YIELD CURVE MODELLING VIA TWO PARAMETER PROCESSES

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YIELD CURVE MODELLING VIA TWO PARAMETER PROCESSES

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Abstract

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Random field models have provided a flexible environment in which the properties of the term structure of interest rates are captured almost as observed. In this study we provide an overview of the forward rate random field models and propose an extension in which the forward rates fluctuate along with a two parameter process represented by a random field. We then provide a mathematical expression of the yield curve under this model and sketch the prospective utilities and applications of this model for interest rate management.

Keywords: Interest Rate Models, Multiparameter Processes, Random Fields, Term Structure, Yield Curve.

Öz

İKİ PARAMETRELİ SÜREÇLERLE GETİRİ EĞRİSİ MODELLENMESİ

Pekerten, Uygar
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Rastgele alanlar faiz haddi ve verim eğrisi modellemelerinde önemli yenilikler getirmiştir. Bu çalışmanın amacı forward oranların rastgele alanlarla modellenmesine genel bir bakış açısıyla yaklaşmak ve bu modellerin uzantısı olarak forward oranları, rastgele alanlarla betimlenen iki parametreli süreçlerle modellemektir. Bu modelin bir uygulaması olarak verim eğrisi için matematiksel bir ifade sunulmuş ve gelecekte faiz haddi yönetiminde kullanılabilecek çalışmalar önerilmiştir.

Anahtar Kelimeler: Faiz Haddi Modelleri, Rastgele Alanlar , Çok Parametreli Süreçler, Verim Eğrisi.

To my family

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Chapter 1

INTRODUCTION

The beginning of modern interest rate modelling is credited to Vasiček in 1977 and Cox, Ingersoll, and Ross in 1985. Their work were extended by other models but no model for the interest rates has been accepted as a whole as it has been for the Black and Scholes model in the stock markets. This is mainly due to the fact that interest rates are driven with the change of the whole yield curve, not by just one single interest rate. The most popular models are the ones which try to describe interest rates by one or more driving factors. Models of this class are tractable and easy to implement, moreover, they produce explicit formulas for prices of options. However, a model calibrated to perform at pricing one type of derivative cannot perform well on another type of derivative. This is a typical example of trying to explain too much with too few information which results in inadequacies to match the expectations that the agents in modern finance nurture.

As the solution of the differential equation in the Black-Scholes model was taken from the science of projectile motions, interest rate theory has adapted random fields which are widely used in an array of disciplines such as meteorology, climatology, hydrology, machine learning, image processing and many more[1]. Random fields are multi-dimensional stochastic processes $Z(x),\ x \in \mathbb{R}^d$, specified by the joint distribution of any subset of $(Z(x_1), Z(x_1), \ldots, Z(x_n))$. Random field models which specified the dynamics of the instantaneous forward rates

[1] See [Adl81], [Fur03], [CDMP02], [LMP01].

were first introduced by Kennedy [Ken94] on the backbone of the Heath-Jarrow-Morton model and carried further by Kennedy [Ken97] and Goldstein [Gol00], having been attracting more attention ever since. Random field models attach every maturity of a forward rate its own stochastic shock, hence, they eliminate the shortcomings of describing the whole term structure with small amounts of factors.

Random field models had their own costs. During the ten years of their life in the interest rate theory, they had been considered as not practical unless they were approximated by a low dimensional model. However, recent studies show that it is possible to construct a feasible random field model without reducing it to lower dimensions. It has been shown empirically that random fields have a superiority in performing accurate estimations of volatility and correlation than their finite factor counterparts. The subject is relatively new in the financial mathematics community and the results are promoting hopes that a standard model for interest rates can be raised, too.

Our aim in this thesis is provide a flexible two-parameter setting that carries the random field framework further. We propose that the dynamics of the forward rates be driven by a drift term and a diffusion term attached to a two-parameter process that is identified by a random field. We also give the proof of the existence of the risk-neutral probability measure of which its existence has been assumed but not proven in the random field framework. We finally assert an expression of the yield curve where the forward rates are driven by a two-parameter Ornstein-Uhlenbeck process.

The thesis is organized as follows. Chapter 2 provides the necessary mathematical and financial background necessary to build the framework of the models. Chapter 3 gives a brief overview of random fields and introduces the forward rate random fields, with applications to the interest rate market. Chapter 4 defines the two-parameter martingale model and contains the proof of the existence of the risk-neutral probability measure. We also provide a mathematical application that produces an expression for the expectation of the yield curve. Chapter 5 concludes this thesis.

Chapter 2

PRELIMINARIES

This chapter consists of two sections. The first section contains introduces the necessary mathematical concepts to build the interest rate models given in the following chapters. The second section introduces the definitions of the elements of the interest rate markets.

2.1 Mathematical Background

To model prices of financial instruments and their derivatives one needs an extensive probabilistic framework. The basic building block is the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where Ω is the sample space, $\mathcal{A}$ is a family of events and $\mathbb{P}$ is a probability measure. The definitions in this section are compiled from [DJ03], [HK01], [Kle01], [LL00], and [Mik99]. We begin with the following assumption.

Assumption 2.1.1. All random objects appearing in this thesis are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and are in continuous time.

The mathematical entity for representing the flow of information through time is *filtration*.

Definition 2.1.2. A **filtration** $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is an increasing family of sub-σ algebras of $\mathcal{A}$.

We will impose some conditions on $\mathbb{F}$.

Assumption 2.1.3. $\mathbb{F}$ satisfies the following conditions called the *usual conditions.*

- $\mathbb{F}$ contains all $\mathbb{P}$-null (or negligible) subsets of Ω, i.e., if $A \in \mathcal{A}$ and if $\mathbb{P}(A) = 0$, then for any t, $A \in \mathcal{F}_t$.
- $\mathbb{F}$ is right continuous, i.e.,
$$\mathcal{F}_t = \bigcap_{\varepsilon \geq 0} \mathcal{F}_{t+\varepsilon}$$
for any $t \geq 0$.

The σ-algebra $\mathcal{F}_t$ represents the information available at time t. As time passes we learn more about prices and rates, hence more information. If the price of an asset indexed by time is characterized by $(S_t)_{t\geq 0}$ at time t the fact that *"the past prices are known"* is represented by the statement *"$(S_u)_{u\geq t}$ is $\mathcal{F}_t$-measurable."*.

We now give the definition of a stochastic process.

Definition 2.1.4. A **stochastic process** is a family $\{X(t,\omega), t \in \mathbb{R}_+, \omega \in \Omega\}$ of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

A stochastic process X is a function of two variables: For a fixed t it is a random variable $X(t) = X(t,\omega)$, $\omega \in \Omega$, whereas, for a fixed random outcome $\omega \in \Omega$, it is a function of time $X(t) = X(t,\omega)$, $t \in \mathbb{R}_+$. We will drop ω in representing stochastic processes for clarity. A common stochastic process is the Brownian Motion.[1]

Definition 2.1.5. The **standard Brownian motion** is a continuous real-valued stochastic process $W = \{W_t,\ t \geq 0\}$ with the following properties:

- $W_0 = 0$ a.s.[2],

[1]The (standard) Brownian motion is widely used as an indicator of the fluctuations in the movements of prices. Brownian motion is also called as the Wiener process, named after Norbert Wiener who made the most meticulous definition of the process among all definitions that were present in his time.

[2]In a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a property which holds for all $\omega \in \Omega$ except on a negligible subset of Ω is said to hold *almost surely (a.s.)* or $\mathbb{P}$-a.s..

- independent increments: if $0 \leq s \leq t, \quad W_t - W_s$ is independent of $\mathcal{F}_s = \sigma(W_u, u \leq s)$.
- stationary increments: if $0 \leq s \leq t, W_t - W_s$ and W_{t-s} have the same probability law
- if $0 \leq s < t, \quad W_t - W_s \sim N(0, t-s)$ where $N(0, t-s)$ is the normal distribution with mean 0 and variance $t - s$.

The portfolio choices of an agent is characterized by *trading strategies.*

Definition 2.1.6. Assume that the assets in a market are indexed by the set I. A **trading strategy** is a progressively measurable[3] stochastic process

$$\phi = ((\phi_t^0, \phi_t^1, \ldots, \phi_t^{\#(I)}))_{t \geq 0}$$

in $\mathbb{R}^{\#(I)}$ where ϕ_t^i denotes the number of shares of asset i that an agent holds at time t.

A trading strategy ϕ over the time interval $[0, T]$ is *self-financing* if its wealth generating process $V(\phi)$ which is defined as

$$V_t(\phi) = S_t^0 \phi_t^0 + S_t^1 \phi_t^1 + \ldots + S_t^{\#(I)} \phi_t^{\#(I)}, \quad \forall t \in [0, T],$$

satisfies the following:

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^0 dS_u^0 + \int_0^t \phi_u^1 dS_u^1 + \ldots + \int_0^t \phi_u^{\#(I)} dS_u^{\#(I)}, \quad \forall t \in [0, T].$$

For a portfolio to be considered as executable its value must always be nonnegative.

