TENNUR BAŞ

METU 2005

A SURVEY OF MATHEMATICAL AND PHILOSOPHICAL PROBLEMS GENERATED BY ZENO'S PARADOXES

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A SURVEY
OF
MATHEMATICAL AND PHILOSOPHICAL
PROBLEMS
GENERATED BY
ZENO'S PARADOXES

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Arts of Philosophy.

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ABSTRACT

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This thesis analyzes the solution attempts of Zeno's paradoxes and its related problems in a historical context. The evolution of calculus and its critiques will also be examined regarding the rigor problem in mathematics. As a conclusion a compound method is proposed.

Keywords: Zeno's Paradoxes, Calculus, Infinitesimals, Mathematical rigor.

ÖZ

ZENO PARADOKSLARININ YOLAÇTIĞI
MATEMATİKSEL VE FELSEFİ
PROBLEMLERİN İNCELEMESİ

Baş, Tennur

Yüksek Lisans, Felsefe Bölümü

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Bu tez Zeno paradokslarına çözüm çabalarını ve ilgili problemleri tarihsel bir çerçeve içinde incelemiştir. Matematiksel katılık gözönünde bulundurularak Calculus'un evrimi ve yöneltilen eleştiriler de incelenecektir. Sonuç olarak bileşik bir metod önerilmiştir.

Anahtar Kelimeler: Zeno Paradoksları, Calculus, Sonsuz Küçükler, Matematiksel katılık.

To My Parents

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CHAPTER 1

INTRODUCTION

The debates that continued for centuries on the nature of time and space has an important place in man's adventure of understanding the environment where he lived. This curiosity that can be traced back to the first sunrise, is very valuable both for understanding the universe and for giving a meaning to the existence of human beings themselves and their experiences. Although the history of investigations on time is nearly equal in length to the history of human thought, there are some issues that still remain mysteries. There are solutions related with the controversial issues, but the missing part is the general acceptance for any one of those solutions. Gathering data from the bare experience and experiments, especially when time is under consideration is so difficult that completely opposite ideas can be proposed in the same age as we will try to examine. Some philosophers, before trying to expose the nature of time, firstly tried to prove the existence of time, which is still under discussion by some of them.

The interval of time between the rise and fall of the sun does not only give to human beings light and heat, but also an opportunity to think about 'the path that followed by the sun' and the 'day' that the sun needs to travel along this path. Motion as *the source of time*, and time as *the measurement of motion* are both dependent on each other. The studies showed us that there are some concepts that cannot be considered separately from time while trying to discover time and expose its characteristics. The concept of motion and the concepts of space, continuity, change, divisibility, multiplicity, infinity, isotropic[1]/anisotropic, absolute/relative are all interrelated.

The effort of mankind to understand the universe can be counted as a relay race in which all contenders try to finish their own courses at the shortest possible time. Even if they fail to reach at the finishing line, their performances help their successors to proceed towards that point. The observation of the opposite and not universally accepted ideas at different levels of a problem solving study can give us some clues about the nature of the progress of the human thought. And if we note that the problem of time has attracted attentions of many researchers from different disciplines and from different schools for a long time, then debates on its solutions and related proposed theories would be so diverse that the account of these debates would be fruitful and successful for a person whose intention is to grasp the progress of human thought and mechanisms that underlies this history.

Considering its importance for understanding the world in which we live, time is not only a matter of philosophy, but also of science. Science, an activity which generally tries to find laws for the events in our everyday world to control these events, uses time as a unit of measurement in its calculations. Furthermore, the concept of time can be counted as a battleground of the debates that take place between science and philosophy. Scientists define time as a linear continuum which consists of instants that has no length. The idea of *continuum* implies that there are other instants between two instants. And time is the sum of these infinitely many instants that have no length. This kind of definition of time was respected as complicated for some of the others, since it is not so easy adding infinitely many terms. But science has been using it for centuries in its theories and calculations. There is no problem about this usage, at least scientifically. On the other hand, philosophers have been asking questions about this continuum by claiming that there is no concrete experiment proving that time is composed of instants. Moreover, some of the philosophers claim that instant theory is not satisfied by our experiences of time. We will try to glance at some of those claims and counter claims in the history of science and philosophy.

[1] Isotropy n. (Physics): Uniformity of physical properties in all directions in a body; absence of all kinds of polarity; specifically, equal elasticity in all directions.

One of the reasons why I chose the paradoxes named after their inventor, Zeno of Elea at about 450 BC as the subject of my dissertation is that they give us the chance to study the subjects such as time, motion and the related main concepts that are dealt with by every actor of the story of understanding the world. Florian Cajori in his article “The History of Zeno’s argument on Motion” states that “The history of these paradoxes is largely the history of concepts of continuity, of the infinite and infinitesimal.”[2] These paradoxes conclude that there is neither motion nor multiplicity in nature, which contradicts with our physical experience.

Secondly, this study gives us the chance to investigate the ‘side’ of the science at this debate. My aim is to sketch the history of scientific approach to the problems of time, space and motion. In other words, by means of a study on Zeno’s paradoxes, we can see the consequences of the attempts of abstraction of the sensible world by science and observe how and where philosophy and science approach differently to the same problem.

Lastly, my aim is to testify the evolution of human thought both before and after Parmenides and Zeno with the help of a study of the solutions or dissolutions of Zeno’s paradoxes in history, while the flag is moving from one relay runner to another. These paradoxes with a history of 2500 years, which have attracted the attention from different philosophers and schools, can provide us with a clear account of this evolution.

Although some scientists and even philosophers argue that the solution of the paradoxes is reached after the invention of calculus, the solutions is still do not satisfy all. And the debate is still going on.

[2] Cajori, F. “The History of Zeno’s Arguments on Motion: Phases in the Development of the Theory of Limits I” *The American Mathematical Monthly*, XXII(1915), p.1.

CHAPTER 2

TIME, SPACE AND MOTION

At the beginning of the history of human thought, time was a god that was the father of other important gods. He was not personified as a human, he appeared as a god, for example as Kāla in Vedic tradition, as Zurvān in Persian civilisation, as Chronos[3] in Greek mythology. "In each case he creates out of his seed, without a consort; at sidon and in Iran he is explicitly said to have had intercourse with himself. In each case he is not himself the builder of the material world, but the progenitor of a divine demiurge."[4]

Before the Greek civilization that can be counted as the source of modern western way of thinking, the understanding of space like understanding of time is also different from our Newtonian space with dimensions. Bradie and explain this primeval thinking by arguing that in those times the distinction between the reality and its symbols, subject and object, appearance and reality was not clear.

> …in the Egyptian creation myths, we find the story of a primeval hill which emerges from the flood and becomes the dry earth. All temples had a place which was this hillock, no matter where (in our sense) the temple was. All such sacred places were, for the Egyptians, the same place[5]

And the notion of space in our minds "is not something which is given to us naturally, but is something which reflects basic cultural conventions"[6]. How space is given to us may not be so clear, but the point is that time and space were approached differently in different cultures. It may be difficult for us to understand

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[3] Chronos was the son of Heaven (Uranus) and Earth (Gaia).

[4] West, M.L. *Early Greek Philosophy and the Orient*, (Oxford: Clarendon Press, 1971), p.4.

[5] "The Evolution Of The Concepts Of Space And Time", by M. Bradie and C. Duncan, http://chandra.bgsu.edu/~gcd/Spacetime1.html, May'03,2005.

the mythological notion of time and space. Since time and space are very important in grasping the universe and life, it can be claimed that the ancients should be living in a world different from ours.

Time and space are regarded as having different natures by different disciplines and schools. In order to give a brief introduction to this huge subject it is better to make a classification with respect to some of their characteristics and related core concepts.

The question concerning the uniqueness of every single instant can be our first question. In other words, as in the above example of the 'sameness' of space, according to the Chinese, Babylonians, Indians and Greeks there are time cycles in which the same events happen infinitely many times. For example, the time of migration of the birds or the time of paying taxes in a year is exactly the same for every year. The latter example is evidently related to a psychological point of view. John Randolph Lucas says that temporal order is not cyclical, but it is difficult to discover its reasons.[7] This difficulty, according to Lucas, comes from confusing time and change. The events in nature occur periodically, but it does not mean these periodic events are the same. If this view is adopted then every event is a repetition of the old ones, which means that there is nothing new under the sun. In other words, there is no unpredictable future or no free will. That will leave us with determinism.

The belief in cyclical occurrences may derive from another belief in the immortality of soul arguing that soul is immortal in contrast to body and reincarnates again and again in different bodies. This idea may suggest that every event is recurrent. The problem here is the confusion between the similarity and the sameness of the events. But this problem is disregarded by some ancient civilizations because their culture, mythology and religion did not require such a distinction.

Whether time has a beginning or an end is a common question to many people. Monotheistic religions assume the creation of the world. According to some views, such as Plato's philosophy, this was also the case with the creation of

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[6] *ibid*.
[7] Lucas, J.R. *A Treatise on Time and Space*, (London: Methuen, 1973), p.57.

time. On the other hand, some thought that time always existed. Again, monotheistic religions claim that time has an end, namely apocalypse. Similarly, about the finitude of space there are controversial ideas. Plato claimed that space is finite that there is nothing beyond the space, i.e. *the cosmos*. Aristotle stated that the earth is at the center of the universe. Then space should be finite, since otherwise it cannot have a center.

Another issue related with time is its direction. In daily life we experience the arrow of time from the future to the past. The notion of flow helps people relate the past events with the future ones, the temporal relation of the events are basically depends on the direction of time. We also experience that events in practical life are irreversible, for example when a match is struck it cannot be undone. Moreover, some claim that it is possible for a man to change the direction of time by the help of his daydreams and memories. For example we can remember what was like the match just before it was struck. As we can see clearly, in our daily life, the reversal of the events can only be possible by the psychological experience of time. Although physical events contain the direction of time, the theories that examine these events are generally time-symmetric.

> With some apparently minor exceptions involving kaon decay and Higgs boson decay, all the basic laws are time symmetric. This means that if a certain process is allowed by the equations, then that process reversed in time is also allowed. In other words, the basic laws of science are insensitive to the distinction between past and future.[8]

The reversibility of time in the equations opens a discussion for people who try to understand the nature of time. The confusion here is that reversible mathematical equations stand for irreversible physical events.

Another question is the following: If there is no object in the world, is it still possible to talk about space? Is it possible to think of an empty space? According to Sir Isaac Newton, space and object are different things and the former can exist without the latter. On the other hand Leibniz disagrees with Newton, since according to him, space was abstraction of the distances between the objects, so the objects are necessary to talk about space. A similar dependence can also be

[8] "Time" by Bradley Dowden, Internet Encyclopedia of Philosophy, http://www.iep.utm.edu/, May'03,2005.

asked for time. Without objects in motion, is it possible to conceive time? Is there any duration of time that contains no change? Again, different answers were given. For example, Aristotle who defined time as the measure of motion claimed that time is object dependent. Newton, by defining absolute time as well as absolute space, claimed that time is object independent. Dependence of time and space on mind is another question. In other words, without a mind, which grasps them, is it possible to conceive their existence? Immanuel Kant claimed that time and space are the mediums of perception and do not have a real existence outside of our minds.

As we saw, there are many controversial points about time and space. Moreover, most of them have no conventionally agreed solutions. Especially as time is more difficult to conceive since we deal with time as an abstract entity. Although time is very evident for us since we experience it, it is not easy to define.[9] We cannot make experiments, or touch or smell or taste it, except by the help of metaphors. It may be possible to claim that some arguments about the total rejection of time can have their sources in lack of 'concrete evidence'.

The Eleatics, such as Parmenides and his pupil Zeno rejected time, but it should not be thought that the idea is old-fashioned. Some philosophers who are called as neo-eleatics, like McTaggart claimed the unreality of time nearly two thousand years after the Eleatics. It should be noted that although plurality and motion is more visible and more *concrete*, Eleatics also rejected them. And our subject, Paradoxes of Zeno is the best example to this rejection. We have more knowledge about motion and plurality than we have about time. We can measure where a stone that is thrown will land by scientific equations. But the paradoxes are still pointing their arrows to those concepts and we can not save ourselves from these arrows by simply saying that we can see motion and plurality with our eyes. Experiencing motion and plurality is rather a beginning point for a solution or a belief that paradoxes are not true. On the other hand, what we exactly need is a method that will cancel the paradoxes. The philosophy that contains this method will be different from Parmenides'.

[9] Like St.Augustine said "What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks, I do not know"

Questions that are asked to reach the information of the objects varied from the features of the object like color, smell and mass to the ones about the characteristics itself, for example, are those characteristics permanent or temporary? If characteristics were changing, do these changes occur with respect to a period? The questions are also changing with respect to ages and needs. There are some questions besides the ones related with the needs of practical life. These arise from the curiosity of man about the origin of universe and life and how they exist. If there is a common substance that underlies all the objects in the world, then human being, by obtaining knowledge of that prime substance, can lead a more harmonious life with the universe. The answers that were the results of this curiosity have been expressed in different contexts, which can be roughly classified into the mythology, religion and science.

These headings have never been so distinct from each other. Their distance to each other changed in different ages of the history. At the beginnings of the history of human thought, answers to the main questions about the origin of the world had their sources in religious beliefs, especially in the eastern civilizations. In those climates, the need for knowledge arose mostly from needs of the practical life. Why man is pursuant of the knowledge is an important question, which is a concern for another discussion. For the sake of our argument, it is better to stress that man tries to gather knowledge according to his needs, which affect the methods he uses. In some cultures, while an approximate answer to a specific problem is sufficient, the others were mostly interested in the abstraction of that problem so that they can express the solution as a law applicable to the same kind of problems. Before examining the Greek philosophy where the abstraction is considered for the first time, we will present an example to expose how the practical needs affect the methods of solving. For example, squaring the circle problem is considered differently by various civilizations. Squaring the circle is one of the famous three construction problems - the others are doubling the cube and trisecting an angle - and has common characteristics with Zeno's paradoxes. Both were tried to be solved by many philosophers from different ages and cultures and in both cases there was no consensus by the researchers. While in the Greek mathematics, the proof of the solution and mathematical rigor were important, the Eastern civilizations were mainly interested in finding a solution. Although many researchers in the Eastern civilizations studied this problem, we see no trace of conceptualizing the elements of the problem, for example, the circle did not have even a name.[10] Both conversions from square to circle and circle to square were made, but "these conversions are reciprocal neither in results nor in the type of procedures"[11]. According to our understanding of science, these gaps and therefore the solution cannot be acceptable. On the other hand, for Babylonians, Hindus or Egyptians the concern was elsewhere, and they obtained their answers according to their needs.

This did not mean that mathematics was not important in the Eastern civilizations. For example, in Hindu culture,

> the study of Ganit i.e mathematics was given considerable importance in the Vedic period. The Vedang Jyotish (1000 BC) includes the statement: "Just as the feathers of a peacock and the jewel-stone of a snake are placed at the highest point of the body (at the forehead), similarly, the position of Ganit is the highest amongst all branches of the Vedas and the Shastras.[12]

We can conclude that the needs shape the approaches and the methods, and result in different solving methods. We will see the similar situations whenever the needs affect the studies, for example in the evolution of calculus, more generally, in the history of scientific progress.

If we come to the shores of the Aegean Sea, we can see some differences about the reasons for gaining knowledge and its use. The same practical needs are also concerned and studied. Moreover, they tried to make abstractions of the physical reality so that the solution can be applicable to the same kind of problems. They also tried to strengthen the solutions to fill the gaps that can be seen in the Eastern civilizations studies. The reason for these attempts was the desire to gain the universal truths, which have an important place in the Greek philosophy.

