ON ASYMPTOTIC PROPERTIES OF POSITIVE OPERATORS ON BANACH LATTICES

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

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Abstract

ON ASYMPTOTIC PROPERTIES OF POSITIVE OPERATORS ON BANACH LATTICES

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In this thesis, we study two problems. The first one is the renorming problem in Banach lattices. We state the problem and give some known results related to it. Then we pass to construct a positive doubly power bounded operator with a non-positive inverse on an infinite dimensional AL-space which generalizes the result of [10].

The second problem is related to the mean ergodicity of positive operators on KB-spaces. We prove that any positive power bounded operator T in a KB-space E which satisfies

$$\lim_{n\to\infty} \operatorname{dist}\Big(\frac{1}{n}\sum_{k=0}^{n-1} T^k x, [-g,g] + \eta B_E\Big) = 0 \quad (\forall x \in E, \|x\| \leq 1), \qquad (*)$$

where B_E is the unit ball of E, $g \in E_+$, and $0 \leq \eta < 1$, is mean ergodic and its fixed space $\operatorname{Fix}(T)$ is finite dimensional. This generalizes the main result of [12]. Moreover, under the assumption that E is a σ-Dedekind complete Banach lattice, we prove that if, for any positive power bounded operator T, the condition $(*)$ implies that T is mean ergodic then E is a KB-space.

Keywords : Positive isometry, (doubly) power bounded operator, renorming problem, AL-spaces, mean ergodicity, KB-spaces.

Öz

BANACH ÖRGÜLERİ ÜZERİNDE POZİTİF DÖNÜŞÜMLERİN ASİMPTOTİK ÖZELLİKLERİ ÜZERİNE

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Bu tezde iki problem ele alınmaktadır. Bunlardan ilki Banach Örgülerinde normun yeniden tanımlanmasıdır. Önce problem tanımlanmakta ve bunun ile ilgili bilinen sonuçlar verilmektedir. Daha sonra sonsuz boyutlu bir AL-uzayında pozitif ve çift kuvvet sınırlı ve tersi pozitif olmayan bir dönüşüm inşa edilmektedir. Bu sonuç [10] daki sonucu genelleştirmektedir.

Ele alınan ikinci problem ise KB uzaylarında tanımlı pozitif dönüşümlerin ortalama ergodikliği ile ilgilidir. Burada bir KB-uzayı E de tanımlı kuvvet sınırlı, pozitif ve

$$\lim_{n\to\infty} \operatorname{dist}\Big(\frac{1}{n}\sum_{k=0}^{n-1} T^k x, [-g, g] + \eta B_E\Big) = 0 \quad (\forall x \in E, \|x\| \leq 1), \tag{$*$}$$

özelliğini sağlayan T dönüşümünün ortalama ergodik olduğu ve sabit uzayının $Fix(T)$ sonlu boyutlu olduğu kanıtlanmıştır. Eşitsizlikte B_E , E'nin kapalı birim yuvarı, $g \in E_+$ ve $0 \leq \eta < 1$ olarak alınmıştır. Bu sonuç [12] nolu kaynağn ana sonucunu genelleştirmektedir. Diğer yandan, E uzayının σ- Dedekind tam olması durumunda, $(*)$ eşitsizliğini sağlayan pozitif kuvvet sınırlı T dönüşümünün ortalama ergodik olmasını gerektirmesi, E'nin KB-uzayı olduğunu gerektirmesi de kanıtlanmıştır.

Anahtar Kelimeler : Pozitif isometri, (çift) kuvvet sınırlı operator, normlama problemi, AL-uzayları, ortalama Ergodiklik, KB-uzayları.

To my family

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Chapter 1

Introduction

The study of the asymptotic behavior of operator semigroups is of a fundamental importance in many disciplines for example this theory has many applications in PDE and, in general, in dynamical systems. Asymptotic behavior refers to the orbits of the initial value under a given semigroup and phenomena such as stability. In this thesis, two problems are treated in connection with asymptotic behavior of positive operator semigroups on Banach lattices. we consider the discrete case only, when a semigroup is generated by a single operator. The general case is more or less similar at least for one-parameter semigroups. The **first problem** is called the *renorming problem in Banach lattices*. It deals with the way by which one can renorm the Banach lattice and make every positive doubly power bounded operator on it an isometry. The **second problem** is related to the *mean ergodicity of positive operators on KB-spaces*, for which the asymptotic behavior of the Cesàro means of a positive operator on a KB-space is studied. **Chapter 1** of this thesis presents the scope of the study as an introduction.

Chapter 2 deals with the Riesz spaces, Banach lattices, positive operators between them, mean ergodic operators and mean ergodic theorems. The first section includes the definitions and basic examples of Riesz spaces and Banach lattices. We give also some definitions and fundamental properties of special elements, sets, and subspaces of Riesz spaces. The remainder of this section is devoted to present important classes of Banach lattices which are the so-called Banach lattices with order continuous norm and KB-spaces. We give their definitions, some examples and collect their most important characterizations in Theorems 2.1.21 and 2.1.27 respectively. This is followed by a section on positive operators on Banach lattices. We start with their definition, examples and give some of their important properties. We pass to introduce the positive operators preserving the lattice operations which are known as Riesz homomorphisms and give some of their properties needed in this thesis and we end this section by giving the definition of AL-spaces, their characterization and some of their important properties. In the third section, we give the definition of mean ergodic operators and some of their examples and others which are not. The section ends with a major characterization of mean ergodic operators known as Eberlein's theorem.

Chapter 3 focuses on the study of the renorming problem in Banach lattices. The main goal of this chapter is to construct a positive doubly power bounded operator with a non-positive inverse on an infinite dimensional AL-space. This generalizes the result of [10]. The first section of this chapter presents the statement of the renorming problem in Banach lattices and gives some basic definitions and theorems related to the problem. The second section begins by the definition of AL-spaces and gives some of their properties. Next we give some lemmata forming the bases to state Theorem 3.2.8, the main theorem of this chapter, in which we construct a positive doubly power bounded operator with a non-positive inverse on an infinite dimensional L_1-space, as a result of this theorem we get corollary 3.2.9 by which we achieve the main goal of this chapter. The results of Chapter 3 were published in [7].

Chapter 4 is devoted to study the mean ergodicity of positive operators on KB-spaces which forms the second problem studied in this thesis. In the first section, we give an introduction about the mean ergodicity of Markov operators and we mention a related result in [12]. In preparation to state our main theorem, we give two lemmata talking about the existence of a non-zero positive fixed element (of maximal support) of a positive power bounded operator. These lemmata put us in a position to state our main Theorem 4.1.3 which gives a condition of a positive power bounded operator on KB-space to be mean ergodic. The rest of this section gives a special result of Theorem 4.1.3 for which we have more than mean ergodicity of a positive power bounded operator on KB-space, whose fixed space is finite dimensional. The third and final section of this chapter begins by discussing the relationship between the mean ergodicity of power bounded operators on a Banach space and the reflexivity of this space. This motivates us to study such a relationship but this time between the mean ergodicity of special power bounded operators on a $\sigma-$Dedekind complete

Banach lattice and the KB-property of this Banach lattice. This idea is stated as Theorem 4.2.2 in which we obtain a characterization of KB-spaces. The results of Chapter 4 will be published in [8].

Chapter 2

Preliminaries

In this chapter, for the convenience of the reader, we present the general background needed in this thesis. We give concise presentation of the basic structural properties of Riesz spaces and Banach lattices and pay a special attention to study Banach lattices with order continuous norm, AL-spaces and KB-spaces. Moreover we discuss the fundamental properties of (positive) operators acting on Banach lattices. We end the chapter by a section in which we state Eberlein's mean ergodic theorem.

2.1 Riesz spaces

A real vector space E is said to be an ***ordered vector space*** whenever it is equipped with an order relation $\geq$ (i.e., $\geq$ is reflexive, antisymmetric, and transitive) that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms:

(1) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$,

(2) If $x \geq y$, then $\alpha x \geq \alpha y$ holds for all $\alpha \in \mathbb{R}^+$.

An alternative notation for $x \geq y$ is $y \leq x$. An element x in an ordered vector space E is called ***positive*** whenever $x \geq 0$ holds. The set of all positive elements of E is called the ***positive cone*** of E and it will be denoted by E_+. i.e.,

$$E_+ = \{x \in E : x \geq 0\}$$

Definition 2.1.1. *A **Riesz space** is an ordered vector space E with the additional property that for each pair of elements $x, y \in E$ the supremum of the set $\{x, y\}$ exists in E, which is equivalent to that the infimum of the set $\{x, y\}$ exists for each x, y. We shall write*

$$x \vee y := \sup\{x, y\} \quad \textit{and} \quad x \wedge y := \inf\{x, y\}.$$

For a vector x in a Riesz space, the positive part x^+, the negative part x^-, and the absolute value $|x|$ are defined by

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \;\; \textit{and} \;\; |x| := x \vee (-x).$$

The functions $(x, y) \to x \vee y$, $(x, y) \to x \wedge y$, $x \to x^+$, $x \to x^-$, *and* $x \to |x|$ are referred to collectively as the ***lattice operations*** of a Riesz space.

Example 2.1.2. Many familiar spaces are Riesz spaces as the following examples show:

(1) The Euclidean space $\mathbb{R}^n$ is a Riesz space under the usual ordering $x = (x_1, \ldots, x_n) \geq y = (y_1, \ldots, y_n)$ whenever $x_i \geq y_i$ for each $i = 1, \ldots, n$. The infimum and supremum of two vectors x and y are given by
$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$ *and* $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$.

(2) If (X, Σ, μ) is a measure space and $0 \leq p < \infty$, then the vector space

$$L_p(\mu) = \left\{ f : X \to \mathbb{R}, f \;\; \textit{is} \;\; \mu - \textit{measurable} \;\; \textit{and} \int_X |f|^p \, d\mu < \infty \right\}$$

is a Riesz space under the almost everywhere pointwise ordering. That is, $f \geq g$ in $L_p(\mu)$ means that $f(x) \geq g(x)$ for $\mu-$almost every x. The infimum and supremum of $f, g \in L_p(\mu)$ are given by

$$(f \vee g)(x) = \max\{f(x), g(x)\} \;\; \textit{and} \;\; (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

Under the same definitions of infimum and supremum as above the vector space

$$L_\infty(\mu) = \{f : X \to \mathbb{R}, \; f \;\; \textit{is} \;\; \mu - \textit{measurable} \;\; \textit{and} \;\; \textit{ess} \sup |f| < \infty\}$$

is a Riesz space.

(3) The vector spaces $l_p(1 \leq p < \infty)$ of all real sequences $(x_1, x_2, \dots)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$, $l_\infty(\Omega)$ the space of all bounded real valued functions on Ω and c_0 the space of null real sequences are all Riesz spaces under the usual pointwise ordering.

(4) Both the vector spaces $C(X)$ of all real continuous functions and the vector space $C_b(X)$ of all bounded continuous functions on the topological space X are Riesz spaces when the ordering is defined pointwise. That is, $f \leq g$ holds whenever $f(x) \leq g(x)$ for all $x \in X$. The infimum and supremum are defined as

$$(f \vee g)(x) = \max\{f(x), g(x)\} \;\; \textit{and} \;\; (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

(5) Let E be the vector space of all real valued functions on the non-empty point set X with the addition and multiplication pointwise, i.e., $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $(\alpha f)(x) = \alpha f(x)$ for all $x \in X$. The ordering is also defined pointwise, i.e., $f \leq g$ whenever $f(x) \leq g(x)$ for all $x \in X$. This makes E a Riesz space.

(6) Let the ordering in $\mathbb{R}^2$ be defined so that $(x_1, x_2) \leq (y_1, y_2)$ whenever either $x_1 < y_1$ or ($x_1 = y_1$ and $x_2 \leq y_2$). This makes $\mathbb{R}^2$ a Riesz space. The ordering is called *lexicographical ordering.*

(7) Let $P = \{f : f : [0,1] \rightarrow \mathbb{R} \text{ is a polynomial}\}$. Then P is an ordered vector space under the pointwise operations and ordering, but not a Riesz space.

The next theorem gives some basic properties of positive part, negative part, and the absolute value of an element in a Riesz space.

Theorem 2.1.3. $[5, Thm.1.3]$ *If x is an element of a Riesz space, then we have*
(1) $x = x^+ - x^-$; (2) $|x| = x^+ + x^-$; *and* (3) $x^+ \wedge x^- = 0$.
Moreover, the decomposition in (1) *is unique in the sense that if $x = y - z$ holds with $y \wedge z = 0$, then $y = x^+$ and $z = x^-$.*

Definition 2.1.4. *Two elements x and y in a Riesz space are said to be **disjoint**, written by $x \perp y$, if $|x| \wedge |y| = 0$. For any non-empty subset A of a Riesz space E the set*

$$A^d = \{x \in E : x \perp y \text{ for all y} \in \text{A}\}$$

*is called the **disjoint complement** of A.*

Note that $A \bigcap A^d = \{0\}$. The disjoint complement $A^{dd} = (A^d)^d$ of A^d is called the ***second disjoint complement*** of A.

We give now some properties of disjoint elements in Riesz spaces.

Theorem 2.1.5. [19, Thm.14.2, Thm.14.3] *For any Riesz space E and $x, y, z \in E$, we have*

(1) *If $x \perp y$ and $|z| \leq |x|$, then $z \perp y$.*

(2) *If $x \perp y$ and $a \in \mathbb{R}$, then $ax \perp y$.*

(3) *If $x \perp y$ and $z \perp y$, then $(x+z) \perp y$.*

(4) *We have $x \perp y$ if and only if $x^+ \perp y$ and $x^- \perp y$.*

(5) *If D is subset of a Riesz space E such that $x_0 = \sup D$ exists in E, and if $x \perp y$ holds for all $x \in D$, then $x_0 \perp y$.*

If x and y are two elements in a Riesz space E with $x \leq y$, then the ***order interval*** $[x, y]$ is the subset defined by

$$[x, y] = \{z \in E : x \leq z \leq y\}.$$

A subset A of a Riesz space is said to be ***order bounded from above*** whenever there exists some x satisfying $y \leq x$ for all $y \in A$. Similarly, a subset A of a Riesz space is said to be ***order bounded from below*** whenever there exists some x satisfying $x \leq y$ for all $y \in A$. Finally, a subset A of a Riesz space is said to be ***order bounded*** if it is order bounded both from above and below (or, equivalently, if it is contained in an order interval).

Definition 2.1.6. *A net $\{x_\alpha\}$ in a Riesz space is* **decreasing***, written $x_\alpha \downarrow$, if $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$. The symbol $x_\alpha \uparrow$ indicates an* **increasing** *net, while $x_\alpha \uparrow \leq x$ (resp. $x_\alpha \downarrow \geq x$.) denotes an increasing (resp. decreasing) net that is order bounded from above (resp. below) by x. The notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$. The meaning of $x_\alpha \uparrow x$ is similar.*

Definition 2.1.7. *The Riesz space E is said to be* ***Archimedean*** *if $n^{-1}x \downarrow 0$ holds for all $x \in E_+$.*

All classical function spaces are examples of Archimedean Riesz spaces. In spite of this there exist non-Archimedean Riesz space. As an example, let $E = \mathbb{R}^2$ with the lexicographical ordering (see Example 2.1.2(6)). The element $(0, 1)$ in E is a lower bound of the sequence $((n^{-1}, n^{-1}))$. Hence, $x = (1, 1)$ does not satisfy the condition that $\inf\{n^{-1}x : n = 1, 2, \dots\} = 0$. Actually, the sequence of all $n^{-1}x$ does not have an infimum at all in this case.