Definition 2.1.7. A strategy ϕ is **admissible** if it is self-financing and if $V_t(\phi) \geq 0$ for any $t \geq 0$.

One of the most interesting dilemmas in the world of finance rises from the concept of *arbitrage.* People hate it in theory but love to do it in practice.

[3]A *progressively measurable* process $(X_t)_{t \geq 0}$ satisfies the following: $\forall t \geq 0$ X_t is $\mathcal{F}_t$-measurable.

Definition 2.1.8. An **arbitrage strategy**[4] is an admissible strategy that has zero initial value and non-zero final value.

Definition 2.1.9. The market is **viable** if it is arbitrage-free.

Arbitrage is mainly profiting from the different prices of the same asset. A model producing arbitrage opportunities is basically useless since it will produce more than one price for an asset. The notion of self-financing strategy is not sufficient to exclude arbitrage opportunities from a model (see [MR97]). Thus we need to introduce the definition of a *martingale*.

Definition 2.1.10. Under the probability space equipped with a filtration $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{F})$, a progressively measurable family $(M_t)_{t\geq 0}$ of integrable random variables (i.e., $\mathbb{E}(|M_t|) < +\infty$ for any t) is

- a **martingale** if, for any $u \leq t$, $\mathbb{E}(M_t|\mathcal{F}_u) = M_u$,
- a **submartingale** if, for any $u \leq t$, $\mathbb{E}(M_t|\mathcal{F}_u) \geq M_u$,
- a **supermartingale** if, for any $u \leq t$, $\mathbb{E}(M_t|\mathcal{F}_u) \leq M_u$.

In a model, if the discounted price processes are rendered martingales, the model satisfies what is known as *the efficient market hypothesis*[5]. That is, intuitively, every agent in the economy will have the same expectation of what the price of a particular asset will become at a future point in time. This enables a unified pricing perspective.

Next, we describe a crucial instrument.

Definition 2.1.11. A **European option of maturity** T is a contract which has a payoff $h \geq 0$ where h is $\mathcal{F}_t$-measurable. A **call** on the underlying instrument S with strike price K has a payoff $h = (S-K)_+ = \max\{0, S-K\}$; a **put** on the same instrument with strike price K is defined by $h = (K-S)_+ = \max\{0, K-S\}$.

[4] Arbitrage is also referred as *free lunch*.

[5] Price processes are discounted to generate *comparable* prices. What is meant by comparable is discussed in the definition of a numeràire in the next section.

An option gives the owner the right to exercise it. For a call, if $S - K < 0$ then an agent may not exercise the option and be left with just the loss occurring from paying the price of the option. The same argument applies to a put when $K - S < 0$. Therefore the payoff of an option is a *contingent claim*, it may or may not be realized and this depends on the future prices of the asset.

Definition 2.1.12. The contingent claim defined by h is **attainable** if there exists an admissible strategy which replicates h. The market is **complete** if every contingent claim is attainable.

The assumption that the market is complete is a strict assumption which may not have an economical justification. Rather, it allows to derive simple models for pricing contingent claims and it is useful for hedging purposes. There are many recent works that build up models where the market is *incomplete*.

The following theorems are the building blocks of pricing contingent claims.

Theorem 2.1.13. A market is viable if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted asset prices are martingales.

Theorem 2.1.14. A viable market is complete if and only if there exists a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted asset prices are martingales.

The latter theorem is referred to as the *Fundamental Theorem of Asset Pricing*. The measure $\mathbb{P}^*$ can be thought in analogy with a pricing function, since in a probabilistic framework prices are related to the probability of the states of the economy. Thus the latter theorem can be interpreted in the following sense: The economy is arbitrage free and every contingent claim can be replicated if and only if for every discounting procedure there exists a unique pricing function in the economy.

2.2 Elements of the Interest Rate Markets

We begin with the most fundamental instrument, the zero-coupon bond (also called the pure discount bond).

Definition 2.2.1. A **zero-coupon bond** is a contract that guarantees the payment of one unit of money at a future date T and makes no payments before T. Its price at time $t \leq T$ is denoted by $P(t, T)$. We will assume, for theoretical reasons, that for every T there exists a zero-coupon bond with maturity T.

The price of a zero-coupon bond reflects the current value of one unit paid with complete certainty at some future point in time. Thus risk is eliminated. This makes $P(t, T)$ representative for the *time value* of money. A translation of the zero-coupon bond prices yields their rate of return $Y(t, T)$, called the *yield to maturity*[6] at time t for the time to maturity $T - t$, which is implicitly formulated as

$$P(t, T) = e^{-Y(t,T)(T-t)} \tag{2.2.1}$$

The above relation yields

$$Y(t, T) = -\frac{\ln P(t, T)}{T - t}. \tag{2.2.2}$$

The zero-coupon bond price and its translation yield to maturity has a wide array of implementations such as bond pricing, discounting future cash flows, pricing fixed-income derivative products, obtaining other types of interest rates and determining risk premia of portfolios of bonds with different maturities [BG02]. The set of yields to maturities $Y(t, T), t < T$, constitute the *yield curve*. The *short rate*, a useful theoretical entity, is derived from the yield to maturity.

Definition 2.2.2. The **short rate at time t** $r(t) = Y(t, t)$ is intuitively defined as the instantaneous rate for borrowing and lending. Obviously there exists no such tangible asset.[7]

[6] $Y(t, T)$ is also called *zero rate* or *spot rate* and often denoted by $R(t, T)$

[7] Practitioners use short term rates ranging from overnight rates to 3-month-rates as substitutes for the instantaneous short rate but they should not be confused with the theoretical entity proposed in this context.

A precisely defined discounting process is crucial for any model. Discounting processes are used to compare prices of an asset having different maturities. Such a discounting process is called a *numeràire*.

Definition 2.2.3. A **numeràire** $(N(t), t \geq 0)$ is a process used to compute and compare relative prices of a particular asset. If the price of an asset at time t is $S(t)$, then its value relative to the numeràire is

$$\tilde{S}(t) = \frac{S(t)}{N(t)}.$$

A numeràire must be strictly positive and self-financing. It can be chosen among many instruments which satisfy these conditions. A choice of a numeràire does not affect the pricing of assets structurally. It only eliminates outside influences (such as inflation, opportunity cost, or simply the time value decay) one is able to compare prices of an asset at different times. Suppose that an agent is to choose between receiving 100 unit of money today and 110 unit next year. These prices are non-comparable, since, if the risk-free bond has a rate of return at 20 percent, receiving 100 today, investing it in the bond, and receiving 120 next year is more profitable: Here, if the bond returns at 5 percent this strategy is less profitable[8]. Similar to the risk free bond choice presented in this example a typical choice of a numeràire is the *bank account process*.

Definition 2.2.4. A sum of 1 unit deposited in a bank and accumulated continuously up to time t is called the **bank account**. Its value is denoted by $B(t)$ and is equal to

$$B(t) = e^{\int_0^t r(s)ds}.$$

We take the bank account as our numeràire for the rest of our dissertation.

Assumption 2.2.5. $N(t) = B(t)$ for all $t \geq 0$.

Definition 2.2.6. The **continuously compounded forward rate for the period** $[T_1, T_2]$ **at time** t, $t \leq T_1 \leq T_2$, fixes the rate assigned to borrowing at T_1 and repaying at T_2, at time t. It is denoted by $f(t, T_1, T_2)$.

[8] Surely this reasoning is subject to the agent's utility function. This is a logical reasoning made to clarify the notion of numeràire presented here.

