CONFORMAL VECTOR FIELDS WITH RESPECT TO THE SASAKI METRIC TENSOR FIELD

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CONFORMAL VECTOR FIELDS WITH RESPECT TO THE SASAKI METRIC TENSOR FIELD

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

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Abstract

CONFORMAL VECTOR FIELDS WITH RESPECT TO THE SASAKI METRIC

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On the tangent bundle of a Riemannian manifold the most natural choice of metric tensor field is the Sasaki metric. This immediately brings up the question of infinitesimal symmetries associated with the inherent geometry of the tangent bundle arising from the Sasaki metric. The elucidation of the form and the classification of the Killing vector fields have already been effected by the Japanese school of Riemannian geometry in the sixties. In this thesis we shall take up the conformal vector fields of the Sasaki metric with the help of relatively advanced techniques.

Keywords: Sasaki metric tensor, Tangent Bundle, Killing vector fields, Conformal vector fields, Lifts of tensor fields.

Öz

SASAKİ METRİĞİNE GÖRE KONFORM VEKTÖR ALANLARI

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Bir Riemann manifolduna ait teğet demeti üzerinde metrik tansör olarak en tabii olanı Sasaki metriğidir. Bu hal hemen teğet demeti üzerinde Sasaki metriği vasıtasıyla bu suretle ortaya çıkan geometriye ait enfinitesimal simetriler sorusunu davet eder. Bu cümleden olmak üzere Killing vektör alanlarının şekillerinin ortaya çıkartılması ve sınıflandırılması çalışmaları Japon Riemann geometrisi ekolünce altmışlı yıllarda bitirilmiştir. Bu tezde Sasaki metriğinin konform vektör alanları nispeten ileri yöntemlerle ele alınacaktır.

Anahtar Kelimeler: Sasaki Metrik tansörü, Teğet demeti, Killing vektör alanları, Konform vektör alanları, Tansör alanlarının kaldırılması.

In the memory of my mother **Cennet Şimşir** and my father **Fatih Şimşir**

"Sadece ve sadece çocuklarının iyi insanlar olmalarını istediler".

Yüreği büyümüş bir çocuktum ben
Gizli gizli ne kadar çok ağladım
Bir gün öleceğini düşünerek onun
Annem yok artık,
Onun yüreğindeki ben de yokum,
Yani annemle tanımlanan ben de öldüm onunla
Şimdi,
Yeni bir tanıma alıştırmalıyım kendimi,
Şimdi ,
Ben kendimi düşünmezken bile
Kim düşür beni ?
A.Behramoğlu

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Chapter 1

Introduction

In its most embryonic form, the idea of working with the tangent bundles occurs in the theory of ordinary differential equations where one routinely transforms systems of second order ordinary differential equations into systems of first order ordinary differential equations in twice as many unknowns. Since the systems of second order differential equations constitute a central issue of classical mechanics, the importance of this technique can not be exaggerated. The natural and nonetheless ingenious sequel to this approach is analytical mechanics in the style of Lagrange in which components of momentum are treated on equal footing with components of position, that is as ”coordinates” not of the physical object but of the ”physical state” of the object. Even at this stage the implicit occurence of the cotangent bundle is clearly recognisable.

Tangent bundle of a manifold has a natural manifold structure. The inherent linearity of tangent fibers has a mathematically simplifying effect on the tangent bundle, bestowing upon it a hybrid structure combining linear and nonlinear features. The modern study of the geometry of tangent bundle may be considered to have commenced with the seminal articles of S. Sasaki ([11]) in 1958 and P. Dombrowski ([3]) in 1962. Of course, the tangent bundle of a Riemannian manifold is an attractive object for at least two reasons: Firstly, in the presence of a Riemannian tensor field on a manifold, the tangent bundle becomes ”paired” with the cotangent bundle and inherits a natural symplectic structure. Secondly, a Riemannian tensor field on a manifold induces a natural flow on the tangent bundle which is called the geodesic flow and is one of the most intensely studied objects of dynamical systems theory ([4], [7], [1], [9]).

Definition of the Sasaki metric was followed by the question of infinitesimal symmetries associated with this natural Riemannian geometry of the tangent bundle. Killing and Conformal vector fields on the one hand ([5], [6], [8]) and the tangent bundle of a manifold ([10]) on the other are natural objects of study in mathematics as well as in theoretical physics. The Killing vector fields with respect to the Sasaki metric tensor field on the tangent bundle of a Riemannian manifold were completely characterized by S. Tanno ([12]) yet his calculations were not so easy to follow. We reobtain the results of S. Tanno ([12]) in Chapter 3 by introducing more streamlined methods in the light of which we shall characterize the conformal vector fields on the tangent bundle with respect to the Sasaki metric tensor field in Chapter 4.

All that follows will be in the smooth category unless explicitly qualified otherwise.

Chapter 2

Vector Fields on Riemannian Manifolds

2.1 Killing Vector Fields

Let g be the metric tensor field on the Riemannian manifold M. A vector field $A \in \mathfrak{X}(M)$ is said to be a Killing vector field with respect to a Riemannian metric tensor field $\mathsf{g} \in \mathfrak{X}^{(0,2)}(M)$ if it satisfies the so called Killing's equation, that is

$$\mathfrak{L}_A \mathsf{g} = o \tag{2.1}$$

In the presence of a chart $x = (x^i)_{1 \leq i \leq n}$ such that $\mathsf{g} \mid_{dom(x)} = g_{ij} dx^i \otimes dx^j$ the equation (1) reduces to

$$\mathfrak{L}_A \mathsf{g} = \nabla_i A_j + \nabla_j A_i = 0 \tag{2.2}$$

in $dom(x)$ where $A \mid_{dom(x)} = A^i \dfrac{\partial}{\partial x^i}$.

Example 2.1. $A \in \mathfrak{X}(\mathbb{R}^2)$, A is a Killing vector field with respect to $\mathsf{g} = dx \otimes dx + dy \otimes dy$ iff

$$A = (-wy + a)\frac{\partial}{\partial x} + (wx + b)\frac{\partial}{\partial y}$$

for some $w, a, b \in \mathbb{R}$. $\diamondsuit$

Example 2.2. Consider the Poincare half plane,
$\{(x,y) \in \mathbb{R}^2 \ \mid y > 0\}$ and $\mathsf{g} = \dfrac{1}{y^2}(dx \otimes dx + dy \otimes dy)$. $A \in \mathfrak{X}(M)$ is a Killing vector field iff

$$A = a\frac{\partial}{\partial x} + b(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + c[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}]$$

for some $a, b, c \in \mathbb{R}$ $\diamondsuit$

Example 2.3. Generalising 2.1 $A \in \mathfrak{X}(\mathbb{R}^n)$, is a Killing vector field with respect to $\mathsf{g} = \delta_{ij} dx^i \otimes dx^j$. Then A is a Killing vector field iff $A_i = Y_i + x^q H_{qi}$ where $H_{qi} + H_{iq} = 0$. $\diamondsuit$

2.2 Conformal Vector Fields

Definition 2.1. A vector field $A \in \mathfrak{X}(M)$ is said to be a *conformal vector field* (or a *conformal Killing vector field*) with respect to a Riemannian metric tensor field $\mathsf{g} \in \mathfrak{X}^{0,2}(M)$ if

$$L_A \mathsf{g} = 2\ \sigma\ \mathsf{g} \tag{2.3}$$

for some scalar field $\sigma \in \mathfrak{F}(M)$.

A conformal vector field A is called an *infinitesimal homothety* if σ is constant. In the presence of a chart $x = (x^i)_{1 \leq i \leq n}$, the equation 2.3 reduces to

$$\nabla_i A_j + \nabla_j A_i = 2\ \sigma\ g_{ij} \tag{2.4}$$

on $dom(x)$ where

$$\mathsf{g}|_{dom(x)} = g_{ij}\ dx^i \otimes dx^j$$

and

$$A|_{dom(x)} = A^i\ \frac{\partial}{\partial x^i} \quad .$$

Raising j and contracting with i in 2.4 it is easily seen that

$$\sigma = \frac{1}{n}\ div(A)$$

where $n = \dim M$. We would like to mention immediately one special instance of a conformal vector field as it strongly bears upon our work in the sequel :

Example 2.4. Let $n \geq 3$. On the n-dimensional Euclidean space, which is simply the manifold $\mathbb{R}^n$ with the Riemannian metric tensor field $\mathsf{g} = g_{ij}\ dx^i \otimes dx^j$ where g_{ij} is a constant for each $1 \leq i, j \leq n$, the equation 2.4 reduces to

$$\frac{\partial A_j}{\partial x^i} + \frac{\partial A_i}{\partial x^j} = 2\ \sigma\ g_{ij} \tag{2.5}$$

from which it can be deduced that σ must be of the form $\sigma = a_q x^q + b$ for some constants $a_q, b \in \mathbb{R}$ and

$$A_i = Y_i + x^q H_{qi} + \frac{1}{2} x^q x^r (a_q g_{ri} + a_r g_{qi} - a_i g_{qr})$$

where $Y_i, H_{qi} \in \mathbb{R}$ are constants and $H_{qi} + H_{iq} = 2\ b\ g_{qi}$ (Chapter 2.7, p.53 [2]) . It is easily discernible that the first term generates a translation whereas the second generates a rotation followed by a dilation . The third term, however, can generate only a local flow which is an inversion followed by a reflection for each non-zero value of the parameter of the local flow. We notice that this conformal vector field is complete iff the vector $\mathsf{a} = (a^i)_{1 \leq i \leq n}$ vanishes . On complete Riemannian manifolds, Killing vector fields are always complete. The above example shows us that this is not always the case for conformal vector fields, [2]. ♢

Chapter 3

Geometry of the Tangent Bundle

The purpose of this chapter is to present the fundamental concepts in the geometry of the tangent bundle.

3.1 Preliminaries

Given a manifold M, let $\mathfrak{F}(M)$ stand for the ring of smooth scalar fields on M and let $\mathfrak{X}^{p,q}(M)$ denote the $\mathfrak{F}(M)$-module of tensor fields of bidegree (p,q) on M. In particular $\mathfrak{X}^{0,0}(M) = \mathfrak{F}(M)$, $\mathfrak{X}^{1,0}(M) = \mathfrak{X}(M), \mathfrak{X}^{0,1}(M) = \mathfrak{X}^*(M)$.

