ON THE WKB ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE HYPERGEOMETRIC TYPE

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ON THE WKB ASYMPTOTIC SOLUTION OF DIFFERENTIAL EQUATIONS OF THE HYPERGEOMETRIC TYPE

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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WKB procedure is used in the study of asymptotic solutions of differential equations of the hypergeometric type. Hence asymptotic forms of classical orthogonal polynomials associated with the names Jacobi, Laguerre and Hermite have been derived. In particular, the asymptotic expansion of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ as n tends to infinity is emphasized.

Keywords: Asymptotic Approximation, WKB Method, Special Functions, Orthogonal Polynomials, Jacobi Polynomials.

ÖZ

HİPERGEOMETRİK TİPTEN DİFERENSİYEL DENKLEMLERİN WKB ASİMPTOTİK ÇÖZÜMLERİ

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Hipergeometrik tipten diferensiyel denklemlerin asimptotik çözümleri WKB yöntemi kullanılarak incelenmiştir. Böylece, Jacobi, Laguerre ve Hermite adlarıyla bilinen klasik ortogonal polinomların asimptotik formları çıkarılmıştır. Jacobi $P_n^{(\alpha,\beta)}(x)$ polinomlarının, yeterince büyük n değerleri için asimptotik açılımları özellikle vurgulanmıştır.

Anahtar Kelimeler: Asimptotik Yaklaşımlar , WKB Metod , Özel Fonksiyonlar, Ortogonal Polinomlar, Jacobi Polinomları.

To my family

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CHAPTER 1

Introduction

Simply stated, asymptotic analysis is that branch of mathematics devoted to the study of the behaviour of functions in the vicinity of given points in their domains of definition. Suppose then that $f(x)$ is a function of a real or a complex variable, which is to be considered near the point $x = x_0$. If f is analytic at $x = x_0$, then the desired behaviour can be determined by studying its Taylor series expansion about $x = x_0$. If $x = x_0$ is a singularity of f, either a pole or a branch point, then again the analysis can be reduced to the investigation of convergent series expansions. However, if $x = x_0$ is an irregular singularity of f, then no such reduction is possible and the analysis is more complicated. Partly for this reason, we shall find that most often our investigations will involve the study of functions near their points of irregular singularity [7].

Let us begin by considering an example to understand how to get meaningful approximations of complicated integral expressions, such as

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\mathrm{d}t \tag{1.1}$$

where x is real and nonnegative [17]. The integral in (1.1) is known as the exponential integral. Let us look for the analytic approximation to $Ei(x)$ for large enough positive x values. Integrating by parts

$$u = \frac{1}{t}, \quad \mathrm{d}w = e^{-t}\mathrm{d}t$$

$$\mathrm{d}u = \frac{-1}{t^2}\mathrm{d}t, \quad w = -e^{-t}$$

we have

$$Ei(x) = -\frac{e^{-t}}{t}\bigg|_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}\mathrm{d}t \tag{1.2}$$

Integrating by parts once more

$$u = \frac{1}{t^2}, \quad \mathrm{d}w = e^{-t}\mathrm{d}t$$

$$\mathrm{d}u = \frac{-2}{t^3}\mathrm{d}t, \quad w = -e^{-t}$$

leads to

$$Ei(x) = \frac{e^{-x}}{x} + \frac{e^{-t}}{t^2}\bigg|_x^\infty + 2\int_x^\infty \frac{e^{-t}}{t^3}\mathrm{d}t.$$

