THE MODULI OF SURFACES ADMITTING GENUS TWO FIBRATIONS OVER ELLIPTIC CURVES

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

THE MODULI OF SURFACES ADMITTING GENUS TWO FIBRATIONS OVER ELLIPTIC CURVES

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In this thesis, we study the structure, deformations and the moduli spaces of complex projective surfaces admitting genus two fibrations over elliptic curves. We observe that, a surface admitting a smooth fibration as above is elliptic and we employ results on the moduli of polarized elliptic surfaces, to construct moduli spaces of these smooth fibrations. In the case of nonsmooth fibrations, we relate the moduli spaces to the Hurwitz schemes $\mathcal{H}(1, X(d), n)$ of morphisms of degree n from elliptic curves to the modular curve $X(d)$, $d \geq 3$. Ultimately, we show that the moduli spaces, considered, are fiber spaces over the affine line $\mathbb{A}^1$ with fibers determined by the components of $\mathcal{H}(1, X(d), n)$.

Keywords: Moduli spaces, fibrations, Hurwitz schemes

ÖZ

ELİPTİK EĞRİLER ÜSTÜNDE CİNS İKİ EĞRİLERLE LİFLENEN YÜZEYLERİN MODÜL UZAYLARI

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Bu tezde eliptik eğriler üstünde cins iki eğrilerle liflenen karmaşık projektif yüzeylerin yapısını, deformasyonlarını ve modül uzaylarını inceledik. Bu şekilde düzgün liflenme kabul eden yüzeylerin eliptik olduklarını gözlemledik ve bu düzgün liflenmelerin modül uzaylarını kurmak için kutuplaşmış eliptik yüzeylerin modül uzayları hakkındaki sonuçları kullandık. Düzgün olmayan liflenmelerin modül uzaylarını ise eliptik eğrilerden modüler eğri $X(d)$'ye olan n dereceli morfizmaların Hurwitz şemaları, $\mathcal{H}(1, X(d), n)$, ile ilişkilendirdik. Sonuç olarak, incelediğimiz modül uzaylarının afin doğru $\mathbb{A}^1$ üstünde, lifleri $\mathcal{H}(1, X(d), n)$'nin bileşenleri ile belirlenmiş olan lif uzayları olduklarını gösterdik.

Anahtar Kelimeler: Modül uzayları, liflenmeler, Hurwitz şemaları

To Oğuz Kaya

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CHAPTER 1

INTRODUCTION

The aim of this thesis is to work out the structure, deformations and the moduli spaces of complex projective surfaces admitting genus two fibrations over elliptic curves.

In the literature, the cases of albanese fibrations with fiber genus two over arbitrary base curves and nonalbanese fibrations over curves of genus $g \geq 2$ have been studied extensively ([14], [15] for the former type and [9], [8], [10] for the latter). We aim at complementing these results by examining the case of fibrations with irregularity $q(S) = 2$ over elliptic curves. These fibrations are of nonalbanese type and have Kodaira dimension $\kappa(S) = 1$ (respectively 2) in case the given fibration is smooth (respectively non-smooth). Thus as a by product of our results on the corresponding moduli spaces we will observe the well-known contrast between the behavior of elliptic surfaces and that of surfaces of general type; to construct moduli spaces for elliptic surfaces, one has to impose a choice of polarization on the surfaces. This leads to, as expected, a weaker result on the moduli problem.

We work over the complex numbers $\mathbb{C}$ and use the following standard notation:

S is a smooth projective surface.

$c_1(S),\ c_2(S)$ denote the first and the second chern classes of S, respectively.

$\kappa(S),\ q(S)$ are the Kodaira dimension and the irregularity of S, respectively.

$K(S),\ \chi(S)$ are the canonical class and the holomorphic Euler characteristic of S.

For fixed K^2 and χ, $\mathcal{M}(g, K^2, \chi)$ is the moduli space of surfaces of general type admitting genus two fibration with irregularity $q = g + 1$ and slope λ which satisfies the slope formula $K^2 = \lambda\chi + (8 - \lambda)(g - 1)$.

In the first section of the thesis we study the structure of the surfaces under consideration and we obtain the following results:

Lemma 1.1. *Let $\pi : S \to C$ be a smooth genus two fibration. Then π is isotrivial, say with monodromy group G and fiber F. Moreover, we have*

(i) If $q(S) = g(C) + 2$, then $S \to C$ is trivial.

(ii) If $q(S) \leq g(C) + 1$, then S admits a second fibration $\varphi : S \to F/G$.

(iii) If $q(S) = g(C) + 1$, then $E'' = F/G$ is an elliptic curve and φ is smooth with fiber C' except for two double fibers of the form $2.C$. Here C' is the Galois étale cover of C with group G.

Corollary 1.2. *Let $\pi : S \to E$ be a smooth genus two fibration over an elliptic curve E with $q(S) = 2$. Then S admits an elliptic fibration with two double fibers of the form $2.E$. All other fibers are smooth and are isomorphic to E'*

(*the double cover of* E *corresponding to the monodromy representation arising from* π).

Lemma 1.3. *Let* $\mathcal{S} \to T$ *be a deformation of* S. *Then there exists an elliptic curve* $\mathcal{E}'' \to T$ *and two sections* $s_1, s_2 : T \to \mathcal{E}''$ *such that* $\mathcal{S} \to T$ *factors through* $\mathcal{E}''$. *Furthermore,* $\mathcal{S} \to \mathcal{E}''$ *is smooth outside* $s_1(T) \cup s_2(T)$ *and for each* $t \in T$ *the restriction of* $\mathcal{S} \to \mathcal{E}''$ *induces an elliptic fibration* $\mathcal{S}_t \to \mathcal{E}''_t$ *with precisely two double fibers* (*over* $s_1(t), s_2(t)$).

We have the following converse of Lemma 1.1, which will be crucial in relating the moduli spaces of our surfaces to the moduli of elliptic surfaces.

