ANALYSIS OF AN INVENTORY SYSTEM WITH ADVANCE DEMAND INFORMATION AND SUPPLY UNCERTAINTY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

EMEL ARIKAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
INDUSTRIAL ENGINEERING

DECEMBER 2005

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Çağlar Güven
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Refik Güllü
Co-Supervisor

Asst. Prof. Dr. Haldun Süral
Supervisor

Examining Committee Members

Assoc. Prof. Dr. Yasemin Serin	(METU,IE)	
Asst. Prof. Dr. Haldun Süral	(METU,IE)	
Prof. Dr. Refik Güllü	(BOUN,IE)	
Prof. Dr. Nur Evin Özdemirel	(METU,IE)	
Assoc. Prof. Dr. Canan Sepil	(METU,IE)	

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name :

Signature :

ABSTRACT

ANALYSIS OF AN INVENTORY SYSTEM WITH ADVANCE DEMAND INFORMATION AND SUPPLY UNCERTAINTY

Arıkan, Emel
M.Sc., Department of Industrial Engineering
Supervisor: Asst. Prof. Dr. Haldun Süral
Co-Supervisor: Prof. Dr. Refik Güllü

December 2005, 72 pages

In this study we address a periodic review capacitated inventory system with supply uncertainty where advance demand information is available. A stochastic dynamic programming formulation is applied with the objective of minimizing the expected inventory related costs over a finite horizon. Three different supply processes are assumed. Under the all-or-nothing type supply process and partially available supply process, the structure of optimal policy is proved to be a base stock policy and numerical examples are given to demonstrate the effects of system parameters. Under Binomially distributed supply process it is shown that a simple base stock policy is not optimal.

Keywords: Inventory, Advance Demand Information, Supply Uncertainty, Dynamic Programming

ÖZ

ÖN TALEP BİLGİSİ VE RASSAL TEDARİK MİKTARI ALTINDA BİR ENVANTER SİSTEMİNİN İNCELENMESİ

Arıkan, Emel
Yüksek Lisans, Endüstri Mühendisliği Bölümü
Tez Yöneticisi: Y. Doç. Dr. Haldun Süral
Ortak Tez Yöneticisi: Prof. Dr. Refik Güllü

Aralık 2005, 72 sayfa

Bu çalışmada dönemsel olarak gözden geçirilen, kapasite kısıtlı bir envanter sistemi ön talep bilgisi ve rassal tedarik miktarı altında incelenmektedir. Envanter masraflarını en aza indiren ısmarlama kuralını bulmak amacıyla bir stokastik dinamik programlama modeli kullanılmıştır. Üç değişik tedarik süreci ele alınmıştır. Model ya hep ya hiç tipi tedarik süreci altında ve kısmi tedarik miktarına izin verildiği durumda incelendiğinde en iyi ısmarlama kuralının baz-stok kuralı olduğu ispatlanmış ve sistem parametrelerinin etkilerini görmek için sayısal örnekler verilmiştir. Tedarik miktarının ikiterimli dağılım altında incelenmesi durumunda baz-stok politikasının eniyi kural olmadığı gösterilmiştir .

Keywords: Envanter, Ön Talep Bilgisi, Rassal Tedarik, Dinamik Programlama

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to my supervisors, Prof. Dr. Refik Güllü and Asst. Prof. Dr. Haldun Süral, for their patience and guidance all through the study.

I would also like to use this opportunity to thank my parents for their endless support and encouragement, and Sinan and Hande for all their help and friendship.

TABLE OF CONTENTS

LIST OF TABLES

TABLE

LIST OF FIGURES

FIGURE

CHAPTER 1

INTRODUCTION

In most production/inventory systems uncertainties in demand is considered as the main problem. The traditional solution to this problem is to keep safety stocks. However, the increasing emphasis on cooperation between customers and suppliers offers an alternative solution, information sharing. Typically the shared information includes point-of-sale data, demand forecasts, and production schedules.

In this study we consider a particular type of information, advance demand information (ADI). ADI refers to the customer orders placed before their due dates. Customers put their orders to be delivered at some point in the future. The time between customer's order and its delivery is called the demand lead time. When each order comes with the same demand lead time and cancelations are not allowed then the supplier has perfect ADI. In this case demand for a number of future periods becomes deterministic. On the other hand, if customers are allowed to put their orders with different lead times, then future demand has both deterministic and probabilistic parts. Perfect ADI, which we assume in this study, is generally observed between supply chain partners and specified with contractual agreements.

The aim of this research is to understand the impact of ADI in a capacitated system with supply uncertainty. Most inventory models concentrate on the modeling of demand uncertainties and treat supply as an infinitely available source which is not subject to any uncertainties. However, in real life situations supply is restricted with the inventory level of the supplier, the machine capacity or the availability of resources. Moreover it can be also random due to many factors such as lots containing defective items, machine breakdowns, or strikes.

Understanding the relation between ADI, capacity and supply uncertainty may give necessary insights about how to decrease costs and increase service levels. Increasing capacity, decreasing supply uncertainty, and acquiring ADI may be substitutes for each other in terms of cost reduction. However, none of them comes for free, so it is important to understand which one is more profitable. In order to do that, we need to quantify the benefits with respect to each other.

In this study, we analyze a periodic review capacitated inventory system with supply uncertainty where advance demand information is available. There are two parties involved: the retailer and the supplier. The retailer observes a stationary demand process from its customers. Demand is observed a number of periods in advance and this is fixed and same for every customer. According to this demand information the retailer puts his order to the supplier. Supplier checks its own inventory level and satisfies the retailer's order if enough inventory is available. Inventory level of the supplier is uncertain, so the retailer faces the possibility of not receiving what he ordered. Moreover, there is an upper bound on the supplier's inventory level above which the retailer cannot order. Under these circumstances the retailer should consider the following while deciding the ordering quantity: 1) demand during the demand lead time which is certain, 2) distribution of demand beyond demand lead time, 3) distribution of supply availability, 4) capacity constraint. Our aim is to find the optimal ordering policies for the retailer with the objective of decreasing inventory related costs, and then present the effects of system parameters on the policies.

In Chapter 2, we present a literature review on inventory systems with supply uncertainties, and systems with demand information. General description of the model is given in Chapter 3. In this chapter we build a dynamic model and describe the basic parameters of the model. In the next three chapters we analyze our model with three different supply processes. In Chapter 4, we assume supply is either fully available or completely unavailable. Under this assumption the optimality of the order-up-to policy is shown and the behavior of the policy with respect to time, capacity and demand information is analyzed. In Chapter 5, partial supply availability is allowed. The characteristic of the optimal policies for one and two period problems are shown and numerical examples are given for longer periods. Chapter 6 deals with the problem when supply availability depends on the order size. A Binomially distributed supply process is considered, and the findings turn out to be different from the previous two models. A simple order-up-to policy is not optimal for this case. Finally, in Chapter 7 we discuss our findings and give the conclusions.

CHAPTER 2

LITERATURE REVIEW

There is a vast literature on inventory modeling with different approaches and assumptions. Simpler models assume deterministic supply processes with unlimited capacity. In these cases a stationary base-stock policy is optimal, such that when the inventory level is below a critical number it is optimal to produce/order enough to bring the inventory up to that number, or nothing should be produced/ordered.

One of the early works assuming limited capacity is Federgruen and Zipkin (1986). They describe a periodic review inventory model with linear production costs and a convex function representing expected holding and penalty costs. Demand is nonnegative and i.i.d. in each period. Stockouts are backordered. In the first model, they consider the infinite horizon problem where demand follows a discrete distribution. Applying the average-cost criterion they prove the optimality of a modified base stock policy: produce enough to bring the inventory up to a critical level when it is possible, produce to capacity when the optimal production quantity exceeds capacity.

In the second model they apply the discounted-cost criterion, and both finite and infinite horizon problems are treated. Different from the first model, demand is assumed to be continuous. The dynamic programming equation and the structure of this equation are the main references to our analysis. Despite the only state variable in their model is the beginning inventory level, they provide guidelines as to how the convexity and monotonicity results should be demonstrated in our model where there is also advance demand information as an additional state variable. They show the optimality of a modified base stock policy as in the first model.

Our model is very similar to Federgruen and Zipkin (1986) in the way we define the dynamic programming formulation, however there are two main differences: including advance demand information as a state variable, and assuming an uncertain supply process. On the other hand, they give a partial characterization of optimal modified base stock policy, which is hard to achieve with a more complicated supply process and a larger state space like ours.

Supply uncertainty and use of demand information are the two critical properties of our model that require special attention. That is why the literature on these topics will be mentioned separately in the following two sections.

2.1 Modeling Supply Uncertainty

In the production/inventory literature, the focus is mostly on modeling demand uncertainties, and uncertainties on the supply process are generally ignored. While variable supply leadtimes are considered, variable production quantities are not taken into account. Generally, whatever ordered is assumed to be received exactly. However there may be many cases in which the quantity supplied is not equal to the quantity ordered because of defective items in a batch, failures in the production system, variable capacity etc.

A comprehensive review of the literature on lot sizing in the presence of random yields is provided by Yano and Lee (1995). They classify models of yield uncertainty in five different groups. The first group of models assumes that the creation of good units is a Bernoulli process, and so the number of good units in a batch has a Binomial distribution. The second way of modeling yield uncertainty is to specify the distribution of the fraction of good units, which is referred to as stochastically proportional yield. Different from the binomial model, both the mean and the variance of good units can be specified under this approach. Under the third approach, the distribution of the number of good units is not stochastically proportional but changes with the batch size. Uncertain yield is a consequence of a failure in the production system. This approach involves specifying the distribution of the time until there is a failure in the production system and starts making defective parts. The fourth group of models assumes uncertain production capacity. Thus the upper limit of the yield quantity is a random variable. The last group includes more general modeling approaches where independence, stationarity, and stochastically decreasing or increasing patterns are inapplicable.

Silver (1976) and Shih (1980) are the two early models that incorporate supply uncertainty in the EOQ formulation. For the EOQ model the basic assumption is to have constant demand rates which is very restrictive in many cases. However, the simple structure of the model makes it practical to use.

Silver (1976) extends the classical EOQ model to include supply uncertainty. Two problems are analyzed: One in which the standard deviation of quantity received is independent of quantity ordered, and another in which it is proportional to quantity ordered. For both cases the optimal order quantity is shown to be a simple modification of the EOQ.

Shih(1980) considered a production system where yield uncertainty is a result of defective items. It is assumed that the percentage defective in a lot is a random variable with a known distribution. A deterministic EOQ model and a stochastic single period model are analyzed. The optimal ordering quantities are proved to be greater for the case of uncertain supply. Moreover, the optimal order-up-to levels decreases with the variance of the yield rate.

Ehrhardt and Taube (1987) also analyzed a stochastic demand single-period model with stochastically proportional yield. In order to minimize the expected cost, they showed the optimality of base stock and (s,S) policies when there is no setup cost and a positive setup cost respectively. For the special case of uniform demand, they obtain closed form analytic expressions for optimal policy. They then evaluate three heuristics for the uniformly distributed demand and negative binomial demand cases, and conclude that a simpler heuristic which accounts for the expected value of supply quantity, and not its variability, performs quite well.

A more general model is considered by Henig and Gerchak (1990) with both finite-horizon and infinite-horizon analysis. First the distribution of yield is taken as independent of the input level or the lot size. For the single period case, they showed that there is an order point below which an order should be placed. Moreover, this order point is proved to be independent of the yield randomness. They then analyzed stochastically proportional yield model, where the yield is a random multiple of lot size. For the multiple period case they developed a stochastic dynamic program. The finite-horizon model turns out to be very complicated but it is proved that there exists a critical point above which nothing is ordered. The solution to the finite-horizon problem is shown to converge to the solution of the infinite-horizon problem. It is also proved that the order point for the random yield model is no smaller than that of the certain yield model.

Uncertain capacity, which is another cause of supply uncertainty, is analyzed by Ciarallo, Akella and Morton (1994), with a model similar to Henig and Gerchak (1990). First, the single-period problem is considered and it is shown that the randomized capacity does not affect the optimal order policy, it is identical to the classical newsboy problem. For the multiple-period problem still an optimal order-up-to level exists but depends on the variability of capacity. For the infinite-horizon case, an extended myopic policy turns out to be optimal, where review periods of stochastic length should be considered.

Wang and Gerchak (1996) further extend the model of Ciarallo et al. (1994), and incorporate both variable capacity and random yield in their model. For the finite horizon problem, the structure of objective function is similar to the problem where only variable capacity is considered. On the other hand, the structure of optimal policy is similar to that of the random yield models. There is a critical point below which it is optimal to order, but this is not an order-up-to type policy. The reorder point of the single-period problem is not affected by neither the variable capacity nor the yield randomness. They also prove that the solution of the finite horizon model converges to the solution of the infinite horizon model.

Güllü, Önol and Erkip (1999) assume a supply process such that in any period supply is either fully available, partially available or completely unavailable. Demand is considered as deterministic and dynamic. For this partial availability model they apply a dynamic programming formulation and show that the optimal ordering policy is of order-up-to type. For a special case of the model where supply is either fully available or completely unavailable, they provide a simple formula to determine the optimal order-up-to levels.

Argon et al.(2001) consider a discrete time, periodic review inventory model where demand pattern changes after stockout realizations. They incorporate a supply process where the time between two supply realizations is random. In any period supply is either fully available or it is completely unavailable. They assume that lead time is zero and capacity is infinite whenever supply is available. They formulated a model that captures the interaction between supply uncertainty and post-backorder demand patterns. Despite the objective function, which is maximizing expected profit per period, does not show a specific structure, they obtain optimum order-up-to levels for the numerical examples.

In this study we present three different supply models. The first one is an all-or-nothing type supply process which is used by Argon et al. (2001), Güllü et al. (1999), and Pentico (1980). The second model incorporates supply uncertainty such that the quantity supplied is distributed with a known probability distribution as in Henig and Gerchak (1990), Wang and Gerchak(1994). Since the distribution of the supply availability does not change with the order size, amount received is not defined as stochastically proportional and falls into the third category of random yield models described by Yano and Lee (1995). Lastly, we analyze a production system where supply follows a binomial distribution. Unlike most of the papers presented above, we incorporate capacity restriction in our model. However, it should be noted that, in our model capacity is fixed, meaning it is not the cause of supply uncertainty.

2.2 Modeling Demand Information

In an early study, Milgrom and Roberts (1988) analyzed a production environment where the manufacturer can gather ADI from any fraction of customers. The firm produces a number of products each of which faces independent and normally distributed demand. The firm has the option of learning the exact demand from any fraction of the market surveyed, but this survey comes with a cost. The problem is to determine what fraction of the market to survey in order to maximize expected profits. Their main result is that depending on system parameters, the firm should either survey the whole market or not conduct any survey and just produce for inventory.

Hariharan and Zipkin (1995) study a continuous review model where customer orders follow a Poisson process. They consider a system where customers place their orders l units of time before they require these orders. The time between customer order and delivery, l, is defined as demand lead time. Their work is the first to introduce the concept of demand lead time. Both constant and stochastic lead times for supply and demand are considered. They do not assume any capacity constraint. They first consider a model where there is no fixed order cost, and show that a base stock policy is optimal, then include fixed costs into the model and for this case (s, S) policies turn out to be optimal. Their work reveals that having demand lead times has the same effect as reducing supply lead times, and that demand information can be considered as a substitute for safety stock.

Like Milgrom and Roberts, Decroix and Mookerjee (1997) considered a problem where there is an option of gathering advance demand information for a cost. At the beginning of each period it should be decided whether to buy ADI or not. They considered two different types of information: perfect information is the exact demand for the upcoming period, imperfect information is signals of demand that help better estimate the future demand. For the perfect information system they characterize the optimal policy, but the imperfect information system is very complex to fully analyze, so they comment on numerical analysis. However, it is observed that the basic qualitative behavior of perfect information case seems to be the same for the imperfect information case. Their main finding is that information is more valuable when the initial inventory level is low, and the length of the planning horizon is long.

In Chen (2001) demand lead time is considered as the delay that the customers accept for their orders to be filled. When a customer puts an order, he is offered a price discount scheme if he accepts to wait for replenishment; the more he waits, the less he pays. When the customer agrees to wait, the firm gains advance demand information. The market consists of several segments which are different in their willingness to wait, so each segment comes with a different demand lead time. It is assumed that the firm has an multi-stage supply chain, and the objective is to minimize the total supply chain costs. The paper shows how to determine the optimal price-delay schedule.

Use of ADI in project environment is analyzed by Donselaar, Kopczak, and Wouters (2001). They consider a manufacturer who gets regular demand from many small orders and irregular lumpy demand from infrequent large orders. These infrequent demands are taken from installers who bid for some large projects. ADI, in this case, is the information on material requirements of the project which is offered for a bid. However, it is not certain that the installer will win the bid or if he wins, it is not certain that he will choose the manufacturer: there is imperfect information. Thus, if the manufacturer cannot get the demand from the project, information will be worthless or even it will cause the manufacturer to be left with excess inventory. Their main conclusion is that imperfect ADI is more useful when the potential demand from large projects is irregular and the probability that the project will turn into an order is high.

Gallego and Özer (2001) analyze a discrete-time, periodic-review inventory problem. N is defined as the information horizon, such that customers place orders $0 \leq i \leq N$ periods in advance. However, demand lead time, i, is different for every customer. That is why, at the beginning of a period, demand for the next N periods are divided into two parts as the observed part which is known, and the unobserved part which is not yet known. Supply lead time is assumed to be fixed, and supply is uncapacitated. They show that a state-dependent base stock policy is optimal when there is no setup cost, and a state dependent (s, S) policy is optimal when there is setup cost.