The idea of abstraction that peaked in the ancient Greece, especially in Plato's philosophy, blossomed in the studies of Thales. Because of the trade relations with

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[10] The definition of the circle in the form we use was made in *The Elements* by Euclid around 300 B.C.

[11] Goldstein, C. "Stories of the Circle" in *A History of Scientific Thoughts: Elements of a History of Science*, ed. M. Serres (Oxford: Blackwell, 1995), p.164.

the East and the journeys of some philosophers to the Eastern countries, the Western civilizations knew the practical knowledge of the East, and started their studies by making rigorous proofs for these solutions.

[12] "History of Mathematics in India", http://members.tripod.com/~INDIA_RESOURCE/mathematics.htm, May'03,2005.

CHAPTER 3

CHANGE OF THE METHOD:
BEGINNINGS OF THE ABSTRACTION ATTEMPTS

Thales, who is known to have a journey to Egypt, is known as the first philosopher and the founder of astronomy and mathematics. Thales claimed that *archē*, the principal matter, was water. It is told that, Thales, in putting forth this idea, was influenced by the Egyptian mythology. In the Egyptian mythology, the world is in the middle of the water. It is known that he learned Egyptian mathematics during his visit and he continued his studies of Egyptian mathematics after his journey. Then, Thales is an important figure for comparing ways of thought in these two cultures, as he had knowledge of both of them. He was not satisfied with the practical use of the knowledge he gained and was interested in theory. Thales tried to go beyond mythological and religious knowledge. This made Aristotle call him the *founder of materialism*.

Thales, with other philosophers who are known as the members of the Milesian school, namely Anaximemes and Anaximander, was investigating the origin of the universe and seeking the 'unit' substance that everything came from. Their methods can be counted as the first rationalist attempts in the history of thought. Although we know that the philosophers that will be mentioned here are affected by the eastern civilizations after their trips to those climates. They tried to free their minds from religion and mythology, which constitute the former knowledge. They only relied on their own reasoning. Their philosophy and the methods used to build their philosophies affected many of the latter philosophers. It can easily be said that the foundations of modern western philosophy and science were set in Milesian school.

3.1 Thales

Although, none of Thales' works survived, the tradition counts Thales as the founder of Greek mathematics and astronomy. He also is remembered with his prediction of a solar eclipse. According to Diogenes Laertius, Thales gave excellent advice on political matters and foresaw that the season would be great for olives. He rented all oil-mills and made a fortune to show it is very easy for a wise man to become rich.[13] These are the examples of Thales' uses of science and knowledge in practical life. What are the reasons for seeing him as the founder of a new philosophy that is different from former kind of knowledge like that the Egyptians have? When we consider Thales' philosophy of cosmology, we come across the idea that 'everything is water' which was interpreted by Aristotle as that everything comes from water. In other words, everything in nature is made up of water. The second interpretation could be that before everything, there was water, then the rest came after it. According to Aristotle, the reason that resulted in this conclusion is

> [S]eeing that the nutriment of all things is moist, and that heat itself is generated from the moist and kept alive by it (and that from which they come to be is a principle of all things). He got his notion from this fact, and from the fact that the seeds of all things have a moist nature, and that water is the origin of the nature of moist things.[14]

This explanation contains the traces of observation of environment to construct the principles of nature. He was living in a seashore city and some commentators say that when he was in Egypt he saw the importance of the river Nile for the life of the residents. The importance of Thales' thought is that he was not explaining his theory by the help of mythology. He tried to verify his theory by observing nature. We can say that the question Thales asked about the nature of the world was more important than the answers and the methods to get these answers. Thales' explanation and the question that stands at the beginning of his inquiry were new. This is the reason why Thales is called the founder of

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[13] Laertius, D. *Lives and Opinions of Eminent Philosophers*, online at http://classicpersuasion.org/pw/diogenes/index.htm, May'03,2005.
[14] Aristotle, *Metaphysics*, trans. by Ross, W. D., online at http://classics.mit.edu//Aristotle/metaphysics.html, May'03,2005.

philosophy and science and his work can be counted as a cornerstone in the history of thought of mankind.

3.2 Anaximander

After Thales, Anaximander, another member of Milesian School, continued asking questions about the nature and origin of the world, and seeking a substance underlying all the sensible things. He is remembered for his predictions of an earthquake and making a gnomon, a kind of sundial, which shows us his relations with Babylonian culture. He is also said to have drawn a map. Anaximander wrote a book called *On Nature* of which some fragments have survived. We have more information about his philosophy than theories of Thales. Anaximander did not agree with Thales about the characteristics of the substance that underlies all sensible things. According to him, that substance cannot be something that we meet in our daily life and was not one of the elements, namely water, air, earth and fire. All these elements can be changed into one or more of the other elements. After observing those changes, Anaximander concluded that none of the elements could underlie all the changes, so they were weak candidates for being the substance that underlies nature. Anaximander called the principal substance *aperion*, or that which has no boundaries. But he did not give a detailed explanation or a proper description of the *aperion*. Being boundless can be interpreted as having no limits in space and time. Some scholars have even argued that the word's meaning is 'that which is not experienced', by relating the Greek word 'apeiron' not to 'peras' ('boundary', 'limit'), but to 'perao' ('to experience', 'to apperceive'). His philosophy differs from the philosophy of Thales by accepting an 'undefined' thing as the principal element. This is a cornerstone in the history of thought, because Anaximander, started by observing the sensible world and ended with an abstract explanation. In the fragments that we owe to Simplicius, Anaximander stated that:

> [A]nd the things from which is the coming into being for the things that exist are also those into which their destruction comes about, in accordance with what must be. …

> For they give justice (*dikê*) and reparation to one another for their offence (*adikia*) in accordance with the ordinance of time.[15]

West claims that Anaximander cannot be counted as a 'rigorous rationalist'. He says that Anaximander

> allowed divinity an important place in his universe, major parts of his system had a visionary rather than a logical foundation, and he explained certain cosmic changes in terms of 'injustice', 'retribution', 'ordinance', language which Simplicius calls 'rather poetic' (DK 12 B I) but which it is more meaningful to classify as theological.[16]

The philosophy of Anaximander can still be seen as affected by the mythological and theological factors and beliefs. The methods he used cannot be named as scientific in the modern sense. But what he has done, is a perfect reasoning and abstraction.

3.3 Anaximenes

Anaximenes, like the other two Milesians, was also interested in the origin of the universe. His candidate substance that everything came from was air. Although his choice for *arche* was something familiar to us like Thales' water, he rejected that water can be the source, since there is another element that cannot have the characteristics of water, namely fire. And air, like *aperion* of Anaximander, was boundless and had no beginning or end. Anaximenes claimed that other elements, namely fire, earth and water can be obtained by changing the density of air. The question of how this change was possible was the missing point in Thales' theory. Anaximenes gave the explanation of this change: "[Air] differs in essence in accordance with its rarity or density. When it is thinned it becomes fire, while when it is condensed it becomes wind, then cloud, when still more condensed it becomes water, then earth, then stones. Everything else comes from these."[17] According to him, by rarefaction, decreasing the density and condensation, increasing the density, one element can change into another one. This theory of change will be further developed by Heraclitus and others.

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[15] West, M.L. *Early Greek Philosophy and the Orient*, (Oxford: Clarendon Press, 1971), p. 76.
[16] *ibid*., p. 77.

3.4 Pythagoras

Like Anaximenes, Pythagoras is said to be a disciple of Anaximander. According to the records of Diogenes Laertius, he had been to Egypt and Crete. It is said that the religious style of his doctrines came from the Egyptian culture. Pythagoras lived in the same period with Buddha and Confucius and that period was an age of development of religion. Pythagoras and his students were also investigating the origin of the universe and the laws of nature. Pythagoras claimed that there was a universal harmony. He described this harmony as limiting the unlimited, bounding the boundless. Anaximander claimed that *apeiron* was the source for the universe. Pythagoras said that as *peras*, which means limit, is imposed on *apeiron*, which is limitless, the result was harmony and beauty of the universe. According to him, the examples of this harmony could be seen in music. Human being could be integrated into this harmony with the help of the human soul. He believed that the human soul was immortal, and that after death of the body it passes into another body. This was occurring in cycles. This conception of soul would affect Plato in his theory of recollection that our knowledge consists of our remembrances of the times when the soul was not in the body. Aristotle in his *Metaphysics*, characterizes the Pythagoreans as the people "who were the first to take up mathematics" and that

> they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being-more than in fire and earth and water.[18]

Pythagoreans were mostly remembered by the motto 'everything is number'. Like the notes in music, which can be represented by numbers, everything in nature could be represented in a similar way. They thought that the odd numbers were limited and the evens are unlimited. Then, the number one is both limited and unlimited since it is even and odd. And all numbers come from the number one. Although this is true only for natural numbers, we know that they meant all numbers. Then, everything comes from one. Let us turn to the boundless air of

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[17] "Anaximenes", by Daniel W. Graham, Internet Encyclopedia of Philosophy, http://www.iep.utm.edu/, May'03,2005.

[18] Aristotle, *Metaphysics*, trans. by Ross, W. D., online at http://classics.mit.edu//Aristotle/metaphysics.html, May'03,2005.

Anaximenes. Imagine a universe that is full of bounded objects and the air that is everywhere, including these objects, and the air is not bounded. That is the harmony of *apeiron* and *peras*, limitless and limit, infinite and finite and even and odd. According to the Pythagoreans, all of these concepts are related with each other, and constitute the rules underlying the life in the universe. These radical ideas will affect the later generations with its power of abstraction.

3.5 Heraclitus

One of the most important figures in ancient philosophy is Heraclitus. He lived in Ephesus, not far very far from Miletus, the motherland of the philosophers mentioned above. Heraclitus, like the other philosophers was behind the structure and laws of nature. He described the world as that "This order, which is the same in all: things no'one of gods or men has made; but it was ever, is now and ever ,shall be an everliving. Fire, fixed measures of it kindling and fixed measures going out."[19] Fire was for Heraclitus, the source of the universe. But this idea should not be taken as identical with the *water* of Thales or *air* of Anaximenes. First of all, it is different from the other main elements by being active and moving. Fire is the source element of the world, and its characteristics could be taken as an example or an analogy of the processes in nature. Heraclitus' prime element that underlies all was change itself, and he used fire to express this change. The river metaphor is very famous: You cannot step twice into the same river. Everything was in a continuous and an infinite change. This understanding of the prime element makes him "a rebel, perhaps, but not a revolutionary."[20] The only thing that stayed constant was the change itself. For example, there is always a flowing traffic in a main street of a metropolis. Milesians thought that elements were changing into other elements, and claimed that the element in question was the source of everything. On the other hand, Heraclitus by claiming that fire is the source was not saying that it is the source of everything, but "all things are exchanged for Fire, and Fire for all things as wares are exchanged for gold, and

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[19] Heraclitus , *Fragments*, translated by John Burnet (1892), online at http://philoctetes.free.fr//heraclitus.htm, May'03,2005.
[20] Barnes, J. *Presocratic Philosophers*, (London: Routledge & Kegan Paul, 1982), p.61.

gold for wares."[21] Heraclitus was not talking about an identical relationship between fire and others, but a potential for a transformation between them, like the relation between gold and wares. Heraclitus called this constant change *logos*. He thought that everything comes about in accordance with logos. Everything moves and flows, *panta rhei*, according to this law, or *logos*. Modern commentators call this view the theory of flux.

All the elements change into one another, all the opposites turn into each other. Heraclitus likened this to a war and war is the father of everything. In Anaximander, the war had a negative meaning. When one element transgresses, then it is punished and its border is surpassed. On the other hand, to Heraclitus who said, "strife is justice", the war was essential for the balance of the nature. The different and opposite things are changing into each other with respect to a law. This is the process of life in nature. The human beings should admit that without this kind of war between the opposites, life would not exist. Fire changes into water then into earth, and similarly earth changes into water and then into earth. There should be a constant flux between the opposite elements so that life in nature can continue.

One of the arguments of Heraclitus is the unity of the opposites. The opposites that turn into one another, like day and night, winter and summer, war and peace, fire and water are 'one'. How can that be possible while we cannot even step into the same river at different successive seconds? Actually, the unity of opposites is closely related to his theory of constant change. He does not claim that, for example, that night and day is equal to each other, but that there exists something that can be night and day at different times. Being night and day is a characteristic of that thing. In other words, we are not talking about two different things like day and night, but only one thing that can be day and night at different times.

It should be pointed out that although Heraclitus use the word 'one' for the relation between the opposites, he did not mean that everything is one and constant. On the contrary he claimed that everything is 'one', since everything is

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[21] Heraclitus , *Fragments*, translated by John Burnet (1892), online at http://philoctetes.free.fr//heraclitus.htm, May'03,2005.

changing into its opposite. This should not be confused with Parmenidean idea of the ‘one’.

3.6 Parmenides

Parmenides who lived in a Greek city named Elea in southern Italy, was the founder of the Eleatic school which was continued by Zeno of Elea and Melissus. Parmenides is mentioned as the pupil of Xenophanes and it is also said that he was also a pupil of Anaximander. According to Diogenes Laertius, he followed another man named Aminias who was a Pythagorean. It is said that Parmenides and his pupil Zeno were members of the Pythagorean School. Guthrie thinks that philosophy of Parmenides shows some traces of Pythagorean thought. But he estimates that “he certainly broke away from it, as from all other previous philosophical systems.”[22] Another view on the relationship of Parmenides and Pythagorean School is that although he followed the Pythagorean way of life, “there is absolutely no trace, either in his writings or in what we are told about him, of his having been in any way affected permanently by the superstitious elements in Pythagoreanism.”[23] ‘Parmenidean life’ was a Greek proverb with a meaning of ‘blessed life’. Moreover, Parmenides also knew the philosophy of Heraclitus. According to some scholars, his works do not include a direct reply to Heraclitus. Although the name of Heraclitus was not mentioned explicitly, Parmenides’ works can be seen as an objection to his philosophy. Parmenides had a very different understanding of the world and its progress.

3.6.1 Parmenides’ Poem

> The mares which carry me, as far as ever my heart may desire, were escorting me,
> when they brought and placed me on the resounding road of the goddess,
> which carries through all places the man who knows.
> On it I was carried; for on it the well-discerning horses were straining the chariot
> And the maidens were leading the way [24]

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[22] Guthrie, W.K.C. *A History of Greek Philosophy Vol. II: The Presocratic Tradition from Parmenides to Democritus*, (Cambridge: Cambridge University Press, 1965), p. 3.
[23] Burnet, J. *Early Greek Philosophy*, (London: A. and C. Black, 1892), p.182.

The fragments of Parmenides, which survived mostly through Simplicius begin with a description of a spiritual journey to a goddess who would reveal the truths. They were written in hexameter verse, so they can be counted as a poem rather than prose. Before him, the other philosophers were using prose or "what they meant for prose."[25] The poem was divided into two parts: Way of Truth and Way of Seeming. One path was the "unshaken heart of the well-rounded truth" and the other was "the opinions of mortals in which there is no true belief."[26]

While the journey was continuing in the dark, there appeared *the gate of day and night*.[27] After unlocking and passing that door, it was *day*. The Goddess, who called him *noble youth*, says that there are *two paths*: the path of being and the path of not-being.

