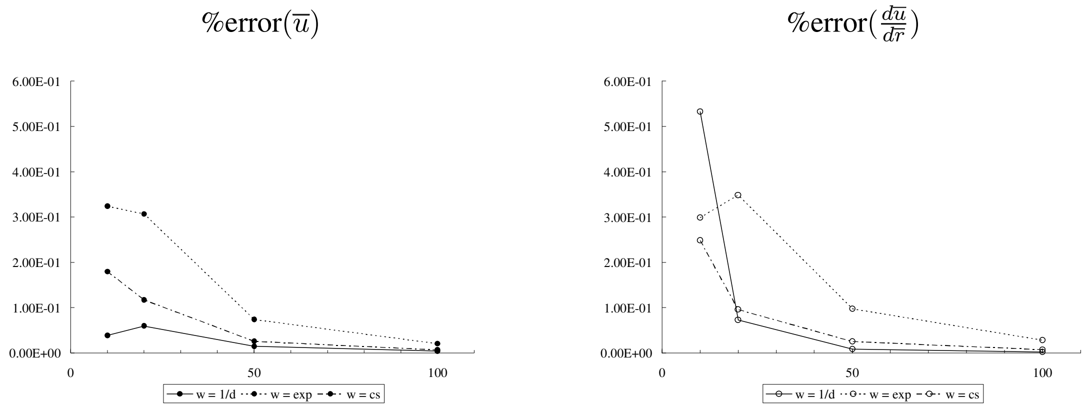


Figure 8: Elastic response of a rotating FGM tube for $n = 1.1$ at $\Omega = \Omega_e = 1.23340$.



Figures 9 & 10. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a rotating FGM tube for $n = 1.5$ at $\Omega = \Omega_e = 1.26734$ versus number of nodes.

In three cases considered, n_{cr} and three Ω_e values calculated analytically in Ref.[9] are used. They can also be obtained by GFD method employing an iteration process on the basis of Tresca's yield criterion which states that plasticity occurs if $\sigma_\theta(r) - \sigma_r(r) = \sigma_0$ with the condition $\sigma_r < \sigma_z < \sigma_\theta$. A stress variable ϕ_Y is included in the calculations in order to determine the stress state at each point , where $\phi_Y(r) = [\sigma_\theta(r) - \sigma_r(r)]/\sigma_0$ (Ref.[10]). Elastic stress state is valid if $\phi_Y(r) < 1.0$, plasticity occurs otherwise.

For the first case, plastic behavior commences at inner and outer surfaces of the FGM tube simultaneously. This behavior necessitates $\phi_Y(r_{in}) = \phi_Y(r_{out}) = 1.0$. Newton's method is used to reach n_{cr} and Ω_e. We start with assuming initial values for n_{cr} and Ω_e. Let these be n_{cr1} and Ω_{e1}. The iteration procedure is as follows:

1. calculate $fin_{11} = \phi_Y(r_{in}) - 1.0$ and $fout_{11} = \phi_Y(r_{out}) - 1.0$ with n_{cr1} and Ω_{e1}
2. calculate $fin_{12} = \phi_Y(r_{in}) - 1.0$ and $fout_{12} = \phi_Y(r_{out}) - 1.0$ with n_{cr1} and $\Omega_{e2} = \Omega_{e1} + \Delta_{\Omega_e}$
3. calculate $fin_{21} = \phi_Y(r_{in}) - 1.0$ and $fout_{21} = \phi_Y(r_{out}) - 1.0$ with $n_{cr2} = n_{cr1} + \Delta_{n_{cr}}$ and Ω_{e1}
4. construct Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial fin}{\partial n_{cr1}} & \frac{\partial fin}{\partial \Omega_{e1}} \\ \frac{\partial fout}{\partial n_{cr1}} & \frac{\partial fout}{\partial \Omega_{e1}} \end{bmatrix}$$

where,

$$\frac{\partial fin}{\partial n_{cr1}} = \frac{fin_{21} - fin_{11}}{\Delta_{n_{cr}}}, \ \frac{\partial fin}{\partial \Omega_{e1}} = \frac{fin_{12} - fin_{11}}{\Delta_{\Omega_e}}, \ \frac{\partial fout}{\partial n_{cr1}} = \frac{fout_{21} - fout_{11}}{\Delta_{n_{cr}}}, \ \frac{\partial fout}{\partial \Omega_{e1}} = \frac{fout_{12} - fout_{11}}{\Delta_{\Omega_e}}.$$

5. construct array F

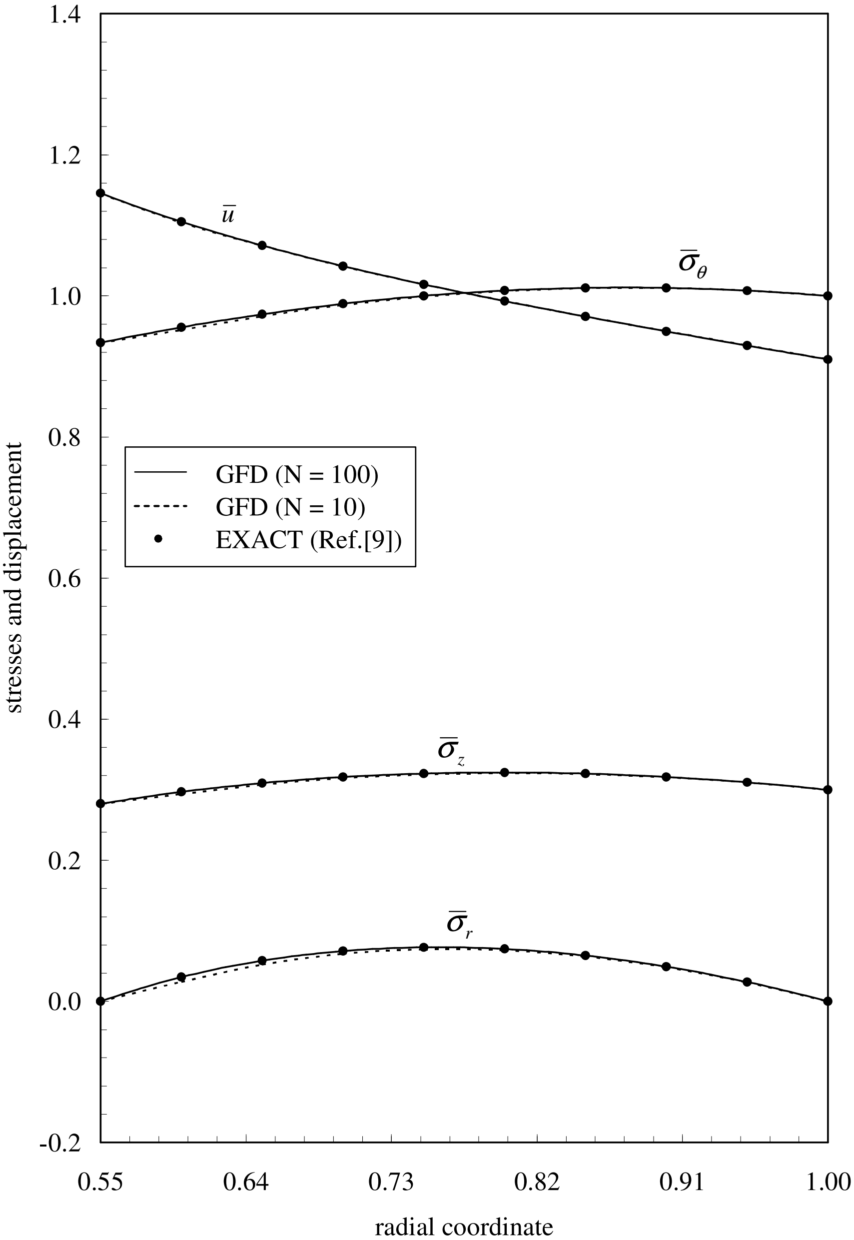


Figure 11: Elastic response of a rotating FGM tube for $n = 1.5$ at $\Omega = \Omega_e = 1.26734$.

$$F^T = \left[\begin{array}{cc} fin_{11} & fout_{11} \end{array} \right]$$

6. calculate the elements of array Δ from the equation, $J.\Delta = -F$
7. calculate $n_{cr1\_new} = n_{cr1} + \Delta_1$ and $\Omega_{e1\_new} = \Omega_{e1} + \Delta_2$

Iterations are repeated until $|n_{cr1\_new} - n_{cr1}|$ and $|\Omega_{e1\_new} - \Omega_{e1}|$ are less than specified error tolerance which is defined as $1\text{x}10^{-10}$. Error for n_{cr} and Ω_e in GFD method with $w = 1/d$ and $dm = 2h + \delta$ employing 100 and 200 nodes in the domain are shown in Table 5.

Table 5. Error in GFD method for n_{cr} and Ω_e in a rotating FGM tube in the case of yielding at both surfaces simultaneously

N	%error(n_{cr})	%error(Ω_e)
100	2.44E-03	2.17E-03
200	6.26E-04	5.73E-04

Plasticity begins at the inner surface of the tube at the second case in which $n = 1.1 < n_{cr}$. $\phi_Y(r_{in}) = 1.0$ due to yielding criteria. Again Newton's method is used in the iterations in order to find Ω_e. The process starts with a first assumption value Ω_{e1} and continues as follows:

1. calculate $f_1 = \phi_Y(r_{in}) - 1.0$ with Ω_{e1}
2. calculate $f_2 = \phi_Y(r_{in}) - 1.0$ with $\Omega_{e2} = \Omega_{e1} + \Delta$
3. calculate $f_3 = \phi_Y(r_{in}) - 1.0$ with $\Omega_{e3} = \Omega_{e1} - \Delta$
4. calculate $\Omega_{e1\_new} = \Omega_{e1} - \frac{2\Delta f_1}{f_2 - f_3}$

Iterations are repeated until $|\Omega_{e1\_new} - \Omega_{e1}|$ is less than specified error tolerance which is defined as $1\text{x}10^{-10}$.

At the third case, plastic behavior commences at the outer surface of the tube for $n = 1.5 > n_{cr}$. The process to reach Ω_e is very similar to the one used in the second case. The only difference is utilizing $\phi_Y(r_{out}) - 1.0$ instead of $\phi_Y(r_{in}) - 1.0$.

Errors in GFD method for Ω_e at the last two cases employing 100 and 200 nodes in the domain are shown in Table 6. GFD formulae are constructed by employing $w = 1/d$ and $dm = 2h + \delta$.

Table 6. Error in GFD method for Ω_e in a rotating FGM tube for $n = 1.1$ and $n = 1.5$

	%error(Ω_e)	
N	$n = 1.1 < n_{cr}$	$n = 1.5 > n_{cr}$
100	2.59E-03	1.76E-03
200	6.84E-04	4.66E-04

CHAPTER 3

ELASTOPLASTIC RESPONSE OF A LONG FUNCTIONALLY GRADED TUBE SUBJECTED TO INTERNAL PRESSURE

3.1 General

In this chapter, elastoplastic response of a long FGM tube subjected to internal pressure is analyzed by GFD method. Parameters which influence radial variations of elastic modulus, yield stress and Poisson's ratio are changed in various problems. Both fixed and free ended tubes are considered.

For FGM tubes with fixed ends, firstly graded elastic modulus only then graded elastic modulus and graded yield stress together are considered in the computations. These cases are taken into account in elastic and elastoplastic stress states. Results are compared to analytical solutions obtained from Refs.[10] & [11].

Functionally graded tubes with free ends are examined only in elastic stress state. Elastic modulus and Poisson's ratio vary radially throughout the tube. Results of the numerical solution are compared to those of shooting method calculated in Ref.[12].

Error definition in the chapter is based on Eq. (25).

Exponential and cubic spline weighting functions utilized in GFD formulae are described in Eqs. (23) and (24), respectively. As stated in previous chapter, in the solution with cubic spline weighting function, for $d = dm$, $w \approx 0$, and corresponding nodes have nearly zero effect in the calculations. Therefore, in the presentation of errors calculated for cubic spline weighting function, dm is taken h more than the ones for the other weighting functions.

Iterations throughout the chapter are performed by Newton's method. A detailed example procedure for iterations is given in the previous chapter.

In all calculations following material properties are considered: $E_0 = 200$ GPa, $\sigma_0 = 430$ MPa, $\nu = 0.3$. Inner and outer radii of the tube are 0.7 m and 1.0 m, respectively.

3.2 Elastoplastic Response of a Pressurized Functionally Graded Tube with Fixed Ends

3.2.1 Solution for Graded Modulus of Elasticity

In this section, the problem solved analytically in Ref.[10] is studied by GFD method, and results obtained are compared. The problem is elastoplastic response of a long FGM tube with fixed ends subjected to internal pressure. The modulus of elasticity of the tube material varies in the radial direction with following relation:

$$E(r) = E_0 \left[1 - n\left(\frac{r}{r_{out}}\right)^k\right] \tag{26}$$

where n and k are material parameters.

In the problem, cylindrical polar coordinates (r,θ,z) are considered and the notation of Timoshenko and Goodier [1] is used. A state of plane strain ($\varepsilon_z = 0$) and infinitesimal deformations are assumed. The equation of equilibrium in the radial direction (Ref.[1]) is as follows:

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0 \tag{27}$$

Following non-dimensional and normalized values are used in the presentation of the results.

$$\overline{r} = \frac{r}{r_{out}}\ ,\ \ \overline{\sigma}_j = \frac{\sigma_j}{\sigma_0}\ ,\ \overline{u} = \frac{uE_0}{\sigma_0 r_{out}}\ ,\ \overline{\epsilon}_j^p = \frac{\epsilon_j^p E_0}{\sigma_0} \tag{28}$$

3.2.1.1 Elastic Stress State

Eqs. (14), (15) and (16) are valid for stress displacement relations in elastic stress state. Substitution of Eq. (26) into Eqs. (14) and (15), and those into Eq. (27) result in the governing differential equation for radial displacement u.

$$r^2\left[1 - n\left(\frac{r}{r_{out}}\right)^k\right]\frac{d^2u}{dr^2} + r\left[1 - n(1+k)\left(\frac{r}{r_{out}}\right)^k\right]\frac{du}{dr} - \frac{1 - \nu - n\left[1 - \nu(1+k)\right]\left(\frac{r}{r_{out}}\right)^k}{1-\nu}u = 0 \tag{29}$$

In the solution, GFD formulae are constructed similar to the example problem. Derivative coefficient matrices are used in order to construct left hand side of the Eq. (29). Right hand side of the equation is an array of length N with zeros. Once the system $K.u = F$ is constructed, then boundary conditions are inserted in the equation. In this problem, boundary conditions read $\sigma_r(r_{in}) = -p_{in}$ and $\sigma_r(r_{out}) = 0$, where p_{in} is the internal pressure. The system is solved to obtain displacement values, and then stresses are calculated according to the Eqs. (14), (15), and (16).

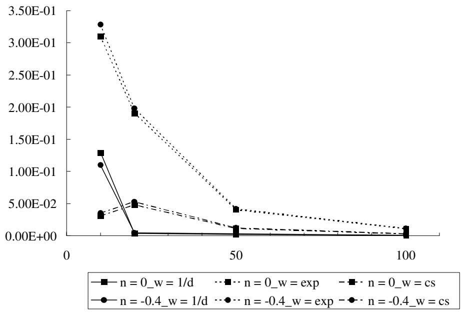
Elastic responses of homogeneous and FGM tubes subjected to nondimensional internal pressure $\overline{p}_{in} = 0.255$ are examined. In the calculations, material parameters are $n = 0$ for homogeneous tube through which elastic modulus is constant, and $n = -0.4$, $k = 0.6$ for nonhomogeneous tube through which elastic modulus varies according to the Eq. (26). Errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in both tubes calculated by utilizing different GFD parameters with $N = 100$ are shown in Table 7.

Variation of errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ with increasing number of nodes in both homogeneous and nonhomogeneous tubes are shown in Figures 12 and 13, respectively. In the calculations, $dm = 2h + \delta$ for $w = 1/d$ and exponential weighting function, and $dm = 5h + \delta$ for cubic spline weighting function.

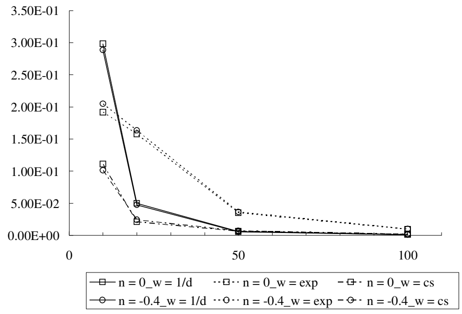
Table 7. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in homogeneous and non-homogeneous tubes in elastic stress state subjected to internal pressure $\overline{p}_{in} = 0.255$

		Homogeneous Tube		Nonhomogeneous Tube	
weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	$2h + \delta$	7.32E-04	1.32E-03	9.71E-04	1.20E-03
$1/d$	$3h + \delta$	1.37E-02	1.34E-02	1.41E-02	1.37E-02
	$4h + \delta$	3.55E-02	3.86E-02	3.60E-02	3.93E-02
	$2h + \delta$	1.10E-02	9.74E-03	1.14E-02	1.01E-02
exponential	$3h + \delta$	3.82E-02	3.97E-02	3.89E-02	4.05E-02
	$4h + \delta$	1.03E-01	1.15E-01	1.04E-01	1.16E-01
	$3h + \delta$	5.02E-03	3.26E-03	5.33E-03	3.47E-03
cubic spline	$4h + \delta$	2.18E-02	2.21E-02	2.23E-02	2.26E-02
	$5h + \delta$	3.16E-03	1.94E-03	3.36E-03	2.08E-03

%error($\overline{u}$)



%error($\frac{d\overline{u}}{d\overline{r}}$)



Figures 12 & 13. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in homogeneous ($n = 0$) and FGM ($n = -0.4$, $k = 0.6$) tubes in elastic stress state under internal pressure $\overline{p}_{in} = 0.255$ versus number of nodes

Stresses and displacements in the tubes calculated with GFD parameters $w = 1/d$, $dm = 2h + \delta$ and $N = 100$ are plotted in Figure 14. Solid and dashed lines show GFD results for FGM and homogeneous tubes, respectively. Analytical results in Ref.[10] are also displayed in the graph by dots for nonhomogeneous tube and by crosses for homogeneous one.

Elastic limit pressure calculations are based on Tresca's yield condition criterion and related stress variable ϕ_Y, where $\phi_Y(r) = (\sigma_\theta(r) - \sigma_r(r))/\sigma_0$. Elastic limit is the pressure under

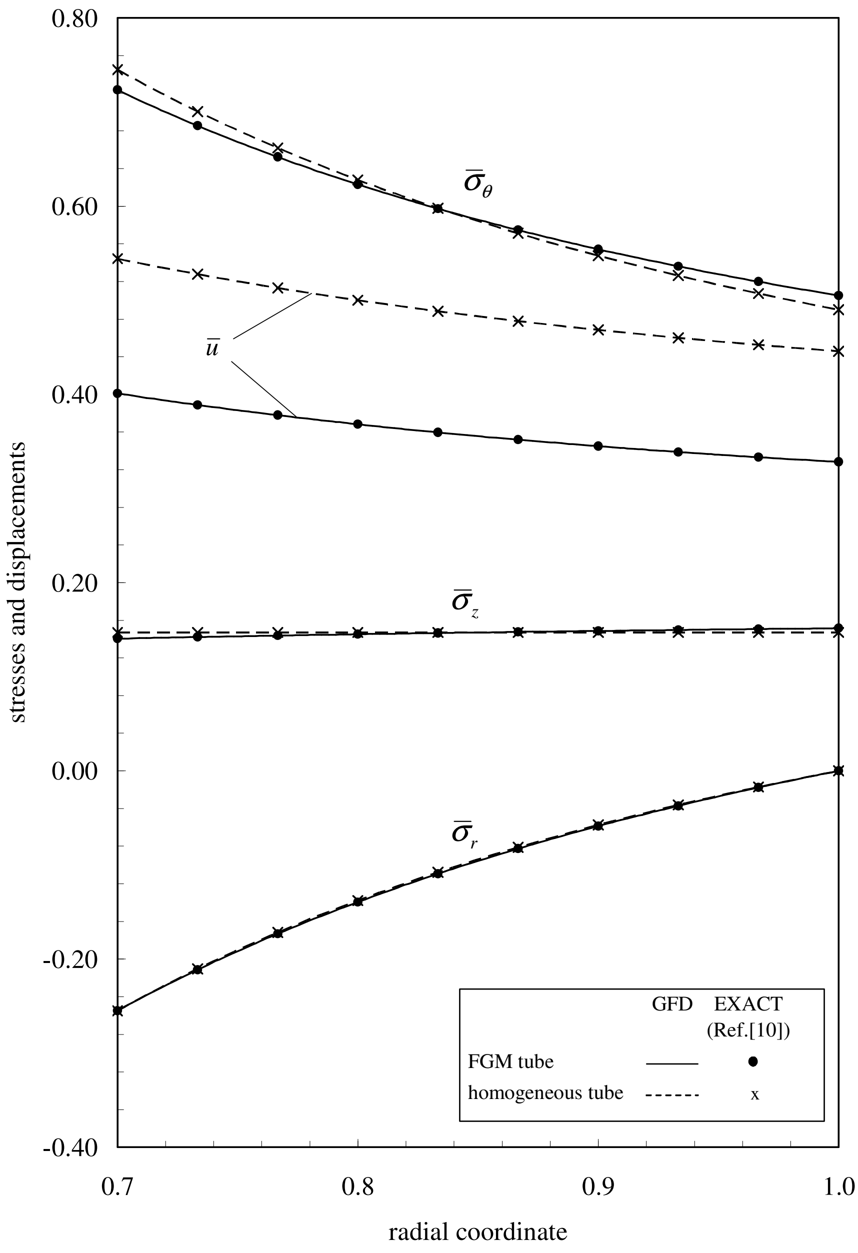


Figure 14: Comparison of stresses and displacement in an FGM tube ($n = -0.4$, $k = 0.6$) to those in a homogeneous tube ($n = 0$) under internal pressure $\overline{p}_{in} = 0.255$.

which $\phi_Y(r_{in})$ reaches 1.0. Newton's method is used in the iterations to obtain elastic limit pressures for given material parameters.

The variation of elastic limit pressure with parameter n is examined for different values of parameter k. Values of n ranging from -0.990 to 0.990 are taken into consideration. In GFD formulae, $w = 1/d$ and $dm = 2h + \delta$ are used. Errors obtained with $N = 100$ and $N = 200$ are shown in Table 8.

Table 8. Error in GFD method ($N = 100$ and $N = 200$) for elastic limit pressures in FGM tubes for different values of parameter k

	%error($\overline{p}_e$)			
N	$k = 0.0$	$k = 0.6$	$k = 1.2$	$k = 1.8$
100	8.73E-04	1.52E-03	1.53E-03	1.37E-03
200	2.38E-04	1.06E-03	6.79E-04	3.51E-04

GFD results with $w = 1/d$, $dm = 2h + \delta$ and $N = 200$, and exact elastic limit pressures are plotted in Figure 15. Solid lines present numerical results, and dots show exact values obtained from Ref.[10].

3.2.1.2 Elastoplastic Stress State

If the tube is exposed to internal pressures greater than elastic limit pressure, it goes into elastoplastic stress state. Inner surface of the tube deforms plastically, and after the elastic-plastic border elastic deformation begins.

As the condition $\sigma_r < \sigma_z < \sigma_\theta$ is satisfied throughout the tube, Tresca's yield criteria reads

$$\sigma_\theta - \sigma_r = \sigma_0 \tag{30}$$

In elastoplastic stress state, total strains are expressed as the sum of elastic and plastic strains.

$$\varepsilon_r = \frac{1}{E(r)}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] + \varepsilon_r^p \tag{31}$$

$$\varepsilon_\theta = \frac{1}{E(r)}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] + \varepsilon_\theta^p \tag{32}$$

$$\varepsilon_z = \frac{1}{E(r)}\left[\sigma_z - \nu(\sigma_\theta + \sigma_r)\right] + \varepsilon_z^p = 0 \tag{33}$$

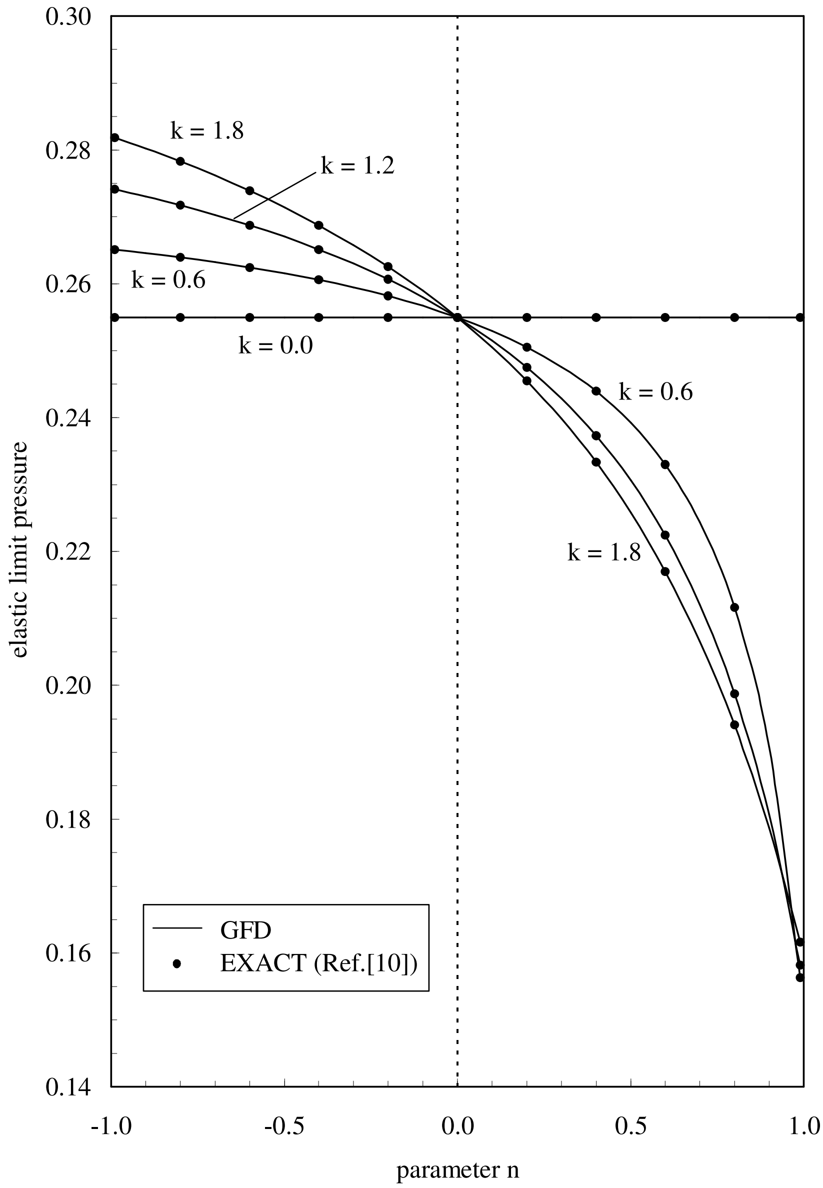


Figure 15: Variation of elastic limit pressure $\overline{p}_e$ in pressurized FGM tubes with material parameters n and k.

