LIFTING FIBRATIONS ON ALGEBRAIC SURFACES TO CHARACTERISTIC ZERO

CELALETTİN KAYA

JANUARY 2005

LIFTING FIBRATIONS ON ALGEBRAIC SURFACES TO CHARACTERISTIC ZERO

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

CELALETTİN KAYA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

JANUARY 2005

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Canan ÖZGEN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Şafak ALPAY
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Hurşit ÖNSİPER
Supervisor

Examining Committee Members

Assoc. Prof. Ali Sinan SERTÖZ	(Bilkent Unv., MATH)	
Prof. Dr. Hurşit ÖNSİPER	(METU, MATH)	
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ABSTRACT

LIFTING FIBRATIONS ON ALGEBRAIC SURFACES TO CHARACTERISTIC ZERO

Kaya, Celalettin

M.Sc., Department of Mathematics

Supervisor: Prof. Dr. Hurşit ÖNSİPER

January 2005, 21 pages

In this thesis, we study the problem of lifting fibrations on surfaces in characteristic p, to characteristic zero. We restrict ourselves mainly to the case of natural fibrations on surfaces with Kodaira dimension -1 or 0. We determine whether such a fibration lifts to characteristic zero. Then, we try to find the smallest ring over which a lifting is possible. Finally, in some favourable cases, we compare the moduli of liftings of the fibration to the moduli of liftings of the surface under consideration.

Keywords: Liftings, Fibrations.

ÖZ

CEBİRSEL YÜZEYLER ÜZERİNDEKİ LİFLENMELERİ KARAKTERİSTİK SIFIRA KALDIRMA

Kaya, Celalettin

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi: Prof. Dr. Hurşit ÖNSİPER

Ocak 2005, 21 sayfa

Bu tezde, karakteristiği sıfırdan büyük olan cisimler üzerindeki yüzeylerde bulunan liflenmelerin kaldırılması problemi incelenmiştir. Genelde, Kodaira boyutu -1 veya 0 olan yüzeyler üzerindeki doğal liflenmeler çalışılmıştır. Öncelikle, böyle bir liflenmenin karakteristik sıfıra kaldırılıp kaldırılamayacağı tespit edilmiş, cevabın olumlu olduğu durumlarda, bu kaldırmanın mümkün olduğu en küçük halka bulunmaya çalışmıştır. Ayrıca bu kaldırmaların moduli uzayı, ele alınan yüzeyin moduli uzayı ile karşılaştırılmıştır.

Anahtar Kelimeler: Kaldırmalar, Lif uzayları.

To My Supervisor

ACKNOWLEDGMENTS

I have two secrets to share. The first one is a “known secret”: How much labor I had to put into this thesis. The other secret which I will share so openly for the first time is this: If it were not for my supervisor I would not know how to travel this long, hard road without losing my sanity; for it was filled with more than its fair share of stress and emotional roller coasters. I hope I will not disappoint my supervisor.

I should also thank many other people who helped me along the way. First of all, I thank Feza Arslan, Özgür Kişisel, Yıldıray Ozan and Ali Sinan Sertöz for reading my thesis and joining the jury. Furthermore, I have to thank Mehmet Turan and Ömür Uğur who helped me with the format of the thesis.

Can Tekinel, Gülay Karadoğan and Pınar Topaloğlu who always shared their materials as well as their experiences with me; İbrahim Çetin and Uğur Gül, you were always with me; Haydar Alıcı, Ali Sait Demir, Abdullah Özbekler, Ali Öztürk, Özer Öztürk and all the others in the department whose friendship I enjoyed all the time; also, all my lecturers and all the staff in the department;

I thank you all, sincerely and truly.

Finally, there are such people in our lives that thanking them sounds strange as we take them for granted, without them life would not be such a rich and meaningful experience. These peoples’ names are synonymous with love: My soul mate Muhammed Ali Alan; my brother-in-law Aydın Ada; my lovely nephews İlke, Zenyel and cute İlhan; my lovely sisters Seda, Sevda, and Ferda; my lovely parents and my supervisor; I send you my love.