Let M be a manifold of dimension n, TM its tangent bundle and $\tau : TM \to M$ be the canonical tangent bundle projection. In the presence of a chart $x = (x^1, \ldots, x^n)$, the vectors

$$\frac{\partial}{\partial x^1} |_p, \ldots, \frac{\partial}{\partial x^n} |_p$$

constitute a basis for the tangent space T_pM at p for each $p \in dom(x)$. For each chart $x = (x^i)_{1\leq i\leq n}$, an open subset A of $dom(x)$ and an open subset U of $\mathbb{R}$ consider the set

$$\mathfrak{N}_{x,A,U} = \{a^i \frac{\partial}{\partial x^i} |_p \in A, (a^1, \ldots, a^n) \in U\} \subseteq TM \ .$$

It can be routinely checked that sets of the form $\mathfrak{N}_{x,A,U}$ constitute a basis for a topology on TM with respect to which TM is locally Euclidean (of dimension $2n$) and Hausdorff, consequently a topological manifold. Given any chart

$$x = (x^i)_{1\leq i\leq n} : \ dom(x) \subseteq_{op} M \longrightarrow x \ dom(x) \subseteq_{op} \mathbb{R}^n$$

on M, we consider the chart

$$\hat{x} = (\hat{x}^\alpha)_{1\leq\alpha\leq 2n} : \ dom(\hat{x}) = \tau^{-1}dom(x) \subseteq_{op} TM \longrightarrow \hat{x}dom(x) \times \mathbb{R}^n \subseteq_{op} \mathbb{R}^{2n}$$

defined for each $\mathsf{u} \in \tau^{-1}dom(x)$ by

$$\begin{aligned} \hat{x}^i(\mathsf{u}) &= x^i(\tau(\mathsf{u})) \\ \hat{x}^{i+n}(\mathsf{u}) &= u^i \end{aligned}$$

for $1 \leq i \leq n$ where $\mathsf{u} = u^i \dfrac{\partial}{\partial x^i} |_{\tau(\mathsf{u})}$.

By abuse of notation and as an obvious aid for memory we allow x^i to stand for $\hat{x}^i = x^i \circ \tau$ for $1 \leq i \leq n$. Again for obvious reasons we shall denote $\hat{x}^{i+n}$ by $\dot{x}^i$ for $1 \leq i \leq n$. Of course, the dot is just a notational device and does not connote differentiation.

The above described chart $(x, \dot{x}) = (x^i, \dot{x}^i)_{1\leq i\leq n}$ will be called the chart associated with the chart $x = (x^i)_{1\leq i\leq n}$.

If the charts $x = (x^i)_{1\leq i\leq n}$ and $y = (y^i)_{1\leq i\leq n}$ on M have overlapping domains, each y^i can be expressed as a function $y^i = y^i(x^1, x^2, \ldots, x^n)$ of x^j s on $dom(x) \cap dom(y)$. In this case $(x, \dot{x})$ and $(y, \dot{y})$ have overlapping domains, too. Moreover, on $dom(x, \dot{x}) \cap dom(y, \dot{y})$ we have

$$\begin{aligned} y^i &= y^i(x^1, x^2, \ldots, x^n) \\ \dot{y}^i &= \frac{\partial y^i}{\partial x^q}(x^1, x^2, \ldots, x^n)\dot{x}^q \end{aligned}$$

We observe that $\dot{y}^i$ is linear in $\dot{x}^q$ s. Indeed the following is well known in classical Lagrangian mechanics:

$$\frac{\partial \dot{y}^i}{\partial \dot{x}^j} = \frac{\partial y^i}{\partial x^j}$$

This shows us, among others thatthe charts on TM associated to smooths charts on M, induce a smooth structure on TM.

3.2 Lifting tensor fields to the tangent bundle

In the study of the tangent bundle it will be important to produce tensor fields on the tangent bundle, from out of tensor fields on the manifold itself. Such a procedure is usually referred to as a "lifting" of tensor fields on M to tensor fields on TM.

Let f be a scalar field on $U \subseteq_{op} M$, the corresponding scalar field $f \circ \tau$ on $\tau^{-1}(U)$ will be written as f for brevity. Indeed, we have already availed ourselves of this simplification in connection with the chart $(x, \dot{x})$ associated with the chart x.

Definition 3.1. Given $A \in \mathfrak{X}(M)$, the <u>vertical lift</u> ${}^{v}A$ of A is the velocity field of the flow

$$\Psi : TM \times \mathbb{R} \longrightarrow TM$$

on TM defined by

$$\Psi(\mathsf{u}, t) = \mathsf{u} + tA_{\tau(\mathsf{u})}.$$

It can be readily checked that in the presence of a chart $x = (x^i)_{1 \leq i \leq n}$ with $A = A^i \dfrac{\partial}{\partial x^i}$

$${}^{v}A|_{dom(x,\dot{x})} = A^i \frac{\partial}{\partial \dot{x}^i} \ .$$

Note that the vertical lift of vector fields is "tensorial". To be precise, the value of ${}^{v}A$ at $\mathsf{u} \in M$ can be determined only on the basis of value of A at $\tau(\mathsf{u})$. In fact

$${}^{v}(fA) = f\ {}^{v}A$$

for any $A \in \mathfrak{X}(M),\ f \in \mathfrak{F}(M)$.

Definition 3.2. Each tensor field $\mathsf{S} \in \mathfrak{X}^{p,q+1}(M)$ can be naturally regarded as a tensor field ${}^{n}\mathsf{S} \in \mathfrak{X}^{p,q}(M)$ ([14]), which we call, for want of a better name, the <u>natural lift</u> of S and define it by

$${}^{n}\mathsf{S}|_{dom(x)} = \dot{x}^k S^{i_1 \ldots i_p}_{k j_1 \ldots j_q} \frac{\partial}{\partial \dot{x}^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \dot{x}^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}$$

where

$$\mathsf{S}|_{dom(x)} = S^{i_1 \ldots i_p}_{k j_1 \ldots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \frac{\partial}{\partial x^{i_p}} \otimes dx^k \otimes dx^{j_1} \otimes \ldots dx^{j_q}$$

Given a tensor field $\mathsf{H} \in \mathfrak{X}^{1,1}(M)$ it is instructive to construct its natural lift ${}^{n}\mathsf{H} \in \mathfrak{X}(TM)$ in three equalivalent ways :

1. For all $\mathsf{u} \in TM$, ${}^{n}\mathsf{H} = {}^{v}(\mathsf{H}_{\tau(\mathsf{u})}(\mathsf{u}))$.

2. H induces a flow

$$\Psi : TM \times \mathbb{R} \longrightarrow TM$$

defined by

$$\Psi(\mathsf{u}, t) = \exp(t\mathsf{H}_{\tau(\mathsf{u})})\mathsf{u}$$

$^{n}\mathsf{H}$ is the velocity field of Ψ.

3. In the presence of a chart $x = (x^i)_{1\leq i\leq n}$ where

$$\mathsf{H} \mid_{dom(x)} = \mathsf{H}^i_q \frac{\partial}{\partial x^i} \otimes dx^q$$

we have

$$\mathsf{H} \mid_{dom(x,\dot{x})} = \dot{x}^q \ \mathsf{H}^i_q \ \frac{\partial}{\partial \dot{x}^i}$$

Definition 3.3. Given $A \in \mathfrak{X}(M)$ which is also understood as a map $A : M \longrightarrow TM$, the complete lift ${}^{c}A$ of A is defined by

$${}^{c}A = TA : TM \longrightarrow T(TM).$$

Again in the presence of a chart $x = (x^i)_{1\leq i\leq n}$ with $A = A^i \dfrac{\partial}{\partial x^i}$ it can be checked that

$${}^{c}A|_{dom(x,\dot{x})} = A^i \frac{\partial}{\partial x^i} + \dot{x}^q \frac{\partial A^i}{\partial x^q} \frac{\partial}{\partial \dot{x}^i} \ .$$

Note that the complete lift of a vector field is not "tensorial" ! Indeed,

$${}^{c}(fA) = f \ {}^{c}A + \ {}^{n}df \ {}^{v}A \ .$$

Definition 3.4. Given $A \in \mathfrak{X}(M)$ and a connection ∇ on M, the horizontal lift ${}^{h}A \in \mathfrak{X}(M)$ of A (with respect to ∇) is defined for each $\mathsf{u} \in TM$ by ${}^{h}A_{\mathsf{u}} = \dot{U}(0)$, where $U : J \longrightarrow TM$ is the parallel vector field along a curve $\gamma : J \longrightarrow M$ with $U(0) = \mathsf{u}$, $\gamma(0) = m$, $\dot{\gamma}(0) = A_m$, J being an open subset of $\mathbb{R}$ with $0 \in J$.

In the presence of a chart $x = (x^i)_{1\leq i\leq n}$ with $A = A^i \dfrac{\partial}{\partial x^i}$ we find easily

$${}^{h}A|_{dom(x,\dot{x})} = A^i \frac{\partial}{\partial x^i} - \dot{x}^q \Gamma^i_{qp} A^p \frac{\partial}{\partial \dot{x}^i} \quad .$$

where $\Gamma^k_{ij} = dx^k(\nabla(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}))$.

Remark 3.1. In the presence of a Connection ∇, ${}^cA - {}^hA = {}^n\nabla A$

On $dom(x)$, instead of the local frame fields that arise from the chart $(x, \dot{x})$ on TM, we shall make consistent use of the non-holonomic frame fields

$$\mathsf{e}_i = {}^h\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \dot{x}^q\,\Gamma^r_{qi}\,\frac{\partial}{\partial \dot{x}^r} \qquad\qquad \mathsf{e}_{\bar{i}} = {}^v\frac{\partial}{\partial x^i} = \frac{\partial}{\partial \dot{x}^i}$$

for $1 \leq i \leq n$, where $\bar{i}$ stands for $i+n$. For $A \in \mathfrak{X}(M)$ with $A|_{dom(x)} = A^i\dfrac{\partial}{\partial x^i}$ we clearly have

$${}^vA|_{dom(x,\dot{x})} = A^i\mathsf{e}_{\bar{i}}\ ,\quad {}^cA|_{dom(x,\dot{x})} = A^i\mathsf{e}_i + \dot{x}^q\nabla_qA^i\mathsf{e}_i\ ,\quad {}^hA\Big|_{dom(x,\dot{x})} = A^i\mathsf{e}_i\ .$$

It should be pointed out that the Lie-Brackets of non-holonomic frame fields do not vanish, in general. By a routine computation,

$$[\mathsf{e}_i, \mathsf{e}_j] = -\,\dot{x}^qR_{ijq}{}^k\ \mathsf{e}_{\bar{k}} \quad , \quad [\mathsf{e}_i, \mathsf{e}_{\bar{j}}] = [\mathsf{e}_j, \mathsf{e}_{\bar{i}}] = -\Gamma^k_{ij}\ \mathsf{e}_{\bar{k}} \quad , \quad [\mathsf{e}_{\bar{i}}, \mathsf{e}_{\bar{j}}] = 0.$$

Definition 3.5. For any $A \in \mathfrak{X}(M)$ with $A|_{dom(x)} = A^i\dfrac{\partial}{\partial x^i}$, the modified vertical lift ${}^{v'}A \in \mathfrak{X}(TM)$ of A will be defined by

$${}^{v'}A\Big|_{dom(x,\dot{x})} A^i\mathsf{e}_{\bar{i}} - \dot{x}^q\nabla^iA_q\mathsf{e}_i$$

We shall also make use of the frame field $(\theta^\alpha)_{1\leq\alpha\leq 2n}$ dual to $(\mathsf{e}_\alpha)_{1\leq\alpha 2n}$. It can be easily checked that

$$\begin{aligned}\theta^i &= dx^i\\ \theta^{\bar{i}} &= d\dot{x}^i + \Gamma^i_{qr}\dot{x}^qdx^r\end{aligned}$$

3.3 The Sasaki Metric Tensor Field

Let M be a Riemannian manifold with tangent bundle TM. The Sasaki metric tensor field G on TM is defined by

$$\mathsf{G}\ |_{dom(x,\dot{x})} = g_{ij}\left(\theta^i \otimes \theta^j + \theta^{\bar{i}} \otimes \theta^{\bar{j}}\right)$$

Equivalently, G can be written in the form $G\ |_{dom(x,\dot{x})} = G_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$ where $G_{ij} = g_{ij}$, $G_{i\bar{j}} = G_{\bar{i}j} = 0$ and $G_{\bar{i}\bar{j}} = g_{ij}$.