The flow rule associated with the yielding criteria is $\varepsilon_\theta^p = -\varepsilon_r^p$ and $\varepsilon_z^p = 0$ (Ref.[2]). Thus $\varepsilon_z^e = 0$, and axial stress is

$$\sigma_z = \nu(\sigma_\theta + \sigma_r) \tag{34}$$

Strain displacement relations are

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r} \tag{35}$$

Utilizing Eqs. (31), (32), (35) and flow rule $\varepsilon_\theta^p + \varepsilon_r^p = 0$, simplification of the sum $\varepsilon_r + \varepsilon_\theta$ gives governing differential equation for radial displacement u.

$$\frac{du}{dr} + \frac{u}{r} = \frac{1}{E(r)} \left[(\sigma_\theta + \sigma_r)(1 - 2\nu)(1 + \nu) \right] \tag{36}$$

Elastic-plastic border radius at the interface of two regions is denoted as r_{ep}. In order to find r_{ep} under a given internal pressure p_{in}, iterations are performed to reach $\sigma_\theta^e(r_{ep1}) - \sigma_\theta^p(r_{ep1}) = 0$ where superscripts e and p denote elastic and plastic regions in the tube. Steps of iterations with an initial assumption value for r_{ep} which is r_{ep1} are as follows:

1. calculate stresses in plastic region by employing Eqs. (27), (30), (34) and boundary condition $\sigma_r(r_{in}) = -p_{in}$.
2. calculate stresses and displacement in elastic region according to the procedure given in Section 3.2.1.1 using boundary conditions $\sigma_r(r_{ep1})$ found in step 1 and $\sigma_r(r_{out}) = 0$.
3. calculate $f_1 = \sigma_\theta^e(r_{ep1}) - \sigma_\theta^p(r_{ep1})$, where $f_1 = f(r_{ep1})$
4. repeat steps 1 through 3 for $r_{ep2} = r_{ep1} + \Delta$ and $r_{ep3} = r_{ep1} - \Delta$
5. calculate $r_{ep1\_new} = r_{ep1} - \frac{2\Delta f_1}{f_2 - f_3}$

Iterations are repeated until $|r_{ep1\_new} - r_{ep1}|$ is less than specified error tolerance which is defined as 1x10^{-10}. With the knowledge of r_{ep}, displacement values in plastic part are obtained from Eq. (36) by the use of boundary condition $u(r_{ep})$ found in step 2. Plastic strains are calculated by Eqs. (31) and (32).

The elastic-plastic border moves towards the outer surface of the tube as internal pressure p_{in} increases. Propagation of elastic-plastic border radius for 3 different values of parameter n is analyzed. Material parameter $k = 0.6$, GFD parameters $w = 1/d$ and $dm = 2h + \delta$ are used in the calculations. Errors in $\overline{r}_{ep}$ calculated with $N = 100$ and $N = 200$ are shown in Table 9.

Figure 16 shows propagation of elastic-plastic border radius by solid lines being numerical results and dots displaying analytical ones in Ref.[10]. Fully plastic limits for different values of parameter n are found by extrapolation with second degree polynomials.

Stresses and displacement values under internal pressure $\overline{p}_{in} = 0.33$, for $n = -0.4$ and $k = 0.6$ are also investigated. Interface point $\overline{r}_{ep} = 0.835838$ which is calculated analytically in Ref.[10] is used in order to present errors by the help of same coordinates. Matching

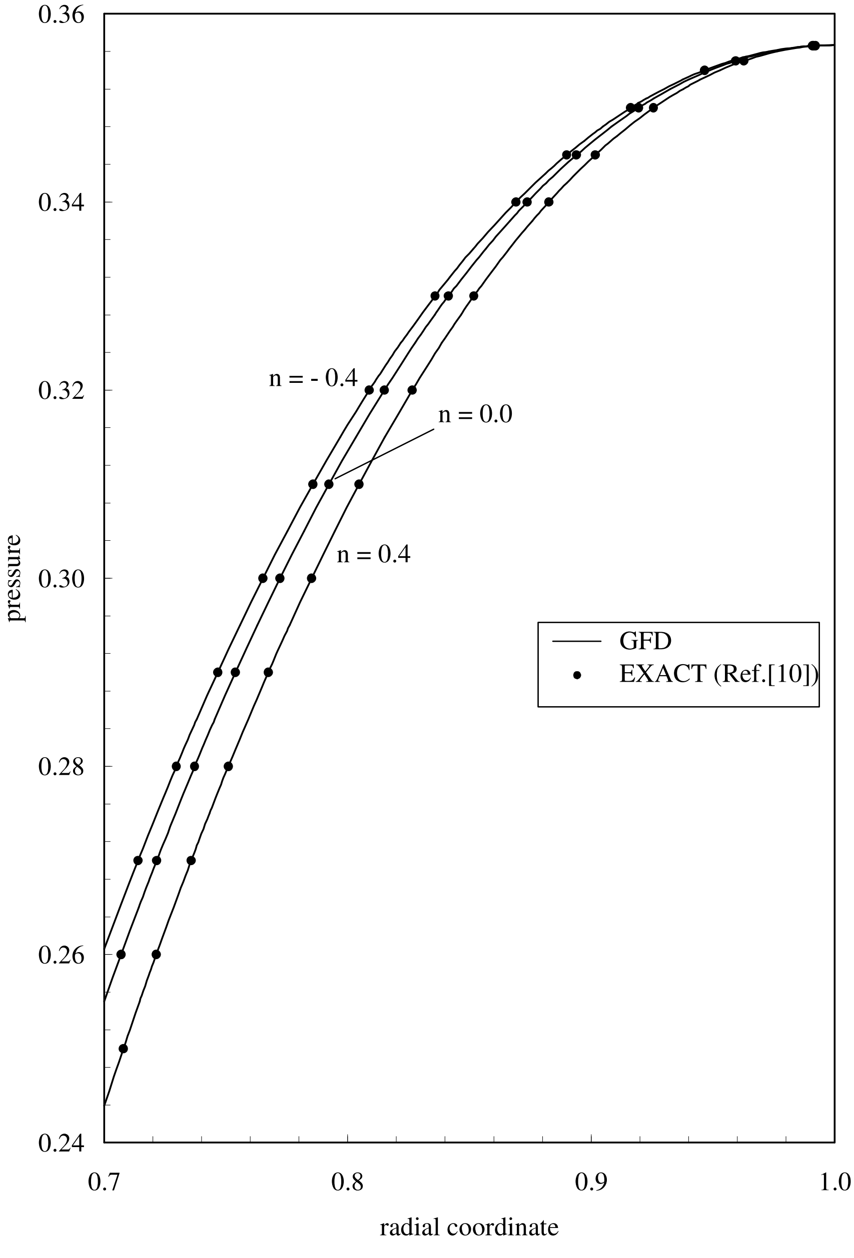


Figure 16: Propagation of elastic-plastic border radius in partially plastic FGM tubes ($k = 0.6$) with increasing internal pressures using n as a parameter.

Table 9. Error in GFD method ($N = 100$ and $N = 200$) for $\overline{r}_{ep}$ in partially plastic FGM tubes subjected to internal pressure for different values of parameter n

	%error($\overline{r}_{ep}$)		
N	n = -0.4	n = 0.0	n = 0.4
100	2.73E-03	2.83E-03	2.40E-03
200	6.71E-04	6.81E-04	5.69E-04

GFD result with $N = 200$ is $\overline{r}_{ep} = 0.835839$. Grid distance in both plastic and elastic regions are kept as close as possible by distribution of number of nodes proportional to the distances between boundary points in each part. For instance, number of nodes in plastic part is calculated as $N_P = 46$ for $N = 100$.

Since elastic behavior is examined in the previous section, plastic part is the main concern in this section. Firstly, error in GFD method for $\overline{\sigma}_r$ values in plastic part, calculated with different GFD parameters considering 200 nodes in the domain is shown in Table 10.

Table 10. Error in GFD method ($N = 200$) for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$

weighting function	dm	%error($\overline{\sigma}_r^p$)
$1/d$	2h + δ	1.49E-04
	3h + δ	2.73E-04
	4h + δ	4.34E-04
exponential	2h + δ	2.02E-04
	3h + δ	4.01E-04
	4h + δ	8.24E-04
cubic spline	3h + δ	1.74E-04
	4h + δ	2.30E-04
	5h + δ	1.98E-04

Variation of errors for $\overline{\sigma}_r^p$ with increasing number of nodes are shown in Figure 17. In the calculations $dm = 2h + \delta$ for $w = 1/d$ and exponential weighting function, $dm = 3h + \delta$ for cubic spline weighting function.

Errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in plastic region for different GFD parameters with $N = 200$ when $dm = 2h + \delta$ and $w = 1/d$ in the calculations of elastic part and plastic stresses are shown in Table 11.

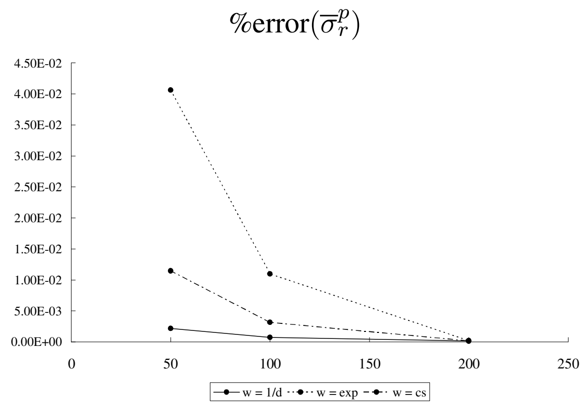


Figure 17. Error in GFD method for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$ versus number of nodes.

Table 11. Error in GFD method ($N = 200$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	$2h+\delta$	3.98E-01	7.07E-01
$1/d$	$3h+\delta$	3.99E-01	5.36E-01
	$4h+\delta$	3.98E-01	5.96E-01
	$2h+\delta$	3.99E-01	5.37E-01
exponential	$3h+\delta$	3.99E-01	5.36E-01
	$4h+\delta$	3.99E-01	5.36E-01
	$3h+\delta$	3.99E-01	5.36E-01
cubic spline	$4h+\delta$	3.99E-01	5.36E-01
	$5h+\delta$	3.99E-01	5.36E-01

Accuracy of the method decreases significantly as seen in Table 11. It must be noted that calculations when $w = 1/d$ and $dm = 3h + \delta$ in GFD formulae gives ε_θ^p and ε_r^p values, which must be zero at the interface point, closest to zero of all mentioned in the table.

Matching stresses, displacement and plastic strains are plotted in Figure 18. In GFD formulae $w = 1/d$ and $dm = 3h + \delta$ for calculations of plastic displacement, $w = 1/d$ and $dm = 2h + \delta$ otherwise. In the figure, ϕ_Y is the stress variable obtained from $\phi_Y(r) = [\sigma_\theta(r) - \sigma_r(r)]/\sigma_0$. It is seen that $\phi_Y = 1$ in plastic region, and $\phi_Y < 1$ in elastic part. Solid lines present numeric results, and dots show exact results in Ref.[10].

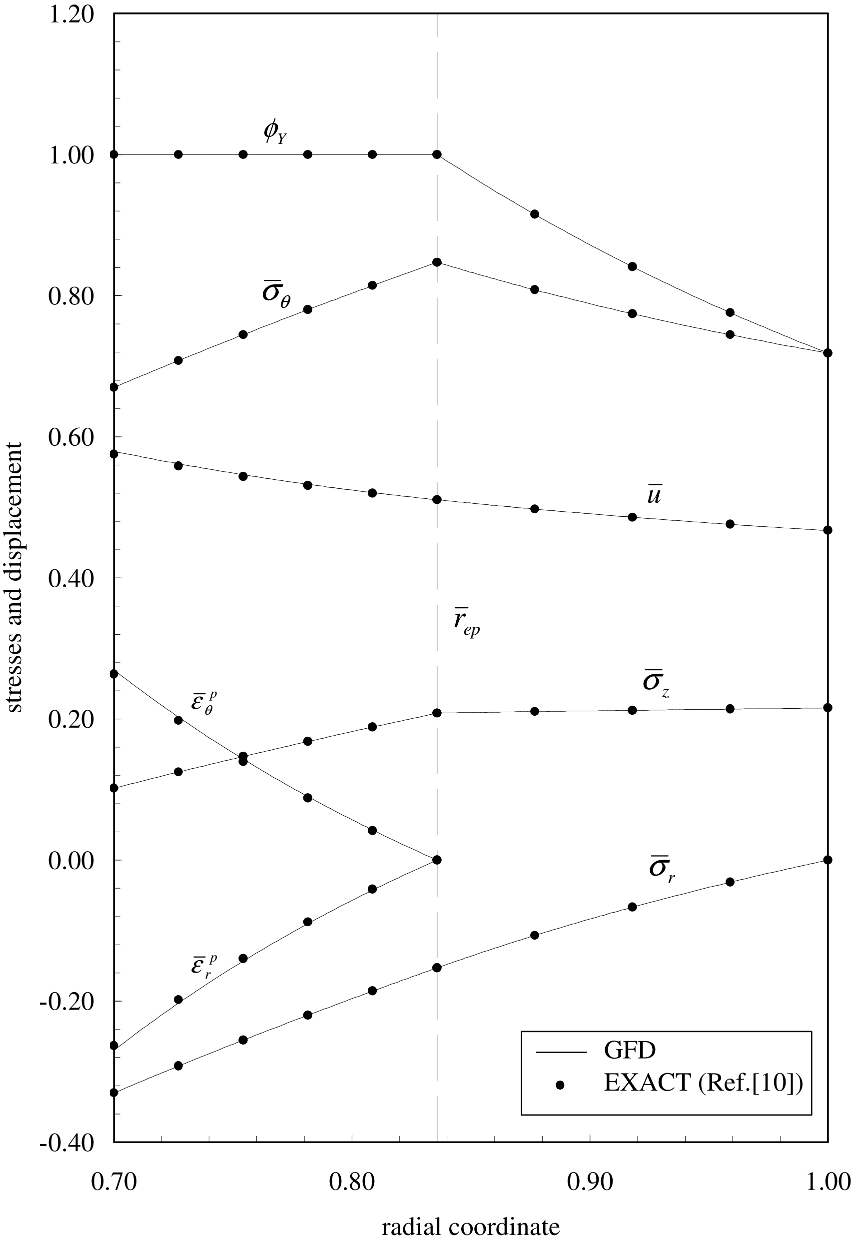


Figure 18: Stresses, displacement and plastic strains in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) under internal pressure $\overline{p}_{in} = 0.33$.

3.2.2 Solution for Graded Modulus of Elasticity and Yield Stress

In this section, elastoplastic response of a long pressurized FGM tube with fixed ends is examined for radial variation of elastic modulus and yield stress. In Ref.[11], it was stated that location of yielding points in an FGM tube may vary with material composition. Four parameter sets are considered in order to observe different modes of plastification. The results obtained are compared to those of analytical solution in Ref.[11].

The modulus of elasticity of the tube varies radially with Eq. (26), and yield limit of the tube material varies in the radial direction with following relation.

$$\sigma_Y(r) = \sigma_0 \left[1 - m\left(\frac{r}{r_{out}}\right)^s\right] \tag{37}$$

where σ_0 is the reference value of σ_Y, and m and s are material parameters.

All equations in Section 3.2.1 are valid here with the substitution of $\sigma_Y(r)$ for σ_0.

Following non-dimensional and normalized values are used in the presentation of the results.

$$\overline{r} = \frac{r}{r_{out}}\ ,\ \ \overline{\sigma}_j = \frac{\sigma_j}{\sigma_Y(r_{in})}\ ,\ \overline{u} = \frac{uE_0}{r_{out}\sigma_Y(r_{in})}\ ,\ \overline{\epsilon}_j^p = \frac{\epsilon_j^p E_0}{\sigma_Y(r_{in})} \tag{38}$$

3.2.2.1 Elastic Stress State

Stresses and displacement in an FGM tube subjected to elastic limit pressure are calculated for different material parameters by GFD method. Exact elastic limit pressures obtained from Ref.[11] are used in the computations. Elastic limit pressures calculated by GFD method are also investigated.

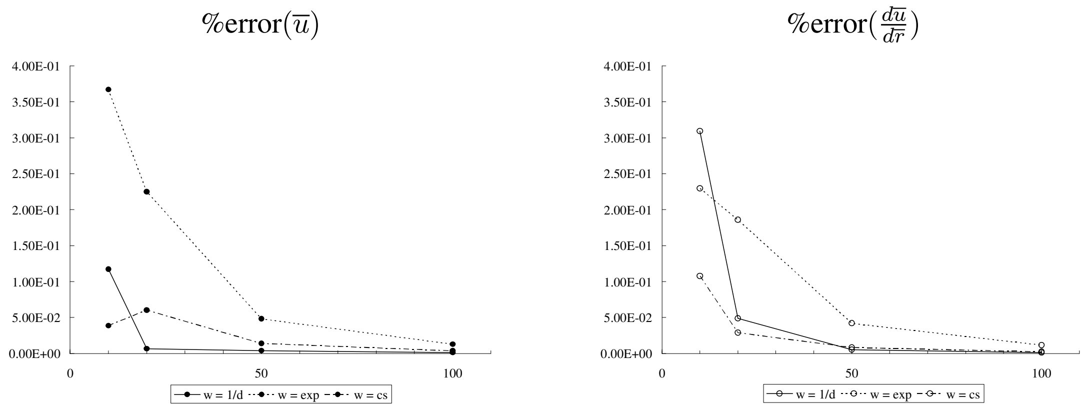
The stress variable ϕ_Y is expressed in the form $\sigma_\theta(r) - \sigma_r(r))/\sigma_Y(r)$, since σ_Y varies in radial direction. For any internal pressure in the elastic limits $\phi_Y(r_y)$, where r_y is the start point of yielding, is the maximum of ϕ_Y values throughout the tube. Yielding commences when $\phi_Y(r_y)$ reaches 1.0. This criteria is used in the iterations with Newton's Method to calculate elastic limit pressure.

In the first case, parameter set: $n = 0.2$, $k = -1.1$, $m = 0.4$ and $s = 0.9$ which brings about initiation of plastic flow at the inner surface of the tube is considered (Ref.[11]). For this case nondimensional elastic limit pressure is calculated as $\overline{p}_e = 0.269497$ in Ref.[11]. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ obtained by employing $N = 100$ for different parameters of the method is shown in Table 12.

Table 12. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.269497$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
$1/d$	$2h + \delta$	1.24E-03	1.11E-03
	$3h + \delta$	1.60E-02	1.55E-02
	$4h + \delta$	4.06E-02	4.38E-02
exponential	$2h + \delta$	1.30E-02	1.15E-02
	$3h + \delta$	4.41E-02	4.55E-02
	$4h + \delta$	1.17E-01	1.30E-01
cubic spline	$3h + \delta$	6.20E-03	4.14E-03
	$4h + \delta$	2.54E-02	2.55E-02
	$5h + \delta$	3.84E-03	2.44E-03

Variation of errors in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ under $\overline{p}_e = 0.269497$ with number of nodes is seen in Figures 19 and 20. In the calculations $dm = 2h + \delta$ for $w = 1/d$ and exponential weighting function. $dm = 5h + \delta$ for cubic spline weighting function.



Figures 19 & 20. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.269497$ versus number of nodes.

Elastic response of the tube for mentioned material parameters under elastic limit pressure is analyzed by GFD formulae constructed with $w = 1/d$, $dm = 2h + \delta$ and $N = 100$. Results obtained are plotted in Figure 21 (continuous lines). In the figure, there are also exact solution of the problem from Ref.[11], presented by dots. $\phi_Y(r_{in}) = 1$ and $\phi_Y < 1$ elsewhere, which proves that for parameter set $n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$, plasticity commences at the inner surface of the tube.

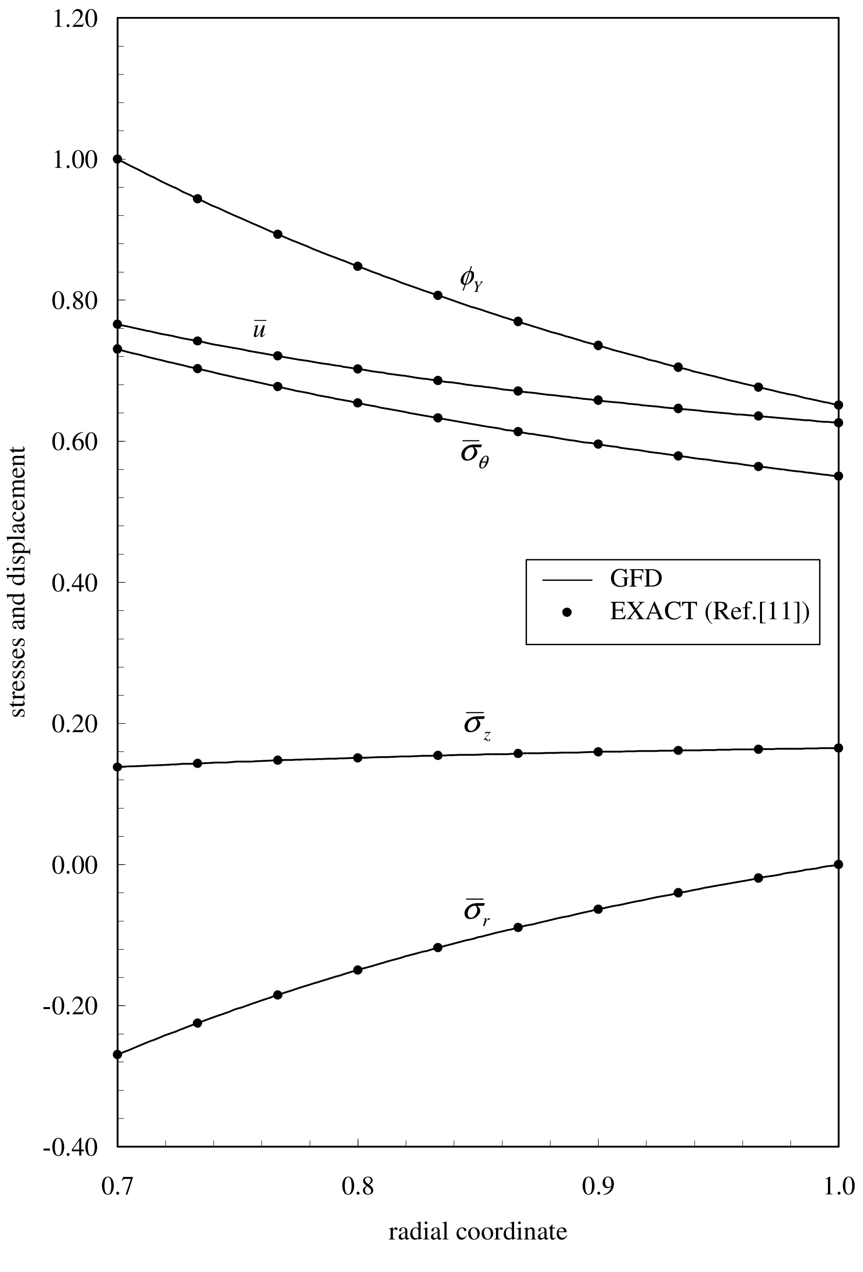


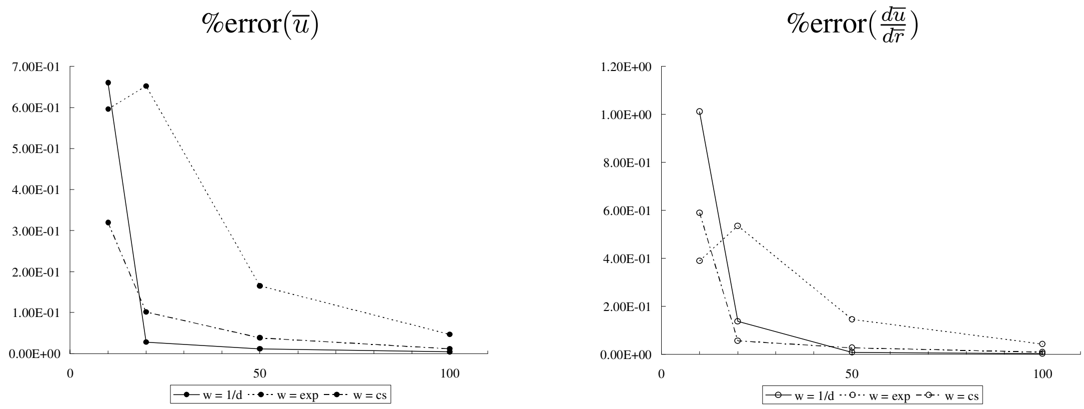
Figure 21: Elastic response of an FGM tube ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.269497$.