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CHAPTER 1

BASIC CONCEPTS

The idea of lifting objects to characteristic zero is one of the main themes of the interplay between characteristic zero and characteristic p geometries. It is the counterpart of the more accessible technique of "reduction mod p" and in its most general form the theory is concerned with the following questions.

- Given an object or a structure in characteristic p, can one find an object/structure of the same type over a discrete valuation ring of mixed characteristic such that the special fiber is the initially given object/structure?

In case one has an affirmative answer to this question, we ask

- What is the "smallest" (a term to be clarified in the sequel) base scheme over which a lifting exists ?
- What is the moduli of the liftings ?

Clearly, these problems fit into the general framework of deformation theory in algebraic geometry and the relevant results are indispensable for instance, in the moduli problems over $Spec(\mathbb{Z})$. To deal with the lifting problems, one needs more sophisticated techniques than those used in equicharacteristic deformation theory. This sophistication, on the other hand, pays off in the form of a series of astonishing results some of which justify the following rather vague statements :

1) A result is true in characteristic zero, if and only if it holds in characteristic p, for almost all p.

2) If a variety lifts to characteristic zero, then it does not have pathological behavior as regards the properties which are deformation invariant.

Before we describe the problem we are concerned with in this thesis, we fix our notation and then we recall some of the most outstanding applications of liftings to characteristic zero.

- k is a field (algebraically closed unless otherwise stated) of characteristic $p > 0$.
- R is a complete discrete valuation ring (dvr) of characteristic zero with residue field k.
- $W(k)$ is the ring of Witt vectors over k and $W_n(k) = W(k)/m^n$ is the ring of Witt vectors of length n.
- X is a projective smooth minimal surface over k.
- c_1, c_2 denote the first and the second Chern classes, χ_{et} is the etale Euler characteristic and χ_{zar} is the Zariski Euler characteristic.
- $\Pi_1(Y)$ denotes the algebraic fundamental group of Y.
- A lifting of X means a projective smooth scheme $\mathcal{X}$ over $S = Spec(R)$ with special fiber $\mathcal{X} \times_S Spec(k) \cong X$.

Now we can recall some well known applications of the theory of liftings. The first and arguably the most outstanding result is the following theorem due to Deligne-Illusie ([7]).

Theorem. *If X lifts to $W_2(k)$, then the deRham-Witt spectral sequence for X degenerates.*

In particular, we obtain the following result.

Corollary. *If X lifts to $W_2(k)$ and if $H^2_{cry}(X/W)$ is torsion free, then X has Hodge symmetry, that is $h^{p,q} = h^{q,p}$ for all pairs (p,q).*

As will be explained in Chapter III of this thesis, this theorem depends on the "size" of the base ring over which a lifting exists. One has counterexamples due to W. Lang in case X lifts to characteristic zero only after some ramification (that is, not over $W(k)$ but over a dvr which is an extension of $W(k)$).

Another important application is the work by Ekedahl ([9]) on foliations in characteristic p. The asymptotic result thus obtained about the Bogomolov inequality is particularly interesting.

After these general remarks and propaganda we now describe the problem we are interested in this thesis.

We consider a separable, connected fibration $\pi : X \to C$ over a smooth projective surface X in characteristic p and as our first question, we ask if π lifts to characteristic zero. More precisely, we look for a discrete valuation ring R of mixed characteristic, a smooth projective surface $\mathcal{X}$ and a smooth curve $\mathcal{C}$ over R which give the following commutative diagram.

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \to & \mathcal{C} \\ \downarrow & & \downarrow \\ Spec(k) & \to & Spec(R) \end{array}$$

More generally, we will allow "weak liftings", that is liftings over integral domains R of characteristic zero which admit surjective homomorphisms $R \to k$.

Then we want to determine the moduli of such liftings (over a fixed base ring R) and in particular, we compare the moduli of liftings of π to the moduli of liftings of the surface X (after forgetting the fibration).