To preclude arbitrage:

$$e^{Y(t,T_2)(T_2-t)} = e^{Y(t,T_1)(T_1-t)} e^{f(t,T_1,T_2)(T_2-T_1)}$$

$\Leftrightarrow$

$$Y(t,T_2)(T_2-t) = Y(t,T_1)(T_1-t) + f(t,T_1,T_2)(T_2-T_1)$$

$\Leftrightarrow$

$$f(t,T_1,T_2) = \frac{1}{(T_2-T_1)} \ln \frac{P(t,T_1)}{P(t,T_2)}.$$

Then, by (2.2.2),

$$f(t,T_1,T_2) = -\frac{1}{(T_2-T_1)}(\ln P(t,T_2) - \ln P(t,T_1)). \tag{2.2.3}$$

Definition 2.2.7. The **instantaneous forward rate** $f(t,T)$ is the instantaneous short rate at time T anticipated at time t.

$$f(t,T) = f(t,T,T) = \lim_{T_2 \to T} f(t,T,T_2). \tag{2.2.4}$$

Remark 2.2.8. Before going on to the next chapter, we should note the following:

- $P(T,T) = 1$ in an arbitrage-free economy.
- $Y(t,T) = f(t,t,T)$ and $r(t) = f(t,t)$
- $f(t,T_1,T_2) = \frac{1}{(T_2-T_1)} \int_{T_1}^{T_2} f(t,u)du$ in analogy with Intermediate Value Theorem from calculus.
- $f(t,T) = \lim_{T_2 \to T} f(t,T,T_2) = \lim_{T_2 \to T} -\frac{(\ln P(t,T_2) - \ln P(t,T))}{T_2 - T}$ by (2.2.3) and setting $h = T_2 - T$,

$$f(t,T) = -\lim_{h \to 0} \frac{\ln P(t,T+h) - \ln P(t,T)}{h}$$

$\Rightarrow$

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T). \tag{2.2.5}$$

- Following (2.2.5) we have

$$Y(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds \tag{2.2.6}$$

  and

$$P(t,T) = e^{-\int_t^T f(t,s)ds}. \tag{2.2.7}$$

Chapter 3

RANDOM FIELD MODELS

The first section of the chapter introduces random fields, the second section provides an overview of the forward-rate-random-field models in the literature, and the third section briefly shows the applications of the random-field models and the comparison of their performance with that of the other term-structure models in the literature.

3.1 Introduction to Random Fields

Definition 3.1.1. A **random field** is a family of random variables $Z(x, \omega)$ indexed by $x \in \mathbb{R}^d$, and $\omega \in \Omega$, together with a collection of marginal distributions[1]

$$F_{x_1,x_2,\ldots,x_n}(b_1, b_2, \ldots, b_n) = \mathbb{P}\{Z(x_1, \omega) \leq b_1, Z(x_2, \omega) \leq b_2, \ldots, Z(x_n, \omega) \leq b_n\},$$

$(b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$. The mean function, the covariance function, and the correlation function of Z, respectively, are defined by

$$\begin{aligned} m(x) &:= \mathbb{E}[Z(x, \omega)], \\ R(x, y) &:= \mathbb{E}[Z(x, \omega)Z(y, \omega)] - m(x)m(y), \\ c(x, y) &:= \frac{R(x, y)}{\sqrt{R(x, x)R(y, y)}}. \end{aligned}$$

[1]The marginal distributions of Z are also called the *finite-dimensional distributions* or *fi-dis* in short.

We will drop the ω parameter in representations of random fields for convenience. A random field, in the most crude meaning, is a multiparameter stochastic process defined by its marginal distributions. Another aspect of the random fields is that, when fixing any set of $d-1$ parameters and allowing one parameter to fluctuate, one obtains a stochastic process in the fluctuating parameter. The following remark is crucial in constructing random fields.

Remark 3.1.2. The covariance function $R(\cdot,\cdot)$ is non-negative definite since for $\zeta_i \in \mathbb{R}$, $i = 1, 2, \dots, d$,

$$\begin{aligned}\sum_{i=1}^{d}\sum_{j=1}^{d}\zeta_i\zeta_j R(x_i,x_j) &= \sum_{i=1}^{d}\sum_{j=1}^{d}\zeta_i\zeta_j \mathbb{E}[Z(x_i)Z(x_j)] - \sum_{i=1}^{d}\sum_{j=1}^{d}\zeta_i\zeta_j m(x_i)m(x_j) \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{d}\zeta_i Z(x_i)\right)^2\right] - \left(\sum_{i=1}^{d}\zeta_i m(x_i)\right)^2 \geq 0.\end{aligned}$$

The property of non-negative definiteness characterizes covariance functions. As Santa-Clara and Sornette (2001) shows, given $m : \mathbb{R}^d \to \mathbb{R}$ and a non-negative definite $R : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ it is possible to construct a random field for which m and R are its mean and covariance functions respectively [SS01].

Stationarity in random fields, which is a central concept, is reflected by homogeneity. A random field is *homogeneous* or *second-order stationary* if

- $m(x) = m \in \mathbb{R}$ is independent of $x \in \mathbb{R}^d$.
- $R(x, y)$ depends only on $x - y$.

The second feature is called the *translation invariance*. A common class of random fields is *isotropic* random fields. An isotropic random field is a random field which has a covariance function satisfying $R(v) = R(\|v\|)$ for a vector $v \in \mathbb{R}^d$. This implies that correlation of any two points depends between the distance between them. Next, we give three examples of random fields.

Definition 3.1.3. A **Gaussian random field** is a random field which has multivariate normal finite-dimensional distributions.

An example, analogous with its one-dimensional counterpart, is the *standard Brownian sheet*.

Definition 3.1.4. A **standard Brownian sheet** $W(t,T)$ is the centered Gaussian random field with the covariance function

$$R(W(t_1,T_1),W(t_2,T_2)) = (t_1 \wedge t_2)(T_1 \wedge T_2), \quad 0 \leq t_1 \leq T_1, \; 0 \leq t_2 \leq T_2 \quad (3.1.1)$$

where "$x \wedge y$" in probability theory jargon denotes "$\inf(x,y)$".

Definition 3.1.5. Let $Z_1, Z_2, \ldots, Z_n$ be independent, homogeneous real-valued Gaussian random fields with mean $m(x) = \mathbb{E}[Z_i(x)] = 0 \; \forall i = 1, 2, \ldots, n$, and common covariance function $R(y) = \mathbb{E}[Z(x)Z(x+y)]$[2] and variance $\sigma^2(x) = R(0)$. For $x \in \mathbb{R}^d$, the process

$$Y(x) := Z_1^2(x) + Z_2^2(x) + \ldots + Z_n^2(x)$$

is called a **chi-squared random field** with parameter n.

Its density function is

$$f_{\chi^2(r,\sigma)}(u) = \frac{1}{\Gamma(\frac{r}{2})(2\sigma)^{(\frac{r}{2})}} u^{\frac{r}{2}-1} e^{-\frac{u}{2\sigma}}$$

where r denotes the degrees of freedom.

3.2 Forward Rate Random Field Models

In this section, we present the forward rate models driven by random fields. The term structure models can be dated back to Vasiček, Cox-Ingersoll, and Ross whose work was modelling the short rate $r(t)$ depending on finite factor processes and constant coefficients [LL00]. Their models were extended to affine models by Dai and Singleton. These models are tractable and used widely in the world of finance. They produce explicit forms for bond prices, yields and

[2]The homogeneity of the fields enables such a covariance function.

forward rates and the process of calibrating them to price interest-rate derivatives is easy but they have serious empirical shortcomings, the most severe one is their inconsistency with the current term structure. Also, they fail to capture the stochastic volatility of short rates. Jagannathan, Kaplin, and Sun (2001) show that a third shortcoming of the affine models is that low dimensional affine models[3] fail to capture the joint dynamics of yields, caps and swaptions, i.e., once calibrated to price a specific derivative, the model performs poorly when pricing another kind of derivative.

The second generation of term structure models either made explicit dependence on the short rate e.g., *log-r* models of Black-Derman-Toy and Black-Karasinski or modelled the forward rates directly with taking the initial forward curve as an input, e.g., Heath-Jarrow-Morton (henceforth HJM) model[4]. The second class of models fit the current yield curve, due to taking the initial forward curve as an input, however they are not able to capture the changes of a new yield curve. Thus, practitioners need to calibrate the model once a new term structure is realized. Although recalibration may seem innocent at a first glance, since the parameters of the models are non-stochastic, it is structurally erroneous. Consequently, the second class of models is inconsistent with empirical data.

The random field models were derived on the HJM framework described below. In the HJM model the innovation of the entire forward rate curve depends on multiple Brownian motions with drifts and volatilities that are arbitrary adapted deterministic processes. Kennedy (1994) carried the HJM framework further by allowing the forward rates to follow a continuous Gaussian random field with two parameters $X(t,T)$ where t represents the calendar time (i.e. time flows in the t-direction) and T is the maturity of the forward rate [Ken94]. For the time being it is necessary to make the following assumption.

Assumption 3.2.1. All assertions in the random field setting are under the risk neutral probability $\mathbb{P}^*$ which satisfies the Fundamental Theorem of Asset Pricing.

We begin with the definition of the HJM model.

[3]By *low dimensional* we mean models with three or less diffusive factors.
[4]For a thorough view of these models, see [JW02].