In the second case, in order to find a composition which leads to plastic deformation at outer surface of the tube, parameter set: $n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$ is considered. This material composition results in elastic limit pressure $\overline{p}_e = 0.264938$ in Ref.[11]. GFD method is applied to the analysis of the tube subjected to elastic limit pressure. Errors for displacement and first derivative values with different GFD parameters and $N = 100$ are shown in Table 13.

Table 13. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.264938$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	$2h + \delta$	4.47E-03	1.99E-03
$1/d$	$3h + \delta$	5.51E-02	5.19E-02
	$4h + \delta$	1.40E-01	1.41E-01
	$2h + \delta$	4.72E-02	4.22E-02
exponential	$3h + \delta$	1.55E-01	1.51E-01
	$4h + \delta$	4.05E-01	4.12E-01
	$3h + \delta$	2.25E-02	1.75E-02
cubic spline	$4h + \delta$	9.08E-02	8.73E-02
	$5h + \delta$	1.18E-02	8.72E-03

Variation of errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ with number of nodes is seen in Figures 22 and 23. In the calculations $dm = 2h + \delta$ for $w = 1/d$ and exponential weighting function. $dm = 5h + \delta$ for cubic spline weighting function.



Figures 22 & 23. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.264938$ versus number of nodes.

Stresses and displacements in the tube calculated by GFD method with $w = 1/d$, $dm = 2h + \delta$ and $N = 100$ are plotted by continuous lines in Figure 24. Analytical solution of the problem obtained from Ref.[11] are also presented by dots. It is seen in the figure that $\phi_Y(r_{out}) = 1$ and $\phi_Y < 1$ in the elastic region.

After the calculations which reveal yielding point at the boundary points separately, a parameter set that concludes yielding at boundary points simultaneously is considered in the third case. Material parameters are $n = 0.348459$, $k = -0.7$, $m = 0.7$ and $s = 1.1$. Under elastic limit pressure $\overline{p}_e = 0.273478$ calculated analytically in Ref.[11], error in GFD method for displacement and first derivative values with different GFD parameters and $N = 100$ are shown in Table 14.

Table 14. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in an FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.273478$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{dr}$)
	$2h + \delta$	1.42E-03	1.03E-03
$1/d$	$3h + \delta$	1.63E-02	1.58E-02
	$4h + \delta$	4.13E-02	4.46E-02
	$2h + \delta$	1.34E-02	1.18E-02
exponential	$3h + \delta$	4.50E-02	4.64E-02
	$4h + \delta$	1.19E-01	1.32E-01
	$3h + \delta$	6.47E-03	4.33E-03
cubic spline	$4h + \delta$	2.59E-02	2.60E-02
	$5h + \delta$	4.01E-03	2.56E-03

Error in the method for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ versus increasing number of nodes are shown in Figures 25 & 26. In the calculations, for $w = 1/d$ and exponential weighting function $dm = 2h + \delta$, and for cubic spline weighting function $dm = 5h + \delta$.

Stresses and displacement obtained in the third case by setting $w = 1/d$, $dm = 2h + \delta$ and $N = 100$ in GFD formulae are plotted in Figure 27. Continuous lines and dots present numerical and analytical results, respectively. It can be seen in the figure that $\phi_Y(r_{in}) = \phi_Y(r_{out}) = 1.0$ which shows plastification at the both ends of the tube simultaneously.

In the fourth case, parameter set $n = 0.4$, $k = -1.6$, $m = 0.2$ and $s = 0.9$ is considered, and yielding commences at a point where $r_{in} < r < r_{out}$ under $\overline{p}_e = 0.336428$ (Ref.[11]). Errors in GFD method for displacement and first derivative values with different GFD parameters and $N = 100$ are shown in Table 15.

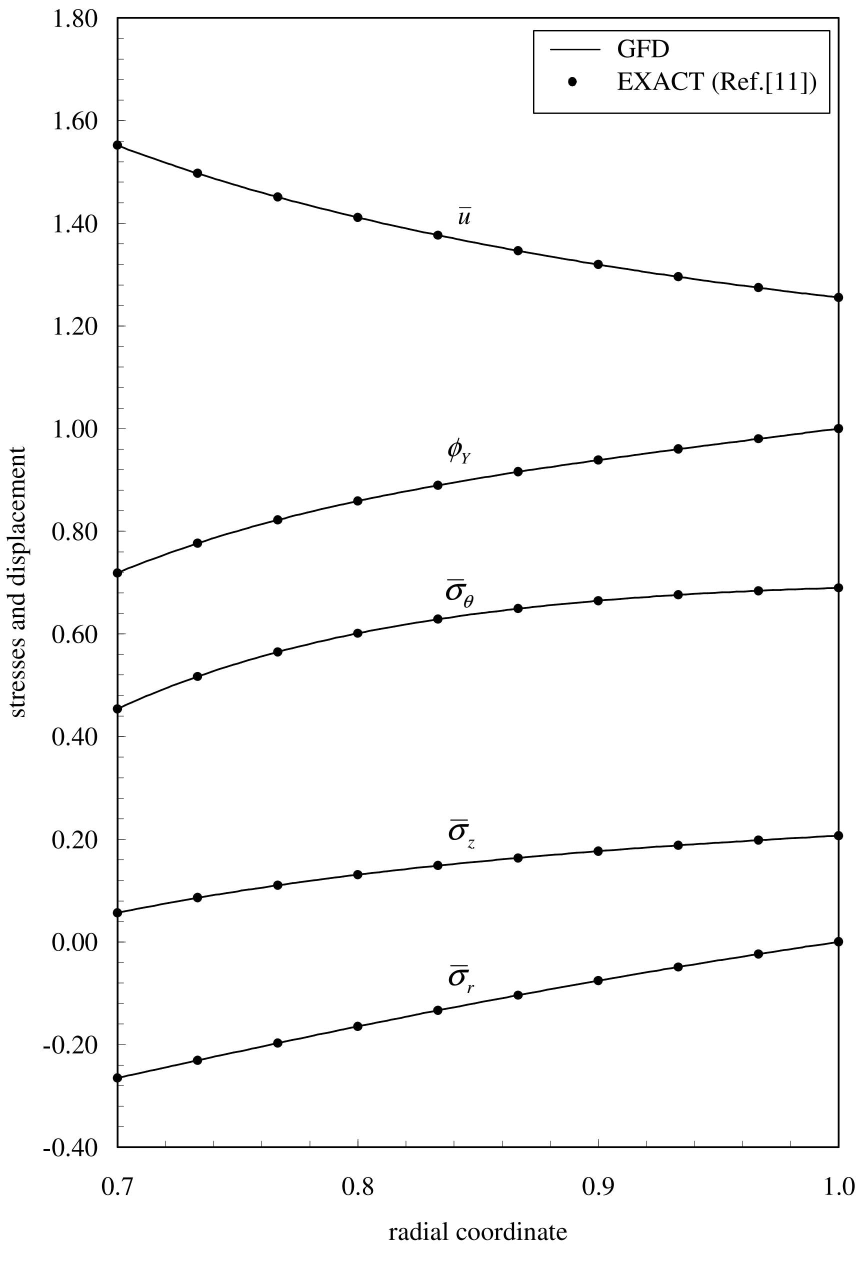
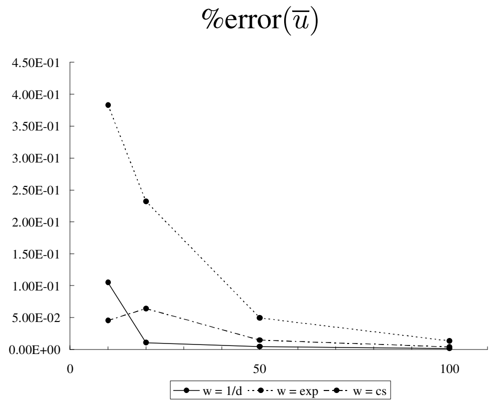
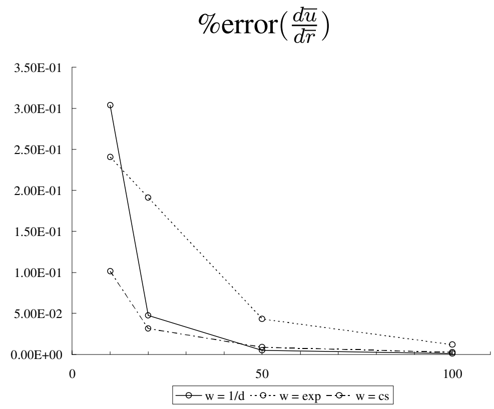


Figure 24: Elastic response of an FGM tube ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.264938$.





Figures 25 & 26. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.273478$ versus number of nodes.

Table 15. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.336428$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	$2h+\delta$	3.95E-03	1.86E-03
$1/d$	$3h+\delta$	5.48E-02	5.18E-02
	$4h+\delta$	1.40E-01	1.41E-01
	$2h+\delta$	4.68E-02	4.20E-02
exponential	$3h+\delta$	1.55E-01	1.51E-01
	$4h+\delta$	4.07E-01	4.14E-01
	$3h+\delta$	2.20E-02	1.72E-02
cubic spline	$4h+\delta$	9.07E-02	8.73E-02
	$5h+\delta$	1.15E-02	8.47E-03

Error for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ versus increasing number of nodes in the fourth case are shown in Figures 28 & 29. In the calculations, for $w = 1/d$ and exponential weighting function, $dm = 2h + \delta$ and for cubic spline weighting function $dm = 5h + \delta$.

Matching stresses and displacement calculated by setting $w = 1/d$, $dm = 2h + \delta$ and $N = 100$ in GFD formulae are plotted by continuous lines in Figure 30. Dots present analytical solution in Ref.[11].

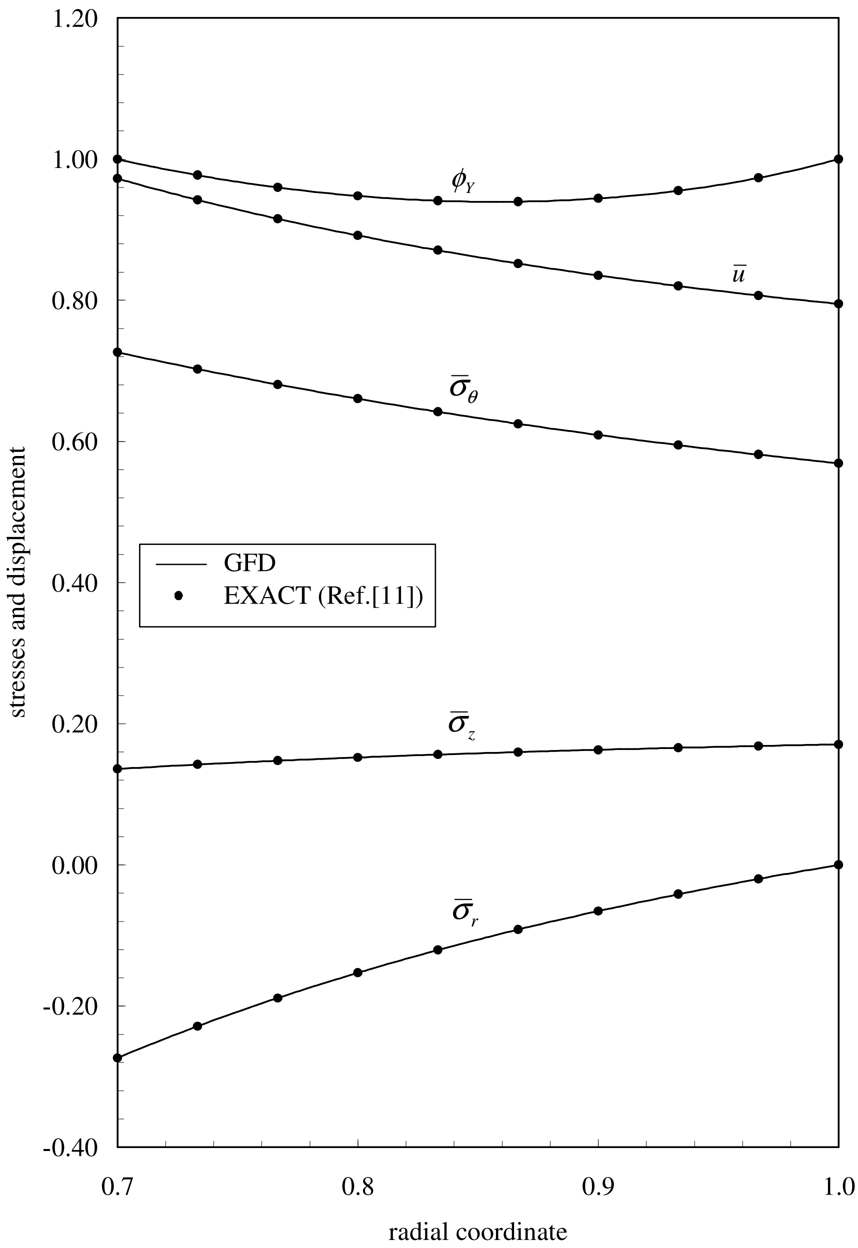
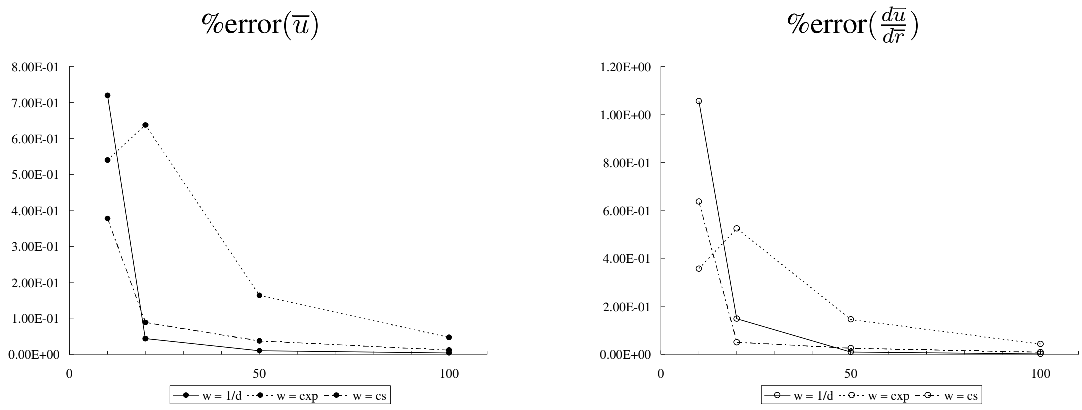


Figure 27: Elastic response of an FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.273478$.

%error($\overline{u}$) and %error($\frac{d\overline{u}}{d\overline{r}}$)



Figures 28 & 29. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in an FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.336428$ versus number of nodes.

For each case considered here, elastic limit pressures are calculated by GFD method and iterations using Newton's Method. Table 16 shows errors for elastic limit pressures in the method with $w = 1/d$ and $dm = 2h + \delta$.

Table 16. Error in GFD method for $\overline{p}_e$ in FGM tubes versus increasing number of nodes in four cases considered

	%error($\overline{p}_e$)			
N	case 1	case 2	case 3	case 4
10	7.42E-02	6.97E-01	1.31E-01	8.76E-01
20	1.22E-02	3.76E-02	4.69E-03	8.49E-02
50	4.49E-03	1.03E-02	3.81E-03	5.10E-03
100	1.32E-03	4.13E-03	1.19E-03	3.04E-03
200	3.54E-04	1.24E-03	3.28E-04	9.24E-04

3.2.2.2 Elastoplastic Stress State

As mentioned earlier the tube goes into elastoplastic stress state under internal pressures $p_{in} > p_e$. Elastic and plastic regions are seen through the tube.

Tresca's yield criteria reads $\sigma_\theta(r) - \sigma_r(r) = \sigma_Y(r)$ since $\sigma_r < \sigma_z < \sigma_\theta$ in the tube. Therefore for plastic region Eq. (27) gives

$$r\frac{d}{dr}\sigma_r(r) = \sigma_Y(r) \tag{39}$$

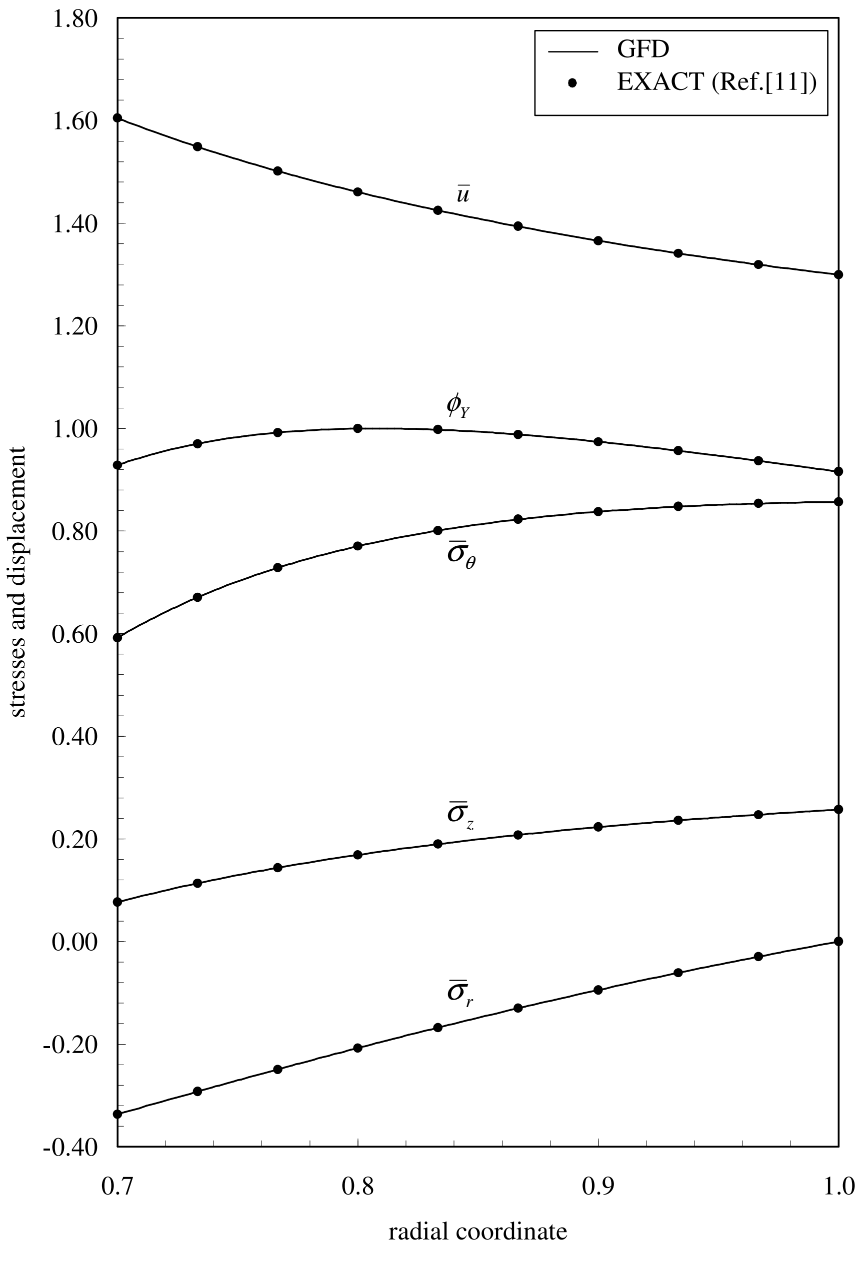


Figure 30: Elastic response of an FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = \overline{p}_e = 0.336428$.

Four cases considered in elastic stress state are studied, and elastoplastic behavior of FGM tube is analyzed. In the figures matching analytical results in Ref.[11] are displayed by dots.

As shown in Figure 21 plastic flow starts at the inner surface of the tube with parameter set: $n = 0.2$, $k = -1.1$, $m = 0.4$ and $s = 0.9$ under $\overline{p}_e = 0.269497$ in the first case. Plastic border propagates towards the outer surface of the tube as internal pressure increases. So an inner plastic region and an outer elastic region come into being. Calculations start with the stresses in plastic region employing boundary condition $\sigma_r(r_{in}) = -p_{in}$. Then $\sigma_r(r_{ep})$ found in plastic part and $\sigma_r(r_{out}) = 0$ are used as boundary conditions to calculate stresses and displacement in elastic part. Finally, displacement values in plastic part are found by utilizing boundary condition $u(r_{ep})$ obtained from elastic computations. Iterations with Newton's Method in order to find r_{ep} are based on the difference $\sigma_\theta^p(r_{ep}) - \sigma_\theta^e(r_{ep}) = 0$ where superscripts e and p denote elastic and plastic parts. Propagation of elastic-plastic border can be seen in Figure 31. In the calculations GFD formulae are constructed with $w = 1/d$, $dm = 2h + \delta$ and $N = 200$.

In the second case for parameter set: $n = 0.5$, $k = -1.2$, $m = 0.7$ and $s = 0.6$, it is shown in Figure 24 that plasticity commences at the outer surface of the tube under $\overline{p}_e = 0.264938$. Elastic-plastic border propagates towards inner surface for increasing internal pressures when $p_{in} > p_e$. In this case there are two regions, elastic and plastic through the outer surface of the tube. Calculations are very much the same as the first case. The difference is the boundary conditions considered due to the location of elastic and plastic regions. In order to calculate the stresses in plastic part, boundary condition $\sigma_r(r_{out}) = 0$ is used. For elastic region, boundary conditions read $\sigma_r(r_{ep})$ found in plastic part and $\sigma_r(r_{in}) = -p_{in}$. As for displacement calculations in plastic part, $u(r_{ep})$ obtained in elastic part is taken into account. Variation of interface points with increasing internal pressures calculated for $w = 1/d$, $dm = 2h + \delta$ and $N = 200$ in GFD formulae is shown in Figure 32.

Plasticity starts at inner and outer surfaces simultaneously with material composition $n = 0.348459$, $k = -0.7$, $m = 0.7$ and $s = 1.1$ under $\overline{p}_e = 0.273478$ in the third case. Two elastic plastic borders come into being if pressure is increased above elastic limit pressure, and consequently three regions appear: plastic region for $r_{in} < r < r_{ep1}$, elastic region for $r_{ep1} < r < r_{ep2}$ and plastic region for $r_{ep2} < r < r_{out}$. In the calculations r_{ep1} is assumed, and then stresses at inner plastic part is calculated. With boundary conditions $\sigma_r(r_{ep1})$ and $\sigma_r(r_{out})$, calculations for the second case in which plasticity commences at the outer surface is valid, and r_{ep2} is found. Finally iterations are performed to reach $\sigma_\theta^p(r_{ep1}) - \sigma_\theta^{e-p}(r_{ep1}) = 0$ where superscripts p and $e - p$ denote inner plastic and outer elastic-plastic parts. Plastic regions move towards each other under increasing internal pressures. Finally the tube goes into pure plastic stress state. The expansion of plastic regions is shown in Figure 33. In the calculations GFD formulae are constructed with $w = 1/d$, $dm = 2h + \delta$ and $N = 200$.

In the fourth case parameter set: $n = 0.4$, $k = -1.6$, $m = 0.2$ and $s = 0.9$ is considered. Plasticity commences somewhere in a middle point r where $r_{in} < r < r_{out}$ under $\overline{p}_e = 0.336428$. As the pressure is increased, the tube undergoes elastic-plastic-elastic stress states from inner surface to outer surface. There are two elastic-plastic borders, r_{ep1} and r_{ep2}.

Different from previous cases, iterations are performed not only for interface values but also for $\sigma_r(r_{ep1})$. For inner elastic part between r_{in} and r_{ep1}, by the use of boundary conditions $\sigma_r(r_{in}) = -p_{in}$ and assumed $\sigma_r(r_{ep1})$, stresses and displacements are calculated. For outer plastic-elastic part steps followed are the same as the ones in the first case in which plasticity commences at inner surface, and r_{ep2} is found. In order to obtain r_{ep1} and $\sigma_r(r_{ep1})$, iterations with Newton's method are performed to reach $\sigma_\theta^e(r_{ep1}) - \sigma_\theta^{p-e}(r_{ep1}) = 0$ and $u^e(r_{ep1}) - u^{p-e}(r_{ep1}) = 0$, respectively, where superscripts e and $p - e$ express inner elastic and outer plastic-elastic parts. As the pressure is increased, plastic region spreads to the surfaces. First, it reaches to inner surface, then there remains an inner plastic region and an outer elastic region. The problem turns into the one in the first case where the plasticity commences at the inner surface. Finally the tube goes into pure plastic stress state. Elastic displacements and plastic stresses are found by GFD formulae constructed with $w = 1/d$ and $dm = 2h + \delta$. Using $dm = 3d + \delta$ with $w = 1/d$ in the calculations for plastic displacements provides faster convergence. In the domain of the problem 200 nodes are considered. Propagation of interface points towards the surfaces of the tube is shown in Figure 34.

For all cases, interface points calculated by GFD method are compared to analytical results in Ref.[11] to get relative percent errors. Table 17 shows error in the method computed by employing GFD formulae with $N = 100$ and $N = 200$.

Table 17. Error in GFD method ($N = 100$ and $N = 200$) for $\overline{r}_{ep}$ in partially plastic FGM tubes subjected to internal pressure in four cases considered

	%error($\overline{r}_{ep}$)			
N	case 1	case 2	case 3	case 4
100	5.82E-03	1.27E-02	2.28E-02	1.17E-02
200	1.42E-03	2.69E-03	1.14E-02	3.69E-03

Stresses and displacement for the last two cases are also examined.