3.6.2 (It) is and (It) is not[28]

> The one, that it is and that it is impossible for it not to be, is the path of Persuasion. The other, that it is not, and that it must necessarily not be[29]

These are the ways of inquiry that goddess informed us. 'It is' can be thought as a tautology, and its subject is not clearly expressed. According to Burnet[30], the reference of 'it is' is something has a spatial existence and he concludes that "the assertion that *It is* amounts to … that the universe is a *plenum*."[31] Other commentators suggest that we should not look for a subject, since there was no definite one. But, to some, this was not an efficient answer, and, for example,

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[24] Taran, L. *Parmenides: A Text with Translation, Commentary and Critical Essays*, (Princeton: Princeton University Press, 1965), p.30.
[25] Burnet, J. *Early Greek Philosophy*, (London: A. and C. Black, 1892), p.183.
[26] Taran, L. *Parmenides: A Text with Translation, Commentary and Critical Essays*, (Princeton: Princeton University Press, 1965), p.9.
[27] "The gates and the paths of Night and Day are conflated from Hesiod and the *Odyssey*, but for him the journey through the gates from Night to Day represents a progress from ignorance to the knowledge or truth which awaits him on the other side. " (Guthrie, 1965, p.12)
[28] Some translated those as 'exists' and 'exists not'. But the main tendency in translations was 'it is' and 'it is not', e.g. Guthrie, Burnet.
[29] Guthrie, W.K.C. *A History of Greek Philosophy Vol. II: The Presocratic Tradition from Parmenides to Democritus*, (Cambridge: Cambridge University Press, 1965), pp. 13-14.
[30] Burnet, J. *Early Greek Philosophy*, (London: A. and C. Black, 1892), p.189.
[31] *ibid*. p.190.

Owen[32] suggests that ‘it is’ stood for anything that can be thought. Actually this was not Owen’s definition, in the poem, the goddess continues by saying that “for it is the same thing that can be thought and that can be.”[33] So, we know now that *the one* always existed, since “it is impossible for it not to be” and it can be thought. This conception was different from what the former philosophers thought since they conceived the universe as having a beginning, and they sought a prime matter that gives birth to other elements.

Burnet, says that the originality of Parmenides’ philosophy comes from his method of argument. According to him, Parmenides investigated the common point of all the former philosophies for the first time, and then attacked to that point, namely the existence of empty space. “The next question is whether this can be thought, and the answer is that it cannot. If you think at all, you must think of something. Empty space is nothing, and you cannot think of nothing. Therefore empty space does not exist.”[34] As a result the one, *it is*, has no beginning, since a beginning assumes coming out of nothing.[35]

After eliminating the second path, namely ‘it is not’, Parmenides continued to investigate the nature of ‘it is’. Before that, the goddess mentions a third path that was followed by mortals, who knows nothing, and like the second path this one should also be avoided. Third path was dependent on the other two, and actually it was the confusion between ‘it is’ and ‘it is not’. It was the belief that ‘what is not is’ and ‘to be and not to be are the same and not the same’. This denies the concept of change, motion, becoming and perishing, although it is in contradiction with our senses. While Parmenides was claiming that our senses cannot perceive the truth, he was not referring to a nature that cannot be perceived by our senses, he was rather saying that our senses were not capable of perceiving the truth.

The critique of third path can be counted as the critique of the philosophy of Heraclitus, although there was no direct reference. Heraclitus claimed that

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[32] Guthrie, W.K.C. *A History of Greek Philosophy Vol. II: The Presocratic Tradition from Parmenides to Democritus*, (Cambridge: Cambridge University Press, 1965), p. 15.
[33] *ibid*., p.14.
[34] Burnet, J. *Early Greek Philosophy*, (London: A. and C. Black, 1892), p.191.
[35] Ex nihilo nihil fit: Generation from the non-existent is impossible.

prime matter, fire, passed into other forms such as water. Identity of opposites implied a continuous change from one to the other. In Parmenides' philosophy, there was no place for change, on the other hand Heraclitus defined life as a continuous change and struggle of the elements.

Now it is time to investigate the characteristic of 'it is'. It is eternal with no creation and it will not perish. Moreover it is one, continuous and indivisible. It does not move or change. And it has a symmetric shape, it is a sphere.

3.6.3 No Motion and No Change

Parmenides' *one* was filling the whole. This assumption cancels the possibility of motion. Since, moving is defined from some point to another. But in the world of Parmenides there is no other place to move. Moreover, since the *one* has no parts, it is impossible to conceive of an internal move of the parts.

There is no change. Because it means changing from one state to another, but as already mentioned, it is only a confusion between '*it is*' and '*it is not*'. And he claimed that we should not follow that path.

Parmenides influenced philosophers that came after him. And with his *truth* that cannot be seen in daily life, he was the originator of an important debate that would last for centuries. He was the first philosopher who clearly made a distinction between grasping the nature through our senses and reason. Guthrie says that Parmenides' originality also comes from the fact that he started from the beginning and continued with only reason.

> All previous thinkers had taken the physical world as a datum and interested themselves in questions of its origin, the kind of basic stuff that might underlie its variegated appearance, and the mechanical processes by which it was produced. Parmenides refused to accept this datum, or any datum.[36]

3.7 Zeno of Elea

Zeno who was an Eleatic in 5 B.C. was widely known as the favorite disciple of Parmenides and called the *founder of dialectic* by Aristotle. Some claimed that he was a teacher of students including Pericles and Socrates, Empedocles.

[36] Guthrie, W.K.C. *A History of Greek Philosophy Vol. II: The Presocratic Tradition from Parmenides to Democritus*, (Cambridge: Cambridge University Press, 1965), p.16.

We have only a part of Zeno's writings. Our knowledge about his paradoxes comes from Plato, Aristotle's *Physics* and Simplicius's *Commentary on Aristotle's Physics*. It is claimed that there are more than forty attributed to Zeno. But four paradoxes about motion and one on plurality are important. Since ancient commentators examined those important ones, they survived to our times.

Zeno's writings can be regarded as negative contributions to Eleatic philosophy. He did not write, at least according to our knowledge about his works, anything that improved the philosophy of Eleatic school. His well-known mission was trying to refute the counter arguments against Parmenides' philosophy. He proposed a set of counter arguments, called Zeno's paradoxes that are still waiting their solutions according to some commentators.

Zeno's attempt to defend the philosophy of Parmenides and Eleatics is seen as the source of his arguments. The view that these arguments were developed against the Pythagorean theory of point-atoms was recently forsaken. According to Parmenides, reality does not contain plurality or change or motion. Owen says that "this is an embarrassment to those who want to portray him as trying to set up a consistent logic for analyzing the structure of space and time. For it means that he thought there was no structure." [37] If one assumes that there is more than one thing, he faces with Zeno's arguments of plurality, which are self-contradictory since they showed that one thing is *p* and *not p* at the same time. Similarly his arguments of motion intended to show no method of dividing anything into spatial or temporal parts could be described without absurdity.

The force of his arguments comes from the contradiction of these arguments with the experience of daily life. We can see this confusion in the sentences of St. Augustine quoted by Cajori as

> When this discourse was concluded, a boy came running from the house to call us to dinner. I then remarked that this boy compels us not only to define motion, but to see it before our very eyes. So let us go, and pass from this place to the another; for that is, if I am not mistaken, nothing else than motion.[38]

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[37] Owen, G.E.L. "Zeno and the Mathematicians" in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p. 140.

[38] Cajori, F. "The History of Zeno's Arguments on Motion: Phases in the Development of the Theory of Limits II-III", *The American Mathematical Monthly*, XXII(1915), p.45.

The arguments of Zeno are not confusing according to certain thinkers such as Charles Peirce, cited by Sainsbury speaking of the paradox of Achilles:

> This ridiculous little catch presents no difficulty at all to a mind adequately trained in mathematics and in logic, but is one of those which is very apt to excite minds of a certain class to an obstinate determination to believe a given proposition.[39]

There are many philosophers who think that escaping from his arguments is not that much easy. Sainsbury mentions the names of various thinkers in this camp such as Hegel, Russell and Aristotle, who said that the arguments cannot be "dismissed as a mere propounder of childish riddles."[40] Benacerraf[41] presents his side by stating that the difficulties raised by the arguments were "far from silly".

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[39] Sainsbury, R.M. *Paradoxes*, (New York: Cambridge University Press, 1995), p.6.
[40] *ibid*. p.6
[41] Benacerraf, P. "Tasks, Super-Tasks, and the Modern Eleatics" in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p.103.

CHAPTER 4

ZENO'S PARADOXES

Zeno's paradoxes can be classified into two groups: paradoxes of motion and paradox of multiplicity. There are four arguments about motion and one for plurality. We will first state them in their simplified forms and then present some critiques and solution/dissolution attempts.

4.1 Paradox of Plurality

This is not a paradox on the existence of motion. It is a defense of Parmenidean monism, which states that there is only the *One* that is indivisible and has no parts. If someone proposes that there are many things, s/he has to face the paradox that offers us two choices: Object has no magnitude or it is infinitely large. In other words one thing is p and not p at the same time, which is a contradiction. We saw in philosophy of Heraclitus that change is crucial to life and one thing can have different states like wet and dry. Paradox of plurality still deserves an examination.

If we assume that an extended thing exists, for example a line segment composed of a multiplicity of points, then it must be composed of parts. Moreover, these parts have also parts themselves. So we can always bisect a line segment, and every bisection leaves us with a line segment that can itself be bisected. Since the process of division can be repeated infinitely many, there must be infinitely many parts. Continuing with the bisection process, we never come to a point where we can stop. In other words the unit parts cannot have magnitude, if they have then the bisection can continue. But the whole cannot be composed of parts that have no magnitude, since the sum of zeros is zero.

On the other hand, if we assume that the ultimate parts have a magnitude greater than zero, since we have infinitely many parts, then the sum of all those infinitely many parts will lead us into an infinite magnitude.

The conclusion is that the line is so small as to have no magnitude and so large as to be infinite at the same time. Again, our physical experience contradicts with the conclusion.

4.2 Paradoxes of Motion

4.2.1 Achilles and Tortoise

Achilles is known as a fast runner and the tortoise, which is known as the slowest. Zeno asserts that when Achilles gives tortoise a head start, he will never catch the tortoise. Since, in order to overtake the tortoise, Achilles must run from his starting point to the tortoise's starting point. But by the time Achilles reaches this point, the tortoise has moved to a forward position and so on. The claim is that although the distance between the two gets shorter and shorter, Achilles will never catch the tortoise.

It is against the ordinary prediction, which arises from our everyday experiences that Achilles will overtake the tortoise.

4.2.2 The Dichotomy (The Racecourse)

An athlete who is wanted to finish a predetermined racecourse will not be able to perform this task. Before the athlete can cover the whole distance to his goal, he must first cover the first half of it. Then he must cover the first half of the remaining distance and so on. Then the athlete should have occupied infinite number of points, in other words should have completed all of an infinite number of journeys. But it is logically absurd that someone should have completed all of an infinite number of tasks. Therefore, it is absurd to suppose that any journey can be completed.

Besides that progressive version, a regressive version is possible. Before he can complete the full distance, he must run half of it. But before he can do that he must first run half of that i.e. the first quarter. But before he can complete the first quarter, he must run the first eighth, and so on. Since before he starts to move he should have completed all of an infinite number of tasks. For the same reason, this is impossible. The athlete cannot even move. In other words, motion is impossible.

4.2.3 The Arrow

Everything must be either at rest or motion. Nothing ‘occupying a space equal to itself’ is in motion. The flying arrow is always ‘at any given moment’. Whatever ‘at any given moment’ is ‘occupying a space equal to itself’. Therefore, the flying arrow is (always) ‘occupying a space equal to itself’. And so it is not in motion. Therefore the flying arrow is at rest.

4.2.4 The Stadium

There are three columns of soldiers in a stadium. Two columns will move with the same velocity to the opposite directions and the third will be stationary as they move. It will be provided that each step brings every soldier in the moving columns into line with the next soldier in the stationary column. With each step, each soldier in each moving column meets one soldier in the stationary column but two soldiers in the oppositely moving column. The soldiers moving with equal velocity must take the same time to pass an equal number of soldiers. Then, time that is required to pass a soldier is equal to time required to pass two soldiers, in other words $t = 2t$. This is impossible.

The argument can be stated in another way as Ray presented.

> Now imagine that each soldier represents an indivisible minimum unit of length and that each step represents an indivisible minimum unit of time … at what instant and in what position did the two moving columns align so that each soldier was alongside the next (rather than the next-but-one) soldier in the adjacent moving column? If we can sub-divide the time for the step and the space between steps there is no problem at all. For they will meet after a step but we have supposed that there is no such thing as half of one of our units of length or time – since they are indivisible minima.[42]

[42] Ray, C. *Time, Space and Philosophy*, (New York: Routledge, 1991), p.8.

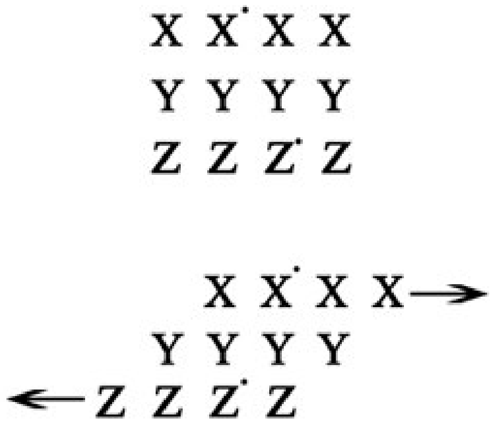

Figure 1. Stadium Paradox

4.2.5 Comments, Critiques, Solutions/Dissolutions

Zeno proposed these paradoxes in the fifth century before the Christ. Thus the attempts of the solutions have a history of nearly 2500 years. Although Zeno defended Parmenides' the *one* and aimed to show that there is no motion and plurality, the studies also had to investigate the concepts other than motion and plurality, because there are some concepts that cannot be considered separately from motion such as time and space. Moreover, the studies of these paradoxes showed us there are some other problematic issues, which also should be added into the discussions, since the solution can also be regarded as the discussion of these concepts, for example, continuity, infinity and infinitesimals. "Zeno's arguments, in some form, have afforded grounds for almost all the theories of space and time and infinity which have been constructed from his day to our own."[43] We will try to examine the studies of these arguments by also considering the additional discussion where it is needed. After exposing the philosophical arguments about the paradoxes we will state the mathematical solution that needs

[43] Russell, B. "The problem of Infinity Considered Historically" in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p.54.

the knowledge of calculus. Then we will see the similar questions, which are asked while studying Zeno's paradoxes, especially on infinitesimals are now asked about the foundations of calculus.

Shimony[44] wrote a *philosophical puppet play* in which the *dramatis personae* are Zeno, pupil and a lion. When the pupil informed Zeno that a lion was approaching, he replied that the lion should cover half the distance between the zoological garden and the Eleatic school. The lion takes half the distance mentioned. When he proved that the lion could not move at all with his method of regressive dichotomy argument, the lion entered the schoolyard. And the curtains were drawn after the lion devoured Zeno. The essential question came from the pupil: "My mind is a daze. Could there be a flaw in the Master's argument?"

We are looking for that answer too.

The arguments have been handled in different ways for more than twenty centuries. Let us try to classify them into groups. It can be said that there are three main tendencies. The first camp tried to show that the theory of time and space that underlies the arguments are wrong and so the paradoxes are based on the false assumptions. We said that time, space and motion are controversial issues along centuries and gave some examples of philosophers' conceptualization. Critics in the second camp tried to show that the arguments were not valid, so there are no paradoxes. Those in the third camp tried to solve the paradoxes by accepting thems as they were stated and examined the ambiguous concepts that the arguments brought about, like infinity, infinitesimals and limit.