Clearly, the problem makes sense only if X lifts as a surface. Then since there is no obstruction to lifting the base curve C, the problem becomes a special case of the general problem of lifting morphisms to characteristic zero. To treat problems of this type, at least in the case of smooth morphisms, there is the classical theory (initiated by Grothendieck in [10]) which combines the obstruction theory (in the cohomology of the relative cotangent complex) to the existence of infinitesimal deformations and the technique of algebraization of formal liftings. The basic results of this theory will be quoted at the end of this chapter. In this thesis, we restrict ourselves mainly to the case of *natural* fibrations on surfaces with Kodaira dimension $\kappa = -1, 0$. Lifting problems related to such fibrations are amenable to straightforward applications of the method described or to elementary and explicit constructions. For general fibrations one incorporates more sophisticated techniques including the recent progress on foliations in characteristic p.

First, we include some basic facts about Henselian rings, which we will refer to in the following chapters. An excellent concise reference for this topic is ([13], pp. 35–39).

1) Any complete discrete valuation ring is Henselian.

2) If R is a Henselian ring with residue field k, then we have:

- For smooth maps $\mathcal{X} \to S = Spec(R)$, the specialization map $\mathcal{X}(S) \to \mathcal{X}(Spec(k))$ is surjective.

- For $\mathcal{X} \to S$ proper, the map $\mathcal{E}t(\mathcal{X}) \to \mathcal{E}t(X)$, $\mathcal{Y} \mapsto \mathcal{Y} \times_S Spec(k)$ induces a bijection between the sets of etale covers.

Finally, we recall some elementary results from infinitesimal deformation theory as developed in ([10], Exp.182).

Let X be a smooth projective scheme over an (algebraically closed) field k and let R be complete discrete valuation ring with residue field k.

Fact 1. *If $H^2(X, \Theta_X) = 0 = H^2(X, \mathcal{O}_X)$ (here Θ_X is the tangent sheaf of X), then there exists a projective flat R-scheme $\mathcal{X}$ such that $\mathcal{X} \times_R Spec(k) \cong X$. ([10], Exp.182, Cor.3 on p.14).*

Fact 2. *If furthermore $H^1(X, \Theta_X) = 0$, then the R-scheme $\mathcal{X}$ is unique up to isomorphism which is identity on the closed fiber. (Loc. cit. Cor.1 on p.14).*

Fact 3. *Suppose X lifts to $\mathcal{X}$ over $Spec(R)$ and let $\mathcal{E}$ be a locally free sheaf on X. We have:*

(i) If $H^2(X, End(\mathcal{E})) = 0$, then $\mathcal{E}$ lifts to a locally free sheaf on $\mathcal{X}$

(ii) If furthermore $H^1(X, End(\mathcal{E})) = 0$, then this lifting is unique.

(Loc. cit. Prop.3 and Thm.1 on p.11).

Fact 4. *Given a smooth morphism $f : X \to Y$ between smooth schemes over k. The obstruction to lifting f vanishes if $H^2(X, \Theta_{X/Y}) = 0$. ($\Theta_{X/Y}$ denotes the relative cotangent sheaf of X/Y).*

Remark 1. *From Fact 1, it follows that any smooth proper curve C over $Spec(k)$ lifts to a smooth projective R-curve $\mathcal{C}$.*

Remark 2. *Any locally free sheaf $\mathcal{E}$ on a smooth proper curve C/k, lifts to a locally free sheaf on $\mathcal{C}$.*

CHAPTER 2

ELEMENTARY OBSERVATIONS

In this chapter we will work out some elementary results which follow more or less directly from the general results of relative algebraic geometry and from the theory of algebraic surfaces.

We first observe that if $\mathcal{X}$ is a lifting of X, then the generic fiber $\mathcal{X}_\eta$ has the same Kodaira dimension as X and $\mathcal{X}_\eta$ is minimal if X is. This is a consequence of specialization in intersection theory ([11]).