In the third case plasticity starts at inner and outer surfaces simultaneously with material composition $n = 0.348459$, $k = -0.7$, $m = 0.7$ and $s = 1.1$. Corresponding stresses and displacements are calculated under internal pressure $\overline{p}_{in} = 0.28 > \overline{p}_e$. Analytical calculations (Ref.[11]) result in interface points $\overline{r}_{ep1} = 0.740051$ and $\overline{r}_{ep2} = 0.966528$. Interface points calculated by GFD method with $N = 200$ are $\overline{r}_{ep1} = 0.740049$ and $\overline{r}_{ep2} = 0.966530$. Analytical results are used in the calculations in order to present errors by the help of same coordinates. Errors in GFD method for plastic regions are investigated. Firstly, error in $\overline{\sigma}_r^p$ values for different GFD parameters and $N = 200$ are shown in Table 18. In the calculations for elastic part $w = 1/d$ and $dm = 2h + \delta$ in GFD formulae.

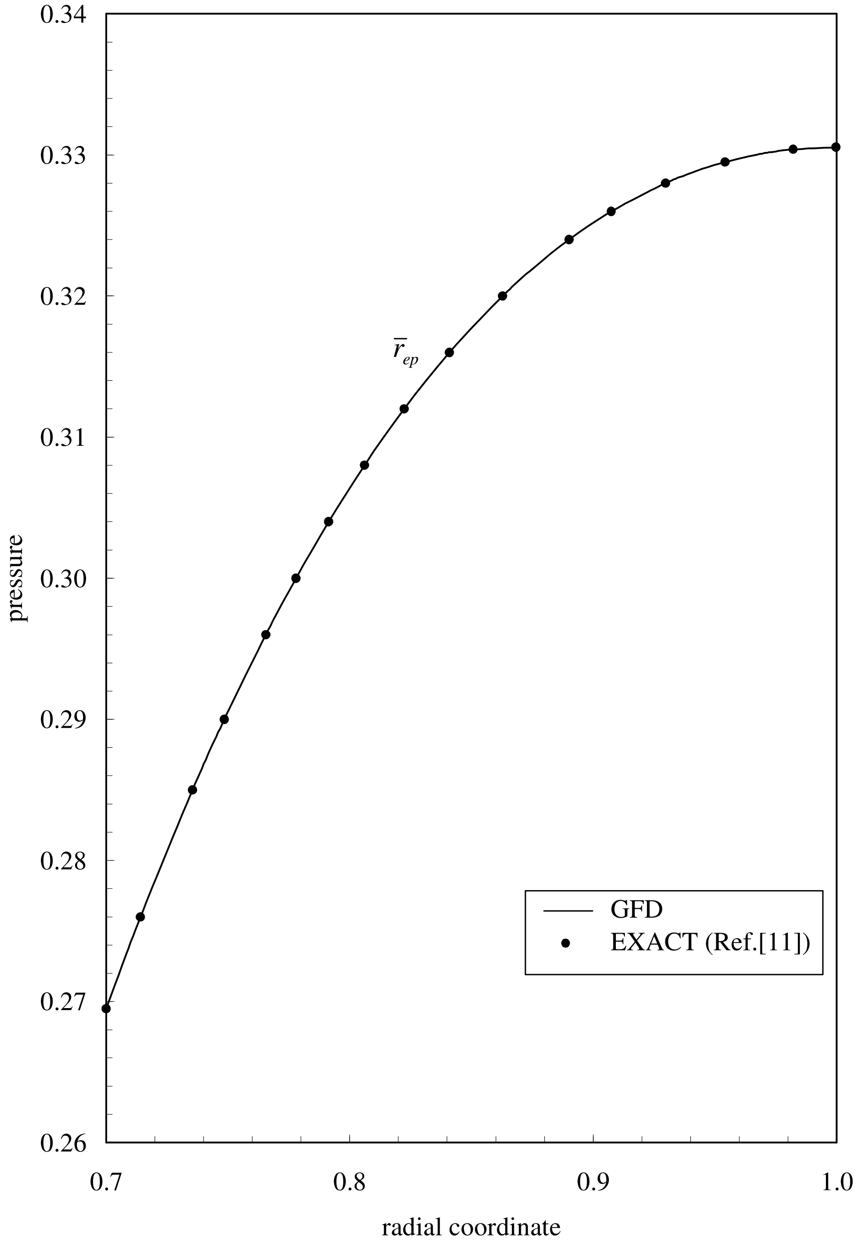


Figure 31: Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.2$, $k = -1.1$, $m = 0.4$, $s = 0.9$) with increasing internal pressures.

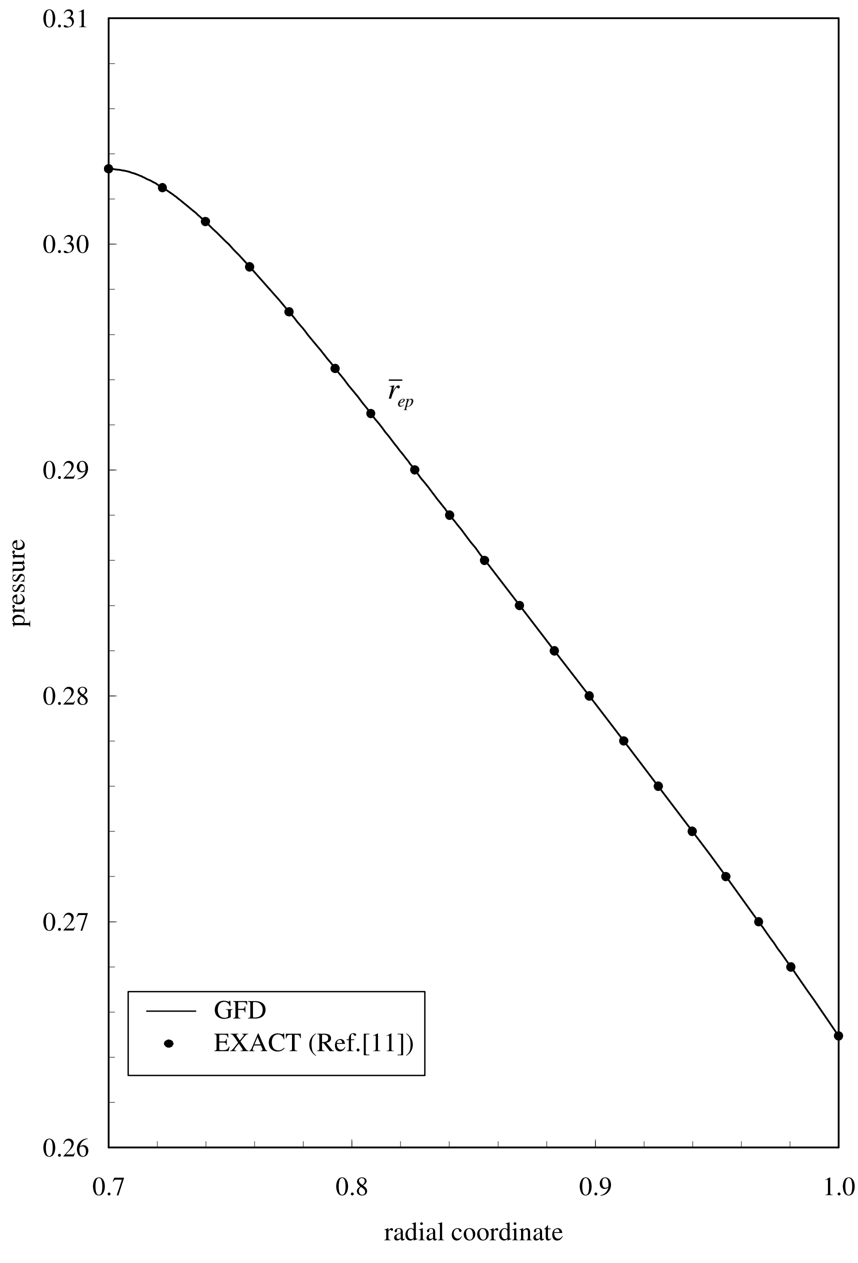


Figure 32: Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.5$, $k = -1.2$, $m = 0.7$, $s = 0.6$) with increasing internal pressures.

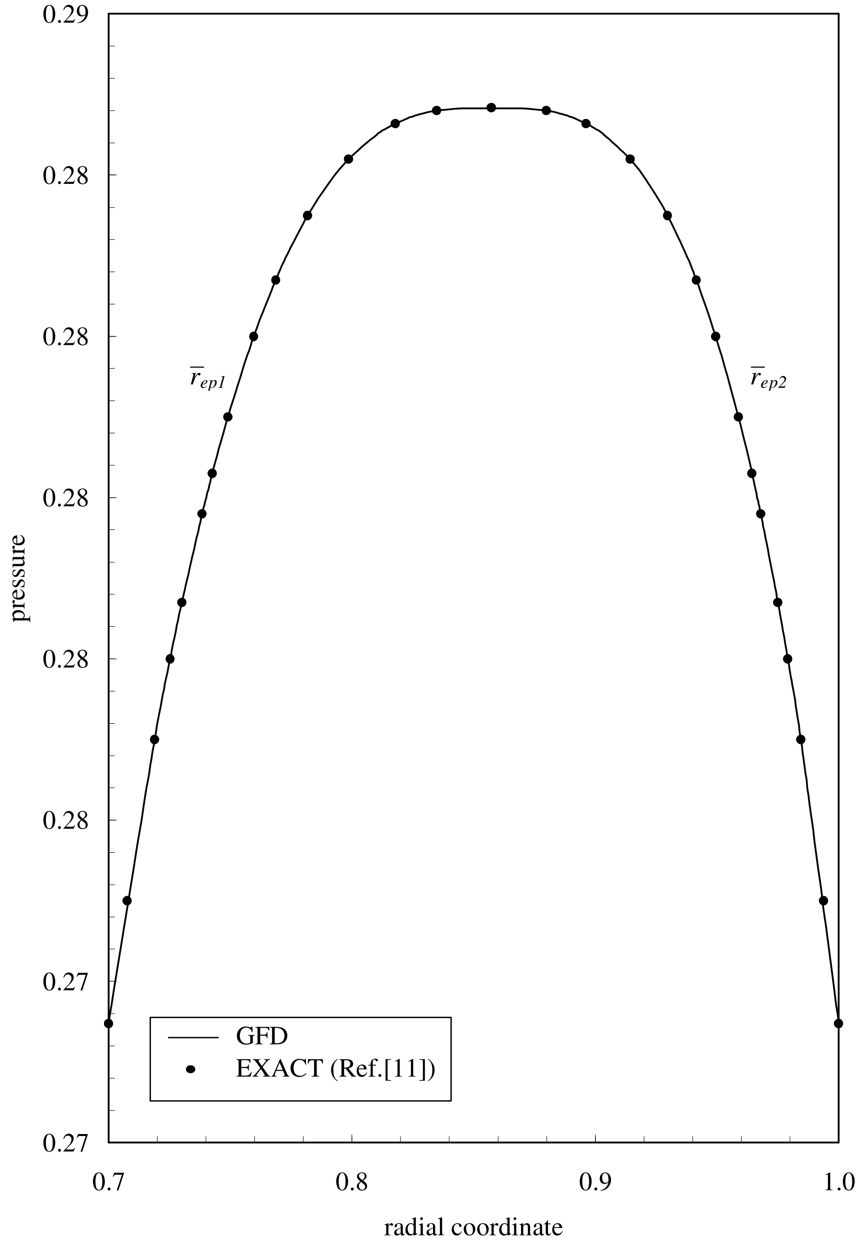


Figure 33: Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) with increasing internal pressures.

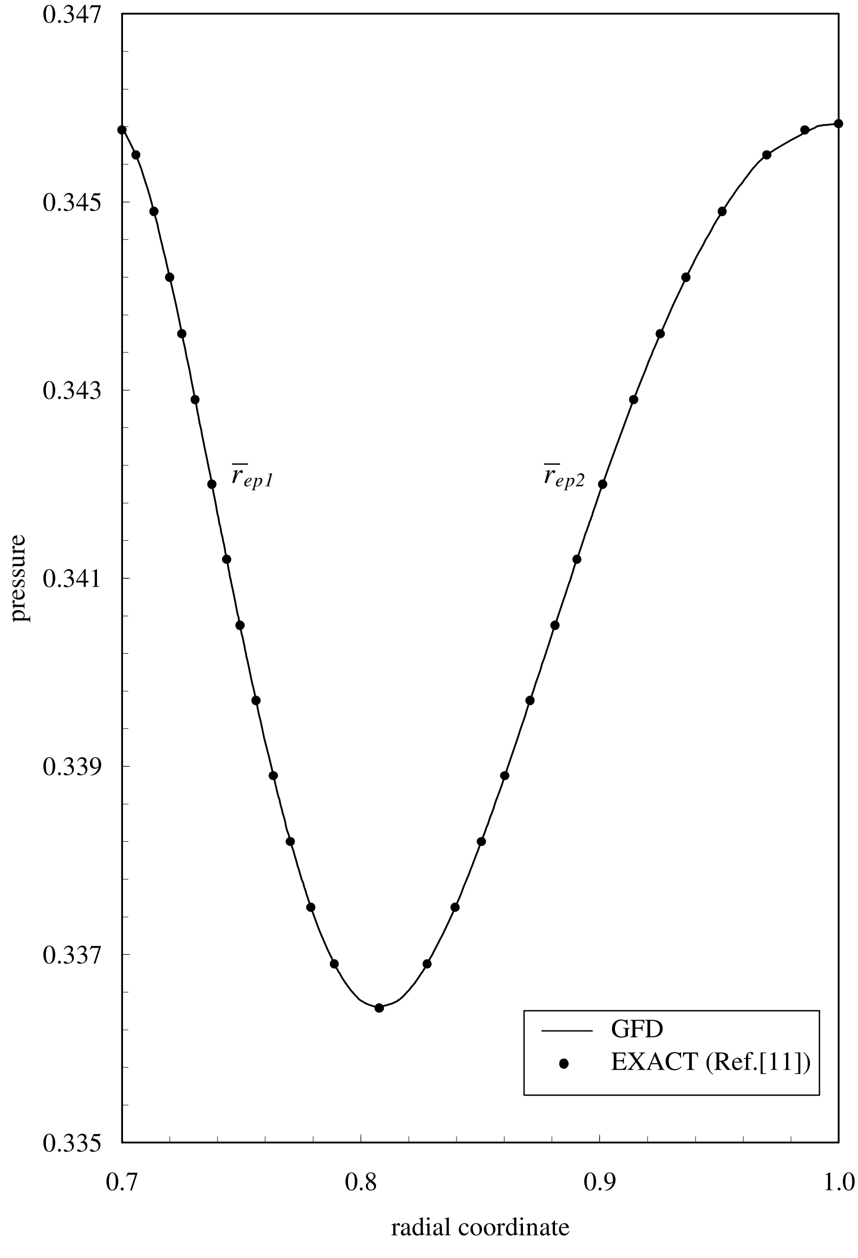


Figure 34: Propagation of elastic-plastic border radius in partially plastic FGM tubes ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) with increasing internal pressures.

Table 18. Error in GFD method ($N = 200$) for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$

weighting function	dm	%error($\overline{\sigma}_r^p$)
	$2h + \delta$	3.01E-04
$1/d$	$3h + \delta$	5.86E-04
	$4h + \delta$	7.43E-04
	$2h + \delta$	3.92E-04
exponential	$3h + \delta$	8.01E-04
	$4h + \delta$	1.87E-03
	$3h + \delta$	3.44E-04
cubic spline	$4h + \delta$	5.67E-04
	$5h + \delta$	3.88E-04

Error for $\overline{\sigma}_r^p$ in GFD method versus increasing number of nodes are shown in Figure 35. In the calculations, for $w = 1/d$ and exponential weighting function, $dm = 2h + \delta$ and for cubic spline weighting function, $dm = 3h + \delta$.

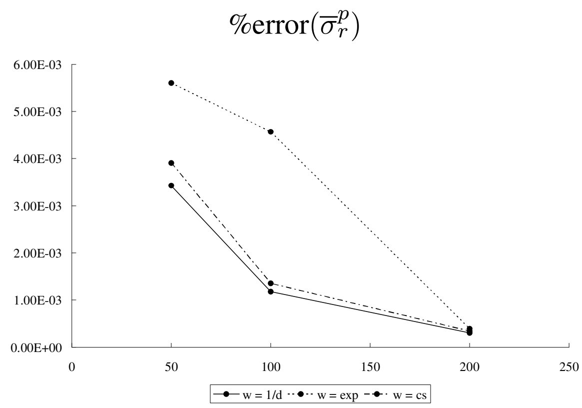


Figure 35. Error in GFD method for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$ versus number of nodes.

Next, errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in plastic regions for different GFD parameters with $N = 200$ when $dm = 2h + \delta$ and $w = 1/d$ in GFD calculations of elastic part and plastic stresses are shown in Table 19.

Table 19. Error in GFD method ($N = 200$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	$2h + \delta$	1.31E-04	7.51E-04
$1/d$	$3h + \delta$	1.43E-04	3.10E-04
	$4h + \delta$	1.65E-04	1.37E-03
	$2h + \delta$	1.38E-04	2.26E-04
exponential	$3h + \delta$	1.59E-04	3.78E-04
	$4h + \delta$	3.11E-04	1.59E-03
	$3h + \delta$	1.35E-04	3.25E-04
cubic spline	$4h + \delta$	1.40E-04	4.17E-04
	$5h + \delta$	1.37E-04	2.75E-04

Finally stresses and displacement for the third case calculated with $dm = 2h+\delta$, $w = 1/d$ and $N = 200$ in GFD formulae are plotted in Figure 36. Plastic strains are presented in Figure 37. In both figures, continuous lines and dots show numerical and analytical results, respectively.

In the fourth case, three regions: elastic-plastic-elastic occur throughout the tube for material parameters $n = 0.4$, $k = -1.6$, $m = 0.2$ and $s = 0.9$. Under $p_{in} = 0.341 > p_e$, elastic-plastic boundaries calculated analytically in Ref.[11] are $\overline{r}_{ep1} = 0.745370$ and $\overline{r}_{ep2} = 0.887791$. Matching GFD solution reads $\overline{r}_{ep1} = 0.745378$ and $\overline{r}_{ep2} = 0.887783$. Analytical results are used in the computations. Firstly error in the method for $\overline{\sigma}_r^p$ with different GFD parameters and $N = 200$ are shown in Table 20. In the calculations for elastic part $w = 1/d$ and $dm = 2h+\delta$.

Table 20. Error in GFD method ($N = 200$) for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$

weighting function	dm	%error($\overline{\sigma}_r^p$)
	$2h + \delta$	2.01E-04
$1/d$	$3h + \delta$	3.75E-04
	$4h + \delta$	5.90E-04
	$2h + \delta$	2.73E-04
exponential	$3h + \delta$	5.62E-04
	$4h + \delta$	9.31E-04
	$3h + \delta$	2.36E-04
cubic spline	$4h + \delta$	3.14E-04
	$5h + \delta$	2.69E-04

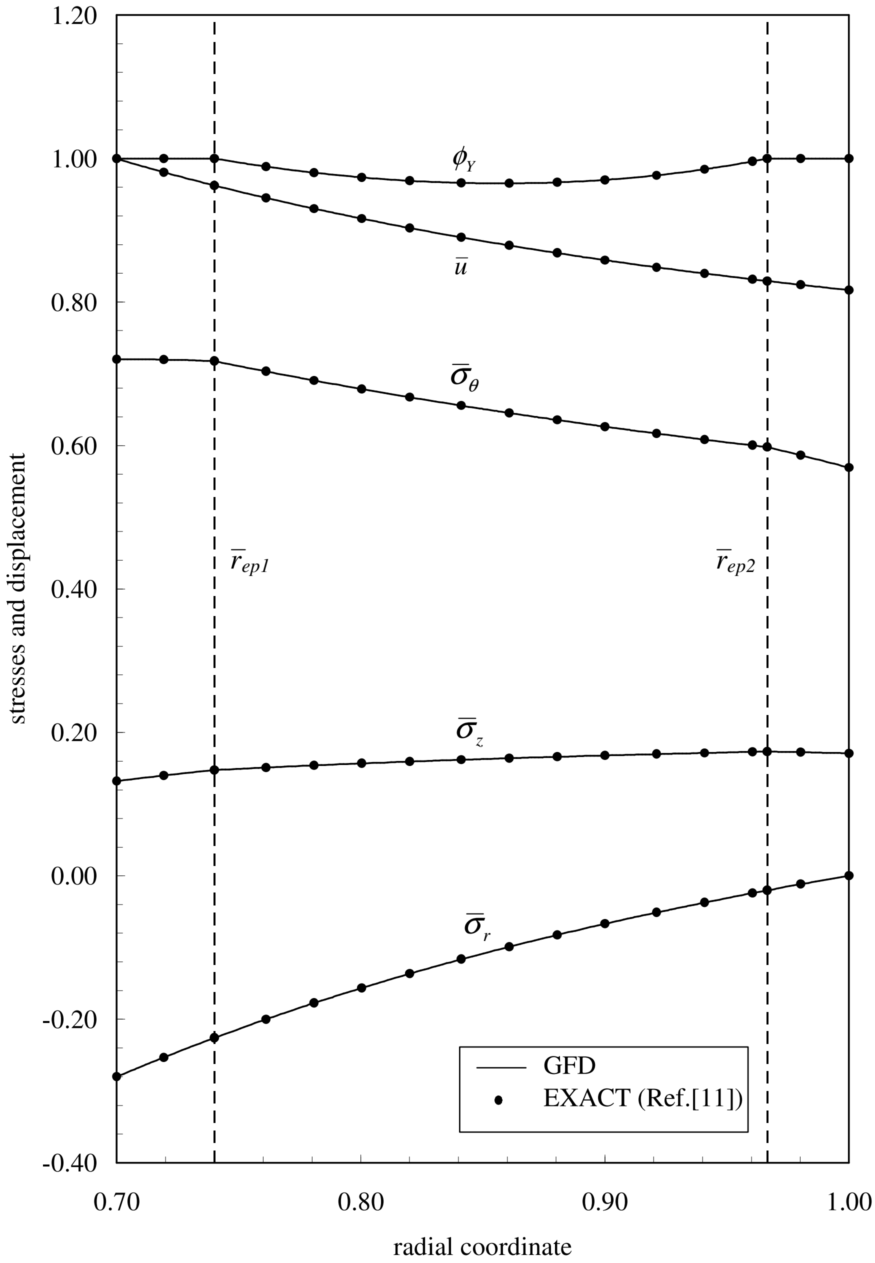


Figure 36: Stresses and displacement in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$.

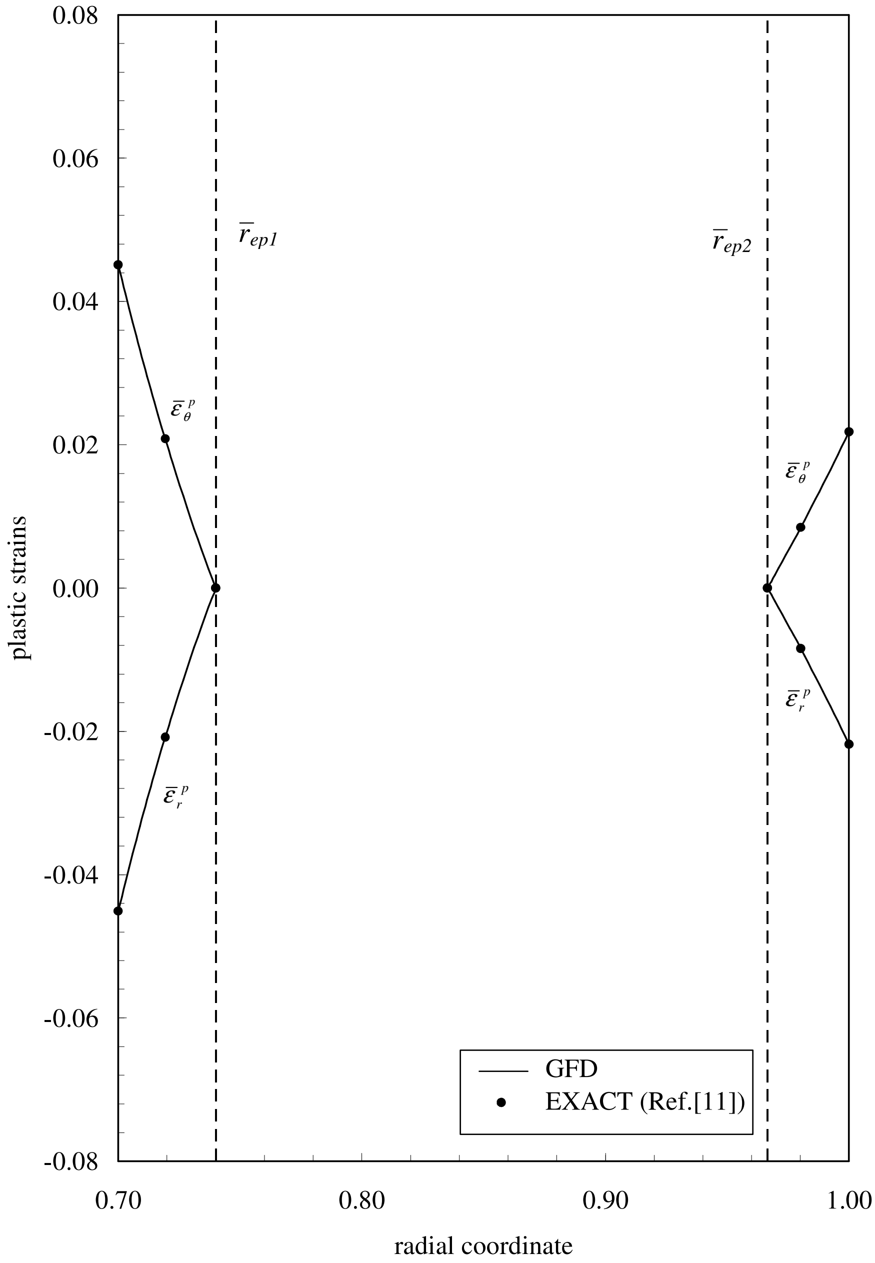


Figure 37: Plastic strains in a partially plastic FGM tube ($n = 0.348459$, $k = -0.7$, $m = 0.7$, $s = 1.1$) under internal pressure $\overline{p}_{in} = 0.28$.