Continuing in this way, we obtain

$$Ei(x) = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2}\right) - 2!\frac{e^{-t}}{t^3}\bigg|_x^\infty - 3!\int_x^\infty \frac{e^{-t}}{t^4}\mathrm{d}t$$

$$= e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3}\right) - 3!\frac{e^{-t}}{t^4}\bigg|_x^\infty + 4!\int_x^\infty \frac{e^{-t}}{t^5}\mathrm{d}t$$

and finally

$$Ei(x) = e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + ... + (-1)^n\frac{(n-1)!}{x^n}\right] + (-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\mathrm{d}t$$

$$= S_n(x) + r_n(x)$$

where the partial sum $S_n(x)$ and the remainder $r_n(x)$ are defined by

$$S_n(x) = e^{-x}\sum_{k=0}^{n-1}\frac{(-1)^k k!}{x^{k+1}}, \quad n = 1, 2, ... \tag{1.3}$$

and

$$r_n(x) = (-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\mathrm{d}t \tag{1.4}$$

respectively. Although it is good since $Ei(x)$ is represented by a series whose terms involve inverse powers of x, $S_n(x)$ is divergent for any fixed x since the general term tends to infinity as $n \to \infty$. Also, $r_n(x)$ is unbounded as $n \to \infty$.

But, it is possible to show that the integral is convergent for all $x > 0$, so that $Ei(x) = S_n(x) + r_n(x)$ must be bounded

$$Ei(x) = \int_x^1 \frac{e^{-t}}{t}\mathrm{d}t + \int_1^\infty \frac{e^{-t}}{t}\mathrm{d}t \leq A(x) + \int_1^\infty e^{-t}\mathrm{d}t$$

$$= A(x) - e^{-1} < \infty.$$

Suppose we consider n fixed, and let x be sufficiently large. From (1.4)

$$|r_n(x)| = n! \int_x^\infty \frac{e^{-t}}{t^{n+1}}\mathrm{d}t \leq \frac{n!}{x^{n+1}} \int_x^\infty e^{-t}\mathrm{d}t = \frac{n!e^{-x}}{x^{n+1}} \tag{1.5}$$

for $x >> 1$. Obviously

$$|r_n(x)| \to 0 \quad \text{as} \quad x \to \infty.$$

Here $\frac{n!e^{-x}}{x^{n+1}}$ is also the magnitude of the first neglected term (the term for $k = n$) in the series for $S_n(x)$. Thus, for n fixed, the ratio of $r_n(x)$ to the last term in $S_n(x)$ is such that

$$\left|\frac{r_n(x)}{(n-1)!e^{-x}x^{-n}}\right| \leq \left|\frac{n!e^{-x}x^{-n-1}}{(n-1)!e^{-x}x^{-n}}\right| = \frac{n}{x} \to 0 \tag{1.6}$$

as $x \to \infty$. The error in approximating $Ei(x)$ by $S_n(x)$ has the order of the first neglected term. Now, we may write that $Ei(x)$ is asymptotically equal to $S_n(x)$

$$Ei(x) \sim \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + ...\right) \quad \text{as} \quad x \to \infty. \tag{1.7}$$

Therefore, we conclude that when x is large, we may use the partial sum $S_n(x)$ which is divergent, to approximate $Ei(x)$. This simple example reflects almost all features of an asymptotic approximation.

The main goal of this study is to find asymptotic representations of polynomial solutions of a differential equation known as equation of the hypergeometric type. Thus, the organization of the thesis is as follows: In Chapter 2, differential equation of special functions are reviewed. Chapter 3 deals with asymptotic WKB approximation and its applications to orthogonal polynomials. Hilb's type asymptotic expansion for the Jacobi poynomials is given in Chapter 4. Finally, we discuss the results in Chapter 5.

CHAPTER 2

Differential Equation of Hypergeometric Type

2.1 A Differential Equation for Special Functions

Consider the second order equation

$$u'' + \frac{\tilde{\tau}(x)}{\sigma(x)}u' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}u = 0 \tag{2.1}$$

where $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials of degree at most 2, $\tilde{\tau}(x)$ is a polynomial of degree at most 1. Here, x denotes, in general, a complex variable. Now, we try to reduce (2.1) to a simpler form by introducing the transformation

$$u = \phi(x)y$$

where $\phi(x)$ is to be determined appropriately. Substituting u, u' and u'' into (2.1) we obtain

$$y'' + \left(2\frac{\phi'}{\phi} + \frac{\tilde{\tau}}{\sigma}\right)y' + \left(\frac{\tilde{\sigma}}{\sigma^2} + \frac{\tilde{\tau}}{\sigma}\frac{\phi'}{\phi} + \frac{\phi''}{\phi}\right)y = 0. \tag{2.2}$$