Definition 2.1.8. *Let E be a Riesz space. Subsets of E are assumed to inherit the ordering from E.*

(1) *The linear subspace V of E is called a* ***Riesz subspace*** *of E if for all members $x, y \in V$, the element $x \vee y$ is likewise a member of V.*

(2) *The linear subspace A of E is said to be an* ***ideal*** *in E if it follows from $x \in A$ and $|y| \leq |x|$ that $y \in A$. Sometimes this is called an order ideal to distinguish it from an algebraic ideal in a ring.*

(3) *The ideal B of E is said to be a* ***band*** *whenever $\{x_\alpha\} \subseteq B$ and $0 \leq x_\alpha \uparrow x$ imply $x \in B$ (or, equivalently, if and only if $D \subseteq B_+$ and $D \uparrow x$ imply $x \in B$) .*

Example 2.1.9. (1) The set $E = \{f : f : [0, 1] \to \mathbb{R}, f(x) = ax + b\}$ is a vector subspace, but not a Riesz subspace of $C[0, 1]$.

(2) For all A, a subset of a Riesz space E, A^d is a band (this is an immediate consequence of Theorem 2.1.5).

(3) If c denotes the space of all convergent real sequences, then c is a Riesz subspace of l_∞, but fails to be an ideal.

(4) c_0 is an ideal in l_∞, but not a band.

Let A be a non-empty subset of a Riesz space E. Then ***the ideal generated by A*** is the smallest (with respect to inclusion) ideal that contains A. A moment's thought reveals that this ideal is

$$\left\{ x \in E : \exists\, x_1, \dots, x_n \;\; \textit{and} \;\; \lambda_1, \dots, \lambda_n \in \mathbb{R}^+ \;\; \textit{with} \;\; |x| \leq \sum_{i=1}^{n} \lambda_i |x_i| \right\}.$$

The ideal generated by an element x will be denoted by A_x. By the above

$$A_x = \{y \in E : \exists\, \lambda > 0 \;\; with \;\; |y| \leq \lambda\, |x|\} = \bigcup_{n=1}^{\infty} [-nx, nx].$$

Every ideal of the form A_x is referred to as a ***principal ideal***. Similarly, ***the band generated* by A** is the smallest band that contains A. Such a band always exists (since it is the intersection of the family of bands that contain A, and E is one of them). The band generated by an element x will be denoted by B_x and it is given as following

$$B_x = \{y \in E : |y| \wedge n|x| \uparrow |y|\}.$$

An element $e > 0$ in a Riesz space E is said to be a ***weak order unit*** (or shortly ***weak unit***) whenever the band B_e generated by e coincides with E, i.e., if $x \wedge ne \uparrow x$ holds for each $x \in E_+$. Clearly every element $x \in E_+$ is a weak unit in the band it generates. As a simple characterization of such elements in Archimedean Riesz spaces we have, an element $e > 0$ is a weak unit if and only if $x \perp e$ implies $x = 0$.

Definition 2.1.10. *A band B in a Riesz space E is said to be a* ***projection band*** *if it satisfies $E = B \oplus B^d$.*

In $C[0,1]$, For each real number a such that $0 < a \leq 1$, the ideal

$$B(a) = \{f \in C[0,1] : f(t) = 0, \forall\, t \geq a\}$$

is a band of E but not a projection band.
By the term operator $T : E \to F$ between two vector spaces, we mean a "linear operator", i.e., that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ holds for all $x, y \in E$ and all $\alpha, \beta \in \mathbb{R}$. A linear operator $P : E \to E$ on a Riesz space E is said to be a ***projection*** whenever $P = P^2$.
Now let B be a projection band in a Riesz space E. Thus $E = B \oplus B^d$ holds, and so every element $x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. Then it is easy to see that a projection $P_B : E \to E$ is defined by $P_B(x) := x_1$. Clearly, P_B is a positive projection. Any projection of the form P_B is called a ***band projection***. This type of projections are characterized as follows.

Theorem 2.1.11. $[5, Thm.3.10]$ *For an operator $T : E \to E$ on a Riesz space the* *following statements are equivalent:*

(1) T *is a projection band.*

(2) T *is a projection satisfying* $0 \leq T \leq I$ *(where* I *is the identity operator on* E*).*

A useful comparison property of band projections is described next.

Theorem 2.1.12. $[5, Thm. 3.12]$ *If* A *and* B *are projection bands in a Riesz space, then the following statements are equivalent:*
(1) $A \subseteq B$; (2) $P_A P_B = P_B P_A = P_A$; *and* (3) $P_A \leq P_B$.

Definition 2.1.13. *Let* E *be a Riesz space.* E *is called* ***Dedekind complete*** *whenever every non-empty subset of* E *that is bounded above (bounded below) has supremum (infimum). Similarly,* E *is called* $\sigma-$***Dedekind complete*** *whenever every countable subset of* E *that is bounded above (bounded below) has supremum (infimum).*

The spaces $C_b[0,1]$, L_p $(1 \leq p \leq \infty)$ and l_p $(1 \leq p \leq \infty)$ are all examples of Dedekind complete Riesz spaces whereas $C[0,1]$ is not. From definition 2.1.13, it is clear that every Dedekind complete Riesz space is $\sigma-$Dedekind complete. The converse is not true that is, there exists Riesz spaces which are $\sigma-$Dedekind complete but not Dedekind complete, as an example assume that (Ω, Σ, μ) is a measure space such that $\Sigma \neq \mathcal{P}(\Omega)$, the power set of Ω, and Σ contains the points of Ω. For every $1 \leq p \leq \infty$ let $\mathcal{L}_p(\mu)$ consist of all μ-measurable functions $f : \Omega \to \mathbb{R}$ such that $\int |f|^p \, d\mu < \infty$ if $p < \infty$ or such that ess sup $|f| < \infty$ if $p = \infty$. Ordered pointwise, $\mathcal{L}_p(\mu)$ is $\sigma-$Dedekind complete, but fails to be Dedekind complete.

The following theorem gives some important properties of Dedekind complete and $\sigma-$Dedekind complete Riesz space and it combines [4, Lemma 6.4] , [26, Thm.IV.1.2, Thm.IV.1.4] and [5, Thm.3.8] .

Theorem 2.1.14. (1) *Every* $\sigma-$*Dedekind complete Riesz space is Archimedean.*

(2) *Every ideal of a Dedekind complete (* $\sigma-$*Dedekind complete) Riesz space is also a Dedekind complete (a* $\sigma-$*Dedekind complete) Riesz space.*

(3) *Every band in a Dedekind complete Riesz space is a projection band.*

Note that the third statement of the above theorem provides many examples of projections bands.

Definition 2.1.15. *Let E be a Riesz space , equipped with a norm. The norm in E is called a* ***Riesz norm*** *if $|x| \leq |y|$ in E implies $\|x\| \leq \|y\|$. Any Riesz space equipped with a lattice norm is called a* ***normed Riesz space****. If a norm Riesz space is also norm complete, then it is called a* ***Banach lattice****.*

Example 2.1.16. Here are some examples of normed Riesz spaces and Banach lattices.

(1) The spaces $\mathbb{R}^n$ with their Euclidean norms are all Banach lattices.

(2) If K is a compact space, then the Riesz space $C(K)$ of all continuous real functions on K under the sup norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in K\}$$

is a Banach lattice.

(3) If X is a topological space, then $C_b(X)$, the Riesz space of all bounded real continuous functions on X under the lattice norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

is a Banach lattice.

(4) The Riesz space $C[0,1]$ under the L_1 lattice norm

$$\|f\| = \int_0^1 |f(x)|\, dx$$

is a normed Riesz space, but not Banach lattice.

(5) If X is an arbitrary non-empty set, then the Riesz space $B(X)$ of all bounded real functions on X under the lattice norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

is a Banach lattice.

(6) The Riesz spaces $L_p(\mu)$, $1 \le p < \infty$, (and hence l_p-spaces) are all Banach lattices when equipped with their L_p-norms

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}} .$$

Similarly, the $L_\infty-$spaces are Banach lattices with their essential sup $-$norms.

(7) The Riesz space c_0 of all real null sequences is a Banach lattice under the sup $-$norm

$$\|(x_1, x_2, \dots)\|_\infty = \{\sup |x_n| : n = 1, 2, \dots\}.$$

Definition 2.1.17. *An operator $T : E \to F$ between two Riesz spaces is said to be **order bounded** whenever it maps order bounded subset of E onto order bounded subsets of F. The operator T is said to be **bounded** (or **continuous**) if there exists a $C > 0$ such that $\|Tx\| \le C\|x\|$ for all $x \in E$.*

The space of all order bounded linear functionals on a Riesz space E, denoted by $\widetilde{E}$, is called the ***order dual*** **of E**, i.e.,

$$\widetilde{E} = \{f : E \to \mathbb{R} : f \textit{ is an order bounded linear functional}\}.$$

Similarly, The space of all continuous linear functionals on a normed Riesz space E, denoted by E', is called the ***norm dual of E***, i.e.,

$$E' = \{f : E \to \mathbb{R} : f \textit{ is a continuous linear functional}\}.$$

By the notation E'' we mean the norm dual of E' i.e., $E'' = (E')'$. We will denote by $L(E, F)$ to the space of all bounded operators between normed spaces E and F and by $L(E)$ to the space $L(E, E)$.

The following theorem shows that any normed Riesz space offers a Banach lattice.

Theorem 2.1.18. $[5, Thm.12.1]$ *The norm dual of a normed Riesz space is a Banach lattice.*

Recall that The ***operator norm*** of a bounded operator $T : E \longrightarrow F$ between normed spaces E and F is defined as

$$\|T\| = \sup\{\|Tx\| : x \in E, \|x\| \le 1\},$$

The ***norm*** of an element x in a normed space E is given as

$$\|x\| = \sup\{|y(x)| : y \in E', \|y\| \le 1\}$$

and the ***dual map*** (***adjoint***) of an operator $T \in L(E, F)$ between normed spaces E and F is defined as a linear operator $T' : F' \longrightarrow E'$ such that

$$T'y = y \circ T \quad for \quad all \quad y \in F'.$$

Next we collect a few elementary properties of dual maps.

Theorem 2.1.19. *For a normed spaces, the following statements are valid:*

(a) $(S \circ T)' = T' \circ S'$, *for* $T \in L(E, F)$ *and* $S \in L(F, G)$.

(b) $(Id_E)' = Id_{E'}$ *and if* $E = F$, *then* $(T')^n = (T^n)'$ *for all* $n \in \mathbb{Z}^+ \bigcup \{0\}$.

(c) *If* $T \in L(E)$ *is invertible, then* $(T')^{-1} = (T^{-1})'$ *and so* $(T')^n = (T^n)'$ *for all* $n \in \mathbb{Z}$.

(d) $\|T\| = \|T'\|$.

Proof. **(a)** For $y \in G'$ we get $(S \circ T)'y = y \circ S \circ T = T'(y \circ S) = (T' \circ S')y$.
(b) For For $y \in E'$ we get $(Id_E)'y = y \circ Id_E = y = Id_{E'}y$ and this proved the statement $(T')^n = (T^n)'$ for $n = 0$. we will use induction now to prove the statement for $n \in \mathbb{Z}^+$. For $n = 1$, the statement is clearly true so assume that it is true for n. Now by the assumption and (a) we have $(T')^{n+1} = (T')^n T' = (T^n)'T' = (TT^n)' = (T^{n+1})'$ and the assertion is proved.
(c) Let $y \in E'$ and $x \in E$ then there exists $z \in E$ such that $Tz = x$ and so $(T^{-1})'y(x) = y \circ T^{-1}(x) = y(z)$. On the other hand,

$$(T')^{-1}y(x) = (T')^{-1}y(Tz) = (T')^{-1}(y \circ T)z = (T')^{-1}(T'y)z = y(z) \ so,$$

$(T')^{-1}y = (T^{-1})'y$ for all $y \in E'$. Thus $(T')^{-1} = (T^{-1})'$. In (b) the statement $(T')^n = (T^n)'$ was proved for $n \in \mathbb{Z}^+ \bigcup \{0\}$ so we only need to prove it for $n \in \mathbb{Z}^-$. If $n \in \mathbb{Z}^-$, then $n = -m$ for some $m \in \mathbb{Z}^+$ and so by the first assertion of this part we have

$$(T')^n = (T')^{-m} = ((T')^{-1})^m = ((T^{-1})')^m = ((T^{-1})^m)' = (T^{-m})' = (T^n)'.$$

Hence $(T')^n = (T^n)'$ for all $n \in \mathbb{Z}$.

(d) From the definitions of the norms of elements and operators we have

$$\begin{aligned}\|T'\| &= \sup_{\|y\|\le 1} \{\|T'y\| : y \in E'\}\\ &= \sup_{\|y\|\le 1} \sup_{\|x\|\le 1} \{\|(T'y)x\| : y \in E', x \in E\}\\ &= \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} \{|y(Tx)| : y \in E', x \in E\}\\ &= \sup_{\|x\|\le 1} \{\|Tx\| : x \in E\}\\ &= \|T\|.\end{aligned}$$

□

An important connection between the order and topological structures of Riesz space is provided by the notion of order continuity. In what follows we are mainly interested in Banach lattices with order continuous norms and we will only give some of their examples and their most important characterizations. For more details in Banach lattices with order continuous norms we refer to [5].

Definition 2.1.20. *A lattice norm $\|\cdot\|$ on a Riesz space is said to be **order continuous** if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$. A Banach lattice **has order continuous norm** if its norm is order continuous.*

Typical examples of Banach lattices with order continuous norms are the $L_p(\mu)-$spaces ($1 \le p < \infty$) and (in general) reflexive Banach lattices. On the other hand, the norm on every space of type $C(K)$ fails to be order continuous unless K is finite for example, let $E = c$ be the space of all convergent real sequences and

$$x_n = (a_{m,n})_{m=1}^{\infty} \ \ such\,that \ \ a_{m,n} = \begin{cases} 0 & \text{if } m \leq n \\ 1 & \text{if } m > n \end{cases}$$

We see that $\inf\{x_n : n \in \mathbb{N}\} = 0$ and $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

We state now the most important characterizations of Banach lattices with order continuous norms collected from [5] and [20].

Theorem 2.1.21. *The following assertions are equivalent.*

(1) *The norm on E is order continuous.*

(2) *If $0 < x_n \uparrow x$ holds in E, then $(x_n)_n$ is a norm Cauchy sequence.*

(3) *E is Dedekind complete (σ-Dedekind complete) satisfying $\|x_n\| \to 0$ as $n \to \infty$ for any sequence $(x_n)_n \subset E_+$ with $x_n \downarrow 0$.*

(4) *Every monotone order bounded sequence of E is norm convergent.*

(5) *Every disjoint order bounded sequence of E_+ is norm convergent to zero.*

(6) *E is an ideal in E''.*

(7) *Every order interval of E is weakly compact.*

(8) *Every Riesz subspace isomorphic to c_0 is the range of a positive projection.*

(9) *Every closed ideal of E is a band.*

(10) *Every closed ideal of E is the range of a positive projection.*

A subset C of a Banach lattice E is said to be ***almost order bounded*** if for each $\epsilon > 0$ there is $x_\epsilon \in E_+$ such that $C \subseteq [-x_\epsilon, x_\epsilon] + \epsilon B_E$, where $B_E$is the closed unit ball of E.