Error in the method for $\overline{\sigma}_r^p$ in plastic region versus increasing number of nodes is shown in Figure 38. In GFD formulae, for $w = 1/d$ and exponential weighting function, $dm = 2h + \delta$, and for cubic spline weighting function $dm = 3h + \delta$.

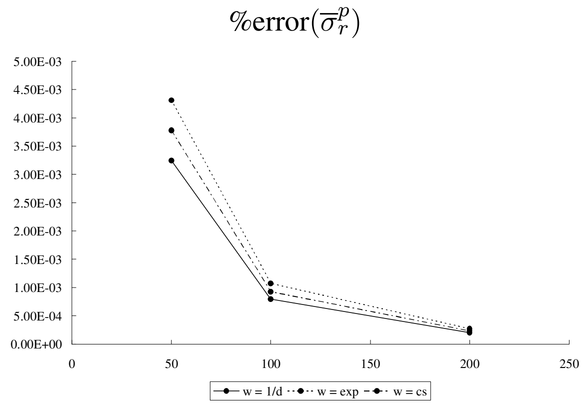


Figure 38. Error in GFD method for $\overline{\sigma}_r^p$ in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$ versus number of nodes

Next errors for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in plastic region for different GFD parameters with $N = 200$ are shown in Table 21. In the calculations $dm = 2h + \delta$ and $w = 1/d$ for elastic region and plastic stresses.

Table 21. Error in GFD method ($N = 200$) for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$

weighting function	dm	%error($\overline{u}$)	%error($\frac{d\overline{u}}{d\overline{r}}$)
	$2h + \delta$	9.50E-04	7.24E-02
$1/d$	$3h + \delta$	7.91E-05	2.52E-04
	$4h + \delta$	1.58E-02	2.62E-01
	$2h + \delta$	5.28E-05	1.59E-04
exponential	$3h + \delta$	1.33E-04	2.64E-04
	$4h + \delta$	2.64E-04	5.71E-04
	$3h + \delta$	4.63E-05	1.93E-04
cubic spline	$4h + \delta$	6.15E-05	2.16E-04
	$5h + \delta$	5.26E-05	2.91E-04

Matching stresses, displacement and plastic strains are plotted in Figures 39 and 40. In the calculations, $dm = 2h + \delta$ and $w = 1/d$ for elastic region and plastic stresses. $dm = 3h + \delta$ and $w = 1/d$ in plastic displacement computations. In the figures, continuous lines and dots show numerical and analytical results, respectively.

3.3 Elastic Response of a Pressurized Functionally Graded Tube with Free Ends

3.3.1 Solution for Graded Modulus of Elasticity and Poisson's Ratio

In this section, elastic response of a long FGM tube with free ends subjected to internal pressure is analyzed by GFD method, and results are compared to the ones obtained by shooting method in Ref.[12]. The modulus of elasticity of the tube material varies radially according to Eq. (26), and Poisson's ratio varies in the radial direction with following relation.

$$\nu(r) = \nu_0 \left(1 + \nu_1 r + \nu_2 r^2\right) \tag{40}$$

where ν_0, ν_1 and ν_2 are material parameters.

In the problem, a state of plane strain (ε_z = constant) is considered. The equation of equilibrium Eq. (27) is valid.

Non-dimensional and normalized values in Eq. (28) are used in the presentation of the results.

3.3.1.1 Elastic Stress State

Stress displacement relations for elastic stress state are expressed in the following forms.

$$\sigma_r = \frac{E(r)}{(1+\nu(r))(1-2\nu(r))}\left[\nu(r)\frac{u}{r} + (1-\nu(r))\frac{du}{dr} + \nu(r)\varepsilon_z\right] \tag{41}$$

$$\sigma_\theta = \frac{E(r)}{(1+\nu(r))(1-2\nu(r))}\left[(1-\nu(r))\frac{u}{r} + \nu(r)\frac{du}{dr} + \nu(r)\varepsilon_z\right] \tag{42}$$

$$\sigma_z = \frac{E(r)}{(1+\nu(r))(1-2\nu(r))}\left[\nu(r)\frac{u}{r} + \nu(r)\frac{du}{dr} + (1-\nu(r))\varepsilon_z\right] \tag{43}$$

Substitution of Eqs. (26) and (40) into Eqs. (41) and (42), and then those into Eq. (27) result in the governing differential equation for radial displacement u.

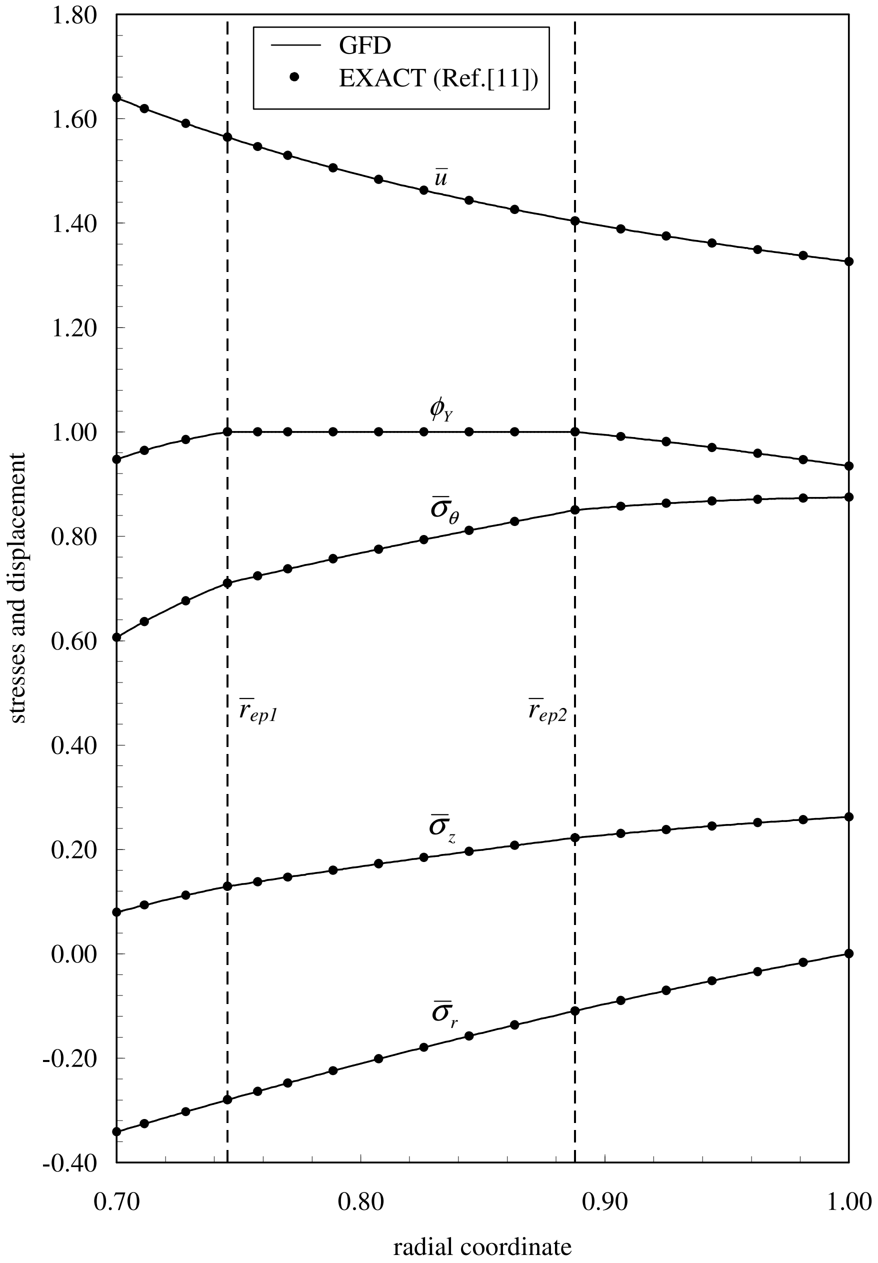


Figure 39: Stresses and displacement in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$.

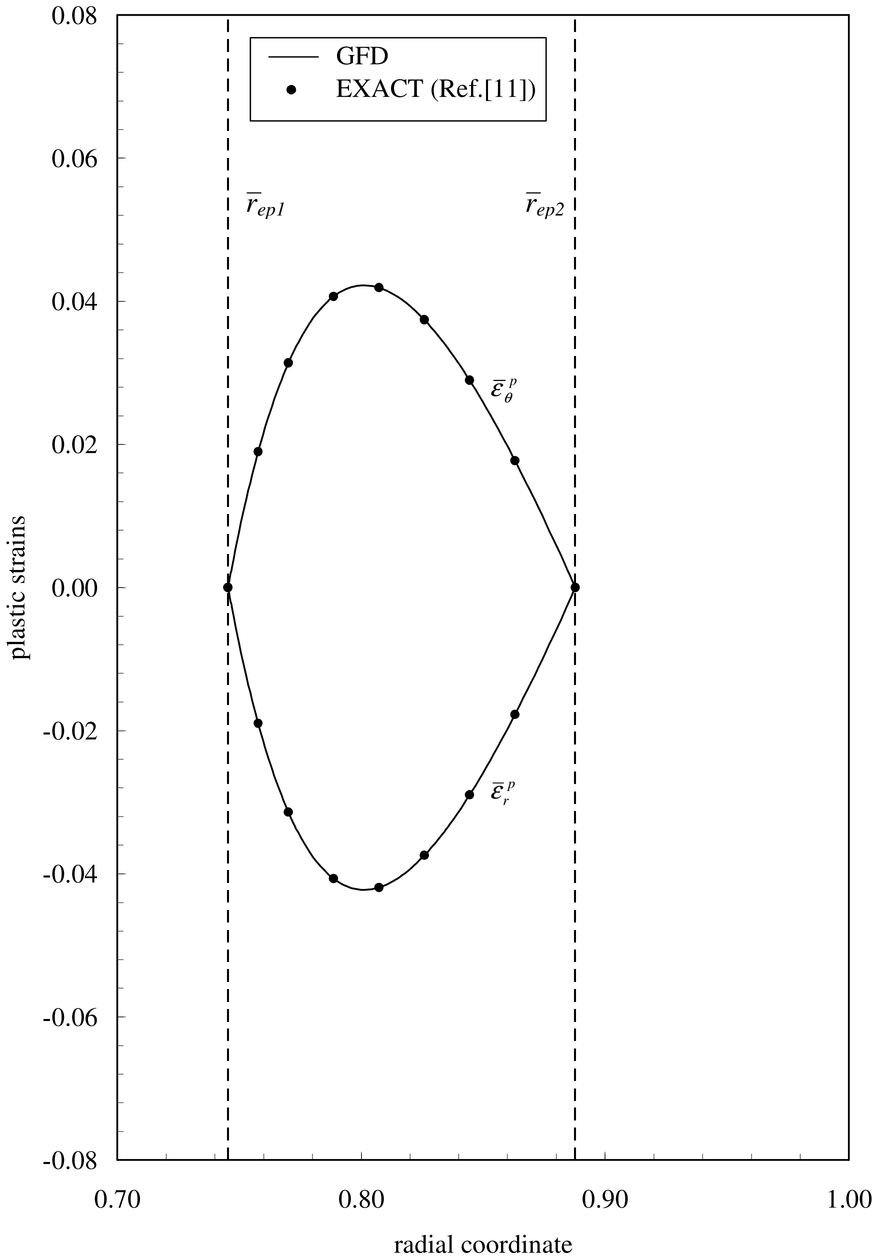


Figure 40: Plastic strains in a partially plastic FGM tube ($n = 0.4$, $k = -1.6$, $m = 0.2$, $s = 0.9$) under internal pressure $\overline{p}_{in} = 0.341$.

$$
\begin{aligned}
&r^2\left[1-n\left(\frac{r}{r_{out}}\right)^k\right]\frac{d^2u}{dr^2}+\\
&\left\{r\left[1-n(1+k)\left(\frac{r}{r_{out}}\right)^k\right]+2r^2\frac{\nu(2-\nu)\frac{d\nu}{dr}}{(1-2\nu)(1-\nu^2)}\left[1-n\left(\frac{r}{r_{out}}\right)^k\right]\right\}\frac{du}{dr}+\\
&\left\{r\frac{(1+2\nu^2)\frac{d\nu}{dr}}{(1-2\nu)(1-\nu^2)}\left[1-n\left(\frac{r}{r_{out}}\right)^k\right]-\frac{1-\nu-n\left[1-\nu(1+k)\right]\left(\frac{r}{r_{out}}\right)^k}{1-\nu}\right\}u\\
&=\left\{rnk\frac{\nu}{1-\nu}\left(\frac{r}{r_{out}}\right)^k-r^2\frac{(1+2\nu^2)\frac{d\nu}{dr}}{(1-2\nu)(1-\nu^2)}\left[1-n\left(\frac{r}{r_{out}}\right)^k\right]\right\}\varepsilon_z. \qquad (44)
\end{aligned}
$$

Since the tube has axially unconstrained ends, axial force $F_z = 0$. F_z is calculated by the following relation.

$$F_z = \int_{r_{in}}^{r_{out}} \sigma_z dA = 2\pi \int_{r_{in}}^{r_{out}} r\sigma_z dr = 0 \qquad (45)$$

GFD formulae are constructed considering 100 nodes in the domain with $dm = 2h + \delta$ and $w = 1/d$, and then derivative coefficient matrices are modified and summed up to form Eq. (44). The right hand side of the equation has term ε_z which is constant and an unknown in the equation. In order to find ε_z iterations are performed by Newton's method, and Eq. (45) is used to reach zero axial force.

Influences of variations of elastic modulus and Poisson's ratio on the elastic response of the tube are investigated. In order to have an increasing distribution of elastic modulus, material parameters $n = -0.4$ and $k = 0.6$ are used. Parameter set: $n = 0.4$, $k = 0.6$ is considered for decreasing distribution. Poisson's ratio parameters $\nu_0 = 0.3$, $\nu_1 = -0.154$ and $\nu_2 = 0.220$ are utilized to observe a distribution in which $\nu(r_{in}) = 0.3$ and $\nu(r_{out}) = 0.3198$. Parameter set $\nu_0 = 0.3$, $\nu_1 = 0.154$, $\nu_2 = -0.220$ results in $\nu(r_{in}) = 0.3$ and $\nu(r_{out}) = 0.2802$, which shows that Poisson's ratio decreases radially through outer surface. $n = 0$ and $\nu_1 = \nu_2 = 0$ are used to obtain constant elastic modulus and Poisson's ratio, respectively.

Figures 41, 42 and 43 show the influence of variation of Poisson's ratio on the elastic response of the tube under internal pressure $\overline{p}_{in} = 0.24$ when elastic modulus decreases, is constant and increases, respectively.

Since it is shown that variation of Poisson's ratio does not have a major effect on stresses and displacement, influence of variation of elastic modulus on the response of the tube is considered together with constant ν. Matching stresses and displacement are plotted in Figure 44.

Results are compared to those of shooting method solution in Ref.[12]. Error in GFD method for $\overline{u}$ and $\frac{d\overline{u}}{d\overline{r}}$ in nine possible cases are shown in Table 22.

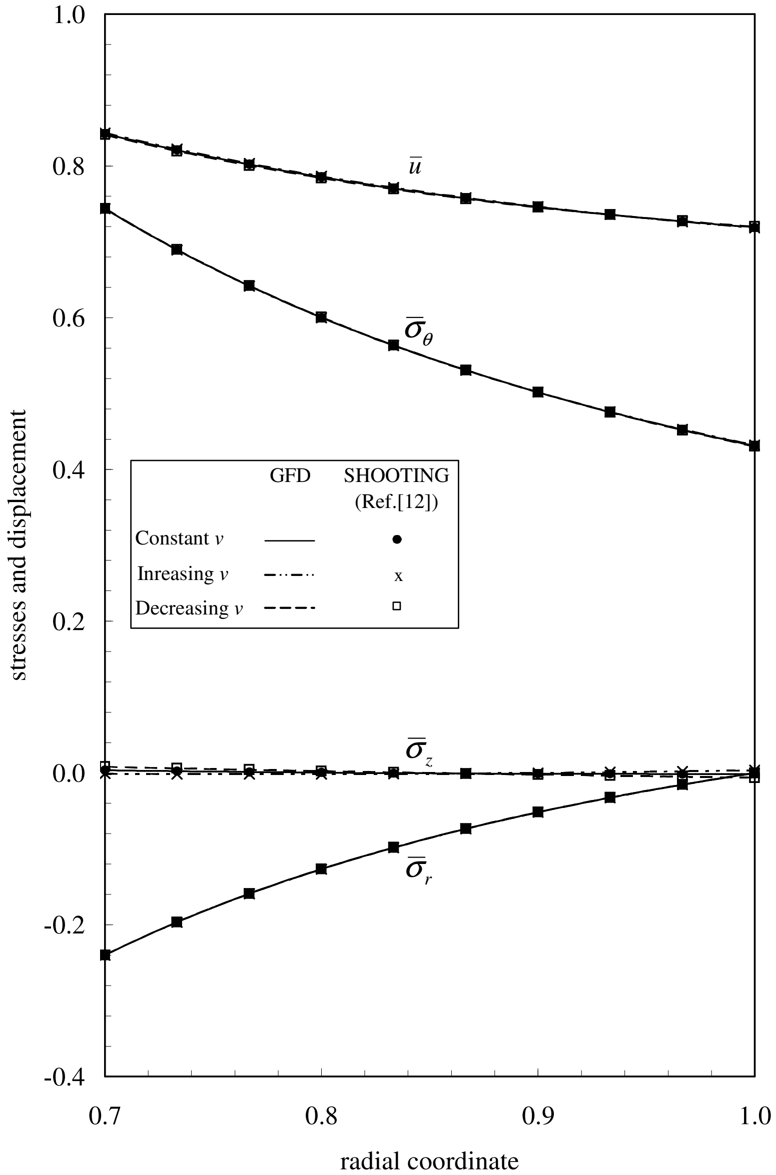


Figure 41: Elastic responses of FGM tubes ($n = 0.4$, $k = 0.6$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of ν as a parameter.

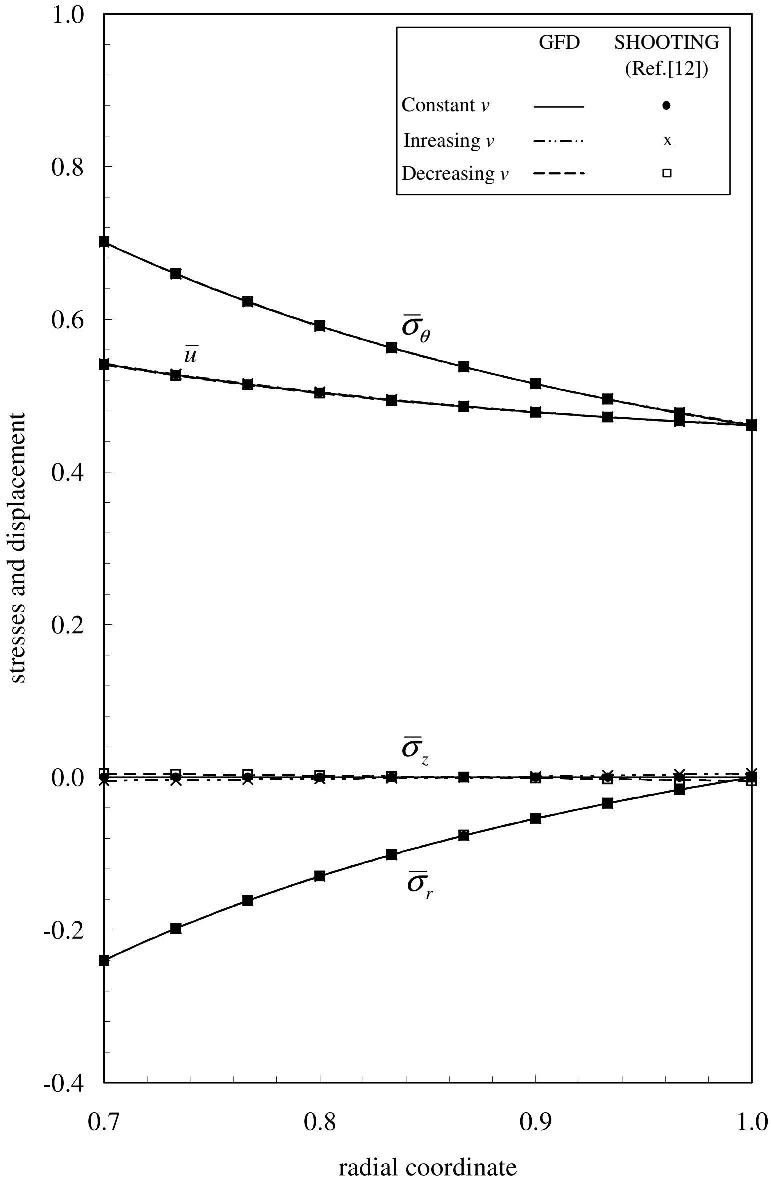


Figure 42: Elastic responses of FGM tubes ($n = 0$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of ν as a parameter.

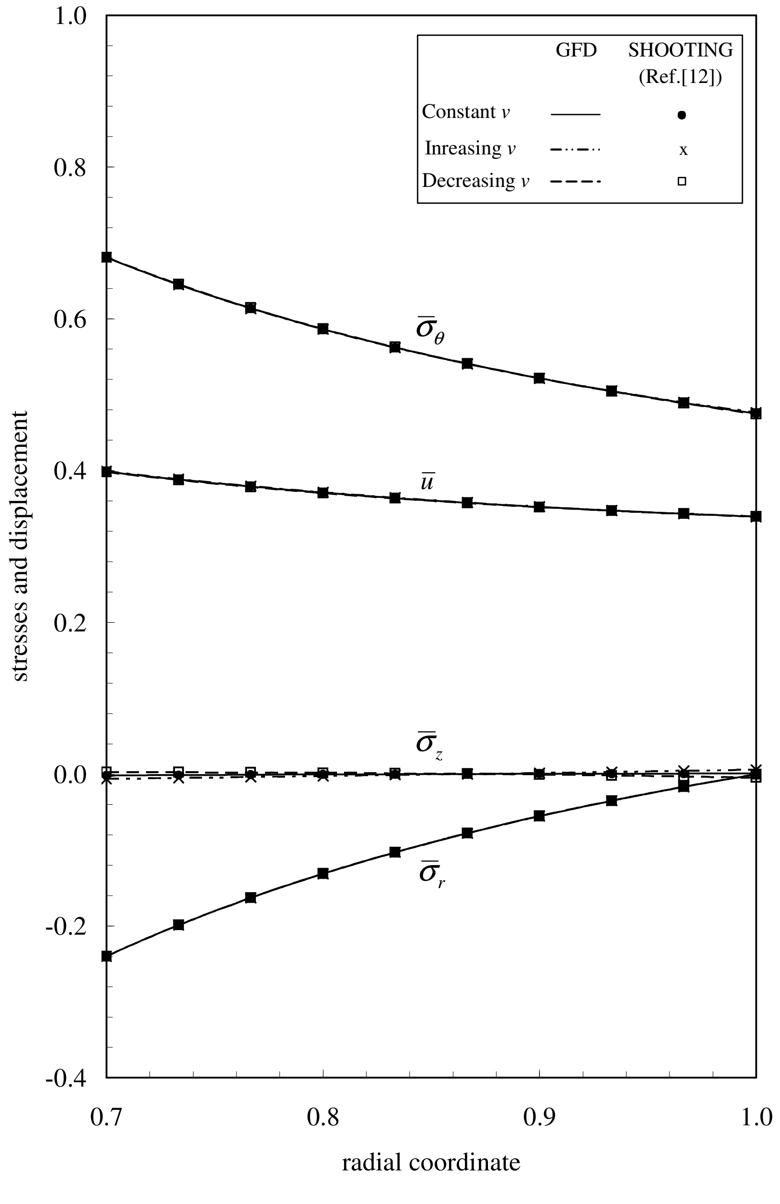


Figure 43: Elastic responses of FGM tubes ($n = -0.4$, $k = 0.6$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of ν as a parameter.

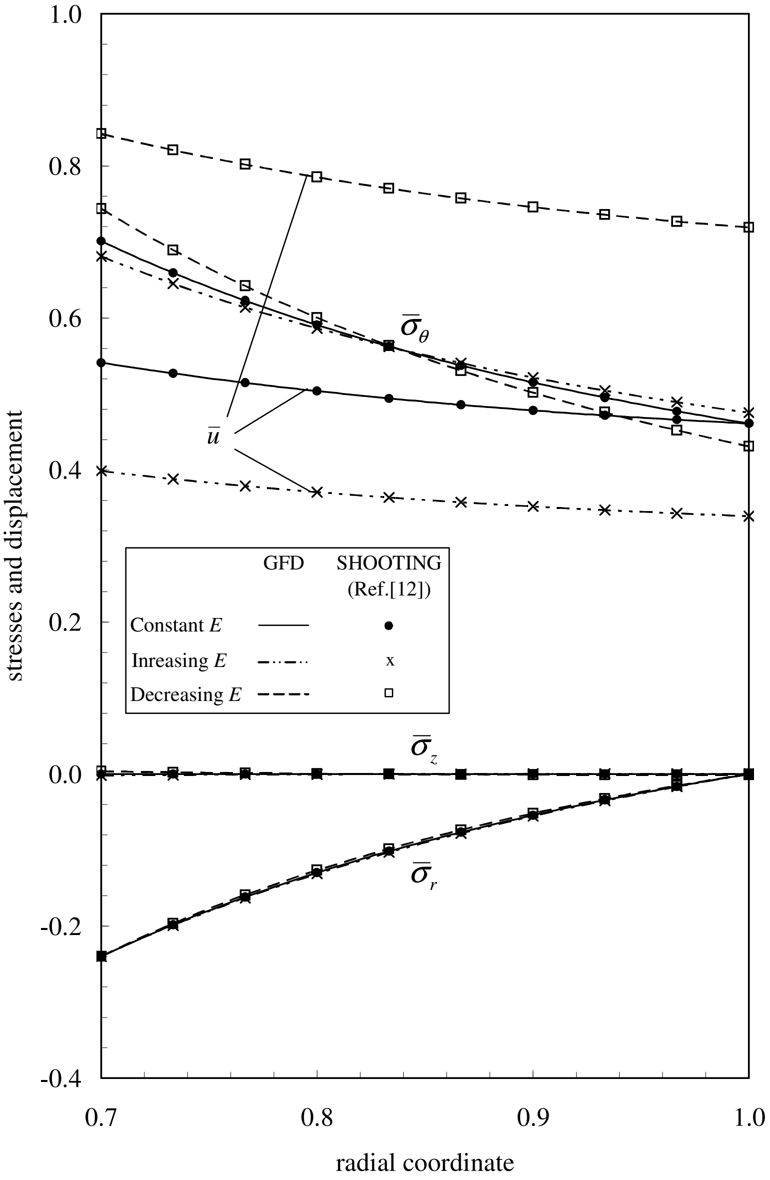


Figure 44: Elastic responses of FGM tubes ($\nu 1 = \nu 2 = 0$) with free ends under internal pressure $\overline{p}_{in} = 0.24$ using variation of E as a parameter.