Definition 3.2.2. The HJM model specifies the instantaneous forward rate dynamics as follows:

$$df(t,T) = \mu(t,T) + \sum_{i=1}^{N} \sigma_i(t,T)dW_i(t),$$

where W_i, $i = 1, \ldots, N$ are independent Brownian motions and $(\mu(t,T))_{0\leq t\leq T}$ and $(\sigma(t,T))_{0\leq t\leq T}$ satisfy the following so-called *the HJM conditions*:

1. $\int_0^T |\mu(u,T)dt| < \infty$ $\mathbb{P}^*$-a.s. for all $0 \leq T$.
2. $\int_0^T \sigma^2(u,T)du < \infty$ $\mathbb{P}^*$-a.s. for all $0 \leq T$.
3. $\int_0^{T'} |f(0,u)|ds < \infty$ $\mathbb{P}^*$-a.s. for $0 \leq T \leq T'$.
4. $\int_0^{T'} \int_0^s |\mu(u,s)|duds < \infty$ $\mathbb{P}^*$-a.s. for $0 \leq T \leq T'$.

Definition 3.2.3. The Gaussian Random Field Model for the instantaneous forward rates is defined as

$$f(t,T) = \mu(t,T) + X(t,T), \tag{3.2.2}$$

where $X(t,T)$ is a centered continuous Gaussian random field with the covariance structure specified by

$$R(X(t_1,T_1), X(t_2,T_2)) = c(t_1 \wedge t_2, T_1, T_2), \;\; 0 \leq t_i \leq T_i, \;\; i = 1, 2. \tag{3.2.3}$$

for a given $c(\cdot,\cdot,\cdot)$.

The drift function $\mu(t,T)$ is deterministic and continuous in $0 \leq t \leq T$. The covariance function $c(t,T_1,T_2)$ is non-negative definite and symmetric in T_1 and T_2. The choice made by defining the covariance function by $c(t_1 \wedge t_2, \cdot, \cdot)$ in t_1 and t_2 ensures that $X(t,T)$ has independent increments in the t-direction. Indeed if $0 \leq t_1 \leq t_2 \leq T$ the random variable $X(t_2,T) - X(t,T)$ is independent of the

σ-field $\mathcal{F}_t = \sigma\{X(u,v) : u \leq t, u \leq v\}$. For $u \leq t$, $u \leq v$,

$$\begin{aligned} R(X(t_2,T) - X(t_1,T), X(u,v)) &= c(t_2 \wedge u, T, v) - c(t_1 \wedge u, T, v) \\ &= c(u,T,v) - c(u,T,v) \\ &= 0, \end{aligned}$$

implying independence by the Gaussian assumption.

Based on this model it is easy to derive a necessary and sufficient condition on the drift surface to obtain the discounted bond price process $\tilde{P}(t,T)$ which is a martingale.

Theorem 3.2.4. The following statements are equivalent:

(a) For each $T \geq 0$, the discounted bond price process $\left(\tilde{P}(t,T)\right)_{0\leq t\leq T}$ is a martingale.

(b) $\mu(t,T) = \mu(0,T) + \int_0^T c(t \wedge v, v, T)dv$ for all $0 \leq t \leq T$.

(c) $P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(u)du|\mathcal{F}_t}\right]$ for all $0 \leq t \leq T$.

Remark 3.2.5. If $\mathcal{F}_0$ is not the trivial σ-field then (b) in Theorem 3.2.4 becomes

$$\mu(t,T) = \mu(0,T) + \int_0^T \left[c(t \wedge v, v, T) - c(0, v, T)\right] dv.$$

Goldstein generalizes Kennedy's results to include non-Gaussian random fields by defining the forward rate dynamics by the stochastic differential equation

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)Z(dt,T), \tag{3.2.4}$$

where $Z(t,T)$ is a random field with a correlation structure specified by the quadratic covariance dynamics $d\langle Z(\cdot,T_1), Z(\cdot,T_2)\rangle_t = c(t,T_1,T_2)$, $0 \leq t \leq T_1 \wedge T_2$.

It is assumed that $(\mu(t,T))_{0\leq t\leq T}$ and $(\sigma(t,T))_{0\leq t \ leqT}$ are progressively measurable processes satisfying the technical conditions imposed by the HJM framework given in Definition 3.2.2. Goldstein reaches the necessary and sufficient conditions given in Theorem 3.2.4, thus his model is a generalized version containing both

Gaussian and non-Gaussian fields [Gol00]. The following lemma will be useful in reaching his conclusion.

Lemma 3.2.6. Assume that the forward rate dynamics are given as in equation (3.2.4) with the drift and volatility satisfying the conditions stated above. Define

$$I(t) := \int_t^T f(t,v)dv$$

so that $P(t,T) = e^{-I_t}$. Then,

(a)

$$\begin{aligned} d(I_t) &= d\left(\int_t^T f(t,v)dv\right) \\ &= \underbrace{\left(\int_t^T \mu(t,s)ds\right)}_{\text{call this } \mu^*(t,T)} dt + \int_t^T dv\sigma(t,v)Z(dt,v) - r(t)dt \\ &= \mu^*(t,T)dt + \int_t^T dv\sigma(t,v)Z(dt,v) - r(t)dt. \end{aligned}$$

(b)

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} = & -\mu^*(t,T) - \textstyle\int_t^T dv\sigma(t,v)Z(dt,v) + r(t)dt \\ & + \tfrac{1}{2}\textstyle\int_t^T \sigma(t,u)\sigma^*(t,T,u)dudt, \end{aligned}$$

where $\sigma^*(t,T_1,T_2) = \int_t^{T_1} \sigma(t,v)c(t,T_2,v)$.

Proof We replicate Furrer's proof [Fur03].

(a) Since $df(t,T) = \mu(t,T)dt + \sigma(t,T)Z(dt,T)$

$$\begin{aligned} I(t) &= \int_t^T \left(f(0,v) + \int_0^t \mu(s,v)ds + \int_0^t \sigma(s,v)Z(ds,v)\right) dv \\ &= \int_t^T f(0,v)dv + \int_t^T \int_0^t \mu(s,v)dsdv + \int_t^T \int_0^t \sigma(s,v)Z(ds,v)dv. \end{aligned}$$

Apply the stochastic version of Fubini's Theorem[5] to get

$$\begin{aligned} I(t) &= \int_0^T f(0,v)dv + \int_0^t \int_0^T \mu(s,v)dvds + \int_0^t \int_s^T \sigma(s,v)dvZ(ds,v) \\ &\quad - \int_0^t f(0,v)dv - \int_0^t \int_0^t \mu(s,v)dvds - \int_0^t \int_s^t \sigma(s,v)dvZ(ds,v) \\ &= I(0) + \int_0^t \mu^*(s,v)ds + \int_0^t \int_s^T dv\sigma(s,v)Z(ds,v) \\ &\quad - \int_0^t \left(f(0,v) + \int_0^v \mu(s,v)ds + \int_0^v \sigma(s,v)Z(ds,v) \right) dv \\ &= I(0) + \int_0^t \mu^*(s,T)ds + \int_0^t \int_s^T dv\sigma(s,v)Z(ds,v) - \int_0^t f(v,v)dv \\ &= I(0) + \int_0^t \mu^*(s,T)ds + \int_0^t \int_s^T dv\sigma(s,v)Z(ds,v) - \int_0^t r(v)dv. \end{aligned}$$

Thus, taking derivative with respect to t yields

$$dI(t) = \mu^*(t,T)dt + \int_t^T dv\sigma(t,v)Z(dt,v) - r(t)dt. \tag{3.2.5}$$

(b) Let $f(x) = e^{-x}$. Apply the Itô formula to $f(X_t)$ where X_t is an Itô process:

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X_\cdot \rangle_s \\ &= e^{-X_0} - \int_0^t e^{-X_s}dX_s + \frac{1}{2}\int_0^t e^{-X_s}d\langle X_\cdot \rangle_s \\ \Rightarrow df(X_t) &= e^{-X_t}\left(-dX_t + \frac{1}{2}d\langle X_\cdot \rangle_t \right) \end{aligned}$$

Then,

$$de^{-I_t} = dP(t,T) = P(t,T)(-dI_t + \frac{1}{2}d\langle I_\cdot \rangle_t) \tag{3.2.6}$$

[5]See [BDKR98].

which implies

$$
\begin{aligned}
d\langle I.\rangle_t &= \int_t^T \int_t^T \sigma(t,v)\sigma(t,u) d\langle Z(\cdot,u), Z(\cdot,v)_t\rangle dudv \\
&= \int_t^T \int_t^T \sigma(t,u)\sigma(t,v)c(t,u,v)dtdudv \\
&= \int_t^T \sigma(t,u) \underbrace{\int_t^T \sigma(t,v)c(t,u,v)dv}_{\sigma^*(t,T,u)} \, dudt. \qquad (3.2.7)
\end{aligned}
$$

Combine (3.2.6) with (3.2.5) and (3.2.7) to get

$$
\begin{aligned}
\frac{dP(t,T)}{P(t,T)} &= -\mu^*(t,T)dt - \int_t^T dv\sigma(t,v)Z(dt,v) + r(t)dt \\
&\quad + \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)dudt.
\end{aligned}
$$

□

The following is the main result of the generalized random field models:

Proposition 3.2.7. Let the dynamics of the forward rates be specified by Lemma 3.2.6. Then the risk neutral drift condition is given by

$$\mu(t,T) = \sigma(t,T)\sigma^*(t,T,T).$$

Proof Under the risk neutral probability $\mathbb{P}^*$ the discounted bond prices are martingales. Let $\tilde{P}(t,T) = \frac{P(t,T)}{B(t)}$ where $B(\cdot)$ is the bank account process given by $B(t) = e^{\int_0^t r(s)ds}$. Notice that $dB(t) = B(t)r(t)dt$. Hence,

$$\begin{aligned}
d\tilde{P}(t,T) &= \frac{d\tilde{P}(t,T)}{B(t,T)} + P(t,T)d\left(\frac{1}{B(t)}\right) \\
&= \frac{dP(t,T)}{B(t)} + P(t,T)\left(-\frac{1}{B^2(t)}dB(t)\right) \\
&= \frac{dP(t,T)}{B(t)} - \frac{P(t,T)}{B(t)}r(t)dt \\
&= \frac{P(t,T)}{P(t,T} \cdot \frac{dP(t,T)}{B(t)} - \tilde{P}(t,T)r(t)dt \\
&= \tilde{P}(t,T)\frac{dP(t,T)}{P(t,T)} - \tilde{P}(t,T)r(t)dt
\end{aligned}$$

$$\Rightarrow \frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = \frac{dP(t,T)}{P(t,T)}r(t)dt.$$

By Lemma (3.2.6)

$$\frac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = -\mu(t,T)dt - \int_t^T dv\sigma(t,v)Z(dt,v) + \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)dudt.$$

If $\tilde{P}(t,T)$ is a martingale then the terms with dt must vanish. Then

$$\begin{aligned}
-\mu(t,T)dt + \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)dudt = 0 \\
\Rightarrow \mu^*(t,T) = \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)dudt \\
\Rightarrow \int_t^T \mu(t,s)ds = \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)du.
\end{aligned}$$

Differentiating with respect to t gives

$$\mu(t,T) = \sigma(t,T)\sigma^*(t,T,T).$$

3.3 Applications of Random Field Models

The forward rate random field models delineated in the previous section provide a more parsimonious and flexible environment compared to their counterparts. Finite factor models have the same finite set of shocks affects all forward rates thus any observed sample of forward rates may have a covariance matrix of rank at most N which imposes severe restrictions on the covariance structures of the factor models. The random field models attribute each forward rate its own drift, diffusion and stochastic shock in consistency with the fact that the forward rates form a continuum[6]. Due to these desirable properties random field models attracted much attention and in this section we will demonstrate some applications of the random field models.

3.3.1 Option Pricing

One application field of interest rate models is pricing interest rate and bond options. Kennedy [Ken94] applies the Gaussian random field model for the forward rates to price an interest rate caplet.
An interest rate caplet at strike rate d for the period $[T_1, T_2]$ is a European option on the forward rate $f(t, T_1, T_2)$ where, if the rate $f(T_1, T_2, T_2)$ exceeds d, it is feasible to exercise the option which then gives a payoff at time T_2 of

$$\left[e^{(T_2-T_1)f(T_1,T_1,T_2)} - e^{(T_2-T_1)d}\right]_+ .$$

A series of caplets is called a *cap* which its price is calculated using the price of its caplets. The discounted payoff of a caplet at time 0 is

$$e^{-\int_0^{T_2} r(u)du} \left[e^{(T_2-T_1)f(T_1,T_1,T_2)} - e^{(T_2-T_1)d}\right]_+ \tag{3.3.8}$$

[6] A *continuum* is a linearly ordered set which is densely ordered, i.e., between any two members of the set there exists another member. The idea that forward rates form a continuum comes from the assumption that for every T there exists a zero coupon bond with maturity T.

The price of this caplet at time 0 is the expectation of (3.3.8):

$$\mathbb{E}\left[e^{-\int_0^{T_2} r(u)du}\left[e^{(T_2-T_1)f(T_1,T_1,T_2)} - e^{(T_2-T_1)d}\right]_+\right].$$

Let $N_1 = (T_2 - T_1)f(T_1, T_1, T_2)$, $N_2 = -\int_0^{T_2} r(u)du$, and $\gamma = (T_2 - T_1)d$. Here, N_1 and N_2 are normally distributed with $\mathbb{E}[N_i] = \mu_i$ and $\mathrm{Var}(N_i) = \sigma_i^2$, $i = 1, 2$. The conditional distribution of N_2 given N_1 is

$$N\left(\mu_2 + \rho\sigma_2\frac{N_1 - \mu_1}{\sigma_1}, \sigma_2^2(1 - \rho^2)\right),$$

where $\rho = Corr(N_1, N_2)$. Then

$$\begin{aligned}\mathbb{E}\left[(e^{N_1} - e^{\gamma})_+ e^{-N_2}\right] &= e^{\mu_1-\mu_2+\mathrm{Var}(N_1-N_2)/2}\Phi\left(\frac{\mu_1 - \gamma + \sigma_1^2 - R(N_1, N_2)}{\sigma_1}\right) \\ &\quad -e^{\gamma-\mu_2+\sigma_2^2/2}\Phi\left(\frac{\mu_1 - \gamma - R(N_1, N_2)}{\sigma_2}\right), \qquad (3.3.9)\end{aligned}$$

where Φ is the standard normal distribution function. Using Theorem 3.2.4(b) and setting

$$g(s, t, T) = \int_{u=t}^{T}\int_{v=t}^{u} c(s, u, v)dudv, \quad h(t, T) = \int_{u=t}^{T}\int_{v=t}^{u} c(v, u, v)dudv,$$

we can write

$$\begin{aligned}\mu_1 &= \int_{T_1}^{T_2} \mu(T_1, u)du = \int_{T_1}^{T_2}\left[\mu(0, u) + \int_0^u c(T_1 \wedge v, v, u)dv\right] du \\ &= \int_{T_1}^{T_2} \mu(0, u)du + \int_{u=T_1}^{T_2}\int_{v=0}^{T_1} c(v, u, vdudv) + \int_{u=T_1}^{T_2}\int_{v=T_1}^{u} c(t, u, v)dudv \\ &= \int_{T_1}^{T_2} \mu(0, u)du + h(0, T_2) - h(0, T_1) - h(T_1, T_2) + g(T_1, T_1, T_2).\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu_2 &= \int_0^{T_2} \mu(0,u)du + h(0,T_2) \\
\sigma_1^2 &= 2g(T_1,T_1,T_2) \\
\sigma_2^2 &= 2h(0,T_2) \\
R(N_1,N_2) &= h(0,T_2) - h(0,T_1) + h(T_1,T_2) + 2g(T_1,T_1,T_2).
\end{aligned}$$

Then,

$$\mu_1 - \mu_2 + \frac{\mathrm{Var}(N_1 - N_2)}{2} = -\int_0^{T_1} \mu(0,u)du \tag{3.3.10}$$

$$-\mu_2 + \frac{\sigma_2^2}{2} = -\int_0^{T_2} \mu(0,u)du \tag{3.3.11}$$

$$\mu_1 + \sigma_1^2 - R(N_1,N_2) = \int_{T_1}^{T_2} \mu(0,u)du + \frac{\sigma_1^2}{2}. \tag{3.3.12}$$

Substituting (3.3.10), (3.3.11), (3.3.12) to (3.3.9) gives the price of the caplet as

$$\begin{aligned}
&e^{-\int_0^{T_2}\mu(0,s)du}\left[e^{(T_2-T_1)\mu(0,T_1,T_2)}\Phi\left(\frac{(T_2-T_1)(\mu(0,T_1,T_2)-d)}{\sigma}+\frac{\sigma}{2}\right)\right] \\
&-e^{-\int_0^{T_2}\mu(0,s)du}\left[e^{(T_2-T_1)d}\Phi\left(\frac{(T_2-T_1)(\mu(0,T_1,T_2)-d)}{\sigma}-\frac{\sigma}{2}\right)\right],
\end{aligned}$$

where

$$\mu(t,T_1,T_2) = \mathbb{E}[f(t,T_1,T_2)] = \frac{1}{(T_2-T_1)}\int_{T_1}^{T_2} \mu(s,u)du$$

and

$$\sigma^2 = \mathrm{Var}((T_2-T_1)f(T_1,T_1,T_2)) = 2\int_{T_1}^{T_2}\int_{T_1}^{u} c(t,u,v)dudv.$$

3.3.2 Random Fields vs. Affine Models

Random field models are not flawless in practicability. As Collin-Dufresne and Goldstein ([CG03] suggest, random field models have been considered as non-implementable unless they are reduced to low dimensions. Longstaff, Santa-Clara, and Schwartz [LSS99] estimate a random field model where the drift and diffusion parameters of the model are calibrated by the principal components analysis approach so that the model produces the first N components of the data. However, such a model is equivalent to a finite factor model, as observed by Kerkhof and Pelsser [KP02]. By using the Markov Chain Monte Carlo (MCMC) method Bester [Bes04] estimates both random field and affine models from the same set of forward rates data and compares them. The finite-factor models fail to capture the volatility towards the end of the term structure, whereas the random field models produce exactly the observed volatilities for nine out of ten long term maturities in the sample. They also accurately reflect the humps and increases in the term structure of volatilities. Another subject that the random fields surpass the finite-factor models is the correlation structure of the forward rates. In the short maturities, both types of models can be considered successful but as the time to maturity gets greater the performance of the finite factor models drop.