Dellaert and Melo (2002) considered a manufacturing system in a make-to-stock environment with partial demand information. In any period t, the manufacturer promises to customers to have their orders finished by the end of period $t + i$, so i is the demand lead time. Demand lead time of each order can vary from 1 to N, i.e. $1 \leq i \leq N$. At the beginning of a period demand for that period is known exactly, and there is partial knowledge on the demands for the next $N - 1$ periods. At the beginning of each period, the manufacturer determines the production lot-size, and the production order arrives immediately at the end of the period. A Markov decision model is developed in order to minimize the long-run average costs, given an unlimited production capacity. Since an optimal policy is too complex, they present two heuristics and compare them with (s, S) and (R, S) inventory policies.

Karaesmen et al. (2002) study a make-to-stock production system consisting of a manufacturing stage and a finished goods inventory. They assume that the supplier receives all orders exactly H periods in advance of their due-date. In any period demands that have to be satisfied in the next H periods are known with certainty. They assume unit order arrivals, the customer orders exactly one unit, with probability q, or does not order anything. They define the vector $\mathbf{D}(t) = (D_1(t), D_2(t)...D_H(t))$ as the ADI vector, all the elements of which are either zero or one. An element $D_i(t)$ of the vector is the number of orders to be satisfied at the end of period $t + i - 1$, so at the end of the current period $D_1(t)$ has to be satisfied. They include $\mathbf{D}(t)$ in the system state with $X(t)$, the inventory level. They do not consider a supply lead time. When a production order is placed, it is supplied at the end of the period with probability p. They reach near optimal policies that are called BSADI. These policies require two parameters, the base stock level, and the release lead time. They conclude that the suppliers can benefit from ADI if production capacity is sufficient.

Özer and Wei (2004) consider the capacitated version of Gallego and Özer's model. For the zero fixed cost case, a state dependent modified base stock policy is proved to be optimal. If the inventory is below the base stock level produce in order to bring it up to the base stock. For the positive fixed cost case, they put a restriction such that either full capacity is produced or nothing is produced. In this case, a state dependent threshold level for the inventory position triggers production. They show that, when there is no fixed cost, advance demand information can be a substitute for capacity and inventory.

A capacitated problem is considered by Wijngaard (2004). A single-product case is assumed where customer order lead time is constant and same for every order. Production takes place in a simple manner, such that it is either on or off. When production is on a fixed amount is produced. Problem is to decide when and how long to produce. A simple policy is analyzed: as long as the net inventory level (on-hand inventory-known demand) is below a critical point keep producing. This policy is shown to be optimal for both the lost-sales and the backlogging cases. The main arguments are that when there is no foreknowledge of demand there is always inventory to buffer the uncertainties in future demand but if there is foreknowledge inventory is kept to satisfy the future demand on time. Moreover, when there is foreknowledge average inventory level decreases by an amount proportional to the demand lead time.

Zhu and Thonemann (2004) analyzed a supply chain with a single retailer and multiple identical customers. Demand of each customer is normally distributed and demands are correlated. The retailer has the chance to obtain demand information from the customers before placing an order to the supplier. This information consists of the forecasts of customers for the coming periods so it is an expectation and not perfect information. Zhu and Thonemann refer this imperfect information as future demand information (FDI). Since collecting FDI is not for free, the problem is to determine the number of customers to acquire

FDI. While it is optimal to either collect information from all customers or not to collect any information at all for the case of independent demand with perfect information (Milgrom and Roberts 1988), this is not true when there is correlated demands and imperfect information. It is often optimal to collect FDI from some customers, and to use this information in order to better estimate the demands from the rest of the customers.

In the above papers benefits of advance demand information are analyzed in two ways: first one is to use this advance information in order to better update the forecast on future demands, or to develop analytical models that incorporate this information as a parameter. We take the second approach and develop a dynamic programming model including ADI as a state variable.

Moreover, our model differs from the above papers by assuming supply uncertainty which is considered by only Karaesmen et al. (2002), and we incorporate a limited production capacity. Capacitated problems are only studied recently by Wijngaard (2004) and Özer and Wei (2004). On the other hand, we do not consider supply lead time, while random supply implies lead time to a certain degree it does not cover the effects of lead time in a complete manner. As a final difference, we assume that all the demand during the information horizon is known exactly, i.e. we do not consider imperfect or partial demand information.

CHAPTER 3

GENERAL DESCRIPTION OF THE MODEL

We consider a single item, periodic review inventory system where advance demand information (ADI) is available. While some authors define ADI as imperfect information on future demand, we assume that we have perfect information on demand for a number of future periods. The objective is to minimize the expected total discounted inventory related costs. We assume linear and fixed holding and backorder costs and we consider a discounting factor so that the time value of money is accounted for. All unmet demand is fully backlogged. Holding and backorder costs are charged on the beginning inventory.

x_t: beginning inventory level when there are t periods remaining in the planning horizon (inventory on hand-backorders)

n: demand lead time

D_t: demand vector when there are t periods remaining in the planning horizon, with dimension n, $D_t = (d_1, d_2, ..., d_n)$

$f(y)$: density function of demand per period.

s: production capacity

r_t: quantity ordered at the beginning of the current period $(0 \leq r \leq s)$

W: amount supplied $(0 \leq w \leq r)$

$L_r(w)$: distribution function of W when quantity ordered is r

$V_t(x, D)$: expected minimum cost of operating the system for t periods, when starting inventory level is x and demand vector for current period is D

b: penalty cost per unit of backorder per period

h: holding cost per unit per period

$c(x)$: one period holding and penalty cost when beginning inventory level is x. $c(x) = max\{hx, -bx\}$

β: discount factor ($0 \leq \beta \leq 1$)

Customers place their orders n periods before they actually need them, so demand lead time is n. D_t represents the demand vector when there are t periods in the planning horizon. It has n entries corresponding to the demands for the next n periods, that is $D_t = (d_1,\ d_2,\ ...,\ d_n)$. Thus, at the beginning of each period demands for the next n periods, including the demand for the current period, d_1 , are known for sure. Say we are in the first period of the planning horizon, and length of the horizon is t. During the period, customer orders for the $n+1^{st}$ period from now on are accumulated, which is a random number with a density function of $f(y)$. At the end of the period, demand for the current period, d_1, is satisfied. Then the demand vector is updated accordingly. At the beginning of the next period we will have $t-1$ periods to go and the new demand vector becomes $D_{t-1} = (d_2,\ d_3,\ ...,\ d_{n+1})$. Since all the orders come exactly n periods in advance, d_2 through d_n are same for both D_t and D_{t-1}. If customers were allowed to put orders with different lead times, every entry of the vector would be updated each period. Demands are stochastic with a common density function, $f(y)$, for every period. However, we know the first n periods' demand, so this process is used to generate the demand for the $n + 1^{st}$ period from now on.

In order to describe the transformation from D_t to D_{t-1}, let us define the following operations: $S(D)$ is the shifting operation such that if $D = (d_1, d_2, ..., d_n)$ then $S(D) = (d_2, d_3, ..., d_n, 0)$. α is a vector of size n, with zeroes in the first $n - 1$ places and the last entry is a random variable Y, i.e. α=(0,0,...,0,Y). At the beginning of the period when our demand vector is D_t, D_{t-1} is a random vector such that $D_{t-1} = S(D_t) + \alpha$. The last entry of D_{t-1} is distributed with $f(Y)$. If customers were allowed to put orders with different lead times, every entry of the vector D_{t-1} would be random. We also define another transformation $T(D)$, such that $T(D)$ truncates the vector D from its first entry, i.e. $T(D) = (d_2, ..., d_n)$. $T(D)$ has a dimension of $n - 1$.

We assume limited capacity on the supply side. There is no supply lead time, but there is uncertainty on the amount supplied. Production orders are placed at the beginning of the period based on the system state which includes the beginning inventory level, x_t, and the current demand vector, D_t. Once an amount $r \geq 0$ is ordered, $w \leq r$ arrives immediately (at the end of the period), before current period's demand is satisfied. The amount which does not arrive immediately $(r - w)$ will never arrive at all and it is lost forever. Amount supplied, w, comes from a distribution function, $L_r(w)$, which is conditioned on the amount ordered, r. This function depends on the ordering quantity because when we order r the most we can receive will be r. Thus, amount ordered will be the upper bound for the variable W.

The order of events that take place during the period is as follows:

- D_t and x_t are observed
- Holding and backorder costs are incurred based on x_t
- r is ordered
- $w \leq r$ arrives
- d_{n+1} is observed
- d_1 is satisfied

We define $V_t(x, D)$ as the expected minimum cost of operating the system for t periods, when starting inventory level is x and the demand vector for current period is D; i.e.,

$$V_t(x, D) = V_t(x, d_1, ..., d_n) \tag{3.1}$$

$$V_t(x, D) = c(x) + \beta \min_{0 \le r \le s} \{J_t(x, D, r)\} \tag{3.2}$$

where

$$J_t(x, D, r) = \int_0^s \int_y V_{t-1}(x - d_1 + w, S(D) + \alpha) f(y) dy dL_r(w) \tag{3.3}$$

Here we describe a new function,

$$G_t(x, T(D)) = \int_y V_{t-1}(x, S(D) + \alpha) f(y) dy \tag{3.4}$$

$V_{t-1}(x, S(D)+\alpha)$ is the expected cost of operating the system for $t-1$ periods applying the best ordering strategy when the inventory level at the beginning of the period is x and the demand vector is $D_{t-1} = S(D) + \alpha$. However, when we have t periods in the planning horizon the last entry of $S(D) + \alpha = (d_2, d_3, ..., y)$ is a random variable which is the demand for the $n + 1^{st}$ period from now on. When we integrate it over y we reach the expected value of V_{t-1} with respect to the demand for the $n + 1^{st}$ period from now on, so $G_t(x, T(D))$ is the expected cost of operating the system for the next $t - 1$ periods when we end the current period with an inventory level of x.

However, the level of inventory that we will end up with, $x - d_1 + w$, is also a random variable. It depends on the supply process. That's why we also have to take the expectation over W. $J_t(x, D, r)$ includes both the expectations on demand and supply. In this case $J_t(x, D, r)$ represents the expected optimal cost of operating the system for the next $t - 1$ periods if we decide to order r in this period.

We do not assume any salvage value or salvage cost, so $V_0(x, D) = c(x)$.

In the following chapters we will present three models with different supply processes.

CHAPTER 4

ANALYSIS OF THE MODEL UNDER ALL OR NOTHING TYPE SUPPLY PROCESS

In this chapter we present a model where demand follows a general distribution with properties described in the previous chapter but supply process is further specified. Supply is either fully available with probability p or it is completely unavailable with probability $1-p$. Partial supply availability is not possible.

At the beginning of the period production order of size r is placed based on the system state which includes the beginning inventory level, x, and the current demand vector, D. Then, we receive all that we ordered with probability p or receive nothing with probability $1-p$. Thus, amount supplied, W, is either r or zero, i.e.

$$W = \begin{cases} r \quad with \quad probability \quad p \\ 0 \quad with \quad probability \quad 1-p \end{cases}$$

The ordering quantity will change for different lengths of planning horizon even when the state of the system, (x, D), is fixed. Thus r depends on the length of the horizon, t, and we represent it as r_t.

After the production order, r_t, is placed, we observe the demand for the $n+1^{st}$ period from now, and we satisfy the demand for the current period. Then,if supply is available, we will begin the next period with an inventory level of $x - d_1 + r_t$. Let us define k_t as the effective inventory level representing this case i.e. $k_t = x - d_1 + r_t$. k_t is the level of inventory we will have when we order and receive all of r_t, which happens with probability p. With probability $1-p$, supply will not be available and the next period's beginning inventory level will be $x - d_1$ instead of k_t.

Since the supply process has a discrete (Bernoulli) distribution, the auxiliary function $J_t(x, D, r)$ changes as,

$$J_t(x, D, r) = (1-p)G_t(x-d_1, T(D)) + pG_t(x-d_1+r_t, T(D)) \tag{4.1}$$

The first term in the equation represents the discounted minimum cost of the next $t-1$ periods when supply is not available, multiplied with its corresponding probability $1-p$. The minimum cost of the next $t-1$ periods is,

$$G_t(x-d_1, T(D)) = \int_y V_{t-1}(x-d_1, S(D)+\alpha)f(y)dy \tag{4.2}$$

The second term is the cost when supply is available, and,

$$G_t(x-d_1+r_t, T(D)) = \int_y V_{t-1}(x-d_t+r_t, S(D)+\alpha)f(y)dy \tag{4.3}$$

Remember, $V_t(x, D) = c(x) + \beta \min_{0 \leq r_t \leq s} J_t(x, D, r)$. However, the first term of $J_t(x, D, r)$, equation 4.2, does not depend on r, so we can take it out of the minimization, and write the objective function as,

$$V_t(x, D) = c(x) + \beta(1-p)G_t(x-d_1, T(D)) + \beta p \min_{0 \leq r_t \leq s} \{G_t(x-d_1+r_t, T(D))\} \tag{4.4}$$

4.1 Characterization of the Optimal Policy

In this section we obtain results on the structure of the model and its optimal solution.

First we will present a model where demand lead time is 1, just the current period's demand is known. System state includes only the beginning inventory level, x, and the current period's demand, d. If supply is available, we will begin the next period with an inventory level of $x-d+r_t$, which happens with probability p. With probability $1-p$, supply will not be available and the next period's beginning inventory level will be $x-d$. Then, k_t is defined as

$k_t = x - d + r_t$ and the functional equation of this problem becomes:

$$V_t(x,d) = c(x) + \beta(1-p)G_t(x-d) + \beta p \min_{x-d \le k_t \le x-d+s} \{G_t(k_t)\}$$

$$G_t(k_t) = \int_y V_{t-1}(k_t, y) f(y) dy$$

We are trying to find the cost minimizing ordering quantity, r_t, but this is same as finding the cost minimizing effective inventory level, k_t. That is why the minimization is over k_t. Since ordering quantity, r_t, can be at least 0 and at most capacity, s, then k_t should be at least $x-d$ and at most $x-d+s$. Here, as we can have the minimization over a single variable, k_t, G_t simplifies as above.

Theorem 1. *$V_t(x,d)$ is convex in x, and optimal ordering policy is of order-up-to type. Optimal ordering quantity is defined as*

$$r_t^*(x,d) = \begin{cases} s & \text{if} \quad x - d < k_t^* - s \\ k_t^* - x + d & \text{if} \quad k_t^* - s \le x - d \le k_t^* \\ 0 & \text{if} \quad x - d > k_t^* \end{cases}$$

where k_t^* minimizes $G_t(k)$.

Proof: Provided in Appendix A.

Theorem 1 proves that the optimal policy is a base stock policy where the optimal base stock level is k_t^*. Whenever system state is (x,d), an amount which will bring the effective inventory position to k_t^* should be ordered. If capacity is not enough to increase the net inventory to its optimal level k_t^*, then capacity should be ordered.

Next we will show the monotonicity of the optimal policy with respect to time.

Theorem 2. $k_{t+1}^* \ge k_t^*$

Proof: Provided in Appendix A.

As can be expected, order-up-to levels increase when there are more periods in the planning horizon. Closer to the end of the horizon, one needs less inventory to cover the uncertain demand of future periods, shortage in capacity, and possibility of no supply in any period.

If we compare two systems with different production capacities, their order-up-to levels are expected to be different. In the following theorem, we show that this expectation holds true, and that the optimal order size decreases as capacity increases.

Theorem 3. *Let us define $k_t^*(s)$ as order-up-to level when capacity is s.*

Then, $k_t^*(s_1) \leq k_t^*(s_2)$ for $s_1 \geq s_2$

Proof: Provided in Appendix A.

If we extend the model to one with a lead time of $n \geq 1$, structure of the optimal policy does not change at all. Still an order-up-to level determines the optimal policy. However, in this case everything is conditioned on the demand vector, D.

$$D = (d_1, d_2, ..., d_n)$$

$$V_t(x, D) = V_t(x, d_1, ..., d_n) = c(x) + \beta(1-p)G_t(x-d_1, T(D)) + \beta p \min_k \{G_t(k, T(D))\}$$

$$G_t(k, T(D)) = G_t(k, d_2, ..., d_n) = \int_y V_{t-1}(k, d_2, ..., d_n, y) f(y) dy$$

Theorem 4. *$V_t(x, D)$ is convex in x, and optimal ordering policy is of order-up-to type which is defined by*

$$r_t^*(x, D) = \begin{cases} s & \text{if} \quad x - d_1 < k_t^*(T(D)) - s \\ k_t^*(T(D)) - x + d_1 & \text{if} \quad k_t^*(T(D)) - s \leq x - d_1 \leq k_t^*(T(D)) \\ 0 & \text{if} \quad x - d_1 > k_t^*(T(D)) \end{cases}$$

where $k_t^*(T(D))$ minimizes $G_t(k, T(D))$.

Proof: Provided in Appendix A.

Note that the first element of the vector, d_1, does not affect $k_t(T(D))$. It is the rest of the vector that specifies the optimal order-up-to level. d_1 only determines the ordering quantity, $r_t(x, D)$.

Theorem 5. $k^*_{t+1}(T(D)) \geq k^*_t(T(D))$

Proof: Provided in Appendix A.

As it was shown in Theorem 2 for $n = 1$, order-up-to points increases with time for any $n \geq 1$. When there are more periods to go it is optimal to order more. However, this is only true for periods with the same information about the future demands. Order-up-to level for $t + 1$ periods can be less than that for t periods, if the elements of the demand vector D_{t+1}, are smaller than that of D_t, excluding the first element.

Theorem 6. $k^*_t(T(D); s_1) \leq k^*_t(T(D); s_2)$ *for all* $s_1 \geq s_2$

Proof: Provided in Appendix A.

With greater capacity one may either choose to produce for the future demands and carry some inventory or to delay production and use the capacity in a future period so as to decrease holding costs. Thus, increasing capacity can be expected to either decrease or increase the order-up-to levels. However, the above theorem shows that it is always optimal to decrease production when capacity increases.

Theorem 7. $k^*_t(T(D_1)) \leq k^*_t(T(D_2))$
$T(D_1) \leq T(D_2)$ *such that,* i^{th} *element of* $T(D_1)$ *is smaller than* i^{th} *element of* $T(D_2)$ *for any* i

Proof: Provided in Appendix A.