Another way of introducing the Sasaki metric is the following: For every $\mathsf{u} \in TM$ we define the vertical subspace $V_{\mathsf{u}}TM$ of $T_{sfu}TM$ to be the subspace generated by vertical lifts of elements of $T_{\tau(\mathsf{u})}$ to u. Clearly, in the presence of a chart $x = (x^i)_{1\leq i\leq n}$ with $\tau(\mathsf{u}) \in \mathsf{dom}(\mathsf{x})$

$$V_{\mathsf{u}}TM = \left\langle \frac{\partial}{\partial \dot{x}^i} \mid_{\mathsf{u}} \right\rangle_{1\leq i\leq n}$$

The distribution $\mathsf{u} \longrightarrow V_{\mathsf{u}}TM$ on TM is called the vertical distribution. There is no inherent distribution that complements the vertical distribution. With respect to a connection ∇ we define the horizontal space $H_{\mathsf{u}}TM$.

$$H_{\mathsf{u}}TM = \left\langle \frac{\partial}{\partial \dot{x}^i} - \Gamma^r_{iq} u^q \frac{\partial}{\partial \dot{x}^r} \right\rangle_{1\leq i\leq n}$$

where, of course $u = u^i \dfrac{\partial}{\partial x^i} \mid_{\tau(\mathsf{u})}$ and Γ^r_{iq}'s are the Christoffel symbols of ∇ with respect to the chart x with $x \in \tau^{-1}(domx)$. H_{u} is independent of the choice of the chart x. The distribution $\mathsf{u} \longrightarrow H_{\mathsf{u}}(TM)$ on TM is called the horizontal distribution with respect to ∇. It can be checked that for all $\mathsf{u} \in T_{\mathsf{u}}TM$, $T_{\mathsf{u}}TM = V_{\mathsf{u}}TM \oplus H_{\mathsf{u}}TM$.

Remark 3.2. Notice that we could complement the (inherently defined) vertical distribution only by introducing a connection. Conversely, each choice of a distribution on TM complementing the vertical distribution (provided a simple linearity assumption is made!) arises from a connection.

Note that:

$$\begin{aligned} T_{\mathsf{u}}\tau \quad &: \quad H_{\mathsf{u}}TM \leq T_{\mathsf{u}}TM \longrightarrow T_mM \quad \text{and} \\ \Psi_{\mathsf{u}} \quad &: \quad V_{\mathsf{u}}TM \longrightarrow T_mM \\ & \quad\;\; \frac{\partial}{\partial \dot{x}^i} \longrightarrow \frac{\partial}{\partial x^i} \end{aligned}$$

for any chart x with $m \in dom(x)$ are isomorphisms. We define G_{u} on $H_{\mathsf{u}}TM$ by pulling g_m on T_mM back by $T_{\mathsf{u}}\tau$. We define G_{u} $V_{\mathsf{u}}TM$ by pulling g_m on T_mM back by Ψ_{u}. To complete the definition of G on $T_{\mathsf{u}}TM$ we declare $V_{\mathsf{u}}TM$ and $H_{\mathsf{u}}TM$ orthogonal.

Let $\Box$ be the Levi-Civita connection of the Sasaki metric tensor G on TM. The associated Christoffel symbols can be calculated from the formula

$$\Upsilon^{\gamma}_{\alpha\beta} = \theta^{\gamma}(\Box(\mathsf{e}_{\alpha}, \mathsf{e}_{\beta}))$$

Theorem 3.1.

$$\Upsilon_{ij}^{k} = \Gamma_{ij}^{k} \quad , \quad \Upsilon_{\bar{i}j}^{k} = \frac{1}{2} R_{qij}{}^{k}\ \dot{x}^q \quad , \quad \Upsilon_{i\bar{j}}^{k} = \frac{1}{2} R_{qji}{}^{k}\ \dot{x}^q \quad , \quad \Upsilon_{\bar{i}\bar{j}}^{k} = 0$$

$$\Upsilon_{ij}^{\bar{k}} = -\frac{1}{2} R_{ijq}{}^{k}\ \dot{x}^q \quad , \quad \Upsilon_{\bar{i}j}^{\bar{k}} = 0 \quad , \quad \Upsilon_{i\bar{j}}^{\bar{k}} = \Gamma_{ij}^{k} \quad , \quad \Upsilon_{\bar{i}\bar{j}}^{\bar{k}} = 0$$

Proof. Applying the formula

$$\begin{aligned} \mathsf{G}(\square(X,Y),Z) &= \frac{1}{2}\{\{X\mathsf{G}(Y,Z) + Y\mathsf{G}(X,Z) - Z\mathsf{G}(X,Y) \\ &+ \mathsf{G}([X,Y],Z) - \mathsf{G}([Y,Z],X) + \mathsf{G}([Z,X],Y)\} \end{aligned}$$

to the non-holonomic frame fields we obtain on the lefthand side of the equation

$$\begin{aligned} \mathsf{G}(\square(\mathsf{e}_i,\mathsf{e}_j),\mathsf{e}_k) &= \mathsf{G}(\Upsilon_{ij}^{r}\ \mathsf{e}_r + \Upsilon_{ij}^{\bar{r}}\ \mathsf{e}_{\bar{r}},\ \mathsf{e}_k) \\ &= \Upsilon_{ij}^{r}\ g_{rk} \end{aligned}$$

On the righthand side we obtain, $\frac{1}{2}\{\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\}$. Hence,

$$\Upsilon_{ij}^{r} = \frac{1}{2}\{\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\}g^{rk} = \Gamma_{ij}^{k}$$

The other formulas can be derived by the same method.

$$\begin{aligned} \mathsf{G}(\square(\mathsf{e}_i,\mathsf{e}_j),\mathsf{e}_{\bar{k}}) &= \mathsf{G}(\Upsilon_{ij}^{r}\ \mathsf{e}_r + \Upsilon_{ij}^{\bar{r}}\ \mathsf{e}_{\bar{r}},\ \mathsf{e}_{\bar{k}}) \\ &= \Upsilon_{ij}^{\bar{r}}\ g_{rk} \\ &= \frac{1}{2}\{\mathsf{e}_i\mathsf{G}(\mathsf{e_j},\mathsf{e_{\bar{k}}}) + \mathsf{e_j}\mathsf{G}(\mathsf{e_i},\mathsf{e_{\bar{k}}}) - \mathsf{e_{\bar{k}}}\mathsf{G}(\mathsf{e_i},\mathsf{e_j}) \\ &+ \mathsf{G}([\mathsf{e}_i,\mathsf{e_j}],\mathsf{e}_{\bar{k}}) - \mathsf{G}([\mathsf{e}_j,\mathsf{e}_{\bar{k}}],\mathsf{e}_i) + \mathsf{G}([\mathsf{e}_{\bar{k}},\mathsf{e}_i],\mathsf{e}_{\bar{j}})\} \\ &= -\frac{1}{2}\dot{x}^q\ R_{ijq}{}^{k}\ g_{rk} \end{aligned}$$

Hence, $\Upsilon_{ij}^{\bar{r}} = -\frac{1}{2}\dot{x}^q\ R_{ijq}{}^{k}$. The remaning formulae can be routinely checked. $\square$

Given a vector field $A \in \mathfrak{X}(TM)$ where $A|_{dom(x,\dot{x})} = A^i\mathsf{e}_i + A^{\bar{i}}\mathsf{e}_{\bar{i}}$ we can work out the following special cases of

$$\square_\alpha A_\beta = \mathsf{e}_\alpha A_\beta - \Upsilon_{\alpha\beta}^{\mu}A_\mu$$

as follows:

Case $\alpha = i, \beta = j$:

$$\begin{aligned}\Box_i A_j &= \mathsf{e}_i A_j - \Upsilon^k_{ij} A_k - \Upsilon^{\bar{k}}_{ij} A_{\bar{k}} \\ &= (\frac{\partial}{\partial x^i} - \dot{x}^q\ \Gamma^r_{qi}\ \frac{\partial}{\partial \dot{x}^r}) A_j - \Gamma^k_{ij} A_k + \frac{1}{2}\ R_{ijq}{}^k\ \dot{x}^q A_{\bar{k}}\end{aligned}$$

Case $\alpha = \bar{i}, \beta = j$:

$$\begin{aligned}\Box_{\bar{i}} A_j &= \mathsf{e}_{\bar{i}} A_j - \Upsilon^k_{\bar{i}j} A_k - \Upsilon^{\bar{k}}_{\bar{i}j} A_{\bar{k}} \\ &= \frac{\partial A_j}{\partial \dot{x}^i} - \frac{1}{2}\ R_{qij}{}^k\ \dot{x}^q A_k\end{aligned}$$

Case $\alpha = i, \beta = \bar{j}$:

$$\begin{aligned}\Box_i A_{\bar{j}} &= \mathsf{e}_i A_{\bar{j}} - \Upsilon^k_{i\bar{j}} A_k - \Upsilon^{\bar{k}}_{i\bar{j}} A_{\bar{k}} \\ &= (\frac{\partial}{\partial x^i} - \dot{x}^q\ \Gamma^r_{qi}\ \frac{\partial}{\partial \dot{x}^r}) A_{\bar{j}} - \frac{1}{2}\ R_{qji}{}^k\ \dot{x}^q A_k - \Gamma^k_{ij} A_{\bar{k}}\end{aligned}$$

Case $\alpha = \bar{i}, \beta = \bar{j}$:

$$\begin{aligned}\Box_{\bar{i}} A_{\bar{j}} &= \mathsf{e}_{\bar{i}} A_{\bar{j}} - \Upsilon^k_{\bar{i}\bar{j}} A_k - \Upsilon^{\bar{k}}_{\bar{i}\bar{j}} A_{\bar{k}} \\ &= \frac{\partial A_{\bar{j}}}{\partial \dot{x}^i}\end{aligned}$$

Chapter 4

Killing Vector Fields with respect to the Sasaki Metric Tensor Field

Definition 4.1. On M with Riemannian metric tensor field g we shall call $\mathsf{P} \in \mathfrak{X}^{1,1}(M)$ *skew symmetric* with respect to g if

$$\mathsf{g}(\mathsf{P}(X), Y) = \mathsf{g}(X, \mathsf{P}(Y))$$

for all $X, Y \in \mathfrak{X}(M)$.

In the presence of a chart $x = (x)_{1 \leq i \leq n}$, if

$$\mathsf{P}|_{dom(x)} = P_i^{\ j} dx^i \otimes \frac{\partial}{\partial x^j}$$

the above condition assumes the very simple local form

$$P_{ij} + P_{ji} = 0$$

using the index raising and lowering convention. Notice that a skew symmetric P has the effect of an infinitesimal Euclidean rotation on each $T_m M$ with respect to the Euclidean geometry induced by the inner product g_m. In this chapter, as well as in the following we shall make use of a tensor field introduced by S. Tanno [12].