4.2.6 Arguments of the First camp: Conceptualization of Space and Time is wrong

The first tendency is the critique of the assumed theory of space, time and motion. Since the debate on the arguments of Zeno started with Aristotle's criticisms, first example came from him. We can name Aristotle as an empiricist. According to Aristotle, science, that is systematic search for knowledge, was the key of understanding the world and our experience of that world. Aristotle also added to

[44] Shimony, A. "Resolution of the Paradox: A Philosophical Puppet Play " in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p. 3.

the history of thought the relationship of potential and actual. This conceptualization became important when discussing the change of an object and trying to explain how the same object can be wet and dry. We met this question in Parmenides, and his answer was to reject change. Aristotle saw potentiality of dryness in a wet object, and defined change as actualizing of potentiality. The concept wetness does not change into the concept dryness, but there is a change that occurs in the object. He distinguished between potential and actual infinity. This was an answer to Zeno's paradoxes, as he rejected Zeno's understanding of the concepts that underlie the paradoxes. When we talk about infinite space and time, or infinite divisibility of space and time, we actually mean a process that is infinite and we can see that it lasts forever, its completion, an actual infinite, is impossible. Then, according to Aristotle, there was no actual infinity in time and space. So, Aristotle disproved the Achilles and the Dichotomy paradoxes by distinguishing between actual and potential infinity. Division of the journey into infinitely many parts can only be potential. Then one question that can be asked here on the side of Zeno is the following: How can we talk about a potential that will never come to an actualization?

Aristotle disproved the arrow in a similar manner by rejecting the conceptualization underlying the problem. He defined 'being at rest' and 'being in motion' only over periods of time. There could be no motion or rest at an unextended instant of time, and to be at rest is to occupy the same space for some period of time. The concept of motion in Zeno's mind was wrong. Aristotle also rejected the view that time is composed of instants. Moreover, Aristotle pointed out a similar wrong assumption in the Stadium, holding that Zeno was in error as he disregarded the relative velocity. In other words, Zeno was falsely assuming that a soldier moves with the same velocity with respect both to the moving soldier and the soldier at rest.

Similar arguments, which suggest that Zeno's conceptualization was wrong, were also proposed after Aristotle. And these suggested different reasons from the Aristotelian critiques. For example, according to C.H.E. Lohse[45] who is a

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[45] Cajori, F. "The History of Zeno's Arguments on Motion: Phases in the Development of the Theory of Limits VII", *The American Mathematical Monthly*, XXII(1915), p. 180.

follower of Kantian philosophy Zeno's theory of time and space is wrong. Time and space are not subjective to our senses, but are forms that affect our senses. We see objects *through* time and space, and we do not sense time and space by the help of objects. Time and space are *a priori* and motion is dependent upon them. Time and space can both be infinitely divisible but that does not mean as Zeno thinks that time is composed of points. Kant himself also rejected that infinity can be thought by human mind. Moreover, according to him rest is not the lack of movement but it is the least velocity of succession. Only a moving body can be perceived. Such understanding of motion is the error that gives rise to all of the mistakes.[46]

Georg Wilhelm Friedrich Hegel's understanding of time and space was also different. According to Hegel[47], halving the space is not possible since space exists only in movement and is different from a piece of wood that can be broken into two. The problem Hegel suggested was the confusion of the objects of physical reality and space. This was stated by Johann Heinrich Loewe as the reasons underlying the paradoxes. The solution of the paradoxes "appears to us in the knowledge that contradictions must arise inevitably, as soon as space, time, and motion are considered at the same time from the stand-point of sensuous presentation and of non-sensuous conceptual reasoning"[48]. Only a part of the infinite division can be followed by our senses, but rest is related with pure reasoning. Again we are facing again a problem raised by the difficulty of grasping the infinity.

The paradoxes contain some concepts that are difficult to grasp since they have no physical applications. But some scientific theories including the mathematical solution of Zeno's paradoxes contain these vague concepts. The solution is criticized since it has potentially possible concepts that we cannot experience in daily life. The problem emerges when we use the concepts that are not elements of the physical reality. The relation between the world around us and the abstraction of that physical reality has been one of the popular debates for

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[46] *ibid.*, p. 180.
[47] *ibid.*, p. 181.
[48] *ibid.*, p. 184.

centuries and involved the discussions about the legitimacy of natural science. This is closely connected with the debates on nature of time and space. Some thinkers see the source of the problems about time, space and motion in the incongruent relationship between the reality and theory. To put it in another way, the assumption that the abstractions are true, may cause problems. For example, Jules Henri Poincaré[49] stated that the axioms of geometry and the principles of mechanics for example were not true, but only convenient. One of the ways, which can reduce the number of such problems, may be to place the 'truths' of natural sciences as an option besides the others, and to admit the existence of such 'truths' without necessarily agreeing with them. Henri-Louis Bergson, for example, distinguished the *experienced time* and time in science, namely the *spatialized time*. The latter was an independent variable and could be calculated by calculus and excluded experience. Bergson constructs his conception of time on his critiques of the concept of time in science. This approach can be a solution for people in the first camp. These critiques were proposing that the conceptualization of time and space that underlies the arguments of Zeno were wrongly stated, and that a proper distinction on these concepts can be helpful in dissolving the paradoxes. If there are two different solutions of the same problem and none of them is universally accepted, using them in different situations as solutions can be a third solution. This way of thinking can be seen in Bergson's classification of time, which I will later investigate in some detail.

4.2.7 Second and Third Camps: Discussions on the Validity of The Arguments

Critiques in the second camp question the validity of the arguments. The first aim of this approach is not asking questions about the concepts of time and space as Zeno thought of them, but, first of all question the validity of the using logic. In this approach, the concepts are questioned after the truth-value of the premises are investigated.

If we accept that the argument is valid and the premises are true, then we have to admit that *unexpected* conclusion is also true. We have two possible ways

[49] Reiser, O.R. "The Problem of Time in Science and Philosophy" *The Philosophical Review*, XXXV(1926), p. 237.

to deal with an argument that has an *unexpected* conclusion and apparently true premises. First, after accepting validity of argument and claiming that the conclusion is false, we should prove that one of the premises is false. Second, we can try to show that the argument is not valid, which means that although the premises are true, the conclusion may not be true.

James Thomson tried to prove that the Dichotomy, was not valid. In his article "Tasks and Super-Tasks", his discussion of invalidity is mainly related to the impossibility of completing infinitely many tasks. This was a common problem for every person who entered the world of Zeno's arguments of motion. Therefore, his discussion also belongs to the third camp in our classification. Thomson will be our intersection and two views will be taken together.

Let us consider Thomson's presentation of the Dichotomy:

> 1. To complete any journey you must complete an infinite number of journeys. For to arrive from A to B you must first go from A to A', the mid-point of A and B, and thence to A'', the mid-point of A' and B, and so on."
> 2. But it is logically absurd that someone should have completed all of an infinite number of journeys, just as it is logically absurd that someone should have completed all of an infinite number of tasks.
> Conclusion: Therefore it is absurd to suppose that anyone has ever completed any journey.[50]

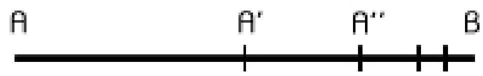

Figure 2. The Dichotomy Paradox

[50] Thomson, J. " Tasks and Super-Tasks " in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p. 89.

Thomson says that the absurdity of the conclusion made people deny one of the premisses. He mentions two groups denying the premisses. Max Black, whose ideas will be discussed later in detail, belongs to the first group. Black stated that *an infinite number of acts* was self-contradictory and declared that the second premiss is true. Therefore, the first premiss is false since you do not need to cover infinite number of journeys to go from A to B. Taylor and Watling who deny the second premiss and Black's rejection discussed that if an uninterrupted journey from A to B was made, then this movement can be analyzed as in the first premiss. The second premiss is false then, since it is seen that covering infinitely many journeys took no more effort and time than completing one, namely distance between A and B.

Thomson continues by saying that we do not have to take a side in this debate since two groups mentioned above assume the validity of the argument, but it is not true. These interpretations could not make both premisses true and he regards the argument as "fallacy of equivocation".

Paul Benacerraf who gave a response to Thomson in his article "Tasks, Super-Tasks, and the Modern Eleatics" did not accept the invalidity claim of Thomson. According to him, these premisses could not show the formal invalidity of the argument.

> At best it shows that the argument could not be used to establish its conclusion. If the phrase in question is indeed ambiguous, then one not to averse to arguing sophistically could well employ the argument in a debate and then, if pushed, admit that one of his premisses was false. He would thereby maintain the purity of his logic, if not his soul.[51]

Thomson continues with the examination of the second premiss by defining the super-task. First of all, by an infinite number of tasks, a super-task, he does not mean an operation that could be infinitely often performed. And, secondly, an infinite number of operations does not mean one infinite operation. It is the set of finite operations whose cardinal[52] is infinite. People who affirm the first premiss might have two reasons. First, it is conceivable that a number of tasks can be performed and infinite numbers are also numbers. Then, it should also be

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[51] Benacerraf, P. "Tasks, Super-Tasks, and the Modern Eleatics" in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p.105.

[52] The number of elements in a mathematical set.

conceivable to complete an infinite number of tasks. Secondly, the performance of a man can be increased, by meditation or drugs, so that he can perform a task in half the time the predecessor task needs. Therefore, the entire task can be finished in only twice of time the first half of the task took. This idea was developed by Bertrand Russell who claimed that completing an infinite number of tasks was only *medically possible*. Thomson criticizes Russell as not being even able to see the difficulty. He concludes with the claim that the reasons to believe that a super-task could be performed were not good. Thomson advances to show the opposite; there are good reasons to believe that a super-task cannot be performed. Benacerraf commented Thomson's view as being a *modern Eleatic*, since Zeno accepted the impossibility of a single task, on the other hand, although Thomson rejected this statement, he thought that "we mustn't have too many often."[53]

To defend the idea that an infinite number of operations could not be performed Thomson gives an example of a super-task:

> There are certain reading lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off. So if the lamp was originally off, and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half minute, and so on, according to Russell's recipe. After I have completed the whole infinite sequence of jabs, i.e. at the end of the two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turn it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turn it off without at once turning it on. But the lamp must be either on or off. This is a contradiction.[54]

According to Thomson the last condition of the lamp could not be known and the contradiction refutes all the possibility of similar *infinity machines*. This kind of examples 'involving' infinity is a common method that the other thinkers also used.

If we try to eliminate this contradiction by calculus, that is, by series and sequences and their limits, as we 'easily' do in the case of Achilles and Dichotomy, another problem arises. Thomson gives a sequence as +1,-1,+1,… where +1 and –1 accordingly denoted that lamp is on and is off. According to

[53] *ibid*., p.106.

mathematicians, this divergent series has a sum which is ½. The question whether the lamp is on or off is not answered by this sum which may mean that the lamp is half-open. Thomson claims by the help of this example that "there is no established method for deciding what is done when a super-task is done. And this at least shows that the concept of a super-task has not been explained."[55]

Benacerraf argues that this example is formally invalid, although it is neat and convincing. He improves the lamp argument by two lamps and two operators. He assumes that after the job has been completed one of the lamps was on and the other was off. According to Benacerraf the contradiction with Thomson's results arises from the fact that Thomson's specifications do not apply to the time when the task is finished, let's say t, but from the first jab to t.

Sainsbury criticizes Thomson in a similar way by describing a *T*-series, where T_1, corresponds to first switching, T_2 to the second and so on and T^* as the first moment after the completion of the series. Since T^* is not an element of the *T*-series, Thomson's instructions do not apply to T^* and the lamp argument "has no consequences, let alone contradictory consequences, for how things are at T^*, which lies outside the series."[56] Moreover, by giving a modified version of the lamp argument, he shows that if this lamp was possible, it would not produce any problem with the Dichotomy argument of Zeno.

Thomson turns to the Dichotomy by claiming that completing the journey which involves infinitely many journeys does not contradict the impossibility of super-tasks. He asks, where the runner was just before reaching to B when covering all midpoints to travel from A to B, and states that the runner must cover a point external to all infinitely many journeys in order to reach to B. The example of lamp and the necessity to cover a point external to all of an infinitely many tasks both show us that super-tasks which have no last task, in other words, sequence of tasks of type ω[57] cannot be performed. The task of completing a journey mentioned in Dichotomy is a task of arriving at a point. This is not of type

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[54] Thomson, J. " Tasks and Super-Tasks " in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), pp. 94-95.
[55] *ibid*., p. 96.
[56] Sainsbury, R.M. *Paradoxes*, (New York: Cambridge University Press, 1995), p. 14.

ω but ω+1, 1 stands for the last task. Then the journey is not a super-task, it is just a task and he concludes by saying that "what is said to be possible by the first premiss is not what is said to be impossible by the second premiss."[58]

Max Black agrees with Thomson on the subject that the Dichotomy was just a task, not a super-task.

> Achilles is not called upon to do the logically impossible; the illusion that he must do is created by our failure to hold separate the finite number of real things that the runner has to accomplish and the infinite series of numbers by which we describe what he actually does. We create the illusion of the infinite tasks by the kind of mathematics that we use to describe space, time, and motion.[59]

Black, in his article "Achilles and the Tortoise" calculated the distance where Achilles met the tortoise not by using the sum of an infinite series. But he used a mathematical solution, in which the shadow of the ambiguous concept that the sum of an infinite series was excluded. He thought that this type of answer could not go to "the heart of the matter."[60] In other words, the time and distance when and where the two would meet could be calculated by mathematics, but this solution could not show why Zeno was wrong. The sum of an infinite series is not the actual sum of all the terms of the series. He explains this by an example of Gottlob Frege, this infinite addition would demand infinite amount of paper, ink and time.[61] He mentions mathematicians who saw performing a super-task possible and who thought that the confusion which infinity causes was the weakness of imagination. Black, then, shows that an infinite number of tasks are self-contradictory and that "failure to see this arises from confusing a series of acts with a series of numbers generated by some mathematical law."[62] The physical world could only be divided to a certain point. The intervals or distances or parts of a matter would not be spatio-temporal after that point.

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[57] Any set ordered as a progression, i.e., having the same order as the natural numbers in their natural order, is said to have order type ω; ω is the first transfinite ordinal number

[58] Thomson, J. " Tasks and Super-Tasks " in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p. 100.

[59] Black, M. "Achilles and Tortoise" in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p. 81.

[60] *ibid*. p. 70.

[61] *ibid*. p. 70.

[62] *ibid*. p. 72.

Owen, in his article “Zeno and the Mathematicians” regards the arguments of motion were also attacks on plurality. Zeno claimed together with Parmenides that there existed only one thing. Zeno asked to the opponents of this view how they could distinguish their individuals from the rest. The answer lied in finding a unit to define each of these individuals. But the solution of gaining a unit was to divide the whole. We know what happens when we start to divide: absurdity. The argument proceeded as follows: if we divide infinitely we reach a quantity having a size or having no size. If it has a size, it should also be divided. If it has no size, on the other hand, it does not exist, since there cannot be any quantity that has no size. We encounter this problematic also in the Achilles and the Dichotomy paradoxes. If we assume that an infinite number of tasks could be performed, another question will arise: What would be the end products of this process?