Theorem 2.1.22. [22, Lemma 3.2] *A Banach lattice E has order continuous norm if and only if every almost order bounded set in E is relatively weakly compact.*

An element $u > 0$ in a normed Riesz space is said to be a ***quasi-interior point*** whenever the ideal A_u generated by u is norm dense in E. As an example of such elements we have, the constant function 1 on $[0, 1]$ is a quasi-interior point in $L_1[0, 1]$. The quasi-interior points are characterized as follows.

Theorem 2.1.23. $[5, Thm.15.3]$ *For a positive element u in a normed Riesz space E the following statements are equivalent:*

(1) *u is a quasi-interior point.*

(2) *For each $x \in E$ we have $\lim_{n\to\infty} \|x - x \wedge nu\| = 0$.*

(3) *u is strictly positive on E'.*

As an immediate consequence of this theorem, we have every quasi-interior point in a normed Riesz space is a weak unit. The other implication is not true, that is , there exist a weak unit which is not a quasi-interior point, for example the function $e(t) = t$ is a weak unit in $C[0, 1]$ but it fails to be a quasi-interior point. However, the converse implication holds in Banach lattices with order continuous norm, namely we have

Theorem 2.1.24. $[3, Prob.4.2.4]$ *In a Banach lattices with order continuous norm, a positive element is a quasi-interior point if and only if it is a weak unit.*

The existence of weak units in a Banach lattices with order continuous norm ensures the existence of strictly positive functionals in E' as the following theorem says.

Theorem 2.1.25. $[18, Thm.1.b.15]$ *In any Banach lattices with order continuous norm which has a weak unit $e > 0$, there exists a functional $e' > 0$ in E' such that $e'(|x|) = 0$ implies $x = 0$.*

We now turn our attention to the important class of KB-spaces which will be the mean object of Section 4.3.

Definition 2.1.26. *A Banach lattice E is said to be a **KB-space** (Kantrovič-Banach space) whenever every increasing norm bounded sequence of E_+ is norm convergent.*

Note that a Banach lattice is a KB-space if and only if $0 \leq x_\alpha \uparrow$ and $\sup\{\|x_\alpha\|\} < \infty$ imply that the net x_α is norm convergent, this easily follows from the fact that if a net $0 < x_\alpha \uparrow$ is not Cauchy, then there exist some $\epsilon > 0$ and a sequence $\{\alpha_n\}$ of indices with $\alpha_n \uparrow$ and $\|x_{\alpha_{n+1}} - x_{\alpha_n}\| > \epsilon$ for all n. In particular, it follows that every KB-space has order continuous norm (see Theorem 2.1.21($1 \Leftrightarrow 2$)). The converse need not be true, for example the norm on c_0 is order continuous, but c_0 fails to be a KB-space as KB-space characterization theorem shows.

Now let $\{x_n\}$ be a sequence in a Banach lattice E satisfying $0 \leq x_n \uparrow$ and $\sup\{\|x_n\|\} < \infty$. Then $0 \leq x_n \uparrow x''$ holds in E'' for some $x'' \in E''$. In case E is reflexive, x'' belongs to E, and the order continuity of the norm implies that $\{x_n\}$ is norm convergent. Therefore, reflexive Banach lattices are examples of KB-spaces. The following theorem gives a characterizations of KB-space.

Theorem 2.1.27. [20, *Thm.*2.4.12] *The following assertions are equivalent.*

(1) *E is a KB-space.*

(2) *E is a band in E''.*

(3) *E does not contain any Riesz subspace isomorphic to c_0.*

As an easy remark on this characterization and the fact that for any Banach lattice E, E'' is Dedekind complete, we get that every KB-space E is a projection band in E''.

2.2 Positive operators

In this section, we discuss some basic properties of positive operators that will be used in next chapters.

Definition 2.2.1. *An operator $T : E \to F$ between two Riesz spaces is said to be* ***positive*** *(in symbols, $T \geq 0$ or $0 \leq T$) whenever it maps positive vectors to positive vectors. That is, T is positive if $x \geq 0$ in E implies $T(x) \geq 0$ in F. T is said to be* ***strictly positive*** *whenever $T(x) > 0$ for all $x > 0$.*

The positive operators are characterized by means of their additivity property on the positive cone [6, Lemma 1.1.64].

Theorem 2.2.2 (Kantorovič). *Suppose that E and F are two Riesz spaces with F Archimedean. Assume also that $T : E_+ \to F_+$ is additive, that is, $T(x + y) = T(x) + T(y)$ holds for all $x, y \in E_+$. Then T has a unique extension to a positive operator from E to F.*

An important property of positive operators between Banach lattices is that they are necessarily continuous.

Theorem 2.2.3. $[5, Thm. 12.3]$ *Every positive operator from a Banach lattice to a normed Riesz space is continuous.*

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a Riesz space are said to be ***equivalent*** whenever there exist constants $K, M > 0$ satisfying

$$K\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \quad for\ all \quad x \in E$$

As a direct corollary of Theorem 2.2.3 we have

Corollary 2.2.4. $[5, Cor. 12.4]$ *All lattice norms that make a Riesz space a Banach lattice are equivalent.*

Since order bounder linear functionals on Riesz space can be written as a difference of two positive linear functionals, the following result should be immediate from the previous theorem.

Corollary 2.2.5. $[5, Cor. 12.5]$ *The norm dual of a Banach lattice E coincides with its order dual, i.e., $\widetilde{E} = E'$.*

The following theorem of S. S. Schaefer exhibits a remarkable property of positive projections defined on Banach lattices.

Theorem 2.2.6. [23, Prop. III.11.5] *Let P be a positive projection in $L(E)$, where E is any Banach lattice. The range $P(E)$ is a Riesz space under the order induced by E and a Banach lattice under a norm equivalent to the norm induced by E. If P is strictly positive, then $P(E)$ is a Riesz subspace of E.*

Now we introduce a special class of positive operators which preserve the lattice operations.

Definition 2.2.7. *An operator $T : E \to F$ between two Riesz spaces is said to be a **Riesz homomorphism** whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$.*

The following theorem gives some properties of Riesz homomorphisms.

Theorem 2.2.8. $[19, Thm.18.3]$ *For any Riesz homomorphisms $T : E \to F$ we have:*

(1) *T is a positive operator.*

(2) *$T(E)$ is a Riesz subspace.*

(3) *The set $\{x \in E : T(x) = 0\}$ is an ideal.*

Example 2.2.9. Here are some examples of Riesz homomorphisms.

(1) Every band projection on a Riesz space is a Riesz homomorphism.

(2) Consider two compact Hausdorff spaces K_1 and K_2, let $\varphi : K_2 \to K_1$ be continuous, and let $g \in C(K_2)_+$. We define

$$T : C(K_1) \to C(K_2) \ \ by \ \ Tf = g \cdot f \circ \varphi.$$

It easy to show that T is a Riesz homomorphism.

The elementary characterization of operators that are Riesz homomorphisms are presented next.

Theorem 2.2.10. $[5, Thm.7.2]$ *For an operator $T : E \to F$ between two Riesz spaces, the following statements are equivalent*

(1) *T is a Riesz homomorphism.*

(2) *$T(x^+) = (Tx)^+$ holds for all $x \in E$.*

(3) *$T(x \wedge y) = T(x) \wedge T(y)$ holds for all $x, y \in E$.*

(4) *If $x \wedge y = 0$ holds in E, then $T(x) \wedge T(y) = 0$ holds in F.*

(5) $T(|x|) = |T(x)|$ *holds for all* $x \in E$.

A Riesz homomorphism which is in addition one-one is referred to as a ***Riesz isomorphism***. Two Riesz spaces E and F are called ***Riesz isomorphic*** whenever there exists a Riesz isomorphism from E onto F. An operator $T : E \to F$ between two normed Riesz spaces is said to be an ***isometry*** whenever $\|T(x)\| = \|x\|$ holds for all $x \in E$.

Among the positive operators that are onto, the Riesz isomorphisms are characterized as follows.

Theorem 2.2.11. [5, *Thm.*7.3] *Assume that an operator $T : E \to F$ between two Riesz spaces is one-to-one and onto. Then T is a Riesz isomorphism if and only if T and T^{-1} are both positive operators.*

We give now the definition of AL-spaces which play a significant role in analysis.

Definition 2.2.12. *A Banach lattice E is said to be an **AL-space** if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E_+$ with $x \wedge y = 0$.*

$L_1(\mu)-$spaces are examples of these spaces. In what follows, we state Kakutani's theorem [2, Thm.3.5] which characterizes the vectors of AL-space as a functions in some familiar function space.

Theorem 2.2.13 (Kakutani). *A Banach lattice is an AL-space if and only if it is isometrically Riesz isomorphic to an $L_1(\mu)-$space.*

The following theorem gives a characterization of AL-spaces and shows their Dedekind completeness property.

Theorem 2.2.14. *If E is a Banach lattice, then*

(1) *E is an AL-space if and only if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E_+$.*

(2) *If E is an AL-space, then E is Dedekind complete.*

Proof. **(1)** If $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E_+$, then E is an AL-space. Now assume that E is an AL-space. Then by Kakutani's theorem E is isometrically Riesz isomorphic to some $L_1(\mu)-$space. Now notice that if $0 \leq x, y \in L_1(\mu)$, then

$$\|x+y\|_1 = \int_E (x+y)\, d\mu = \int_E x\, d\mu + \int_E y\, d\mu = \|x\|_1 + \|y\|_1.$$

(2) The proof of this assertion follows from Kakutani's theorem and the fact that every $L_1(\mu)$-space is Dedekind complete. □

It should be noted that every AL-space E has order continuous norm. Indeed, if $\{x_n\} \subseteq [0, x]$ is a disjoint sequence, then from the inequality

$$\sum_{n=1}^{k} \|x_n\| = \|\sum_{n=1}^{k} x_n\| = \| \vee_{n=1}^{k} x_n\| \leq \|x\|,$$

it follows that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, and so $\lim_{n\to\infty} \|x_n\| = 0$ holds. By Theorem 2.1.21 ($1 \Leftrightarrow 5$), the norm of E is order continuous. Moreover, AL-spaces are also examples of KB-spaces. For, if $\{x_n\}$ is a sequence in a Banach lattice E satisfying $0 \leq x_n \uparrow$ and $\sup\{\|x_n\|\} < \infty$. Then $0 \leq x_n \uparrow x''$ holds in E'' for some $x'' \in E''$. In case E is an AL-space, then so is E'', and so E'' has order continuous norm, from which it follows that $\{x_n\}$ is norm convergent.

The following theorem gives a condition under which Archimedean Riesz spaces become finite dimensional.

Theorem 2.2.15 (Judin). [6, *Exer.*13, *p.*46] *If every subset of pairwise disjoint elements in an Archimedean Riesz space E is finite, then E is Riesz isomorphic to some $\mathbb{R}^n$.*

2.3 The mean ergodic theorem

This section is devoted to state Eberlein's mean ergodic theorem which will play an important role in Chapter 4. For this purpose, we need some basic definitions and lemmata which are important not only for stating the mean ergodic theorem but also for many other discussions throughout this thesis.
Let T be an operator on a Banach space X, we define its ***n-th Cesàro mean*** (or

average) by

$$\mathcal{A}_n^T = \frac{1}{n}\sum_{k=0}^{n-1} T^k.$$

It is easy to see that if T is a continuous operator, then so is $\mathcal{A}_n^T$ for all $n \in \mathbb{N}$. If, in addition, X is a Banach lattice and T is a positive operator, then each $\mathcal{A}_n^T$ is also a positive operator.

The next lemma presents some elementary identities for the averaging operators.

Lemma 2.3.1. *For a linear operator T on a Banach space X and arbitrary numbers n, j, k and i we have*

(1) $\mathcal{A}_n^T T = T\mathcal{A}_n^T = \frac{n+1}{n}\mathcal{A}_{n+1}^T - \frac{1}{n}I.$

(2) $\mathcal{A}_{nk}^T = \frac{1}{k}\left(\mathcal{A}_n^T + T^n\mathcal{A}_n^T + T^{2n}\mathcal{A}_n^T + \ldots + T^{(k-1)n}\mathcal{A}_n^T\right).$

(3) $\mathcal{A}_{j+i}^T - \mathcal{A}_j^T = (j+i)^{-1}(T^j + T^{j+1} + \ldots + T^{j+i-1}) - i(j+i)^{-1}\mathcal{A}_j^T.$

Proof. **(1)** From the definition of Cesàro means of an operator we have

$$\begin{aligned}
\mathcal{A}_n^T T &= \frac{1}{n}\sum_{k=0}^{n-1} T^k T = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})T \\
&= \frac{1}{n}(T + T^2 + T^3 + \cdots + T^n) = \frac{1}{n}T(I + T + T^2 + \cdots + T^{n-1}) \\
&= T\left(\frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})\right) \\
&= T\frac{1}{n}\sum_{k=0}^{n-1} T^k \\
&= T\mathcal{A}_n^T.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{n+1}{n}\mathcal{A}_{n+1}^T - T\mathcal{A}_n^T &= \frac{1}{n}\sum_{k=0}^{n} T^k - T\frac{1}{n}\sum_{k=0}^{n-1} T^k \\
&= \frac{1}{n}(I + T + T^2 + \cdots + T^n) - \frac{1}{n}(T + T^2 + T^3 + \ldots + T^n) \\
&= \frac{1}{n}I.
\end{aligned}$$

That is, $\mathcal{A}_n^T T = T\mathcal{A}_n^T = \frac{n+1}{n}\mathcal{A}_{n+1}^T - \frac{1}{n}I$.

$$\textbf{(2)}\quad \begin{aligned} \mathcal{A}_{nk}^T &= \frac{1}{nk}\sum_{j=0}^{nk-1} T^j \\ &= \frac{1}{k}\left(\frac{1}{n}\left[I + T + T^2 + \cdots + T^{n-1} + T^n + \cdots + T^{2n-1} + \cdots + T^{kn-1}\right]\right) \\ &= \frac{1}{k}\left(\frac{1}{n}\left[\sum_{j=0}^{n-1} T^j + \sum_{j=0}^{n-1} T^{j+n} + \sum_{j=0}^{n-1} T^{j+2n} + \cdots + \cdots + \sum_{j=0}^{n-1} T^{j+(k-1)n}\right]\right) \\ &= \frac{1}{k}\left(\frac{1}{n}\left[\sum_{j=0}^{n-1} T^j + T^n\sum_{j=0}^{n-1} T^j + T^{2n}\sum_{j=0}^{n-1} T^j + \cdots + \cdots + T^{(k-1)n}\sum_{j=0}^{n-1} T^j\right]\right) \\ &= \frac{1}{k}(\mathcal{A}_n^T + T^n\mathcal{A}_n^T + T^{2n}\mathcal{A}_n^T + \ldots + T^{(k-1)n}\mathcal{A}_n^T). \end{aligned}$$

$$\textbf{(3)}\quad \begin{aligned} \mathcal{A}_{j+i}^T - \mathcal{A}_j^T &= \frac{1}{j+i}\sum_{k=0}^{j+i-1} T^k - \frac{1}{j}\sum_{k=0}^{j-1} T^k = \frac{1}{j+i}\sum_{k=0}^{(j-1)+i} T^k - \frac{1}{j}\sum_{k=1}^{j-1} T^k \\ &= \frac{1}{j+i}\sum_{k=0}^{j-1} T^k + \frac{1}{j+i}\sum_{k=1}^{i} T^{j-1+k} - \frac{1}{j}\sum_{k=0}^{j-1} T^k \\ &= \frac{j-(j+i)}{j(j+i)}\sum_{k=0}^{j-1} T^k + \frac{1}{j+i}\sum_{k=1}^{i} T^{j-1+k} \\ &= \frac{-i}{j(j+i)}\sum_{k=0}^{j-1} T^k + \frac{1}{j+i}\sum_{k=1}^{i} T^{j-1+k} \\ &= (j+i)^{-1}(T^j + T^{j+1} + \ldots + T^{j+i-1}) - i(j+i)^{-1}\mathcal{A}_j^T. \end{aligned}$$

□

Definition 2.3.2. *An operator T on a Banach space X is called* ***power bounded*** *whenever* $\sup_{n\geq 0}\|T^n\| < \infty$.