Table 22. Error in GFD method ($N = 100$) for $\overline{u}$ and $\frac{d\overline{u}}{dr}$ in FGM tubes with free ends under internal pressure $\overline{p}_{in} = 0.24$ in various cases for variation of E and ν

$CASE$		error	
E	ν	%error($\overline{u}$)	%error($\frac{d\overline{u}}{dr}$)
INCREASING	DECREASING	9.13E-04	2.19E-03
INCREASING	CONSTANT	1.17E-03	2.02E-03
INCREASING	INCREASING	1.50E-03	1.82E-03
CONSTANT	DECREASING	6.76E-04	2.27E-03
CONSTANT	CONSTANT	9.23E-04	2.11E-03
CONSTANT	INCREASING	1.23E-03	1.92E-03
DECREASING	DECREASING	2.07E-04	2.44E-03
DECREASING	CONSTANT	4.30E-04	2.29E-03
DECREASING	INCREASING	7.11E-04	2.13E-03

Nondimensional axial strain values calculated by shooting method (Ref.[12]) and GFD method for the cases mentioned in Table 22 are shown in Table 23.

Table 23. Nondimensional axial strains in FGM tubes under internal pressure $\overline{p}_{in} = 0.24$ calculated by shooting method (Ref.[12]) and GFD method in various cases for variation of E and ν

$CASE$		$\overline{\varepsilon}_z$	
E	ν	Shooting Method	GFD Method
INCREASING	DECREASING	-0.098137	-0.098134
INCREASING	CONSTANT	-0.101384	-0.101380
INCREASING	INCREASING	-0.104634	-0.104629
CONSTANT	DECREASING	-0.133977	-0.133973
CONSTANT	CONSTANT	-0.138353	-0.138348
CONSTANT	INCREASING	-0.142733	-0.142727
DECREASING	DECREASING	-0.211053	-0.211048
DECREASING	CONSTANT	-0.217757	-0.217752
DECREASING	INCREASING	-0.224468	-0.224461

CHAPTER 4

CONCLUSIONS

4.1 General

The objective of this study was to evaluate the accuracy of GFD method in one dimensional problems of nonhomogeneous materials. Several problems were solved for different parameters of the method. Potential, exponential and cubic spline weighting functions were utilized in the calculations. Distance of influence was changed between $2h + \delta$ and $4h + \delta$.

First of all, the method was explained in detail through a sample problem: elastic response of a rotating FGM tube. Ten regular nodes were considered in the domain, and each step leading to derivative coefficient matrices of the method was provided. The numerical results were compared to those of analytical solution in Ref.[9].

Elastoplastic response of a long pressurized FGM tube with fixed ends was analyzed for graded modulus of elasticity only and for graded elastic modulus and graded yield stress together. The results were compared to those of analytical solutions obtained from Refs.[10] and [11].

Elastic stress analysis of a pressurized FGM tube with free ends was also performed. Material nonhomogenity was based on variation of elastic modulus and Poisson's ratio. The results obtained were compared to those of shooting method in Ref.[12].

In the light of analyses and comparisons stated above, it was shown that GFD method can give precise results. In elastic deformation calculations and computations for plastic stresses in elastic-plastic deformations, best results were obtained with $w = 1/d$ and $dm = 2h + \delta$. Error in the method increased as dm was increased when $w = 1/d$. Cubic spline weighting function gave more accurate results than exponential weighting function. But both show inconsistent behavior in terms of dm. Error decreased as the number of nodes in the domain was increased as expected. However, in case of elastic-plastic deformations, the discrepancies between numerical and analytical results increased in determining plastic displacements. The predictions for tubes with two graded properties, i. e. the modulus of elasticity and the yield limit, turned out to be better than those with one graded property in this regard.

As stated earlier GFD method is a meshless method which uses a set of arbitrarily scattered nodes within the problem domain. Meshless methods do not require connectivity between grids which is utilized to construct elements for mesh-based methods like Finite Element Method. With a mesh-based method, a problem domain with moving discontinuities or boundaries can be treated by remeshing whenever it is needed in order to keep the mesh edges coincident with changing domain throughout the evolution of the problem. Mesh generation in these cases can be a time-consuming and expansive task. On the other hand, meshless methods have the flexibility in adding or deleting points where necessary. Besides having the advantage of its meshless property, GFD method provides accurate approximations as shown in this thesis and other related studies.

4.2 Recommendations for Future Work

In this study the basics of GFD method were given, and 1D problems with regular domains were solved by the method. The equations herein can simply be extended to multidimensional problems by considering appropriate form of Taylor series expansion.

As shown in Chapter 2, Taylor series expansion up to second derivatives was used in the formulation of the method. It is known as a basic rule that taking more terms in Taylor series expansion provides better approximations. For this reason, in order to obtain more precise results, more terms of Taylor series expansion can be utilized in GFD formulae. It was stated in Ref.[7] that including third order terms in the expansion for the solution of a 2D problem by GFD method reduced the error in the method but this reduction did not justify the additional amount of the calculations required. It must be noted that number of additional terms to the second order Taylor series expansion to get third order approximation is only one for a 1D function whereas it is four for 2D functions. Therefore, the influence of additional terms in Taylor series expansion on the accuracy of GFD method for the approximation of 1D functions can be examined in the future.

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APPENDIX A

SUMMARY OF COMPUTER PROGRAMS

A.1 Subroutine GFD

The program constructs GFD formulae and calculates GFD coefficient matrices for first and second derivatives of any sufficiently differentiable function of one variable. Firstly, coordinate array is formed by given boundary points and number of nodes in the domain. Then for every node being the central node in turn, number of nodes utilized in Eq. (10) is found, and these nodes are stored in a connectivity array. This array is used to construct matrices A_P and B_P in Eqs. (7) and (9). Matrix C_P is then found by Eq. (11). Weighting functions and distance of influence to be used in the calculations are determined by parameters IWF and $NUMDM$. Afterwards, first row of matrix C_p for each node is inserted in coefficient matrix for first derivative K_r. Central node and nodes in the connectivity array determine row and column numbers for insertion, respectively. Similarly second row of matrix C_p is placed in the coefficient matrix for second derivative K_{rr}.

A.2 Subroutine ELASTIC_ROT

The program calculates stresses and displacement in a rotating FGM tube with fixed ends in elastic stress state. Modulus of elasticity of tube material varies radially according to Eq. (12). Firstly radial coordinates are defined, and then coefficient matrices K_r and K_{rr} are obtained from subroutine GFD. These matrices are modified and summed up in the name of K in order to construct left hand side of Eq. (17). Boundary conditions $\sigma_r(r_{in}) = \sigma_r(r_{out}) = 0$ are inserted in the system $K.u = F$, and the system is solved by subroutines $DGBFA$ and $DGBSL$ in $LINPACK$ library (Ref.[13]). Displacement array u is multiplied by K_r and K_{rr} to give first and second derivatives, respectively. Stresses are calculated by Eqs. (14), (15) and (16). Finally stresses and displacements normalized according to Eq. (18) are stored in matrices S and D.

A.3 Subroutine EPLSTC

The program calculates stresses and displacement in a pressurized FGM tube with fixed ends in elastic, plastic and elastoplastic stress states. Elastic modulus and yield stress vary in radial direction with Eqs. (26) and (37), respectively. Variation of elastic modulus and yield stress can be controlled by changing material parameters.

Parameter array IP on entry defines not only stress state of the problem, but also the locations of pressure and displacement boundary conditions in plastic stress state, and whether the plastic region is at the inner or outer parts of the tube in elastoplastic state of stress.

In elastic stress state, matrices K_r and K_{rr} are calculated by subroutine GFD following construction of coordinate array, and Eq. (29) is formed. Then boundary conditions $\sigma_r(r_1) = -P_1$ and $\sigma_r(r_n) = -P_2$ are inserted in the system where P_1 and P_2 are internal and external pressures, respectively. Displacement values are calculated by subroutines $DGBFA$ and $DGBSL$, and then used to obtain first and second derivatives by Eqs. (20) and (21). Also stresses are found according to Eqs. (14), (15), and (16).

In plastic stress state, calculations begin with radial coordinates as before. Then derivative coefficient matrices K_r and K_{rr} are obtained as the output of subroutine GFD. Firstly stresses are to be calculated according to Eq. (39) utilizing K_r. Boundary condition, $\sigma_r(r_1)$ or $\sigma_r(r_N)$ is inserted in the equation. The location of boundary condition used in the calculations is obtained from parameter array IP. The value of radial pressure is also an input parameter. The system is solved by subroutines $DGBFA$ and $DGBSL$. With the knowledge of radial stresses, circumferential and axial stresses are found by relation $\sigma_\theta(r) - \sigma_r(r) = \sigma_Y(r)$ and Eq. (34), respectively. For displacement calculations, Eq. (36) is constructed, and then boundary condition is inserted into the equation. The value of displacement at specified boundary is an input variable. Displacements are found by subroutines $DGBFA$ and $DGBSL$. Multiplication of displacement array u by K_r and K_{rr} gives first and second derivatives of function $u(r)$, respectively. Plastic strains, ε_r^p and ε_θ^p are calculated by the help of Eqs. (31), (32) and (35). Stress variable ϕ_Y is also found by relation $\phi_Y(r) = [\sigma_\theta(r) - \sigma_r(r)]/\sigma_Y(r)$. Displacement calculations in plastic stress state can be skipped by putting appropriate parameters in array IP.

In elastoplastic stress state, with a given interface point and total number of nodes, firstly number of nodes in elastic and plastic parts are found by subroutine MOVINGGRIDS written by Ahmet Eraslan. Number of nodes in each part is proportional to the distances between interface point and corresponding boundary points. The criteria herein is to set grid distance in elastic and plastic regions as close as possible. Whether the plasticity starts at the inner or outer surfaces of the tube can be given as an input parameter in the program within array IP. The calculations start dealing with the plastic region. After radial coordinates are set, radial stresses are found by the help of Eq. (39) and boundary condition. For instance if plasticity commences at the inner surface, boundary condition is $\sigma_r(r_1) = -P_1$, otherwise it is $\sigma_r(r_N) = -P_2$. Circumferential and axial stresses are calculated by the same way in plastic stress state. Elastic calculations begin at this point where radial stress at interface point $\sigma_r(r_{ep})$ is known and can be used as boundary condition. Eq. (29) is constructed and solved to find displacements in elastic part, and then those values are substituted in Eqs. (14), (15) and (16) to obtain stresses. Last step includes the calculations for plastic displacement and strains. $u(r_{ep})$ found in elastic calculations takes the part of boundary condition and Eq. (36) is used to calculate plastic displacements. Plastic strains are found by Eqs. (31), (32) and (35).

For all stress states, the results are normalized according to Eq. (38) and stored in matrices S and D. Matrix S is of length N by 4 . Radial, circumferential and axial stresses are stored in first three columns. The last column is for stress variable ϕ_Y. Matrix D is of length N by 5 and contains displacement, first and second derivative values in first three columns and plastic strains in last two columns.

A.4 Subroutine ELASTIC_FREE

The program calculates stresses and displacement in an FGM tube with free ends subjected to internal pressure in elastic stress state. The procedure is quite similar to previous subroutines. Radial coordinates are placed in an array, and with matrices K_r and K_{rr} they are used to form Eq. (44). Boundary conditions are $\sigma_r(r_{in}) = -p_{in}$ and $\sigma_r(r_{out}) = 0$. After the insertion of boundary conditions, system is solved by subroutines $DGBFA$ and $DGBSL$. Displacement values are multiplied with K_r and K_{rr} to give first and second derivative values, respectively. Then Eqs. (41), (42) and (43) are utilized to compute stresses. Finally results are normalized according to Eq. (28), and stored in matrices S and D. The main difference of the program is such that generalized plane strain (ε_z = constant) states axial force, $F_z = 0$. Recall Eq. (45)

$$F_z = \int_{r_{in}}^{r_{out}} \sigma_z dA = 2\pi \int_{r_{in}}^{r_{out}} r\sigma_z dr = 0$$

The integral herein is calculated by utilizing subroutine DAVINT written by Ahmet Eraslan**,** and resulting value of F_z is used in iterations for axial strain ε_z.

APPENDIX B

THE CODES OF COMPUTER PROGRAMS

B.1 Subroutine GFD

```
C     ==============================================
      SUBROUTINE GFD (X1, XN, N, IWF, NUMDM, D1M, D2M)
C     ==============================================
      IMPLICIT NONE
      INTEGER N, NUMDM, IWF
      DOUBLE PRECISION X1, XN, D1M(N,N), D2M(N,N)
C
C     GFD COMPUTES COEFFICIENT MATRICES FOR FIRST AND SECOND
C     DERIVATIVES BY GENERALIZED FINITE DIFFERENCE METHOD.
C
C     ON ENTRY
C
C     X1          DOUBLE PRECISION
C                 THE START POINT
C
C     XN          DOUBLE PRECISION
C                 THE END POINT
C
C     N           INTEGER
C                 THE NUMBER OF NODES
C
C     IWF         INTEGER
C                 PARAMETER WHICH DETERMINES WEIGHTING FUNCTION
C
C                       1 FOR WF = 1.0 / DABS(DLTXJ)
C
C                       2 FOR WF = DEXP(-10.0*DABS(DLTXJ))
C
C                       3 FOR
C                       IF DABS(DLTXJ) ≤ DM/2 THEN
C                       WF = 2.0D0 / 3.0D0 - 4.0D0 * (DABS(DLTXJ)/DM) ** 2 -
C                       4.0D0 * (DABS(DLTXJ)/DM) ** 3
C                       ELSE
C                       WF = 4.0D0 / 3.0D0 - 4.0D0 * (DABS(DLTXJ)/DM) +
C                       4.0D0 * (DABS(DLTXJ)/DM) ** 2 - 4.0D0 / 3.0D0 *
C                       (DABS(DLTXJ)/DM) ** 3
C
C                       WHERE DLTXJ IS THE DIFFERENCE BETWEEN
C                       CENTRAL NODE IN THE EQUATION AND ANY
```

```
C                                 OTHER NODE WITHIN THE DISTANCE OF
C                                 INFLUENCE DM.
C
C       NUMDM           INTEGER
C                       PARAMETER OF DISTANCE OF INFLUENCE
C
C       ON RETURN
C
C       D1M             DOUBLE PRECISION ARRAY OF LENGTH N BY N
C                       CARRYING THE COEFFICIENT MATRIX FOR THE FIRST
C                       DERIVATIVE  (Kr IN THE TEXT)
C
C       D2M             DOUBLE PRECISION ARRAY OF LENGTH N BY N
C                       CARRYING THE COEFFICIENT MATRIX FOR THE SECOND
C                       DERIVATIVE (Krr IN THE TEXT)
C
C
C       SUBROUTINES AND FUNCTIONS CALLED
C
C       LINPACK ROUTINES DGBFA AND DGBSL
C
C       INTERNAL VARIABLES
C
        INTEGER I, J, K ,IFAC, NUMNODE, CENTNODE
        INTEGER LDA, NN, INFO, JOB, IPVT(2)
        DOUBLE PRECISION DLTX, DLTXJ, DM, WF
        DOUBLE PRECISION X(N), A(2,2), I1(2), I2(2)
        INTEGER, ALLOCATABLE :: CON(:)
        DOUBLE PRECISION, ALLOCATABLE :: B(:,:), C(:,:)
C
C       COORDINATES
C
        DLTX = (XN - X1) / (N - 1)
        X(1) = X1
        DO 10 I = 2, N
                X(I) = X(I-1) + DLTX
  10    CONTINUE
C
C       COEFFICIENT MATRICES
C
        D1M = 0.0D0
        D2M = 0.0D0
        DO 20 J = 1, N
C
C               NUMBER OF NODES IN EACH EQUATION
C
                DM = NUMDM * DLTX + 1.0D-10
                NUMNODE = 0
                DO 30 I = 1, N
                        DLTXJ = X(I) - X(J)
                        IF (DABS(DLTXJ) .LE. DM) THEN
                        NUMNODE = NUMNODE + 1
```

```
                    END IF
   30          CONTINUE
C
C              CONNECTIVITY VECTOR
C
               ALLOCATE (CON(NUMNODE), B(2,NUMNODE), C(2,NUMNODE))
               IFAC = 1
               DO 40 I = 1, N
                    DLTXJ = X(I) - X(J)
                    IF (DABS(DLTXJ) .LE. DM) THEN
                    CON(IFAC) = I
                    IF (I.EQ.J) CENTNODE = IFAC
                    IFAC = IFAC + 1
                    END IF
   40          CONTINUE
C
C              PRIMARY MATRICES A,B,C
C
               DO 50 I = 1, NUMNODE
C
C                   WEIGHTING FUNCTION
C
                    DLTXJ = X(CON(I)) - X(J)
                    IF (DLTXJ .EQ. 0.0D0) THEN
                    WF = 0.0D0
                    ELSE
                    IF (IWF .EQ. 1) THEN
                    WF = 1.0D0 / DABS(DLTXJ)
                    END IF
                    IF (IWF .EQ. 2) THEN
                    WF = DEXP(-10.0D0*DABS(DLTXJ))
                    END IF
                    IF (IWF .EQ. 3) THEN
                    IF (DABS(DLTXJ) .LE. DM/2) THEN
                    WF = 2.0D0 / 3.0D0 - 4.0D0 * (DABS(DLTXJ)/DM) ** 2 -
    *               4.0D0 * (DABS(DLTXJ)/DM) ** 3
                    ELSE
                    WF = 4.0D0 / 3.0D0 - 4.0D0 * (DABS(DLTXJ)/DM) +
    *               4.0D0 * (DABS(DLTXJ)/DM) ** 2 -
    *               4.0D0 / 3.0D0 * (DABS(DLTXJ)/DM) ** 3
                    ENDIF
                    ENDIF
                    ENDIF
C
C                   MATRIX A
C
                    A(1,1) = A(1,1) + (WF**2) * (DLTXJ**2)
                    A(2,2) = A(2,2) + (WF**2) * (DLTXJ**4) / 4
                    A(1,2) = A(1,2) + (WF**2) * (DLTXJ**3) / 2
                    A(2,1) = A(1,2)
C
C                   MATRIX B PART 1 (ALL ELEMENTS BUT COLUMN J)
```

```
C
                  B(1,I) = (WF**2) * DLTXJ
                  B(2,I) = (WF**2) * (DLTXJ**2) / 2
  50          CONTINUE
C
C             MATRIX B PART 2 (COLUMN J)
C
              DO 60 I = 1, NUMNODE
                  IF (I .NE. CENTNODE) THEN
                  B(1,CENTNODE) = B(1,CENTNODE) - B(1,I)
                  B(2,CENTNODE) = B(2,CENTNODE) - B(2,I)
                  ENDIF
  60          CONTINUE
C
C             INVERSE OF MATRIX A
C
              I1 = (/1.0D0,0.0D0/)
              I2 = (/0.0D0,1.0D0/)
              LDA = 2
              NN = 2
              JOB = 0
              CALL DGEFA (A, LDA, NN, IPVT, INFO)
              CALL DGESL (A, LDA, NN, IPVT, I1, JOB)
              CALL DGESL (A, LDA, NN, IPVT, I2, JOB)
              A(1,1) = I1(1)
              A(1,2) = I2(1)
              A(2,1) = I1(2)
              A(2,2) = I2(2)
C
C             MATRIX C
C
              C = MATMUL(A,B)
C
C             MATRICES D1M AND D2M
C
              DO 70 I=1,NUMNODE
                  D1M(J,CON(I)) = C(1,I)
                  D2M(J,CON(I)) = C(2,I)
  70          CONTINUE
              A = 0.0D0
              B = 0.0D0
              DEALLOCATE (CON, B, C)
  20      CONTINUE
          RETURN
          END
```

B.2 Subroutine ELASTIC_ROT

```
C   ========================================
    SUBROUTINE ELASTIC_ROT (N, ANGV, R, S, D)
C   ========================================
        IMPLICIT NONE
        INTEGER N
        DOUBLE PRECISION ANGV
        DOUBLE PRECISION R(N), S(N,4), D(N,3)
C
C       SUBROUTINE ELASTIC_ROT CALCULATES STRESSES AND
C       DISPLACEMENT IN A ROTATING FUNCTIONALLY GRADED TUBE IN
C       PURELY ELASTIC STRESS STATE USING GFD METHOD. MODULUS OF
C       ELASTICITY VARIES RADIALLY ACCORDING TO FOLLOWING FORM
C
C       E(R) = E0 * (R/ROUT) ** NN.
C
C       ON ENTRY
C
C       N               INTEGER
C                       THE NUMBER OF NODES
C
C       ANGV            DOUBLE PRECISION
C                       NONDIMENSIONAL ANGULAR VELOCITY
C
C
C       ON RETURN
C
C       R               DOUBLE PRECISION ARRAY OF LENGTH N CARRYING
C                       RADIAL COORDINATES
C
C       S               DOUBLE PRECISION ARRAY OF LENGTH N BY 4
C                       CARRYING STRESS VALUES
C
C                       FIRST COLUMN
C                       NONDIMENSIONAL RADIAL STRESS VALUES (SR)
C
C                       SECOND COLUMN
C                       NONDIMENSIONAL CIRCUMFERENTIAL STRESS VALUES
C                       (ST)
C
C                       THIRD COLUMN
C                       NONDIMENSIONAL AXIAL STRESS VALUES (SZ)
C
C                       FOURTH COLUMN
C                       NONDIMENSIONAL STRESS VARIABLE CALCULATED
C                       ACCORDING TO THE FORM
C                       SV(R) = (ST(R) - SR(R)) / S0
C                       WHERE S0 IS THE YIELD STRESS
C
C       D               DOUBLE PRECISION ARRAY OF LENGTH N BY 3
```

```
C                       CARRYING DISPLACEMENT VALUES
C
C                       FIRST COLUMN
C                       NONDIMENSIONAL DISPLACEMENT VALUES (U)
C
C                       SECOND COLUMN
C                       NONDIMENSIONAL FIRST DERIVATIVE VALUES OF
C                       DISPLACEMENT FUNCTION U. (D1U) (Dr IN THE TEXT)
C
C                       THIRD COLUMN
C                       NONDIMENSIONAL SECOND DERIVATIVE VALUES OF
C                       DISPLACEMENT FUNCTION U. (D2U) (Drr IN THE TEXT)
C
C
C       SUBROUTINES AND FUNCTIONS CALLED
C
C       GFD, IDNT, LINPACK ROUTINES DGBFA AND DGBSL
C
C
C       INTERNAL VARIABLES
C
        INTEGER I, I1, I2, ID(N,N), IPVT(N), INFO, IWF
        INTEGER J, JOB, K, LDA, M, ML, MU, NUMDM
        DOUBLE PRECISION DLTR, RIN, ROUT
        DOUBLE PRECISION E0, S0, NN, NU, RHO
        DOUBLE PRECISION D1M(N,N), D2M(N,N), DM(N,N)
        DOUBLE PRECISION F(N), SR(N), ST(N), SZ(N), SV(N)
        DOUBLE PRECISION U(N), D1U(N), D2U(N)
        DOUBLE PRECISION, ALLOCATABLE :: DMBD(:,:)
        COMMON / GEO / RIN, ROUT
        COMMON / MAT / E0, S0, NN, NU, RHO
C
C       DIMENSIONAL ANGULAR VELOCITY
C
        ANGV = ANGV * DSQRT(S0/RHO) / ROUT
C
C       RADIAL COORDINATES
C
        DLTR = (ROUT-RIN) / (N-1)
        R(1) = RIN
        DO 10 I = 2, N
            R(I) = R(I-1) + DLTR
  10    CONTINUE
C
C       GFD COEFFICIENT MATRICES
C
        IWF = 1
        NUMDM = 2
        CALL GFD (R(1), R(N), N, IWF, NUMDM, D1M, D2M)
C
C       GOVERNING DIFFERENTIAL EQUATION
C
```

```
C       LEFT HAND SIDE
C
        CALL IDNT(N,ID)
        DO 20 I = 1, N
                DO 30 J = 1, N
                DM(I,J) = (R(I)**2) * D2M(I,J) + R(I) * (1.0D0+NN) *
     *          D1M(I,J) - ((1.0D0 - NU * (1.0D0+NN)) / (1.0D0-NU)) * ID(I,J)
  30            CONTINUE
  20    CONTINUE
C
C       RIGHT HAND SIDE
C
        DO 40 I = 1, N
                F(I) = - (ROUT**NN) * (R(I)**(3.0D0-NN)) * (1.0D0+NU) *
     *          (1.0D0-2*NU) * RHO * (ANGV**2) / (E0 * (1.0D0-NU))
  40    CONTINUE
C
C       BOUNDARY CONDITIONS
C
        DO 50 I = 1, N
                DM(1,I) = E0 * ((R(1)/ROUT) ** NN) * (1.0D0-NU)/
     *          ((1.0D0+NU) * (1.0D0-2*NU)) * D1M(1,I)
                DM(N,I) = E0 * ((R(N)/ROUT) ** NN) * (1.0D0-NU)/
     *          ((1.0D0+NU) * (1.0D0-2*NU)) * D1M(N,I)
  50    CONTINUE
        DM(1,1) = DM(1,1) + (E0 * ((R(1)/ROUT) ** NN) * NU
     *  / (R(1) * (1.0D0+NU) * (1.0D0-2*NU)))
        DM(N,N) = DM(N,N) + (E0 * ((R(N)/ROUT) ** NN) * NU
     *  /( R(N) * (1.0D0+NU) * (1.0D0-2*NU)))
        F(1) = 0.0D0
        F(N) = 0.0D0
C
C       DISPLACEMENT VALUES
C
        JOB = 0
        ML = NUMDM
        MU = NUMDM
        LDA = 2 * ML + MU + 1
        ALLOCATE (DMBD(LDA,N))
        M = ML + MU + 1
        DO 60 J = 1, N
                I1 = MAX0(1, J-MU)
                I2 = MIN0(N, J+ML)
                DO 70 I = I1, I2
                        K = I - J + M
                        DMBD(K,J) = DM(I,J)
  70            CONTINUE
  60    CONTINUE
        CALL DGBFA (DMBD, LDA, N, ML, MU, IPVT, INFO)
        CALL DGBSL (DMBD, LDA, N, ML, MU, IPVT, F, JOB)
        U = F
```

```
C
C       FIRST AND SECOND DERIVATIVES
C
        D1U = MATMUL(D1M,U)
        D2U = MATMUL(D2M,U)
C
C       STRESSES
C
        DO 80 I = 1, N
              SR(I) = (E0 * ((R(I)/ROUT) ** NN) / ((1.0D0+NU) *
   *          (1.0D0-2*NU))) * (NU * (U(I)/R(I)) + (1.0D0-NU) * D1U(I))
              ST(I) = (E0 * ((R(I)/ROUT) ** NN) / ((1.0D0+NU) *
   *          (1.0D0-2*NU))) * ((1.0D0-NU) * (U(I)/R(I)) + NU * D1U(I))
              SZ(I) = (E0 * ((R(I)/ROUT) ** NN) / ((1.0D0+NU)  *
   *          (1.0D0-2*NU))) * (NU * (U(I)/R(I)) + NU* D1U(I))
  80    CONTINUE
C
C       NONDIMENSIONAL VALUES
C
        DO 90 I=1,N
              U(I) = U(I) * E0 / (ROUT*S0)
              D1U(I) = D1U(I) * E0 / (ROUT*S0)
              D2U(I) = D2U(I) * E0 / (ROUT*S0)
              SR(I) = SR(I) / S0
              ST(I) = ST(I) / S0
              SZ(I) = SZ(I) / S0
              SV(I) = ST(I) - SR(I)
  90    CONTINUE
C
C       ARRAYS S AND D
C
        DO 100 I = 1, N
              S(I,1) = SR(I)
              S(I,2) = ST(I)
              S(I,3) = SZ(I)
              S(I,4) = SV(I)
              D(I,1) = U(I)
              D(I,2) = D1U(I)
              D(I,3) = D2U(I)
 100    CONTINUE
C
C       NONDIMENSIONAL ANGULAR VELOCITY
C
        ANGV = ANGV * ROUT * DSQRT(RHO/S0)
        RETURN
        END
```