Definition 4.2. To each vector field $B \in \mathfrak{X}(M)$, let the tensor field $\mathfrak{T}_B \in \mathfrak{X}^{0,4}(M)$ be defined by

$$\mathfrak{T}_B(X, Y, Z, W) = \mathsf{G}(\nabla((R(X, Z, W), B), Y)$$

If $B|_{dom(x)} = B^i \dfrac{\partial}{\partial x^i}$ in the presence of a chart $x = (x)_{1\leq i\leq n}$, then

$$\mathfrak{T}_B|_{dom(x)} = R_{qij}{}^{p} \ \nabla_p B_r \ \ dx^q \otimes dx^i \otimes dx^j \otimes dx^r \ .$$

A vector field $A \in \mathfrak{X}(TM)$ where

$$A \ |_{dom(x,\dot{x})} = A^i \frac{\partial}{\partial x^i} + A^{\bar{i}} \frac{\partial}{\partial x^{\bar{i}}} \ .$$

is a Killing vector field iff

$$\Box_\alpha A_\beta + \Box_\beta A_\alpha = 0 \tag{4.1}$$

for each choice of chart x on M.

The equation (4.1) reduces in the respective cases $\alpha = i, \beta = j$ and $\alpha = \bar{i}, \beta = j$ and $\alpha = \bar{i}, \beta = \bar{j}$ to

$$\frac{\partial A_j}{\partial x^i} - \Gamma^p_{ij} A_p + \frac{\partial A_i}{\partial x^j} - \Gamma^p_{ji} A_p - \dot{x}^q \left(\Gamma^r_{qi} \ \frac{\partial A_j}{\partial \dot{x}^r} + \Gamma^r_{qj} \ \frac{\partial A_i}{\partial \dot{x}^r} \right) = 0 \tag{4.2}$$

and

$$\frac{\partial A_j}{\partial \dot{x}^i} + \frac{\partial A_{\bar{i}}}{\partial x^j} - \Gamma^p_{ij} A_{\bar{p}} - \dot{x}^q \left(R_{qij}{}^{p} \ A_p + \Gamma^r_{qj} \ \frac{\partial A_{\bar{i}}}{\partial \dot{x}^r} \right) = 0 \tag{4.3}$$

and

$$\frac{\partial A_{\bar{j}}}{\partial \dot{x}^i} + \frac{\partial A_{\bar{i}}}{\partial \dot{x}^j} = 0 \tag{4.4}$$

From (4.4) we immediately conclude that

$$A_{\bar{i}} = Y_i + \dot{x}^q H_{qi}$$

where Y_i and H_{qi} are "vertically constant " functions on $dom(x, \dot{x})$ and

$$H_{ij} + H_{ji} = 0$$

Substituting this expression for $A_{\bar{i}}$ in (4.3) and regrouping we obtain

$$\frac{\partial A_j}{\partial \dot{x}^i} + \nabla_j Y_i + \dot{x}^q \left(-R_{qij}{}^{p} \ A_p + \nabla_j H_{qi} \right) = 0 \tag{4.5}$$

Lemma 4.1. *$A_j s$ in (4.5) is an analytic function of $\dot{x}^i s$.*

Proof. Regarding x^i's as constants, the above assertion is seen to be a special case of the following : Given analtic functions K_{ijq}, L_{ij} where $1 \leq i, j, q \leq N$, of $z = (z^1, \ldots, z^N) \in \Omega \subseteq \mathbb{R}^N$, defined on the open set Ω, each solution $F^i : \Omega \to \mathbb{R}$ of the system

$$\frac{\partial F^i}{\partial z^j} = K_{ijq} F^q + L_{ij}$$

is analytic in $z = (z^1, \ldots, z^N)$. To see this, consider a point $a = (a^1 \ldots a^N) \in \Omega$ and take any convex neighbourhood V of a in Ω . Putting

$$\varphi^i = \varphi^i_z(t) = F^i((1-t)a + tz)$$

for each $z = (z^1 \ldots z^N) \in V$ we obtain

$$\begin{aligned} \frac{d\varphi^i}{dt} &= \frac{\partial F^i}{\partial z^j}(z^j - a^j) \\ &= (z^j - a^j) K_{ijq}((1-t)a + tz)\varphi^q(t) + (z^j - a^j) L_{ij}((1-t)a + tz) \end{aligned}$$

and hence we find that for each $1 \leq i \leq N$, φ^i is the solution of

$$\begin{aligned} \frac{d\varphi^i}{dt} &= k_{iq}\varphi^q + \rho_i \\ \varphi^i(0) &= F^i(a) \end{aligned}$$

where the functions $k_{iq} = k_{iq,z}(t)$, $\rho_i = \rho_{i,z}(t)$ defined on an open interval containing $[0,1] \subseteq \mathbb{R}$ depend analtically on the parameter $z = (z^1, \ldots, z^N)$. Since the above system is linear it has a solution on an open interval containing $[0,1]$ that depends analytically on $z = (z^1, \ldots, z^N)$. Consequently, $F^i(z^1, \ldots, z^N) = \varphi^i(1)$ is an analytic function of $z = (z^1, \ldots, z^N) \in V$. $\square$

Lemma 4.2.

$$\frac{\partial^N A_j}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \cdots \partial \dot{x}^{i_N}} = \frac{1}{N} \sum_{k=1}^{N} \dot{x}^q \; R_{q i_k j}{}^p \frac{\partial^{N-1} A_p}{\partial \dot{x}^{i_1} \cdots \partial \dot{x}^{i_{k-1}} \partial \dot{x}^{i_{k+1}} \cdots \partial \dot{x}^{i_N}}$$

for all $N \geq 2$ *. In particular*

$$\left. \frac{\partial^N A_j}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \cdots \partial \dot{x}^{i_N}} \right|_{\dot{x}=0} = 0$$

for all $N \geq 2$ *.*

Proof. Assuming $A_j \in \mathfrak{F}(TM)$ we first differentiate (4.5) with respect to $\dot{x}^r$.

$$\frac{\partial^2 A_j}{\partial \dot{x}^r \dot{x}^i} - \ R_{rij}{}^p \ A_p + \nabla_j H_{ri} - \dot{x}^q \ R_{qji}{}^p \ \frac{\partial A_p}{\partial \dot{x}^r} = 0$$

Interchanging i and r we find

$$\frac{\partial A_j}{\partial \dot{x}^i \dot{x}^r} - \ R_{irj}{}^p \ A_p + \nabla_j H_{ir} - \dot{x}^q \ R_{qrj}{}^p \ \frac{\partial A_p}{\partial \dot{x}^i} = 0$$

Adding up these equations, in view of the antisymmetry of R in its first two arguments and that of H we obtain,

$$\frac{\partial^2 A_j}{\partial \dot{x}^r \dot{x}^i} = \frac{1}{2}\dot{x}^q \left(R_{qij}{}^p \frac{\partial A_p}{\partial \dot{x}^r} + R_{qrj}{}^p \frac{\partial A_p}{\partial \dot{x}^i} \right)$$

Now, differentiating this equation with respect to $\dot{x}^s$ we get,

$$\frac{\partial^3 A_j}{\partial \dot{x}^s \partial \dot{x}^r \partial \dot{x}^i} = \frac{1}{2} \left(R_{sij}{}^p \frac{\partial A_p}{\partial \dot{x}^r} + R_{srj}{}^p \frac{\partial A_p}{\partial \dot{x}^i} \right) + \frac{1}{2}\dot{x}^q \left(R_{qij}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^r} + R_{qrj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^i} \right)$$

Permutating i, s, t cyclically and adding up the resulting three equations we obtain

$$\frac{\partial^3 A_j}{\partial \dot{x}^s \partial \dot{x}^r \partial \dot{x}^i} = \frac{\dot{x}^q}{3} \left(R_{qij}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^r} + R_{qrj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^i \partial \dot{x}^s} + R_{qij}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^r} + R_{qsj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^r \partial \dot{x}^i} \right)$$

Therefore, the assertion true for $N = 3$. Differentiating that identity with respect to $\dot{x}^{i_{N+1}}$ we obtain

$$\begin{aligned} \frac{\partial^{N+1} A_j}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}} &= \frac{1}{N} \sum_{k=1}^{N} R_{i_{N+1} i_k j}{}^p \ \frac{\partial^{N-1} A_p}{\partial \dot{x}^{i_1} \cdots \widehat{\partial \dot{x}^{i_k}} \cdots \partial \dot{x}^{i_N}} \\ &+ \frac{\dot{x}^q}{N} \sum_{k=1}^{N} R_{q i_k j}{}^p \ \frac{\partial^N A_p}{\partial \dot{x}^{i_1} \cdots \widehat{\partial \dot{x}^{i_k}} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}} \end{aligned}$$

Permuting $i_1, i_2, \cdots, i_N, i_{N+1}$ cyclically and adding up the resulting equations will give rise to

$$\frac{\partial^{N+1} A_j}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}} = \frac{\dot{x}^q}{N+1} \sum_{k=1}^{N+1} R_{q i_k j}{}^p \ \frac{\partial^N A_p}{\partial \dot{x}^{i_1} \cdots \widehat{\partial \dot{x}^{i_k}} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}}$$

which was to be proven. $\square$

Since A_i is "vertically analytic" we find that

$$A_i = X_i + \dot{x}^q V_{qi}$$

where X_i and V_{qi} are "vertically constant " functions on $dom(x, \dot{x})$. Substituting these expressions in (4.2) and (4.3) respectively we conclude that

$$V_{ij} + \nabla_j Y_i + (-\ R_{qij}{}^p X_p + \nabla_j H_{qi})\dot{x}^q + (-\ R_{qij}{}^p V_{rp})\dot{x}^q \dot{x}^r = 0 \tag{4.6}$$

$$\nabla_i X_j + \nabla_j X_i + \dot{x}^q(\nabla_i V_{qj} + \nabla_j V_{qi}) = 0 \tag{4.7}$$

Lemma 4.3. *If* $\nabla_i B_j + \nabla_j B_i = 0$, *then* $R_{kji}{}^p B_p = -\nabla_i \nabla_j B_k$.