Chapter 4

THE TWO-PARAMETER-PROCESS MODEL

In this chapter we propose a model which is an extension to the random field models. The first section introduces the model and its features –along with a setting which allows to operate under any probability measure, a distinct attribute of our extension. The second section brings forth an expression of the yield curve under the model proposed in the first section.

4.1 Introduction

We will operate under a modified version of the filtration introduced in Definition 2.1.2.

Assumption 4.1.1. The filtration $\mathbb{F}$ is modified for our model as follows:

$$\mathbb{F} = (\mathcal{F}_t = \sigma\{W(u,v) : u \leq t, u \leq v\})_{t \geq 0} \, .$$

We now define a two parameter process represented by a Brownian sheet and construct our model.

Definition 4.1.2. Let

$$M(t,T) = \int_0^t \int_0^T F(u,v) W(du,dv)$$

be a martingale with respect to $\mathcal{F}_t$ where $W(du,dv)$ is a Brownian sheet. The dynamics of the forward rates is defined as

$$f(t,T) = f(0,T) + \int_0^t \mu(u,T)du + \int_0^t \varphi(u,T)M(du,T), \tag{4.1.1}$$

where $\mu(t,T)$ and $\varphi(t,T)$ are adapted processes satisfying the HJM conditions given in Definition 3.2.2.

Differentiating (4.1.1) with respect to t yields

$$df(t,T) = \mu(t,T)dt + \varphi(t,T)M(dt,T) \tag{4.1.2}$$

The zero coupon bond prices satisfy $P(t,T) = \exp\{-\int_t^T f(t,s)ds\}$ and the spot rate is $r(s) = f(s,s)$, thus the bank account process is

$$B(t) = \exp\left\{\int_0^t \left(f(0,s) + \int_0^s \mu(u,s)du + \int_0^s \varphi(u,s)M(du,s)\right)\right\}.$$

Then, the discounted bond price process $\tilde{P}(t,T)$ becomes

$$\begin{aligned} \tilde{P}(t,T) &= e^{\left\{-\left[\int_t^T f(0,s)ds + \int_t^T\left(\int_0^t \mu(u,s)du\right) + \int_t^T\left(\int_0^t \varphi(u,s)M(du,s)\right)ds\right]\right\}} \\ &\quad \cdot \underbrace{e^{\left\{-\left[\int_0^t f(0,s)ds + \int_0^t\left(\int_0^s \mu(u,s)du\right)ds + \int_0^t\left(\int_0^s \varphi(u,s)M(du,s)\right)ds\right]\right\}}}_{\text{Discount factor}}. \end{aligned} \tag{4.1.3}$$

Let

$$K_t = \int_{s=t}^T \left(\int_{u=0}^t \varphi(u,s)M(du,s)\right) ds + \int_{s=0}^t \left(\int_{u=0}^s \varphi(u,s)M(du,s)\right) ds \tag{4.1.4}$$

denote the terms with $M(du,s)$ in (4.1.3). Differentiating (4.1.3) with respect to t delineates the necessary condition for $d\tilde{P}/\tilde{P}$ to be a martingale (see

Definition 2.1.10):

$$\frac{d\tilde{P}}{\tilde{P}} = \underbrace{\text{non-martingale terms}}_{A_t dt} + \underbrace{\text{martingale terms}}_{dM_t} + \frac{1}{2} d\langle K, K\rangle_t$$

Here, $d\tilde{P}/\tilde{P}$ is a martingale if and only if $A_t dt = -\frac{1}{2}d\langle K, K\rangle_t$. The following lemma will be useful in proving the subsequent proposition.

Lemma 4.1.3. For the two process M described above the following relation holds:

$$\int_0^t \int_0^T \varphi(u,s) M(du,s) = \int_{u=0}^t \int_{v=0}^T \left(\int_{s=v}^T \varphi(u,s) ds \right) M(du,dv). \tag{4.1.5}$$

Proof Apply the stochastic version of Fubini's Theorem to the left hand side of (4.1.5):

$$\begin{aligned} \int_0^t \int_0^T \varphi(u,s) M(du,s) &= \int_0^T \left(\int_0^t \varphi(u,s) M(du,s) \right) ds \\ &= \int_{s=0}^T \left(\int_{u=0}^t \int_{v=0}^s \varphi(u,s) M(du,dv) \right) ds \end{aligned}$$

since $0 \le v \le s$ and $0 \le s \le T$,

$$= \int_{u=0}^t \int_{v=0}^T \left(\int_{s=v}^T \varphi(u,s) ds \right) M(du,dv).$$

□

The following proposition will be decisive in distinguishing the martingale terms in (4.1.3).

Proposition 4.1.4. K_t, defined as in (4.1.4) is an $\mathbb{F}$-martingale.

Proof Consider the first integral appearing in (4.1.4):

$$\int_{s=t}^{T}\left(\int_{u=0}^{t}\varphi(u,s)M(du,s)\right)ds = \int_{u=0}^{t}\left(\int_{s=0}^{T}\varphi(u,s)M(du,s)\right)ds$$

$$= \int_{u=0}^{t}\int_{s=0}^{T}\varphi(u,s)M(du,s)ds - \int_{u=0}^{t}\int_{s=0}^{t}\varphi(u,s)M(du,s)ds \quad (4.1.6)$$

$$= \int_{u=0}^{t}\int_{v=0}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,s) - \int_{u=0}^{t}\int_{s=0}^{t}\varphi(u,s)M(du,s)ds. \quad (4.1.7)$$

The expression in (4.1.6) equals (4.1.7) by Lemma 4.1.3. Then,

$$\begin{aligned} K_t &= \int_{u=0}^{t}\int_{v=0}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,dv) - \int_{u=0}^{t}\int_{s=0}^{t}\varphi(u,s)M(du,s)ds \\ &\quad + \int_{s=0}^{t}\left(\int_{u=0}^{s}\varphi(u,s)M(du,s)\right)ds \\ &= \int_{u=0}^{t}\int_{v=0}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,dv) \\ &\quad - \int_{s=0}^{t}\left(\int_{u=s}^{t}\varphi(u,s)M(du,s)\right)ds. \end{aligned} \quad (4.1.8)$$

Let

$$\begin{aligned} Y_t &:= \int_{s=0}^{t}\left(\int_{u=s}^{t}\varphi(u,s)M(du,s)\right)ds, \quad 0 \le\ v \le s \le u \le t. \\ &= \int_{s=0}^{t}\left(\int_{u=s}^{t}\varphi(u,s)\int_{v=0}^{s}M(du,dv)\right)ds. \end{aligned} \quad (4.1.9)$$

Examining the limits of the above integrals reveals that $0 \le v \le s$, $s \le u \le t$, and $0 \le s \le t$. Then, it is possible to reconstruct (4.1.9) with s, v, u so that $v \le s \le u$, $0 \le v \le u$, and $0 \le u \le t$.

$$Y_t = \int_{u=0}^{t}\int_{v=0}^{u}\left(\int_{s=v}^{u}\varphi(u,s)ds\right)M(du,dv). \quad (4.1.10)$$

Combine (4.1.8) with (4.1.10),

$$K_t = \int_{u=0}^{t}\int_{v=0}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,dv)ds - \int_{u=0}^{t}\int_{v=0}^{u}\left(\int_{s=v}^{u}\varphi(u,s)ds\right)M(du,dv).$$

By rearranging the first integral we can write

$$\begin{aligned} K_t &= \int_{u=0}^{t}\int_{v=u}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,dv)+\int_{u=0}^{t}\int_{v=0}^{u}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,dv) \\ &\quad -\int_{u=0}^{t}\int_{v=0}^{u}\left(\int_{s=v}^{u}\varphi(u,s)ds\right)M(du,dv) \\ &= \underbrace{\int_{u=0}^{t}\int_{v=u}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)M(du,dv)}_{\mathbf{I}}+\underbrace{\int_{u=0}^{t}\int_{v=0}^{u}\left(\int_{s=u}^{T}\varphi(u,s)ds\right)M(du,dv)}_{\mathbf{II}}. \end{aligned}$$

The integrals **I** and **II** are martingales since $\varphi(u,s) =$ is $\mathbb{F}$-adapted and $M(du,\cdot)$ is an $\mathbb{F}$-martingale by construction. Therefore, K_t is an $\mathbb{F}$-martingale.