Note that $T(D)$ is the truncated version of the demand vector D, such that for $D = (d_1, d_2, d_3, ..., d_n)$ truncated D is $T(D) = (d_2, d_3, ..., d_n)$. Theorem 7 shows that if the future demand is known to be greater the ordering quantity should also be greater.

4.2 Numerical Study for All-or-Nothing Type Supply Process

In this section we present some numerical examples to demonstrate the results established in the previous section. We also want to get an understanding of the impact of demand information, capacity and supply uncertainty on system performance.

We use a backward induction algorithm to solve the functional equation. For a five period problem we find the optimal ordering quantities and the corresponding inventory related costs. We try different demand lead times, capacity levels, and supply availability probabilities to see the affect of these parameters.

Under our problem setting a discrete Uniform demand distribution between 0 and 3 is assumed, and the problem is solved for each combination of the following system parameters: s=2, 3, 4; p=0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1; n=1,2,3; h=3 and b=7.

First we consider the problem when one period of demand information is available, i.e. n=1, and solve for $V_5(x,0)$. Without using the structural results of the previous section we observe the optimality of a base stock policy and find the base stock levels for every problem instance.

For Figure 4.1, we solve the problem assuming capacity level is 3, and illustrate the base stock levels under three different supply availability levels.

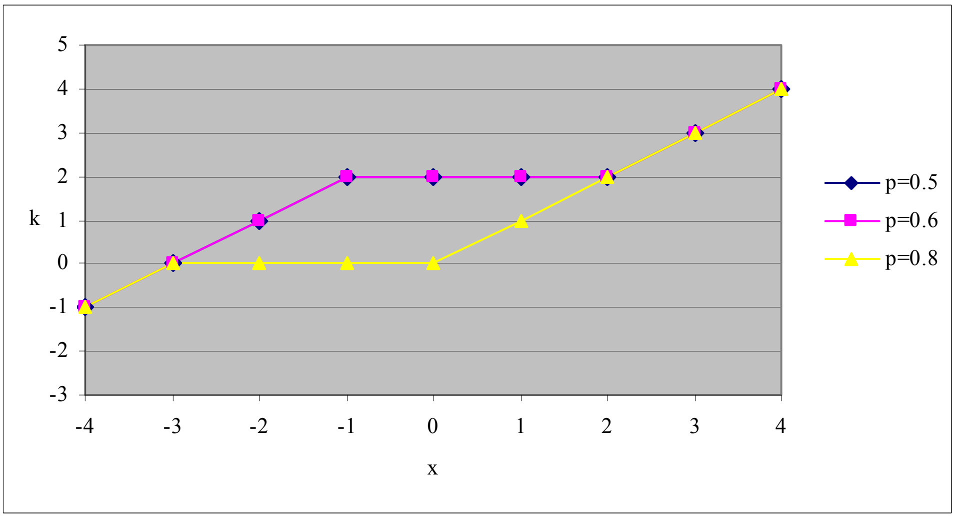

Figure 4.1: Base stock levels, k, for s=3

From Figure 4.1 we can see the optimality of a base stock policy. For *p*=0.5 and *p*=0.6, the optimal base stock level is 2, and for *p*=0.8 it is 0.

Likewise, in Figure 4.2, we observe the base stock levels for a capacity of 4.

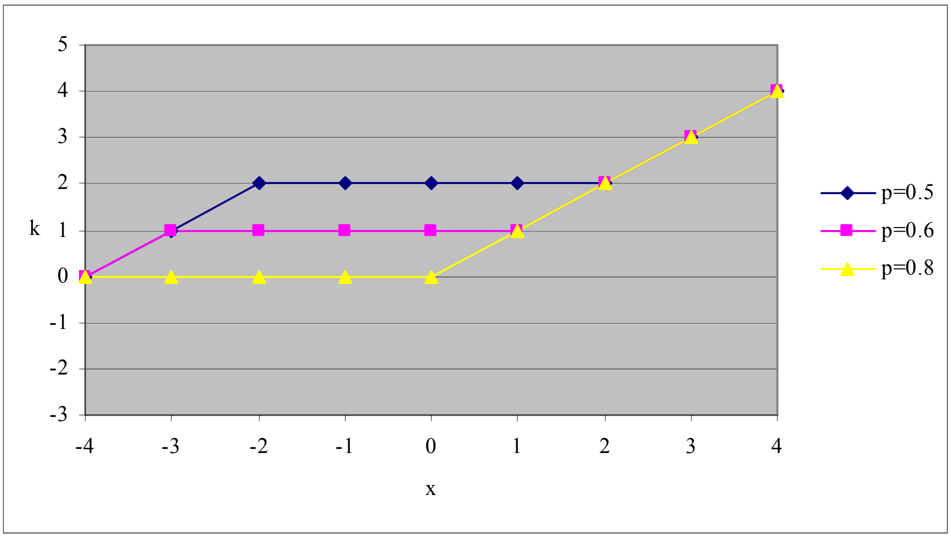

Figure 4.2: Base stock levels, *k*, for *s*=4

In this case, the optimal base stock level for *p*=0.5 is 2, it is 1 for *p*=0.6, and 0 for *p*=0.8.

When the two figures are compared we can observe that the optimal base stock level decreases as capacity increases. For example with a capacity level of 3, optimal base stock level is 2 with an availability level of 0.6, whereas with a capacity level of 4 it drops to 1.

In Figure 4.3 we can see the effect of supply availability level, *p*, on the optimum base stock levels.

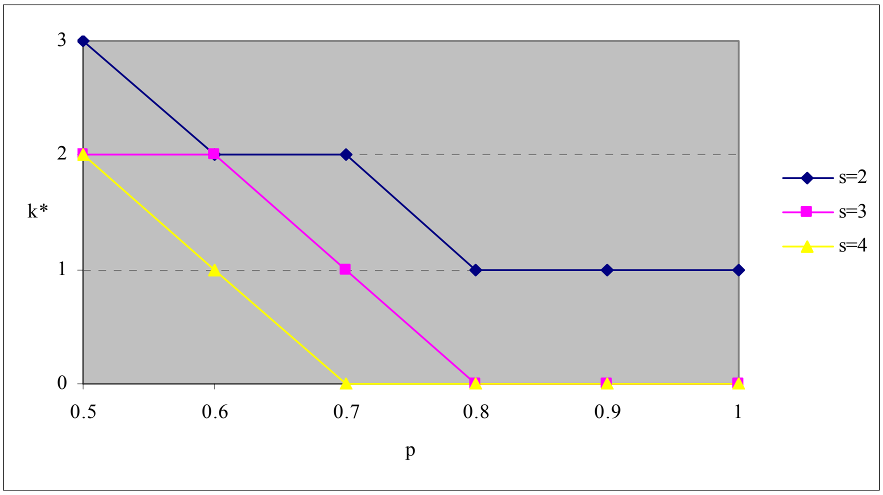

Figure 4.3: Optimum base stock levels, k^*, versus p

From Figure 4.3, we can see that the optimal base stock level decreases as p increases under each capacity level that we assumed. For example, with a capacity level of 3, optimal base stock level is 2 when p=0.6, it is 1 when p=0.7, and it is 0 for higher p values.

Same properties hold for n>1. Below we present the optimum ordering quantities of the 5-period problem where capacity is 3. The first row of Table 4.1 shows the base stock levels when n=1 and the demand vector D=(0), i.e. we know that the current period's demand is 0. In the next four rows we have 2 periods of demand information, i.e. $D=(0,d_2)$, and in the last four rows we have 3 periods of demand information, i.e. $D=(0,0,d_3)$. From Table 4.1 we can observe the change in the optimum ordering quantities when more demand information is available.

Table 4.1: Optimum ordering quantities

		p=0.3	p=0.4	p=0.5	p=0.6	p=0.7	p=0.8
***n*=1**	***D*=(0)**	2	2	1	0	0	0
***n*=2**	***D*=(0,0)**	1	1	0	0	0	0
	***D*=(0,1)**	2	2	1	1	1	0
	***D*=(0,2)**	3	3	2	2	2	1
	***D*=(0,3)**	3	3	3	3	3	2
***n*=3**	***D*=(0,0,0)**	0	0	0	0	0	0
	***D*=(0,0,1)**	1	1	0	0	0	0
	***D*=(0,0,2)**	2	2	1	0	0	0
	***D*=(0,0,3)**	3	3	2	0	0	0

First we look at the difference in optimum ordering quantities of n=1 and n=2. For both cases we know that the current period's demand, d_1, will be 0. For all p values we see that optimal base stock level decreases when d_2=1, when d_2=2 it is same as the case where n=1, and it increases when d_2=3. Thus, under this problem setting, when we know that the next period's demand will be lower than the expected quantity of demand, it is optimal to order less now. In this case we can say that one more period of demand information serves like a substitute for safety stock. On the other hand when we know that the demand will be over its expected level, then we should order more now.

When n=3 still an order of size d_3 is optimum if p=0.3 and p=0.4. When p=0.6, 0.7, 0.8 if we know that the next two demands will be 0, no order is given whatever d_3 is, thus it is not optimal to take protection for the uncertainties in demand for more than 2 periods. Thus, we observe that if p is low demand information has more effect on optimal policy whereas if p is high demand information beyond 2 periods does not make any difference.

In Table 4.2 and Table 4.3 we see the expected minimum costs of operating the system for 5 periods when x=0 and d_1=0, and the capacity levels are 3 and 2 respectively. The first rows of each table shows the costs when n=1, and the next four rows show the costs when n=2, for d_2=0,1,2,3. Since demand is Uniformly distributed each value of d_2 is equally likely. "**$\sum p_i * V_5(0,0,i)$**" row shows the average costs of operating the system with two periods of demand information.

Table 4.2: Expected minimum costs when s=3

	p=0.5	p=0.6	p=0.7	p=0.8	p=0.9	p=1
$V_5(0,0)$	34.8	27.8	21.1	13.6	6.3	0
$V_5(0,0,0)$	20.4	16.2	12.9	9.2	4.4	0
$V_5(0,0,1)$	26.8	20.7	16.3	11.6	5.5	0
$V_5(0,0,2)$	34.2	26.0	20.1	14.6	6.9	0
$V_5(0,0,3)$	42.8	32.0	24.3	17.8	8.6	0
$\sum p_i * V_5(0,0,i)$	31.05	23.72	18.40	13.30	6.35	0
$V_5(0,0) - \sum p_i * V_5(0,0,i)$	3.75	4.08	2.70	0.30	0.05	0

Table 4.3: Expected minimum costs when *s*=2

	p=0.5	*p*=0.6	*p*=0.7	*p*=0.8	*p*=0.9	*p*=1
$\boldsymbol{V_5(0,0)}$	42.5	34.6	27.9	21.7	15.9	10.6
$\boldsymbol{V_5(0,0,0)}$	25.0	20.6	16.3	12.4	8.2	3.7
$\boldsymbol{V_5(0,0,1)}$	33.0	26.4	20.4	15.6	9.9	3.7
$\boldsymbol{V_5(0,0,2)}$	43.3	33.5	25.6	19.5	13.1	6.2
$\boldsymbol{V_5(0,0,3)}$	62.0	48.6	36.7	26.5	17.5	9.2
$\boldsymbol{\sum p_i * V_5(0,0,i)}$	40.82	32.27	24.75	18.50	12.17	5.7
$\boldsymbol{V_5(0,0) - \sum p_i * V_5(0,0,i)}$	1.67	2.32	3.15	3.20	3.72	4.9

Since "$\boldsymbol{\sum p_i * V_5(0,0,i)}$" rows are always smaller than the costs in the first row, i.e. $\boldsymbol{V_5(0,0)}$**,** we observe that increasing demand lead time is beneficial for all values of *p*.

In Table 4.4, we see the percent decrease in the expected cost when we increase the demand lead time from 1 to 2. For example, **12.80**= $[(V_5(0,0) - \sum p_i * V_5(0,0,i))/ V_5(0,0)] * 100$, when *p*=0.7 and *s*=3. Results are shown for each capacity level, *s*, and five different supply availability levels, *p*, in order to demonstrate their effect on the value of information.

Table 4.4: Percent decrease in the expected cost

	p=0.5	*p*=0.6	*p*=0.7	*p*=0.8	*p*=0.9
s=2	3.94	6.72	11.29	14.75	23.43
s=3	10.78	14.66	**12.80**	2.21	0
s=4	12.97	14.62	7.70	0.27	0

When *s*=2 value of information increases when *p* increases. However, when s=3 and *s*=4 value of information first increases and then decreases as *p* increases. In these cases the benefit of increasing demand information is observed to be higher for intermediate supply availability levels.

CHAPTER 5

ANALYSIS OF THE MODEL UNDER PARTIAL SUPPLY AVAILABILITY

In this chapter we analyze a more complicated system where partial supply availability is possible. Quantity produced is independent of the quantity ordered but the quantity received is at most what we ordered. The supplier produces to inventory and we are not the only customer, so the production system is independent of our order. If the supplier's inventory level is less than what we ordered we get whatever is available. If it is equal to or greater than the quantity ordered then we get exactly what we ordered. However, there is still a capacity constraint as an upper bound for the ordering quantity.

Inventory availability of the supplier, say U, is stochastic with a general distribution function, $L(u)$. When the quantity ordered is, r, quantity received, W, is stochastic with distribution function $L_r(w)$. The realization of amount supplied, w, is either r or less than r. Thus, when the quantity ordered is r, expected supply is,

$$E_r[W] = \int_0^r w dL(w) + \int_r^s r dL(w)$$

In this case, expected minimum cost of operating the system for t periods, when starting inventory level is x and the demand vector for the current period is D can be defined with the following equation:

$$V_t(x, D) = c(x) + \beta \min_{0 \leq r \leq s} \{J(x, D, r)\}$$

where

$$J_t(x, D, r) = \int_0^r G_t(x - d_1 + w, T(D)) dL(w) + \int_r^s G_t(x - d_1 + r, T(D)) dL(w) \quad (5.1)$$

and

$$G_t(x - d_1 + w, T(D)) = \int_y V_{t-1}(x - d_t + w, S(D) + \alpha) f(y) dy$$

Remember that α is a vector of size n, with zeroes in the first $n-1$ places and the last entry is a random variable y i.e. $\alpha = (0, 0, ..., y)$

5.1 Solution of the One Period Problem

Since it is more difficult to show the structure of the optimal policy for this supply process, we first take the one-period problem. Cost function for the one-period problem is,

$$V_1(x, D) = c(x) + \beta \min_{0 \le r \le s} \{J_1(x, D, r)\}$$

The last period cost function is still the same, $V_0(x, d) = c(x)$.

$$G_1(x, T(D)) = \int_y V_0(x, S(D) + \alpha) f(y) dy = c(x)$$

Then $J_1(x, D, r)$ becomes,

$$J_1(x, D, r) = \int_0^r c(x - d_1 + w) dL(w) + \int_r^s c(x - d_1 + r) dL(w) \tag{5.2}$$

Note that $J_1(x, D, r)$ does not change according to the demand lead time. Thus, for all $n \ge 1$, $V_1(x, D)$ has the same properties and has the same solution. That's why we take demand lead time as 1, and the results are applicable for all n.

In this case instead of a demand vector D we have one period of demand information and define it as d, so the functions are,

$$V_1(x, d) = c(x) + \beta \min_{0 \le r \le s} \{J_1(x, d, r)\}$$

$$G_1(x,d) = \int_y V_0(x,y)f(y)dy = c(x)$$

$$J_1(x,d,r) = \int_0^r c(x-d+w)dL(w) + \int_r^s c(x-d+r)dL(w) \tag{5.3}$$

Although we cannot show convexity for J_1 we can show that it is decreasing up-to a critical point and then it is increasing. Thus, this critical point minimizes J_1 and it is the optimal ordering quantity.

Theorem 8. *$J_1(x,d,r)$ is not convex in r but it has a unique minimum where $r = r_1^*(x,d) = d - x$. Optimal ordering quantity is then,*

$$r_1^*(x,d) = \begin{cases} s & \text{if} \quad d-x > s \\ d-x & \text{if} \quad s \geq d-x \geq 0 \\ 0 & \text{if} \quad d-x < 0 \end{cases}$$

Proof: Provided in Appendix A.

Remember effective inventory level k_t is the inventory level after the production order is placed and the current period's demand is satisfied, $k_t = x - d + r_t$. In fact this is not the level of inventory we will have at the beginning of the next period. This level can only be reached when supply is fully available.

For the one-period problem optimal ordering quantity is $r_1^*(x,d) = d - x$ so the optimal level of effective inventory is $k_1^* = 0$. When there is only one period to operate then the optimal base stock level is 0.

Since we know the optimal ordering quantity for the one-period problem, we can write the minimum cost function as,

$$V_1(x,d) = c(x) + \begin{cases} \int_0^s c(x-d+w)dL(w) & \text{if} \quad d-x > s \\ \int_0^{d-x} c(x-d+w)dL(w) + \int_{d-x}^s c(0)dL(w) & \text{if} \quad 0 \leq d-x \leq s \\ c(x-d) & \text{if} \quad d-x < 0 \end{cases} \tag{5.4}$$

Theorem 9. *$V_1(x, d)$ is convex in x, and it has a unique minimum where $x = x_1^*(d)$. $x_1^*(d) = 0$ or $max\{x|bL(d-x) \leq h\}$*

Proof:

First we evaluate $\partial V_1(x, d)/\partial x = V_1'(x, d)$.

For simplicity assume $\beta = 1$ without loss of generality.

$$V_1'(x, d) = c'(x) + \beta \begin{cases} -b & \text{if} \quad d - x > s \\ -bL(d-x) & \text{if} \quad 0 \leq d - x \leq s \\ h & \text{if} \quad d - x < 0 \end{cases}$$

Let's define $\bar{x}(d) = max\{x|bL(d-x) \leq h\}$. Note that $\bar{x}(d)$ can be at most d and at least $d - s$. For $x = d$ then $bL(d-x) = 0$, for $x = d - s$, $bL(d-x) = b$.