Example 2.3.3. (1) All contractions are power bounded operators.
(2) Consider the Volterra operator

$$(Vf)(t) = \int_0^t f(s)\,ds.$$

on $L_2(0,1)$. Define $A = (I+V)^{-1}$. Then $\|A^n\| = 1$ for all $n \in \mathbb{N}$, that is A is power bounded.

The following lemma provide some elementary properties of power bounded operators.

Lemma 2.3.4. *If T is a power bounded operator on a Banach space X.Then*

(1) $\sup_{n\geq 0} \left\|\mathcal{A}_n^T\right\| < \infty$.

(2) $\lim_{n\to\infty} \frac{1}{n}T^{n-1}x = 0$ *for all $x \in X$.*

Proof. Since T is power bounded, there exists $M \in \mathbb{R}$ such that $\sup_{k\geq 0} \|T^k\| \leq M$
(1) From the Cesàro means definition we have

$$\begin{aligned}\sup_{n\geq 0} \left\|\mathcal{A}_n^T\right\| &= \sup_{n\geq 0} \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^k\right\| \\ &\leq \sup_{n\geq 0} \frac{1}{n}\sum_{k=0}^{n-1} \|T^k\| \\ &\leq \sup_{n\geq 0} \frac{1}{n}\sum_{k=0}^{n-1} M = M < \infty\end{aligned}$$

(2) For the second assertion we have

$$\|\frac{1}{n}T^{n-1}\| = \frac{1}{n}\|T^{n-1}\| \leq \frac{1}{n}\sup_{k\geq 0}\|T^k\| \leq \frac{1}{n}M \to 0 \quad as \quad n\to\infty.$$

So, $\lim_{n\to\infty} \frac{1}{n}T^{n-1}x = 0$ for all $x \in X$. □

Definition 2.3.5. *An operator T on a Banach space X is called* ***Cesàro bounded*** *whenever* $\sup_{n\geq 0} \left\|\mathcal{A}_n^T\right\| < \infty$.

Remark 2.3.6. (1) Every power bounded operator is Cesàro bounded (see Lemma 2.3.4(1)). The converse is not true and for counterexamples the reader may consult [25].
(2) There exist Cesàro bounded operators which may fail to satisfy (2) of Lemma 2.3.4, as a counterexample, one may consider the Assani's well-known example

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

Definition 2.3.7. *An operator T on a Banach space X is called **mean ergodic** whenever the sequence $\left(\mathcal{A}_n^T x\right)_n$ is norm convergent for all $x \in X$.*

If X is a Banach space and $T \in L(X)$ is mean ergodic, then we introduce the operator $P_T : X \to X$ via

$$P_T(x) = \lim_{n\to\infty} \mathcal{A}_n^T x.$$

The following theorem shows that P_T is a projection and gives some of it's basic properties. We call this projection P_T as the ***mean ergodic projection***. For any operator T on a Banach space X, we denote by Fix(T) it's fixed space. That is, Fix(T) $= \{x \in X : Tx = x\}$.

Theorem 2.3.8. [15, *Thm.*2.1.3] *If $T \in L(X)$ is mean ergodic, then P_T is a continuous projection whose range is* Fix(T). *Moreover, we have $P_T T = T P_T = P_T$.*

Eberlein's well-known mean ergodic theorem (cf.[15, Thm.2.1.5]) gives the major characterizations of mean ergodic operator semigroups and here we restrict ourselves to it's special case [15, Thm.2.1.1].

Theorem 2.3.9 (Eberlein). *Let T be a Cesàro bounded operator in a Banach space X. For any $x \in X$ satisfying $\lim_{n\to\infty} \frac{1}{n} T^{n-1} x = 0$ and for any $y \in X$ the following assertions are equivalent:*

(i) *$Ty = y$ and $y \in$ closed convex hull of $\{x, Tx, T^2 x, \ldots\}$.*

(ii) *$y = \lim_{n\to\infty} \mathcal{A}_n^T x$.*

(iii) *$y = w - \lim_{n\to\infty} \mathcal{A}_n^T x$.*

(iv) *y is a weak cluster point of the sequence $(\mathcal{A}_n^T x)_{n\in\mathbb{N}}$.*

Theorem 2.3.10. [15, *Thm.*2.1.2] *Every power bounded operator in a reflexive Banach space is mean ergodic.*

Example 2.3.11. (1) Left and right shift operators on L_p-spaces where $1 < p < \infty$ are mean ergodic (cf. Theorem 2.3.10).
(2) The operator $R : c_0 \to c_0$, $R((a_n)_n) = (a_1, a_1, a_2, \ldots)$ *for every* $(a_n)_n \in c_0$, is not mean ergodic. For, let $u \in c_0$, $u = (1, 0, 0, \ldots)$. Then $\left(\mathcal{A}_n^R u\right)_n$ does not converge

in c_0. Thus R is not mean ergodic.
(3) The left shift operator $L : l_\infty \to l_\infty$, $L((a_n)_n) = (a_2, a_3, a_4, \ldots)$ for every $(a_n)_n \in l_\infty$, is not mean ergodic. For, let $u = (a_n)_n \in l_\infty$ be defined as follows

$$a_n = \begin{cases} 1, & \text{if n=1 or if there exists } k \in \mathbb{N}\bigcup\{0\} \text{ such that} \\ & 1 + 2 + 2^2 + \cdots + 2^{3k} < n \leq 1 + 2 + 2^2 + \cdots + 2^{3k+1}; \\ 0, & \text{otherwise (i.e., if there exists } k \in \mathbb{N}\bigcup\{0\} \text{ such that} \\ & 1 + 2 + 2^2 + \cdots + 2^{3k+1} < n \leq 1 + 2 + 2^2 + \cdots + 2^{3k+3}). \end{cases}$$

If we note for every $m \in \mathbb{N}$

$$\frac{1}{m} \sum_{i=0}^{m-1} L^i u = (b_1^{(m)}, b_2^{(m)}, \cdots),$$

then it follows that for every $k \in \mathbb{N}$

$$b^{(\sum_{i=0}^{3k+1} 2^i)} \geq \frac{2^{3k+1}}{\sum_{i=0}^{3k+1} 2^i} = \frac{2^{3k+1}}{2^{3k+2} - 1} \geq \frac{1}{2}$$

and

$$b^{(\sum_{i=0}^{3k+1} 2^i)} \leq \frac{\sum_{i=0}^{3k+1} 2^i}{2^{3k+2} + 2^{3k+3}} = \frac{2^{3k+2} - 1}{3 \cdot 2^{3k+2}} \leq \frac{1}{3}.$$

It follows that L is not mean ergodic.
(4) The operator $P : l_1 \to l_1$, $P((a_n)_n) = (0, a_1, a_2, \ldots)$ for every $(a_n)_n \in l_1$, is not mean ergodic. For, the dual map of P is the operator $L : l_\infty \to l_\infty$ defined in Example (3) above. Let $u \in l_\infty$ be the sequence defined in Example (3) and let $v \in l_1$ be the sequence $v = (1, 0, 0, \ldots)$. It follows that the sequence $\left(\mathcal{A}_n^P v\right)_n$ does not converge in l_1 since for every $n \in \mathbb{N}$,

$$\left\langle u, \mathcal{A}_n^P v \right\rangle = \left\langle \mathcal{A}_n^L u, v \right\rangle$$

and since the sequence $\left(\left\langle \mathcal{A}_n^L u, v \right\rangle\right)_n$ does not converge in $\mathbb{R}$. It follows that P is not mean ergodic.
(5) Denote by e_k the element of c_0 such that its k-th coordinate is equal to 1, and all other coordinates are zero. Fix η, $0 < \eta < 1$, and define the operator $S_\eta : c_0 \to c_0$ as

$$S_\eta e_k = \begin{cases} e_1 + \eta\, e_2 & k = 1 \\ e_{k+1} & k > 1 \end{cases}$$

and let $T_\eta := (I + S_\eta)/2$. The sequence $(\mathcal{A}_n^{T_\eta} e_1)_n$ does not converge to any element of c_0. Hence T_η is not mean ergodic.

Note : Examples (2), (3) and (4) played an important role in [29], in which it was shown that if a Banach lattice E has a sublattice which is lattice isomorphic to l_∞ , then there exists a (power bounded positive) operator $T : E \to E$ which is not mean ergodic, and if a $\sigma-$Dedekind complete Banach lattice E has a sublattice which is lattice isomorphic to l_1 (or c_0), then also a (power bounded positive) operator $T : E \to E$ which is not mean ergodic exists.

Chapter 3

The renorming problem in banach lattices

3.1 Statement of the problem

Isometries are, in the most general sense, transformations which preserve distance between elements. Such transformations are basic in the study of geometry which is concerned with rigid motions and properties preserved by them. In particular, positive isometries of Banach lattices possess many attractive properties. For instance, due to the well-known result of J. Lamperti [17], positive isometries on L_p-spaces $(1 \leq p < \infty)$ have representations as weighted shift operators with positive weight.

The renorming problem that we are going to discuss in this section states the following.

If T is a positive operator on a Banach lattice X, when one can renorm the Banach lattice X to make T invertible isometry?

This question has a trivial answer in the Banach space setting, namely, the necessary and the sufficient condition for an operator T in a Banach space X to be an invertible isometry with respect to some equivalent norm is that T be ***doubly power bounded***, i.e., $\sup \{\|T^n\| : n \in \mathbb{Z}\} < \infty$, in this case an equivalent norm $\|\cdot\|_T$ under which T

and T^{-1} are isometries can be defined as :

$$\|x\|_T := \sup\{\|T^n x\| : n \in \mathbb{Z}\} \qquad (\forall x \in X). \tag{1}$$

However, the situation is different if we consider a positive operator on a Banach lattice that is, the norm in (1) may not be a Banach lattice norm, since we do not know if $T^{-1} \geq 0$. If this occurs, then according to Theorem 2.2.11 the operator T is a lattice automorphism and this will make the norm defined in (1) really an equivalent lattice norm. For

$$\begin{aligned}
|x| \leq |y| &\Longrightarrow T^n|x| \leq T^n|y| && (\text{ since } T^n \text{ is positive })\\
&\Longrightarrow |T^n x| \leq |T^n y| && (\text{ since } T^n \text{ is a Riesz automorphism })\\
&\Longrightarrow \|T^n x\| \leq \|T^n y\| && (\text{ since } \|.\| \text{ is a Riesz norm })\\
&\Longrightarrow \|x\|_T \leq \|y\|_T && (\text{ by the definition of } \|.\|_T)
\end{aligned}$$

So, in the Banach lattice case the doubly power boundedness is not enough for a positive operator to be invertible isometry with respect to the norm in (1), that is the positivity of T^{-1} is essential here.

It is well-known (see for example [22, Lemma 2.4]) that the positivity of T^{-1} occurs for any doubly power bounded positive operator defined on a finite dimensional Banach lattice, that is :

Theorem 3.1.1. *Every positive doubly power bounded operator on a finite dimensional Banach lattice has a positive inverse.*

The proof of this result depends on the very important Jacobs-Deleeuw-Glicksberg Decomposition Theorem [15, Thm.2.4.4] for which we need some preliminaries.

Definition 3.1.2. *Let X be a Banach space. A subset $\mathcal{T} = \{T_t : t \in \mathbb{R}_+\}$ of $L(X)$ is said to be (**one-parameter**) **semigroup** of bounded linear operators on X, usually written $(T_t)_{t\geq 0}$, whenever it satisfies*
(1) $T_0 = Id$, the identity operator on X.
(2) $T_{s+t} = T_s \circ T_t$ for all $t, s \in \mathbb{R}_+$.
*A semigroup $\mathcal{T}$ is said to be **abelian** whenever $TS = ST$ for all $T, S \in \mathcal{T}$.*

Example 3.1.3. (1) Let T be a power bounded operator on a Banach space E, i.e., $\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty$. Then an abelian semigroup of bounded operators on E may be defined as $(T^n)_{n\geq 0}$ and it is called the discrete semigroup.

(2) It follows from elementary operator theory that for every bounded operator A on a Banach space, the sum

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} := e^{tA}$$

exists and determines a semigroup $(e^{tA})_{t\geq 0}$.

(3) Let X be one of the following function spaces $C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity, or $L_p(\mathbb{R})$ for $1 \leq p < \infty$. Define T_t to be the translation operator

$$T_t f(x) := f(x+t)$$

for $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Then $(T_t)_{t\geq 0}$ is a semigroup of bounded operators on X.

Definition 3.1.4. *A semigroup $\mathcal{T}$ of bounded operators on a Banach space X is said to be **almost periodic** (resp. **weakly almost periodic**) if for all $x \in X$, the orbit $\mathcal{T}x = \{Tx : T \in \mathcal{T}\}$ is relatively compact (resp. relatively weakly compact).*

Let $\widetilde{\mathcal{T}}$ denote the closure of the semigroup $\mathcal{T}$ in the weak operator topology and denote by $\widetilde{\mathcal{T}x}$ for the weak closure of $\mathcal{T}x$ for all $x \in X$. The sets of ***reversible*** and ***flight*** vectors of the semigroup $\mathcal{T}$ are defined as

$$X_{rev}(\mathcal{T}) = \{x \in X \ : \ \forall\, T \in \widetilde{\mathcal{T}}, \ \ \exists\, R \in \widetilde{\mathcal{T}}\,; \ \ RTx = x\}$$

and

$$X_{fl}(\mathcal{T}) = \{x \in X \ : \ \exists\, S \in \widetilde{\mathcal{T}} \ \ ; Sx = 0\}$$

We are now in a position to formulate the Jacobs-Deleeuw-Glicksberg Decomposition Theorem [15, Thm.2.4.4].