> If Achilles claims to have finished his task we can ask about the positions of these marks, and in particular of the last two. If that are in the same place there is no stage determined by them, and if there is any distance between them, however small, this distance is the smallest stage in an infinite set of diminishing stages and therefore the course is infinitely long and not just infinitely divisible.[63]

Some thinkers put forward *atomic distances* which cannot be further divisible and have some size. Hence, the last move of the Dichotomy could be taken as an *atomic distance*. Let’s say the solution of these two paradoxes was reached by introducing *atomic distances*. However, at this point, it is high time for the second version of the Stadium argument to bring about problems. The step of a soldier cannot be divided since it is assumed that it is an *atomic distance*.

In mathematics, the idea of infinitely small is named as ‘infinitesimal’. Although, according to Owen, an infinitesimal and an atomic distance are the same, Ray warns us that by accepting that infinitesimals have size, no matter how small, we fall into the traps of Zeno. In addition, Cauchy, by bringing the notion of limit, provided a more coherent theory where the infinitesimal was absorbed. The transfer of the physical reality to mathematics was mainly attributed to Weierstrass:

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[63] Owen, G.E.L. “Zeno and the Mathematicians” in *Zeno’s Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p.145.

> [Weierstrass] showed that we could move the debate from the realm of geometry to that of arithmetic, from ideas of spatial and temporal distances to those of functions. Instead of talking about ever-decreasing distances along a straight line, we could talk with a little rigour about infinite series converging on limiting values in terms of functions and real numbers.[64]

And Ray continues by asking "to what extent are mathematical and geometrical concepts and structures *strictly true* of the physical world? Why should abstractions apply literally to the physical world."[65] With this question we turns to the beginning. Thomson states that a mathematical solution "does not resolve all the hesitations one might feel about the premiss of the paradox…these hesitations are not merely frivolous, and that insofar as they spring from misunderstandings these misunderstandings are shared by those who support the 'mathematical solution'"[66]

Up to this point, we have been examining the proposed solutions on the Zeno's paradoxes, and we saw that there are concepts closely related with time, space and motion, such as infinity, infinitesimal and limit. These core concepts are not easier to grasp than the former ones. Some interpreters claim that the solution of the paradoxes is easy for people who have mathematical knowledge. Moreover, after *calculus*, Zeno's paradoxes are said to be solved. Although there are who were not satisfied this answer, the discovery of *calculus* was a great cornerstone of human thought and history of mathematics and science. We will briefly look at the progress of mathematics, and try to understand that why some were not satisfied by the invention of *calculus*. These critics have doubts about the abstraction of reality. We will try to see whether the problem is really due to a lack of imagination of the people who still have doubts. Or perhaps we should ask more questions about the nature of mathematics and experience.

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[64] Ray, C. *Time, Space and Philosophy*, (New York: Routledge, 1991), p.11.
[65] *ibid*., p.14.
[66] Thomson, J. " Tasks and Super-Tasks " in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p.102.

CHAPTER 5

CALCULUS: A NEW APPROACH TO THE PROBLEM

Examining Zeno's paradoxes and trying to discover the nature of time, space and motion, we are faced with other concepts closely connected with the former ones. Two of them were infinite and infinitesimal. Some commentators claim the impossibility of grasping those concepts, since they are out of our capacity of mind. But this did not prevent others to work on those issues. Moreover, although even grasping them is still seen as a problem, mathematicians claim that calculus can provide us with answers. These answers employ the concepts infinity and infinitesimal.

Zeno raised questions about the nature of the universe, and his conclusions contradicted our daily experiences. The proposed solution by mathematics has also brought difficulties for people who had little knowledge of mathematics. One can approach to these problems in various ways. First of all, sometimes it may be convenient to ask naïve questions, like a child who has no background of higher mathematics and who knows nothing about the abstraction of physical reality to mathematical conceptualization and notation. This is not how we will proceed, but there are times that this view may be helpful in order not to be lost in the very abstract world of mathematical ideas. The second approach is the view of a person who has talented skills of mathematics. In this view, there is no place for questions about mathematical ideas like infinity and limit. This is the view of these who think that after a point in the history of the progress of mathematical ideas there is nothing left mysterious and unexplained about Zeno's paradoxes. There is another approach that will be a combination of these two. This is the approach of a person who has proper knowledge to understand the problem and to perform the needed study for exposing the difficulties of the issue. This is a person who does not undervalue the problems that arise while the study is going on, and does not avoid asking further questions. This approach can be called philosophical.

To expose the differences between these approaches, a survey of the history of the progress of calculus is a very useful since in this history there are controversies on the abstraction of reality that are treated differently by these different approaches.

The discovery of calculus is attributed to two men: Isaac Newton and Gottfried Leibniz. But the former works was very helpful to them while they were developing their theories. The history of calculus contains the history of the progress of the human thought on the disputatious concepts, namely infinite, infinitesimal and limit. Mathematical solutions of some of Zeno's paradoxes will be stated here to expose how calculus solves the problem. The infinitely many tasks, for example, completing the infinitely many journeys in the Dichotomy paradox is solved by the sum of the infinite series which will be criticized by some scholars since it contains infinity which cannot be conceived easily.

5.1 Mathematical Solutions of Zeno's Paradoxes

5.1.1 The Arrow

First important critique of arrow paradox came from Aristotle, who claimed that motion could only be possible over a period of time. This critique is important since it was the chief comment on the solution of arrow paradox and the notion of time until new insights were brought to mathematics. If we talk about velocity, then it should be the average velocity over a period of time. Aristotle simply refused the atomic view of time that time is composed of instants, of instantaneous velocities of an object along a motion, if we put it in modern language. Average velocity which is the only type of velocity that appears in Aristotle's works can be easily calculated by dividing the total distance covered to total time elapsed. Since an object cannot take any distance at an instant then this division that is 0/0, which is absurd. Frank Arntzenius in his article examining the existence of instantaneous velocity comments on the view of Aristotle as follows:

> the development of the position of an object over a period of time cannot be represented by a point-valued function X(t) from the real numbers to the real numbers. Indeed it is not immediately clear how one could do mathematical physics at all in anything like the manner that we are accustomed to. Aristotle himself certainly did not develop the mathematical tools to implement this view at any level of rigor. Such tools were not developed until the twentieth century[67]

If the trajectory of the arrow is drawn, it is seen that although there is no motion at an instant, the arrow is in motion at all instants. This comes from the notion of *continuous function*. The value of a function at a time t can be constant but it does not necessarily entail that the function itself is constant at time t. If we adopt the Aristotelian view, then we cannot make mathematical physics that are needed at least for our practical life. If we inclined towards the mathematical solution then we have serious questions that have no clear answers except for the mathematicians.

5.1.2 The Dichotomy

If we assume length of the road covered as 1 unit length then it follows. First the runner have to cover the half of it, that means ½. Similarly half of ½ is ¼. The related series with the problem is

½ + ¼ + 1/8 + 1/16 + 1/32 + …

This type of a series is called geometric. A geometric series is a series that begins with one term and then each successive term is found by multiplying the previous term by some fixed amount, say x. x in the above series is ½. And if x is smaller than 1, then the series converges, that means it has a finite sum. This sum can be calculated by dividing the first term of the series by 1-x. In our case this is ½ / 1 – ½ = 1. Time elapsed can be written as a infinite geometric series, and x in that series is less than 1, and we have a finite sum of an infinite series. We saw that while the runner is approaching the goal, the distance that has to be covered at each step and the time elapsed at each step are geometrically decreasing. So the total distance is finite. Another mathematical solution, actually the easiest one is the direct proportion. If the runner took the first half of the road in t seconds, then the whole road is covered in 2t seconds.

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[67] Arntzenius, F. “Are there really instantaneous velocities?” *Monist,* 83/2(2000), p.187.

After examining these solutions the dichotomy can easily be solved with a little knowledge of calculus sounds reasonable. But the possibility of completing infinitely many tasks in a finite time is still an issue open to discussion.

5.1.3 Achilles and The Tortoise

Since the case is similar to dichotomy, we will only state the solution. If we assume that Achilles runs ten times as fast as the tortoise, and when Achilles is at the starting point tortoise has a hundred meters head start, then the series of the distances that Achilles has to cover to reach the tortoise is 100 + 10 + 1 + 1/10 + … This is again a geometric series with x equal to 1/10. The same formula applies: 100 / (1 – 1/10) = 1000/9. The result is again a finite number. Moreover we have again a very easy mathematical solution that does not contain an infinite series. Let us say the Achilles runs 10 meters in a second, i.e. the Achilles' velocity is 10 m/s and the tortoise's is 1 m/s with a 100 meters head start. After the first second, the tortoise will be the point 101 meters while Achilles is at 10 meters. After ten seconds tortoise will be at 110 meters. As a result, they meet at the end of 11^{th} second.

5.2 A Survey of the Evolution of Calculus

Newton came with new ideas for the history of human thought. From the time of Aristotle to Newton there are many philosophers who made attributions to the development of the ideas we are interested. During all the ages, including those in which the Greek science declined, those in which the Greek works were preserved and translated by Arabs, and those in which, like the Renaissance, the Greek works were rediscovered, it can be said that Greek science always preserved its importance. It had beautiful basic questions about the universe, and controversial answers to those questions. New ideas, especially after the rediscovery of Greek works, emerged from the reconstruction or critiques of those texts.

Although calculus is said to be a product of the seventeenth century, we saw that the questions that let it emerge were asked very long time ago. Although these questions were common to different civilizations including the eastern ones, they were handled adequately only by Greeks. I have noted before that the existence of different approaches in different civilizations should not be considered as separate from practical needs and culture.

Discovery of incommensurable[68] magnitudes was the first important crisis in Greek mathematics. It is attributed to Hippasus and according to some accounts, as a result of this discovery he was expelled from the Pythagorean School. According to Pythagorean thought everything in the universe can be explained by the help of numbers. This does not include only mathematical relations, but also the essence of life. Numbers, in this sense can be compared to other philosophers' prime matters, such as air, fire, etc. Hippasus showed the existence of incommensurable magnitudes by proposing that the diagonal and one side of a square (or cone or pentagon) are not comparable in terms of integers. To find a unit of magnitude with which comparing is possible, Greek mathematicians realized that they had to convey a search that has infinitely many steps. Boyer says the following:

> It was a shock to the Greek mathematical community to learn that there are such things as incommensurable line segments and that the occurrence of such situations is appallingly commonplace-that is, that concepts akin to the calculus arise in the most elementary situations. The dialogues of Plato show that mathematicians of the time were deeply disturbed by this discovery.[69]

Here, I want to emphasize to that when a new discovery is made, the reactions can be very severe as in this case. Since it shook the bases of a whole philosophy the supporters were not pleased with this change. What could a Pythagorean in this situation? If he still believes that the discovery is wrong, then he may try to disprove it. But in this case it is impossible, since a mathematical proof exists. Approximately at the same period, the Pythagoreans were confronted with a new attack to their philosophy by Zeno's paradoxes. Similarly, these paradoxes are related to problems of infinity that made Greek mathematicians uncomfortable. "Aristotle and other Greek Philosophers sought to answer the paradoxes of Zeno, but the replies were so unconvincing that mathematicians of the time concluded

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[68] Quantities are incommensurable when no third quantity can be found that is an aliquot part of both.

[69] Boyer, C.B. "The History of the Calculus" *The Two Year College Mathematics Journal*, I/1(1970), p. 62

that it was best to shun infinite processes altogether."[70] Boyer argues that this approach was an obstacle to the development of Greek calculus.

Arguments proposed by Hippasus and Zeno affected the development of Greek mathematics. These arguments had two important results which can be seen in the first mathematics textbook, the *Elements* of Euclid. It was a shift from numbers to geometrical objects. "The realm of number continued to have the property of discreteness, but the world of continuous magnitudes was a thing apart from number and had to be treated through geometrical method."[71] Secondly, with the inspiration of these two problems, the deductive method is adopted. Geometrical algebra took place of the arithmetical algebra, and mathematical abstraction and deductive method were adopted instead of mathematical prescriptions.

Archimedes is called the father of mathematical physics. It can be said that the ideas that underlie infinitesimal calculus blossomed in the texts of Archimedes. His book named *Method* contains the calculation of the volumes and areas of some geometrical objects by the help of infinitesimals that will be invented in 1600s. Archimedes describes this method in a letter as "a certain method by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics."[72] Measurement was an important issue in his works and he changed the nature of Greek mathematics by using concepts such as "related rates, limits, tangents and the evaluation of Riemann sums"[73] akin to differential calculus while he was working on spirals, and to integral calculus when he used the method of exhaustion, in for example, solving the problem of squaring the circle. The idea was that a line is composed of infinitely many points, a surface is made of infinitely many parallel lines, and a solid is composed of parallel planes. Then the volume of a solid, for example, is the sum of those infinitely many planes.

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[70] *ibid* p.63
[71] Boyer, C.B. *History of Mathematics*, (Brooklyn: John Wiley & Sons, Inc., 1968), p.84
[72] Boyer, C.B. "The History of the Calculus" *The Two Year College Mathematics Journal*, I/1(1970), p. 64.
[73] Vardi, I. "What is Ancient Mathematics?" *Mathematical Intelligencer*, 21/3(1999), p.40.

After Archimedes we see a decline of Greek civilization, and therefore a decline in ancient mathematics. Romans who have been gaining power were different from the Greek civilization in their approaches to mathematics. They were trying to build a new empire, and hence were interested in and needed practical uses of mathematics. Therefore, they needed measurement and calculation, so arithmetic was more important than abstract ideas. With the decline of Roman Empire, and the rise of Christianity a new period began. The conditions that affected human thought changed once again. The most important characteristic of this new era was the dominance of theology in every branch of thought. Since God revealed all the necessary truths, there is no need for further examination especially in natural sciences. Some philosophers took the middle course by writing scientific works that verify the revelation of God. The existence of actual infinite was accepted by St. Augustine who claimed that it was a constant rather than a variable by indicating that any positive integer is an example of that type. St. Augustine's view is very controversial and actually was not accepted. For example, Thomas Aquinas, another important figure of Medieval philosophy, while exposing the nature of continuum argued that a continuous line could be infinitely divisible only potentially. Moreover, a point is not a part that constitutes a line so it does not have the same properties with the line, namely infinite divisibility. The construction of the line was described as the trajectory of the motion of a point. This approach to continuum is generally accepted until the nineteenth century.

To the questions asked in the fourteenth century the most convenient answer was the whole of the works which is called calculus in modern terminology. And these works were a step beyond the *static* works of Archimedes by including the variables. An object with changing velocity or changing heat was under examination. Nicole Oresme, while trying to find the total distance of an object that moves with varying velocity paired the instants and the velocity of those instants in an early example of a graph by drawing a line passing through these pairs. Then the distance was equal to the area under that graph which means *integration* in modern terms. This is a great step, and Boyer stresses its importance by claiming that "one runs up against all the philosophical and logical difficulties that led to the paradoxes of Zeno and caused the careful Greek mathematicians to avoid the study of variations as such."[74] Concepts related to infinity were not problematic anymore, at least mathematically.

In 1600s, during the Renaissance, the translations of the texts of the former philosophers especially the works of Archimedes made the hundreds of years of mathematical heritage available to Europeans. The two important figures of this period used the notion of infinity in their works: Kepler and Galileo. Kepler defined circle as a polygon with an infinite number of sides and its area as the sum of infinitely many triangles. Galileo, who organized the ideas of Oresme in mathematical precision, made the same definition for the circle in his works. In his book *Two New Sciences*, Galileo make two characters, Salviati and Simplicio talk about infinity. Salviati who was scientifically informed, showed the way from the infinite in geometry to the infinite in arithmetic by indicating one-to-one correspondence between the set of integers which has an infinite cardinal and the perfect squares. The old methods of Greek mathematics are adapted to the conditions of the period and turned into arithmetical ones.