If $x \leq \bar{x}(d)$ then $bL(d-x) \geq h$

If $x \geq \bar{x}(d)$ then $bL(d-x) \leq h$

Then $V_1'(x, d)$ should be evaluated for three cases:

1. $\mathbf{0 \leq d - s \leq \bar{x}(d) \leq d}$
2. $\mathbf{d - s \leq 0 \leq \bar{x}(d) \leq d}$
3. $\mathbf{d - s \leq \bar{x}(d) \leq 0 \leq d}$

For each case we define five intervals.

For the first case:

1. $\mathbf{0 \leq d - s \leq \bar{x}(d) \leq d}$
    1. $x \leq 0$ $V_1'(x, d) = -b - b$
    2. $0 \leq x \leq d - s$ $V_1'(x, d) = h - b$
    3. $d - s \leq x \leq \bar{x}(d)$ $V_1'(x, d) = h - bL(d-x)$
    4. $\bar{x}(d) \leq x \leq d$ $V_1'(x, d) = h - bL(d-x)$
    5. $d \leq x$ $V_1'(x, y) = h + h$

$V_1'(x, d)$ is always negative in the first interval and positive in the fifth interval. Thus, it is never optimal to have a beginning inventory level smaller than

0 or greater than d. Since there is only one period to go and the only demand to be satisfied is d, there is no need to keep stock more than d.

In order to see the sign of $V_1'(x,d)$ in the intervals 2, 3, and 4 we have to specify the cost parameters h and b.

If $h \geq b$,

1. $\mathbf{0 \leq d - s \leq \bar{x}(d) \leq d}$

1. $x \leq 0$ $V_1'(x,d) = -b - b \leq 0$
2. $0 \leq x \leq d - s$ $V_1'(x,d) = h - b \geq 0$
3. $d - s \leq x \leq \bar{x}(d)$ $V_1'(x,d) = h - bL(d-x) \geq 0$ since $bL(d-x) \leq b \leq h$
4. $\bar{x}(d) \leq x \leq d$ $V_1'(x,d) = h - bL(d-x) \geq 0$ since $bL(d-x) \leq b \leq h$
5. $d \leq x$ $V_1'(x,y) = h + h \geq 0$

If h is greater than b, $V_1'(x,d)$ is negative in the first interval, i.e. $x \leq 0$, and positive in the other intervals. Then the optimal beginning inventory level, $x_1^*(d)$, is 0. It is optimal to charge no inventory holding cost now and take the risk of having a backorder cost next period.

If $h \leq b$,

1. $\mathbf{0 \leq d - s \leq \bar{x}(d) \leq d}$

1. $x \leq 0$ $V_1'(x,d) = -b - b \leq 0$
2. $0 \leq x \leq d - s$ $V_1'(x,d) = h - b \leq 0$
3. $d - s \leq x \leq \bar{x}(d)$ $V_1'(x,d) = h - bL(d-x) \leq 0$ since $bL(d-x) \geq h$ for $x \leq \bar{x}(d)$
4. $\bar{x}(d) \leq x \leq d$ $V_1'(x,d) = h - bL(d-x) \geq 0$ since $bL(d-x) \leq h$ for $x \geq \bar{x}(d)$
5. $d \leq x$ $V_1'(x,y) = h + h \geq 0$

If holding cost, h, is less than backorder cost, b, $V_1'(x,d)$ is always negative in the first three intervals and positive in intervals 4 and 5. Then, $V_1'(x,d)$ is negative upto $\bar{x}(d)$ and then positive, so the minimizing point for $V_1(x,d)$ is $x = x_1^*(d) = \bar{x}(d)$. As b increases $x_1^*(d) = \bar{x}(d)$ gets closer to d, and as it decreases $x_1^*(d) = \bar{x}$ gets closer to $d - s$. This makes sense because when b is large it is better to have inventory now and satisfy as much of demand as we can rather than taking the risk of having a backorder next period. Likewise, if b is not much higher than h, it is optimal to have only an inventory level of almost $d - s$, less than d, but still can be satisfied by ordering full capacity, $r_1^*(x,d) = d - x = d - (d - s) = s$.

Following the same method we can show that the conclusion holds also for the second and third cases. Whatever the assumptions about the values of b, h and $L(d-x)$, above property remains same: as x increases $V_1'(x,d)$ is first negative then it becomes positive, so it has a unique minimum where $x = x_1^*(d)$. (Details are provided in Appendix A)

To show the convexity of $V_1(x,d)$, second derivatives are analyzed: $\partial^2 V_1(x,d)/\partial x^2 = V_1''(x,d)$

$$V_1''(x,d) = \begin{cases} 0 & \text{if} \quad \mathrm{d-x>s} \\ bl(d-x) & \text{if} \quad 0 \leq \mathrm{d-x} \leq \mathrm{s} \\ 0 & \text{if} \quad \mathrm{d-x<0} \end{cases}$$

Since $V_1''(x,d) \geq 0$ for all x, $V_1(x,d)$ is convex in x. Moreover, we see that in the first region, $d - x > s$, $V_1(x,d)$ decreases linearly and in the third region $d - x \leq 0$ it increases linearly as x increases.

In this section we show that the optimal ordering quantity for the one period problem is $r_1^*(x,d) = d - x$, and that we can find the optimal beginning inventory level for different values of cost parameters.

5.2 Analysis of the Multi Period Problem

When analyzing the multi period problem we take demand lead time as $n \geq 1$. Functional equations of the problem are,

$$V_t(x, D) = c(x) + \beta \min_{0 \leq r \leq s} \{J(x, D, r)\}$$

where

$$J_t(x, D, r) = \int_0^r G_t(x - d_1 + w, T(D)) dL(w) + \int_r^s G_t(x - d_1 + r, T(D)) dL(w) \quad (5.5)$$

and

$$G_t(x - d_1 + w, T(D)) = \int_y V_{t-1}(x - d_t + w, S(D) + \alpha) f(y) dy$$

Theorem 10. *$J_t(x, D, r)$ is decreasing with r up to a point and then it is increasing so it has a unique minimum with respect to r.*

Proof:

The proof of the Theorem is obtained by induction. From Theorem 9 we know the convexity of $V_1(x, D)$.

If we assume convexity of $V_{t-1}(x, D)$ that implies the convexity of $G_t(x - d_1 + r, T(D)) = \int_y V_{t-1}(x - d_1 + r, S(D) + \alpha) f(y) dy$ with respect to both x and r. So, for a specific $r_t^*(x, D)$,

$$\partial G_t(x - d_1 + r, T(D))/\partial r \leq 0 \text{ for } r \leq r_t^*(x, D)$$
$$\partial G_t(x - d_1 + r, T(D))/\partial r \geq 0 \text{ for } r \geq r_t^*(x, D)$$

To see the behavior of $J_t(x, D, r)$ we should analyze,

$$\begin{aligned} \partial J_t(x, D, r)/\partial r &= \int_r^s [\partial G_t(x - d_1 + r, T(D))/\partial r] dL(w) \\ &= [\partial G_t(x - d_1 + r, T(D))/\partial r][1 - L(r)] \end{aligned}$$

Since $[1 - L(r)]$ is always positive,

$\partial J_t(x, D, r)/\partial r \leq 0$ for $r \leq r_t^*(x, D)$

$\partial J_t(x, D, r)/\partial r \geq 0$ for $r \geq r_t^*(x, D)$

When there are t periods in the planning horizon it is optimal to order $r_t^*(x, D)$ where x is the beginning inventory level and D is the current demand vector.

Now we will show the convexity of $V_t(x, D)$ in x in order to complete the proof. Since we know that $J_t(x, D, r)$ is minimized when $r = r_t^*(x, D)$

$$V_t(x, D) = c(x) + \begin{cases} J_t(x, D, s) & \text{if} \quad \mathrm{r_t^*(x, D) > s} \\ J_t(x, D, r_t^*(x, D)) & \text{if} \quad \mathrm{0 \leq r_t^*(x, D) \leq s} \\ J_t(x, D, 0) & \text{if} \quad \mathrm{r_t^*(x, D) < 0} \end{cases}$$

$$J_t(x, D, r) = \int_0^r G_t(x - d_1 + w, T(D))dL(w) + \int_r^s G_t(x - d_1 + r, T(D))dL(w) \quad (5.6)$$

Since $G_t(x - d_1 + w, T(D))$ and $G_t(x - d_1 + w, T(D))$ are convex in x, $J_t(x, D, r)$ is convex in x, and so $V_t(x, D)$ is.

5.3 Numerical study Under Partially Available Supply Process

In the previous two sections, we established the structure of the optimal policy analytically, and proved that it is a base stock policy. In this section we present some numerical examples in order to demonstrate the affects of demand and supply distributions, to see the value of demand information, and to observe the structure of the optimal policy numerically.

The 5-period problem is solved, where holding cost is fixed as 1, and backorder cost is 5 and 10. Demand lead time is assumed to be 1 and 2.

We tried 3 different demand distributions: F1, F2, and F3. Under each distribution demand can take values between 0 and 3, and f_i is the probability that i units will be demanded. Expected value of demand under each distribution is the same,1.5, but the variance changes. Table 5.1 shows the f_i values under each demand distribution and the corresponding variances.

Table 5.1: Parameters of the demand distribution

	f_0	f_1	f_2	f_3	E[Y]	Var(Y)
F1	0.10	0.40	0.40	0.10	1.5	0.65
F2	0.25	0.25	0.25	0.25	1.5	1.25
F3	0.40	0.10	0.10	0.40	1.5	1.85

Supply availability level is distributed with P1, P2, P3, and P4 with increasing variances. Let p_i be the probability that available supply level is i where $0 \leq i \leq 4$. Parameters of each supply distribution, i.e. the p_i values, expected supply level, and the variances, are shown in Table 5.2.

Table 5.2: Parameters of the supply distribution

	p_0	p_1	p_2	p_3	p_4	E[W]	Var(W)
P1	0	0	1	0	0	2	0
P2	0	0.50	0	0.50	0	2	1
P3	0.30	0.15	0.10	0.15	0.30	2	2.7
P4	0.50	0	0	0	0.50	2	4

Thus we have the results for h=1; b=5,10; n=1,2 under 3 different demand distributions and 4 supply distributions. The problem is solved under each combination of these parameters and the observations are summarized in this section.

The following table summarizes the results when the problem is solved without using the analytical findings of the previous section in order to demonstrate the optimality of the base stock policy. Results for all supply availability distributions are included while demand lead time is 1, backorder cost is 5, and the beginning inventory level, x, is 0. d_1 is the demand information for the current period, and the next period's demand information is generated by using F1. The results for all demand distributions and two different backorder costs are given in Table B.1 and Table B.2 of Appendix B.

Table 5.3: Optimal base stock levels and costs under F1 when b=5, x=0

	P1			**P2**			**P3**			**P4**		
d_1	r^*	k	V	r^*	k	V	r^*	k	V	r^*	k	V
0	0	0	3.1	1	1	6.4	2	2	12.2	3	3	16.5
1	1	0	3.1	2	1	7.5	3	2	16.6	4	3	22.7
2	2	0	3.1	3	1	13.3	4	2	23.6	4	2	30.3
3	2	-1	13.6	3	0	22.3	4	1	33.5	4	1	40.3

In Table 5.3, r^* is the optimum ordering quantity and k is the modified base stock level, i.e. $k=x-d_1+r^*$. V is the minimum expected cost when the optimum policy is applied. For all levels of d_1, we observe that both r^* and V increase with the variance of supply availability. When the supply level is more variable it is better to order more now so that to take precaution against uncertainties on future supply. We can also observe the optimality of a base-stock policy from the above table. For example for d_1=0 optimal base stock level, k^*, is 0 under P1, 1 under P2, 2 under P3, and 3 under P4. Thus the optimal policy is to order the quantity that brings the modified inventory level, $x-d_1+r$, to k^*, but if the capacity is not enough it is optimal to order up to capacity.

Under the first supply distribution, P1, the minimum expected cost is significantly higher when d_1 is 3: it is 13.6 whereas it is 3.1 for all other d_1 values.

Such an increase can also be observed from Table B.1 and Table B.2 under all demand distributions and for different backorder costs. Under P1 supply availability level is 2 with probability 1. Thus when the current period's demand, d_1, is known to be 0, 1, or 2 it can be satisfied for sure. However, when it is 3 a backorder of 1 unit is certain, and increases the cost immediately.

Figure 5.1 shows the modified inventory levels with respect to the beginning inventory level, x for different values of d_1 under F1 and P3. From Table 5.3 the structure of the optimal policy is observed and from Figure 5.1 we can better visualize it.

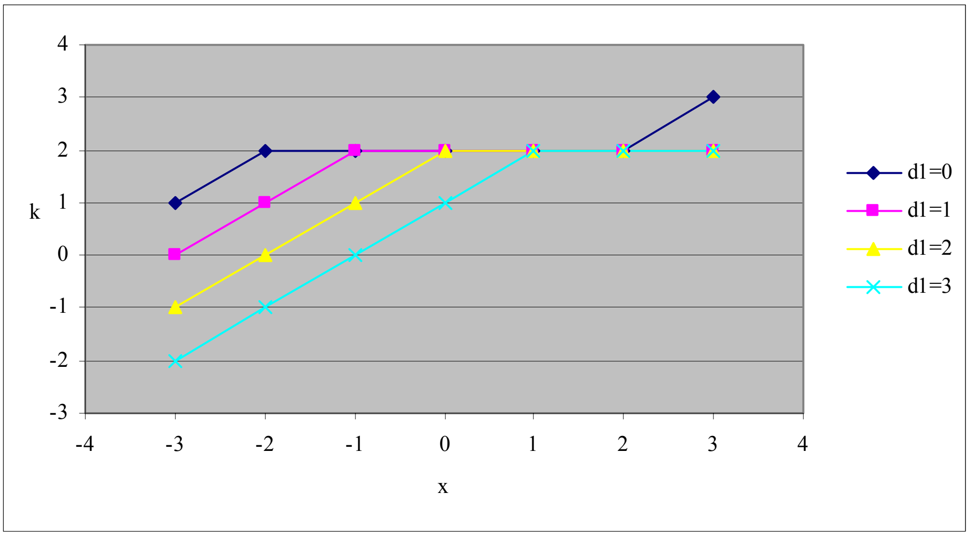

Figure 5.1: Modified inventory levels, k, versus beginning inventory level, x when n=1 under F1 and P3.

The optimality of a base stock policy and the optimal base stock level for the problem can also be observed from Figure 5.1. When supply availability levels are generated from P3 the optimum base stock level is 2.

The following table gives the results under P3 for each demand distribution in order to observe the changes in the policy parameters as the variance of demand changes.

Table 5.4: Optimal base stock levels and costs under P3 when *b*=5

	F1			**F2**			**F3**		
d_1	***r****	***k***	***V***	***r****	***k***	***V***	***r****	***k***	***V***
0	2	2	12.2	3	3	14	3	3	15.7
1	3	2	16.6	4	3	18.5	4	3	20.3
2	4	2	23.6	4	2	25.6	4	2	27.6
3	4	1	33.5	4	1	35.5	4	1	37.5

Table 5.4 shows the values when *x*=0, *b*=5, *n*=1, and the supply distribution is P3. However, we observe the same structure for every combination of the parameters in our problem setting: as the variance of demand distribution increases optimal ordering quantities, *r**, and minimum expected costs, *V*, increase so under a more variable demand distribution it is optimal to order more now.

When the demand lead time is 2, i.e. 2 periods of demand information is available, structure of the optimal policy remains the same. In Table 5.5 the optimum ordering quantities are given when demand lead time is 2, backorder cost is 5, and the beginning inventory level is 0. The effect of second period's demand information on the ordering quantities can be observed from the table.

Table 5.5: Optimum ordering quantities when *b*=5, *n*=2

		F1				**F2**				**F3**			
		P1	**P2**	**P3**	**P4**	**P1**	**P2**	**P3**	**P4**	**P1**	**P2**	**P3**	**P4**
d_1	d_2	***r****	***r****	***r****	***r****	***r****	***r****	***r****	***r****	***r****	***r****	***r****	***r****
0	0	0	0	1	1	0	0	1	1	0	0	1	1
0	1	0	1	2	2	0	1	2	2	0	1	2	2
0	2	0	2	3	3	1	2	3	3	1	2	3	3
0	3	1	3	4	4	2	3	4	4	2	3	4	4
1	0	1	1	2	2	1	1	2	2	1	1	2	2
1	1	1	2	3	3	1	2	3	3	1	2	3	3
1	2	1	3	4	4	2	3	4	4	2	3	4	4
1	3	2	3	4	4	2	3	4	4	2	3	4	4
2	0	2	2	3	3	2	2	3	3	2	2	3	3
2	1	2	3	4	4	2	3	4	4	2	3	4	4
2	2	2	3	4	4	2	3	4	4	2	3	4	4
2	3	2	3	4	4	2	3	4	4	2	3	4	4
3	0	2	3	4	4	2	3	4	4	2	3	4	4
3	1	2	3	4	4	2	3	4	4	2	3	4	4
3	2	2	3	4	4	2	3	4	4	2	3	4	4
3	3	2	3	4	4	2	3	4	4	2	3	4	4

Since there is no on-hand inventory, i.e. x=0, the optimum ordering quantity is at least the current period's demand, d_1. However, the second period's demand information does not always triggers an order. For example under F2 and P1, when current period's demand is known to be 0, it is optimal to order 0 even if the second period's demand information is 1.

Figure 5.2 shows the modified inventory levels with respect to x for different values of d_1 under F1 and P3. Second period's demand information is assumed to be 1, i.e. d_2=1. The figure shows the optimality of a base-stock policy, and the optimum base-stock level, k*, is 2.