Theorem 3.1.5 (Jacobs-Deleeuw-Glicksberg). *Given a weakly almost periodic semigroup $\mathcal{T} = (T_\tau)_{\tau\in J}$ in a Banach space X, then X can be decomposed into the* *direct sum* $X = X_{fl}(\mathcal{T}) \oplus X_{rev}(\mathcal{T})$ *and the restriction of* $\widetilde{\mathcal{T}}$ *to* $X_{rev}(\mathcal{T})$ *is a group. Moreover if* $\mathcal{T} = (T^n)_{n\geq 0}$ *is a discrete almost periodic semigroup, then*

$$X_{fl}(\mathcal{T}) = \{x \in X : \lim_{t\to\infty} \|T^n x\| = 0\}.$$

Proof of Theorem 3.1.1:
Let E be a finite dimensional Banach lattice, T be a positive doubly power bounded operator on E and $\mathcal{T} = (T^n)_{n\in\mathbb{N}}$, then the semigroup $\mathcal{T}$ is almost periodic since $\dim(E) < \infty$ and $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. By Jacobs – Deleeuw – Glicksberg's theorem,

$$E = E_{rev}(\mathcal{T}) \oplus E_{fl}(\mathcal{T}) \;\;\&\;\; E_{fl}(\mathcal{T}) = \{x \in E : \lim_{n\to\infty} \|T^n x\| = 0\}.$$

The last condition together with the doubly power boundedness of T imply that $E_{fl}(\mathcal{T}) = \{0\}$, and hence $E = E_{rev}(\mathcal{T})$. Then T^{-1} belongs to the closure (in the wo-topology) of the set $(T^n)_{n\in\mathbb{N}}$ of positive operators. Then T^{-1} is positive as well. $\square$

Another case providing the positivity for the inverse of a positive doubly power bounded operator was given by Abromovich in [1]. Indeed in the paper [1] a slightly more general result was proved, but we restrict our attention only on the following particle case of it.

Theorem 3.1.6 (Abromovich). *Any surjective positive isometry on a Banach lattice has a positive inverse.*

Proof. Let T be a surjective positive isometry on a Banach lattice E. We only have to show that $T(E_+) \supseteq E_+$. Assume that is not true, then there exists $y \in E_+$ such that $y \notin T(E_+)$. Let $\|y\| = 1$, now by the surjectivity of T there exists $x \in E$ such that $Tx = y$. Since T is an isometry, $\|x\| = 1$ also we have $x \notin E_+$. Consequently, $x^- > 0$ and $x^+ \neq 0$ (for, if $x^+ = 0$ then $y = Tx = -T(x^-) < 0$ which contradicts $y \in E_+$). Set $y_1 = T(x^+)$ and $y_2 = T(x^-)$, then $y_1 > 0$, $y_2 > 0$ and $y = y_1 - y_2 > 0$. Moreover, we have

$$\|y_1 + y_2\| = \|T(x^+ + x^-)\| = \|x^+ + x^-\| = \||x|\| = \|x\| = 1 \tag{2}$$

Now we will prove by induction that

$$\|x^+ + kx^-\| = 1 \quad for \quad all \quad k \in \mathbb{N} \tag{3}$$

For $k = 1$, (3) follows from (2). Let (3) be true for k. We prove the statement for $k+1$. Let us consider the element $x^+ - (k+1)x^-$. Then

$$T(x^+ - (k+1)x^-) = y_1 - (k+1)y_2 \quad and \quad -(y_1 + ky_2) \leq y_1 - (k+1)y_2 \leq y_1 + ky_2$$

(for, $(k+1)y_2 \geq ky_2 \Rightarrow -(k+1)y_2 \leq -ky_2 \Rightarrow y_1 - (k+1)y_2 \leq y_1 - ky_2 \leq y_1 + ky_2$ *also* $y_1 - (k+1)y_2 + y_1 + ky_2 = y_1 - y_2 + y_1 = y + y_1 \geq 0$). Then

$$|y_1 - (k+1)y_2| \leq y_1 + ky_2 \tag{4}$$

and

$$\begin{aligned}
\|x^+ + (k+1)x^-\| &= \|x^+ - (k+1)x^-\| = \|T^{-1}(y_1 - (k+1)y_2)\| \\
&= \|y_1 - (k+1)y_2\| \\
&\leq \|y_1 + ky_2\| \qquad\qquad (By\ (4)) \\
&= \|T(x^+ + kx^-)\| \\
&= \|x^+ + kx^-\| = 1
\end{aligned}$$

also $1 = \|x\| = \|x^+ + x^-\| \leq \|x^+ + (k+1)x^-\|$, so $\|x^+ + (k+1)x^-\| = 1$
Now for all $k \in \mathbb{N}$, $k\|x^-\| = \|kx^-\| \leq \|x^+ + kx^-\| = 1 \Rightarrow \|x^-\| \leq \frac{1}{k}$ and as $k \to \infty$ we get $\|x^-\| \leq 0 \Rightarrow x^- = 0$ which is a contradiction. Thus there is no $y \in E_+$ such that $y \notin T(E_+)$, that is $T(E_+) \supseteq E_+$ and the proof is finished. □

Because of Theorem 3.1.1 and Theorem 3.1.6 the renorming problem stated above will be reduced to the case of positive doubly power bounded operators, which are not surjective isometries, on infinite dimensional Banach lattices. The discussion above motivates the following substantial question which appeared in [10]:

Is T^{-1} positive for any positive doubly power bounded operator on a Banach lattice?

i.e can the Abromovich's theorem be generalized to the case of doubly power bounded positive operators?

In [10] a negative answer was given for the preceding question by means of a counterexample on a special L_1- space as following :

Theorem 3.1.7 (Emel'yanov). *Given $\omega \notin \mathbb{R}$, take $\Omega = \mathbb{R} \bigcup \{\omega\}$ and let a measure μ on the Borel algebra $\beta = \beta(\Omega)$ be defined as the Lebesgue measure on $\beta(\mathbb{R})$ and $\mu(\omega) = 1$. Then for any $\varepsilon > 0$ there exists a positive operator on $L_1(\Omega, \beta, \mu)$ with nonpositive inverse that satisfies $\sup_{n \in \mathbb{Z}} \|T^n\| \leq 1 + \epsilon$.*

Proof. Let $\varepsilon > 0$. Consider a measure preserving automorphism S in Ω defined by $S(\omega) = \omega$ and $S(t) = t - 1$, whenever $t \in \mathbb{R}$, and define a positive operator T in $L_1(\Omega, \beta, \mu)$ by

$$Tf := f \circ S + \varepsilon \cdot \Big[\int_0^1 f d\mu \Big] \cdot \chi_{\{\omega\}},$$

where χ_A, as usual, is the indicator function of a subset A. Then it is easy to see that

$$\begin{aligned} T^n f &= f \circ S^n + \varepsilon \cdot \Big[\sum_{i=1}^{n} \int_0^1 f \circ S^{i-1} d\mu \Big] \cdot \chi_{\{\omega\}} \\ &= f \circ S^n + \varepsilon \cdot \Big[\sum_{i=1}^{n} \int_{n-i}^{n-i+1} f \circ S^{n-1} d\mu \Big] \cdot \chi_{\{\omega\}} \\ &= f \circ S^n + \varepsilon \cdot \Big[\int_0^n f \circ S^{n-1} d\mu \Big] \cdot \chi_{\{\omega\}} \end{aligned}$$

for all $n \in \mathbb{Z}_+$ and $f \in L_1(\Omega, \beta, \mu)$. A similar computation shows that

$$T^n f = f \circ S^n - \varepsilon \cdot \Big[\int_{-n}^0 f \circ S^{n-1} d\mu \Big] \cdot \chi_{\{\omega\}}$$

for all $n \in \mathbb{Z} \setminus \mathbb{Z}_+$ and $f \in L_1(\Omega, \beta, \mu)$, in particular, T^{-1} is not positive. Moreover,

$$\|T^n f\| \leq \|f \circ S^n\| + \varepsilon \cdot \Big[\int_{-\infty}^{\infty} |f \circ S^{n-1}| d\mu \Big] \cdot \|\chi_{\{\omega\}}\| = (1 + \varepsilon)\|f\|$$

for all $n \in \mathbb{Z}$ and $f \in L_1(\Omega, \beta, \mu)$, which provides the required property $\sup\{\|T^n\| : n \in \mathbb{Z}\} \leq 1 + \varepsilon$. □

3.2 A positive doubly power bounded operator with a non-positive inverse on an AL-space

The purpose of this section, whose results were published in [7], is to generalize the result of Theorem 3.1.7. More precisely we show that an operator as the one in Theorem 3.1.7 can be constructed not only on L_1- spaces but also on any infinite dimensional AL-space. To that end we give some lemmata and a corollary as a preparation to state our main theorem (Theorem 3.2.8) .

Lemma 3.2.1. [7, *Lemma* 2.2] *Let E be an infinite dimensional AL-space and $\{E_\alpha\}_{\alpha\in A}$ be a countable family of pairwise disjoint nonzero bands of E such that E_α is isometrically Riesz isomorphic to E_β for all $\alpha, \beta \in A$. Then for any $\epsilon > 0$ there exists a positive operator $T : E \longrightarrow E$ with non-positive inverse that satisfies* $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1 + \epsilon$.

Proof. Take one of $\{E_\alpha\}_{\alpha\in A}$ and call it $\breve{E}$ and reorder the others in a countable way as $\{E_i\}_{i=-\infty}^{\infty}$, let Ψ_i be the isometric isomorphism of E_i on E_{i+1} for all i. Since E is an AL-space, E is Dedekind complete (see Theorem 2.2.14(2)) and so, by Theorem 2.1.14, E_i is a projection band for all i. Let P_i be the band projection whose range is E_i. Let X be the band generated by $\{E_\alpha\}_{\alpha\in A}$, so $E = X \bigoplus X^d$ and denote by P the band projection whose range is X^d. Let $\xi \in \breve{E}$, $\xi \geq 0$, $\xi \neq 0$, $\|\xi\| \leq 1$, now for all $g \in E_+$ and $\epsilon > 0$, define $T : E \longrightarrow E$ as follows :

$$T(g) = P(g) + \sum_{k=-\infty}^{\infty} \Psi_k P_k(g) + \epsilon \cdot \Phi P_1(g) \cdot \xi$$

where Φ is the unique linear extension on E of the additive function $\|.\| : E_+ \longrightarrow \mathbb{R}^+$ (see Theorem 2.2.2). It is clear that T is a positive operator.

Claim (1) : For all $m \in \mathbb{Z}^+$ we have

$$T^m g = P(g) + \sum_{k=-\infty}^{\infty} \Psi_k \dots \Psi_{k-(m-1)} P_{k-(m-1)}(g) + \epsilon \cdot \Phi\left(\sum_{i=1}^{m} P_{2-i}(g)\right) \cdot \xi$$

We prove this statement by induction, clearly it is true for $m = 1$, assume that it is true for $m = n$, to prove it for $m = n + 1$.

$$
\begin{aligned}
T^{n+1}g &= T(T^n g) \\
&= P(g) + T\left(\sum_{k=-\infty}^{\infty} \Psi_k \dots \Psi_{k-(n-1)} P_{k-(n-1)}(g)\right) + \epsilon \cdot \Phi\left(\sum_{i=1}^{n} P_{2-i}(g)\right) \cdot \xi \\
&= P(g) + \sum_{i=-\infty}^{\infty} \Psi_i P_i \left(\sum_{k=-\infty}^{\infty} \Psi_k \dots \Psi_{k-(n-1)} P_{k-(n-1)}(g)\right) \\
&+ \epsilon \cdot \Phi P_1 \left(\sum_{k=-\infty}^{\infty} \Psi_k \dots \Psi_{k-(n-1)} P_{k-(n-1)}(g)\right) \cdot \xi + \epsilon \cdot \Phi\left(\sum_{i=1}^{n} P_{2-i}(g)\right) \cdot \xi \\
&= P(g) + \sum_{i=-\infty}^{\infty} \Psi_i \dots \Psi_{i-n} P_{i-n}(g) + \epsilon \cdot \Phi\big(\Psi_0 \dots \Psi_{-(n-1)} P_{-(n-1)}(g)\big) \cdot \xi \\
&+ \epsilon \cdot \Phi\left(\sum_{i=1}^{n} P_{2-i}(g)\right) \cdot \xi
\end{aligned}
$$

Now from the definition of Φ and the isometric property of $\Psi_i' s$ we have

$$
\begin{aligned}
T^{n+1}g &= P(g) + \sum_{i=-\infty}^{\infty} \Psi_i \dots \Psi_{i-n} P_{i-n}(g) + \epsilon \cdot \Phi P_{-(n-1)}(g) \cdot \xi + \epsilon \cdot \Phi\left(\sum_{i=1}^{n} P_{2-i}(g)\right) \cdot \xi \\
&= P(g) + \sum_{i=-\infty}^{\infty} \Psi_i \dots \Psi_{i-((n+1)-1)} P_{i-((n+1)-1)}(g) + \epsilon \cdot \Phi\left(\sum_{i=1}^{n+1} P_{2-i}(g)\right) \cdot \xi
\end{aligned}
$$

so the statement is proved for $m = n + 1$. Similarly, for all $m \in \mathbb{Z}^+$ we have

$$
T^{-m}g = P(g) + \sum_{k=-\infty}^{\infty} \Psi_k^{-1} \dots \Psi_{k+(m-1)}^{-1} P_{k+m}(g) - \epsilon \cdot \Phi\left(\sum_{i=1}^{m} P_{i+1}(g)\right) \cdot \xi
$$

Claim (2) : T is a doubly power bounded operator. For, let $g \in E_+$:

$$
\begin{aligned}
\|T^m g\| &= \|P(g)\| + \left\|\sum_{k=-\infty}^{\infty} \Psi_k \dots \Psi_{k-(m-1)} P_{k-(m-1)}(g)\right\| + \epsilon \cdot \left\|\sum_{i=1}^{m} P_{2-i}(g)\right\| \cdot \|\xi\| \\
&= \|P(g)\| + \sum_{k=-\infty}^{\infty} \left\|\Psi_k \dots \Psi_{k-(m-1)} P_{k-(m-1)}(g)\right\| + \epsilon \cdot \left\|\sum_{i=1}^{m} P_{2-i}(g)\right\| \cdot \|\xi\|
\end{aligned}
$$

$$
\begin{aligned}
&= \|P(g)\| + \sum_{k=-\infty}^{\infty} \left\|P_{k-(m-1)}(g)\right\| + \epsilon \cdot \left\|\sum_{i=1}^{m} P_{2-i}(g)\right\| \cdot \|\xi\| \\
&= \|P(g)\| + \left\|\sum_{k=-\infty}^{\infty} P_{k-(m-1)}(g)\right\| + \epsilon \cdot \left\|\sum_{i=1}^{m} P_{2-i}(g)\right\| \cdot \|\xi\| \\
&= \left\|P(g) + \sum_{k=-\infty}^{\infty} P_{k-(m-1)}(g)\right\| + \epsilon \cdot \left\|\sum_{i=1}^{m} P_{2-i}(g)\right\| \cdot \|\xi\| \\
&\leq \|g\| + \epsilon \cdot \|g\| \cdot \|\xi\| \leq \|g\|(1+\epsilon).
\end{aligned}
$$

So, $\|T^m\| \leq (1+\epsilon)$ for all $m \in \mathbb{Z}^+$. A similar computation shows that $\|T^m\| \leq (1+\epsilon)$ for all $m \in \mathbb{Z}^-$, that is $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1+\epsilon$ and hence T is a doubly power bounded operator as claimed.