Require that the coefficient of y' is of the form

$$\frac{\tau(x)}{\sigma(x)}$$

with $\tau(x)$ a polynomial of degree at most 1, it follows that

$$2\frac{\phi'}{\phi} = \frac{\tau(x) - \tilde{\tau}(x)}{\sigma(x)}$$

and that

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)} \tag{2.3}$$

where

$$\pi(x) = \frac{1}{2}[\tau(x) - \tilde{\tau}(x)] \tag{2.4}$$

is also a polynomial of degree at most 1. Since

$$\frac{\phi''}{\phi} = \left(\frac{\phi'}{\phi}\right)' + \left(\frac{\phi'}{\phi}\right)^2 = \left(\frac{\pi}{\sigma}\right)' + \left(\frac{\pi}{\sigma}\right)^2$$

equation (2.2) takes the form,

$$y'' + \frac{\tau(x)}{\sigma(x)}y' + \frac{\tilde{\tilde{\sigma}}(x)}{\sigma^2(x)}y = 0 \tag{2.5}$$

in which

$$\tau(x) = 2\pi(x) + \tilde{\tau}(x) \tag{2.6}$$

and

$$\tilde{\tilde{\sigma}}(x) = \pi^2(x) + [\tilde{\tau}(x) - \sigma'(x)]\pi(x) + [\tilde{\sigma}(x) + \pi'(x)\sigma(x)] \tag{2.7}$$

where and $\tilde{\tilde{\sigma}}(x)$ is a polynomial of degree at most 2. Hence, (2.5) is an equation of the same type of (2.1), so that we have found a class of transformations that does not change the type of the equation. In other words, equation (2.1) remains invariant under the transformations induced by the substitution $u = \phi(x)y$, where $\phi(x)$ satisfies (2.3), with an arbitrary linear polynomial $\pi(x)$. Now, we try to choose $\pi(x)$ so as to make (2.5) as simple as possible for studying the properties of solutions. We may choose the coefficients of $\pi(x)$ so that $\tilde{\tilde{\sigma}}(x)$ in (2.7) is divisible by $\sigma(x)$, that is,

$$\tilde{\tilde{\sigma}}(x) = \lambda\sigma(x), \tag{2.8}$$

where λ is a constant. Then (2.5) can be written as,

$$y'' + \frac{\tau(x)}{\sigma(x)}y' + \lambda\frac{\sigma(x)}{\sigma^2(x)}y = 0$$

or as

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0 \tag{2.9}$$

upon multiplication by $\sigma(x)$.

Equation (2.9) is referred to as a differential equation of the hypergeometric type (EHT) and its solutions are called functions of the hypergeometric type. When solutions are polynomials, they are called polynomials of the hypergeometric type. Now, equation (2.1) may be called a generalized differential equation of the hypergeometric type.

2.1.1 Polynomial Solutions of EHT

It is possible to show that the EHT has polynomial solutions if and only if

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma'', \quad n = 0, 1, \ldots. \tag{2.10}$$

These polynomials are given explicitly by the Rodriguez formula,

$$y_n(x) = \frac{B_n}{\rho(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\sigma^n(x)\rho(x)] \tag{2.11}$$

where B_n is a normalization constant, and $\rho(x)$ satisfies the equation

$$[\sigma(x)\rho(x)]' = \tau(x)\rho(x)$$

which is used to make (2.9) self-adjoint [16].