Proof. Let

$$\begin{aligned} H_{ijk} &= \nabla_i \nabla_j B_k + R_{kji}{}^p B_p \\ &= \nabla_i \nabla_j B_k + R_{kjip} B^p \\ &= \nabla_i \nabla_j B_k - \nabla_k \nabla_j B_i + \nabla_j \nabla_k B_i \\ &= \nabla_i \nabla_j B_k + \nabla_k \nabla_i B_j + \nabla_j \nabla_k B_i \end{aligned}$$

Furthermore

$$\begin{aligned} H_{ijk} &= \nabla_i \nabla_j B_k + \nabla_k \nabla_i B_j + \nabla_j \nabla_k B_i \\ &= -\nabla_i \nabla_k B_j - \nabla_k \nabla_j B_i - \nabla_j \nabla_i B_k \\ &= -H_{jik} \end{aligned}$$

yet

$$\begin{aligned} H_{ijk} &= \nabla_i \nabla_j B_k + \nabla_k \nabla_i B_j + \nabla_j \nabla_k B_i \\ &= \nabla_j \nabla_i B_k + \nabla_i \nabla_k B_j + \nabla_k \nabla_j B_i + (R_{ijkp} + R_{jkip} + R_{kijp}) B^p = H_{jik} \end{aligned}$$

from which we conclude that $H_{ijk} = 0$. □

Theorem 4.1. *(S. Tanno [12])* $A \in \mathfrak{X}(TM)$ *is a Killing vector field for* G *iff*

$$A = {}^c X + {}^{v'} Y + {}^n \mathsf{P}$$

where $X, Y \in \mathfrak{X}(M), \mathsf{P} \in \mathfrak{X}^{1,1}(M)$ *such that*

1. *X is a Killing vector with respect to* g ,

2. *The tensor fields* $\nabla\nabla Y \in \mathfrak{X}^{1,2}(M)$ *and and* $\mathfrak{T}_Y \in \mathfrak{X}^{0,4}(M)$ *are antisymmetric in the first two components,*

3. P *is a parallel tensor field that is skew symmetric with respect to* g.

Proof. We know that

$$A|_{dom(x,\dot{x})} = \left(X^i + \dot{x}^q V_q{}^i\right) \mathsf{e}_i + \left(Y^i + \dot{x}^q H_q{}^i\right) \mathsf{e}_{\bar{i}}$$

such that employing the above lemma and considering the coefficients of $\dot{x}^q$ and $\dot{x}^q\dot{x}^r$ in the equations (4.6) and (4.7), we find that $V_{ij} = -\nabla_j Y_i$ and

$$\nabla_i X_j + \nabla_j X_i = 0$$

$$\nabla_i \nabla_j Y_q + \nabla_j \nabla_i Y_q = 0$$

$$\nabla_j \left(\nabla_i X_q + H_{qi}\right) = 0$$

$$R_{qijp} \nabla^p Y_r + R_{rijp} \nabla^p Y_q = 0.$$

Putting $P_{ij} = \nabla_i X_j + H_{ji}$ we find

$$A|_{dom(x,\dot{x})} = \left(X^i - \dot{x}^q \nabla^i Y_q\right) \mathsf{e}_i + \left(Y^i + \dot{x}^q (P^i{}_q - \nabla^i X_q)\right) \mathsf{e}_{\bar{i}}$$

and in view of $\nabla^i X_q = -\nabla_q X^i$

$$A|_{dom(x,\dot{x})} = \left(X^i \mathsf{e}_i + \dot{x}^q \nabla_q X^i \mathsf{e}_{\bar{i}}\right) + Y^i \mathsf{e}_{\bar{i}} + \dot{x}^q P^i{}_q \mathsf{e}_{\bar{i}} - \dot{x}^q \nabla^i Y_q \mathsf{e}_i$$

□

The vague and intuitively rather unappealing condition (2) is drastically simplified on a compact manifold. Indeed when M is compact,

Corollary 4.1. *(S. Tanno[12]) If M is compact and orientable, $A \in \mathfrak{X}(TM)$ is a Killing vector field with respect to the Sasaki metric tensor field on TM iff*

$$A = {}^c X + {}^v Y + {}^n \mathsf{P}$$

with $X, Y \in \mathfrak{X}(M)$, $\mathsf{P} \in \mathfrak{X}^{1,1}(M)$ *where X is a Killing vector field on M, Y and* P *are parallel and* $\mathsf{g}(\mathsf{P}(.), .)$ *is antisymmetric.*

Proof. Since $\nabla\nabla Y$ is antisymmetric, it can be immediately seen that $\nabla_i \nabla^i Y_q = \nabla^i \nabla_i Y_q = 0$ and

$$\| \nabla Y \|^2\Big|_{dom(x)} = \nabla_i Y_j \nabla^i Y^j = \nabla_i (Y_j \nabla^i Y^j) \quad .$$

Consequently, $\partial \nabla Y \partial^2 = \mathrm{div} B$ where $B|_{dom(c)} = Y_j \nabla^i Y^j \dfrac{\partial}{\partial x^i}$ and thus

$$\int_M \| \nabla Y \|^2 \, \mathrm{vol}_{\mathsf{g}} = \int_M \mathrm{div} B \, \mathrm{vol}_{\mathsf{g}} = 0$$

which shows that $\nabla Y = 0$. $\square$

Chapter 5

Conformal Vector Fields with respect to the the Sasaki Metric Tensor Field

We shall be particularly interested in the natural lift ${}^{n}\alpha \in \mathfrak{F}(TM)$ of a covector field $\alpha \in \mathfrak{X}^*(TM)$ and ${}^{n}\mathsf{I} \in \mathfrak{X}(TM)$ for the identity tensor field $\mathsf{I} \in \mathfrak{X}^{1,1}(M)$. Clearly

$$ {}^{n}\alpha|_{dom(x,\dot{x})} = \dot{x}^q \alpha_q $$

where $\alpha|_{dom(x)} = \alpha_i dx^i$ and

$$ {}^{n}\mathsf{I}|_{dom(x,\dot{x})} = \dot{x}^q \mathsf{e}_{\bar{q}} \quad . $$

Also the natural scalar field $E \in \mathfrak{F}(TM)$ defined by

$$ E|_{dom(x,\dot{x})} = \frac{1}{2}\; g_{qr}\; \dot{x}^q \dot{x}^r $$

will make its appearance in our calculations.

The symbol $\mathfrak{S}_{i,j,k}$ will indicate summation under cyclic permutation of i, j, k in the given order.

5.1 Statement of Results

For each $B \in \mathfrak{X}(M)$ we shall consider its dual $B_\flat = \mathsf{g}(B, \bullet) \in \mathfrak{X}^*(M)$. Clearly,

$$ B_\flat|_{dom(x)} = g_{ij}\; B^j\; dx^i = B_i\; dx^i $$

where

$$B|_{dom(x)} = B^i \frac{\partial}{\partial x^i} \quad .$$

We shall also make use of a rather unusual if very straightforward extension of the gradient and divergence operators to arbitrary tensor fields. If $\mathsf{T} \in \mathfrak{X}^{p,q}(M)$ with

$$\mathsf{T}|_{dom(x)} = T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}$$

then $grad(\mathsf{T}) \in \mathfrak{X}^{p+1,q}(M)$ and $div(\mathsf{T}) \in \mathfrak{X}^{p-1,q}(M)$ are defined by

$$\begin{aligned} grad(\mathsf{T})|_{dom(x)} &= \nabla^k T^{i_1 \cdots i_p}_{j_1 \ldots j_q} \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q} \\ div(\mathsf{T})|_{dom(x)} &= \nabla_k T^{k i_2 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}. \end{aligned}$$

We employ the natural method proposed by S. Tanno with some modifications and prove the following result :

Theorem 5.1. *Given a Riemannian manifold (M, g) with $n = dim(M) \geq 3$, $A \in \mathfrak{X}(TM)$ is a conformal vector field with respect to the Sasaki metric tensor field on TM iff*

$$A = {}^cX + {}^{v'}Y + {}^n\mathsf{P} - E\ {}^cgrad\ (b) + E\ {}^v\mathsf{a} + {}^n\mathsf{a}_\flat\ {}^n\mathsf{I} - \frac{1}{3}\ E\ {}^ngrad\ (\mathsf{a}_\flat)$$

where $X, Y \in \mathfrak{X}(M)$, $\mathsf{P} \in \mathfrak{X}^{1,1}(M)$ and

$$\begin{aligned} b = \frac{1}{n}\ div(X) &\quad \in \quad \mathfrak{F}(M) \\ \mathsf{a} = -\frac{1}{n}\ div(grad(Y)) &\quad \in \quad \mathfrak{X}(M) \end{aligned}$$

such that

1. *X is a conformal vector field with respect to g and*

$$\nabla\nabla b = 0\ ,$$

2. *the tensor fields*

$$\begin{aligned} \nabla\nabla Y + \mathsf{g} \otimes \mathsf{a} &\in \mathfrak{X}^{1,2}(M) \\ \nabla\nabla \mathsf{a} &\in \mathfrak{X}^{1,2}(M) \\ \mathfrak{T}_Y + \frac{2}{3}\ \mathsf{g} \otimes \nabla \mathsf{a}_\flat - \frac{2}{3}\ (\mathsf{g} \otimes \nabla \mathsf{a}_\flat)_{[2,4]} &\in \mathfrak{X}^{0,4}(M) \\ \mathfrak{T}_\mathsf{a} &\in \mathfrak{X}^{0,4}(M) \end{aligned}$$

are antisymmetric in the first two arguments , *3.* P *is skew symmetric with respect to* g *and satisfies*

$$\nabla \mathsf{P} = \ \mathsf{I} \otimes db - \mathsf{g} \otimes grad\ b \quad .$$

This is admittedly not quite pretty. However, it assumes a simple and surprising form on compact manidifolds.

Theorem 5.2. *Let* (M, g) *be a* <u>*compact*</u> *Riemannian manifold with* $n = dim(M) \geq 3$ *. A vector field* $A \in \mathfrak{X}(TM)$ *, is a conformal vector field with respect to the Sasaki metric tensor field on* TM *iff it is a Killing vector field with respect to the Sasaki metric tensor field on* TM*.*

We note once again that the above results are valid for manifolds of dimension at least three. Two dimensional manifolds present certain peculiarities and they will be treated elsewhere.

5.2 Proofs of the Results

<u>Case $\alpha = i, \beta = j$:</u>

$$\Box_i A_j = \mathsf{e}_i A_j - \Upsilon^k_{ij} A_k - \Upsilon^{\bar{k}}_{ij} A_{\bar{k}} = (\frac{\partial}{\partial x^i} - \dot{x}^q\ \Gamma^r_{qi}\ \frac{\partial}{\partial \dot{x}^r}) A_j - \Gamma^k_{ij} A_k + \frac{1}{2}\ R_{ijq}{}^k\ \dot{x}^q A_{\bar{k}}$$

<u>Case $\alpha = \bar{i}, \beta = j$:</u>

$$\Box_{\bar{i}} A_j = \mathsf{e}_{\bar{i}} A_j - \Upsilon^k_{\bar{i}j} A_k - \Upsilon^{\bar{k}}_{\bar{i}j} A_{\bar{k}} = \frac{\partial A_j}{\partial \dot{x}^i} - \frac{1}{2}\ R_{qij}{}^k\ \dot{x}^q A_k$$

<u>Case $\alpha = i, \beta = \bar{j}$:</u>

$$\Box_i A_{\bar{j}} = \mathsf{e}_i A_{\bar{j}} - \Upsilon^k_{i\bar{j}} A_k - \Upsilon^{\bar{k}}_{i\bar{j}} A_{\bar{k}} = (\frac{\partial}{\partial x^i} - \dot{x}^q\ \Gamma^r_{qi}\ \frac{\partial}{\partial \dot{x}^r}) A_{\bar{j}} - \frac{1}{2}\ R_{qji}{}^k\ \dot{x}^q A_k - \Gamma^k_{ij} A_{\bar{k}}$$

<u>Case $\alpha = \bar{i}, \beta = \bar{j}$:</u>

$$\Box_{\bar{i}} A_{\bar{j}} = \mathsf{e}_{\bar{i}} A_{\bar{j}} - \Upsilon^k_{\bar{i}\bar{j}} A_k - \Upsilon^{\bar{k}}_{\bar{i}\bar{j}} A_{\bar{k}} = \frac{\partial A_{\bar{j}}}{\partial \dot{x}^i}$$