Lemma 1. *If a surface X lifts to characteristic zero, then the Bogomolov inequality holds for X.*

Proof. Let $\mathcal{X} \to S$ be a lifting. Then, since the map $\mathcal{X} \to S$ is proper, the Zariski Euler characteristic χ_{zar} and the etale characteristic χ_{et} of the fibers are fixed ([13]). Therefore, since $12\chi_{zar} = c_1^2 + c_2$, $\chi_{et} = c_2$ and since Bogomolov's inequality holds for all surfaces in characteristic zero, we obtain $c_1^2(X) = 12\chi_{zar}(X) - c_2(X) = 12\chi_{zar}(\mathcal{X}_\eta) - c_2(\mathcal{X}_\eta) = [c_1^2(\mathcal{X}_\eta) + c_2(\mathcal{X}_\eta)] - c_2(\mathcal{X}_\eta) = c_1^2(\mathcal{X}_\eta) \leq 3c_2(\mathcal{X}_\eta) = c_2(X)$ □

It follows from this lemma that there are surfaces of general type admitting smooth nonisotrivial fibrations, which do not lift even as surfaces. For instance, one may take the smooth fibrations $X \to C$ constructed by Szpiro in ([18]) for which $c_1^2(X) > 3c_2(X)$. There are similar examples with fiber genus 2, constructed by Parshin in ([16]).

Another obstruction to lifting fibrations comes from the comparison of the contributions of singular fibers to c_2. We recall that if $\pi : X \to C$ is a (connected) fibration, then in etale cohomology we have the following formula

$$\chi_{et}(X, l) = \chi_{et}(C, l).\chi_{et}(\overline{F}, l) + t$$

which relates the etale Euler characteristics of X, C and of the geometric generic fiber $\overline{F}$. ([13], Remark 2.15(c) on p.192). Here t is the contributions of the singular fibers.

It is well known that

(i) in characteristic zero,

$$t = \sum_{p \in C} [\chi(F_p, l) - \chi(\overline{F}, l)] \qquad (1)$$

(ii) in characteristic $p > 0$,

$$t = \sum_{p \in C} [\chi(F_p, l) - \chi(\overline{F}, l) - \sum_{n=0}^{2} (-1)^n \alpha_p(\mathcal{F}^n)] \qquad (2)$$

In (ii), $\mathcal{F}^n = R^n \pi_* \mathbb{F}_l$ and α_p is the associated (exponent of) the wild conductor at p $\in C$, defined in ([13], p.188).

If the fibration lifts, then clearly, the base genus (resp. the fiber genus) of the fibrations on $\mathcal{X}_\eta$ and on X are equal. Therefore, since c_2 is fixed, if the given fibration is generically smooth then we see that the contributions t of the singular fibers on X and on $\mathcal{X}_\eta$ are equal, too. Recalling that a singular fiber of $X \to C$ does not lift to a smooth fiber on $\mathcal{X}_\eta$ ([18]) and by comparing (1) and (2) we obtain a restriction on the Milnor numbers of singular fibers on X (compare with the formula in [8]). This observation, however, is hard to apply in practical situations because of the difficulty of calculating the wild conductors.

On the positive side, for some special fibrations one has the following result for the existence of liftings.

Lemma 2. *Let $\mathcal{X}$ be a lifting of X over a complete discrete valuation ring and assume that the fibration $\pi : X \to C$ is either*

a) the albanese fibration $X \to Alb_X$, or

b) the n-th canonical fibration.

Then π lifts to a fibration $\mathcal{X} \to \mathcal{C}$.

Proof.

a) We consider the dual of the reduced component of the Picard scheme $Pic_{\mathcal{X}/S}$ containing the identity. Under the given hypothesis this is an abelian scheme and is the relative albanese scheme $Alb_{\mathcal{X}/S}$ of $\mathcal{X}/S$. As the base scheme is Henselian, the point in $X(k)$ used in defining the albanese map of the special fiber, lifts to a section in $\mathcal{X}(S)$ (Chapter I). Thus the relative albanese map $\mathcal{X} \to Alb_{\mathcal{X}/S}$ is defined over S and gives the required curve $\mathcal{C}$.

b) If the given fibration corresponds to the n-th canonical map, then clearly the image of the map $\mathcal{X} \to \mathbb{P}(\pi_{S*}(w_{\mathcal{X}/S}^{\otimes n}))$ is a curve $\mathcal{C}$; the result follows. □

Lemma 2(b) applies in particular to surfaces with Kodaira dimension $\kappa = 1$, because elliptic fibrations on surfaces with $\kappa = 1$ arise from n-th canonical mappings ([14]). Details of the lifting problem concerning these surfaces will not be covered in this thesis.