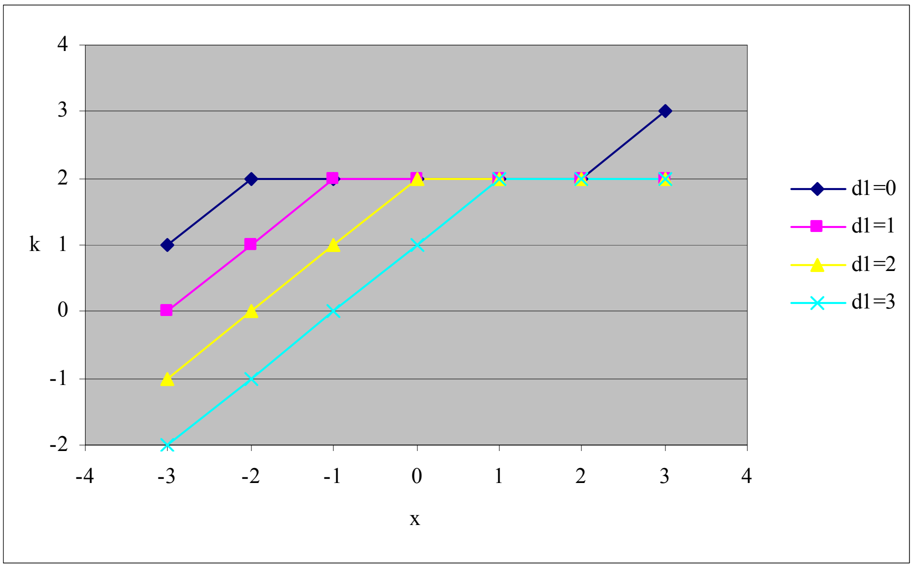

Figure 5.2: Modified inventory levels, k, versus beginning inventory level, x when n=2, d_2=1 under F1 and P3.

Figure 5.3 shows the modified inventory levels with respect to x like Figure 5.2. However, now the second period's demand information is assumed to be 0, i.e. d_2=0, and it is observed that the optimum base stock level, k*, decreases to 1.

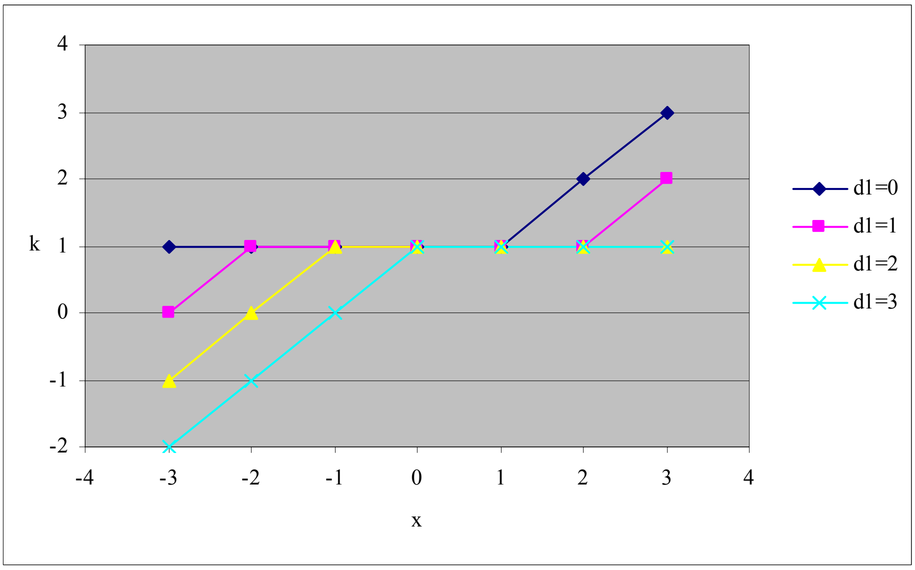

Figure 5.3: Modified inventory levels, *k*, versus beginning inventory level, *x* when n=2, d_2=0 under F1 and P3.

From Figure 5.2 and 5.3, it is observed that k^* decreases by 1 unit when d_1 decreases by 1, and this is true for all cases under our problem setting. Thus it is meaningful to define k^* as a simple function of d_1, like we define it as a function of x.

In order to understand the value of advance demand information, we compare the results of the problems with n=1 and n=2. In Table 5.6 and 5.7, we can see the decrease in the expected cost when we increase the demand lead time from 1 to 2. Values are found by using the below formula:

$$\frac{V_5(0,0) - \sum_i f_i * V_5(0,0,i)}{V_5(0,0)} * 100$$

Beginning inventory level is assumed to be 0 and the current period's demand is known to be 0. The figures show the change in the expected cost if one period of demand information is available when d_1=0. Results for other d_1 values can be found in Table B.5 and Table B.6 of Appendix B.

Table 5.6: Percent change in the expected cost when x=0, d_1=0, b=5

	P1	**P2**	**P3**	**P4**
F1	58.06	22.97	7.87	5.82
F2	45.31	25.87	11.96	8.56
F3	45.15	22.75	13.76	9.85

Table 5.7: Percent change in the expected cost when x=0, d_1=0, b=10

	P1	**P2**	**P3**	**P4**
F1	54.76	18.41	5.89	4.14
F2	48.83	17.92	8.30	5.02
F3	41.25	16.22	8.37	5.39

There are two main observations from Table 5.6 and 5.7. For a given demand distribution, value of ADI decreases when the variance of supply availability increases. However, for a given supply distribution it may decrease, increase, or first increases then decreases depending on the demand distribution. When supply variability is high, i.e. P3 and P4, value of ADI increases with the variance of demand. On the other hand, it decreases for P1 and for P2 it either decreases or first increases then decreases.

We can also observe that as d_1 increases value of ADI decreases. d_1 is the current period's demand and when it is high supply is totally used to satisfy it. Then there is not much chance to take action according to the next period's demand. In this case, knowing next period's demand does not mean much especially if it is known to be high.

Percent change in the expected cost is significantly higher under P1 than other supply distributions. However, from Table B.5 and Table B.6 of Appendix B, we see that this property holds when d_1 is 0, 1, or 2. When d_1 is 3, value of ADI decreases sharply under P1, because it is only possible to receive 2 units and a backorder cost is incurred for sure.

When we compare Table 5.6 and Table 5.7 we observe that the value of ADI decreases as b increases from 5 to 10, but the above properties are the same for both cases.

CHAPTER 6

ANALYSIS OF THE MODEL UNDER BINOMIALLY DISTRIBUTED SUPPLY PROCESS

In this chapter we assume that the level of available supply is distributed according to a Binomial process which depends on the retailer's order size. The supplier has a make-to-order system and produces according to the retailer's order. When the retailer puts an order of size, r, the supplier produces exactly r units. However, each unit is acceptable with probability p and unacceptable with probability $1-p$. In this case quantity received by the retailer, W, has a Binomial distribution with parameters r and p. This model differs from the previous two models in the sense that not only the quantity received but also the supply availability depends on the quantity ordered. There is still the capacity constraint on the supplier side. Supplier can produce at most capacity, s, so r should be lower than s.

Probability of receiving w when r is ordered, i.e. density function of quantity received, is:

$$l_r(w) = \binom{r}{w} p^w (1-p)^{r-w}$$

Then

$$V_t(x, D) = c(x) + \beta \min_{0 \leq r \leq s} \{J_t(x, D, r)\}$$

where

$$J_t(x, D, r) = \sum_{0}^{r} G_t(x - d_1 + w, T(D)) l_r(w)$$

and

$$G_t(x - d_1 + w, T(D)) = E_Y[V_{t-1}(x - d_1 + w, S(D) + \alpha)]$$

Remember d_1 is the current period's demand and D is the current demand vector with n entries starting with d_1. Y is the random variable representing the $n + 1^st$ period's demand and y is its realization. α is a vector of size n, with zeroes in the first $n - 1$ places and y in the n^th place i.e. $\alpha = (0, 0, ..., y)$. Then S(D)+α is the next period's demand vector with n entries starting with d_2 and the last entry is a random variable.

Note that, assuming quantity received, W, has a Binomial distribution implies that it takes integer values. That's why we also assume that the demand has a discrete distribution and inventory level is also integer.

6.1 Analysis of the One Period Problem

Remember, when there is only one period to operate, size of the demand vector does not affect the analysis as long as it is larger than or equal to 1. In order to simplify the notation we assume we have one period of demand information and denote it by d. Then $D = \{d\}$ and $T(D)$ has no entries and $S(D) + \alpha = y$, and

$$G_1(x - d + w, T(D)) = G_1(x - d + w) = E_Y[V_0(x - d + w, y)]$$

Since $V_0(x, d) = c(x)$,

$$G_1(x - d + w) = E_Y[V_0(x - d + w, y)] = c(x - d + w)$$

and

$$J_1(x, d, r) = \sum_0^r c(x - d + w) l_r(w)$$

$$V_1(x, d) = c(x) + \beta \min_{0 \le r \le s} \{J_1(x, d, r)\}$$

Now, assuming $s = 2$ we will show that a simple base-stock policy is not optimal.

When $s = 2$, optimum ordering quantity is either 0, 1, or 2.

$$\min_{0 \le r \le s} J_1(x,d,r) = min \begin{cases} J_1(x,d,0) = & c(x-d) \\ J_1(x,d,1) = & pc(x-d+1) + (1-p)c(x-d) \\ J_1(x,d,2) = & p^2 c(x-d+2) + 2p(1-p)c(x-d+1) \\ & +(1-p)^2 c(x-d) \end{cases}$$

We should consider three cases:

1) When $x \ge d$ $J_1(x,d,0)$ is smaller than both $J_1(x,d,0)$ and $J_1(x,d,0)$, so optimum ordering quantity is $r_1^*(x,d) = 0$.

2) When $x \le d-2$, $c(x-d+2) \le c(x-d+1) \le c(x-d)$, so $J_1(x,d,2)$ has the smallest value and $r_1^*(x,d) = 2$.

3) When $x = d-1$, $r_1^*(x,d)$ is either 1 or 2 depending on the cost parameters. If $J_1(x,d,1) - J_1(x,d,2) \le 0$ then $r_1^*(x,d) = 1$ otherwise $r_1^*(x,d) = 2$

$$J_1(x,d,1) - J_1(x,d,2) = p^2[c(x-d+1) - c(x-d+2)] + p(1-p)[c(x-d) - c(x-d+1)]$$

Since $x = d-1$, $c(x-d+1) = 0$, $c(x-d+2) = c(1) = h$, and $c(x-d) = c(-1) = b$.

$$J_1(x,d,1) - J_1(x,d,2) = p^2(-h) + p(1-p)(b)$$

If $p \ge b/(h+b)$ then $J_1(x,d,1) - J_1(x,d,2) \le 0$ and $r_1^*(x,d) = 1$.
If $p < b/(h+b)$ then $J_1(x,d,1) - J_1(x,d,2) > 0$ and $r_1^*(x,d) = 2$.

Thus, for $p < b/(h+b)$, when $x = d$ $r_1^*(x,d) = 0$ but when $x = d-1$ $r_1^*(x,d) = 2$. When x decreases by 1 unit optimum ordering quantity may increase by 2 units. This shows that a base-stock policy is not the optimal policy.

Moreover this result does not change for larger capacity levels.

We solve the one-period problem assuming $h = 3$ and $b = 7$. In Table 6.1 and Table 6.2 we present the optimum ordering quantities when capacity is 2 and 5 respectively. From the tables it can be observed that the optimal policy is not a base stock policy.

Table 6.1: Optimum ordering quantities for the one-period problem when s=2

	p=0.5	p=0.6	p=0.7	p=0.8
$x \leq d-2$	2	2	2	2
$x = d-1$	2	2	1	1
$x \geq d$	0	0	0	0

Note that $b/(h+b) = 0.7$, and Table 6.1 shows that for $p < 0.7$ the optimum ordering quantity increases 2 units when x decreases 1 unit from d to $d-1$.

Table 6.2: Optimum ordering quantities for the one-period problem when s=5

	p=0.5	p=0.6	p=0.7	p=0.8
$x \leq d-4$	5	5	5	5
$x = d-3$	5	5	5	4
$x = d-2$	5	4	3	3
$x = d-1$	2	2	1	1
$x \geq d$	0	0	0	0

Same property can be observed in Table 6.2 when $s = 5$. For example when $p = 0.5$, $r_1^*(x,d) = 5$ when $x = d-2$ and it is 2 when $x = d-1$. One unit of decrease in x results in 3 units of increase in $r_1^*(x,d)$.

CHAPTER 7

CONCLUSION

In this research, a periodic review capacitated inventory model is analyzed under supply uncertainty and advance demand information. Uncertainties in the supply structure are defined by three different models. In the first model an all-or-nothing type supply process is assumed, in the second model partial supply availability is allowed, and the third model assumes a Binomially distributed supply process.

For the first model, the structure of the cost function and the form of the optimal policy are analyzed and the optimality of a base stock policy is established. It is observed that the costs and optimal base stock levels decrease when capacity, expected supply level, and demand lead time increases. Numerical examples show that Advance Demand Information can decrease costs as increase in capacity or decrease in supply uncertainty does. If it is too costly to increase capacity, then it can be a good practice to search for acquiring demand information which may result in the same benefit. We also illustrate that in order the benefits of Advance Demand Information to be significant, enough capacity is required. However, when capacity is too high then the improvement is most valuable for systems with lower levels of expected supply availability. Similarly, ADI is most valuable for systems with intermediate levels of supply availability.

The supply process assumed in the second model is more complicated to present analytical solutions in terms of the effects of system parameters. However, it is still possible to show that the optimal policy is a base stock policy. With numerical observations, we reached two main conclusions. The first one is that costs and optimal base stock levels increase, and value of ADI decreases while the variance of the supply process increases. Secondly we observe that the change in the value of ADI with respect to the variance of demand is not so straightforward. As the variance of demand increases, value of ADI may be increasing or decreasing under different supply processes. Since it is generally the supply process that we can control and demand process is assumed to be given the first observation has more practical value.

The observations from these two models show that ADI decreases inventory related costs. Under capacitated and/or uncertain supply systems, the immediate solution to the problem of decreasing costs is to increase capacity or decrease uncertainty in the supply process. However, we can now conclude that ADI can also be a solution to the problem. Then depending on the values of the system parameters, increasing capacity, decreasing supply uncertainty, and collecting ADI may be substitutes for each other.

Under Binomially distributed supply process, a simple base stock policy is not optimal. The main difference of this model is that the mean and the variance of supply availability level depend on the quantity ordered by the retailer. This property makes the model so complex that it is not possible to apply a base stock policy. Working on the structure of the optimal policy and its behavior under such a distribution can be a further study.

Another extension might be introducing supply lead time in the model. Both deterministic and probabilistic supply lead times can be assumed and with such a model effects of supply lead time and demand lead time can be compared.

In this research we assume perfect Advance Demand Information: demand for a number of future periods are known with certainty. Further studies can be performed when each customer is willing to pay for different demand lead times, or given the possibility of cancelations, or when the information is the signals of demand rather than the exact quantity of future demand. Studies under such information (e.g. Decroix and Mookerjee (1997), Gallego and Özer (2001), Özer and Wei (2004)) specify the structure of the optimal policy but do not comment on the effects of variance of supply and demand. Since optimal policies derived in these studies are similar to the ones in our study, similar conclusions can be expected. However the effect of uncertainty included in the information itself coupled with the uncertainty in the supply and demand processes needs further analysis.

REFERENCES

1. Argon, N.T., Güllü and Erkip, N., 2001, "Analysis of an Inventory System under Backorder Correlated Deterministic Demand and Geometric Supply Process", International Journal of Production Economics, Vol.71, 247-254.

2. Chen, F., 2001, "Market Segmentation, Advanced Demand Information, and Supply Chain Performance", Manufacturing and Service Operations Management, Vol.3, 53-67.

3. Ciarallo, W., Akella, R. and Morton, T.E., 1994, "A Periodic Review, Production Planning Model with Uncertain Capacity and Uncertain Demand-Optimality of Extended Myopic Policies", Management Science, Vol.40, 320-332.

4. De Croix, G.A. and Mookerjee, V.S, 1997, "Purchasing Demand Information in a Stochastic Demand Inventory System", European Journal of Operational Research, Vol.102, 36-57.

5. Dellaert, N.P. and Melo, M.T., 2003, "Approximate Solutions for a Stochastic Lot Sizing Problem with Partial Customer Order Information", European Journal of Operational Research, Vol.150, 163-180.

6. van Donselaar, K., Kopczak, L.R. and Wouters, M., 2001, "The Use of Advance Demand Information in a Project Based Supply Chain", European Journal of Operational Research, Vol.130, 519-528.

7. Ehrhardt, R. and Taube, L., 1987, "An Inventory Model with Random Replenishment Quantities", International Journal of Production Research,

Vol.25, 1795-1803.

8. Federgruen, A. and Zipkin, P., 1986, “An Inventory Model with Limited Production Capacity and Uncertain Demands II: The Discounted Cost Criterion”, Mathematics of Operations Research, Vol.11, 208-215.

9. Gallego, G. and Özer, Ö., 2001, “Integrating Replenishment Decisions with Advance Order Information”, Management Science, Vol.47, 1344-1360.

10. Güllü, R., Önol, E. and Erkip, N., 1999, “Analysis of an Inventory System Under Supply Uncertainty”, International Journal of Production Economics, Vol.59, 377-385.

11. Hariharan, R. and Zipkin, P., 1995,“Customer-order Information, Leadtimes and Inventories”, Management Science, Vol.41, 1599-1607.

12. Henig, M. and Gerchak, Y., 1990, “The Structure of Periodic Review Policies in the Presence of Random Yield”, Operations Research, Vol.38, 634-643.

13. Karaesmen, F., Buzacott, J.A. and Dallery, Y., 2002,“Integrating Advance Order Information in Production Control”, IIE Transactions, Vol.34, 649-662.

14. Milgrom, P. and Roberts, J., 1998, “Communication and Inventory as Substitutes in Organizing Production”, Scandinavian Journal of Economics,

Vol.90, 275-289.