A vector field $A \in \mathfrak{X}(TM)$ where $A|_{dom(x,\dot{x})} = A^\alpha \mathsf{e}_\alpha = A^i \mathsf{e}_i + A^{\bar{i}} \mathsf{e}_{\bar{i}}$ is a conformal vector field with respect to G iff

$$\Box_\alpha A_\beta + \Box_\beta A_\alpha = 2\ \sigma\ G_{\alpha\beta} \tag{5.1}$$

for each choice of chart x and $\alpha, \beta \in \{i, \bar{i}\}$, where $G_{ij} = G_{\bar{i}\bar{j}} = g_{ij}$ and $G_{i\bar{j}} = G_{\bar{i}j} = 0$ for all $1 \le i, j \le n$. The equation 5.1 reduces in the respective cases $(\alpha, \beta) = (i, j)$, $(\alpha, \beta) = (\bar{i}, j)$, $(\alpha, \beta) = (\bar{i}, \bar{j})$ to

$$\frac{\partial A_j}{\partial x^i} - \Gamma^p_{ij} A_p + \frac{\partial A_i}{\partial x^j} - \Gamma^p_{ji} A_p - \dot{x}^q \left(\Gamma^r_{qi} \frac{\partial A_j}{\partial \dot{x}^r} + \Gamma^r_{qj} \frac{\partial A_i}{\partial \dot{x}^r} \right) = 2\, \sigma\, g_{ij} \tag{5.2}$$

$$\frac{\partial A_j}{\partial \dot{x}^i} + \frac{\partial A_{\bar{i}}}{\partial x^j} - \Gamma^p_{ij} A_{\bar{p}} - \dot{x}^q \left(R_{qij}{}^p\ A_p + \Gamma^r_{qj} \frac{\partial A_{\bar{i}}}{\partial \dot{x}^r} \right) = 0 \tag{5.3}$$

$$\frac{\partial A_{\bar{j}}}{\partial \dot{x}^i} + \frac{\partial A_{\bar{i}}}{\partial \dot{x}^j} = 2\, \sigma\, g_{ij} \tag{5.4}$$

Treating functions of x^i's as constants it is seen that 5.4 is nothing but the equation 2.5 in unknowns $\dot{x}^i$'s and

$$\sigma = a_q \dot{x}^q + b \tag{5.5}$$

$$A_{\bar{i}} = Y_i + \dot{x}^q H_{qi} + \frac{1}{2} \dot{x}^q \dot{x}^r S_{qri} \tag{5.6}$$

where $a_q, b, Y_j, H_{qi}, S_{qri} \in \mathfrak{F}(dom(x, \dot{x}))$ are functions of x^is alone and

$$H_{qr} + H_{rq} = 2\ b\ g_{qr} \tag{5.7}$$

$$S_{qri} = a_q g_{ri} + a_r g_{qi} - a_i g_{qr} \quad . \tag{5.8}$$

We avail ourselves of the conventional abuse of notation and understand a_q, $b, Y_j, H_{qi}, S_{qri} \in \mathfrak{F}(dom(x))$. In fact these functions are clearly components of tensor fields restricted to $dom(x)$. In particular there exist $Y \in \mathfrak{X}(M)$ and $\mathsf{H} \in \mathfrak{X}^{1,1}(M)$ such that

$$Y|_{\ dom(x)} = Y^i \frac{\partial}{\partial x^i}$$

and

$$\mathsf{H}|_{\ dom(x)} = H_q{}^i\ \frac{\partial}{\partial x^i} \otimes dx^q \quad .$$

Substituting 5.5 and 5.6 in the equation 5.3 and regrouping we obtain

$$\frac{\partial A_j}{\partial \dot{x}^i} + \nabla_j Y_i + \dot{x}^q \left(\nabla_j H_{qi} - R_{qij}{}^k\ A_k \right) + \frac{1}{2} \dot{x}^q \dot{x}^r \nabla_j S_{qri} = 0 \tag{5.9}$$

Lemma 5.1. *A_j in 5.9 is an analytic function of $\dot{x}^i$s.*

Proof. Regarding x^i s as constants, the above assertion is seen to be a special case of the following : Given analytic functions $K^i{}_{jq}, L^i{}_j$ where $1 \le i, j, q \le N$, of

$z=(z^1,\ldots,z^N)\in\Omega\subseteq\mathbb{R}^N$, defined on the open set Ω, each solution $F^i:\Omega\to\mathbb{R}$ of the system

$$\frac{\partial F^i}{\partial z^j}={K^i}_{jq}F^q+{L^i}_j$$

is analytic in $z=(z^1,\ldots,z^N)$. To see this, consider a point $a=(a^1\ldots a^N)\in\Omega$ and take any convex neighbourhood V of a in Ω . Putting

$$\varphi^i=\varphi^i_z(t)=F^i((1-t)a+tz)$$

for each $z=(z^1\ldots z^N)\in V$ we obtain

$$\begin{aligned}\frac{d\varphi^i}{dt}&=\frac{\partial F^i}{\partial z^j}(z^j-a^j)\\&=(z^j-a^j){K^i}_{jq}((1-t)a+tz)\varphi^q(t)+(z^j-a^j){L^i}_j((1-t)a+tz)\end{aligned}$$

and hence we find that for each $1\le i\le N$, φ^i is the solution of

$$\begin{aligned}\frac{d\varphi^i}{dt}&=&{k^i}_q\varphi^q+\rho^i\\ \varphi^i(0)&=&F^i(a)\end{aligned}$$

where the functions ${k^i}_q={k^i}_{q,z}(t)$, $\rho^i=\rho^i_z(t)$ defined on an open interval containing $[0,1]\subseteq\mathbb{R}$ depend analtically on the parameter $z=(z^1,\ldots,z^N)$. Since the above system is linear, it has a solution on an open interval containing $[0,1]$ that depends analytically on $z=(z^1,\ldots,z^N)$. Consequently, $F^i(z^1,\ldots,z^N)=\varphi^i(1)$ is an analytic function of $z=(z^1,\ldots,z^N)\in V$. □

Lemma 5.2.

$$\frac{\partial^N A_j}{\partial\dot{x}^{i_1}\partial\dot{x}^{i_2}\cdots\partial\dot{x}^{i_N}}=\frac{\dot{x}^q}{N}\sum_{k=1}^{N}{R_{qi_kj}}^{p}\frac{\partial^{N-1}A_p}{\partial\dot{x}^{i_1}\cdots\widehat{\partial\dot{x}^{i_k}}\cdots\partial\dot{x}^{i_N}}$$

for all $N\ge4$. *In particular*

$$\left.\frac{\partial^N A_j}{\partial\dot{x}^{i_1}\partial\dot{x}^{i_2}\cdots\partial\dot{x}^{i_N}}\right|_{\dot{x}=0}=0$$

for all $N\ge4$.

Proof. Assuming $A_j\in\mathfrak{F}(TM)$ we first differentiate 5.9 with respect to $\dot{x}^s$ to obtain

$$\frac{\partial^2A_j}{\partial\dot{x}^i\partial\dot{x}^s}+(\nabla_jH_{si}-{R_{sij}}^pA_p)+\dot{x}^q\left(-{R_{qij}}^p\frac{\partial A_p}{\partial\dot{x}^s}+\nabla_jS_{sqi}\right)=0$$

in view the symmetry of S_{sqi} in the first two indices. Interchanging i and s we find

$$\frac{\partial^2 A_j}{\partial \dot{x}^s \partial \dot{x}^i} + (\nabla_j H_{is} - R_{isj}{}^p A_p) + \dot{x}^q \left(-R_{qsj}{}^p \frac{\partial A_p}{\partial \dot{x}^i} + \nabla_j S_{iqs} \right) = 0 \ .$$

Adding up these equations, we obtain

$$\frac{\partial^2 A_j}{\partial \dot{x}^i \partial \dot{x}^s} = \frac{\dot{x}^q}{2} \left(R_{qij}{}^p \frac{\partial A_p}{\partial \dot{x}^s} + R_{qsj}{}^p \frac{\partial A_p}{\partial \dot{x}^i} \right) - \frac{1}{2} \dot{x}^q (\nabla_j S_{sqi} + \nabla_j S_{iqs}) - \nabla_j \ b \ g_{is}$$

in view of 5.5 and the antisymmetry of the Riemannian curvature tensor in the first two arguments. Differentiating this equation with respect to $\dot{x}^t$ we find

$$\begin{aligned} \frac{\partial^3 A_j}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^t} &= \frac{1}{2} \left(R_{tij}{}^p \frac{\partial A_p}{\partial \dot{x}^s} + R_{tsj}{}^p \frac{\partial A_p}{\partial \dot{x}^i} \right) \\ &+ \frac{\dot{x}^q}{2} \left(R_{qij}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^t} + R_{qsj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^i \partial \dot{x}^t} \right) - \frac{1}{2} (\nabla_j S_{sti} + \nabla_j S_{its}) \ . \end{aligned}$$

Permutating i, s, t cyclically and adding up the resulting three equations we obtain

$$\begin{aligned} \frac{\partial^3 A_j}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^t} &= \frac{\dot{x}^q}{3} \left(R_{qij}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^t} + R_{qsj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^i \partial \dot{x}^t} + R_{qtj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^i \partial \dot{x}^s} \right) \\ &- \frac{1}{3} (\nabla_j S_{ist} + \nabla_j S_{sti} + \nabla_j S_{tis}) \ . \end{aligned}$$

Now, differentiate this with respect to $\dot{x}^r$ to get

$$\begin{aligned} \frac{\partial^4 A_j}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^t \partial \dot{x}^r} &= \frac{1}{3} \left(R_{rij}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^s \partial \dot{x}^i} + R_{rsj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^i \partial \dot{x}^t} + R_{rtj}{}^p \frac{\partial^2 A_p}{\partial \dot{x}^i \partial \dot{x}^s} \right) \\ &+ \frac{\dot{x}^q}{3} \left(R_{qij}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^s \partial \dot{x}^t \partial \dot{x}^r} + R_{qsj}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^i \partial \dot{x}^t \partial \dot{x}^r} + R_{qtj}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^r} \right) \end{aligned}$$

Finally, permutating i, s, t, r cyclically and adding up the resulting four equations we obtain

$$\begin{aligned} \frac{\partial^4 A_j}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^t \partial \dot{x}^r} &= \frac{\dot{x}^q}{4} \left(R_{qij}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^s \partial \dot{x}^t \partial \dot{x}^r} + R_{qsj}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^i \partial \dot{x}^t \partial \dot{x}^r} \right. \\ &+ \left. R_{qtj}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^r} + R_{qrj}{}^p \frac{\partial^3 A_p}{\partial \dot{x}^i \partial \dot{x}^s \partial \dot{x}^t} \right) \ . \end{aligned}$$

Therefore, the assertion true for $N = 4$. Differentiating the asserted identity for N with respect to $\dot{x}^{i_{N+1}}$ we obtain

$$\begin{aligned} \frac{\partial^{N+1} A_j}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}} &= \frac{1}{N} \sum_{k=1}^{N} R_{i_{N+1} i_k j}{}^p \frac{\partial^{N-1} A_p}{\partial \dot{x}^{i_1} \cdots \widehat{\partial \dot{x}^{i_k}} \cdots \partial \dot{x}^{i_N}} \\ &+ \frac{\dot{x}^q}{N} \sum_{k=1}^{N} R_{q i_k j}{}^p \frac{\partial^N A_p}{\partial \dot{x}^{i_1} \cdots \widehat{\partial \dot{x}^{i_k}} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}} \end{aligned}$$