□

Let $K_t^1 = \mathbf{I}$ and $K_t^2 = \mathbf{II}$. The quadratic variations of K_t^1 and K_t^1 are:

$$\langle K_\cdot^1, K_\cdot^1\rangle_t = \int_{u=0}^{t}\int_{v=u}^{T}\left(\int_{s=v}^{T}\varphi(u,s)ds\right)^2 F^2(u,v)dudv \tag{4.1.11}$$

$$\langle K_\cdot^2, K_\cdot^2\rangle_t = \int_{u=0}^{t}\int_{v=0}^{u}\left(\int_{s=u}^{T}\varphi(u,s)ds\right)^2 F^2(u,v)dudv \tag{4.1.12}$$

Notice that in (4.1.11) v varies between u and T, whereas in (4.1.12) it varies between 0 and u. Thus, $\langle K_\cdot^1, K_\cdot^1\rangle_t$ and $\langle K_\cdot^2, K_\cdot^2\rangle_t$ are orthogonal. Hence, the quadratic variation of K_t is:

$$\langle K_\cdot, K_\cdot\rangle_t = \langle K_\cdot^1, K_\cdot^1\rangle_t + \langle K_\cdot^2, K_\cdot^2\rangle_t$$

The derivative of $\langle K_\cdot, K_\cdot\rangle_t$ with respect to t is:

$$\kappa = \frac{d}{dt}\langle K_\cdot, K_\cdot\rangle_t = \int_{v=t}^{T}\left(\int_{s=v}^{T}\varphi(t,s)ds\right)^2 F^2(t,v)dv + \int_{v=0}^{t}\left(\int_{s=t}^{T}\varphi(t,s)ds\right)^2 F^2(t,v)dv.$$

The derivative of the non-martingale terms in (4.1.3) with respect to t is:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\int_t^T f(0,s)ds + \int_0^t f(0,s)ds + \int_t^T \left(\int_0^t \mu(u,s)du\right) ds + \int_0^t \left(\int_0^t \mu(u,s)du\right) ds\right] \\
&+\frac{1}{2}\kappa \\
=\ & \frac{\partial}{\partial t}\left[\int_0^T f(0,s)ds + \int_t^T \left(\int_0^t \mu(u,s)du\right) ds + \int_0^t \left(\int_0^t \mu(u,s)du\right) ds\right] \\
&+\frac{1}{2}\kappa \\
=\ & -\int_0^t \mu(u,t)du + \int_t^T \mu(t,s)ds + \int_0^t \mu(u,t)du + \frac{1}{2}\kappa \\
=\ & 0.
\end{aligned}$$

The last equality is necessary to ensure that $\tilde{P}(t,T)/\tilde{P}$ is a martingale. Taking the derivative of the last equation with respect to T leaves an expression for the drift surface of the forward rates:

$$\mu(t,T) = \varphi(t,T)\left[\int_t^T \left(\int_v^T \varphi(t,s)ds\right) F^2(t,v)dv + \int_0^t \left(\int_t^T \varphi(t,s)ds\right) F^2(t,v)dv\right]. \tag{4.1.13}$$

In (4.1.13) the variables of the first integral satisfy $t \leq v \leq s \leq T$ and the variables in the second integral satisfy $v \leq t \leq s \leq T$. Therefore, a change in the bounds of the integral and application of Fubini's Theorem to both integrals yield:

$$\mu(t,T) = \varphi(t,T)\left[\int_{s=t}^T \left(\int_{v=t}^s \varphi(t,s)F^2(t,v)dv\right) ds + \int_{s=t}^T \left(\int_{v=0}^t \varphi(t,s)F^2(t,v)dv\right) ds\right].$$

Hence,

$$\mu(t,T) = \varphi(t,T)\int_t^T \varphi(t,s)\left(\int_0^s F^2(t,v)dv\right) ds. \tag{4.1.14}$$

Compare this with the drift surface obtained in the generalized random field model:

$$\mu(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)c(t,T,v)dv.$$

Up to this point, all results concerning the drift surface were derived under the risk neutral probability along with Assumption 3.2.1. Goldstein states that in an N-factor model $(N+1)$ bonds are needed in order to perfectly hedge a portfolio where this hedging strategy allows to identify the risk-neutral measure, whereas, since the random field models imply that there are an infinite number of economic factors that derive the interest rates, hedging with a finite number of bonds is impossible [Gol00]. So, rather than proving the existence of the risk-neutral measure, its existence is taken as granted. This approach is convenient in the random field setting, whereas, the flexibility in our proposed model leads to a different scheme. This is illustrated in the following setting: Let $X(t,s) = \varphi(t,s)M(t,s)$, then forward rates can be written as:

$$f(t,s) = \mu(t,s) + X(t,s). \tag{4.1.15}$$

Thus the dynamics of the forward rates are:

$$\begin{aligned} df(t,s) &= \frac{\partial\mu(t,s)}{\partial t}dt + \frac{\partial\varphi(t,s)}{\partial t}M(t,s)dt + \varphi(t,s)M(dt,s) \qquad (4.1.16) \\ &= \frac{\partial\mu(t,s)}{\partial t}dt + \frac{\partial\varphi(t,s)}{\partial t}\varphi^{-1}(t,s)X(t,s)dt + \varphi(t,s)M(dt,s) \\ &= A(t,s)dt + \varphi(t,s)M(dt,s). \end{aligned}$$

Here, $A(t,s) = \frac{\partial\mu(t,s)}{\partial t} + \frac{\partial\varphi t,s}{\partial t}\varphi^{-1}(t,s)X(t,s)$ is the new drift surface. This drift surface contains a term with $X(t,s)$ hence it is no longer deterministic, giving rise to a need for a change of probability measure under which, the drift term is rendered riskless. This change of measure can be constructed with an algorithm similar to Girsanov's Theorem in one-parameter framework[1].

Let $H(u,s)$ be a process that satisfies the following *Novikov condition*:

$$\mathbb{E}^*\left[e^{\frac{1}{2}\int_0^t H^2(u,s)du}\right] < \infty \;\; \forall s \in [0,T],$$

where $\mathbb{E}^*$ is the expectation operator under the risk neutral probability $\mathbb{P}^*$. Let

[1]See [LL00].

us put

$$B(dt,s) := \left(\int_0^s F^2(t,v)dv\right)^{-\frac{1}{2}} M(dt,s).$$

Observe that $\forall s$, $B(t,s)_{0\leq t\leq T}$ is a Brownian motion. The entity $B(dt,s)$ defined above is rather a Gaussian random field. Therefore,

$$L_t(s) := \exp\left\{\int_0^t H(u,s)B(du,s) - \frac{1}{2}\int_0^t H^2(u,s)du\right\} \tag{4.1.17}$$

is a $(\mathbb{F},\mathbb{P}^*)$ martingale in t for any fixed s. Consider the probability measure $\mathbb{P}_s$ defined by

$$d\mathbb{P}_s = L_T(s)d\mathbb{P}^* \tag{4.1.18}$$

Notice that the probability measure $\mathbb{P}_s$ is dependent on s which forms a continuum, thus this probability measure can be thought as a coordinate of an infinite dimensional probability measure. According to Girsanov's Theorem,

$$\hat{B}(t,s) = B(t,s) - \int_0^t H(u,s)du$$

is a $\mathbb{P}_s$-Brownian motion in t for any fixed s. We have

$$M(dt,s) = \left(\int_0^s F^2(t,v)dv\right)^{\frac{1}{2}} B(dt,s).$$

Define

$$\hat{M}(dt,s) := M(dt,s) - \left(\int_0^s F^2(t,v)dv\right)^{\frac{1}{2}} H(t,s)dt. \tag{4.1.19}$$

Taking into account the relation (4.1.19), we can write

$$\begin{aligned} df(t,s) &= \frac{\partial\mu(t,s)}{\partial t}dt + \varphi(t,s)M(dt,s) \\ &= \frac{\partial\mu(t,s)}{\partial t}dt + \varphi(t,s)\left[\hat{M}(dt,s) + \left(\int_0^s F^2(t,v)dv\right)^{\frac{1}{2}} H(t,s)dt\right]. \end{aligned}$$

We impose H to satisfy the equation

$$\varphi(t,s)\left(\int_0^s F^2(t,v)dv\right)^{\frac{1}{2}} H(t,s) = \frac{\partial\varphi(t,s)}{\partial t}\hat{M}(t,s), \tag{4.1.20}$$

in order to have

$$df(t,s) = \frac{\partial\mu(t,s)}{\partial t}dt + \frac{\partial\varphi(t,s)}{\partial t}\hat{M}(t,s)dt + \varphi(t,s)\hat{M}(dt,s). \tag{4.1.21}$$