15. Özer, Ö. and Wei, W., 2004, "Inventory Control with Limited Capacity and Advance Demand Information", Operations Research, Vol.52, 988-1000.

16. Shih, W., 1980, "Optimal Inventory Policies when Stockouts Results from Defective Products", International Journal of Production Research, Vol.18, 677-685.

17. Silver, E.A., 1976, "Establishing the Order Quantity When the Amount Received is Uncertain", INFOR, Vol.14, 32-39.

18. Wang, Y. and Gerchak, Y., 1996, "Periodic Review Production Models with Variable Capacity, Random Yield, and Uncertain Demand", Management Science, Vol.42, 130-137.

19. Wijngaard, J., 2004, "The Affect of Foreknowledge of Demand in case of a Restricted Capacity: The Single-Stage, Single-Product Case", European Journal of Operational Research, Vol.159, 95-109.

20. Yano, C.A. and Lee, H.L, 1995, "Lot Sizing with Random Yields: A Review", Operations Research, Vol.43, 311-334.

21. Zhu, K. and Thonemann, U., 2004, "Modeling the Benefits of Sharing Future Demand Information", Operations Research, Vol.52, 136-147.

APPENDIX A

PROOF OF THEOREMS

Proof of Theorem 1:

$V_0(x,d) = c(x) = max(hx, -bx)$ is convex in x. If we assume that the convexity holds for $V_{t-1}(x,d)$, then $G_t(x) = \int_y V_{t-1}(x,y)f(y)dy$ is convex in x and it has a finite minimizer k_t^*. Then optimal policy is to order up to k_t^*, and we can write the following:

$$V_t(x,d) = c(x)+\beta(1-p)G_t(x-d)+\beta p \begin{cases} G_t(x-d+s) & \text{if} \quad \mathrm{x-d<k_t^*-s} \\ G_t(k_t^*) & \text{if} \quad \mathrm{k_t^*-s \leq x-d \leq k_t^*} \\ G_t(x-d) & \text{if} \quad \mathrm{x-d>k_t^*} \end{cases}$$

Since G_t(x) is convex, above definition is a summation of convex functions and it is also convex in x.

Proof of Theorem 2:

It is sufficient to show that $G'_{t+1}(k) \leq G'_t(k)$ $\forall$ $k \leq k_t^*$, where

$$G_t(k) = \int_y V_{t-1}(k,y)f(y)dy$$

For t=1, k_1^* minimizes G_1(k)=c(k)=min(hk,-bk), so k_1^*=0. When $k < k_1^*$=0,

$$\mathrm{V'_1(k,y) = c'(k) + \beta(1-p)c'(k-y) + \beta p} \begin{cases} \mathrm{c'(k-y+s)} & \text{if} \quad \mathrm{k-y<-s} \\ \mathrm{c'(0)} & \text{if} \quad \mathrm{-s \leq k-y \leq 0} \end{cases}$$

$$\mathrm{V'_0(k,y) = c'(k)}$$

$$\mathrm{G'_2(k) = \int_y V'_1(k,y)f(y)dy \leq G'_1(k) = \int_y V'_0(k,y)f(y)dy}$$

Assume $G'_t(k) \leq G'_{t-1}(k)$ $\forall$ $k \leq k_{t-1}^*$, so $k_{t-1}^* \leq k_t^*$. Then;

$$G'_t(k) \leq G'_{t-1}(k) \leq 0 \quad for \quad k \leq k^*_{t-1} \leq k^*_t$$

$$G'_t(k) \leq 0 \leq G'_{t-1}(k) \quad for \quad k^*_{t-1} \leq k \leq k^*_t$$

$$V'_t(k,y) = c'(k)+\beta(1-p)G'_t(k-y)+\beta p \begin{cases} G'_t(k-y+s) & \text{if} \quad \mathrm{k} < \mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y} \\ G'_t(k^*_t) & \text{if} \quad \mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_\mathrm{t} + \mathrm{y} \\ G'_t(k-y) & \text{if} \quad \mathrm{k} > \mathrm{k}^*_\mathrm{t} + \mathrm{y} \end{cases}$$

$$A_t = G'_t(k-y)$$

$$B_t = \begin{cases} G'_t(k-y+s) & \text{if} \quad \mathrm{k} < \mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y} \\ G'_t(k^*_t) & \text{if} \quad \mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_\mathrm{t} + \mathrm{y} \\ G'_t(k-y) & \text{if} \quad \mathrm{k} > \mathrm{k}^*_\mathrm{t} + \mathrm{y} \end{cases}$$

As a result of the above assumption, A_t is smaller than A_{t-1} for all $k \leq k^*_t$, since $k - y \leq k^*_t$.

For B_t we should consider the following two cases, and eight intervals, where $k \leq k^*_t$:

1. $\mathbf{k^*_{t-1} - s + y \leq k^*_{t-1} + y \leq k^*_t - s + y \leq k^*_t + y}$

1. $\mathrm{k} \leq \mathrm{k}^*_{\mathrm{t}-1} - \mathrm{s} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}-\mathrm{y}+\mathrm{s}) \leq \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}-\mathrm{y}+\mathrm{s}) = \mathrm{B}_{\mathrm{t}-1} \leq 0$
2. $\mathrm{k}^*_{\mathrm{t}-1} - \mathrm{s} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_{\mathrm{t}-1} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}-\mathrm{y}+\mathrm{s}) \leq 0 = \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}^*_{\mathrm{t}-1}) = \mathrm{B}_{\mathrm{t}-1}$
3. $\mathrm{k}^*_{\mathrm{t}-1} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}-\mathrm{y}+\mathrm{s}) \leq 0 \leq \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}-\mathrm{y}) = \mathrm{B}_{\mathrm{t}-1}$
4. $\mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_\mathrm{t} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}^*_\mathrm{t}) = 0 \leq \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}-\mathrm{y}) = \mathrm{B}_{\mathrm{t}-1}$

2. $\mathbf{k^*_{t-1} - s + y \leq k^*_t - s + y \leq k^*_{t-1} + y \leq k^*_t + y}$

1. $\mathrm{k} \leq \mathrm{k}^*_{\mathrm{t}-1} - \mathrm{s} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}-\mathrm{y}+\mathrm{s}) \leq \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}-\mathrm{y}+\mathrm{s}) = \mathrm{B}_{\mathrm{t}-1} \leq 0$
2. $\mathrm{k}^*_{\mathrm{t}-1} - \mathrm{s} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}-\mathrm{y}+\mathrm{s}) \leq 0 = \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}^*_{\mathrm{t}-1}) = \mathrm{B}_{\mathrm{t}-1}$
3. $\mathrm{k}^*_\mathrm{t} - \mathrm{s} + \mathrm{y} \leq \mathrm{k} \leq \mathrm{k}^*_{\mathrm{t}-1} + \mathrm{y}$ $\quad \mathrm{B_t} = \mathrm{G}'_\mathrm{t}(\mathrm{k}^*_\mathrm{t}) = 0 = \mathrm{G}'_{\mathrm{t}-1}(\mathrm{k}^*_{\mathrm{t}-1}) = \mathrm{B}_{\mathrm{t}-1}$

4. $\mathrm{k}^*_{t-1} + y \le k \le k^*_t + y$ $\qquad B_t = G'_t(k^*_t) = 0 \le G'_{t-1}(k-y) = B_{t-1}$

For all possible values of $k \le k^*_t$, $A_t \le A_{t-1}$ and $B_t \le B_{t-1}$, so $V_t'(k,y) \le V'_{t-1}(k,y)$, and $G'_{t+1}(k) \le G'_t(k)$.

Proof of Theorem 3:

$V_t(x,d;s)$ defines the optimal cost function when capacity is s. We will show $G'_t(k;s_1) \ge G'_t(k;s_2)\ \forall k$, and $s_1 \ge s_2$.

For t=1 $G'_1(k;s_1) = c'(k) = G'_1(k;s_2)$ so $k^*_1(s_1) = k^*_1(s_2) = 0$

Now assume $G'_{t-1}(k;s_1) \ge G'_{t-1}(k;s_2)\ \forall k$. Then $k^*_{t-1}(s_1) \le k^*_{t-1}(s_2)$ Since $G_t(k;s)$ is convex , $G'_{t-1}(k_1;s) \ge G'_{t-1}(k_2;s)$ where $k_1 \ge k_2$ and $G'_{t-1}(k_1;s_1) \ge G'_{t-1}(k_2;s_1)$ from the convexity of G and $G'_{t-1}(k_2;s_1) \ge G'_{t-1}(k_2;s_2)$ from the induction assumption. Thus,

$$G'_{t-1}(k_1;s_1) \ge G'_{t-1}(k_2;s_1) \ge G'_{t-1}(k_2;s_2)$$

$$V'_{t-1}(k,y;s) = c'(k) + \beta(1-p)A_{t-1}(s) + \beta p B_{t-1}(s)$$

$$A_{t-1}(s) = G'_{t-1}(k-y;s)$$

$$B_{t-1}(s) = \begin{cases} G'_{t-1}(k-y+s;s) & \text{if} \quad k < k_t(s) - s + y \\ G'_{t-1}(k_t;s) & \text{if} \quad k_t(s) - s + y \le k \le k_t(s) + y \\ G'_{t-1}(k-y;s) & \text{if} \quad k > k_t(s) + y \end{cases}$$

From the assumption, $A_{t-1}(s_1) \ge A_{t-1}(s_2)$.

1. $\mathbf{k^*_{t-1}(s_1) - s_1 + y \le k^*_{t-1}(s_2) - s_2 + y \le k^*_{t-1}(s_1) + y \le k^*_{t-1}(s_2) + y}$

1. $k \le k^*_{t-1}(s_1) - s_1 + y$

$B_{t-1}(s_1) = G'_{t-1}(k-y+s_1;s_1) \ge G'_{t-1}(k-y+s_2;s_2) = B_{t-1}(s_2)$ since $s_1 \ge s_2$

2. $k^*_{t-1}(s_1) - s_1 + y \le k \le k^*_{t-1}(s_2) - s_2 + y$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k^*_{t-1}(s_1); s_1) = 0 \geq G'_{t-1}(k - y + s_2; s_2) = B_{t-1}(s_2)}$

3. $\mathrm{k^*_{t-1}(s_2) - s_2 + y \leq k \leq k^*_{t-1}(s_1) + y}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k^*_{t-1}(s1); s1) = 0 = G'_{t-1}(k^*_{t-1}(s_2); s_2) = B_{t-1}(s_2)}$

4. $\mathrm{k^*_{t-1}(s_1) + y \leq k \leq k^*_{t-1}(s_2) + y}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - y; s_1) \geq 0 = G'_{t-1}(k^*_{t-1}(s_2); s_2) = B_{t-1}(s_2)}$

5. $\mathrm{k^*_{t-1}(s_2) + y \leq k}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - y; s_1) \geq G'_{t-1}(k - y; s_2) = B_{t-1}(s_2) \geq 0}$

2. $\mathbf{k^*_{t-1}(s_1) - s_1 + y \leq k^*_{t-1}(s_1) + y \leq k^*_{t-1}(s_2) - s_2 + y \leq k^*_{t-1}(s_2) + y}$

1. $\mathrm{k \leq k^*_{t-1}(s_1) - s_1 + y}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - y + s_1; s_1) \geq G'_{t-1}(k - y + s_2; s_2) = B_{t-1}(s_2) \quad since \quad s_1 \geq s_2}$

2. $\mathrm{k^*_{t-1}(s_1) - s_1 + y \leq k \leq k^*_{t-1}(s_1) + y}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k^*_{t-1}(s_1); s_1) = 0 \geq G'_{t-1}(k - y + s_2; s_2) = B_{t-1}(s_2)}$

3. $\mathrm{k^*_{t-1}(s_1) + y \leq k \leq k^*_{t-1}(s_2) - s_2 + y}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - y; s_1) \geq 0 \geq G'_{t-1}(k - y + s_2; s_2) = B_{t-1}(s_2)}$

4. $\mathrm{k^*_{t-1}(s_2) - s_2 + y \leq k \leq k^*_{t-1}(s_2) + y}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - y; s_1) \geq 0 = G'_{t-1}(k^*_{t-1}(s_2); s_2) = B_{t-1}(s_2)}$

5. $\mathrm{k^*_{t-1}(s_2) + y \leq k}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - y; s_1) \geq G'_{t-1}(k - y; s_2) = B_{t-1}(s_2) \geq 0}$

$A_{t-1}(s_1) \geq A_{t-1}(s_2)$ and $B_{t-1}(s_1) \geq B_{t-1}(s_2)$ for all k,
so $V'_{t-1}(k, y; s_1) \geq V'_{t-1}(k, y; s_2)$ and $G'_t(k; s_1) \geq G'_t(k; s_2)$

Proof of Theorem 4:

$V_0(x, D) = c(x)$ is convex in x.

Assume, $V_{t-1}(x, D)$ is convex. Then, $G_t(x, T(D))$ is convex and it has a finite minimizer $k^*_t(T(D))$. Then optimal policy is to order up to $k^*_t(T(D))$, and we can write the following:

$$V_t(x,D) = c(x) + \beta(1-p)G_t(x-d_1,T(D))$$

$$+\beta p \begin{cases} G_t(x-d_1+s,T(D)) & \text{if} \quad \mathrm{x-d_1 < k_t^*(T(D)) - s} \\ G_t(k_t(T(D)),T(D)) & \text{if} \quad \mathrm{k_t^*(T(D)) - s \le x - d_1 \le k_t^*(T(D))} \\ G_t(x-d_1,T(D)) & \text{if} \quad \mathrm{x-d_1 > k_t^*(T(D))} \end{cases}$$

Since G_t(x,T(D)) is convex, the above definition is summation of three convex functions so it is also convex in x.

Proof of Theorem 5:

It is sufficient to show that $G'_{t+1}(k,T(D)) \le G'_t(k,T(D)) \; \forall \; k \le k_t(T(D))$, and for all D.

For t=1, $k_1(T(D))$ minimizes $\mathrm{G_1}$(k,T(D))=c(k)=min(hk,-bk), so $k_1(T(D))$=0. When $k < k_1(T(D))$=0,

$$\mathrm{V'_1(k,S(D)+\alpha) = c'(k) + \beta(1-p)c'(k-d_2) + \beta p} \begin{cases} \mathrm{c'(k-d_2+s)} & \text{if} \quad {\scriptstyle k-d_2<-s} \\ \mathrm{c'(0)} & \text{if} \quad {\scriptstyle -s\le k-d_2\le 0} \end{cases}$$

$$\mathrm{V'_0(k,S(D)+\alpha) = c'(k)}$$

$$\mathrm{G'_2(k,T(D))} = \int_{\mathrm{y}} \mathrm{V'_1(k,S(D))f(y)dy} \le \mathrm{G'_1(k,T(D))} = \int_{\mathrm{y}} \mathrm{V'_0(k,S(D)+\alpha)f(y)dy} \quad \forall S(D)$$

so, $k_2(T(D)) \ge k_1(T(D))$ for all D

Assume $G'_t(k,T(D)) \le G'_{t-1}(k,T(D)) \; \forall \; D$ and $k \le k_{t-1}(T(D))$, so $k_{t-1}(T(D)) \le k_t(T(D)) \; \forall D$. Then, *for all D*;

$$G'_t(k,T(D)) \le G'_{t-1}(k,T(D)) \le 0 \quad for \quad k \le k_{t-1}(T(D)) \le k_t(T(D))$$

$$G'_t(k,T(D)) \le 0 \le G'_{t-1}(k,T(D)) \quad for \quad k_{t-1}(T(D)) \le k \le k_t(T(D))$$

$$G'_t(k, T(D)) \geq 0 \quad G'_{t-1}(k, T(D)) \geq 0 \quad for \quad k_{t-1}(T(D)) \leq k_t(T(D)) \leq k$$

Let's define the following;

$$G'_t(k, T(D)) = \int_y V'_{t-1}(k, S(D) + \alpha) f(y) dy$$

$$V'_t(k, S(D) + \alpha) = c'(k) + \beta(1-p)A_t + \beta p B_t$$

$$A_t = G'_t(k - d_2, T(S(D) + \alpha))$$

$$B_t = \begin{cases} G'_t(k - d_2 + s, T(S(D) + \alpha)) & \text{if} \quad k < k_t(T(S(D) + \alpha)) + d_2 - s \\ G'_t(k_t(D), T(S(D) + \alpha)) & \text{if} \quad k_t(T(S(D) + \alpha)) + d_2 - s \leq k \\ & \quad \leq k_t(T(S(D) + \alpha)) + d_2 \\ G'_t(k - d_2, T(S(D) + \alpha)) & \text{if} \quad k > k_t(T(S(D) + \alpha)) + d_2 \end{cases}$$

As a result of the above assumption, A_t is smaller than A_{t-1} for all $k \leq k_t(S(D)+\alpha)$, since $k - d_2 \leq k_t(S(D) + \alpha)$.