Claim (3) : T^{-1} is a non-positive operator. For, let $0 < h \in E_2$, then

$$T^{-1}(h) = \Psi_1^{-1} P_2(h) - \epsilon \cdot \Phi P_2(h) \cdot \xi$$

the terms on the right hand side of the above equation are disjoint positive elements and different from zero. By the uniqueness of the decomposition of Riesz space elements as a difference of two disjoint positive elements (cf. Theorem 2.1.3), we have

$$\Psi_1^{-1} P_2(h) = (T^{-1}(h))^+ \quad \textit{and} \quad \epsilon \cdot \Phi P_2(h) \cdot \xi = (T^{-1}(h))^-$$

that is, $T^{-1}(h) \ngeq 0$, and the proof of the lemma is complete. □

Before we pass the next result, we need to define the concept of an *atom* in Riesz spaces and give some of its important properties.

Definition 3.2.2. *A positive element u in a Riesz space E is called an* ***atom*** *whenever $x \wedge y = 0$ and $x, y \in [0, u]$ imply either $x = 0$ or $y = 0$*

It follows from the definition that 0 is an atom and also if u is an atom then so is λu for arbitrary $\lambda \in \mathbb{R}^+$. The Riesz space which has no atoms is referred to as ***atomless*** and as examples of such spaces we may take $C[0,1]$ and $L_1[0,1]$.

The following lemma establishes in (3) a simple characterization of atoms in Archimedean Riesz spaces [2, Lemma 2.30]. In (1) and (2) we give some properties of atoms collected from [26, p.72-73].

Lemma 3.2.3. (1) *If u is an atom and $0 < x < u$, then $u = \lambda x$ for some λ.*

(2) *If u and v are atoms, then either $u \perp v$ or $u = \lambda v$ for some λ.*

(3) *A positive element u of an Archimedean Riesz space is an atom if and only if the ideal A_u generated by u coincides with the vector subspace generated by u. Moreover, if u is an atom then $< u >$, the vector subspace generated by u, is a projection band.*

Now with the help of this lemma and Lemma 3.2.1, one may obtain the following lemma which will play the crucial role in the proof of Theorem 3.2.8.

Lemma 3.2.4. $[7, \textit{Lemma}\, 2.4]$ *Let E be an infinite dimensional AL-space and $\{e_\alpha\}_{\alpha \in A}$ be a countable family of pairwise distinct atoms in E. Then for any $\epsilon > 0$ there exists a positive operator $T : E \longrightarrow E$ with non-positive inverse that satisfies $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1 + \epsilon$.*

Proof. If $\{e_\alpha\}_{\alpha \in A}$ is a countable family of pairwise distinct atoms in the AL-space E and $B_\alpha =< e_\alpha >$ for all $\alpha \in A$, then according to Lemma 3.2.3 (2,3) for all $\alpha, \beta \in A$ we have either $e_\alpha \perp e_\beta$ or $e_\alpha = \lambda e_\beta$ for some $\lambda > 0$ and B_α is a projection band.Therefore $\{B_\alpha\}_{\alpha \in A}$ is a countable family of pairwise disjoint bands of E. Moreover, these bands are isometrically Riesz isomorphic. For, let $\alpha, \beta \in A$, define $G_{\alpha\beta} : B_\alpha \to B_\beta$ as

$$G_{\alpha\beta}(\lambda e_\alpha) = \frac{\lambda \|e_\alpha\|}{\|e_\beta\|} \cdot e_\beta.$$

Clearly that $G_{\alpha\beta}$ is an isometric Riesz isomorphism. Thus the conditions of Lemma 3.2.1 are satisfied and so the required operator exists. □

A useful corollary of this lemma may be given as

Corollary 3.2.5. $[7, \textit{Cor}.2.5]$ *Let m be an infinite cardinal number.Then for any $\epsilon > 0$, there exists a positive operator T on $L_1[0,1]^m$ with non-positive inverse that satisfies $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1 + \epsilon$.*

Proof. Let $Y_k = [0,1]^k$ for all $k \in \mathbb{N}$, the subsets of $[0,1] \times Y_{m-1}$ have the form $[x,y] \times A_1 \times A_2 \ldots A_n \times Y_{m-n-1}$ where $A_i \subseteq [0,1]$ are measurable sets for all $i \in \mathbb{N}$ and $0 \leq x \leq y \leq 1$. Now $\forall\, a, b \in \mathbb{N}$ define

$$\Psi_{ab} : L_1([a,b] \times Y_{m-1}) \longrightarrow L_1([0,1] \times Y_{m-1})$$

$$\chi_{[x,y]\times A_1\times A_2 \ldots A_n \times Y_{m-n-1}} \longmapsto (b-a).\chi_{[\frac{x-a}{b-a}, \frac{y-a}{b-a}]\times A_1\times A_2 \ldots A_n \times Y_{m-n-1}}$$

for all $a \le x \le y \le b$. Ψ_{ab} is an isometric Riesz isomorphism. Now the density of the simple functions in L_1-spaces enables us to extend Ψ_{ab} to the whole space $L_1([a,b] \times Y_{m-1})$. Let $c, d \in \mathbb{N}$ then we get that $\Psi_{cd}^{-1}\Psi_{ab}$ is an isometric Riesz isomorphism between $L_1([a,b] \times Y_{m-1})$ and $L_1([c,d] \times Y_{m-1})$. So $L_1([a,b] \times Y_{m-1})$ $\cong L_1([c,d] \times Y_{m-1})$ for all $a, b, c, d \in \mathbb{N}$.

Let $I_n = (\frac{1}{1+n}, \frac{1}{n}]$ for all $n \in \mathbb{N}$, then the discussion above shows that $L_1(I_n \times Y_{m-1}) \cong L_1(I_{n+1} \times Y_{m-1})$ for all $n \in \mathbb{N}$. Now $\{L_1(I_n \times Y_{m-1})\}_{n=1}^{\infty}$is a countable family of pairwise disjoint nonempty bands in $L_1[0,1]^m$ and satisfies the conditions of Lemma 3.2.1, so the required operator exists. □

Our next result deals with the measure spaces and before that we need to introduce the concept of *atom* in measures and we assume that the reader is familiar with the main concepts of measure theory.

Definition 3.2.6. *Let (X, Σ, μ) be a measure space. A measurable set A is called an* ***atom*** *for μ if $0 < \mu(A) < \infty$ and for every measurable subset B of A either $\mu(B) = 0$ or $\mu(A) = \mu(B)$.The measure μ is called* ***purely atomic****, if every measurable set of positive measure has a subset which is an atom for μ. If μ has no atoms then it is called* ***purely nonatomic****.*

If μ is the counting measure on a set X then every point of X is an atom. The Lebesgue measure on $\mathbb{R}$ is purely nonatomic.

Obviously, identifying almost equal sets as usual, different atoms may be considered as disjoint, and it follows that any set of finite measure (and hence of σ-finite measure) contains at most countable number of atoms. Hence, any set E of σ-finite measure is the disjoint union of two sets E_1 and E_2 such that E_1 does not contain any atom and E_2 is a countable union of atoms.

If (X, Σ, μ) is a finite purely nonatomic measure space, then for all $1 \le p < \infty$, the space $L_p(X, \Sigma, \mu)$ can be represented as a Banach lattice in terms of countable direct sums of spaces $L_p([0,1]^m)$, where m is an infinite cardinal number and $[0,1]^m$

is m product of $[0,1]$ with product Lebesgue measure. More precisely we have :

Theorem 3.2.7. $[16, Thm.5.14.9]$ *Let (X,Σ,μ) be a finite measure space, $1 \leq p < \infty$, and μ be purely nonatomic. Then there is a countable set $\{m_\beta : \beta < \alpha\}$ of distinct cardinals ($m_\beta \geq \aleph_0$) such that $L_p(X,\Sigma,\mu)$ is isometrically Riesz isomorphic to $\left[\bigoplus \sum_{\beta<\alpha} L_p[0,1]^{m_\beta}\right]_p$.*

We are now ready to state our main theorem in this section which forms the core of this chapter.

Theorem 3.2.8. $[7, Thm.2.1]$ *Let (X,Σ,μ) be a measure space, such that $L_1(X,\Sigma,\mu)$ is of infinite dimension. Then for any $\epsilon > 0$, there exists a positive operator $T : L_1(X,\Sigma,\mu) \longrightarrow L_1(X,\Sigma,\mu)$ with non-positive inverse that satisfies $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1+\epsilon$.*

Proof. We discuss the following two cases :

Case (1) : If μ has infinitely many atoms. Then there exists a countable family of pairwise disjoint nonzero atoms $\{E_i\}_{i=1}^\infty$ for μ. If χ_i is the characteristic function on E_i for all i, then $\{\chi_i\}_{i=1}^\infty$ is a countable family of pairwise distinct nonzero atoms in $L_1(X,\Sigma,\mu)$. Now apply Lemma 3.2.4 to get the required operator.

Case (2) : If μ has at most finitely many atoms. Since $\dim L_1(X,\Sigma,\mu) = \infty$, we may suppose that μ is purely nonatomic. Now there are two subcases to consider :
(i) There exists $A \in \Sigma$ such that $0 < \mu(A) < \infty$. Denote by Σ (again) for the induced $\sigma-$algebras in A and in $X \setminus A$, then we have

$$L_1(X,\Sigma,\mu) = L_1(A,\Sigma,\mu) \oplus L_1(X \setminus A,\Sigma,\mu).$$

Now from Theorem 3.2.7 we have

$$L_1(A,\Sigma,\mu) \cong \left[\bigoplus \sum_{\beta<\alpha} L_1[0,1]^{m_\beta}\right]_1$$

where $\{m_\beta : \beta < \alpha\}$ is the countable set of distinct cardinals as in Theorem 3.2.7. Take one of these cardinals and call it m_{β_1}. Now from Corollary 3.2.5 we have that for each $\epsilon > 0$, there exists a positive operator $S_\circ : L_1[0,1]^{m_{\beta_1}} \longrightarrow L_1[0,1]^{m_{\beta_1}}$ with non-positive inverse and satisfies $\sup_{n\in\mathbb{Z}} \|S_\circ^n\| \leq 1+\epsilon$.

Now we define an operator $S : L_1(A, \Sigma, \mu) \longrightarrow L_1(A, \Sigma, \mu)$ as :

1. On $\bigoplus \sum_{\beta_i < \alpha, i \neq 1} L_1[0,1]^{m_{\beta_i}}$ we define S as the identity operator.
2. On $L_1[0,1]^{m_{\beta_1}}$ we define S as $S = S_\circ$.

So, S is a positive operator on $L_1(A, \Sigma, \mu)$ with non-positive inverse and satisfies $\sup_{n \in Z} \|S^n\| \leq 1 + \epsilon$. Finally the required operator

$$T : L_1(X, \Sigma, \mu) \longrightarrow L_1(X, \Sigma, \mu)$$

can be defined as follows : $T = S$ on $L_1(A, \Sigma, \mu)$ and $T = I$, the identity operator on $L_1(X \setminus A, \Sigma, \mu)$.
(ii) For each $A \in \Sigma$, $\mu(A) = \infty$ or $\mu(A) = 0$. In this case we have $L_1(X, \Sigma, \mu) = \{0\}$ which is of dimension one but in our case $\dim(L_1(X, \Sigma, \mu)) = \infty$. □

In view of Kakutani's Theorem 2.2.13, the following corollary is an easy consequence of Theorem 3.2.8.

Corollary 3.2.9. *Let E be an infinite dimensional AL-space. Then for any $\epsilon > 0$, there exists a positive operator $T : E \longrightarrow E$ with non-positive inverse that satisfies $\sup_{n \in \mathbb{Z}} \|T^n\| \leq 1 + \epsilon$.*

If E and F are normed Riesz spaces and $T : E \longrightarrow F$ is a positive operator, then so is T'. Indeed, let $y \in (F')_+$, to show that $T'y \geq 0$. Let $x \in E_+$, since T is positive, $Tx \geq 0$ and the positivity of y implies that $y \circ Tx \geq 0$ hence $T'y(x) \geq 0$, so $T'y$ is positive. On the other hand, If $T \in L(E)$ satisfies $\sup_{n \in \mathbb{Z}} \|T^n\| \leq M < \infty$, then from Theorem 2.1.19(c) we have $(T')^n = (T^n)'$ for all $n \in \mathbb{Z}$ so, $\|(T')^n\| = \|(T^n)'\|$ for all $n \in \mathbb{Z}$ and by Theorem 2.1.19(d) we have $\|(T')^n\| = \|(T^n)'\| = \|T^n\|$ for all $n \in \mathbb{Z}$. Thus $\sup_{n \in \mathbb{Z}} \|(T')^n\| = \sup_{n \in \mathbb{Z}} \|T^n\| \leq M < \infty$.

Using these remarks, another corollary of Theorem 3.2.8 may be obtained as following.

Corollary 3.2.10. *Let F be an infinite dimensional Banach lattice such that $F = E'$ for some infinite dimensional AL-space E. Then for any $\epsilon > 0$, there exists a positive operator $S : F \longrightarrow F$ with non-positive inverse that satisfies $\sup_{n \in \mathbb{Z}} \|S^n\| \leq 1 + \epsilon$.*

Proof. By Corollary 3.2.9, for any $\epsilon > 0$, there exists a positive operator $T : E \longrightarrow E$ with non-positive inverse that satisfies $\sup_{n\in\mathbb{Z}} \|T^n\| \leq 1 + \epsilon$. If we take $S = T'$, then S is positive operator on E' and satisfies $\sup_{n\in\mathbb{Z}} \|S^n\| \leq 1 + \epsilon$. Moreover, its inverse $S^{-1} = (T')^{-1}$ is non-positive. To show that we need to find $y \in E'_+$ such that $S^{-1}y$ is non-positive. Since T^{-1} is non-positive, there exists $x \in E_+$ such that $T^{-1}x$ is non-positive. Now according to the fact that (If F is a normed Riesz space and $z \in F$, then $z \geq 0$ if and only if $y(z) \geq 0$ for all $y \in E'_+$) there exists $y \in E'_+$ such that $y(T^{-1}x) < 0$ and so $S^{-1}y(x) = y(T^{-1}x) < 0$. Hence S^{-1} is non-positive. $\square$

Chapter 4

Mean ergodicity of positive operators

4.1 Mean ergodicity of positive power bounded operators in KB-spaces

The mean goal of this section, whose results are presented in [8], is to study the mean ergodicity of positive power bounded operators in KB-spaces. We begin the section by an introduction about the mean ergodicity of Markov operators.

Let (Ω, Σ, μ) be a $\sigma-$finite measure space, and $L_1 = L_1(\Omega, \Sigma, \mu)$ be the space of all real-valued Lebesgue-integrable functions on (Ω, Σ, μ). Let $\mathcal{D} = \mathcal{D}(\Omega, \Sigma, \mu)$ be the set of all densities on Ω, that is

$$\mathcal{D} = \left\{ f \in L_1 : f \geq 0 \ \ and \ \ \|f\| = 1 \right\}.$$

A linear operator $T : L_1 \to L_1$ is called a ***Markov operator*** if $T(\mathcal{D}) \subseteq \mathcal{D}$. Obviously, any Markov operator is positive and has norm one.

It was proved [12] that if T is a Markov operator on an L_1-space, then T is mean ergodic and satisfies $\dim \mathrm{Fix}(T) < \infty$ whenever there exist a positive function $h \in L_1$ and a real number η such that $0 \leq \eta < 1$ such that

$$\limsup_{n\to\infty} \left\| \left(\mathcal{A}_n^T f - h \right)_+ \right\| \leq \eta$$

for every density f.