B.3 Subroutine EPLSTC

```
C   ===============================================
    SUBROUTINE EPLSTC(N,IP,P1,P2,R1,R2,REP,IUP,R,S,D,ERR)
C   ===============================================
      IMPLICIT NONE
      INTEGER N, IP(3)
      DOUBLE PRECISION P1, P2, R1, R2, REP, IUP, ERR
      DOUBLE PRECISION R(N), S(N,4), D(N,5)
C
C     SUBROUTINE EPLSTC CALCULATES STRESSES AND DISPLACEMENT
C     IN A FUNCTIONALLY GRADED TUBE WITH FIXED ENDS SUBJECTED TO
C     PRESSURE IN ELASTIC, PLASTIC AND ELASTOPLASTIC STRESS STATES.
C     CALCULATIONS ARE BASED ON GFD METHOD. MODULUS OF
C     ELASTICITY AND YIELD STRESS VARY RADIALLY ACCORDING TO
C     FOLLOWING FORMS.
C
C     E(R) = E0 * (1.0 - NN * (R/ROUT) ** KK)
C     SY(R) = S0 * (1.0 - MM * (R/ROUT) ** SS)
C
C     ON ENTRY
C
C     N             INTEGER
C
C                   IN ELASTIC AND PLASTIC STRESS STATES
C                   THE NUMBER OF NODES
C
C                   IN ELASTOPLASTIC STRESS STATE
C                   THE NUMBER OF NODES PLUS 1
C
C     IP            INTEGER ARRAY OF LENGTH 3 CARRYING PARAMETERS
C                   WHICH SHOWS THE STRESS STATE OF THE PROBLEM, THE
C                   PLACE OF PLASTIC PART IN ELASTOPLASTIC STRESS
C                   STATE AND THE PLACE OF BOUNDARY CONDITION IN
C                   PLASTIC STRESS STATE.
C
C                   FIRST COLUMN INDICATES STRESS STATE
C                   0 FOR ELASTIC STRESS STATE
C                   1 FOR PLASTIC STRESS STATE
C                   2 FOR ELASTOPLASTIC STRESS STATE
C
C                   SECOND COLUMN INDICATES THE PLACE OF PRESSURE
C                   BOUNDARY CONDITION IN PLASTIC STRESS STATE AND
C                   THE PLACE OF PLASTIC PART IN ELASTOPLASTIC STRESS
C                   STATE
C
C                   IN PLASTIC STRESS STATE
C                   1 FOR INITIAL PRESSURE BOUNDARY CONDITION
C                   2 FOR END PRESSURE BOUNDARY CONDITION
C
C                   IN ELASTOPLASTIC STRESS STATE
```

```
C                    1 FOR PLASTICITY AT THE INNER PART
C                    2 FOR PLASTICITY AT THE OUTER PART
C
C                    THIRD COLUMN INDICATES THE PLACE OF
C                    DISPLACEMENT BOUNDARY CONDITION IN PLASTIC
C                    STRESS STATE
C
C                    0 TO SKIP DISPLACEMENT CALCULATIONS
C                    1 FOR INITIAL DISPLACEMENT BOUNDARY CONDITION
C                    2 FOR END DISPLACEMENT BOUNDARY CONDITION
C
C     P1             DOUBLE PRECISION
C                    NONDIMENSIONAL PRESSURE AT THE START POINT
C
C     P2             DOUBLE PRECISION
C                    NONDIMENSIONAL PRESSURE AT THE END POINT
C
C     R1             DOUBLE PRECISION
C                    THE START POINT
C
C     R2             DOUBLE PRECISION
C                    THE END POINT
C
C     REP            DOUBLE PRECISION
C                    THE INTERFACE POINT IN ELASTOPLASTIC STRESS STATE
C
C     IUP            DOUBLE PRECISION
C                    NONDIMENSIONAL DISPLACEMENT VALUE AT
C                    BOUNDARY POINT IN PLASTIC STRESS STATE
C
C
C     ON RETURN
C
C     R              DOUBLE PRECISION ARRAY OF LENGTH N CARRYING
C                    RADIAL COORDINATES
C
C     S              DOUBLE PRECISION ARRAY OF LENGTH N BY 4
C                    CARRYING STRESS VALUES
C
C                    FIRST COLUMN
C                    NONDIMENSIONAL RADIAL STRESS VALUES (SR)
C
C                    SECOND COLUMN
C                    NONDIMENSIONAL CIRCUMFERENTIAL STRESS VALUES
C                    (ST)
C
C                    THIRD COLUMN
C                    NONDIMENSIONAL AXIAL STRESS VALUES (SZ)
C
C                    FOURTH COLUMN
C                    NONDIMENSIONAL STRESS VARIABLE CALCULATED
C                    ACCORDING TO THE FORM
```

```
C                     SV(R) = (ST(R) - SR(R)) / SY(R)
C                     WHERE SY IS THE ARRAY OF LENGTH N CARRYING
C                     YIELD STRESS VALUES
C
C     D               DOUBLE PRECISION ARRAY OF LENGTH N BY 5
C                     CARRYING DISPLACEMENT AND PLASTIC STRAIN
C                     VALUES
C
C                     FIRST COLUMN
C                     NONDIMENSIONAL DISPLACEMENT VALUES (U)
C
C                     SECOND COLUMN
C                     NONDIMENSIONAL FIRST DERIVATIVE VALUES OF
C                     DISPLACEMENT FUNCTION U. (D1U) (Dr IN THE TEXT)
C
C                     THIRD COLUMN
C                     NONDIMENSIONAL SECOND DERIVATIVE VALUES OF
C                     DISPLACEMENT FUNCTION U. (D2U) (Drr IN THE TEXT)
C
C                     FOURTH COLUMN
C                     NONDIMENSIONAL RADIAL STRAIN VALUES IN
C                     PLASTIC AND ELASTOPLASTIC STRESS STATES (ERP)
C
C                     FIFTH COLUMN
C                     NONDIMENSIONAL CIRCUMFERENTIAL STRAIN VALUES
C                     IN PLASTIC AND ELASTOPLASTIC STRESS STATES (ETP)
C
C     ERR             DOUBLE PRECISION
C
C                     IN ELASTIC STRESS STATE
C                     DIFFERENCE BETWEEN MAXIMUM VALUE OF THE ARRAY
C                     SV AND 1.0.
C
C                     IN ELASTOPLASTIC STRESS STATE
C                     DIFFERENCE BETWEEN CIRCUMFERENTIAL STRESS
C                     VALUES AT THE INTERFACE
C
C
C     SUBROUTINES CALLED
C
C     GFD, MOVINGGRIDS, IDNT, LINPACK ROUTINES DGBFA AND DGBSL
C
C     INTERNAL VARIABLES
C
      INTEGER I, I1, I2, IBC, INFO, IWF, J, JOB, K
      INTEGER LDA, M, ML, MU, NP, NE, NUMDM
      DOUBLE PRECISION E0, S0, KK, MM, NN, SS, NU
      DOUBLE PRECISION RIN, ROUT, RT(N)
      DOUBLE PRECISION DLTP, DLTE, P, SYRIN, UEP
      DOUBLE PRECISION SR(N), ST(N), SZ(N), SV(N), SY(N)
      DOUBLE PRECISION U(N), D1U(N), D2U(N), ERP(N), ETP(N)
      INTEGER, ALLOCATABLE :: IDE(:,:), IDP(:,:), IPVTE(:), IPVTP(:)
```

```
      DOUBLE PRECISION, ALLOCATABLE :: REPL(:), RE(:), RP(:), FE(:), FP(:)
      DOUBLE PRECISION, ALLOCATABLE :: SRE(:), SRP(:), STE(:), STP(:)
      DOUBLE PRECISION, ALLOCATABLE :: SZE(:), SZP(:), UE(:), UP(:)
      DOUBLE PRECISION, ALLOCATABLE :: D1E(:), D1P(:), D2E(:), D2P(:)
      DOUBLE PRECISION, ALLOCATABLE :: D1ME(:,:), D1MP(:,:)
      DOUBLE PRECISION, ALLOCATABLE :: D2ME(:,:), D2MP(:,:)
      DOUBLE PRECISION, ALLOCATABLE :: DME(:,:), DMP(:,:)
      DOUBLE PRECISION, ALLOCATABLE :: DMBDE(:,:), DMBDP(:,:)
      COMMON / GEO / RIN, ROUT
      COMMON / MAT / E0, S0, KK, MM, NN, SS, NU
C
C     DIMENSIONAL PRESSURE VALUES
C
      SYRIN = S0 * (1.0D0 - MM * (RIN/ROUT) **SS)
      P1 = -P1 * SYRIN
      P2 = -P2 * SYRIN
      IF (IP(1) .EQ. 0) GO TO 300
      IF (IP(1) .EQ. 1) GO TO 400
C
C     NUMBER OF NODES AT ELASTIC AND PLASTIC PARTS IN
C     ELASTOPLASTIC STRESS STATE
C
      CALL GRIDS(N-1, R1, R2, REP, NP, NE, DLTP, DLTE, RT)
      IF (IP(2) .EQ. 2) THEN
      I = NE
      NE = NP
      NP = I
      END IF
C
C     ALLOCATION OF ARRAYS IN ELASTOPLASTIC STRESS STATE
C
      ALLOCATE (IDE(NE,NE), IDP(NP,NP), IPVTE(NE), IPVTP(NP))
      ALLOCATE (RE(NE), RP(NP), FE(NE), FP(NP))
      ALLOCATE (SRE(NE), SRP(NP), STE(NE), STP(NP))
      ALLOCATE (SZE(NE), SZP(NP), UE(NE), UP(NP))
      ALLOCATE (D1E(NE), D1P(NP), D2E(NE), D2P(NP))
      ALLOCATE (D1ME(NE,NE), D1MP(NP,NP))
      ALLOCATE (D2ME(NE,NE), D2MP(NP,NP))
      ALLOCATE (DME(NE,NE), DMP(NP,NP))
C
C     STRESS VALUES IN PLASTIC PART
C
C     RADIAL COORDINATES FOR ELASTOPLASTIC STRESS STATE
C
      DO 10 I = 1, NP
            IF (IP(2) .EQ. 1) THEN
            RP(I) = RT(I)
            END IF
            IF (IP(2) .EQ. 2) THEN
            RP(I) = RT(I+NE-1)
            END IF
   10 CONTINUE
```

```
      GO TO 410
C
C     ALLOCATION OF ARRAYS IN PLASTIC STRESS STATE
C
 400  NP = N
      ALLOCATE (IPVTP(NP), RP(NP), FP(NP), IDP(NP,NP))
      ALLOCATE (SRP(NP), STP(NP), SZP(NP))
      ALLOCATE (UP(NP), D1P(NP), D2P(NP))
      ALLOCATE (D1MP(NP,NP), D2MP(NP,NP), DMP(NP,NP))
C
C     RADIAL COORDINATES FOR PLASTIC STRESS STATE
C
      DLTP = (R2-R1) / (NP-1)
      RP(1) = R1
      DO 11 I = 2, NP
          RP(I) = RP(I-1) + DLTP
  11  CONTINUE
C
C     GFD COEFFICIENT MATRICES
C
 410  IWF = 1
      NUMDM = 2
      CALL GFD (RP(1), RP(NP), NP, IWF, NUMDM, D1MP, D2MP)
C
C     GOVERNING DIFFERENTIAL EQUATION
C
C     LEFT HAND SIDE
C
      DO 20 I = 1, NP
          DO 30 J = 1, NP
              DMP(I,J) = RP(I) * D1MP(I,J)
  30      CONTINUE
  20  CONTINUE
C
C     RIGHT HAND SIDE
C
      DO 40 I = 1, NP
          FP(I) = S0 * (1.0D0 - MM * (RP(I)/ROUT) ** SS)
  40  CONTINUE
C
C     BOUNDARY CONDITIONS
C
      IF (IP(2) .EQ. 1) THEN
          IBC = 1
          P = P1
      END IF
      IF (IP(2) .EQ. 2) THEN
          IBC = NP
          P = P2
      END IF
      DO 50 I=1, NP
          FP(I) = FP(I) - P *DMP(I,IBC)
```

```
  50    CONTINUE
        DO 60 I=1,NP
              DMP(IBC,I)=0.0D0
              DMP(I,IBC)=0.0D0
  60    CONTINUE
        DMP(IBC,IBC) = 1.0D0
        FP(IBC) = P
C
C       DIMENSIONAL STRESS VALUES
C
        JOB = 0
        ML = NUMDM
        MU = NUMDM
        LDA = 2 * ML + MU + 1
        ALLOCATE (DMBDP(LDA,NP))
        M = ML + MU + 1
        DO 80 J = 1, NP
              I1 = MAX0(1, J-MU)
              I2 = MIN0(NP, J+ML)
              DO 70 I = I1, I2
                    K = I - J + M
                    DMBDP(K,J) = DMP(I,J)
  70          CONTINUE
  80    CONTINUE
        CALL DGBFA (DMBDP, LDA, NP, ML, MU, IPVTP, INFO)
        CALL DGBSL (DMBDP, LDA, NP, ML, MU, IPVTP, FP, JOB)
        DEALLOCATE (DMBDP)
        SRP = FP
        DO 90 I=1, NP
              STP(I) = SRP(I) + S0 * (1.0D0 - MM * (RP(I)/ROUT) ** SS)
              SZP(I) = NU * (STP(I) + SRP(I))
  90    CONTINUE
        IF (IP(1) .EQ. 1) GO TO 420
C
C       STRESS AND DISPLACEMENT VALUES IN ELASTIC PART
C
C       RADIAL COORDINATES FOR ELASTOPLASTIC STRESS STATE
C
        DO 100 I = 1, NE
              IF (IP(2) .EQ. 1) THEN
              RE(I) = RT(I+NP-1)
              END IF
              IF (IP(2) .EQ. 2) THEN
              RE(I) = RT(I)
              END IF
 100    CONTINUE
        GO TO 310
C
C       ALLOCATION OF ARRAYS IN ELASTIC STRESS STATE
C
 300    NE = N
        ALLOCATE (IDE(NE,NE), IPVTE(NE), RE(NE), FE(NE))
```

```
      ALLOCATE (SRE(NE), STE(NE), SZE(NE), UE(NE), D1E(NE), D2E(NE))
      ALLOCATE (D1ME(NE,NE), D2ME(NE,NE), DME(NE,NE))
C
C     RADIAL COORDINATES FOR ELASTIC STRESS STATE
C
      DLTE = (R2-R1) / (NE-1)
      RE(1) = R1
      DO 101 I = 2, NE
            RE(I) = RE(I-1) + DLTE
 101  CONTINUE
C
C     GFD COEFFICIENT MATRICES
C
 310  IWF = 1
      NUMDM = 2
      CALL GFD (RE(1), RE(NE), NE, IWF, NUMDM, D1ME, D2ME)
C
C     GOVERNING DIFFERENTIAL EQUATION
C
C     LEFT HAND SIDE
C
      CALL IDNT(NE,IDE)
      DO 110 I = 1, NE
            DO 120 J = 1, NE
            DME(I,J) = (RE(I)**2) * (1.0D0 - NN * (RE(I)/ROUT) ** KK)
   *        * D2ME(I,J) + RE(I) * (1.0D0 - NN * (1.0D0+KK) * (RE(I)/ROUT)
   *        ** KK) * D1ME(I,J) - (1.0D0 - NU - NN * (1.0D0 - NU *
   *        (1.0D0+KK)) * (RE(I)/ROUT) ** KK) / (1.0D0-NU) * DFLOAT(IDE(I,J))
 120        CONTINUE
 110  CONTINUE
C
C     RIGHT HAND SIDE
C
      DO 130 I = 1, NE
            FE(I) = 0.0D0
 130  CONTINUE
C
C     BOUNDARY CONDITIONS
C
      DO 140 I = 1, NE
            DME(1,I) = E0 * (1.0D0 - NN * (RE(1)/ROUT) ** KK) *
   *        (1.0D0-NU) / ((1.0D0+NU) * (1.0D0-2*NU)) * D1ME(1,I)
            DME(NE,I) = E0 * (1.0D0 - NN * (RE(NE)/ROUT) ** KK) *
   *        (1.0D0-NU) / ((1.0D0+NU) * (1.0D0-2*NU)) * D1ME(NE,I)
 140  CONTINUE
      DME(1,1) = DME(1,1) + (E0 * (1.0D0 - NN * (RE(1)/ROUT) ** KK)
   *  * NU / (RE(1) * (1.0D0+NU) * (1.0D0-2*NU)))
      DME(NE,NE) = DME(NE,NE) + (E0 * (1.0D0 - NN * (RE(NE)/ROUT) ** KK)
   *  * NU / ( RE(NE) * (1.0D0+NU) * (1.0D0-2*NU)))
      IF (IP(1). EQ. 0) THEN
      FE(1) = P1
      FE(NE) = P2
```

```
      END IF
      IF (IP(1). EQ. 2) THEN
            IF (IP(2). EQ. 1) THEN
            FE(1) = SRP(NP)
            FE(NE) = P2
            END IF
            IF (IP(2). EQ. 2) THEN
            FE(1) = P1
            FE(NE) = SRP(1)
            END IF
      END IF
C
C     DIMENSIONAL DISPLACEMENT VALUES
C
      JOB = 0
      ML = NUMDM
      MU = NUMDM
      LDA = 2 * ML + MU + 1
      ALLOCATE (DMBDE(LDA,NE))
      M = ML + MU + 1
      DO 150 J = 1, NE
            I1 = MAX0(1, J-MU)
            I2 = MIN0(NE, J+ML)
            DO 160 I = I1, I2
                  K = I - J + M
                  DMBDE(K,J) = DME(I,J)
 160        CONTINUE
 150  CONTINUE
      CALL DGBFA (DMBDE, LDA, NE, ML, MU, IPVTE, INFO)
      CALL DGBSL (DMBDE, LDA, NE, ML, MU, IPVTE, FE, JOB)
      UE = FE
      D1E = MATMUL(D1ME,UE)
      D2E = MATMUL(D2ME,UE)
C
C     DIMENSIONAL STRESS VALUES
C
      DO 170 I = 1, NE
            SRE(I) = (E0 * (1.0D0-NN*(RE(I)/ROUT)**KK) / ((1.0D0+NU) *
   *        (1.0D0-2*NU))) * (NU * (UE(I)/RE(I)) + (1.0D0-NU) * D1E(I))
            STE(I) = (E0 * (1.0D0-NN*(RE(I)/ROUT)**KK) / ((1.0D0+NU) *
   *        (1.0D0-2*NU))) * ((1.0D0-NU) * (UE(I)/RE(I)) + NU * D1E(I))
            SZE(I) = (E0 * (1.0D0-NN*(RE(I)/ROUT)**KK) / ((1.0D0+NU) *
   *        (1.0D0-2*NU))) * (NU * (UE(I)/RE(I)) + NU* D1E(I))
 170  CONTINUE
      UP = 0.0D0
      ERP = 0.0D0
      ETP = 0.0D0
      IF (IP(1). EQ. 0) GO TO 320
      IF (IP(2). EQ. 1) ERR = (STP(NP)-STE(1)) / SYRIN
      IF (IP(2). EQ. 2) ERR = (STP(1)-STE(NE)) / SYRIN
C
C     DISPLACEMENT VALUES IN PLASTIC PART
```

```
C
 420    IF (IP(1). EQ. 1 .AND. IP(3). EQ. 0) GO TO 430
C
C       GFD COEFFICIENT MATRICES
C
        IWF = 1
        NUMDM = 2
        CALL GFD (RP(1), RP(NP), NP, IWF, NUMDM, D1MP, D2MP)
C
C       GOVERNING DIFFERENTIAL EQUATION
C
C       LEFT HAND SIDE
C
        CALL IDNT(NP,IDP)
        DO 180 I = 1, NP
                DO 190 J = 1, NP
                DMP(I,J) = D1MP(I,J) + DFLOAT(IDP(I,J)) / RP(I)
 190            CONTINUE
 180    CONTINUE
C
C       RIGHT HAND SIDE
C
        DO 200 I = 1, NP
                FP(I) = (1.0D0 + NU) * (1.0D0 - 2*NU) * (SRP(I) + STP(I)) /
     *          (E0 * (1.0D0 - NN * ((RP(I)/ROUT) ** KK)))
 200    CONTINUE
C
C       BOUNDARY CONDITIONS
C
        IF (IP(1) .EQ. 1) THEN
        UEP = IUP * (ROUT*SYRIN) / E0
        IF (IP(3) .EQ. 1) THEN
                IBC = 1
        END IF
        IF (IP(3) .EQ. 2) THEN
                IBC = NP
        END IF
        END IF
        IF (IP(1) .EQ. 2) THEN
        IF (IP(2) .EQ. 1) THEN
                IBC = NP
                UEP = UE(1)
        END IF
        IF (IP(2) .EQ. 2) THEN
                IBC = 1
                UEP = UE(NE)
        END IF
        END IF
        DO 210 I=1, NP
                FP(I) = FP(I) - UEP * DMP(I,IBC)
 210    CONTINUE
        DO 220 I=1,NP
```

```
            DMP(IBC,I)=0.0D0
            DMP(I,IBC)=0.0D0
 220    CONTINUE
        DMP(IBC,IBC) = 1.0D0
        FP(IBC) = UEP
C
C       DIMENSIONAL DISPLACEMENT VALUES
C
        ML = NUMDM
        MU = NUMDM
        LDA = 2 * ML + MU + 1
        ALLOCATE (DMBDP(LDA,NP))
        M = ML + MU + 1
        DO 230 J = 1, NP
            I1 = MAX0(1, J-MU)
            I2 = MIN0(NP, J+ML)
            DO 240 I = I1, I2
                K = I - J + M
                DMBDP(K,J) = DMP(I,J)
 240        CONTINUE
 230    CONTINUE
        CALL DGBFA (DMBDP, LDA, NP, ML, MU, IPVTP, INFO)
        CALL DGBSL (DMBDP, LDA, NP, ML, MU, IPVTP, FP, JOB)
        UP = FP
        D1P = MATMUL(D1MP,UP)
        D2P = MATMUL(D2MP,UP)
C
C       DIMENSIONAL STRAIN VALUES
C
        DO 250 I = 1, NP
            ERP(I) = D1P(I) - (SRP(I) - NU * (STP(I)+SZP(I))) /
   *        (E0 * (1.0D0 - NN * ((RP(I)/ROUT) ** KK)))
            ETP(I) = (UP(I)/RP(I)) - (STP(I) - NU * (SRP(I)+SZP(I))) /
   *        (E0 * (1.0D0 - NN * ((RP(I)/ROUT) ** KK)))
 250    CONTINUE
        IF (IP(1). EQ. 2 .AND. IP(2). EQ. 2) THEN
        DO 251 I = 1, NP
            ERP(N-NP+I) = ERP(I)
            ETP(N-NP+I) = ETP(I)
            ERP(I) = 0.0D0
            ETP(I) = 0.0D0
 251    CONTINUE
        END IF
C
C       COMBINATION OF ELASTIC AND PLASTIC PARTS
C       IN ELASTOPLASTIC STRESS STATE
C
        IF (IP(1). EQ. 2) THEN
        IF (IP(2). EQ. 1) THEN
        DO 260 I = 1, N
            IF (I .LE. NP) THEN
            R(I) = RP(I)
```

```
                U(I) = UP(I)
                D1U(I) = D1P(I)
                D2U(I) = D2P(I)
                SR(I) = SRP(I)
                ST(I) = STP(I)
                SZ(I) = SZP(I)
                ELSE
                R(I) = RE(I-NP)
                U(I) = UE(I-NP)
                D1U(I) = D1E(I-NP)
                D2U(I) = D2E(I-NP)
                SR(I) = SRE(I-NP)
                ST(I) = STE(I-NP)
                SZ(I) = SZE(I-NP)
                END IF
  260   CONTINUE
        END IF
        IF (IP(2). EQ. 2) THEN
        DO 261 I = 1, N
                IF (I .LE. NE) THEN
                R(I) = RE(I)
                U(I) = UE(I)
                D1U(I) = D1E(I)
                D2U(I) = D2E(I)
                SR(I) = SRE(I)
                ST(I) = STE(I)
                SZ(I) = SZE(I)
                ELSE
                R(I) = RP(I-NE)
                U(I) = UP(I-NE)
                D1U(I) = D1P(I-NE)
                D2U(I) = D2P(I-NE)
                SR(I) = SRP(I-NE)
                ST(I) = STP(I-NE)
                SZ(I) = SZP(I-NE)
                END IF
  261   CONTINUE
        END IF
        END IF
C
C       STRESSES AND DISPLACEMENT IN ELASTIC STRESS STATE
C
  320   IF (IP(1). EQ. 0) THEN
                DO 262 I = 1, N
                        R(I) = RE(I)
                        U(I) = UE(I)
                        D1U(I) = D1E(I)
                        D2U(I) = D2E(I)
                        SR(I) = SRE(I)
                        ST(I) = STE(I)
                        SZ(I) = SZE(I)
  262           CONTINUE
```

```
      END IF
C
C     STRESSES AND DISPLACEMENT IN PLASTIC STRESS STATE
C
 430  IF (IP(1). EQ. 1) THEN
            DO 263 I = 1, N
                  R(I) = RP(I)
                  U(I) = UP(I)
                  D1U(I) = D1P(I)
                  D2U(I) = D2P(I)
                  SR(I) = SRP(I)
                  ST(I) = STP(I)
                  SZ(I) = SZP(I)
 263        CONTINUE
      END IF
C
C     YIELD STRESS AND STRESS VARIABLE VALUES
C
      DO 270 I = 1, N
            SY(I) = S0 * (1.0D0 - MM * (R(I)/ROUT) ** SS)
            SV(I) = (ST(I) - SR(I)) / SY(I)
 270  CONTINUE
C
C     ERROR IN ELASTIC STRESS STATE
C
      IF (IP(1) . EQ. 0) THEN
            ERR = 1.0D0 - MAXVAL (SV)
      END IF
C
C     NONDIMENSIONAL VALUES
C
      DO 280 I = 1, N
            U(I) = U(I) * E0 / (ROUT*SYRIN)
            D1U(I) = D1U(I) * E0 / (ROUT*SYRIN)
            D2U(I) = D2U(I) * E0 / (ROUT*SYRIN)
            SR(I) = SR(I) / SYRIN
            ST(I) = ST(I) / SYRIN
            SZ(I) = SZ(I) / SYRIN
            ERP(I) = ERP(I) * E0 / SYRIN
            ETP(I) = ETP(I) * E0 / SYRIN
 280  CONTINUE
C
C     ARRAYS S AND D
C
      DO 290 I = 1, N
            S(I,1) = SR(I)
            S(I,2) = ST(I)
            S(I,3) = SZ(I)
            S(I,4) = SV(I)
            D(I,1) = U(I)
            D(I,2) = D1U(I)
            D(I,3) = D2U(I)
```

```
            D(I,4) = ERP(I)
            D(I,5) = ETP(I)
 290    CONTINUE
C
C       NONDIMENSIONAL PRESSURE VALUES
C
        P1 = -P1 / SYRIN
        P2 = -P2 / SYRIN
        RETURN
        END
```