The relation (4.1.21) is, in turn, equivalent to

$$f(t,s) = \mu(t,s) + \hat{X}(t,s),$$

with $\hat{X}(t,s) = \varphi(t,s)\hat{M}(t,s)$. Under this setting it is possible to obtain the drift surface as given by (4.1.14) without the restriction of operating under the risk neutral probability measure. In fact, with any H satisfying (4.1.20), one can calculate the drift surface using the model in hand with parameters estimated with respect to the market rates. Equation (4.1.20) can be written as:

$$\varphi(t,s)\left(\int_0^s F^2(t,v)dv\right)^{\frac{1}{2}} H(t,s) = \frac{\partial\varphi(t,s)}{\partial t}\left[M(t,s) - \int_0^t \left(\int_0^s F^2(u,v)dv\right)^{\frac{1}{2}} H(u,s)du\right].$$

Let

$$G(t,s) := \int_0^t \left(\int_0^s F^2(u,v)dv\right)^{\frac{1}{2}} H(u,s)du. \tag{4.1.22}$$

Then, Equation (4.1.22) becomes

$$\frac{\partial G(t,s)}{\partial t} = \frac{\partial \ln\varphi(t,s)}{\partial t}[M(t,s) - G(t,s)]. \tag{4.1.23}$$

Equation (4.1.23) can then be written as

$$\frac{\partial G(t,s)}{\partial t} = -A(t,s)G(t,s) + A(t,s)M(t,s) \tag{4.1.24}$$

We use the method of *variation of constants* and put

$$G(t,s) = C(t,s)e^{-\int_0^t A(u,s)du}. \tag{4.1.25}$$

We obtain

$$\frac{\partial G(t,s)}{\partial t} = \frac{\partial C(t,s)}{\partial t}e^{-\int_0^t A(u,s)du} - C(t,s)A(t,s)e^{-\int_0^t A(u,s)du} \tag{4.1.26}$$

Combining (4.1.24) and (4.1.26) we get

$$\frac{\partial C(t,s)}{\partial t} = A(t,s)M(t,s)e^{\int_0^t A(u,s)du},$$

which is equivalent to

$$C(t,s) = C(0,s) + \int_0^s A(u,s)M(u,s)e^{\int_0^u A(v,s)dv}du.$$

By using the relation above, (4.1.25) can be rearranged as

$$G(t,s) = \left[C(0,s) + \int_0^s A(u,s)M(u,s)e^{\int_0^u A(v,s)dv}du\right]e^{-\int_0^t A(u,s)du}. \tag{4.1.27}$$

Combining (4.1.22), (4.1.23), (4.1.24), and (4.1.27) we deduce

$$\begin{aligned}
\frac{\partial G(t,s)}{\partial t} &= \left(\int_0^s F^2(t,v)dv\right)^{\frac{1}{2}} H(t,s) \\
&= \left[A(t,s)M(t,s)e^{\int_0^t A(v,s)dv}\right]e^{\int_0^t A(u,s)du} \\
&\quad -A(t,s)\left[C(0,s) + \int_0^s A(u,s)M(u,s)e^{\int_0^u A(v,s)dv}du\right]e^{-\int_0^t A(u,s)du}
\end{aligned}$$

Since $A(t,s) = \frac{\partial \ln\varphi(t,s)}{\partial t}$, $\int_0^t A(v,s)dv = \ln\varphi(t,s) - \ln\varphi(0,s)$. Using this fact, we can obtain an expression for $H(t,s)$:

$$\begin{aligned} H(t,s) &= \left(\int_0^s F^2(t,v)dv\right)^{-\frac{1}{2}} \frac{\partial \ln \varphi(t,s)}{\partial t}\left[M(t,s)\left(\frac{\varphi(t,s)}{\varphi(0,s)}\right)^2\right. \\ &+ \left.\left(C(0,s)+\int_0^t \frac{\partial \ln\varphi(u,s)}{\partial u}M(u,s)\frac{\ln\varphi(u,s)}{\ln\varphi(0,s)}du\right)\frac{\varphi(t,s)}{\varphi(0,s)}\right] \end{aligned}$$

We had assumed that H satisfied the Novikov condition. Indeed, since H depends linearly on $M(t,s)$ –which is Gaussian itself– the Novikov condition is automatically satisfied.

4.2 An Expression for the Yield Curve

Let the forward rate model be given by (4.1.15). We know that

$$Y(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds.$$

Then,

$$\begin{aligned} \mathbb{E}[Y(t,T)] &= \frac{1}{T-t}\int_t^T \mathbb{E}[f(t,s)]ds \\ &= \frac{1}{T-t}\int_t^T \mu(t,s)ds. \end{aligned}$$

The drift condition (4.1.14) yields

$$\mu(t,T) = \mu(0,T) + \int_0^t \varphi(u,T)\left[\int_u^T \left(\varphi(u,s)\int_0^s F^2(u,v)dv\right)ds\right]du. \quad (4.2.28)$$

Let $\varphi(t,s) = e^{-at-bs}$ which makes X a two parameter Ornstein-Uhlenbeck process. In this framework, $F^2(u,v) = e^{2au}e^{2bv}$. The equation (4.2.28) implies

$$\begin{aligned}
\mu(t,s) &= \mu(0,s) + \int_0^t e^{-bs}\left[\int_u^s \left(e^{-bq}\int_0^q e^{2bv}dv\right)dq\right]du \\
&= \mu(0,s) + \int_0^t e^{-bs}\left(\frac{e^{bs} - e^{bu} + e^{-bs} - e^{-bu}}{2b^2}\right)du \\
&= \mu(0,s) + \frac{t}{2b^2} - \frac{e^{-bs}}{2b^3}e^{bt} + \frac{e^{-bs}}{2b^3} + \frac{e^{-bs}}{2b^2}t + \frac{e^{-bs}}{2b^3}e^{-bt} - \frac{e^{-bs}}{2b^3} \\
&= \mu(0,s) + \frac{t}{2b^2} - \frac{e^{-bs}}{2b^3}e^{bt} + \frac{e^{-2bs}}{2b^2}t + \frac{e^{-bs}}{2b^3}e^{-bt}.
\end{aligned}$$

Then, $\mathbb{E}[Y(t,T)]$ becomes

$$\begin{aligned}
\mathbb{E}[Y(t,T)] &= \frac{1}{T-t}\int_t^T \left(\mu(0,s) + \frac{t}{2b^2} - \frac{e^{-bs}}{2b^3}e^{bt} + \frac{e^{-2bs}}{2b^2}t + \frac{e^{-bs}}{2b^3}e^{-bt}\right)ds \\
&= \frac{1}{T-t}\left[\int_t^T \mu(0,s)ds + \frac{t}{2b^2} + \frac{e^{-bt}}{2b^4}\left(\frac{e^{-bT} - e^{-bt}}{T-t}\right)\right] \\
&\quad -\frac{1}{T-t}\left[-\frac{t}{4b^3}\left(\frac{e^{-2bT} - e^{-2bt}}{T-t}\right) - \frac{e^{-bt}}{2b^4}\left(\frac{e^{-bT} - e^{-bt}}{T-t}\right)\right].
\end{aligned}$$

Notice that $\mu(0,s)$ is the initial term structure of the forward rates and can be estimated via various methods such as the Nelson-Siegel method, spline-based methods and their variations.

4.3 Further Work

The applications of this model is only limited with the boundaries of the usage of any interest rate model. It is open to experimentation, both mathematically and empirically. Option pricing is one issue. One may implement any two parameter process in the model, thus any two parameter process will bring its own advantages and disadvantages in pricing options. Also, the existence of the risk-neutral probability measure is a delicate issue in option pricing, thus the problem of exposing the infinite-dimensional probability measure stated in Section 4.1 must be tackled.

Another experiment is taking an approach similar to Kimmel's stochastic volatility concept to define the process M [Kim03]. The stochastic volatility topic itself is a wide area which can provide good opportunities to produce models consistent with empirical data. A third field of interest is the parameter estimation of the model. Bester's work is inspiring in the sense that random field models perform much better in reproducing the observed volatility and correlation of the data than affine models [Bes04].

Chapter 5

CONCLUSION

In this study we tried to give a general overview of the random fields and their applications in interest rate modelling and formulated a flexible interpretation of the forward rate random field models. In this model, rather than taking a random field as a driving source for the innovation of the forward rates, we attached a two parameter process $M(t,T)$. This process provides flexibility in choosing both the random field that derives the process and its incremental process. Any two parameter process can be imposed in the model. Another feature of this model is that one can calculate the drift surface under different probability measures. Also, the model can be reduced into any random field model. Thus this model provides a broad area of mobility for researches in forward rate random field models.

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