In 1635, Bonaventura Cavalieri who was a disciple of Galileo introduced "the theory of indivisibles" that turned the controversial issue of indivisibles to a method to measure the areas and volumes of the object. It was simply reformulating a geometrical object as a sum of infinitely many lower dimensional indivisibles. For example, a surface was made of infinitely many parallel lines. Similarly an object was the sum of infinitely many parallel planes. The method of finding the area of a plane, named Cavalieri's Principle, was comparing it with another plane whose area is known. After establishing one-to-one correspondence between the indivisibles and finding their ratios we can say, according to his principle that their areas are in the same ratio. It is not a completely new idea. His contribution can be said as the application of the method that can be traced back to Archimedes. Cavalieri, "unlike Archimedes, felt no compunction about the logical deficiencies behind such procedures."[75]

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[74] Boyer, C.B. "The History of the Calculus" *The Two Year College Mathematics Journal*, I/1(1970), p. 68.
[75] Boyer, C.B. *History of Mathematics*, (Brooklyn: John Wiley & Sons, Inc., 1968), p.362

I think it can be said that, the change of mentality of dealing with the problems has its source in Archimedes. The new methods of calculation caused a shift from geometrical approaches to arithmetical answers. "Here at the end of Renaissance, supported by the ease of algebraic calculation, the passage from finite to infinite was hardly worrying."[76] While new methods for solutions were gained, some points of views were lost. And this loss would cause new debates.

It is the time of a new shift. Descartes just after Cavalieri's "theory of indivisibles" wrote his book, *La Géométrie* and introduced the theory of analytic geometry, which combines geometry and algebra. His scientific and philosophic views, unlike mathematical ones are regarded as revolutionary attempts that made him the father of the modern philosophy. He is distinguished from the former philosophers in proposing that we should break the relation with past works and their commentaries, and we should progress with our reason alone. With this in mind, he tried to put forth a completely mechanical explanation of the natural world with matter and motion, two favorite concepts of his philosophy. His contribution to mathematics was to expose the relations between the arithmetical calculations and geometrical representations, in other words, he tried to pair the calculations such as multiplication and division with their geometrical equivalents. After a geometrical problem transformed into algebraic notation, then the algebraic equation is simplified and solved. One of the results of his method is a decrease in the use of diagrams in geometry books and in minds. His book is "the earliest mathematical text that a present-day student of algebra can follow without encountering difficulties in notation."[77]

5.3 Newton and Leibniz

Now the basic concepts of calculus are ready so that we can solve Zeno's paradoxes mathematically. The next step is to bring all the ideas and theories related to infinitesimals into a general theory, which we call *calculus*. Calculus provides us with three kinds of data: a set of algorithms to solve the problems, a

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[76] Goldstein, C. "Stories of the Circle" in *A History of Scientific Thoughts: Elements of a History of Science*, ed. M. Serres (Oxford: Blackwell, 1995), p.180.
[77] Boyer, C.B. *History of Mathematics*, (Brooklyn: John Wiley & Sons, Inc., 1968), p.372

set of proofs of the proposed theories that make them theorems and applications of the rules to problems.

This work is attributed to Sir Isaac Newton and Gottfried Wilhelm von Leibniz. One may ask, if all the needed rules to solve the related problems are found and published, then what is these two men's contribution to the field? Boyer says the following:

> Newton and Leibniz were not the first ones to use methods equivalent to differentiation or integration or to notice the inverse relation between these, nor were they first in the use of infinite series. The contribution of these men, … , lay in gathering together devices of limited applicability and developing, from them, methods of universal scope.[78]

The equations and formulae that discovered by former mathematicians satisfied the needs of their ages. Because of the rapid mathematical progress of the era and the discovery of new functions, for example transcendental functions, the solutions are not sufficiently qualified to meet emerging needs. What was needed was a method to deal with problems indifferently rather than a particular answer to a particular problem. What was needed was a tool for examination of the issues related with the controversial concepts such as variability, continuity, acceleration, motion and infinite.

5.3.1 Newton

Newton, in 1665 and 1666, signed very important theories such as binomial theorem, tangent method, method of fluxions or what is now called differential calculus, inverse method of fluxions or integral calculus and theory of colors. It is said that falling of an apple from a tree was the initiator of these works including *Principia*, *The Mathematical Principles of Natural Philosophy*. Cajori says the following:

> …Newton's thoughts began to dwell upon the earth's attraction for the moon. Was the moon attracted to the earth by a force which varied according to the law of the inverse square of its distance from the earth? If so can this be checked by mathematical computation? A simple process of computation was published twenty-one years later in Newton's "Principia".[79]

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[78] Boyer, C.B. "The History of the Calculus" *The Two Year College Mathematics Journal*, I/1(1970), pp. 79-80.

[79] Cajori, F. "Sir Isaac Newton on Gravity" *The Scientific Monthly*, 27/1(1928), p.49.

Newton had an aim to constitute that kind of method when he was working on his calculus. He explains his method and aim at the end of Book II Lemma II of *Principia* as follows:

> This is one particular, or rather a Corollary, of a general method, which extends itself, without any troublesome calculation, not only to the drawing of tangents to any curved lines whether geometrical or mechanical…but also to the resolving other abstruser kinds of problems about the crookedness, areas, lengths, centres of gravity of curves, etc.; nor is it (as Hudden's method de maximis et minimis) limited to equations which are free from surd quantities. This method I have interwoven with that other of working in equations, by reducing them to infinite series.[80]

Newton made a distinction between absolute and relative time (space, place and motion). Absolute time (space, place and motion) was true and mathematical. On the other hand, the second group measurable by us was common and apparent. They were relative, since measuring is a process of comparing one with the other. He gave no explicit definitions of time, space and motion "as being well known to all"[81] and proposed that the reason of this division was removing the prejudices because "the common people conceive those quantities under no other notions but from the relation they bear to sensible objects."[82]

According to this distinction, absolute time and space were homogenous, infinite, immutable, object and mind independent. It is clear that this kind of conceptualization meet the needs of science, for example, since every instant of time would be the same for every person at every corner of the world even in space, then it would be possible to determine the simultaneity of events. Similarly only absolute motion can be true and useful for science since relative motions of an object varies according to reference objects. Then one can ask, what is the reference point for the absolute motion? Let us recall that motion entails a change of the position. Although its name is absolute we still need a reference object to grasp the motion. This should not be confused with Descartes' view of motion, which was rejected by Newton. According to Descartes, motion was dependent upon the surrounding bodies' positions and motions. Reference of Newtonian

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[80] Boyer, C.B. *History of Mathematics*, (Brooklyn: John Wiley & Sons, Inc., 1968), p.436.
[81] Huggett, N. (ed) *Space from Zeno to Einstein: Classic Readings with a Contemporary Commentary*, (Cambridge: MIT Press, 1999), p118.
[82] *ibid*. p. 118

absolute motion was the absolute space itself. On the other hand, relative motion could only be measured with respect to other material objects.

Let us remember that in the Achilles and the Dichotomy, the runner could reach the goal point only if he can traverse the last distance to that point. When we express the task of the runner with a function in mathematical language, then we need that the function should reach its limit. Newton argued that "quantities and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."[83] In other words, if the ratio converges to zero, then the runner can reach his goal. But, together with some other conceptualizations inside Newton's calculus this point is under discussion because of vagueness of equalizing the infinitely small to zero. Making anything vanish even in theory results in important attacks against the foundation of calculus in the following centuries.

5.3.2 Critiques of Newtonian Absolute Space and Time

Newton succeeded in explaining and measuring the physical reality and solving the problems related to sensible world by his approach to space and time. His method generalized the differential and integral calculus. But the absolute components of his theories were criticized just after they were published. He was an empiricist and these concepts were not experienceable. We could only experience temporal and spatial relations of the events and objects. Two important critics of Newton, Leibniz and Berkeley rejected the existence of absolute space and time. These critics were aiming at the metaphysical and philosophical background of Newton's theories rather than their mathematical and physical sides. And later similar criticisms were directed towards Newton's theories by Ernest Mach in whose works Einstein found the necessary inspiration to construct the *relativity theory*. Value of a philosophical system is not measured only by its success in the field, but also its effect in the history of thought although this effect is due to its critiques. Then, Newton with the success of his physical theories and

[83] Cajori, F. "The History of Zeno's Arguments on Motion: Phases in the Development of the Theory of Limits VI", *The American Mathematical Monthly*, XXII(1915), p.143.

questions related to their metaphysical grounds, was a very important and well-known figure in the history of science.

Let us recall that some critics' claims that our concepts of time and space are the reasons underlie Zeno's paradoxes. Newtonian space and time are continuous. Continuity entails divisibility. Therefore, paradoxes follow if we adopt his understanding of time and space.

5.3.3 Leibniz

Historians tell us that there was a debate between Newton and Leibniz who are both remembered as the discoverers[84] of calculus independently. Although they have similar backgrounds on the issues under discussion, their understanding of nature and the ways of grasping the nature were different. While Newton's tools were experiment and observation mostly, Leibniz was a rationalist like Descartes, who was also criticized by Newton. Leibniz claimed that we cannot trust empirical sciences, we have only reason to understand the reality of nature, because the sensible world of appearances and the real world are different. Deduction should be more respectable than the senses in this sense. Moreover, according to Leibniz, pure geometry does not need experiments and observations. In short, deduction from irrefutable rules is the method and reason is the major tool. He characterized the world outside as *phenomenal* and proposed that it consists of real structures, *monads* that are not experiencable by our senses.

Leibniz offered a relational account of space and time and rejects the Newtonian absolute concepts. One event occurs before or after another, and events have temporal relations with other events. Then, according to Leibniz, time can be regarded as the collection of all these temporal relations. Therefore, if there are no events and no relations, then time does not exist. Similarly, there are spatial relations like distances and directions between the objects. Space is the collection of these spatial relations and there is no absolute space that exists independently

[84] Is calculus an invention or discovery? Although this is another matter for discussion, majority of the articles surveyed consulted for this text the term *discovery* is used. *Discovery* seems more appropriate since the relations exposed by mathematics are generally supposed to exist before their *discovery*. In fact, the term *evolution of calculus* seems to be a better expression for the approach in this thesis.

from these relations. If the world consists of spatial and temporal relations and there is nothing further, then one can ask about the moments when nothing happens and the places where nothing exists? It is hard to imagine, but let us try to understand it by means of an example. “Consider the empty space between here and a star. There is nothing that bears the spatial relationship to us of being halfway between us and the star.”[85] I think that Sklar omits the physical reality and that the existence of these two objects entails the existence of the relation between them and so the existence of space according to Leibniz’s theory. He could have given this example merely for the sake of nothing but giving an example. “So we might think, then, of unoccupied places as spatial relations that something might have to the objects of the world but that nothing actually does have…So the family of relations contains possible as well as actual spatial relations.”[86] Hence the concept of totally empty space can be reconstructed by a relationist in terms of potential and actual existence of the relations. This seems to contradict the former claim that if there is no relation then there is no space. A further question on motion can be directed to this approach. If all the objects in the universe *determined* to move then what will be the reference point of an object? If the answer is the relative motions with respect to other, then we can consider the case that every object is moving to the same direction with the same constant velocity. Then all objects will be at rest in case no reference different from the moving object itself is adopted. Actually Leibniz accepted the existence of absolute motion and absolute acceleration in some cases when there is a force on the object. Some critics commented that Leibniz was not a consistent relationist.

The relationist rejection of Newtonian space and time can be seen in the discussions occurring in the letters between Leibniz and Samuel Clarke who was close to Newton. Leibniz proposed two important principles while rejecting absolute space and time.[87] The first one is the *Principle of Sufficient Reason*, that is, nothing happens without there being a cause. Second one, the *Principle of Identity of Indiscernibles*, that is, no two objects have exactly the same properties,

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[85] Sklar, L. *Philosophy of Physics,* (New York: Oxford University Press, 1992), p.20.
[86] *ibid*. p.20.
[87] Huggett, N. (ed) *Space from Zeno to Einstein: Classic Readings with a Contemporary Commentary*, (Cambridge: MIT Press, 1999), p159.

if they have, then they should be the same. The argument begins with the Newtonian proposition that space is absolute and can be regarded as an infinite container of every existing object. In creation, then, God chose the place where the material universe will be located. Leibniz proposed that even if God located the universe to another place different from the former and provided that he located the second universe differently for example upside down, the relative positions of those universes with respect to infinite and homogenous space would be the same. In other words they have no differences, and that makes the two universes the same according to the *Principle of Identity of Indiscernibles*. Therefore, God had no choice for placing the universe since all his choices would lead to the same result. But we know from the *Principle of Sufficient Reason* that nothing happens without a cause. Then one of our assumptions should be wrong. Leibniz concluded that there is no absolute space that serves as a reference for objects in the material universe. A similar reasoning is valid for absolute time. Leibniz showed that there would be no difference for the moment that God created the world, since the histories of the possible universes would be the same for both creation dates.

Clarke replied to Leibniz's proposition that the universes will be the same after a rotation of 180 degrees that the two universes will not be the same since the first is rotated upside down. This is an entry in the list of differences between the two approaches rather than an answer.

Leibniz died without giving very powerful answers to Clarke's claims. But he prepared the bases for the other relationist attacks with his questions on the absolute space and time. "On the continent the metaphysical rationalism of Leibniz was neglected by his followers, who freely attempted to interpret the differentials as actual infinitesimals or even as zeros, and who criticized Leibniz for his hesitancy in this respect."[88]

[88] Boyer, C.B. *The History of Calculus and Its Conceptual Development*, (New York: Dover Publications, 1949), p.224.

5.3.4 Berkeley

Bishop George Berkeley (1685-1753) did not reject the benefits of the methods that calculus provides. "His objections were not to the mathematician as an artist and computist, but as a "man of science and demonstration" and "in so far forth as he reasons.""[89]

According to Berkeley, we can only have knowledge through our senses. No object exists independently from the mind. As an assumption, the mind is real. Entities in the universe are not material objects but impressions through our senses. This approach rejects both abstract conceptualizations of absolute time and space of Newton and causality of Leibniz since we cannot perceive them. All motion is relative. Berkeley tried to construct a theory on space and time that is evident to everyone, in other words his explanations are intended to be true and clear to everyone. Offering the fixed stars as reference points instead of absolute space can be regarded as an example of this attitude. Similarly he rejected infinite divisibility, since mind cannot grasp those infinitely small quantities and they are "dismissed from his philosophy as void of meaning or involving contradictions."[90] Boyer commented on this severe attacks of calculus as that

> Berkeley was unable to appreciate that mathematics was not concerned with a world of "real" sense impressions. In much the same manner today some philosophers criticize the mathematical conceptions of infinity and the continuum, failing to realize that since mathematics deals with relations rather than with physical existence, its criterion of truth is inner consistency rather than plausibility in the light of sense perception or intuition.[91]

5.3.5 Mach

Ernst Mach, inspired by the ideas of Berkeley, was not comfortable with the idea of absolute space and time since according to his understanding of science this kind of unobservable and non-verified conceptualizations cannot be allowed. All we can observe was only relative motion and relative order of events: "Suppose we observe a test body moving in a relative reference frame in some way; how can we

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[89] i*bid*. p.225.
[90] Cajori, F. "The History of Zeno's Arguments on Motion: Phases in the Development of the Theory of Limits VI", *The American Mathematical Monthly*, XXII(1915), p.144.
[91] Boyer, C.B. *The History of Calculus and Its Conceptual Development*, (New York: Dover Publications, 1949), p.227.

legitimately infer that it would behave the same way without the reference bodies?"[92] This question actually aims at the heart of the Newtonian absolute concepts since it cancels the possibility of talking about absolute. No one, including Newton experienced the space without objects, therefore no one could make claims containing these unobservable concepts, at least scientifically. Mach, like Berkeley, offered fixed stars as the reference point instead of a vague container idea.