Lemma 1.4. *An elliptic surface with exactly two double fibers admits a smooth genus two fibration.*

As for the nonsmooth fibrations $\pi : S \to E$ with $q(S) = 2$, we first recall the following fundamental result which is a special case of ([16], Théorème 3.10, p.44)

Theorem 1.5. *Let* E *be an elliptic curve,* d *an integer* ≥ 3. *There exists a genus two fibration of type* (E, d)

$$\Phi : S(E, d) \to X(d)$$

on the modular curve $X(d)$ *which is universal in the following sense: any genus two fibration* $\pi : S \to C$ *with slope* $\lambda = 7 - \dfrac{6}{d}$ *and with* E *as the fixed part of the Jacobian fibration corresponding to* π (*i.e.,* π *is of type* (E, d)) *is the minimal desingularization of the pullback* $f^*(S(E, d))$ *via a surjective holomorphic map* $X(d)$.

Remark: Given $f : C \to X(d)$, the surface $f^*(S(E,d))$ has singularities only if f ramifies over some points in the singular locus of $\Phi : S(E,d) \to X(d)$. A singular fiber of Φ is either an elliptic curve with a single node or two smooth elliptic curves intersecting transversally at a single point ([16], Lemme 3.11, Théorème 3.16). Hence singularities of $f^*(S(E,d))$ are all type A_k for some k depending on the singular point.

We prove

Lemma 1.6. *For a fibration $\pi : S \to C$ over a curve C of genus ≥ 1 arising from a map $f : C \to X(d)$ of degree n we have $c_2(S) > 0$ and $K^2 = {c_1}^2(S) > 0$. In particular, since S is minimal, it is a surface of general type.*

Lemma 1.7. *S with such a fibration over an elliptic curve exists if and only if K^2 and χ have the following values:*

λ	K^2	χ
5	$5n$	n
11/2	$11n$	$2n$
29/5	$29n$	$5n$
6	$36n$	$6n$

where $n \geq 2$ in the first three rows and $n \geq 1$ in the last row.

Corollary 1.8. *(i) The fibration on S is unique up to isomorphism of fibered surfaces.*

(ii) Let $\mathcal{S} \to T$ be a family of surfaces admitting genus two fibrations with λ as above, over elliptic curves. For $t \in T$, let f_t be the map onto $X(d)$

inducing the fibration on $\mathcal{S}_t$. If T is connected, then $deg(f_t)$ is constant.

With these results on the structure and deformations of the surfaces considered available, in Chapter 2 we proceed with the construction of the moduli spaces.

a) The smooth case:

We use the results of Seiler ([12], [13]) on the moduli of elliptic surfaces. We first observe that the type ([13], Definition, p.210) of our surfaces is given by $\tau = (1, 0; 2, 2, 2; 2, 1)$. Then we consider the moduli problem for surfaces S of type τ as given. We obtain

Proposition 1.9. *The corresponding functor is coarsely represented by an irreducible scheme $\mathcal{M}$ of dimension* 3.

To get a more natural description of this functor we observe the following. Roughly, forgetting τ, the moduli of the surfaces we consider is closely related to the moduli of isogenies of degree two of elliptic curves (the base curves) and the moduli of smooth genus two curves C admitting an elliptic subcover $C \to E$ of degree two. The functor corresponding to the first moduli is coarsely represented by affine modular curve $Y_0(2)$. The functor corresponding to the latter one is an open subscheme $\mathcal{H}$ of $\mathcal{A}_{2,1}(X(2) \times X(2))/SL_2(\mathbb{Z})$ ([4], p.210).

Proposition 1.10. *There exists a natural surjective morphism $\phi : \mathcal{M} \to \mathcal{H}$.*

b) The non-smooth case:

We see that a surface S admitting a fibration of type (E', d) induced by a map $f : C \to X(d)$ can be deformed in two ways; we can deform E' to other elliptic curves and we can deform the map f. Therefore, in describing the moduli spaces of surfaces under consideration, we need to clarify the relation of these moduli spaces to the Hurwitz spaces $\mathcal{H}(g, X(d), n)$ of morphisms of degree n from curves of genus g to the modular curve $X(d)$.

We fix an elliptic curve E' and we consider the morphism

$$\Psi_{E'} : \mathcal{H}(g, X(d), n) \to \mathcal{M}(g, K^2, \chi)$$

which corresponds to the morphism of functors mapping $f \in \mathcal{H}(g, X(d), n)(T)$ to the family over T, obtained from $f^*(S(E', d))$ by simultaneous desingularization.

Even though our main interest is in fibrations over elliptic curves, since we will adopt the methods used in [9], we will quote the essential part of [9] which will not be published. As a by product, we will obtain the following algebraic version of the main result in [10] describing the structure of the moduli spaces of surfaces fibered over curves of genus $g \geq 2$. We note that the result for base curves of genus $g \geq 2$ is stronger than the result in the case of $g = 1$. This is due to the fact that we can not prove Lemma 2.9 in full strength when $g = 1$.

Theorem 1.11. *Let K^2, χ and $g \geq 2$ be given and let $\mathcal{H}(g, X(d), n)$ be the Hurwitz scheme of morphisms of degree n from curves of genus g onto $X(d)$. Then we have morphisms $\Phi : \mathcal{M}(g, K^2, \chi) \to \mathbb{A}^1$ and $\Psi_{E'} : \mathcal{H}(g, X(d), n) \to \mathcal{M}(g, K^2, \chi)$ for any elliptic curve E' such that*

(i) $\Psi_{E'}$ establishes a one-to-one correspondence between the components $\mathcal{H}_i$ of $\mathcal{H}(g, X(d), n)$ and the components $\mathcal{M}_i$ of $\mathcal{M}(g, K^2, \chi)$,

(ii) $\Phi : \mathcal{M}_i \to \mathbb{A}^1$ *is a fibration with* $\Psi_{E'}(\mathcal{H}_i)$ *as the fiber over* $[E'] \in \mathbb{A}^1$.