Lemma 2(a) applies to ruled surfaces and hyperelliptic surfaces; because for a ruled surface $X \to C$, the albanese map is simply given by $X \to \mathcal{J}_C$ which factors through $X \to C$ because on an abelian variety there exists no curve of genus zero. On the other hand it is classical that, for a hyperelliptic surface X, we have two elliptic fibrations on X, one of which is the albanese fibering. However, a ruled surface lifts over any complete discrete valuation ring (see Chapter III) whereas this is not the case for hyperelliptic surfaces. In fact W. Lang ([12]) constructed hyperelliptic surfaces in characteristic $p = 2, 3$ which lift only after ramification (of degree 2) is allowed. This example was already mentioned in Chapter I and is related to lifting actions of groups whose order is divisible by the characteristic (see Chapter III).

To state and prove our final result in this chapter we recall the well-known result in characteristic zero to the effect that a surface admits a fibration over a curve of genus $g \geq g_0 \geq 2$ if and only if its topological fundamental group maps onto the fundamental group of a curve of genus g_0. This in particular implies that any deformation of such a surface also admits a fibration with base genus $g \geq 2$. In the proof of the following analogous result one uses the algebraic fundamental group instead of the topological fundamental group.

Lemma 3. *Let $\pi : X \to C$ be a fibration with $g(C) \geq 2$ and suppose that $\mathcal{X}/S$ lifts X. Then the geometric generic fiber $\overline{\mathcal{X}}_\eta$ admits a fibration over a curve of genus $g \geq g(C)$.*

Proof. Fixing a point $x \in X(k)$ and its image $c \in C$, we have the sequence of algebraic fundamental groups $\Pi_1(X, x) \to \Pi_1(C, c) \to 0$ and the specialization map $\Pi_1(\mathcal{X}_\eta, x_\eta) \to \Pi_1(X, x) \to 0$ which give a surjective homomorphism $\Pi_1(\mathcal{X}_\eta, x_\eta) \to \Pi_1(C, c) \to 0$. Then one checks that the proof of (Theorem in [2]) goes through with topological fundamental groups replaced by algebraic fundamental groups. □

Remark 1. *One can give an alternative proof for Lemma 3 in case the lifting is over a complete discrete valuation ring. The essential step in the proof of (Theorem in [2]) is the construction of enough number of cyclic coverings of some fixed degree of* $\mathcal{X}_\eta$*. For this, we consider cyclic covers* $C' \to C$ *of prime degree* $n \neq char(k)$*. Each such cover gives, by base extension, a cylic cover of degree* n*,* $X' \to X$ *of the special fiber of* $\mathcal{X}/S$*. As the base scheme of the lifting is Henselian, this cover lifts to an etale cover* $\mathcal{X}' \to \mathcal{X}$ *(Chapter I) which when restricted to* $\mathcal{X}_\eta$ *gives a cover of the required type.*

Remark 2. *Clearly, the result in Lemma 3 provides an affirmative answer to the main question for fibrations satisfying some additional restrictions (for instance, if the fibration on the generic fiber is defined over the quotient field of* R *and has good reduction and if the fibration on* X *is unique modulo the automorphisms of* π*).*

Remark 3. *The restriction on the genus of the base curve is essential; one can construct simple abelian surfaces in characteristic zero with non-simple reductions mod* p *(see Chapter III).*

CHAPTER 3

THE CASE OF SURFACES WITH $\kappa = -1, 0$

In this chapter we will exploit the classification of surfaces in characteristic p as worked out by Mumford and Bombieri ([3],[14]). We will start with geometrically ruled surfaces and then take up surfaces with Kodaira dimension $\kappa = 0$ where the first examples indicating the subtlety of the problems related to lifting appear. In the $\kappa = 0$ case, rather than repeating the proofs of well-known results, we will refer to the relevant article in the literature.