Permuting $i_1, i_2, \cdots, i_N, i_{N+1}$ cyclically and adding up the resulting equations will give rise to

$$\frac{\partial^{N+1} A_j}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}} = \frac{\dot{x}^q}{N+1} \sum_{k=1}^{N+1} R_{q i_k j}{}^{p} \frac{\partial^N A_p}{\partial \dot{x}^{i_1} \cdots \widehat{\partial \dot{x}^{i_k}} \cdots \partial \dot{x}^{i_N} \partial \dot{x}^{i_{N+1}}}$$

which was to be proven. □

Lemma 5.3. *If $\nabla_i X_j + \nabla_j X_i = 2\ b\ g_{ij}$, then*

$$R_{qij}{}^{p} X_p = \nabla_j \nabla_q X_i - g_{ji} \nabla_q b - g_{qi} \nabla_j b + g_{jq} \nabla_i b \tag{5.10}$$

Proof. Let $N_{ijq} = R_{qij}{}^{p} X_p - \nabla_j \nabla_q X_i$ and notice that

$$\begin{aligned}
N_{ijq} &= -\nabla_q \nabla_i X_j + \nabla_i \nabla_q X_j - \nabla_j \nabla_q X_i \\
&= -\nabla_q \nabla_i X_j + \nabla_i (-\nabla_j X_q + 2\ g_{jq}\ b) - \nabla_j \nabla_q X_i \\
&= -\mathfrak{S}_{i,j,q} (\nabla_q \nabla_i X_j) + 2\ g_{jq}\ \nabla_i b \\
&= \ \mathfrak{S}_{i,j,q} (R_{qij}{}^{p} X_p - \nabla_i \nabla_q X_j) + 2\ g_{jq} \nabla_i b \\
&= -\mathfrak{S}_{i,j,q} (\nabla_i \nabla_q X_j) + 2\ g_{jq}\ \nabla_i b \\
&= N_{iqj}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
N_{ijq} &= -\mathfrak{S}_{i,j,q} (\nabla_q \nabla_i X_j) + 2\ g_{jq}\ \nabla_i b \\
&= -\nabla_q \nabla_i X_j - \nabla_i \nabla_j X_q - \nabla_j \nabla_q X_i + 2\ b\ g_{jq}\ \nabla_i b \\
&= -\nabla_q (-\nabla_j X_i + 2\ b\ g_{ij}) - \nabla_i (-\nabla_q X_j + 2\ b\ g_{jq}) \\
&\qquad - \nabla_j (-\nabla_i X_q + 2\ b\ g_{qi}) + 2\ g_{jq}\ \nabla_i b \\
&= -N_{iqj} - 2\ g_{ji}\ \nabla_q b - 2\ g_{qi}\ \nabla_j b + 2\ g_{jq} \nabla_i b
\end{aligned}$$

hence

$$N_{ijq} = -g_{ji} \nabla_q b - g_{qi} \nabla_j b + g_{jq} \nabla_i b$$

□

Lemma 5.4. *An infinitesimal homothety on a compact manifold is a Killing vector field.*

Proof. Let (M, g) be a compact Riemannian manifold of dimension n, which we assume to be orientable by taking its orientable double cover if necessary and denote its volume form with respect to g by Vol_{g} . If $X \in \mathfrak{X}(M)$ is an infinitesimal homothety, then $L_X \mathsf{g} = 2\sigma \mathsf{g}$ where

$$\sigma = \frac{1}{n}\ div(X)$$

is a constant. We have

$$\sigma^2 = \frac{1}{n}\ \sigma\ div(X) = \frac{1}{n}\ div(\sigma\ X)$$

and

$$\int_M \sigma^2\ Vol_{\mathsf{g}} = \frac{1}{n} \int_M div(\sigma\ X)\ Vol_{\mathsf{g}} = 0$$

hence we conclude that $\sigma \equiv 0$. □

5.2.1 Proof of theorem 2.1

Proof. By Lemma 5.1 and Lemma 5.2 we conclude that

$$A_j = X_j + \dot{x}^q V_{qj} + \frac{1}{2}\dot{x}^q\dot{x}^r K_{qrj} + \frac{1}{6}\dot{x}^q\dot{x}^r\dot{x}^s L_{qrsj} \tag{5.11}$$

where $X_j, V_{qj}, K_{qrj}, L_{qrsj} \in \mathfrak{F}(dom(x, \dot{x}))$ are functions of x^i's alone and K_{qrj} is symmetric in q, r and L_{qrsj} is symmetric in q, r, s. Again by the conventional abuse of notation we understand $X_j, V_{qj}, K_{qrj}, L_{qrsj} \in \mathfrak{F}(dom(x))$. Again we note that these functions are components of tensor fields restricted to $dom(x)$. In particular there exists $X \in \mathfrak{X}(M)$ such that

$$X|_{\ dom(x)} = X^i \frac{\partial}{\partial x^i} \quad .$$

Substituting 5.11 in 5.2 and 5.9 we obtain

$$\begin{aligned}
\nabla_i X_j + \nabla_j X_i &+ \dot{x}^q \left(\nabla_i V_{qj} + \nabla_j V_{qi}\right) \\
&+ \frac{1}{2}\dot{x}^q \dot{x}^r \left(\nabla_i K_{qrj} + \nabla_j K_{qri}\right) \\
&+ \frac{1}{6}\dot{x}^q \dot{x}^r \dot{x}^s \left(\nabla_i L_{qrsj} + \nabla_j L_{qrsi}\right) \\
&= 2bg_{ij} + 2\dot{x}^q a_q g_{ij} \qquad (5.12) \\
V_{ij} + \nabla_j Y_i &+ \dot{x}^q \left(K_{qij} + \nabla_j H_{qi} - R_{qij}{}^p\ X_p\right) \\
&+ \frac{1}{2}\dot{x}^q \dot{x}^r \left(L_{qrij} + \nabla_j S_{qri} - 2\ R_{qij}{}^p\ V_{rp}\right) \\
&+ \dot{x}^q \dot{x}^r \dot{x}^s \left(-\frac{1}{2}R_{qij}{}^p\ K_{rsp}\right) \\
&+ \dot{x}^q \dot{x}^r \dot{x}^s \dot{x}^t \left(-\frac{1}{6}R_{qij}{}^p\ L_{rstp}\right) = 0 \qquad (5.13)
\end{aligned}$$

Since 5.12 is valid for all values of $\dot{x}^q$ we obtain

$$\nabla_i X_j + \nabla_j X_i = 2\ b\ g_{ij} \tag{5.14}$$

and notice that

$$b = \frac{1}{n}\ div(X) \quad .$$

Similarly from 5.13 we obtain $V_{ij} = -\nabla_j Y_i$. Again in 5.12 the coefficient of $\dot{x}^q$ gives us $\nabla_i V_{qj} + \nabla_j V_{qi} = 2\ a_q\ g_{ij}$ from which we obtain

$$\nabla_i \nabla_j Y_q + \nabla_j \nabla_i Y_q = -2\ a_q\ g_{ij} \tag{5.15}$$

Raising j and contracting with i we find

$$a_q = -\frac{1}{n}\nabla_i \nabla^i Y_q \tag{5.16}$$

which shows that a_q's are components of the restriction of a vector field $\mathsf{a} \in \mathfrak{X}(M)$ and

$$\mathsf{a} = -\frac{1}{n}\ div(grad(Y)) \quad .$$

The coefficient of $\dot{x}^q$ in 5.13 gives

$$K_{qij} + \nabla_j H_{qi} - R_{qij}{}^p\ X_p = 0 \tag{5.17}$$

Using 5.7 and the symmetry of K_{qij} in q, i and antisymmetry of $R_{qij}{}^p$ in q, i we have

$$K_{qij} = -g_{qi}\nabla_j b \tag{5.18}$$

In view of the Lemma 5.3 we find from 5.17 and 5.18

$$-g_{qi}\nabla_j b + \nabla_j H_{qi} - \nabla_j \nabla_q X_i + g_{iq}\nabla_j b + g_{ji}\nabla_q b - g_{jq}\nabla_i b = 0$$

and

$$\nabla_j P_{qi} = -g_{ji}\nabla_q b + g_{qj}\nabla_i b \tag{5.19}$$

where

$$\begin{aligned} P_{qi} &= H_{qi} - \nabla_q X_i \\ P_{qi} + P_{iq} &= 0 \end{aligned} \tag{5.20}$$

hence

$$\nabla_j P_{iq} = g_{ij}\nabla_q b - \nabla_i b - g_{qj}\nabla_i b$$

and

$$\nabla_j P_q{}^i = \delta^j{}_i \nabla_q b - g_{qj}\nabla^i b$$

which shows us that there exists a tensor field $\mathsf{P} \in \mathfrak{X}^{1,1}(M)$ having the form $P\mid_{dom(x)}= P^i_q dx^i \otimes \dfrac{\partial}{\partial x^i}$ in local coordinates such that

$$\nabla \mathsf{P} = \mathsf{I} \otimes db - \mathsf{g} \otimes grad\ b \tag{5.21}$$

Using 5.18 we find from the coefficient of $\dot{x}^q \dot{x}^r$ in 5.12

$$\nabla_i\left(-g_{qr}\nabla_j b\right) + \nabla_j\left(-g_{qr}\nabla_i b\right) = 0$$

hence,

$$\nabla_i \nabla_j b = 0 \tag{5.22}$$

Since,

$$R_{qij}{}^p\ K_{rsp} = g_{rs} R_{qij}{}^p\ \nabla_p b = -g_{rs}\left(\nabla_q \nabla_i \nabla_j b - \nabla_i \nabla_q \nabla_j b\right) = 0$$

the coefficient of $\dot{x}^q \dot{x}^r \dot{x}^s$ in 5.12 is automatically zero. Now, considering the coefficient of $\dot{x}^q \dot{x}^r$ in 5.13 we find

$$L_{qrij} + \nabla_j S_{qri} - R_{qij}{}^p\ \nabla_p Y_r - R_{rij}{}^p\ \nabla_p Y_q = 0\ .$$

If we interchange q and i

$$L_{qrij} + \nabla_j S_{irq} - R_{iqj}{}^{p} \nabla_p Y_r - R_{rqj}{}^{p} \nabla_p Y_i = 0$$

and if we interchange r and i

$$L_{qrij} + \nabla_j S_{qir} - R_{qrj}{}^{p} \nabla_p Y_i - R_{irj}{}^{p} \nabla_p Y_q = 0$$

hence,

$$\begin{aligned}
3\ L_{qrij} &= -\ \nabla_j (S_{qri} + S_{riq} + S_{iqr}) \\
&= -\ \nabla_j (a_q g_{ri} + a_r g_{qi} - a_i g_{qr} + a_r g_{iq} + a_i g_{rq} - a_q g_{ri} + a_i g_{qr} + a_q g_{ir} - a_r g_{iq}) \\
&= -\nabla_j (a_q g_{ri} + a_r g_{iq} a_i g_{qr})
\end{aligned}$$

and thus,

$$L_{qrij} \;=\; -\frac{1}{3} \left(\mathfrak{S}_{q,r,i}\ \nabla_j a_q g_{ri} \right) \tag{5.23}$$

As for the coefficient of $\dot{x}^q \dot{x}^r \dot{x}^s$ in 5.12

$$\nabla_i L_{qrsj} + \nabla_j L_{qrsi} = 0$$

which gives in view of 5.23

$$\mathfrak{S}_{q,r,s} \left(\nabla_i \nabla_j a_q + \nabla_j \nabla_i a_q \right) = 0$$