Applying the same methods we obtain the following weaker result:

Theorem 1.12. *Let* $\mathcal{M}_i$ *be a connected component of* $\mathcal{M}(1, K^2, \chi)$*. Then we have a morphism* $\Phi : \mathcal{M}_i \to \mathbb{A}^1$ *(given on closed points by* $[X] \to [E]$ *if* X *is of type* (E, d)*) such that the fiber over* $[E] \in \mathbb{A}^1$ *is a disjoint union*

$$\bigsqcup_j \Psi_E(\mathcal{H}(1, X(d), n)_j).$$

In the appendix we discuss some questions which arise naturally in the context of the problems studied in this thesis. These are related to

(i) moduli problem for genus two fibrations over $\mathbb{P}^1$,

(ii) the moduli spaces in the $d = 2$ case of nonsmooth genus two fibrations and

(iii) the compactification of the moduli spaces via "minimal" degenerations of the surfaces studied.

CHAPTER 2

STRUCTURE OF GENUS TWO FIBRATIONS

Let $\pi : S \to C$ be a connected genus two fibration on a projective minimal smooth surface over a smooth curve C. It is well-known that $g(C) \leq q(S) \leq g(C) + 2$. For reasons explained in the introduction we will consider the following cases: C is an elliptic curve and $q(S) = 3$ or 2, i.e., the given fibration is of nonalbanese type.

First we discuss the case of smooth fibrations. Occasionally we will include proofs of some basic results which are well-known and hold true for base curves of arbitrary genus. In the special case of smooth nontrivial fibrations over elliptic curves, we will see that such surfaces admit elliptic fibrations and this observation will play a crucial role in the discussion of the moduli of these surfaces.

Lemma 2.1. *Let $\pi : S \to C$ be a smooth genus two fibration over a curve C of arbitrary genus. Then we have:*

(i) If $q(S) = g(C) + 2$, then $S \to C$ is trivial.

(ii) If $q(S) \leq g(C) + 1$, then S admits a second fibration $\varphi : S \to F/G$.

(iii) If $q(S) = g(C) + 1$, then $E'' = F/G$ is an elliptic curve and φ is smooth with fiber C' except for two double fibers of the form $2.C$. Here, C' is the Galois étale cover of C with group G.

Proof. **(i)** Since $\mathcal{M}_2$, the moduli space of genus two curves, is affine the modulus map $C \to \mathcal{M}_2$ induced by π is constant. Hence π is an analytic fiber bundle say with fiber F. As $Aut(F)$ is a finite group, this fiber bundle corresponds to a (monodromy) representation $\rho : \pi_1(C) \to Aut(F)$ and we obtain a normal unramified cover $C' \to C$ with covering group $G = Im(\rho)$ from the exact sequence

$$0 \to Kernel(\rho) \to \pi_1(C) \to Im(\rho) \to 0.$$

Clearly the smooth fibration $S' = S \times_C C' \to C'$ has trivial monodromy and therefore is trivial. To prove the statement we show that G is trivial. For this we calculate $q(S)$ using the Galois cover $C' \times F = S' \to S \cong S'/G$ to get

$$\begin{aligned} q(S) &= dim H^0(S, \Omega) \\ &= dim H^0(C' \times F, \Omega)^G \\ &= dim(H^0(C', \Omega)^G \oplus H^0(F, \Omega)^G) \\ &= dim(H^0(C, \Omega) \oplus H^0(F/G, \Omega) \\ &= g(C) + g(F/G). \end{aligned}$$

Therefore, if $q(S) = g(C) + 2$, then $g(F/G) = 2$ and we obtain $F/G \cong F$ (using Riemann-Hurwitz formula), i.e., G is trivial.

(ii) If $q(S) \leq g(C) + 1$, then by the calculation in (i), we see that G is not trivial. Therefore, the trivial fibration $C' \times F \to F$ induces a second fibration $S \cong (C' \times F)/G \to F/G$.

(iii) If $q(S) = g(C) + 1$, then by the calculation in (ii), F/G is an elliptic curve, say E''. Then $G \subset Aut(F)$ is cyclic of order two ([16], Proposition 2.12, p.30) and $F \to E''$ ramifies precisely over two points $p_1, p_2 \in E''$ (by Riemann-Hurwitz formula). Hence the composite map $C' \times F \to F \to E''$ induces a natural fibration $S \to E''$ with generic fiber C' and with two double fibers of the form $2.(C'/G) = 2.C$ over p_1, p_2. This proves (iii). $\square$

As an immediate result of Lemma 2.1(iii) we have

Corollary 2.2. *Let $\pi : S \to E$ be a smooth genus two fibration over an elliptic curve E with $q(S) = 2$. Then S admits an elliptic fibration with two double fibers of the form $2.E$. All other fibers are smooth and are isomorphic to E' (the double cover of E corresponding to the monodromy representation arising from π).*

Remarks:

1) Lemma 2.1(i) says that $S' = C' \times F$ is a Galois cover of S. Then S is algebraic, since S' is algebraic.

2) Since $\mathbb{P}^1$ has no nontrivial étale covers, Lemma 2.1(i) shows that over $\mathbb{P}^1$ all smooth genus two fibrations are trivial.

3) Lemma 2.1(i) is known to be valid more generally, for smooth fibrations with hyperelliptic fibers.

4) Lemma 2.1(ii) is a special case of a more general statement: If $\pi : S \to C$ is a fibration with fiber of genus $g \geq 2$ and if $q(S) = g(C) + g$, then π is trivial.

5) Lemma 2.1(iii) can also be generalized; see ([2], E.8.6, p.151).

Now we consider the case of smooth fibrations $S \to E$ over elliptic curves with $q(S) = 2$. By Lemma 2.1(iii), we know that S admits an elliptic fibration $S \to E''$ with two double fibers of the form $2.E$ and smooth fibers all isomorphic to E' which is the double cover of E corresponding to the monodromy representation. In particular, we see that $\kappa(S) = 1$ ([1], Proposition 12.5(iii), p.215).