Solving (2.11), we obtain, up to constant factors, the possible forms of $\rho(x)$

corresponding to the possible degrees of $\sigma(x)$

$$\rho(x) = \begin{cases} (b-x)^{\alpha}(x-a)^{\beta} & \text{for} \quad \sigma(x) = (b-x)(x-a) \\ (x-a)^{\alpha}e^{\beta x} & \text{for} \quad \sigma(x) = x-a \\ e^{\alpha x^2+\beta x} & \text{for} \quad \sigma(x) = 1 \end{cases}$$

where a, b, α and β are constants. By linear changes of variable, the expressions for $\rho(x)$ and $\sigma(x)$ can be reduced to the following canonical forms:

$$\rho(x) = \begin{cases} (1-x)^{\alpha}(1+x)^{\beta} & \text{for} \quad \sigma(x) = 1-x^2 \\ x^{\alpha}e^{-x} & \text{for} \quad \sigma(x) = x \\ e^{-x^2} & \text{for} \quad \sigma(x) = 1 \end{cases} .$$

2.1.2 Orthogonality of the Polynomials of Hypergeometric Type

Let us consider the polynomial solutions $y_n(x)$ of the EHT for a real variable x. In this case, the following theorem is of fundamental importance.

Theorem 2.1.1. *Let the coefficients in (2.9) be such that*

$$\sigma(x)\rho(x)x^k\Big|_{x=a,b} = 0 \quad for \quad k = 0, 1, \ldots \tag{2.12}$$

at the boundaries of x-interval (a, b). Then the polynomials of the hypergeometric type, which constitute a sequence $\{p_0(x), p_1(x), \ldots, p_m(x) \ldots, p_n(x), \ldots\}$ of real functions of the real argument x, corresponding to the different values of $\lambda = \lambda_n$, i.e. $\lambda_0, \lambda_1, \ldots, \lambda_m, \ldots, \lambda_n, \ldots$ are orthogonal on (a, b) in the sense that

$$\int_a^b \rho(x)p_m(x)p_n(x)\mathrm{d}x = 0 \tag{2.13}$$

for $m \neq n$, where $\rho(x)$ is now called the weighting function.

Proof. The polynomials $p_n(x)$ and $p_m(x)$ satisfy

$$\left[\sigma(x)\rho(x)p_n'(x)\right]' + \lambda_n\rho(x)p_n(x) = 0$$

and

$$\big[\sigma(x)\rho(x)p'_m(x)\big]' + \lambda_m \rho(x) p_m(x) = 0$$

respectively. Multiplying the first by p_m and the second by p_n and subtracting we get

$$p_m(x)\big[\sigma(x)\rho(x)p'_n(x)\big]' - p_n(x)\big[\sigma(x)\rho(x)p'_m(x)\big]' = (\lambda_m - \lambda_n)\rho(x)p_m(x)p_n(x) \tag{2.14}$$

which is equal to

$$\big[\sigma(x)\rho(x)\big]'\big[p_m(x)p'_n(x) - p'_m(x)p_n(x)\big] + \big[\sigma(x)\rho(x)\big]\big[p_m(x)p''_n(x) - p''_m(x)p_n(x)\big] = (\lambda_m - \lambda_n)\rho(x)p_m(x)p_n(x)$$

on rearranging the left hand side. A careful inspection shows that it can be written in a more compact form

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\sigma(x)\rho(x)W(p_m, p_n)(x)\Big] = (\lambda_m - \lambda_n)\rho(x)p_m(x)p_n(x)$$

where $W(p_m, p_n)(x) = p_m(x)p'_n(x) - p'_m(x)p_n(x)$ is the Wronsky determinant of the solutions $p_m(x)$ and $p_n(x)$. Now, integrating both sides from a to b, we obtain

$$(\lambda_m - \lambda_n)\int_a^b \rho(x)p_m(x)p_n(x)\mathrm{d}x = \sigma(x)\rho(x)W(p_m, p_n)(x)\Big|_a^b$$

whose right hand side is equal to zero by hypothesis. Hence, for $m \neq n$, $(\lambda_m \neq \lambda_n)$ we must have $\int_a^b \rho(x)p_m(x)p_n(x)\mathrm{d}x = 0$. More specifically, we may write

$$\int_a^b \rho(x)p_m(x)p_n(x)\mathrm{d}x = \mathcal{N}_n^2\delta_{mn} \tag{2.15}$$