B.4 Subroutine ELASTIC_FREE

```
C     ================================================
      SUBROUTINE ELASTIC_FREE (N, PIN, EPSZ, R, S, D, FZ)
C     ================================================
        IMPLICIT NONE
        INTEGER N
        DOUBLE PRECISION PIN, EPSZ, FZ
        DOUBLE PRECISION R(N), S(N,4), D(N,3)
C
C       SUBROUTINE ELASTIC_FREE CALCULATES STRESSES AND
C       DISPLACEMENT IN A FUNCTIONALLY GRADED TUBE WITH FREE ENDS
C       SUBJECTED TO INTERNAL PRESSURE IN PURELY ELASTIC STRESS
C       STATE. CALCULATIONS ARE BASED ON GFD METHOD. MODULUS OF
C       ELASTICITY AND POISSON'S RATIO VARY RADIALLY ACCORDING
C       TO FOLLOWING FORMS.
C
C       E(R) = E0 * (1.0 - NN * (R/ROUT) ** KK)
C       NU(R)= NU0 * (1.0 + NU1 * R + NU2 * R ** 2)
C
C       ON ENTRY
C
C       N               INTEGER
C                       THE NUMBER OF NODES
C
C       PIN             DOUBLE PRECISION
C                       NONDIMENSIONAL INTERNAL PRESSURE
C
C       EPSZ            DOUBLE PRECISION
C                       NONDIMENSIONAL AXIAL STRAIN
C
C       ON RETURN
C
C       R               DOUBLE PRECISION ARRAY OF LENGTH N CARRYING
C                       RADIAL COORDINATES
C
C       S               DOUBLE PRECISION ARRAY OF LENGTH N BY 4
C                       CARRYING STRESS VALUES
C
```

```
C                     FIRST COLUMN
C                     NONDIMENSIONAL RADIAL STRESS VALUES (SR)
C
C                     SECOND COLUMN
C                     NONDIMENSIONAL CIRCUMFERENTIAL STRESS VALUES
C                     (ST)
C
C                     THIRD COLUMN
C                     NONDIMENSIONAL AXIAL STRESS VALUES (SZ)
C
C                     FOURTH COLUMN
C                     NONDIMENSIONAL STRESS VARIABLE CALCULATED
C                     ACCORDING TO THE FORM
C                     SV(R) = (ST(R) - SR(R)) / S0
C                     WHERE S0 IS THE YIELD STRESS
C
C      D              DOUBLE PRECISION ARRAY OF LENGTH N BY 3
C                     CARRYING DISPLACEMENT AND PLASTIC STRAIN
C                     VALUES
C
C                     FIRST COLUMN
C                     NONDIMENSIONAL DISPLACEMENT VALUES (U)
C
C                     SECOND COLUMN
C                     NONDIMENSIONAL FIRST DERIVATIVE VALUES OF
C                     DISPLACEMENT FUNCTION U. (D1U) (Dr IN THE TEXT)
C
C                     THIRD COLUMN
C                     NONDIMENSIONAL SECOND DERIVATIVE VALUES OF
C                     DISPLACEMENT FUNCTION U. (D2U) (Drr IN THE TEXT)
C
C      FZ             DOUBLE PRECISION
C                     AXIAL FORCE
C
C      SUBROUTINES AND FUNCTIONS
C
C      GFD, DAVINT, IDNT, LINPACK ROUTINES DGBFA AND DGBSL
C
C      INTERNAL VARIABLES
C
       INTEGER I, I1, I2, ID(N,N), IERR, IPVT(N), INFO
       INTEGER IWF, J, JOB, K, LDA, M, ML, MU, NUMDM
       DOUBLE PRECISION DLTR, RIN, ROUT
       DOUBLE PRECISION E0, S0, KK, NN
       DOUBLE PRECISION NU0, NU1, NU2, NU(N), DNU(N)
       DOUBLE PRECISION F(N), INTZ(N)
       DOUBLE PRECISION D1M(N,N), D2M(N,N), DM(N,N)
       DOUBLE PRECISION SR(N), ST(N), SZ(N), SV(N), SY(N)
       DOUBLE PRECISION U(N), D1U(N), D2U(N)
       DOUBLE PRECISION, ALLOCATABLE :: DMBD(:,:)
       COMMON / GEO / RIN, ROUT
       COMMON / MAT / E0, S0, KK, NN, NU0, NU1, NU2
```

```fortran
C
C       DIMENSIONAL INTERNAL PRESSURE
C
        PIN = -PIN * S0
C
C       DIMENSIONAL AXIAL STRAIN
C
        EPSZ = EPSZ * S0 / E0
C
C       RADIAL COORDINATES
C
        DLTR = (ROUT-RIN) / (N-1)
        R(1) = RIN
        DO 10 I = 2, N
                R(I) = R(I-1) + DLTR
  10    CONTINUE
C
C       ARRAYS CARRYING YOUNG'S MODULUS AND ITS FIRST DERIVATIVE
C
        DO 11 I = 1, N
        NU(I) = NU0 * (1.0D0 + NU1*R(I) + NU2*R(I)**2)
        DNU(I) = NU0 * (NU1 + 2*NU2*R(I))
  11    CONTINUE
C
C       GFD COEFFICIENT MATRICES
C
        IWF = 1
        NUMDM = 2
        CALL GFD (R(1), R(N), N, IWF, NUMDM, D1M, D2M)
C
C       GOVERNING DIFFERENTIAL EQUATION
C
C       LEFT HAND SIDE
C
        CALL IDNT(N,ID)
        DO 20 I = 1, N
                DO 30 J = 1, N
                DM(I,J) = R(I) ** 2 * (1.0D0 - NN * (R(I)/ROUT) ** KK) *
   *            D2M(I,J) + (R(I) * (1.0D0 - NN * (R(I)/ROUT) ** KK *
   *            (1.0D0+KK)) + 2 * R(I)** 2 * NU(I) * DNU(I) * (2.0D0-NU(I))
   *            / ((1.0D0-2*NU(I)) * (1.0D0-NU(I)**2)) * (1.0D0 - NN *
   *            (R(I)/ROUT) ** KK)) * D1M(I,J) + (-(1.0D0 - NU(I) - NN *
   *            (1.0D0 - NU(I) * (1.0D0+KK)) * (R(I) / ROUT) ** KK) /
   *            (1.0D0-NU(I)) + R(I) * DNU(I) * (1.0D0+2*NU(I)**2) /
   *            ((1.0D0-2*NU(I)) * (1.0D0-NU(I)**2)) * (1.0D0 - NN *
   *            (R(I)/ROUT) ** KK)) * ID(I,J)
  30            CONTINUE
  20    CONTINUE
C
C       RIGHT HAND SIDE
C
        DO 40 I = 1, N
```

```
               F(I) = EPSZ * (R(I) * NN * KK * NU(I) * (R(I)/ROUT) ** KK /
     *         (1.0D0-NU(I)) - R(I) ** 2 * DNU(I) * (1.0D0 + 2*NU(I)**2) /
     *         ((1.0D0 - 2*NU(I)) * (1.0D0 - NU(I)**2)) * (1.0D0 - NN *
     *         (R(I)/ROUT) ** KK))
  40   CONTINUE
C
C      BOUNDARY CONDITIONS
C
       DO 50 I = 1, N
               DM(1,I) = E0 * (1.0D0 - NN * (R(1)/ROUT) ** KK) *
     *         (1.0D0-NU(1)) / ((1.0D0+NU(1)) * (1.0D0-2*NU(1))) * D1M(1,I)
               DM(N,I) = E0 * (1.0D0 - NN * (R(N)/ROUT) ** KK) *
     *         (1.0D0-NU(N)) / ((1.0D0+NU(N)) * (1.0D0-2*NU(N))) * D1M(N,I)
  50   CONTINUE
       DM(1,1) = DM(1,1) + (E0 * (1.0D0 - NN * (R(1)/ROUT) ** KK) * NU(1)
     * / (R(1) * (1.0D0+NU(1)) * (1.0D0-2*NU(1))))
       DM(N,N) = DM(N,N) + (E0 * (1.0D0 - NN * (R(N)/ROUT) ** KK) * NU(N)
     * / (R(N) * (1.0D0+NU(N)) * (1.0D0-2*NU(N))))
       F(1) = PIN - EPSZ * E0 * (1.0D0 - NN * (R(1)/ROUT) ** KK) * NU(1)
     * / ((1.0D0+NU(1))* (1.0D0-2*NU(1)))
       F(N) = 0.0D0 - EPSZ * E0 * (1.0D0 -NN * (R(N)/ROUT) ** KK) * NU(N)
     * / ((1.0D0+NU(N))* (1.0D0-2*NU(N)))
C
C      DIMENSIONAL DISPLACEMENT VALUES
C
       JOB = 0
       ML = NUMDM
       MU = NUMDM
       LDA = 2 * ML + MU + 1
       ALLOCATE (DMBD(LDA,N))
       M = ML + MU + 1
       DO 60 J = 1, N
               I1 = MAX0(1, J-MU)
               I2 = MIN0(N, J+ML)
               DO 70 I = I1, I2
                       K = I - J + M
                       DMBD(K,J) = DM(I,J)
  70           CONTINUE
  60   CONTINUE
       CALL DGBFA (DMBD, LDA, N, ML, MU, IPVT, INFO)
       CALL DGBSL (DMBD, LDA, N, ML, MU, IPVT, F, JOB)
       U = F
       D1U = MATMUL(D1M,U)
       D2U = MATMUL(D2M,U)
C
C      DIMENSIONAL STRESS VALUES
C
       DO 80 I = 1, N
               SR(I) = E0 * (1.0D0 - NN*(R(I)/ROUT)** KK) / ((1.0D0+NU(I))
     *         * (1.0D0-2*NU(I))) * (NU(I)*EPSZ + NU(I)*(U(I)/R(I))
     *         + (1.0D0-NU(I))*D1U(I))
               ST(I) = E0 * (1.0D0 - NN*(R(I)/ROUT)**KK) / ((1.0D0+NU(I))
```

```
     *          * (1.0D0-2*NU(I)))*(NU(I)*EPSZ + (1.0D0-NU(I))*(U(I)/R(I))
     *          + NU(I)*D1U(I))
                SZ(I) = E0 * (1.0D0 - NN*(R(I)/ROUT)**KK) / ((1.0D0+NU(I))
     *          * (1.0D0-2*NU(I))) * ((1.0D0-NU(I)) * EPSZ + NU(I)*(U(I)/R(I))
     *          + NU(I)*D1U(I))
                SV(I) = (ST(I)-SR(I))/S0
   80   CONTINUE
C
C       NONDIMENSIONAL VALUES
C
        DO 90 I=1,N
                U(I) = U(I) * E0 / (ROUT*S0)
                D1U(I) = D1U(I) * E0 / (ROUT*S0)
                D2U(I) = D2U(I) * E0 / (ROUT*S0)
                SR(I) = SR(I) / S0
                ST(I) = ST(I) / S0
                SZ(I) = SZ(I) / S0
   90   CONTINUE
C
C       ARRAYS S AND D
C
        DO 290 I = 1, N
                S(I,1) = SR(I)
                S(I,2) = ST(I)
                S(I,3) = SZ(I)
                S(I,4) = SV(I)
                D(I,1) = U(I)
                D(I,2) = D1U(I)
                D(I,3) = D2U(I)
  290   CONTINUE
C
C       FORCE INTEGRAL
C
        DO 100 I = 1, N
                INTZ(I) = SZ(I) * R(I)
  100   CONTINUE
        CALL DAVINT (R, INTZ, N, RIN, ROUT, FZ, IERR)
C
C       NONDIMENSIONAL INTERNAL PRESSURE
C
        PIN = -PIN / S0
C
C       NONDIMENSIONAL AXIAL STRAIN
C
        EPSZ = EPSZ * E0 / S0
        RETURN
        END
```

B.5 Subroutine GRIDS

```
C   ==================================================
    SUBROUTINE GRIDS (NTOT, A, B, R1, NPL, NEL, DLTPL, DLTEL, R)
C   ==================================================
C
C      BY AHMET ERASLAN
C
       IMPLICIT DOUBLE PRECISION (A-H , O-Z)
       DIMENSION R(NTOT)
C
       IF (R1 .EQ. A) THEN
       NEL = NTOT
       NPL = 0
       DLTPL = (R1 - A) / (NPL - 1)
       DLTEL = (B - R1) / (NEL - 1)
       R(1)  = A
       DO I = 2, NTOT
       R(I) = R(I-1) + DLTEL
       END DO
       RETURN
       END IF
C
       IF (R1 .EQ. B) THEN
       NEL = 0
       NPL = NTOT
       DLTPL = (R1 - A) / (NPL - 1)
       DLTEL = (B - R1) / (NEL - 1)
       R(1)  = A
       DO I = 2, NTOT
       R(I) = R(I-1) + DLTPL
       END DO
       RETURN
       END IF
C
       R0    = (R1 - A) / (B - R1)
       T     = DFLOAT(NTOT)
       NEL = (T - 1.0D0) / (R0 + 1.0D0)
       NPL   = NTOT - 1 - NEL
       DLTPL = (R1 - A) / NPL
       DLTEL = (B - R1) / NEL
C
       R(1) = A
       DO I = 1, NPL
       R(I+1) = A + DLTPL*I
       END DO
C
       IFAC = 0
       DO I = NPL+1, NTOT-1
       IFAC   = IFAC + 1
       R(I+1) = R1 + DLTEL*IFAC
```

```
      END DO
      NPL = NPL + 1
      NEL = NEL + 1
   RETURN
   END
```

B.6 Subroutine DAVINT

```
C  ==========================================
   SUBROUTINE DAVINT (X, Y, N, XLO, XUP, ANS, IERR)
C  ==========================================
   IMPLICIT DOUBLE PRECISION (A-H , O-Z)
   INTEGER I, IERR, INLFT, INRT, ISTART, ISTOP, N
   DIMENSION X(N),Y(N)
C-----------------------------------------------------------------
C
C      BY AHMET ERASLAN
C
C      DAVINT INTEGRATES A FUNCTION TABULATED AT ARBITRARILY
C      SPACED ABSCISSAS.  THE LIMITS OF INTEGRATION NEED NOT
C      COINCIDE WITH THE TABULATED ABSCISSAS.
C
C      A METHOD OF OVERLAPPING PARABOLAS FITTED TO THE DATA IS
C      USED PROVIDED THAT THERE ARE AT LEAST 3 ABSCISSAS BETWEEN
C      THE LIMITS OF INTEGRATION.  DAVINT ALSO HANDLES TWO SPECIAL
C      CASES. IF THE LIMITS OF INTEGRATION ARE EQUAL, DAVINT RETURNS
C      A RESULT OF ZERO REGARDLESS OF THE NUMBER OF TABULATED
C      VALUES. IF THERE ARE ONLY TWO FUNCTION VALUES, DAVINT USES
C      THE TRAPEZOID RULE.
C
C      DESCRIPTION OF PARAMETERS
C      THE USER MUST DIMENSION ALL ARRAYS APPEARING IN THE CALL
C      LIST X(N), Y(N)
C
C      INPUT
C      X      DOUBLE PRECISION ARRAY OF ABSCISSAS, WHICH MUST BE IN
C             INCREASING ORDER.
C      Y      DOUBLE PRECISION ARRAY OF FUNCTION VALUES. I.E.,
C             Y(I)=FUNC(X(I))
C      N      THE INTEGER NUMBER OF FUNCTION VALUES SUPPLIED.
C             N .GE. 2 UNLESS XLO = XUP.
C      XLO    DOUBLE PRECISION LOWER LIMIT OF INTEGRATION
C      XUP    DOUBLE PRECISION UPPER LIMIT OF INTEGRATION.
C             MUST HAVE XLO.LE.XUP
C
C      OUTPUT
C      ANS    DOUBLE PRECISION COMPUTED APPROXIMATE VALUE OF
C             INTEGRAL
C      IERR   A STATUS CODE
C             NORMAL CODE
```

```
C                =1 MEANS THE REQUESTED INTEGRATION WAS PERFORMED.
C                ABNORMAL CODES
C                =2 MEANS XUP WAS LESS THAN XLO.
C                =3 MEANS THE NUMBER OF X(I) BETWEEN XLO AND XUP
C                (İNCLUSİVE) WAS LESS THAN 3 AND NEITHER OF THE TWO
C                SPECIAL CASES DESCRIBED IN THE ABSTRACT OCCURRED.
C                NO INTEGRATION WAS PERFORMED.
C                =4 MEANS THE RESTRICTION X(I+1).GT.X(I) WAS VIOLATED.
C                =5 MEANS THE NUMBER N OF FUNCTION VALUES WAS .LT. 2.
C                ANS IS SET TO ZERO IF IERR=2,3,4,OR 5.
C-----------------------------------------------------------------------
      IERR = 1
      ANS = 0.0D0
      IF (XLO .GT. XUP) GO TO 160
         IF (XLO .EQ. XUP) GO TO 150
            IF (N .GE. 2) GO TO 10
            IERR = 5
            GO TO 190
 10      CONTINUE
         DO 20 I = 2, N
               IF (X(I) .LE. X(I-1)) GO TO 180
               IF (X(I) .GT. XUP) GO TO 30
 20         CONTINUE
 30         CONTINUE
         IF (N .GE. 3) GO TO 40
               SLOPE = (Y(2) - Y(1))/(X(2) - X(1))
               FL = Y(1) + SLOPE*(XLO - X(1))
               FR = Y(2) + SLOPE*(XUP - X(2))
               ANS = 0.5D0*(FL + FR)*(XUP - XLO)
               GO TO 190
 40         CONTINUE
         IF (X(N-2) .GE. XLO) GO TO 50
               IERR = 3
               GO TO 190
 50       CONTINUE
         IF (X(3) .LE. XUP) GO TO 60
               IERR = 3
               GO TO 190
 60       CONTINUE
         I = 1
 70         IF (X(I) .GE. XLO) GO TO 80
               I = I + 1
               GO TO 70
 80      CONTINUE
         INLFT = I
         I = N
 90      IF (X(I) .LE. XUP) GO TO 100
               I = I - 1
            GO TO 90
100         CONTINUE
         INRT = I
         IF ((INRT - INLFT) .GE. 2) GO TO 110
```

```
                  IERR = 3
                  GO TO 190
110         CONTINUE
            ISTART = INLFT
            IF (INLFT .EQ. 1) ISTART = 2
            ISTOP = INRT
            IF (INRT .EQ. N) ISTOP = N - 1
            R3 = 3.0D0
            RP5 = 0.5D0
            SUM = 0.0D0
            SYL = XLO
            SYL2 = SYL*SYL
            SYL3 = SYL2*SYL
            DO 140 I = ISTART, ISTOP
             X1 = X(I-1)
             X2 = X(I)
             X3 = X(I+1)
             X12 = X1 - X2
             X13 = X1 - X3
             X23 = X2 - X3
             TERM1 = Y(I-1)/(X12*X13)
             TERM2 = -Y(I)/(X12*X23)
             TERM3 = Y(I+1)/(X13*X23)
             A = TERM1 + TERM2 + TERM3
             B = -(X2 + X3)*TERM1 - (X1 + X3)*TERM2
  1           - (X1 + X2)*TERM3
             C = X2*X3*TERM1 + X1*X3*TERM2 + X1*X2*TERM3
             IF (I .GT. ISTART) GO TO 120
                CA = A
                CB = B
                CC = C
             GO TO 130
120          CONTINUE
                CA = 0.5D0*(A + CA)
                CB = 0.5D0*(B + CB)
                CC = 0.5D0*(C + CC)
130          CONTINUE
             SYU = X2
             SYU2 = SYU*SYU
             SYU3 = SYU2*SYU
             SUM = SUM + CA*(SYU3 - SYL3)/R3
  1          + CB*RP5*(SYU2 - SYL2) + CC*(SYU - SYL)
             CA = A
             CB = B
             CC = C
             SYL = SYU
             SYL2 = SYU2
             SYL3 = SYU3
140          CONTINUE
            SYU = XUP
            ANS = SUM + CA*(SYU**3 - SYL3)/R3
  1         + CB*RP5*(SYU**2 - SYL2) + CC*(SYU - SYL)
```

```
 150          CONTINUE
              GO TO 170
 160          CONTINUE
                    IERR = 2
 170          CONTINUE
          GO TO 190
 180      CONTINUE
          IERR = 4
 190  CONTINUE
      RETURN
      END
```

B.7 Subroutine IDNT

```
C    ====================
     SUBROUTINE IDNT(N,ID)
C    ====================
      IMPLICIT NONE
      INTEGER N, ID(N,N)
C
C     SUBROUTINE IDNT GS IDENTITY MATRIX OF ORDER N
C
C     ON ENTRY
C
C     N      INTEGER
C            ORDER OF IDENTITY MATRIX
C
C     ON RETURN
C
C     ID     INTEGER (N,N)
C            IDENTITY MATRIX
C
      INTEGER I, J
      DO 10 I=1,N
        DO 20 J=1,N
            IF (I.EQ.J) THEN
            ID(I,J)=1
            ELSE
            ID(I,J)=0
            END IF
20      CONTINUE
10    CONTINUE
      RETURN
      END
```