Lemma 2.3. *Let $\psi : \mathcal{S} \to T$ be a deformation of S. Then there exists an elliptic curve $\mathcal{E}'' \to T$ and two sections $s_1, s_2 : T \to \mathcal{E}''$ such that $\mathcal{S} \to T$ factors through $\mathcal{E}''$. Furthermore, $\mathcal{S} \to \mathcal{E}''$ is smooth outside $s_1(T) \cup s_2(T)$ and for each $t \in T$ the restriction of $\mathcal{S} \to \mathcal{E}''$ induces an elliptic fibration $\mathcal{S}_t \to \mathcal{E}''_t$ with precisely two double fibers* (*over* $s_1(t), s_2(t)$)*.*

Proof. By standard results in deformation theory, we know that for all $t \in T$, $\mathcal{S}_t$ is a minimal surface and $\kappa(\mathcal{S}_t) = 1$. Furthermore, each $\mathcal{S}_t$ admits an elliptic fibration exactly of the same type as S ([6], Proposition 7.1, p.111) and it follows from ([6], Proposition 7.11(iii), p.128) that there exists an elliptic curve $\mathcal{E}'' \to T$ through which $\mathcal{S} \to T$ factors. Since each surface $\mathcal{S}_t$ is an "elliptic surface of general type" in the terminology of [3] (i.e., $P(n) \geq 2$ for some n), Proposition 10 in [3] applies to prove the existence of two sections $s_1, s_2 : T \to \mathcal{E}''$ ([13], Lemma 1.9) with the properties stated in the Lemma. $\square$

Remarks:

1) The existence of $\mathcal{E}''$ over T can be proved simply by observing that the fibrations in the family $\mathcal{S}$ are induced from m-th canonical map for m sufficiently large ([13], p.194). More precisely, we take $\mathbb{P}(\psi_*\omega_{\mathcal{S}/T}^{\otimes m})$ over T and the morphism $\mathcal{S} \to \mathbb{P}(\psi_*\omega_{\mathcal{S}/T}^{\otimes m})$ induced by the homomorphism $\psi^*(\psi_*\omega_{\mathcal{S}/T}^{\otimes m}) \to \omega_{\mathcal{S}/T}^{\otimes m} \to 0$. $\mathcal{E}''$ is the image of this m-th canonical map.

2) Lemma 2.3 indicates a relation between the moduli of smooth genus two fibrations over elliptic curves with irregularity $q = 2$ and the moduli of elliptic surfaces. In order to apply the results on the latter ([12], [13]), we need the following observation:

Lemma 2.4. *An elliptic surface with exactly two double fibers admits a smooth genus two fibration.*

Proof. If $\pi : S \to E''$ is an elliptic fibration over an elliptic curve E'' with two double fibers over $p_1, p_2 \in E''$. The j-invariant of the fiber is bounded and therefore constant. Indeed, $j_S : E'' - \{p_1, p_2\} \to \mathbb{C}$ is defined by $j_S(t) = j(E')$ for all $t \in E'' - \{p_1, p_2\}$, where E' is the general fiber of π. Let $\pi_J : B = J(S) \to E''$ be the Jacobian fibration and $B' = E' \times E'' \to E''$ be the trivial elliptic fibration. Both, B and B', are elliptic fibrations with sections. Since the associated j-invariants of them are equal and constant, we can find isomorphic compatible lifts ρ and ρ' defined by B and B', respectively ([6], p.41). Hence the Jacobian surface $J(S)$ of S is trivial, being isomorphic to $B' = E' \times E''$ ([6], Theorem 3.14(ii), p.45). Moreover, $R^1\pi_*\mathcal{O}_S \cong R^1(\pi_J)_*\mathcal{O}_{J(S)}$ is trivial and so $L = (R^1\pi_*\mathcal{O}_S)^\vee$ is trivial. Therefore, $\chi(\mathcal{O}_S) = degL = 0$ ([6], Proposition 3.18, p.48), $p_g(S) = g(E'') = 1$ ([6], Proposition 3.22(i), p.49) and so $q(S) = 1 + p_g(S) - \chi(\mathcal{O}_S) = 2$. Then $\pi : S \to E''$ is of nonalbanese type and by the universal property of albanese varieties there is a morphism $Alb(S) \to E''$. Hence $Alb(S)$ is a reducible abelian variety and by the complete reducibility property of abelian varieties ([7], Theorem 1, p.173) it has a projection to an elliptic curve E, $\pi_1 : Alb(S) \to E$, which restricts to a nonconstant morphism from E'. The pullback $S \times_E E' \cong F \times E'$ where F is the general fiber of $\pi' = \pi_1 \circ \alpha : S \to E$ and $\alpha : S \to Alb(S)$ is the albanese map ([2], E.8.6, p151). $g(F) = 2$ since F is a double cover of E'', ramified at two points. □

Remark: Combining the results in Corollary 2.2, Lemma 2.3 and Lemma 2.4, we see that a surface S admitting a smooth genus two fibration deforms only to surfaces of the same type.

Next, we consider the case of nonsmooth fibrations. Let $\pi : S \to C$ be a nonsmooth genus two fibration with $q(S) = g(C) + 1$. Then there is a unique rational number $\lambda = \lambda(\pi)$, which is called the slope of π, such that $K^2 = \lambda\chi + (8 - \lambda)(g(C) - 1)$, and one has $2 \leq \lambda \leq 7$ for nontrivial nonsmooth fibrations ([16], p.22). Let F be a fiber of π and $J(F)$ its Jacobian. We have an injection $t_F : F \to J(F)$ that is uniquely determined up to translation and a projection $p_F : J(F) \to E$ onto the fixed part of the connected fibers. There is a morphism $\tau_F = p_F \circ t_F : F \to E$ ([16], p.34). Let d be the degree of τ_F. Then d is called the degree associated to π and $\pi : S \to C$ is said to be of type

(E, d). In particular, we have $\lambda = 7 - \frac{6}{d}$ ([16], Corollaire 2, p.50). Since we will consider only semistable fibrations, we have $d \geq 3$ ([16], Corollaire, p.47). As an immediate result of ([16], Théorème 3.10, p.44) we have

Theorem 2.5. *Let E be an elliptic curve, d an integer ≥ 3. There exists a genus two fibration of type (E, d)*

$$\Phi : S(E, d) \to X(d)$$

over the modular curve $X(d)$ which is universal in the following sense: any genus two fibration $\pi : S \to C$ with slope $\lambda = 7 - \frac{6}{d}$ and with E as the fixed part of the Jacobian fibration corresponding to π (i.e., π is of type (E, d)) is the minimal desingularization of the pullback $f^*(S(E, d))$ via a surjective holomorphic map $f : C \to X(d)$.