where δ_{mn} is Kronecker delta and $\mathcal{N}_n$ is a normalization constant defined by [16]

$$\mathcal{N}_n^2 = \frac{(-1)^n}{\mathcal{A}_{nn}}\mathcal{N}_{nn}^2 = \frac{(-1)^n}{\mathcal{A}_{nn}}(n!a_n)^2\int_a^b \rho_n(x)\mathrm{d}x,$$

$$\mathcal{A}_{mn} = \frac{n!}{(n-m)!}\prod_{k=0}^{m-1}\left[\tau' + \tfrac{1}{2}\ (n+k-1)\sigma''\right]$$

.

2.2 Gamma Function

Before starting our analysis of polynomials of hypergeometric type, it is convenient to review some properties of the well-known Euler Gamma function which, for any complex number x, such that $\mathrm{Re}(x) > 0$, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t. \tag{2.16}$$

Gamma Function satisfies the recurrence relation

$$\Gamma(x+1) = x\Gamma(x) \tag{2.17}$$

which reduces to

$$\Gamma(n+1) = n! \tag{2.18}$$

if $x = n$ is an integer [9]. Another important functional equation of type

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{2.19}$$

is known as the reflection (addition) formula, which gives the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.20}$$

when $x = 1/2$. The formula

$$2\Gamma(2x) = \frac{2^{2x}}{\Gamma(\frac{1}{2})}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) \tag{2.21}$$

is called the duplication formula, and leads to

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!} \tag{2.22}$$

when $x = n$. From the definition of Gamma function in (2.16) one can easily see that $\Gamma(1) = 1$ and using the relation in (2.17) one obtains $\Gamma(2) = 1$.

A useful equality, which can be deduced from the definition of the beta function, is

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \tag{2.23}$$

where $B(x, y)$ is defined as

$$B(x, y) = \int_{-1}^{1} t^{x-1}(1 - t)^{y-1}\mathrm{d}t. \tag{2.24}$$

Finally, the so-called Pochhammer's symbol defined by

$$(\beta)_n = \beta(\beta + 1)\ldots(\beta + n - 1)\ ,\ (\beta)_0 = 1 \tag{2.25}$$

can be written as

$$(\beta)_n = \frac{\Gamma(\beta + n)}{\Gamma(\beta)} \tag{2.26}$$

in terms of the Gamma function.

2.3 Classical Orthogonal Polynomials

Some special choices of $\sigma(x)$, $\tau(x)$, and λ in (2.9) lead to the well-known families, such as Jacobi, Laguerre, and Hermite polynomials. Important properties of these polynomials will be introduced.

2.3.1 Jacobi Polynomials

Let $\sigma(x) = 1 - x^2$ and $\rho(x) = (1 - x)^\alpha(1 + x)^\beta$ in the differential equation (2.11). Then from (2.11), $\tau(x) = -(\alpha + \beta + 2) + \beta - \alpha$, $\lambda_n = n(n + \alpha + \beta + 1)$. Corresponding polynomials are denoted and defined by the Rodriguez formula in (2.11)

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1 - x)^{-\alpha}(1 + x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1 - x)^{n+\alpha}(1 + x)^{n+\beta}\right] \tag{2.27}$$

where $B_n = \frac{(-1)^n}{2^n n!}$ is chosen for historical reasons [12]. It is clear from (2.9) that Jacobi polynomials satisfy the differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{2.28}$$

Applying Leibniz's rule for the derivatives of a product, it can be seen from (2.27)

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha+1)_n}{(\alpha+1)_{n-k}}\frac{(\beta+1)_n}{(\beta+1)_k}(x-1)^{n-k}(x+1)^k$$

or equivalently, using (2.26) we have

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(x-1)^{n-k}(x+1)^k}{\Gamma(n-k+\alpha+1)\Gamma(k+\beta+1)} \tag{2.29}$$

and

$$P_n^{(\alpha,\beta)}(1) = \frac{1}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \tag{2.30}$$