For B_t we should consider the following two cases, and eight intervals, where $k \leq k_t(S(D) + \alpha)$:

1. $\mathbf{k_{t-1}(\bar{D}) - s + d_2 \leq k_{t-1}(\bar{D}) + d_2 \leq k_t(\bar{D}) - s + d_2 \leq k_t(\bar{D}) + d_2}$

1. $\mathrm{k \leq k_{t-1}(\bar{D}) - s + d_2}$

$\mathrm{B_t = G'_t(k - d_2 + s, S(D) + \alpha) \leq G'_{t-1}(k - d_2 + s, S(D) + \alpha) = B_{t-1} \leq 0}$

2. $\mathrm{k_{t-1}(\bar{D}) - s + d_2 \leq k \leq k_{t-1}(\bar{D}) + d_2}$

$\mathrm{B_t = G'_t(k - d_2 + s, S(D) + \alpha) \leq 0 = G'_{t-1}(k_{t-1}(S(D) + \alpha), S(D) + \alpha) = B_{t-1}}$

3. $\mathrm{k_{t-1}(\bar{D}) + d_2 \leq k \leq k_t(\bar{D}) - s + d_2}$

$\mathrm{B_t = G'_t(k - d_2 + s, S(D) + \alpha) \leq 0 \leq G'_{t-1}(k - d_2, S(D) + \alpha) = B_{t-1}}$

4. $\mathrm{k_t(\bar{D}) - s + d_2 \leq k \leq k_t(\bar{D}) + d_2}$

$\mathrm{B_t = G'_t(k_t(\bar{D}), S(D) + \alpha) = 0 \leq G'_{t-1}(k - d_2, S(D) + \alpha) = B_{t-1}}$

2. $\mathbf{k_{t-1}(\bar{D}) - s + d_2 \leq k_t(\bar{D}) - s + d_2 \leq k_{t-1}(\bar{D}) + d_2 \leq k_t(\bar{D}) + d_2}$

1. $\mathrm{k \leq k_{t-1}(\bar{D}) - s + d_2}$

$\mathrm{B_t = G'_t(k - d_2 + s, S(D) + \alpha) \leq G'_{t-1}(k - d_2 + s, S(D) + \alpha) = B_{t-1} \leq 0}$

2. $\mathrm{k_{t-1}(\bar{D}) - s + d_2 \leq k \leq k_t(\bar{D}) - s + d_2}$

$\mathrm{B_t = G'_t(k - d_2 + s, S(D) + \alpha) \leq 0 = G'_{t-1}(k_{t-1}(S(D) + \alpha), S(D) + \alpha) = B_{t-1}}$

3. $\mathrm{k_t(\bar{D}) - s + d_2 \leq k \leq k_{t-1}(\bar{D}) + d_2}$

$\mathrm{B_t = G'_t(k_t(\bar{D}), S(D) + \alpha) = 0 = G'_{t-1}(k_{t-1}(S(D) + \alpha), S(D) + \alpha) = B_{t-1}}$

4. $\mathrm{k_{t-1}(\bar{D}) + d_2 \leq k \leq k_t(\bar{D}) + d_2}$

$\mathrm{B_t = G'_t(k_t(\bar{D}), S(D) + \alpha) = 0 \leq G'_{t-1}(k - d_2, S(D) + \alpha) = B_{t-1}}$

For all possible values of k$\leq$ $\mathrm{k}_t(S(D)+\alpha)$, and for all $S(D)+\alpha$ $\mathrm{A}_t\leq$ A_{t-1} and $\mathrm{B}_t\leq$ B_{t-1}, so $V'_t(k, S(D)+\alpha) \leq V'_{t-1}(k, S(D)+\alpha)$ $\forall S(D)+\alpha$, and $G'_{t+1}(k, T(D)) \leq G'_t(k, T(D))$ $\forall D$.

Proof of Theorem 6:

We need to show that $G'_t(k, T(D); s_1) \geq G'_t(k, T(D); s_2)$ for all k, D, and $s_1 \geq s_2$.

For t=1,$G'_1(k, T(D); s_1) = c'(k) = G'_1(k, T(D); s_2)$ and $k^*_1(T(D); s_1) = k^*_1(T(D); s_2) = 0$

Assume $G'_{t-1}(k, T(D); s_1) \geq G'_{t-1}(k, T(D); s_2) \forall k, s_1 \geq s_2$ Then, $k^*_{t-1}(T(D); s_1) \leq k^*_{t-1}(T(D); s_2)$

Note that, since $k^*_{t-1}(T(D); s_1) \leq k^*_{t-1}(T(D); s_2)$;

$G'_{t_1}(k^*_{t-1}(T(D); s_1), T(D); s_2) \leq G'_{t_1}(k^*_{t-1}(T(D); s_1), T(D); s_1)$

$\leq G'_{t_1}(k^*_{t-1}(T(D); s_2), T(D); s_1)$

In order show $G'_t(k, T(D); s_1) \geq G'_t(k, T(D); s_2)$ Let's define the following;

$$G'_t(k, T(D); s) = \int_y V'_{t-1}(k, S(D) + \alpha; s) f(y) dy$$

$$V'_{t-1}(k, S(D) + \alpha; s) = c'(k) + \beta(1-p) A_{t-1}(s) + \beta p B_{t-1}(s)$$

$$A_{t-1}(s) = G'_{t-1}(k - d_2, T(S(D) + \alpha); s)$$

$$B_{t-1}(s) = \begin{cases} G'_{t-1}(k-d_2+s, T(S(D)+\alpha); s) \\ \quad if \quad k < k_{t-1}(T(S(D)+\alpha); s) + d_2 - s \\ G'_{t-1}(k_{t-1}(T(S(D)+\alpha, T(S(D)+\alpha); s) \\ \quad if \quad k_{t-1}(T(S(D)+\alpha_n); s) + d_2 - s < k < k_{t-1}(T(S(D)+\alpha); s) + d_2 \\ G'_{t-1}(k-d_2, T(S(D)+\alpha); s) \\ \quad if \quad k > k_{t-1}(T(S(D)+\alpha); s) + d_2 \end{cases}$$

If $A_{t-1}(s_1) \geq A_{t-1}(s_2)$ and $B_{t-1}(s_1) \geq B_{t-1}(s_2)$, then $V'_{t-1}(k, S(D)+\alpha; s_1) \geq V'_{t-1}(k, S(D)+\alpha; s_2)$ which we are searching for. From the induction assumption the first inequality, $A_{t-1}(s_1) \geq A_{t-1}(s_2)$, holds true.

For B_{t-1} we should consider the following two cases with five intervals for each case:

1. $\mathbf{k^*_{t-1}(T(S(D)+\alpha); s_1) + d_2 - s_1 \leq k^*_{t-1}(T(S(D)+\alpha); s_2) + d_2 - s_2 \leq k^*_{t-1}(T(S(D)+\alpha); s_1) + d_2 \leq k^*_{t-1}(T(S(D)+\alpha); s_2) + d_2}$

1. $\mathrm{k \leq k^*_{t-1}(T(S(D)+\alpha); s_1) + d_2 - s_1}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - d_2 + s_1, T(S(D)+\alpha); s_1)}$

$\quad \mathrm{\geq G'_{t-1}(k - d_2 + s_2, T(S(D)+\alpha); s_2) = B_{t-1}(s_2)}$

2. $\mathrm{k^*_{t-1}(T(S(D)+\alpha); s_1) + d_2 - s_1 \leq k \leq k^*_{t-1}(T(S(D)+\alpha); s_2) + d_2 - s_2}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha); s_1) = 0}$

$\quad \mathrm{\geq G'_{t-1}(k - d_2 + s_2, T(S(D)+\alpha); s_2) = B_{t-1}(s_2)}$

3. $\mathrm{k^*_{t-1}(T(S(D)+\alpha); s_2) + d_2 - s_2 \leq k \leq k^*_{t-1}(T(S(D)+\alpha); s_1) + d_2}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha); s_1) = 0}$

$\quad \mathrm{= G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha); s_2) = B_{t-1}(s_2)}$

4. $\mathrm{k^*_{t-1}(T(S(D)+\alpha); s_1) + d_2 \leq k \leq k^*_{t-1}(T(S(D)+\alpha); s_2) + d_2}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - d_2, T(S(D)+\alpha); s_1) \geq 0}$

$\mathrm{= G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha); s_2) = B_{t-1}(s_2)}$

5. $\mathrm{k^*_{t-1}(T(S(D)+\alpha); s_2) + d_2 \leq k}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k - d_2, T(S(D)+\alpha); s_1) \geq G'_{t-1}(k - d_2, T(S(D)+\alpha); s_2) = B_{t-1}(s_2) \geq 0}$

$$\mathbf{1.}\quad \mathbf{k^*_{t-1}(T(S(D)+\alpha);s_1)+d_2-s_1 \le k^*_{t-1}(T(S(D)+\alpha);s_1)+d_2}$$
$$\mathbf{\le k^*_{t-1}(T(S(D)+\alpha);s_2)+d_2-s_2 \le k^*_{t-1}(T(S(D)+\alpha);s_2)+d_2}$$

1. $\mathrm{k \le k^*_{t-1}(T(S(D)+\alpha);s_1)+d_2-s_1}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k-d_2+s_1, T(S(D)+\alpha);s_1)}$

$\mathrm{\ge G'_{t-1}(k-d_2+s_2, T(S(D)+\alpha);s_2) = B_{t-1}(s_2)}$

2. $\mathrm{k^*_{t-1}(T(S(D)+\alpha);s_1)+d_2-s_1 \le k \le k^*_{t-1}(T(S(D)+\alpha);s_2)+d_2}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha);s_1) = 0}$

$\mathrm{\ge G'_{t-1}(k-d_2+s_2, T(S(D)+\alpha);s_2) = B_{t-1}(s_2)}$

3. $\mathrm{k^*_{t-1}(T(S(D)+\alpha);s_2)+d_2 \le k \le k^*_{t-1}(T(S(D)+\alpha);s_1)+d_2-s_2}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k-d_2, T(S(D)+\alpha);s_1) \ge 0}$

$\mathrm{\ge G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha);s_2) = B_{t-1}(s_2)}$

4. $\mathrm{k^*_{t-1}(T(S(D)+\alpha);s_1)+d_2-s_2 \le k \le k^*_{t-1}(T(S(D)+\alpha);s_2)+d_2}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k-d_2, T(S(D)+\alpha);s_1) \ge 0}$

$\mathrm{= G'_{t-1}(k_{t-1}(T(S(D)+\alpha), T(S(D)+\alpha);s_1) = B_{t-1}(s_2)}$

5. $\mathrm{k^*_{t-1}(T(S(D)+\alpha);s_2)+d_2 \le k}$

$\mathrm{B_{t-1}(s_1) = G'_{t-1}(k-d_2, T(S(D)+\alpha);s_1) \ge G'_{t-1}(k-d_2, T(S(D)+\alpha);s_1) = B_{t-1}(s_2) \ge 0}$

For all possible cases $B_{t-1}(s_1) \ge B_{t-1}(s_2)$.
As a result $V'_{t-1}(k, S(D)+\alpha_n; s_1) \ge V'_{t-1}(k, S(D)+\alpha; s_2)$ and $G'_t(k, T(D); s_1) \ge G'_t(k, T(D); s_2)$, meaning $k^*_t(T(D); s_1) \le k^*_t(T(D); s_2)$ for all k, D, and $s_1 \ge s_2$

Proof of Theorem 7:

We need to show if $G'_t(k, T(D_2)) \le G'_t(k, T(D_1))$

For $i = 2$, $T(D_1) = (d_2^-, d_3, ..., d_n)$ and $T(D_2) = (d_2^+, d_3, ..., d_n)$ where $d_2^- \le d_2^+$

$$G_t(k, T(D)) = \int_y V_{t-1}(k, S(D)+\alpha) f(y) dy$$

remember $S(D)+\alpha = (d_2, d_3, ..., d_n, y)$ where y is a random number representing the demand for the $n+1^{st}$ period from now on.

For $G'_t(k,T(D_2)) \leq G'_t(k,T(D_1))$ we should have $V'_{t-1}(k,S(D_2)+\alpha) \leq V'_{t-1}(k,S(D_1)+\alpha)$

$V'_{t-1}(k,S(D)+\alpha) = c'(k) + \beta(1-p)A_{t-1}(d_2) + \beta p B_{t-1}(d_2)$

where

$$A_{t-1}(d_2) = G'_{t-1}(k-d_2, T(S(D)+\alpha))$$

$$B_{t-1}(d_2) = \begin{cases} G'_{t-1}(k-d_2+s, T(S(D)+\alpha)) \\ \quad if \quad k < k^*_{t-1}(T(S(D)+\alpha)) + d_2 - s \\ G'_{t-1}(k^*_{t-1}(T(S(D)+\alpha, T(S(D)+\alpha)) \\ \quad if \quad k^*_{t-1}(T(S(D)+\alpha)) + d_2 - s \leq k \leq k^*_{t-1}(T(S(D)+\alpha)) + d_2 \\ G'_{t-1}(k-d_2, T(S(D)+\alpha)) \\ \quad if \quad k > k^*_{t-1}(T(S(D)+\alpha)) + d_2 \end{cases}$$

From Theorem 4, $G_{t-1}(x,T(D))$ is convex in x, so $G'_{t-1}(k-d_2^+, T(S(D)+\alpha)) \leq G'_{t-1}(k-d_2^-, T(S(D)+\alpha))$ since $k-d_2^+ \leq k-d_2^-$ and $A_{t-1}(d_2^+) \leq A_{t-1}(d_2^-)$

Now we should show that $B_{t-1}(d_2^+) \leq B_{t-1}(d_2^-)$

Note that $G_{t-1}(k,T(S(D)+\alpha))$ is minimized where $k = k^*_{t-1}(T(S(D)+\alpha))$ since $T(S(D_1)+\alpha) = T(S(D_2)+\alpha) = T(S(D)+\alpha) = T(d_2,d_3,...,d_n,y) = (d_3,...,d_n,y)$ $k^*_{t-1}(T(S(D_1)+\alpha)) = k^*_{t-1}(T(S(D_2)+\alpha)) = k^*$

1. $\mathbf{k^* + d_2^- - s \leq k^* + d_2^- \leq k^* + d_2^+ - s \leq k^* + d_2^+}$

1. $\mathrm{k \leq k^* + d_2^- - s}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k-d_2^-+s, T(S(D)+\alpha)) \geq G'_{t-1}(k-d_2^+, T(S(D)+\alpha)) = B_{t-1}(d_2^+)}$

2. $\mathrm{k^* + d_2^- - s \leq k \leq k^* + d_2^-}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k^*, T(S(D)+\alpha)) = 0 \geq G'_{t-1}(k-d_2^+ + s, T(S(D)+\alpha)) = B_{t-1}(d_2^+)}$

3. $\mathrm{k^* + d_2^- \leq k \leq k^* + d_2^+ - s}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k-d_2^-, T(S(D)+\alpha)) \geq 0 \geq G'_{t-1}(k-d_2^+ + s, T(S(D)+\alpha)) = B_{t-1}(d_2^+)}$

4. $\mathrm{k^* + d_2^+ - s \leq k \leq k^* + d_2^+}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k-d_2^-, T(S(D)+\alpha)) \geq 0 = G'_{t-1}(k^*, T(S(D)+\alpha)) = B_{t-1}(d_2^+)}$

5. $\mathrm{k^* + d_2^+ \leq k}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k - d_2^-, T(S(D) + \alpha)) \geq G'_{t-1}(k - d_2^+, T(S(D) + \alpha)) = B_{t-1}(d_2^+) \geq 0}$

2. $\mathbf{k^* + d_2^- - s \leq k^* + d_2^+ - s \leq k^* + d_2^- \leq k^* + d_2^+}$

1. $\mathrm{k \leq k^* + d_2^- - s}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k - d_2^- + s, T(S(D) + \alpha)) \geq G'_{t-1}(k - d_2^+, T(S(D) + \alpha)) = B_{t-1}(d_2^+)}$

2. $\mathrm{k^* + d_2^- - s \leq k \leq k^* + d_2^+ - s}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k^*, T(S(D) + \alpha_)) = 0 \geq G'_{t-1}(k - d_2^+ + s, T(S(D) + \alpha)) = B_{t-1}(d_2^+)}$

3. $\mathrm{k^* + d_2^+ - s \leq k \leq k^* + d_2^-}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k^*, T(S(D) + \alpha)) = 0 = G'_{t-1}(k^*, T(S(D) + \alpha)) = B_{t-1}(d_2^+)}$

4. $\mathrm{k^* + d_2^- \leq k \leq k^* + d_2^+}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k - d_2^-, T(S(D) + \alpha)) \geq 0 = G'_{t-1}(k^*, T(S(D) + \alpha)) = B_{t-1}(d_2^+)}$

5. $\mathrm{k^* + d_2^+ \leq k}$

$\mathrm{B_{t-1}(d_2^-) = G'_{t-1}(k - d_2^-, T(S(D) + \alpha)) \geq G'_{t-1}(k - d_2^+, T(S(D) + \alpha)) = B_{t-1}(d_2^+) \geq 0}$

For all possible intervals for k we showed that $B_{t-1}(d_2^+) \leq B_{t-1}(d_2^-)$, and we know that $A_{t-1}(d_2^+) \leq A_{t-1}(d_2^-)$ so $V'_{t-1}(k, S(D_2) + \alpha) \leq V'_{t-1}(k, S(D_1) + \alpha)$, $G'_t(k, T(D_2)) \leq G'_t(k, T(D_1))$, and $k^*_t(T(D_1)) \leq k^*_t(T(D_2))$.

That proves the theorem for $i = 2$. Based on this prove, we will show that the theorem holds for any i, using the induction method.

Induction Assumption: $G'_t(k, T(D))$ decreases when the i^th element of the demand vector, d_i, increases, i.e,

$G'_t(k, T(D_2)) \leq G'_t(k, T(D_1))$ where $T(D_1) = (d_2, ..., d_i^-, ..., d_n)$ and $T(D_2) = (d_2, ..., d_i^+, ..., d_n)$ for $d_i^- \leq d_i^+$.

If the inequality holds for $i + 1$ we can say that it holds for any i. That is to say, $G'_t(k, T(D_2)) \leq G'_t(k, T(D_1))$ where $T(D_1) = (d_2, ..., d_i, d_{i+1}^-, ..., d_n)$ and $T(D_2) = (d_2, ..., d_i, d_{i+1}^+, ..., d_n)$ for $d_{i+1}^- \leq d_{i+1}^+$.