Remarks:

1) Since $g(X(d)) \geq 3$ for $d \geq 7$, it follows that fibrations over elliptic curves have $d \in \{3, 4, 5, 6\}$. We recall that $X(d) \cong \mathbb{P}^1$ for $d = 3, 4, 5$ and $X(6)$ is the elliptic curve with $j(X(6)) = 0$.

2) There exists a genus two fibration of type (E, d) with base C if and only if one has a surjective morphism $C \to X(d)$ ([16], Corollaire, p.46).

Given $f : C \to X(d)$, the surface $f^*(S(E, d))$ has singularities only if f ramifies over some points in the singular locus of $\Phi : S(E, d) \to X(d)$. A singular fiber of Φ is either an elliptic curve with a single node or two smooth elliptic curves intersecting transversally at a single point ([16], Lemme 3.11, Théorème 3.16). Hence singularities of $f^*(S(E,d))$ are all type A_k for some k depending on the singular point.

As a consequence of this observation we see that we can apply simultaneous desingularization to a family of surfaces obtained via a family of surjective morphisms onto $X(d)$.

Lemma 2.6. *For a fibration $\pi : S \to C$ over a curve C of genus ≥ 1 arising from a map $f : C \to X(d)$ of degree n we have $c_2(S) > 0$ and $K^2 = {c_1}^2(S) > 0$. In particular, since S is minimal, it is a surface of general type.*

Proof. Let $\phi : S(E', d) \to X(d)$ be the corresponding fibration. Then $c_2(S) = -n deg(R^1\phi_*\mathcal{O}_{S(E',d)}) > 0$. Using the relations $c_1^2(S) = \lambda\chi(S)+(8-\lambda)(g(C)-1)$ and $12\chi(S) = c_1^2(S) + c_2(S)$, we have $c_1^2(S) > 0$. □

Lemma 2.7. *Let $S_i \to C_i$, $i = 1, 2$ be two fibrations of the same type (E, d), corresponding to morphisms $f_i : C_i \to X(d)$. Then*

(i) S_i have the same invariants K^2, χ if and only if $deg(f_1) = deg(f_2)$,

(ii) S_1 and S_2 are isomorphic as surfaces if and only if $C_1 = C_2$ and there exist automorphisms $\alpha \in Aut(C_1)$, $\beta \in Aut(X(d))$ such that $f_1 \circ \alpha = \beta \circ f_2$.

Proof. **(i)** This is Lemma 1 in [10].

(ii) That $C_1 = C_2$ follows from the uniqueness of such a fibration on a given surface ([10], Lemma 2(i)). Then the rest of the statement is a consequence of the minimality of the surfaces S_1 and S_2, since the given condition is necessary and sufficient for the surfaces $f_i^*(S(E, d))$ to be birationally equivalent. □

Lemma 2.8. *S with such a fibration over an elliptic curve exists if and only if K^2 and χ have the following values:*

λ	K^2	χ
5	$5n$	n
$11/2$	$11n$	$2n$
$29/5$	$29n$	$5n$
6	$36n$	$6n$

where $n \geq 2$ in the first three rows and $n \geq 1$ in the last row.

Proof. For $n \geq 2$ and for any elliptic curve E we have morphisms $E \to \mathbb{P}^1$ of degree n and for any such a map, using the formulae in the proof of Lemma 2.6 and observing that $-deg(R^1\phi_*\mathcal{O}_{S(E',d)}) = 7, 13, 31$ for $d = 3, 4, 5$, respectively, ([16], p.52), we find the values of K^2 and χ given in the first three rows of the table.

Since $X(6)$ is an elliptic curve, by the same computation, this time using the existence of isogenies of any order and the fact that $-deg(R^1\phi_*\mathcal{O}_{S(E',6)}) = 6$ we obtain the last row. □

We will need the following Lemma ([10], Lemma 2).

Lemma 2.9. *Let $\psi : \mathcal{S} \to T$ be a deformation over a connected base, of a surface S admitting a genus 2 fibration with slope λ over a curve C of genus $g \geq 2$. Then*

(i) each fiber $\mathcal{S}_t$ of ψ admits such a fibration $\mathcal{S}_t \to C_t$ which is unique,

(ii) the slope λ is constant on T,

(iii) the degree of the map $C_t \to X(d)$ inducing the fiber space $\mathcal{S}_t \to C_t$ is constant.

In case of elliptic base curves (ii) and (iii) of Lemma 2.9 remain unchanged when we consider a family $\mathcal{S} \to T$ of surfaces having a fibration of the given form. Moreover, the fibration over any such curve is also unique. However, we do not know if (i) holds, too.

CHAPTER 3

MODULI PROBLEM OF GENUS TWO FIBRATIONS

In Chapter 2, we have seen that if a surface S admits a smooth genus two fibration then it is algebraic and it deforms only to algebraic surfaces of the same type. In the study of the corresponding moduli space, we will use a natural polarization on these surfaces and apply the consequences of Seiler's work ([12], [13]).