$\kappa = -1$ Case :

The following lemma answers completely the existence and moduli questions concerning the lifting of ruled surfaces.

Lemma 4. *Let $\pi : X = \mathbb{P}(E) \to C$ be a ruled surface over a smooth projective curve C and let R be a complete discrete valuation ring with residue field k. Then*

a) π lifts to a $\mathbb{P}^1$-bundle over a curve $\mathcal{C}/S$.

b) Any lifting $\mathcal{X}$ of X over $S = Spec(R)$, is a $\mathbb{P}^1$-bundle over a suitable lifting $\mathcal{C}$ of C.

Proof.

a) We know that any smooth curve over k lifts to R (Chapter I, Remark 1). Moreover, since $H^2(C, End(E)) = 0$, there is no obstruction to lifting E to a rank two locally free sheaf $\mathcal{E}$ on $\mathcal{C}$ (Chapter I, Remark 2). We take $\mathcal{X} = \mathbb{P}(\mathcal{E})$.

b) This is a restatement of the remark following Lemma 2 in Chapter II. □

$\kappa = 0$ Case :

Hyperelliptic Surfaces :

We know that $X = (E_1 \times E_2)/G$ for a group G of automorphisms whose type and action on E_1 and E_2 were worked out in ([3]). Since there is no ramification in the action of G on E_1, the pair $(E_1,\ G)$ lifts over $W(k)$ ([17]). The second pair $(E_2,\ G)$ lifts over $W(k)$ if there is no wild ramification in the action of G on E_2 ([17]). If this is the case, then the fibrations $X \to E_i/G$, i=1,2 clearly lift over $W(k)$ to give $\mathcal{X} = (\mathcal{E}_1 \times_S \mathcal{E}_2)/G \to \mathcal{E}_i/G$. In fact, since $X \to E_1/G$ is the albanese fibration, this fibration lifts automatically (remark after Lemma 2 in Chapter II).

W. Lang's example ([12]) is a hyperelliptic surface in characteristic $p = 2$ and with $G \cong \mathbb{Z}_2$ which acts on the second component with wild ramification. Lang proves that X does not lift over $W(k)$, but lifts over an extension of degree 2 of $W(k)$. Thus, for lifting the fibrations in this example we obtain the same conclusion as in the preceding paragraph, only after we allow ramification of degree 2 (minimum possible !). In fact, this example is a special case of a general result ([17]) which shows that if $\sigma \in Aut(Y)$, where Y is a smooth complete curve, then the pair (Y, σ) lifts to $W[\zeta_p]$ if p^2 does not divide $|\sigma|$.

We note that Lang shows also that ([12], Theorem 1) $b_1^{DR}(X) \neq h^{1,0} + h^{0,1}$ for this surface and thus proves that the hypothesis of the theorem of Deligne-Illusie quoted in Chapter I is optimal.

Quasi-hyperelliptic Surfaces :

This case occurs in characteristic 2 and 3 ([3]).

These surfaces are of the form $X = (E_1 \times C_0)/G$, where C_0 is a cuspidal curve, G is a finite subgroup scheme of E_1 and the action is given by $g.(u, v) = (u + g, \alpha(g)v)$ for some injective homomorphism $\alpha : G \to Aut(C_0)$. We have two fibrations $X \to C_0/\alpha(G) \cong \mathbb{P}^1$ and $X \to E_1/G \cong E$ where $E = Alb(X)$. It follows that, if the surface lifts then the second fibration (the albanese fibration) also lifts.

Abelian Surfaces :

It is well known that any abelian surface lifts in the weak sense ([15]). However fibrations on abelian surfaces need not lift; we give an example constructed using the relative jacobian of a suitable curve of genus 2.