By saying that "the fundamental conception of the nature of science [is the] economy of thought" and science's "goal is the simplest and most economical abstract expression of the facts"[93] Mach was trying to adopt a method of simplification in science. That was the Occam's razor in his hands, and any unnecessary entity would be removed from scientific works. Our reasoning should be limited to experiment and observation. Reality can be grasped through our sense experience, the rest is just speculation and should be excluded from science. Similar approaches were seen in the works of Berkeley and David Hume and would affect the later positivist philosophers. Although Mach was not a relativist, he is also credited as having given inspiration to Einstein for the theory of relativity.

In the nineteenth century Cauchy by making clear "the concepts of functions, limits and continuity, at the same time showing how the concepts of derivatives, differentials, anti-derivatives, and definite integrals could be based upon them."[94] There was one missing point in his theories. He failed to construct the real number system with irrationals. In the same century this gap filled by Dedekind, Weierstrass and Cantor and as a result infinitesimals are banished from calculus.[95] But today discussions on Zeno's paradoxes do not come to an end.

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[92] Huggett, N. (ed) *Space from Zeno to Einstein: Classic Readings with a Contemporary Commentary*, (Cambridge: MIT Press, 1999), p.182.
[93] Ray, C. *Time, Space and Philosophy*, (New York: Routledge, 1991), p.116.
[94] Salmon, W.C. "Introduction", in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc, 1970), p.20.

5.4 Rigor in Mathematics

As we saw, that some of the critiques of calculus and the Newtonian universe were under attacks from other philosophers who wanted to clarify the vagueness of the foundations of calculus. They questioned the fundamental concepts of the system and tried to reconstruct the foundations of the calculus to reach mathematical rigor. The reason for further questions on Zeno's paradoxes was the same approach even after the declaration of the mathematical solution of the paradoxes. Now, with the help of the mathematical solution, we know the time the runner will cover the course. But it is claimed that the answer to how the runner can accomplish the infinitely many tasks is still missing. It should be noted that the some of the above-mentioned discussions on solution/dissolution of the paradoxes are contemporary, the participants know the end of the story, namely the evolution of calculus. It seems that for some people the answers are not sufficient, while others think that the paradoxes are solved and the solutions are clear to everyone who has sufficient mathematical knowledge. On the other hand, "philosophers –even scientifically minded philosophers– cannot, it seems, remain permanently satisfied with an answer to a problem. Zeno's paradoxes were scrutinized again and again."[96] In this sense, Zeno's paradoxes are fruitful, bringing about new and important discussions. We saw that there are different camps regarding the discussions on the solutions of the paradoxes. We also see similar camps in the discussions on the characteristics of science, mathematical modeling and mathematical rigor, concepts one has to consider when discussing the paradoxes. We will try to investigate the approaches and methods of these different camps on the issue and the reasons underlying them.

All the critiques defending the mathematical rigor can be regarded under a single general sentence: Vagueness should be eliminated. There are some questions related to this approach. Why are there some still insisting on the vagueness of the solutions, while others are claiming that they are clear? Why is mathematical rigor in theories important although the method is sufficient to solve the problems necessary for our practical life? Why should the borders of science

[95] *ibid.*, p.21.
[96] *ibid.*, p.27.

be drawn clearly? Is mathematical rigor a concern for a group of people rather than for everyone? Can we distinguish the people involved in the discussions of a scientific theory according to their approaches to the rigor of those theories? In other words, does rigor attract attention from different disciplines at different levels? If so, what are the reasons behind these different approaches? If rigor is important, then why can sometimes rigor be undervalued?

The first issue that will be considered here is the differences of the methods of different disciplines. We have at least three different figures that work in three different disciplines, a mathematician, a physicist and a philosopher, although in some situations the distinction is hardly recognizable. In modern times, this division transformed into the distinctions between philosophers and engineers. There is an anecdote where the same mission was given both to a philosopher and an engineer. The goal was to reach to a stick that was erected at the end of a given course. It was seen that the engineer traversed the course and touched the stick. On the other hand philosopher did not move. When they asked him the reason why he did not accomplish the task he replied by telling the regressive dichotomy argument which claims the motion was impossible. When they asked the engineer the answer was simple: ‘I was close enough to touch’. This is the answer of calculus. We divide, for example, the area under the graph into many parts and after expressing these small areas as functions, we let the variable go to infinity. Therefore, the ratio goes to zero. Although we are very close to accomplish the intended task there are still some doubts about how that can happen.

In calculus, we saw that since the solution can be reached by a mathematical method, the rigor of this method can be disregarded or left to others to be concerned. Generally rigor comes after the theory is constructed.

> Rigor, formalism, and the logical development of a concept, result or theory usually come at the end of a process of mathematical evolution. In the case of calculus, mathematicians achieved very impressive results during the 17^{th} and 18^{th} centuries by intuitive, heuristic reasoning, and therefore had no compelling reasons to put their subjects on firm foundations.[97]

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[97] Kleiner, I. “History of the Infinitely Small and the Infinitely Large in Calculus” *Educational Studies in Mathematics*, 48(2001), p.153.

The questions on the rigor of the methods generally come from the philosophers. Especially the branch named philosophy of science deals with filling the gaps of the scientific theories to make them more 'scientific'. We saw that the problems related with the infinitesimals, the continuum, unobservables etc. are all proposed by philosophers. Moreover until they get a sufficient answer they do not give up asking questions. What is the mission of these questions then? What is the concern of a philosopher of science? Klemke says that "Philosophy of science is the attempt to understand the meaning, method, and logical structure of science by means of a logical and methodological analysis of the aims, methods, criteria, concepts, laws, and theories of science."[98] In other words, philosophy of science tries to draw the borders of science so that the theories that do not satisfy the proposed requirements will be expelled from the world of science.

Someone can ask why such requirements are necessary. Assume that we will construct a theory, at the very moment, which contains fairies and genies as the explanation of the observable events in the physical world. Moreover, we define the change in the universe by the eternal fight between the demons and the angels. That kind of picture of the universe may be convincing for some people. But arguing that it is science is not that much easy because science should satisfy some requirements. The problem of demarcation, as Karl Popper called it, tries to distinguish the science from pseudo-science. According to Popper

> …science, unlike superstition, is at least falsifiable even it is not provable…For example, Newton's laws tell us exactly where certain planets will appear at certain times. And this means that if such predictions fail, we can be sure that the theory behind them is false.[99]

On the other hand, pseudo-sciences cannot be falsified. For example, astrology, a theory that explains human behavior according to the position of the heavenly bodies at the birthday of the person cannot be falsified. Therefore it is not a science. It can be asked at this point what science is? In other words, how can we

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[98] Klemke, E.D. Hollinger, R., Kline A. D. (eds), *Introductory Readings in the Philosophy of Science* (Buffalo: Prometheus Books, 1980), p.2.

[99] Papineau, D. "Methodology: The Elements of the Philosophy of Science" in *Philosophy: A Guide through the Subject*, ed. A.C. Grayling (Oxford: Oxford University Press, 1995), p.129.

be sure that our theories are scientific? In history of science different answers were given to these questions.

We mentioned that Mach criticized the Newtonian universe with Occam's razor in hand, and offered a more materialistic understanding of the world. He tried to make science simple and clear by remarking the unnecessary entities from it. According to him, the criterion for achieving simplicity in science is to expel the concepts that cannot be observed through our senses. The aim is to clean science from any metaphysical problems and to have a clear picture of the universe, which does not contain the entities we cannot experience. Pursuing the same approach, Mach criticized the atomistic theory as well as the Newtonian universe. Since they cannot be verified by sense experience we should not construct our scientific theories on unobservable entities like atoms. Although Mach did not write directly on Zeno's paradoxes, we can estimate his criticism. Some philosophers claimed that infinite divisibility or infinite tasks are out of reach of human experience. Mach's ideas affected the later philosophers and were regarded as the source for positivism. Philosophers like Mach always have questions on vague concepts that scientists freely use. A philosopher of science tries to reformulate vague concepts in terms of observable entities if possible. The philosopher "attempts to reduce or trace such "theoretical constructs" to a lower level in the realm of the observable. Why? Because unless this is done, the doors all open to arbitrarily postulating entities such as gremlins, vital forces, and whatnot."[100] But that does not mean that scientists do not care about strengthening foundations. They have also questions in their minds related with the mathematical rigor of their theories. For example, we know that both Newton and Leibniz wrote more than one copy of calculus to make their theories more rigorous. But in the cases where achieving the correct answers to the critiques is not possible, then the next move of a scientist is to carry on trying to improve his works. For instance, Cavalieri admitted that his methods cannot be rigorous and his defense against the critiques was based on the idea that rigor is the affair of philosophy, not of mathematics. His master Galileo was teaching Cavalieri's methods to his students, underlining the fact that the use of these methods is not permitted in mathematics because of the vagueness of infinitesimals but the methods can still solve the problems. This motto, 'use it but do not believe', is not enough for a philosopher of science. At this point we may consider a critique towards philosophy arguing that there is nothing new in philosophy but science is an example of a rapid progress by rejecting, reconstructing and replacing the theories one after the other. We noticed that the critiques of Zeno's paradoxes are as old as the paradoxes themselves. It can be said that the philosophical questions on paradoxes did not change considerably from the beginning. Actually it should be the case since problematic issues about time and space are still meaningful. The progress in philosophy may be different from the progress in science.

> Skeptics, many of whom are or were students in philosophy classes, complain that the philosophers of twentieth century are discussing essentially the same questions that their predecessors were discussing thousands years ago. Some would go further and claim that they are also coming up with the same answers. Whitehead's famous remark that all of Western philosophy consists of "footnotes of Plato" was certainly not intended to disparage the discipline, but there are many –even within the community of academic philosophers– who see it as a condemnation.[101]

Then these skeptics can claim that philosophy is not very useful for practical life. Philosophy asks the same questions for centuries and further it sometimes tries to block the way of science. Actually, this is an exaggeration. Philosophy's critiques of science are not useless. Questions raised against the foundations of a theory, where they are taken granted by the scientists, may result in developments in science. For example,

> Determinism –the doctrine that each event is completely determined by causes– was not seriously questioned in physics until the quantum theorists needed to come up with a theory of what appeared to be spontaneous radioactive decay. Then the philosophy of physics became a prominent area of interest for some of the world's leading physicists.[102]

When we consider these questions with respect to studies on Zeno's paradoxes we notice that a considerable part of our knowledge on space, time and continuity has been gained by the help of these detailed discussions. Similarly, the critiques of calculus especially on infinitesimals resulted in the theory of limit,

[100] *ibid.*, p.3.
[101] Moody, T.C. *Philosophy and Artificial Intelligence*, (Englewood Cliffs: Prentice Hall, 1993), p.1.

which has an important place in modern mathematics. Let us recall that in our discussion on ancient philosophy we observed that the curiosity about the universe and its mechanism triggered philosophical questions. It was actually the starting point of philosophy, and philosophy could not survive without asking further questions. Philosophy seeks the answers to make universe meaningful to human beings.

[102] *ibid*., p.3.

CHAPTER 6

CHOOSING A METHOD

We have so far considered discussions including a number of methods in the pursuit of understanding the world we live in. Human beings feel more comfortable when they know more about the environment and its mechanism. Science provides us with the rules for explaining the mechanism of the world. Scientific laws, definitions and theories make life easier. Philosophy on the other hand tries to strengthen the foundations and the methods of science to help us have more proper knowledge. Moreover, by considering abstract concepts, philosophy also helps us to create an intellectual universe which human being needs as much as the practical knowledge. But we also saw that we do not have a universal consensus on the theories proposed. This applies to both science and philosophy. Then it can be asked how one can be comfortable, comfort was the aim of gaining knowledge, when there are more than one theories that are convincing and conflicting at the same time. For the time being, we have a simple answer: We can choose the most appealing one. But sometimes answer may not be so easy as it sounds. For example, while we were analyzing the studies on Zeno's paradoxes, we faced a situation that is very much confusing, let alone comforting. First we investigate the concepts that paradoxes were attacking. This leads to another discussion on related concepts. Even after mathematicians came with a solution claiming to cancel the paradoxes, the paradoxes are still criticized and are under investigation by philosophers. Then the critiques of mathematical theories turn into a discussion on mathematical rigor and the methods of abstraction of physical reality. Philosophical critiques are also criticized. And these studies are still continuing in contemporary discussions whose participants had the former knowledge of all precedent studies. Since we have no universally accepted facts after all those attempts and the resultant explanations, one can interpret this account as either frightening or useless. But this is not the case. The method we choose will represent our understanding of life. Therefore, the confusion will disappear from our lives. Let's assume that we made a choice, and for example we agree with Mach's ideas on scientific theories. And our assumption entails the rejection of the other methods, because eliminating confusion is our ultimate aim, which is only possible by adopting a single method by excluding others.

Mach claimed that the theories containing entities that cannot be experienced through our senses should be expelled from the area of science. His theory includes the critiques of both Newtonian understanding of absolute space, which contains the heavenly bodies, and atomic theory, which aims to explain a narrower scale of the world. Mach suggested us to use our senses as a tool to investigate the nature of the universe. Mach is also criticized by many who tried to expose the gaps in his theory. But we will consider only one critique for the sake of our argument. It is not only related with Mach's ideas but targets a wider area, namely, the methods of scientific studies. Thomas Kuhn was an educated theoretical physicist, who was later interested in history and philosophy of science. In *the Structure of Scientific Revolution* he tried to expose the characteristics of the scientific progress by considering its history.

Kuhn claimed that our observations are not independent of our education and beliefs. According to him, people who stresses the importance of observation and sensible experience in science and who suggest that our senses is the only tool to investigate the nature of the universe, do not notice that our senses are not as objective as they believed. He saw an evolution in our sense behavior and observed that the change is due to change in social conditions.

> Kuhn, considers Wittgenstein's example of a drawing which might be seen as a duck or a rabbit:
>
> > What were ducks in the scientist's world before the revolution are rabbits afterwards. … Transformations like these, though usually more gradual and almost always irreversible, are concomitants of scientific training. Looking at a contour map, the student sees lines on paper, the cartographer a picture of a terrain. Looking at a bubble-chamber photograph, the student sees confused and broken lines, the physicist a record of familiar subnuclear events. Only after a

> number of such transformations of vision does the student become an inhabitant of the scientist's world, seeing what the scientist sees and responding as the scientist does. The world that the student then enters is … determined jointly by the environment and the particular normal-scientific tradition that the student has been trained to pursue. [Kuhn, T. (1970), *The Structure of Scientific Revolution*, pp.111-112]

> Kuhn's message is: what we see and certainly our reports about what we see are dependent upon us – upon our education, our social context, our culture, our general beliefs, our scientific theories.[103]

While Mach was seeking a tool to find out whether the theories are scientific and was claiming that we can trust only our senses. It seems that he did not consider the varying characteristic of our observations. Since, at least according to his approach, if the tool is changing then how can we be sure that our results are accurate? Mach claimed that unobservables are problematic and should be excluded from the realm of science but we saw that observables can be as problematic as the unobservables if we consider Kuhn's critiques. Observations cannot be independent from beliefs. Therefore, a scientist can occasionally manipulate the observation according to the needs of his theory. In other words, after developing his theory, he makes observations that support it.