Let $\pi : S \to E$ be a smooth genus two fibration with fiber F and $\pi' : S \to E''$ be the corresponding elliptic fibration with two double fibers. In fact, these double fibers are sections of π, say s_1, s_2. Consider the divisor $s_1(E) + F$ on S. We have

$$(s_1(E) + F)^2 = 2s_1(E).F = 2 > 0$$

and for any irreducible curve C in S

$$(s_1(E) + F).C = s_1(E).C + F.C > 0.$$

Hence $s_1(E) + F$ is an ample divisor on S, by Nakai's ampleness criterion, so is the corresponding line bundle. Let η be the numerical equivalence class of the line bundle corresponding to this ample divisor in $Num(S)$ (the group of numerical equivalence classes of line bundles on S). Then

$$d := \eta^2 = (s_1(E) + F)^2 = 2$$

and

$$e := \eta.f = (s_1(E) + F).E' = F.E' = 1$$

where f is the class of the general fiber E' of π_1. Hence our surfaces are of type $\tau = (1, 0; 2, 2, 2; 2, 1)$ according to the generalized definition of polarized elliptic surfaces given in ([13], p.210). Moreover, any surface of type τ is one of our surfaces. Then we consider the moduli problem for surfaces of type τ as given. We consider the functor $G_\tau : Sch \to Sets$ defined by $G_\tau(T) =$ set of all isomorphism classes of families of polarized elliptic surfaces of type τ over T. We have

Proposition 3.1. *G_τ is coarsely represented by an irreducible scheme $\mathcal{M}$ of dimension 3.*

Proof. Existence of $\mathcal{M}$ follows from ([13], Theorem 2.15, p.211). The proof of this theorem shows that there is a finite map $\mathcal{M} \to Y''$, where Y'' is an open subscheme of $Y' = E_{1,0} \times_{\mathbb{A}^1} M_{1,2}$. Here $E_{1,0}$ denotes the moduli scheme for Weierstrass surfaces with base genus $g = 1$ and $\chi = 0$ which exists by [12], and $M_{1,2}$ is the moduli scheme for elliptic curves with two distinguished points. $E_{1,0}$ splits into a disjoint union of irreducible subschemes $E^n_{1,0}$ for $n = 1, 2, 3, 4, 6$ ([12], p.182) where each $E^n_{1,0}$ represents the subfunctor corresponding to Weierstrass surfaces for which the order of the module $L = (R^1 p_* \mathcal{O})^\vee$ is n. Let S be an elliptic surface of type τ. Since all fibers of the elliptic fibration on S are irreducible, the Weierstrass fibration associated to S is the Jacobian fibration of S ([13], p.191). In the proof of Lemma 2.4, we have seen that the Jacobian of such a surface is a trivial product of two elliptic curves. So the relevant part of $E_{1,0}$ is $E^1_{1,0}$ which corresponds to trivial L. Hence Y'' is an open subscheme of $E^1_{1,0} \times_{\mathbb{A}^1} M_{1,2}$. By ([12], Lemma 10, p.182) $E^1_{1,0} \cong \mathbb{A}^2$. Therefore $\dim(\mathcal{M}) = \dim(E^1_{1,0} \times_{\mathbb{A}^1} M_{1,2}) = 3$. $\square$

In the preceding chapter we have observed that the moduli of the surfaces we consider is closely related to the moduli of isogenies of degree two of elliptic curves (the base curves) and the moduli of smooth genus two curves C admitting an elliptic subcover $C \to E$ of degree two. The functor $\mathcal{Y}_0$ corresponding to the first moduli is coarsely represented by affine modular curve $Y_0(2)$. As for the latter, we have the affine surface $\mathcal{A}_{2,1} = (X(2) \times X(2))/SL_2(\mathbb{Z})$ ([4], p.210) which coarsely represents the functor associated to the triplets $\{(A, \Theta, E)\}$ where A is an abelian surface, Θ is a principal polarization, E is an elliptic subgroup of A and $deg(\Theta|_E) = 2$. Let C be a curve of genus two with Jacobian J_C and canonical polarization Θ. Then there is a bijective correspondence between the set of isomorphism classes of (minimal) elliptic subcovers $f : C \to E$ of degree $deg(f) = 2$ and the set of elliptic subgroups $E \leq J_C$ of J_C of degree $deg_\Theta(E) = 2$ ([4], Theorem 1.9, p.202). Therefore, the functor $\mathcal{M}'_2$ of isomorphism classes of pairs $(\mathcal{C}, \mathcal{E})$ of (relative) smooth curves of genus two and elliptic subcovers $(\mathcal{C} \to \mathcal{E})$ of degree two is coarsely represented by the open subscheme $\mathcal{H} = \Phi^{-1}(t(\mathcal{M}_2))$ of $\mathcal{A}_{2,1}$, where $t : \mathcal{M}_2 \to \mathcal{A}_2$ is the Torelli map which associates to a curve its polarized Jacobian and $\Phi : \mathcal{A}_{2,1} \to \mathcal{A}_2$ is the map which forgets E in the triplets.

Proposition 3.2. *There exists a natural surjective morphism $\phi : \mathcal{M} \to \mathcal{H}$.*

Proof. Consider an object in $G_\tau(T)$ for some T; i.e. a family $\mathcal{S} \to T$ which factors over $\mathcal{E}''$ (Lemma 2.3). Let $f_1, f_2 \in \mathcal{O}_{\mathcal{E}''/T}$ such that $(f_i) = s_i(T),\ i = 1, 2$. Then the natural injection $f : \mathcal{O}_{\mathcal{E}''/T} \to \mathcal{O}_{\mathcal{E}''/T}[\sqrt{f_1 f_2}\ ]$ gives a double cover $(f) : \mathcal{F}/T \to \mathcal{E}''/T$ over T ramified along $(f_1) \cup (f_2) = s_1(T) \cup s_2(T)$, where $\mathcal{F} = Spec\ (\mathcal{O}_{\mathcal{E}''}[t]/(t^2 - f_1 f_2))$. Moreover, since for any $t \in T,\ \mathcal{F}_t \to \mathcal{E}''_t$ is a double cover ramified at two points, we have $g(\mathcal{F}_t) = 2$ by Riemann-Hurwitz formula and so $\mathcal{F}_t$ is a smooth genus two curve. Hence the pair $(\mathcal{F}, \mathcal{E}'')$ corresponds to a point in $\mathcal{H}(T)$. By functoriality of this construction, we obtain natural morphism $\phi : \mathcal{M} \to \mathcal{H}$. $\square$