Explicitly, for each q, r, s, i, j

$$(\nabla_i \nabla_j a_q + \nabla_j \nabla_i a_q)\ g_{rs} + (\nabla_i \nabla_j a_r + \nabla_j \nabla_i a_r)\ g_{sq} + (\nabla_i \nabla_j a_s + \nabla_j \nabla_i a_s)\ g_{qr} = 0\ .$$

Given q, if $g_{qq} \neq 0$ we put $q = r = s$ in this equation to obtain

$$3\ g_{qq} \left(\nabla_i \nabla_j a_q + \nabla_j \nabla_i a_q \right) = 0$$

If $g_{qq} = 0$, we choose r with $g_{qr} \neq 0$ and put $s = q$ to obtain

$$2\ g_{qr} \left(\nabla_i \nabla_j a_q + \nabla_j \nabla_i a_q \right) = 0$$

In any case

$$\nabla_i \nabla_j a_q + \nabla_j \nabla_i a_q = 0\ \ . \tag{5.24}$$

As the coefficient of $\dot{x}^q\dot{x}^r\dot{x}^s\dot{x}^t$ in 5.13 must reduce to 0, we use 5.23 to obtain

$$\begin{aligned} & R_{qij}{}^p\ \nabla_p\ a_r\ g_{st} + R_{qij}{}^p\ \nabla_p\ a_s\ g_{tr} + R_{qij}{}^p\ \nabla_p\ a_t\ g_{rs} \\ + & R_{rij}{}^p\ \nabla_p\ a_s\ g_{tq} + R_{rij}{}^p\ \nabla_p\ a_t\ g_{qs} + R_{rij}{}^p\ \nabla_p\ a_q\ g_{st} \\ + & R_{sij}{}^p\ \nabla_p\ a_t\ g_{qr} + R_{sij}{}^p\ \nabla_p\ a_q\ g_{rt} + R_{sij}{}^p\ \nabla_p\ a_r\ g_{tq} \\ + & R_{tij}{}^p\ \nabla_p\ a_q\ g_{rs} + R_{tij}{}^p\ \nabla_p\ a_r\ g_{sq} + R_{tij}{}^p\ \nabla_p\ a_s\ g_{qr} = 0\ . \end{aligned} \tag{5.25}$$

We shall show that 5.25 is equivalent to

$$R_{qij}{}^p\ \nabla_p\ a_r + R_{rij}{}^p\ \nabla_p\ a_q = 0 \tag{5.26}$$

for all q, r, i, j. It is easy to see that 5.26 implies 5.25 Let us conversely suppose that 5.25 holds. Having fixed i, j for any given q if $g_{qq} \neq 0$ putting $s = t = q = r$ we find that 16 $R_{qij}{}^p\ \nabla_p\ a_q\ g_{qq} = 0$ hence $R_{qij}{}^p\ \nabla_p\ a_q = 0$. If $g_{qq} = 0$, $\exists q'$ such that $g_{qq'} \neq 0$. Put $s = t = q$ and $r = q'$ to obtain 6 $R_{qij}{}^p\ \nabla_p\ a_q g_{qq'} = 0$ hence $R_{qij}{}^p\ \nabla_p\ a_q = 0$ in any case. We note in particular that 5.26 is valid if $q = r$. Now, consider distinct $q, r :$ If $g_{qr} \neq 0$ put $s = q, t = r$ to obtain 4 $(R_{qij}{}^p\ \nabla_p\ a_r + R_{rij}{}^p\ \nabla_p\ a_q)g_{qr} = 0$ and $R_{qij}{}^p\ \nabla_p\ a_r + R_{rij}{}^p\ \nabla_p\ a_q = 0$. If $g_{qr} = 0$, multiply with g^{st} and contract over s, t to obtain $(n + 4)(R_{qij}{}^p\ \nabla_p\ a_r + R_{rij}{}^p\ \nabla_p\ a_q) = 0$. Therefore, in any case 5.26 holds. Finally, we consider the coefficient of $\dot{x}^q\dot{x}^r$ in 5.13

$$L_{qrij} + \nabla_j S_{qri} + R_{qij}{}^p\ \nabla_p Y_r + R_{rij}{}^p\ \nabla_p Y_q = 0$$

which gives in view of 5.8 and 5.23

$$\begin{aligned} - & \frac{1}{3}(\nabla_j\ a_q\ g_{ri} + \nabla_j\ a_r\ g_{iq} + \nabla_j\ a_i\ g_{qr}) + (\nabla_j\ a_q\ g_{ri} + \nabla_j\ a_r\ g_{iq} - \nabla_j\ a_i\ g_{qr}) \\ + & R_{qij}{}^p\ \nabla_p Y_r + R_{rij}{}^p\ \nabla_p Y_q = 0. \end{aligned}$$

Equivalently,

$$R_{qij}{}^p\ \nabla_p Y_r + R_{rij}{}^p\ \nabla_p Y_q = \frac{2}{3}(2\ g_{qr}\ \nabla_j\ a_i - g_{qi}\ \nabla_j\ a_r - g_{ri}\ \nabla_j\ a_q) \tag{5.27}$$

Summing up we find that $A \in \mathfrak{X}(TM)$ is a conformal vector field with respect to the

Sasaki metric tensor field iff

$$
\begin{aligned}
A|_{dom(x,\dot{x})} &= A^i \mathsf{e}_i + A^{\bar{i}} \mathsf{e}_{\bar{i}} \\
&= \left(X^i + \dot{x}^q V_q{}^i + \frac{1}{2}\dot{x}^q \dot{x}^r K_{qr}{}^i + \frac{1}{6}\dot{x}^q \dot{x}^r \dot{x}^s L_{qrs}{}^i \right) \mathsf{e}_i \\
&\qquad + \left(Y^i + \dot{x}^q H_q{}^i + \frac{1}{2}\dot{x}^q \dot{x}^r S_{qr}{}^i \right) \mathsf{e}_{\bar{i}} \\
&= \left(X^i - \dot{x}^q \nabla^i Y_q - \frac{1}{2}\dot{x}^q \dot{x}^r g_{qr} \nabla^i b - \frac{1}{6}\dot{x}^q \dot{x}^r \dot{x}^s \frac{1}{3}\mathfrak{S}_{q,r,s}(\nabla^i a_q g_{rs}) \right) \mathsf{e}_i \\
&\qquad + \left(Y^i + \dot{x}^q H_q{}^i + \frac{1}{2}\dot{x}^q \dot{x}^r (a_q \delta^i_r + a_r \delta^i_q - a^i g_{qr}) \right) \mathsf{e}_{\bar{i}} \\
&= (X^i \mathsf{e}_i + \dot{x}^q \nabla_q X^i \mathsf{e}_{\bar{i}}) + \dot{x}^q (H_q{}^i - \nabla_q X^i) \mathsf{e}_{\bar{i}} + (Y^i \mathsf{e}_{\bar{i}} - \dot{x}^q \nabla^i Y_q \mathsf{e}_i) - E\ \nabla^i b\ \mathsf{e}_i \\
&\qquad + \frac{1}{2}\dot{x}^q a_q \dot{x}^i \mathsf{e}_{\bar{i}} + \frac{1}{2}\dot{x}^r a_r \dot{x}^i \mathsf{e}_{\bar{i}} - E(a^i\ \mathsf{e}_{\bar{i}}) - \frac{1}{3}\left(E\ \dot{x}^q \nabla^i a_q\right) \mathsf{e}_i \\
&= \Bigg({}^c X - \frac{1}{n} E\ {}^c grad(b) + {}^{v'} Y + \frac{1}{n} E\ {}^v \mathsf{a} \\
&+ \frac{1}{n}\ {}^n \mathsf{a}_\flat\ {}^n \mathsf{I} + \frac{1}{3} E\ {}^n grad(\mathsf{a}_\flat) + {}^n \mathsf{P} \Bigg) |_{dom(x,\dot{x})}
\end{aligned}
$$

for each chart $x = (x^i)_{1\le i\le n}$ on M . This proves the assertion and the conditions (1) , (2) , (3) are direct consequences of 5.22, 5.15, 5.24, 5.27, 5.26 , 5.19. 5.20, 5.21.

□

5.2.2 Proof of theorem 2.2

Proof. Taking the orientable double covering space if necessary, we may assume without loss of generality, that M is orientable. Clearly

$$
\begin{aligned}
\|\nabla b\|^2 &= \nabla_i b \nabla^i b \\
&= \nabla_i (b \nabla^i b) - b\ \nabla_i \nabla^i b \\
&= \nabla_i (b \nabla^i b)
\end{aligned}
$$

showing that

$$\|\nabla b\|^2 = div(b\ grad(b))$$

Consequently,

$$\int_M \|\nabla b\|^2 Vol_{\mathsf{g}} = \int_M div(b\ grad(b)) Vol_{\mathsf{g}} = 0$$

by the divergence theorem and hence $\nabla b \equiv 0$ showing that b is a constant and X is an infinitesimal homothety. Consequently $b \equiv 0$ by Lemma 4 and X is a Killing vector field. On the other hand , in view of

$$\begin{aligned} \nabla_i \nabla^i a_j &= g^{ip} \nabla_i \nabla_p a_j \\ &= -g^{ip} \nabla_p \nabla_i a_j = -\nabla_p \nabla^p a_j = 0 \end{aligned}$$

we find

$$\begin{aligned} \|\nabla \mathsf{a}\|^2|_{dom(x)} &= \nabla_i a_j \nabla^i a^j \\ &= \nabla_i (a_j \nabla^i a^j) - a_j \nabla_i \nabla^i a_j \\ &= \nabla_i (a_j \nabla^i a^j) \quad . \end{aligned}$$

Consequently, $\|\nabla \mathsf{a}\|^2 = div(W)$ for $W \in \mathfrak{X}(M)$ where

$$W|_{dom(x)} = a_j \nabla^i a^j \frac{\partial}{\partial x^i} \quad .$$

Therefore,

$$\int_M \|\nabla a\|^2 \; Vol_{\mathsf{g}} = \int_M div(W) \; Vol_{\mathsf{g}} = 0$$

and we conclude $\nabla \mathsf{a} \equiv 0$. Finally,

$$\begin{aligned} \| \mathsf{a} \|^2 \big|_{dom(x)} &= a_q a^q \\ &= -\frac{1}{n} (\nabla_i \nabla^i Y_q) a^q \\ &= -\frac{1}{n} (\nabla_i ((\nabla^i Y_q) a^q) - \nabla^i Y_q \nabla_i a^q) \\ &= -\frac{1}{n} \nabla_i ((\nabla^i Y_q) a^q) \; . \end{aligned}$$

Consequently $\|\mathsf{a}\|^2 = div(Z)$ for $Z \in \mathfrak{X}(M)$ where

$$Z|_{dom(x)} = -\frac{1}{n} \; (\nabla^i Y_q) a^q \; \frac{\partial}{\partial x^i} \quad .$$

Hence

$$\int_M \|\mathsf{a}\|^2 \; Vol_{\mathsf{g}} = \int_M div(Z) \; Vol_{\mathsf{g}} = 0$$

and we conclude $\mathsf{a} \equiv 0$. $\square$

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Vita

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