In this section, we extend this result to any power bounded positive operator on a KB-space. More precisely, we prove that any positive power bounded operator T in a KB-space E which satisfies

$$\lim_{n\to\infty} \operatorname{dist}\Big(\mathcal{A}_n^T x, W + \eta B_E\Big) = 0 \qquad (\forall x \in E, \|x\| \leq 1)$$

where W is a weakly compact subset of E and $0 \leq \eta < 1$, is mean ergodic.

The principal tool in the proof of the main results of [12] was additivity of the norm on the positive part of the L_1-space. Since this is no longer the case for a general KB-space, we use different ideas in present work. First we fix some necessary notion and definitions. Let E be a Banach lattice. We denote by $B_E = \{z \in E : \|z\| \leq 1\}$ the closed unit ball of E. Given an element $x \in E$ and a nonempty subset $A \subseteq E$,

$$\operatorname{dist}(x, A) := \inf\{\|x - a\| : a \in A\}$$

denotes the distance between x and A.

Lemma 4.1.1. *Let E be a KB-space, T be a positive power bounded operator in E, W be a weakly compact subset of E, and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ be such that*

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T x, W + \eta B_E) = 0$$

for any $x \in B_E$. If T' has a nonzero positive fixed element, then so does T.

Proof. Let y' be the nonzero positive $T'-$fixed element. Fix $\epsilon > 0$ satisfying $\eta+\epsilon < 1$, choose $x \in B_E \cap E_+$ such that $\langle y', x\rangle > 1-\epsilon$. Let $x'' \in E''_+$ be a $\sigma(E'', E')$-cluster point of $(\mathcal{A}_n^T x)_n$. Then $T''x'' = x''$. Since W is weakly compact in E and $\lim_n \operatorname{dist}(\mathcal{A}_n^T x, W + \eta B_E) = 0$, we obtain that $x'' \in W + \eta B_{E''}$. Moreover, $\langle y', x''\rangle = \langle y', x\rangle > 1 - \epsilon$ $\big($since x'' is a $\sigma(E'', E')$-cluster point of $(\mathcal{A}_n^T x)_n$ then, for every $\delta > 0$, there exists n_δ such that $\langle y', x''\rangle - \langle y', \mathcal{A}_{n_\delta}^T x\rangle < \delta$. Thus we have $\langle y', x''\rangle - \langle \mathcal{A}_{n_\delta}^{T'} y', x\rangle < \delta$, and since $T'y' = y'$, $\langle y', x''\rangle - \langle y', x\rangle < \delta$. By arbitrariness of δ, $\langle y', x''\rangle = \langle y', x\rangle\big)$.

Let P be the band projection from E'' onto E (such a projection exists because E is a KB-space). Then $(Id_{E''} - P)x'' \in \eta B_{E''}$ (by $Id_{E''}$ we denote the identity operator on E''), and hence

$$\langle y', Px''\rangle = \langle y', x''\rangle - \langle y', (Id_{E''} - P)x''\rangle > 1 - \epsilon - \eta > 0.$$

From

$$Px'' + (Id_{E''} - P)x'' = x'' = T''x'' = x'' = TPx'' + T''(Id_{E''} - P)x'' \in E_+ + E''_+,$$

and the fact that Px'' is the biggest part of x'' in E_+, we get $0 \leq TPx'' \leq Px''$. Hence, $(T^n Px'')_n$ is a decreasing sequence in E_+. Since E has order continuous norm, $z := \lim_n T^n Px'' \in E_+$ exists. Clearly $Tz = z$, and from

$$\langle y', z\rangle = \langle y', Px''\rangle > 0$$

it follows $z \neq 0$. Hence $\text{Fix}(T) \cap E_+ \neq \{0\}$. □

If T is a positive operator on a Banach lattice E, then $x \in E$ is called a ***positive fixed element of maximal support*** if $x \in \text{Fix}(T) \cap E_+$ and every $y \in \text{Fix}(T) \cap E_+$ is contained in the band generated by x.

Lemma 4.1.2. *Let E be a a Banach lattice with order continuous norm, E has a quasi-interior point of E, and T be a positive operator on E with $\text{Fix}(T) \cap E_+ \neq \{0\}$. Then $\text{Fix}(T) \cap E_+$ has an element u of maximal support.*

Proof. Let e be a quasi-interior point of E. Then by Theorem 2.1.25 there exists a strictly positive linear functional ψ on E. For $x \in E$, let P_x be the band projection from E onto the band generated by x. Set

$$\alpha := \sup_{x \in \text{Fix}(T) \cap E_+} \langle \psi, P_x e\rangle > 0.$$

Choose $x_n \in \text{Fix}(T) \cap E_+$, $n \in \mathbb{N}$, $\|x_n\| \leq 1$ with $\alpha = \lim_n \langle \psi, P_{x_n} e\rangle$. Let $u := \sum_n 2^{-n} x_n$. Then $u \in \text{Fix}(T) \cap E_+$. If B_u (resp. B_{x_n}) is the band generated by u (resp. x_n for all $n \in \mathbb{N}$), then $B_u \supseteq B_{x_n}$ and so $P_u \geq P_{x_n}$ for all $n \in \mathbb{N}$ (see Theorem 2.1.12), and hence $\langle \psi, P_u e\rangle = \alpha$. Let now $x \in \text{Fix}(T) \cap E_+$. Clearly $P_{u+x} \geq P_x$ and $P_{u+x} \geq P_u$. From

$$\alpha = \langle \psi, P_u e\rangle \leq \langle \psi, P_{u+x} e\rangle \leq \alpha$$

and the strict positivity of ψ we obtain $P_u e = P_{u+x} e$. Since e is a quasi-interior point then $P_u = P_{u+x}$. From $P_{u+x} \geq P_x$ it follows that $P_u \geq P_x$. Thus $u \in \mathrm{Fix}(T) \cap E_+$ has a maximal support. $\square$

We are now in a position to state our main theorem.

Theorem 4.1.3. [8, *Thm.*1] *Let E be a KB-space, T be a positive power bounded operator in E, W be a weakly compact subset of E, and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$ be such that*

$$\lim_{n\to\infty} \mathrm{dist}(\mathcal{A}_n^T x, W + \eta B_E) = 0$$

for any $x \in B_E$. Then T is mean ergodic.

Proof. Without lost of generality we may assume that E has a quasi-interior point. Indeed, for any $x \in E$, $x \neq 0$, we consider the closed ideal F generated by $\{T^n|x| : n \geq 0\}$, instead of E. Then F is a KB-space (cf. Definition 2.1.26) with a quasi-interior point $\sum_{n\geq 0} 2^{-n} T^n |x|$ and $T(F) \subseteq F$. Moreover, F is a projection band in E (cf. Theorem 2.1.14). If $P : E \longrightarrow F$ denotes the corresponding band projection, then

$$\lim_{n\to\infty} \mathrm{dist}(\mathcal{A}_n^T z, P(W) + \eta B_F) = 0 \qquad (\forall z \in B_F).$$

Since $\|P\| = 1$, and $P(W)$ is weakly compact in F, the restriction $T|_F$ satisfies the assumptions of the theorem. Thus, to show that $(\mathcal{A}_n^T x)_n$ converges, it is enough to show that $T|_F$ is mean ergodic. Hence we may assume that E has a quasi-interior point, say e.

There are two alternative cases:

Case (1): $(\mathcal{A}_n^{T'} x')_n$ is a $\sigma(E', E)$-nullsequence for each $x' \in E'$. Then $(\mathcal{A}_n^T x)_n$ converges weakly to 0 for each $x \in E$ and hence, by Eberlein's mean ergodic theorem (Theorem 2.3.9), $(\mathcal{A}_n^T)_n$ converges strongly to 0. Hence T is mean ergodic.

Case (2): There is $x' \in E'_+$ such that $(\mathcal{A}_n^{T'} x')_n$ is not $\sigma(E', E)$-convergent to 0. Let $0 \neq y' \in E'_+$ be a $\sigma(E', E)$-cluster point of $(\mathcal{A}_n^{T'} x')_n$. We may assume $\|y'\| = 1$. Then, for all $\epsilon > 0$, there exists n with $\langle y', x\rangle - \langle \mathcal{A}_n^{T'} x', x\rangle < \epsilon$ and $\langle T'y', x\rangle - \langle T'\mathcal{A}_n^{T'} x', x\rangle < \epsilon$. Combining these estimates, we arrive at $\langle y', x\rangle - \langle T'y', x\rangle < 2\epsilon$, but ϵ and x were chosen arbitrary, so $T'y' = y'$.
By Lemma 4.1.1 , $\mathrm{Fix}(T) \cap E_+ \neq \{0\}$ and so Lemma 4.1.2 implies that there exists

$u \in \mathrm{Fix}(T) \cap E_+$ such that u has a maximal support. Denote by B_u the projection band generated by u. $B_u = \mathrm{cl}\bigcup_{n=1}^{\infty}[-nu, nu]$ by the order continuity of the norm in E. Denote $Q = Id_E - P_u$ and $S = QT$. Since $TP_u = P_uTP_u$ then easy calculations show that $QT = QTQ$, (and then $(QTQ)^n = (QT)^n = QT^n$ for all n).

We show that the sequence $(Q\mathcal{A}_n^T)_n$ is strongly convergent to 0. If not, then $\mathcal{A}_n^S \not\to 0$, and as in **Case (2)**, there exists $y' \in \mathrm{Fix}(S') \cap E'_+$, $y' \neq 0$. From

$$y' = T'Q'y' = Q'T'Q'y' = Q'S'y' = Q'y'$$

we obtain that $y' \in \mathrm{Fix}(T') \cap E'_+$. By Lemma 4.1.1, there exists $y \in \mathrm{Fix}(T) \cap E_+$ such that $\langle y', y\rangle > 0$. Then

$$\langle y', Qy\rangle = \langle Q'y', y\rangle = \langle y', y\rangle > 0$$

Hence $(Id_E - P_u)y = Qy \neq 0$, i.e. $y \notin B_u$. This contradicts the fact that u has a maximal support. Thus $Q\mathcal{A}_n^T \to 0$ strongly.

Since T is power bounded, $M := \sup_{n\geq 0}\|T^n\| < \infty$. We shall use the following two elementary formulae (cf.,Theorem 2.3.1(2,3))

$$\mathcal{A}_{nk}^T = \frac{1}{k}(\mathcal{A}_n^T + T^n\mathcal{A}_n^T + T^{2n}\mathcal{A}_n^T + \ldots + T^{(k-1)n}\mathcal{A}_n^T) \qquad (1)$$

and

$$\mathcal{A}_{j+i}^T - \mathcal{A}_j^T = (j+i)^{-1}(T^j + T^{j+1} + \ldots + T^{j+i-1}) - i(j+i)^{-1}\mathcal{A}_j^T\ . \qquad (2)$$

Let $x \in E$ and $\epsilon > 0$. Since $\lim\limits_{n\to\infty}\|(Id_E - P_u)\mathcal{A}_n^T x\| = 0$, there exists n_ϵ such that $\mathrm{dist}(\mathcal{A}_{n_\epsilon}^T x, B_u) \leq (3M)^{-1}\epsilon$. Then there exist $c_\epsilon \in \mathbb{R}_+$ and $w \in [-c_\epsilon u, c_\epsilon u]$ satisfying $\|\mathcal{A}_{n_\epsilon}^T x - w\| \leq (2M)^{-1}\epsilon$. Then, for any $l \geq 0$,

$$\|T^l\mathcal{A}_{n_\epsilon}^T x - T^l w\| \leq \|T^l\|\|\mathcal{A}_{n_\epsilon}^T x - w\| \leq M\|\mathcal{A}_{n_\epsilon}^T x - w\| \leq 2^{-1}\epsilon\ . \qquad (3)$$

$T[-u, u] \subseteq [-u, u]$ implies $T^l w \in [-c_\epsilon u, c_\epsilon u]$ for all l. Combining (1) and (3) we obtain that

$$\mathrm{dist}(\mathcal{A}_{n_\epsilon k}^T x, [-c_\epsilon u, c_\epsilon u]) \leq 2^{-1}\epsilon \quad (\forall k \in \mathbb{R})\ . \qquad (4)$$

By (2), there exists $k_\epsilon \in \mathbb{N}$ satisfying

$$\|\mathcal{A}^T_{n_\epsilon k+i}x - \mathcal{A}^T_{n_\epsilon k}x\| \leq 2^{-1}\epsilon \quad (\forall k \geq k_\epsilon,\ i = 1, 2, \dots, n_\epsilon). \tag{5}$$

From (4) and (5) it follows that

$$\text{dist}(\mathcal{A}^T_p x, [-c_\epsilon u, c_\epsilon u]) \leq \epsilon \quad (\forall p \geq n_\epsilon k_\epsilon)\ . \tag{6}$$

By (6) the sequence $(\mathcal{A}^T_n x)_n$ is almost order bounded and hence it is relatively weakly compact (cf. Theorem 2.1.22). Therefore $(\mathcal{A}^T_n x)_n$ has a weak cluster point, and then by Eberlein's theorem the sequence $(\mathcal{A}^T_n x)_n$ is norm convergent for any $x \in E$. Thus T is mean ergodic. □

Remark that, from the proof above, we can see even more, namely that $\text{Fix}(T) \subseteq B_u$. Indeed, if $x \in \text{Fix}(T)$ then

$$P^d_u x = (Id_E - P_u)x = (Id_E - P_u)\mathcal{A}^T_n x \to 0.$$

So $P^d_u x = 0$, and hence $x \in B_u$.

Since order intervals in any KB-space are weakly compact, the theorem is true if we replace a weakly compact subset W of E by an order interval $[-g, g]$ for any $g \in E_+$. In this case, we have even more, the fixed space $\text{Fix}(T)$ of T is finite dimensional and this is what the next theorem shows.

Theorem 4.1.4. [8, *Thm.*2] *Let E be a KB-space, T be a positive power bounded operator on E, $g \in E_+$, and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$, be such that*

$$\lim_{n\to\infty} \text{dist}(\mathcal{A}^T_n x, [-g, g] + \eta B_E) = 0 \tag{7}$$

for any $x \in B_E$. Then T is mean ergodic and $\text{Fix}(T)$ *is finite dimensional.*

Proof. The mean ergodicity of T follows from the preceding theorem.

Let us denote by C the order ideal generated by the set $\{x \in E_+ : \|\mathcal{A}^T_n x\| \to 0\}$ then, for any $c \in C$, $\|\mathcal{A}^T_n c\| \to 0$. By the power boundedness of T, $\|\mathcal{A}^T_n x\| \to 0$ for any $x \in \overline{C}$, and hence the norm closure $\overline{C}$ of C coincides with C. Since any norm closed ideal in a Banach lattice with order continuous norm is a band (cf., Theorem 2.1.21 ($1 \Leftrightarrow 9$)), C is a band, and since every band in E is a projection band, $E = C \oplus C^d$.