Next we consider moduli of surfaces with nonsmooth genus two fibrations of nonalbanese type. In Chapter 2 we have seen that a surface of type (E', d) is the desingularization of $f^*(S(E', d))$ for some morphism $f : C \to X(d)$. Hence such a surface S can be deformed in two ways; we can deform E' to other elliptic curves and we can deform the map f. Therefore, in describing the moduli spaces of such surfaces under consideration, we need to clarify the relation of these spaces to the Hurwitz spaces $\mathcal{H}(g, X(d), n)$ of morphisms of degree n from curves of genus g to the modular curve $X(d)$.

Theorem 3.3. *Let K^2, χ and $g \geq 2$ be given and let $\mathcal{H}(g, X(d), n)$ be the Hurwitz scheme of morphisms of degree n from curves of genus g onto $X(d)$.*

Then we have morphisms $\Phi : \mathcal{M}(g, K^2, \chi) \to \mathbb{A}^1$ *and* $\Psi_{E'} : \mathcal{H}(g, X(d), n) \to \mathcal{M}(g, K^2, \chi)$ *for any elliptic curve* E' *such that*

(i) $\Psi_{E'}$ *establishes a one-to-one correspondence between the components* $\mathcal{H}_i$ *of* $\mathcal{H}(g, X(d), n)$ *and the components* $\mathcal{M}_i$ *of* $\mathcal{M}(g, K^2, \chi)$*,*

(ii) $\Phi : \mathcal{M}_i \to \mathbb{A}^1$ *is a fibration with* $\Psi_{E'}(\mathcal{H}_i)$ *as the fiber over* $[E'] \in \mathbb{A}^1$*.*

This result is a consequence of Lemma 3.4 and Lemma 3.5.

Lemma 3.4. *There exists a morphism* $\Phi : \mathcal{M}(g, K^2, \chi) \to \mathbb{A}^1$ *which maps the class* $[S] \in \mathcal{M}(g, K^2, \chi)$ *to the class* $[E] \in \mathbb{A}^1$ *of the elliptic curve associated to the fibration on* S*.* Φ *is surjective on each component of* $\mathcal{M}(g, K^2, \chi)$*.*

Proof. Let $M(g,\ \lambda) : Sch/\mathbb{C} \to Sets$ be the functor defined by $M(g,\ \lambda)(T) =$ isomorphism classes of families of surfaces over T admitting genus 2 fibrations over curves of genus g, with slope λ. To prove the lemma, it suffices to construct a morphism of functors $M(g,\ \lambda) \to h_{\mathbb{A}^1}$ as described in the lemma. This, on the other hand, follows once we prove that for any $T \in Sch/\mathbb{C}$ and for a given family of surfaces $\mathcal{S} \to T$, the map $T \to \mathbb{A}^1$ defined by $t \mapsto [E_t]$ where $[E_t]$ is the fixed part of the jacobian fibration on $\mathcal{S}_t$, is a morphism.

This last claim being local over the base, we assume that $\mathcal{S} \to T$ is projective and we consider the relative albanese morphism $\alpha : \mathcal{S} \to Alb_{\mathcal{S}/T}$; the image $\mathcal{E} = \alpha(\mathcal{S})$ is a family of smooth isotrivial elliptic surfaces over T and the base of the fibration on $\mathcal{E}_t$ is $C_t =$ the base of the fibration on $\mathcal{S}_t$. It is well known that for such a family of elliptic surfaces, the base curves glue to give a relative curve $\mathcal{C}$ and the map $\mathcal{E} \to T$ factors over $\mathcal{C}$. Since the fibres of $\mathcal{E}_t \to C_t$ are constant, the morphism $\mathcal{C} \to \mathbb{A}^1$ corresponding to the elliptic curves $\mathcal{E}/\mathcal{C}$ coincides with the map $T \to \mathbb{A}^1$ defined above, which completes the proof of the claim.

To prove the surjectivity of Φ, we take any connected component of $\mathcal{M}(g, K^2, \chi)$ and a surface S of type (E, d) corresponding to a point in this component. We let $f : C \to X(d)$ be the map inducing the fibration on S. For any family of elliptic curves $\mathcal{E} \to T$, we have a genus two curve $\mathcal{F} \to H_{\mathcal{E}/T,d,-1}$, where $H_{\mathcal{E}/T,d,-1}$ is an open subscheme of $X(d) \times_{\mathbb{C}} T$, universal for normalized genus two covers ([1], Definition on p.13) of degree d of $\mathcal{E}/T$ ([1], Thm. 1.1). From $(f, id) : C \times T \to X(d) \times_{\mathbb{C}} T$ we obtain a T-morphism $F : U \to H_{\mathcal{E}/T,d,-1}$ where U is an open subscheme of $C \times T$. Completing the family of genus two curves $F^*(\mathcal{F})/U$ to a family over $C \times T$, and then applying simultaneous desingularization we get a family of smooth surfaces $\mathcal{S} \to T'$ where $T' \to T$ is a finite Galois base extension. Since $\mathcal{S}$ contains S as one of the fibers, its moduli lies in the same component of $\mathcal{M}(g, K^2, \chi)$ as the modulus of S. For an arbitrary elliptic curve E', choosing $\mathcal{E} \to T$ as a deformation of E to E', we see that Φ restricted to this moduli has $[E'] \in \mathbb{A}^1$ in its image. This completes the proof of the lemma. $\square$