$V'_{t-1}(k, S(D_2) + \alpha) \leq V'_{t-1}(k, S(D_1) + \alpha)$

Proof of Theorem 8:

$J_1(x,d,r) = \{\int_0^r c(x-d+w)dL(w) + \int_r^s c(x-d+r)dL(w)\}$

$$\delta J_1(x,d,r)/\delta r = J_1'(x,d,r) = c'(x-d+r)[1-L(r)] = \begin{cases} -b[1-L(r)] & \text{if} \quad \mathrm{r < d - x} \\ h[1-L(r)] & \text{if} \quad \mathrm{r \geq d - x} \end{cases}$$

Since $-b[1-L(r)]$ is always negative and $h[1-L(r)]$ is always positive, $J_1'(x,d,r)$ is decreasing up to $d-x$ and then it is increasing.

$$\delta^2 J_1(x,d,r)/\delta r^2 = J_1''(x,d,r) = \begin{cases} bl(r) & \text{if} \quad \mathrm{r < d - x} \\ -hl(r) & \text{if} \quad \mathrm{r \geq d - x} \end{cases}$$

Since $bl(r)$ is positive $J_1(x,d,r)$ is convex in r up to $r = d-x$, but over $d-x$, it is concave since $-hl(r)$ is negative, but still has a unique minimum where $r = r_1 * (x,d) = d-x$.

Proof of Theorem 9:(cont'd)

For the second case:

2. $\mathbf{d - s \leq \bar{x}(d) \leq 0 \leq d}$

1. $x \leq d-s$ $\quad V_1'(x,d) = -b-b \leq 0$
2. $d-s \leq x \leq \bar{x}(d)$ $\quad V_1'(x,d) = -b - bL(d-x) \leq 0$
3. $\bar{x}(d) \leq x \leq 0$ $\quad V_1'(x,d) = -b - bL(d-x) \leq 0$
4. $0 \leq x \leq d$ $\quad V_1'(x,d) = h - bL(d-x) \geq 0$ since $bL(d-x) \leq h$ for $x \geq \bar{x}(d)$
5. $d \leq x$ $\quad V_1'(x,y) = h + h \geq 0$

Both for $h \leq b$ and $h \geq b$, $V_1'(x,d)$ is negative in the first three intervals and it's positive in the last two intervals. Thus as x increases $V_1(x,d)$ decreases upto a critical point, $x_1^*(d) = 0$, and then it increases.

For the third case:

3. $\mathbf{d - s \leq 0 \leq \bar{x}(d) \leq d}$

1. $x \leq d - s$ $V_1'(x,d) = -b - b \leq 0$

2. $d - s \leq x \leq 0$ $V_1'(x,d) = -b - bL(d-x) \leq 0$

3. $0 \leq x \leq \bar{x}(d)$ $V_1'(x,d) = h - bL(d-x) \leq 0$ if $h \leq b$

$V_1'(x,d) = h - bL(d-x) \geq 0$ if $h \geq b$

4. $\bar{x}(d) \leq x \leq d$ $V_1'(x,d) = h - bL(d-x) \geq 0$ since $bL(d-x) \leq h$ for $x \geq \bar{x}(d)$

5. $d \leq x$ $V_1'(x,y) = h + h \geq 0$

In this case if $h \leq b$, $x_1^*(d) = \bar{x}(d)$, and if $h \geq b$, $x_1^*(d) = 0$

APPENDIX B

RESULTS FOR THE PARTIALLY AVAILABLE SUPPLY PROCESS

Table B.1: Optimum ordering quantities and costs when x=0, b=5, n=1

	Demand distribution=F1, var(demand)=0.65											
	P1			**P2**			**P3**			**P4**		
d_1	**r***	**k**	**V**	**r***	**k**	**V**	**r***	**k**	**V**	**r***	**k**	**V**
0	0	0	3.1	1	1	6.4	2	2	12.2	3	3	16.5
1	1	0	3.1	2	1	7.5	3	2	16.6	4	3	22.7
2	2	0	3.1	3	1	13.3	4	2	23.6	4	2	30.3
3	2	-1	13.6	3	0	22.3	4	1	33.5	4	1	40.3
	Demand distribution=F2, var(demand)=1.25											
	P1			**P2**			**P3**			**P4**		
d_1	**r***	**k**	**V**	**r***	**k**	**V**	**r***	**k**	**V**	**r***	**k**	**V**
0	1	1	4.8	2	2	8.6	3	3	14.0	3	3	18.1
1	2	1	4.8	3	2	9.9	4	3	18.5	4	3	24.3
2	2	0	6.8	3	1	16.1	4	2	25.6	4	2	32.2
3	2	-1	18.2	3	0	25.2	4	1	35.5	4	1	42.1
	Demand distribution=F3, var(demand)=1.85											
	P1			**P2**			**P3**			**P4**		
d_1	**r***	**k**	**V**	**r***	**k**	**V**	**r***	**k**	**V**	**r***	**k**	**V**
0	1	1	6.6	2	2	10.2	3	3	15.7	3	3	19.4
1	2	1	6.6	3	2	11.7	4	3	20.3	4	3	25.6
2	2	0	10	3	1	18.4	4	2	27.6	4	2	34.1
3	2	-1	21.7	3	0	27.8	4	1	37.5	4	1	44.1

Table B.2: Optimum ordering quantities and costs when x=0, b=10, n=1

	Demand distribution=F1, var(demand)=0.65											
	P1			P2			P3			P4		
d_1	r*	k	V	r*	k	V	r*	k	V	r*	k	V
0	1	1	4.2	2	2	8.8	3	3	19	4	4	26.8
1	2	1	4.2	3	2	11.8	4	3	28.7	4	3	40.1
2	2	0	5.5	3	1	24.4	4	2	43.8	4	2	56.5
3	2	-1	26.9	3	0	43.4	4	1	63.7	4	1	77.5
	Demand distribution=F2, var(demand)=1.25											
	P1			P2			P3			P4		
d_1	r*	k	V	r*	k	V	r*	k	V	r*	k	V
0	1	1	6.4	2	2	12.0	3	3	22.0	4	4	29.4
1	2	1	6.4	3	2	15.5	4	3	31.9	4	3	42.8
2	2	0	11.8	3	1	29.2	4	2	47.5	4	2	60.0
3	2	-1	35.3	3	0	48.7	4	1	68.4	4	1	80.9
	Demand distribution=F3, var(demand)=1.85											
	P1			P2			P3			P4		
d_1	r*	k	V	r*	k	V	r*	k	V	r*	k	V
0	2	2	8.0	3	3	14.8	4	4	24.6	4	4	31.7
1	2	1	9.8	3	2	18.9	4	3	34.8	4	3	44.9
2	2	0	18.1	3	1	33.5	4	2	50.9	4	2	63.1
3	2	-1	42.1	3	0	53.5	4	1	72.0	4	1	84.4

Table B.3: Optimum ordering quantities and costs when x=0, b=5, n=2

		Demand distribution=F1, var(demand)=0.65											
		P1			P2			P3			P4		
d_1	d_2	r*	k	v	r*	k	v	r*	k	v	r*	k	v
0	**0**	0	0	0.8	0	0	3.2	1	1	7.6	1	1	10.2
0	**1**	0	0	0.8	1	1	4.2	2	2	9.5	2	2	13.5
0	**2**	0	0	1.6	2	2	5.2	3	3	12.5	3	3	17.3
0	**3**	1	1	2.6	3	3	8.5	4	4	16.8	4	4	22.0
1	**0**	1	0	0.8	1	0	3.2	2	1	10.3	2	1	15.5
1	**1**	1	0	0.8	2	1	4.2	3	2	13.3	3	2	19.3
1	**2**	1	0	1.6	3	2	7.5	4	3	17.6	4	3	24.0
1	**3**	2	1	2.6	3	2	12.1	4	3	23.9	4	3	30.1
2	**0**	2	0	0.8	2	0	6.2	3	1	15.0	3	1	21.3
2	**1**	2	0	0.8	3	1	9.5	4	2	19.3	4	2	26.0
2	**2**	2	0	1.6	3	1	14.1	4	2	25.6	4	2	32.1
2	**3**	2	0	11.8	3	1	23.8	4	2	34.8	4	2	42.2
3	**0**	2	-1	5.8	3	0	11.5	4	1	21.6	4	1	28.0
3	**1**	2	-1	6.6	3	0	16.1	4	1	27.9	4	1	34.1
3	**2**	2	-1	16.8	3	0	25.8	4	1	37.1	4	1	44.2
3	**3**	2	-1	32.1	3	0	38.0	4	1	48.6	4	1	55.4

		Demand distribution=F2, var(demand)=1.25											
		P1			P2			P3			P4		
d_1	d_2	r*	k	v	r*	k	v	r*	k	v	r*	k	v
0	**0**	0	0	1.8	0	0	4.2	1	1	8.2	1	1	10.9
0	**1**	0	0	1.9	1	1	5.2	2	2	10.2	2	2	14.2
0	**2**	1	1	2.9	2	2	6.3	3	3	13.2	3	3	18.2
0	**3**	2	2	3.9	3	3	9.8	4	4	17.7	4	4	22.9
1	**0**	1	0	1.8	1	0	4.2	2	1	11.0	2	1	16.2
1	**1**	1	0	1.9	2	1	5.3	3	2	14.0	3	2	20.2
1	**2**	2	1	2.9	3	2	8.8	4	3	18.5	4	3	24.9
1	**3**	2	1	5.1	3	2	13.5	4	3	24.8	4	3	30.7
2	**0**	2	0	1.8	2	0	7.3	3	1	15.7	3	1	22.2
2	**1**	2	0	1.9	3	1	10.8	4	2	20.2	4	2	26.9
2	**2**	2	0	4.1	3	1	15.5	4	2	26.5	4	2	32.7
2	**3**	2	0	14.8	3	1	25.3	4	2	35.7	4	2	42.9
3	**0**	2	-1	6.9	3	0	12.8	4	1	22.5	4	1	28.9
3	**1**	2	-1	9.1	3	0	17.5	4	1	28.8	4	1	34.7
3	**2**	2	-1	19.8	3	0	27.3	4	1	38.0	4	1	44.9
3	**3**	2	-1	34.0	3	0	39.4	4	1	49.5	4	1	56.3

Table B.3: Cont'd

		Demand distribution=F3, var(demand)=1.85											
		P1			P2			P3			P4		
d_1	d_2	r*	k	v	r*	k	v	r*	k	v	r*	k	v
0	**0**	0	0	2.3	0	0	5.2	1	1	9.0	1	1	11.5
0	**1**	0	0	3.0	1	1	6.2	2	2	10.9	2	2	14.7
0	**2**	1	1	4.0	2	2	7.4	3	3	14.1	3	3	19.0
0	**3**	2	2	5.0	3	3	11.1	4	4	18.6	4	4	23.8
1	**0**	1	0	2.3	1	0	5.2	2	1	11.7	2	1	16.7
1	**1**	1	0	3.0	2	1	6.4	3	2	14.9	3	2	21.0
1	**2**	2	1	4.0	3	2	10.1	4	3	19.4	4	3	25.8
1	**3**	2	1	7.4	3	2	14.9	4	3	25.6	4	3	31.3
2	**0**	2	0	2.3	2	0	8.4	3	1	16.6	3	1	23.0
2	**1**	2	0	3.0	3	1	12.1	4	2	21.1	4	2	27.8
2	**2**	2	0	6.4	3	1	16.9	4	2	27.3	4	2	33.3
2	**3**	2	0	17.2	3	1	26.7	4	2	36.6	4	2	43.3
3	**0**	2	-1	8.0	3	0	14.1	4	1	23.4	4	1	29.8
3	**1**	2	-1	11.4	3	0	18.9	4	1	29.6	4	1	35.3
3	**2**	2	-1	22.2	3	0	28.7	4	1	38.9	4	1	45.3
3	**3**	2	-1	35.8	3	0	40.9	4	1	50.4	4	1	57.1

Table B.4: Optimum ordering quantities and costs when x=0, b=10, n=2

		Demand distribution=F1, var(demand)=0.65											
		P1			P2			P3			P4		
d_1	d_2	r*	k	v	r*	k	v	r*	k	v	r*	k	v
0	**0**	0	0	1.3	0	0	4.8	2	2	10.8	2	2	15.3
0	**1**	0	0	1.3	1	1	5.8	3	3	14.4	3	3	21.6
0	**2**	1	1	2.3	2	2	7.5	4	4	20.3	4	4	29.0
0	**3**	2	2	3.3	3	3	13.8	4	4	29.2	4	4	39.2
1	**0**	1	0	1.3	1	0	4.8	3	2	16.7	3	2	26.1
1	**1**	1	0	1.3	2	1	6.5	4	3	22.6	4	3	33.5
1	**2**	2	1	2.3	3	2	12.8	4	3	31.5	4	3	43.7
1	**3**	2	1	4.2	3	2	22.4	4	3	44.7	4	3	56.6
2	**0**	2	0	1.3	2	0	11.0	4	2	26.5	4	2	38.0
2	**1**	2	0	1.3	3	1	17.3	4	2	35.5	4	2	48.2
2	**2**	2	0	3.2	3	1	26.9	4	2	48.6	4	2	61.1
2	**3**	2	0	23.6	3	1	46.6	4	2	67.4	4	2	81.6
3	**0**	2	-1	11.3	3	0	21.8	4	1	40.5	4	1	52.7
3	**1**	2	-1	13.2	3	0	31.4	4	1	53.7	4	1	65.6
3	**2**	2	-1	33.6	3	0	51.1	4	1	72.5	4	1	86.1
3	**3**	2	-1	64.2	3	0	75.8	4	1	96	4	1	108.9

		Demand distribution=F2, var(demand)=1.25											
		P1			P2			P3			P4		
d_1	d_2	r*	k	v	r*	k	v	r*	k	v	r*	k	v
0	**0**	0	0	2.0	1	1	6.4	2	2	12.1	3	3	16.7
0	**1**	0	0	2.7	2	2	7.4	3	3	15.7	4	4	23.1
0	**2**	1	1	3.7	3	3	9.4	4	4	21.9	4	4	30.9
0	**3**	2	2	4.7	3	3	16.2	4	4	31.0	4	4	41.0
1	**0**	1	0	2.0	2	1	6.4	3	2	18.0	4	3	27.6
1	**1**	1	0	2.7	3	2	8.4	4	3	24.2	4	3	35.4
1	**2**	2	1	3.7	3	2	15.2	4	3	33.3	4	3	45.5
1	**3**	2	1	8.8	3	2	25.2	4	3	46.5	4	3	58.1
2	**0**	2	0	2.0	3	1	12.9	4	2	28.1	4	2	39.9
2	**1**	2	0	2.7	3	1	19.7	4	2	37.2	4	2	50.0
2	**2**	2	0	7.8	3	1	29.7	4	2	50.4	4	2	62.6
2	**3**	2	0	29.4	3	1	49.5	4	2	69.4	4	2	83.1
3	**0**	2	-1	12.7	3	0	24.2	4	1	42.3	4	1	54.5
3	**1**	2	-1	17.8	3	0	34.2	4	1	55.5	4	1	67.1
3	**2**	2	-1	39.4	3	0	54.0	4	1	74.4	4	1	87.6
3	**3**	2	-1	68.0	3	0	78.6	4	1	97.9	4	1	110.8

Table B.4: Cont'd

		Demand distribution=F3, var(demand)=1.85											
		P1			P2			P3			P4		
d_1	d_2	r*	k	v	r*	k	v	r*	k	v	r*	k	v
0	**0**	0	0	2.8	1	1	7.4	2	2	13.4	3	3	17.8
0	**1**	1	1	3.8	2	2	8.9	3	3	17.1	4	4	24.1
0	**2**	2	2	4.8	3	3	11.1	4	4	23.5	4	4	32.6
0	**3**	2	2	6.8	3	3	18.6	4	4	32.8	4	4	43.0
1	**0**	1	0	2.8	2	1	7.9	3	2	19.4	4	3	28.6
1	**1**	2	1	3.8	3	2	10.1	4	3	25.8	4	3	37.1
1	**2**	2	1	5.8	3	2	17.6	4	3	35.1	4	3	47.5
1	**3**	2	1	13.2	3	2	27.9	4	3	48.3	4	3	59.4
2	**0**	2	0	2.8	3	1	14.6	4	2	29.7	4	2	41.6
2	**1**	2	0	4.8	3	1	22.1	4	2	39.1	4	2	52.0
2	**2**	2	0	12.2	3	1	32.4	4	2	52.3	4	2	63.9
2	**3**	2	0	34.0	3	1	52.3	4	2	71.2	4	2	84.2
3	**0**	2	-1	14.8	3	0	26.6	4	1	44.1	4	1	56.5
3	**1**	2	-1	22.2	3	0	36.9	4	1	57.3	4	1	68.4
3	**2**	2	-1	44.0	3	0	56.8	4	1	76.3	4	1	88.7
3	**3**	2	-1	71.5	3	0	81.4	4	1	99.8	4	1	112.6

Table B.5: Percent change in the expected cost when x=0, b=5

	F1				F2				F3			
d_1	**P1**	**P2**	**P3**	**P4**	**P1**	**P2**	**P3**	**P4**	**P1**	**P2**	**P3**	**P4**
0	58.06	22.97	7.87	5.82	45.31	25.87	11.96	8.56	45.15	22.75	13.76	9.85
1	58.06	17.20	4.94	3.61	39.06	19.70	7.70	5.35	30.61	17.18	9.61	6.72
2	28.39	6.47	2.80	2.34	16.91	8.54	4.20	3.18	12.60	7.93	5.36	4.31
3	3.31	2.65	1.43	1.59	4.12	3.77	2.25	2.14	3.78	3.74	3.01	2.90

Table B.6: Percent change in the expected cost when x=0, b=10

	F1				F2				F3			
d_1	**P1**	**P2**	**P3**	**P4**	**P1**	**P2**	**P3**	**P4**	**P1**	**P2**	**P3**	**P4**
0	54.76	18.41	5.89	4.14	48.83	17.92	8.30	5.02	41.25	16.22	8.37	5.39
1	52.62	11.53	3.21	2.37	32.81	10.97	4.39	2.69	24.90	9.58	4.68	2.76
2	22.00	3.93	1.76	1.45	11.23	4.28	2.58	1.83	9.28	3.85	2.75	1.89
3	2.34	1.47	0.88	0.85	2.34	1.95	1.28	1.11	2.28	1.74	1.50	1.24