> … once scientists have embraced a theory about the essential nature of their subject matter, such as geocentrism, or Newtonian dynamics, or the wave theory of light they will interpret all observational judgements in the light of that theory, and so will never be forced to recognize the kind of negative observational evidence that might show them that their theory is mistaken.[104]

Mach sought a tool to decide which entity should be excluded from the scientific. According to Kuhn, for example, his first mistake was the tool he chose. But we are more interested in the second one: an attempt to choose only one tool to perform a job. Actually, a scientist uses different methods to construct and verify the theory. A philosopher who is a defender of a specific theory is supposed to have only one method to evaluate the world, both practically and intellectually. As we saw in the discussions on Zeno's paradoxes when a problem emerges the situation can be complex to be solved with only one method. It is generally believed that the underlying reason of the paradoxes was the assumption that time

[103] Ray, C. *Time, Space and Philosophy*, (New York: Routledge, 1991), p.124.
[104] Papineau, D. "Philosophy of Science" in *The Blackwell Companion to Philosophy*, eds. N. Bunnin and E.P. Tsui-James Oxford (U.K. and Malden: Blackwell, 1996), p. 301.

is a continuum. Then we are lost in the infinitely many parts of the continuum. On the other hand, our experience of time, or space and motion tells us that Zeno's arguments cannot be valid, since the motion is *before our very eyes,* as St. Augustine said when he was puzzled with those paradoxes. But our psychological experience of time cannot solve the problem. On the other hand, the discussions on the shortcomings of the scientific approach to time and space do not seem to come to an end. At this point, we have two choices: first we may take a side which is the most convenient for us. It is better than being lost in all of those diverse approaches. It may be a suitable way for having a stable life with no confusion. But this approach must face criticism. We adopt a single method to avoid confusion, but as a result, we see that peace can only be possible if we turn a blind eye to the critiques of the method that we have chosen, which is in fact impossible for a person whose first intention was curiosity. As in the case of Hippasus the Pythagorean, who discovered a number that cannot be written in terms of unity, peace can be destroyed not only by exterior factors, but also by our own efforts, if we are open-minded enough. The second choice which seems more convenient to us than the first. Before elaborating this option, it is better to turn to Kuhn's ideas about scientific method to make a comparison with the philosophical ones.

Kuhn thought that the history of science was usually neglected or undervalued: "History, if viewed as a repository for more than anecdote or chronology, could produce a decisive transformation in the image of science by which we are now possessed."[105] But we cannot say that nobody is interested in the historical account of science. He tried to underscore that history of science is not a simple collection of the answers of many 'who's, 'when's or 'where's. A careful investigation of the history of science can show us some characteristics of the progress of human thought. If we recall the above-mentioned discussions on Zeno's paradoxes and infinitesimals in calculus, it can be claimed that history can teach us more than places, dates and names of the persons. The change of the ideas, the reasons behind those changes, the social situation when the change occurs and the relation between the successive theories may also expose the

[105] Kuhn, T. "The Structure of Scientific Revolutions" in *Introduction to the Philosophy of Science*, ed. A. Zucker (Upper Saddle River: Prentice Hall, 1996), p.160.

characteristics of the situations which require changes and the nature of developing a counter approach. More generally, we can learn the ways that can be followed during developing a counter theory by investigating how the formers achieved it. Kuhn continues by defining a *paradigm*. In the history of science while scientists work on a theory "there is a consensus (often unstated) among scientists about which problems ought to be studied, the ways to study them, and the range of acceptable answers. These things are themselves determined by assumptions about what sorts of things exist and what we take to be well-founded knowledge."[106] Kuhn names it *normal science*. But sometimes a discovery can conflict with the common *paradigm*. Kuhn calls these discoveries as *anomalies*. *Anomalies* are not welcome and are undervalued at the beginning. Only when they do not disappear by themselves and lead to questioning the *paradigm, paradigm* should be changed. He calls this shift in the *paradigm*, a *scientific revolution* and claims that the term *scientific progress* has no meaning since it is the revolution what changes a *paradigm*. There will of course be some resistance to the new paradigm, but after some time they will fade: "A new paradigm is not fully accepted until the holdouts for the old paradigm die. Dying always happens, whereas seeing the world differently may not."[107] Is it possible to make an analogy between the similar characteristics of the *progress* in science and philosophy? Are the terms such as paradigm, normal philosophy or philosophical revolution also convenient for philosophy?

We mentioned that there may be skeptical ideas on whether there is any progress in philosophy, if philosophy is footnotes to a philosopher who lived more than two thousands years ago. It is seen in the discussions of Zeno's paradoxes that the first critiques to the paradoxes came from Aristotle. There are contemporary philosophers who are still studying the Aristotelian point of view on Zeno's paradoxes. They are trying to expose what Aristotle said, establish relations between his approach and the contemporary developments and make a critique of his critiques. Whatever they examine, it seems that the philosophy do not become

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[106] Zucker, A. *Introduction to the Philosophy of Science*, (Upper Saddle River: Prentice Hall, 1996), p.159.
[107] *ibid*., p.159

old-fashioned. Once a "good" theory is developed, it subsists. This makes philosophy different from science in which the new *paradigm* cancels the old one, expressed in Kuhnian terms. *School* may be more convenient than *paradigm* when philosophy is the issue. Turning to *old* theories is not a problem for philosophy. When a scientific revolution occurs, this new scientific theory can be analyzed and criticized by means of the *old* theories. Here philosophical theory stands for a method of reasoning and a way of criticizing. And they are changing like everything under the sun. Can we see this change as a revolution? In Kuhn's terms this does not seem like a revolution since no philosophical ideas cancel the former ones. A new theory, even when it is presented as revolutionary, is generally based on the classical views or at least inherits some ideas from them. In history there are some revolutionary attempts to construct a new way of thinking. For example, Descartes tried to build a philosophy free of the former ideas by forgetting the past and using the reason as his only tool. But many seem to disagree with his ideas: "If barbarism may be characterized by the systematic refusal of history, we should not be afraid to say that the appearance … is the act of barbarism. … All his predecessors' work was wiped out, its usefulness denied."[108] The problem Authier stresses is that although Descartes claimed to have made a revolution in the methods, he was aware of the former philosophies and actually he used them in constructing his ideas but forgot to give references them. The ideas were not revolutionary but their presentation was.[109] We can say that our way of thinking must be affected by the former ideas and methods to examine the universe. Whether it is intended or not, our method can be regarded as a combination of the former methods.

6.1 Conclusions

Let us return to our second option. We just proposed that the new ideas must be affected by the former ones in their construction process. They may be a combination of diverse or even controversial ideas. Then, it can be claimed that

[108] Authier, M. "Refraction and Cartesian Forgetfulness" in *A History of Scientific Thoughts: Elements of a History of Science*, ed. M. Serres (Oxford: Blackwell, 1995), pp. 335-336.
[109] *ibid.*, p.337.

our approach to a controversial issue should be a combination of proposed controversial solutions to the problem instead of adopting a single view. In other words, if there is no anomaly then there will be no important problem with the unique approach that we use to grasp the reality. But the history shows us that there are always anomalies. In addition to that, in some situations, none of the solutions is universally accepted. Then we should choose either having no solution or adopting a compound approach to the problem. This case is different from what science does actually. As Kuhn claimed, new paradigm will replace with the old one and science continues on its path. We are suggesting a compound approach that can be a composition of the old and new ideas of the history of philosophy.

Let us consider the possibility of having such a solution in a philosophical anomaly. Zeno's paradoxes are good examples as anomalies. We saw these arguments attracted attention from nearly every school of philosophy for more than 2500 years, from Aristotle to the contemporaries. Proposed mathematical solution could not end questions especially for the philosophers. Moreover, the same kinds of questions that come to our minds when examining Zeno's paradoxes also apply to this mathematical solution. We mentioned the ideas of Black who saw the problem as an illusion created by failing to make a distinction between the finitely many real things and infinitely many tasks in definition: "We create the illusion of the infinite tasks by the kind of mathematics that we use to describe space, time, and motion."[110] The real task of the runner, for example, was not to traverse a distance by taking half of the course and continue with shorter and shorter movements. The problem lies in the mathematical modeling of the physical world. Science makes this abstraction and we saw that its consistency is not guaranteed: "We do use mathematics in our physical theories with a tremendous degree of success. But even an enthusiastic acknowledgement of the utility of mathematics should not carry with it the presumption that because it works it must *strictly* apply to the physical."[111] We do not want to abandon the benefits of the mathematical solution, but we cannot undervalue the problems it creates. Then,

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[110] Black, M. "Achilles and Tortoise" in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc., 1970), p.81.
[111] Ray, C. *Time, Space and Philosophy*, (New York: Routledge, 1991), p.21.

maybe it is suitable to adopt an approach that does not exclude the mathematical solution but tries to solve the metaphysical problems of Zeno's paradoxes outside the domain of science. This is a division that accepts the characteristics and benefits of diverse approaches and uses them in different parts of the problem. We consider the Bergsonian understanding of time as an example of the proposed compound method.

6.2 An Example: Bergsonian Understanding of Time

Bergson, who was uncomfortable with the notion of time in science, stressed the difference between time in science and the time we experience. Since, according to him, experienced time, which he called duration or *durée*, cannot be understood by means of the apparatus of calculus: "If time, as immediate consciousness perceives it, were, like space, a homogeneous medium, science would be able to deal with it, as it can with space. Now we have tried to prove that duration, as duration, and motion, as motion, elude the grasp of mathematics."[112] The time we experience is not homogenous and cannot be divided into parts since it is unitary, therefore cannot be composed of the instants. He likens duration to a melody, which can be only heard as a melody if we hear it as a whole. Whereas, time, as science takes into consideration, consists of an infinitely many instants, and science uses calculus to study the world at these particular instants. Change is the difference of the states of the world at different instants. According to Bergson, the understanding of time with the imagery experienced in space, and the spatialization of time could cause problems like the arguments of Zeno. He described the source of "[t]he arguments of Zeno of Elea have no other origin than this illusion. They all consist in making time and movement coincide with the line which underlies them, in attributing to them the same subdivisions as to the line, in short, in treating them like that line."[113]

Movement occurs in space, for example the racecourse in the Dichotomy or the ground where Achilles runs. Space can be divided into parts, for example the

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[112] Bergson, H.L. (1889), *Time and Free Will: An Essay on the Immediate Data of Consciousness*, (New York: Harper Torchbooks, 1960), p.234.
[113] Bergson, H.L. (1896), *Matter and Memory*, (New York: Zone Books, 1991), p.191.

steps of both the runner and Achilles. But according to Bergson, this does not mean that the motion is also divisible. "The movement itself is a single unit (we are considering a simple continuous movement in one direction) and it is not divisible."[114] Bergson thought that the reason of contradiction in the arguments of motion is derived from seeing motion as the same with its trajectory.

> But the movement is the only thing that has to happen in time – the trajectory doesn't 'happen' at all – and so the movement simply occurs as a single unity. It may *take* time, but this does not interfere with its unity as a moment, for duration […] does not contain instants in the way that space contains points.[115]

Since the infinite process of division only applies to the trajectory, which is spatial, but not to the motion and duration, Bergson's understanding of time and motion does not meet the assumptions of the paradoxes of motion. Therefore, there is no application of these paradoxes in a Bergsonian universe. In Achilles and the Dichotomy, it is the trajectory that can be divided, not the motion. In the Arrow, the arrow cannot be at a position at an instant. Since there exists no position on its course and no instant in the duration of flight.

> The arrow never *is* in any point of its course. The most we can say is that it might be there, in any sense, that it passes there and might stop there. It is true that if it did stop there, it would be at rest there, and at this point it is no longer movement that we have to do with.[…] At bottom, Zeno's illusion arises from this, that the movement, once *effected*, has laid along its course a motionless trajectory on which we can count as many immobilities as we will. From this we conclude that the movement, *whilst being effected,* lays at each instant beneath it a position with which it coincides. We do not see that the trajectory is created in one stroke, although a certain time is required for it; and that though we can divide at will the trajectory once created, we cannot divide its creation, which is an act in progress and not a thing. To suppose that the moving body *is* at a point of its course is to attribute to the course of itself of the arrow everything that can be said of the interval that the arrow has traversed, that is to say, to admit *a priori* the absurdity that movement coincides with mobility.[116]

Any effort to reconstruct motion and change from the states or immobilities results in the claim of the Arrow: Motion is made up of rests. Then contradictions arise. Bergson used *cinematographic illusion*[117] to describe how science handles the world. Science explores and 'knows' the world through the limit of these

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[114] Lacey, A.R. *Bergson*, (London: Routledge, 1989), p.30.
[115] *ibid*., p.33.
[116] Bergson, H.L. (1907), *Creative Evolution*, (New York: Dover, 1998), p.308.
[117] *Matter and Memory* is published in the same era in which cinema was also born. Publishing date of *Creative Evolution* coincides with the becoming widespread of cinema.

immobilities, points and instants. Our knowledge about the external world is the recomposed snapshots of the environment.

Bergson gave the example of a screening of marching of soldiers. First way we follow to perform screening is to cut out the figures of the individual soldiers and to try to move them individually. The other way, we can take a series of snapshots, in cinema 24 snapshot for a second, and show these instantaneous photographs on the screen one after the other. The result on the screen will be a regiment marching although what we saw on the screen is just the photographs that have no motion individually. This is what Bergson called *cinematographic illusion*. When we gather knowledge "we hardly do anything else than set going a kind of cinematograph inside us."[118] This illusion, according to him, is used by science and can be applied only to the space. We cannot grasp time and motion with this method. When we try to use scientific method to understand time we face with an absurdity which can also be seen in the case of Zeno's paradoxes. "Such insight cannot be achieved by mathematical analysis or by logical reasoning; metaphysical intuition is the only way."[119]

Here it is suggested here that our understanding of time can only be reached by intuition. On the other hand, we need scientific methods to survive in our practical life. Scientists defend their solution because their theories are consistent. Philosophers argue that consistence does not guarantee convenience. Then maybe it is better to accept a distinction between grasping time as it is and using it in a mathematical problem as a variable, which inevitably obeys the rule of a function. We can use the relevant part of our compound theory according to the needs of the issue. "Science really is rag-bag of different instruments" and it uses different instruments to solve a problem. It may be a good idea that philosophy can also use more than one tool or view, when only one is not sufficient to solve a problematic issue. This does not only count for an anomaly. We also need a compound view when we encounter problems with many variables. Sometimes these variables can be so different that the problem becomes very difficult to handle.

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[118] *ibid*. p.306.

[119] Salmon, W.C. "Introduction", in *Zeno's Paradoxes,* ed. W.C. Salmon (Indianapolis: The Bobbs-Merrill Company Inc, 1970), p.19.

History of philosophy shows us that methods chosen are change according to the social conditions and historical backgrounds. Our era will be remembered mostly by the advanced technologies, new scientific theories and related philosophical problems. While the situation is getting more and more complicated the emerging philosophies should also have more complicated methods to cope with the changed situation. "Since the world drives to a delirious state of things, we must drive to a delirious point of view."[120] The new method may be the combination of the old ones and the new ones shaped by the era.

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[120] Baudrillard, J. *Kötülüğün Şeffaflığı*, (Istanbul: Ayrıntı, 1998), p.5.

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