Example. Consider the affine "plane" curve $\mathcal{C}$ given by

$$y^2 = x(x-1)(x-2)(x-5)(x-6) \text{ over } S = Spec(W(\overline{\mathbb{F}}_7)).$$

The complete nonsingular model is a curve of genus 2. We take the jacobian scheme $\mathcal{J}_{\mathcal{C}/S}$. The generic fiber is a geometrically simple abelian surface (cf. [4], p. 159), but the special fiber is the jacobian of the curve birational to the plane curve

$$y^2 = x(x-1)(x-2)(x+2)(x+1) \text{ (since } 6 \equiv -1, 5 \equiv -2 \pmod 7))$$

which admits an elliptic fibration ([4], Thm. 14.1.1(iii)) over an elliptic curve E.

This result, clearly is in conformity with the obstruction theory for liftings. The obstruction to *infinitesimal* lifting of the fibration is in the cohomology group $H^2(J, \Theta_{J/E}) \otimes \pi^*(I)$ (Chapter I, notes on infinitesimal deformation theory) where J is the special fiber of $\mathcal{J}_{\mathcal{C}/S}$ and $\Theta_{J/E}$ is the relative tangent bundle. Since $\Theta_{J/E} \cong \mathcal{O}_J$ we have $H^2(J, \Theta_{J/E}) = \overline{\mathbb{F}}_7$. Therefore, it is not surprising to find out that the obstruction does not vanish.

$K3$ Surfaces :

We know ([6]) that any $K3$ surface lifts. It is also known that this lifting is in the strong sense if X is not nonelliptic superspecial (conjecturally, such $K3$ surfaces do not exist !) and that if $p > 2$, then any $K3$ surface lifts over $W(k)[\sqrt{p}]$.

Lemma 5. *A generically smooth fibration on a $K3$ surface X is necessarily elliptic with base $\mathbb{P}^1$.*

Proof. Since $H^0(X, \Omega_X) = 0$ in all characteristics, the base is $\mathbb{P}^1$. If F is the generic fiber, then $2g(F) - 2 = F.(K_X + F) = 0$ since $K_X = 0$. Thus F is an elliptic curve. □

Then our problem is to see if an elliptic fibration on a $K3$ surface lifts. In some special cases (suitable Kummer surfaces) this problem is related to lifting/reduction of Shioda-Inose structures. We have certain finite groups giving rise to generalized Kummer surfaces which appear only in characteristic $p > 0$. It seems interesting to study the lifting problem relevant to fibrations and Shioda-Inose structures arising from these special groups.

The example we will treat is the fibration on a $K3$ surface which covers an Enriques surface; these we will discuss in the following paragraph.

Enriques Surfaces :

These surfaces admit elliptic fibrations $X \to \mathbb{P}^1$, which may be quasi-elliptic in characteristic 2 and 3 ([3]).

We have the following results as regards the lifting of Enriques surfaces.

1) An Enriques surface X lifts to characteristic zero, if $p \neq 2$ or if (i) X is a μ_2-surface, (ii) classical with a regular 1-form with only isolated singularities ([5], Corollary 1.4.1, p. 93).

2) If X is of α_2-type, then $b_1^{DR}(X) \neq h^{1,0} + h^{0,1}$, hence X does not lift even to $W_2(k)$. (For the relevant terminology, see [3], [5]).

Question :

Does an Enriques surface of α_2-type lift after some ramification ?

Example. We consider an Enriques surface X of classical type over a field k with $char(k) \neq 2$ ([3], Part III). Let Y be the degree 2 etale covering of X, and consider the situation

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ & & \downarrow \pi \\ & & \mathbb{P}^1 \end{array}$$

where φ is the "universal covering map".

Claim: *$\pi o \varphi$ is not connected.*

Proof: If $\pi o \varphi$ is connected, then we obtain an elliptic fibration $\pi o \varphi : Y \to \mathbb{P}^1$ on the $K3$ surface Y, with precisely two double fibers, say over p_1 and p_2. Then by the canonical bundle formula for elliptic fibrations, we get $w_Y = (\pi o \varphi)^*(\mathcal{L}) \otimes \mathcal{O}_Y(F_1' + F_2')$ where $\mathcal{L}$ is a line bundle on $\mathbb{P}^1$ of degree, $deg(\mathcal{L}) = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_{\mathbb{P}^1}) = 0$, since $\chi(\mathcal{O}_X) = 2$. Therefore, $w_Y \cong \mathcal{O}_Y(F_1' + F_2') \neq \mathcal{O}_Y$; contradiction since $K_Y = 0$. $\square$