Let $\mathcal{C} \to T$ be a family of smooth curves of genus g and let $F : \mathcal{C} \to X(d) \times T$ be a family of morphisms of degree n. For a fixed elliptic curve E', applying simultaneous desingularization to the family of surfaces $F^*(S(E', d))$ we obtain a family of fibered surfaces $\mathcal{S} \to T'$ over a Galois extension $T' \to T$

with group G, which defines a morphism $\alpha : T' \to \mathcal{M}(g, K^2, \chi)$. α, being G-invariant, descends to a morphism $T \to \mathcal{M}(g, K^2, \chi)$. Clearly, this construction is functorial and by the defining property of coarse moduli spaces we get a morphism $\Psi_{E'} : \mathcal{H}(g, X(d), n) \to \mathcal{M}(g, K^2, \chi)$. To a given connected component $\mathcal{H}_i$ of $\mathcal{H}(g, X(d), n)$ we assign the component $\mathcal{M}_i$ of $\mathcal{M}(g, K^2, \chi)$ which contains $\Psi_{E'}(\mathcal{H}_i)$.

Lemma 3.5. *The above assignment induces a one-to-one correspondence between the connected components of $\mathcal{M}(g, K^2, \chi)$ and those of $\mathcal{H}(g, X(d), n)$. Morever, we have $\Psi_{E'}(\mathcal{H}_i) = \Phi^{-1}|_{\mathcal{M}_i}([E'])$.*

Proof. Since by Lemma 3.4, each component $\mathcal{M}_i$ of $\mathcal{M}(g, K^2, \chi)$ contains the modulus of a surface of type (E', d), it suffices to check that in each $\mathcal{M}_i$ we have the image under $\Psi_{E'}$ of a unique component of $\mathcal{H}(g, X(d), n)$.

Let $\mathcal{M}_i$ be the component of $\mathcal{M}(g, K^2, \chi)$ which contains $\Psi_{E'}(\mathcal{H}_i)$. Fix $[S_1] \in \Psi_{E'}(\mathcal{H}_i)$ and let $[S_2] \in \mathcal{M}_i$ be an arbitrary point and let C_i, $j = 1, 2$ be the base curves of the corresponding fibrations. Then, the surfaces S_1 and S_2 deform to each other. Since deformations of the surfaces under consideration are induced from deformations of the fibrations (proof of Lemma 3.4), it follows that $f_1 : C_1 \to X(d)$ deforms to a morphism $\overline{f}_2 : C_2 \to X(d)$. Therefore f_1, $\overline{f}_2$ belong to $\mathcal{H}(g, X(d), n)_i$. On the other hand, by (Lemma 2.7 (ii)), f_2 and $\overline{f}_2$ satisfy a relation of the form $f_2 \circ \alpha = \beta \circ \overline{f}_2$ for some $\alpha \in Aut(C_1)$, $\beta \in Aut(X(d))$. Therefore, f_2 and $\overline{f}_2$, hence f_1 and f_2 lie in $\mathcal{H}_i$. This proves the first part of the lemma. The second statement is obvious. □

In the case of elliptic base curves we can not prove that a given deformation of our surfaces $\mathcal{S} \to T$ arises from the deformation of the associated maps $f_t : C_t \to X(d),\ t \in T$. Therefore, by exactly the same proof we obtain the following weaker result:

Theorem 3.6. *Let $\mathcal{M}_i$ be a connected component of $\mathcal{M}(1, K^2, \chi)$. Then we have a morphism $\Phi : \mathcal{M}_i \to \mathbb{A}^1$ (given on closed points by $[X] \to [E]$ if X is of type (E, d)) such that the fiber over $[E] \in \mathbb{A}^1$ is a disjoint union*

$$\bigsqcup_j \Psi_E(\mathcal{H}(1, X(d), n)_j).$$

Remarks :

1) When $\lambda = 6$, one can prove that $\mathcal{M}(g, K^2, \chi) = \bigsqcup_{i=1}^{N} \mathbb{A}^1_i$ where N is the number of distinct étale covers of degree n of the elliptic curve $X(6)$ ([5], Theorem 2.3).

2) Another shortcoming of the result in case of base genus $g = 1$ is that we do not know if each $\mathcal{M}_i$ is a connected component of the corresponding moduli space $\mathcal{M}_{K^2,\chi}$ of surfaces of general type.

APPENDIX

In this appendix, we discuss some problems relevant to the moduli of genus two fibrations, which are not studied in the literature.

Problem 1: Fibrations over $\mathbb{P}^1$

In this case, since the monodromy is trivial, all smooth fibrations are trivial products. Hence we see that the moduli space of smooth genus two fibrations over $\mathbb{P}^1$ is $\mathcal{M}_2$ which is the moduli space of curves of genus two.

For nonsmooth fibrations we have $q(X) = 1$ and $d = 3, 4, 5$. Using the same analysis as in the case of elliptic base curves, one obtains the analog of Theorem 3.6 and also the values of the invariants for which the corresponding moduli space is nonempty (cf. Lemma 2.8 in Chapter 2). We note that in this case we have a second fibration, namely the albanese fibration on the given surface; geometry of this fibration and applications to the study of the moduli spaces seems to be an interesting problem.

Problem 2: $d = 2$ case

This is the main missing case in all the relevant work in the literature. The difficulty of the analysis of this case stems basically from the following phenomena.

(1) We do not have a universal surface $S(E, 2)$ on $X(2)$. This is related to the fact that the curves of genus two with elliptic quotients of degree $d = 2$ do not have a fine moduli space.

(2) One does not know if a genus two fibration with $d = 2$ is a semistable fibration.

(3) $d = 2$ case is the only case where we have isotrivial fibrations.

Problem 3: Compactification of the moduli spaces

This problem requires understanding the degenerations of the surfaces considered. Hence is related to the singularities of surfaces. Even though a general choice of "minimal" degenerations to compactify the moduli spaces of surfaces is not available, in this case one may try using special degenerations where each component is either rational or a surface fibered over a base curve of genus $\leq$ initial base genus ([11]).

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