Obviously, C is T-invariant. Denote by P_C the band projection $P_C : E \to C$, and by P_{C^d} the band projection $P_{C^d} : E \to C^d$. Let $T_1 := P_{C^d}T$ then $0 \leq T_1 \leq T$, and the band C^d is T_1-invariant. The operator T_1 is power bounded, and

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^{T_1} x, [-g, g] + \eta B_E) = 0 \quad (\forall x \in B_E).$$

Thus T_1 satisfies all conditions of Theorem 4.1.3 then, by this theorem, T_1 is mean ergodic. Consider the mean ergodic projections $P_T, P_{T_1} : E \to E$ defined as

$$P_T x = \lim_{n\to\infty} \mathcal{A}_n^T x, \quad P_{T_1} x = \lim_{n\to\infty} \mathcal{A}_n^{T_1} x \quad (\forall x \in E).$$

By Theorem 2.3.8, $\text{Fix}(T) = P_T(E)$ and $\text{Fix}(T_1) = P_{T_1}(E)$. Obviously

$$P_T, P_{T_1} \geq 0 \quad \& \quad \text{Fix}(T_1) \subseteq C^d.$$

Now we show that P_{T_1} is strictly positive on C^d. Since C is T-invariant, we obtain by induction, that $P_{C^d}T^n = P_{C^d}T_1^n$ for all $n \geq 0$. Then $P_{C^d}\mathcal{A}_n^T = P_{C^d}\mathcal{A}_n^{T_1}$ for all $n \geq 0$, and hence

$$P_{C^d}P_T = P_{C^d}P_{T_1}. \tag{8}$$

Let $x \in C^d_+$, $x \neq 0$, then, by the construction of C, $P_T x \neq 0$ and $P_{C^d}P_T x \neq 0$ since $P_T x \in \text{Fix}(T)$. Then, by (8), $P_{C^d}P_{T_1}x \neq 0$, and hence $P_{T_1}x \neq 0$, and so P_{T_1} is strictly positive on C^d. By Theorem 2.2.6, $\text{Fix}(T_1)$ is a Banach Riesz subspace in C^d and hence in E.

As it was shown in the proof of Theorem 4.1.3, there is a positive T_1-fixed vector u_1 of a maximal support, and $(Id_E - P_{u_1})\mathcal{A}_n^{T_1} \to 0$ strongly as $n \to \infty$. Hence

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^{T_1} x, [-P_{u_1}g, P_{u_1}g] + \eta B_E) = 0 \quad (\forall x \in E, \|x\| \leq 1). \tag{9}$$

Assume that $\dim \text{Fix}(T_1) = \infty$ then by Judin's theorem (cf.,Theorem 2.2.15) there exists a sequence $(x_i)_i \subseteq \text{Fix}(T_1)_+$ such that $x_i \wedge_{\text{Fix}(T_1)} x_j = 0$ for all $i \neq j$, and hence $x_i \wedge x_j = 0$ for all $i \neq j$, since $\text{Fix}(T_1)$ is a sublattice in E. We may assume that

$\|x_i\| = 1$. Set $y_i = P_{u_1} g \wedge x_i$ for any i. We obtain From (9)

$$\|y_i\| = \|P_{u_1} g \wedge x_i\| = \|x_i - (x_i - P_{u_1} g)_+\| \geq 1 - \eta > 0$$

for all i. On the other hand, $(y_i)_i$ is an order bounded (by the element $P_{u_1} g$) disjoint sequence in E, so the order continuity of the norm in E implies that $\|y_i\| \to 0$ (cf.,Theorem 2.1.21 ($1 \Leftrightarrow 5$)) which contradicts to the inequality above. Hence $\mathrm{Fix}(T_1)$ is finite dimensional.

Now we shall show that $\mathrm{Fix}(T) \subseteq P_T(\mathrm{Fix}(T_1))$. From this it will follow that $\dim \mathrm{Fix}(T) \leq \dim \mathrm{Fix}(T_1) < \infty$, what is required.

Indeed, let $f \in \mathrm{Fix}(T)$, then

$$f = P_C f + P_{C^d} f = Tf = TP_C f + TP_{C^d} f \quad \textit{and}$$

$$P_{C^d} f = P_{C^d} T P_C f + P_{C^d} T P_{C^d} f = P_{C^d} T P_{C^d} f = T_1 P_{C^d} f,$$

since C is T-invariant. Hence $P_{C^d} f \in \mathrm{Fix}(T_1)$. To finish the proof of the theorem it is enough to show that $f = P_T(P_{C^d} f)$. It follows directly from

$$f = \mathcal{A}_n^T f = \mathcal{A}_n^T(P_C f) + \mathcal{A}_n^T(P_{C^d} f) \to P_T(P_{C^d} f) \quad (n \to \infty).$$

□

Remark that any mean ergodic positive operator T, such that $\dim \mathrm{Fix}(T) < \infty$, satisfies the condition (7) for some $g \in E_+$ and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$. Moreover, η can be taken arbitrary small.

Example 2.3.11(5) shows that the condition that E is a KB-space cannot be omitted in Theorem 4.1.3 even for Banach lattices with order continuous norm. Indeed, $\|T_\eta\| = 1$ and for $k \geq 2$, we have

$$T_\eta^n e_k = 2^{-n} \sum_{l=0}^{n} \binom{n}{l} e_{k+l}.$$

So $\|T_\eta^n e_k\| = 2^{-n} \binom{n}{[n/2]}$, where $[q]$ is the integer part of q. But

$$2^{-n}\binom{n}{[n/2]} \sim 1/\sqrt{\pi[n/2]} \quad ,$$

so $T_\eta^n e_k$ converges in norm to 0 for all $k \geq 2$. Moreover $T_\eta^n e_1 \in [0, e_1] + \eta B_{c_0}$ for all $n \in \mathbb{N}$, and hence

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^{T_\eta} x, [-e_1, e_1] + \eta B_{c_0}) = 0 \quad (\forall\ x \in B_{c_0}).$$

On the other hand, T_η is not mean ergodic (see Example 2.3.11(5)).

4.2 A characterization of KB-spaces

It is well-known, (cf. Theorem 2.3.10), that every power bounded operator in a reflexive Banach space is mean ergodic. An old problem in the theory of Banach spaces is the converse of the above fact, that is : Let E be a Banach space such that every power bounded operator is mean ergodic. *Is E reflexive*? This problem was formulated by Sucheston [24], and it was solved for $\sigma-$Dedekind complete Banach lattices by Zaharopol [29], and for arbitrary Banach lattice by Emely'anov [9] and for Banach spaces with bases by Fonf, Lin and Wojtaszczyk [14]. The question arises here is : which other properties of Banach spaces or Banach lattices may be characterized by the mean ergodicity of power bounded operators belonging to special classes of operators?

In this direction some Banach lattices properties were characterized in [11]. For example, KB-property for $\sigma-$Dedekind complete Banach lattices was characterized there as follows :

Theorem 4.2.1. *Let E be a $\sigma-$Dedekind complete Banach lattice. Then the following conditions are equivalent.*

(a) *Every positive contraction on E which satisfies*

$$\lim_{n\to\infty} \operatorname{dist}(T^n x, [-g, g] + \eta B_E) = 0 \tag{10}$$

where $g \in E_+$, and $\eta \in \mathbb{R}$, $0 \leq \eta < 1$, is mean ergodic.

(b) *E is a KB-space.*

Our idea here is to characterize KB-property for $\sigma-$Dedekind complete Banach lattices by replacing condition (10) by the weaker one (7) and by working on positive power bounded operators instead of positive contractions.

Theorem 4.2.2. [8, $Thm.3$] *Let E be a σ-Dedekind complete Banach lattice. Then the following conditions are equivalent:*

(a) *E is a KB-space.*

(b) *Any positive power bounded operator T on E, which satisfies*

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n^T x, [-g,g] + \eta B_E) = 0 \quad (\forall x \in B_E)$$

for some $g \in E_+$ and $0 \leq \eta < 1$, is mean ergodic.

(c) *Any positive operator T on E, which satisfies*

$$\lim_{n\to\infty} \operatorname{dist}(T^n x, [-g,g] + \eta B_E) = 0 \quad (\forall x \in B_E)$$

for some $g \in E_+$ and $0 \leq \eta < 1$, is mean ergodic.

Proof. $(a) \Rightarrow (b)$: It follows from Theorem 4.1.4.
$(b) \Rightarrow (c)$: It is obvious.
$(c) \Rightarrow (a)$: It follows from [11, Thm.2.2]. We repeat the arguments from [11], in a simple form, for convenience of the reader.

Assume that E is not a KB-space. If the norm on E is not order continuous then there exists a disjoint order bounded sequence $(e_n)_n$ of E_+ which does not converge to 0 in norm (see Theorem 2.1.21 ($1 \Leftrightarrow 5$)). Without lost of generality we may assume that $\|e_n\| = 1$ and $e_n \leq u$ for some $u \in E$ and all n. By [23, Exer.II.18.b] there exists a disjoint normalized sequence $(\psi_n)_n$ in E'_+ such that $\psi_n(e_m) = 0$ for $m \neq n$ and $\psi_n(e_n) \geq 1/2$. We set $\varphi_n = \dfrac{\psi_n}{\psi_n(e_n)}$. Then $\|\varphi_n\| \leq 2$ and $\varphi_n(e_m) = \delta_{n,m}$. The map $U : \ell^\infty \to E$, given by

$$Uf = \sup\{f_n e_n : n \in \mathbb{N}\},$$

is a well defined topological Riesz isomorphism [20, Lemma 2.3.10(ii)]. Define

$V : E \to \ell^\infty$ by

$$(Vx)_n := \varphi_n(x).$$

Then $\|V\| \le 2$ and $VU = I$ on ℓ^∞. Consider the left shift L on ℓ^∞. L is not mean ergodic (cf. Example 2.3.11(3)) and satisfies

$$\lim_{n\to\infty} \operatorname{dist}(L^n x, [-(1)_i, (1)_i]) = 0 \quad (\forall\, x \in B_{\ell^\infty}),$$

where $(1)_i$ is the sequence in ℓ^∞ which is identically equal to 1. Then $T := ULV$ is a positive power bounded operator on E which is not mean ergodic and satisfies

$$\lim_{n\to\infty} \operatorname{dist}(T^n x, [-U((1)_i), U((1)_i)]) = 0 \quad (\forall\, x \in B_E).$$

Thus the norm on E is order continuous. By Theorem 2.1.27, there exists a Riesz subspace F of E and a Riesz isomorphism V_0 from F onto c_0, and by Theorem 2.1.21 ($1 \Leftrightarrow 8$), F is the range of a positive projection P. Set

$$S = V_0^{-1} T_\eta V_0 P,$$

where T_η is the operator on c_0 constructed as in the end of the previous section, and η satisfies

$$0 < \eta \|V_0^{-1}\| \|V_0\| \|P\| < 1.$$

Then S is positive power bounded operator and

$$\lim_{n\to\infty} \operatorname{dist}(S^n x, [-V_0^{-1} e_1, V_0^{-1} e_1] + \eta \|V_0^{-1}\| \|V_0\| \|P\| \cdot B_E) = 0 \quad (\forall\, x \in B_E).$$

The operator T_η is not mean ergodic in c_0. Hence the operator S in E is also not mean ergodic. $\square$

References

[1] Yu.A. Abramovich, "*Isometries of normed lattices*", (Russian), Optimizatsiya **43**(60) (1988), 74–80.

[2] Yu.A. Abramovich and C.D. Aliprantis, *An Invitation to Operator Theory,* Graduate Studies in Mathematics Vol. 50, AMS, Providence, Rhode Island, 2002.

[3] Yu.A. Abramovich and C.D. Aliprantis, *Problems in Operator Theory,* Graduate Studies in Mathematics Vol. 51, AMS, Providence, Rhode Island, 2002.

[4] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis,* Springer–Verlag, Berlin, Heidelberg, 1994.

[5] C.D. Aliprantis and O. Burkinshaw, *Positive Operators,* Academic Press, 1985.

[6] C.D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics,* 2nd ed., AMS, Mathematical Surveys and Monographs, Vol. 105, Providence, RI, 2003.

[7] Ş. Alpay; A. Binhadjah and E.Yu. Emel'yanov, "*A Positive doubly power bounded operator with a nonpositive inverse exists on any infinite dimensional AL-space*", Positivity **10** (2006), 105-110.

[8] Ş. Alpay; A. Binhadjah; E.Yu. Emel'yanov and Z. Ercan, "*Mean ergodicity of positive operators in KB-spaces*", J. Math. Anal. Appl. (to appear).

[9] E.Yu. Emel'yanov, "*Banach lattices on which every power bounded operator is mean ergodic* ", Positivity **1** (1997), 291-296.

[10] E.Yu. Emel'yanov, "*A remark to a theorem of Yu. Abramovich* ", Proc. Amer. Math. Soc. **132** (2003), 781-782.

[11] E.Yu. Emel'yanov and M.P.H. Wolff, "*Mean ergodicity on Banach lattices and Banach spaces*", Arch. Math. (Basel) **72** (1999), no.3, 214–218.

[12] E.Yu. Emel'yanov and M.P.H. Wolff, "*Mean lower bounds for Markov operators*", Ann. Polon. Math. **83** (2004), 11–19.

[13] E.Yu. Emel'yanov and M.P.H. Wolff, "*Asymptotic behavior of Markov semigroups on preduals of von Neumann algebras*", J. Math. Anal. Appl. **314** (2006), 749–763.

[14] V.P. Fonf; M. Lin and P. Wojtaszczyk, "*Ergodic characterization of reflexivity of Banach spaces*", J. Funct. Anal. **187** (2001), 146–162.

[15] U. Krengel, *Ergodic Theorems,* De Gruyter, Berlin, New York, 1985.

[16] H.E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin, New York, 1974.

[17] J. Lamperti, "*On the isometries of some function spaces*", Pacific J. Math. **8** (1958), 459-466.

[18] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II,* Springer–Verlag, Berlin, Heidelberg, New York, 1979.

[19] W.A.J. Luxemburg and A.C. Zaanen, *Riesz Spaces I,* North–Holland, Amesterdam, 1971.

[20] P. Meyer-Nieberg, *Banach Lattices,* Universitext, Springer–Verlag, Berlin, 1991.

[21] R.J. Nagel, *One-Parameter Semigroups of Positive Operators,* Lecture Notes in Math.1184, Springer–Verlag, Berlin, Heidelberg, New York, 1985.

[22] F. Räbiger, "*Attractors and asymptotic periodicity of positive operators on Banach lattices*", Forum Math. **7** (1995), 665–683.

[23] H.H. Schaefer, *Banach Lattices and Positive Operators,* Springer–Verlag, New York, Heidelberg, 1974.

[24] L. Sucheston, "*Problems, Probability in Banach spaces*", Oberwolfach 1975, Lecture notes in Math. **526**, pp. 285-289, Springer–Verlag, Berlin, Heidelberg, New York, 1974.

[25] Yuri. Tomilov and J. Zemànek, "*A new way of constructing examples in operator ergodic theory*", Math. Proc. Camb. Phil. Soc. **137** (2004), 209–225.

[26] B.Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces,*" Wolters–Noordhoff, Groningen, Netherlands, 1967 (English translation from the Russian).

[27] A.C. Zaanen, *Integration,* North–Holland, Amesterdam, 1967.

[28] A.C. Zaanen, *Riesz Spaces II,* North–Holland, Amesterdam, 1983.

[29] R. Zaharopol, "*Mean ergodicity of power-bounded operators in countably order complete Banach lattices*", Math. Z. **192** (1986), 81–88.

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