Therefore, we need "Stein factorization" obtained from the double cover $\mathbb{P}^1 \to \mathbb{P}^1$, to get a connected fibration $Y \to \mathbb{P}^1$ (cf. [1], p.274, Remarks). (We note that this covering $\mathbb{P}^1 \to \mathbb{P}^1$ corresponds to the line bundle $\mathcal{O}(p_1+p_2) \cong \mathcal{O}(1)^{\otimes 2}$ on $\mathbb{P}_1$).

Now, let $\mathcal{X}$ be a lifting of X over a Henselian ring (for instance $W(k)$). We first verify that the fibration $\pi : X \to \mathbb{P}^1$ lifts to a fibration on $\mathcal{X}$. For this, we construct $\mathbb{P}(\mathcal{E})$ over S for a suitable rank 2 locally free sheaf and the map $\mathcal{X} \to \mathbb{P}(\mathcal{E})$ lifting π.

Construction: The map $X \to \mathbb{P}^1$ corresponds to the linear system determined by the line bundle $L = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}(2F_i^{'})$ where $2F_i^{'}$ is one of the double fibers of π lying over $p_1, p_2 \in \mathbb{P}^1$. Since $End(L) \cong \mathcal{O}_X$, $H^2(X, End(L)) \cong H^2(X, \mathcal{O}_X) = 0$ and $H^1(X, End(L)) \cong H^1(X, \mathcal{O}_X) = 0$, because X is an Enriques surface. Therefore by (Fact 3, Chapter I) L lifts to a unique line bundle $\mathcal{L}$ on $\mathcal{X}$. Taking $\mathcal{E} = \varphi_*(\mathcal{L})$ we obtain $\mathcal{X} \to \mathbb{P}(\mathcal{E})$ (corresponding to the natural map $\varphi^*(\varphi_*(\mathcal{L})) \to \mathcal{L} \to 0$) which lifts $\pi : X \to \mathbb{P}^1$.

One checks that the induced fibration on the generic fiber $\mathcal{X}_\eta$ is connected and has precisely two double fibers; in fact these double fibers lie over the generic points of the sections $s_i : S \to \mathbb{P}(\mathcal{E})$ which lift the points $p_1, p_2 \in \mathbb{P}^1(k)$ in the special fiber (Henselian base!). $\square$

The degree 2 etale covering Y of X is a $K3$ surface and since the base scheme is Henselian the covering map $Y \to X$ lifts to give $\mathcal{Y} \to \mathcal{X}$. The composite map $\mathcal{Y} \to \mathcal{X} \to \mathbb{P}(\mathcal{E})$ induces an elliptic fibration which we proved is not connected. The "Stein factorization" $\mathcal{Y} \to \mathbb{P}(\mathcal{E})$ obtained from the double cover $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ which ramifies precisely over $s_1 \cup s_2$, lifts the elliptic fibration on the $K3$ surface Y. This is the example referred to in the preceding paragraph.

Finally, we state some problems which are inspired by and generalize the questions considered in this thesis.

1) Find good criteria for lifting (algebraic) group actions. This problem is related to class field theory as applied in ([17]) with considerable success in case of curves. The most interesting cases are those in which one encounters infitesimal groups and groups with order divisible by the characteristic.

2) Find criteria for lifting varieties with given covers. This problem, in the case of Galois covers is related to the preceding problem.

3) Study the relation of uniformization with lifting problems. In relation to this approach of constructing fibrations, to develop the characteristic p counterpart of L^2-cohomology techniques of Gromov seems to be a very interesting problem. This approach is clearly also related to rigidity theory which proved to be extremely effective in studying deformations of fibered surfaces in characteristic zero.

4) Apply the full power of the theory of foliations in characteristic p, as developed by Ekedahl ([9]), to the problem of lifting fibrations.

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