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n}{n!}\frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)}. \tag{2.31}$$

Putting $x = \mp 1$ in (2.28) and using (2.30) and (2.31) we get

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(1) = \frac{1}{2}(n+\alpha+\beta+1)\frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+2)}, \tag{2.32}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(-1) = (-1)^n\frac{1}{2}(n+\alpha+\beta+1)\frac{\Gamma(n+\beta+1)}{(n-1)!\Gamma(\beta+2)}. \tag{2.33}$$

Similarly differentiating (2.28) and using (2.32) and (2.33) one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n^{(\alpha,\beta)}(1) = \frac{(n-1)(n+\alpha+\beta+2)}{2(\alpha+2)}\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(1), \tag{2.34}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n^{(\alpha,\beta)}(-1) = \frac{(n-1)(n+\alpha+\beta+2)}{2(\beta+2)}\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(-1). \tag{2.35}$$

The condition (2.12) in theorem (2.1) is satisfied when $(a,b) = (-1,1)$, provided $\alpha > -1$, $\beta > -1$. Thus, Jacobi polynomials are orthogonal on $(-1,1)$ with respect to the weighting function $\rho(x) = (1-x)^\alpha(1+x)^\beta$. That is,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \mathrm{d}x = \mathcal{N}_n^2 \delta_{mn} \tag{2.36}$$

where

$$\mathcal{N}_n^2 = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)n!} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}. \tag{2.37}$$

Thus, the recurrence relation is given by [14],

$$\begin{aligned} &2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) = \\ &(n+\alpha+\beta+1)\left[(n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2\right]P_n^{(\alpha,\beta)}(x) \\ &-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x) \end{aligned} \tag{2.38}$$

in which

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).$$

2.3.2 Legendre Polynomials

The class of the Legendre polynomials is a subclass of Jacobi polynomials with $\alpha = \beta = 0$. To simplify the notation it is standard to set $P_n := P_n^{(0,0)}$. We now review the basic properties. According to (2.28) we have the differential equation

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \tag{2.39}$$

The recursion formula in (2.38) takes the form

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{2.40}$$

with $P_0(x) = 1$, $P_1(x) = x$ [13]. Special values in (2.30) and (2.31) give respectively

$$P_n(1) = 1 \ , \ P_n(-1) = (-1)^n, \tag{2.41}$$

and similarly (2.32) and (2.33) imply

$$P_n'(\pm 1) = (\pm 1)^{n-1}\frac{1}{2}n(n+1). \tag{2.42}$$

By taking $\alpha = \beta = 0$ in (2.36) and (2.37) we get the orthogonality property for Legendre polynomials. That is,

$$\int_a^b P_n(x)P_m(x)\mathrm{d}x = \mathcal{N}_n^2\delta_{mn} \tag{2.43}$$

where

$$\mathcal{N}_n^2 = \frac{2}{2n+1}. \tag{2.44}$$

2.3.3 Laguerre Polynomials

Now we introduce another family of polynomial solutions of the hypergeometric differential equation in (2.9). Let $\sigma(x) = x$ and $\rho(x) = x^\alpha e^{-x}$ in the differential equation (2.9). Then, from (2.11), we have $\tau(x) = \alpha + 1 - x$ and $\lambda_n = n$. Corresponding polynomials are called generalized Laguerre polynomials denoted and defined by the Rodriguez formula (2.11)

$$L_n^{(\alpha)}(x) = \frac{1}{n!}e^x x^\alpha \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(x^{n+\alpha}e^{-x}\Big). \tag{2.45}$$

It can be seen from (2.9) that Laguerre polynomials satisfy the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{2.46}$$

Applying Leibniz's rule for derivatives of a product, we have, from (2.45) and (2.26)

$$L_n^{(\alpha)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}(-x)^k \tag{2.47}$$

which gives

$$L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}. \tag{2.48}$$

In (2.46), putting $x = 0$ we get,

$$\frac{\mathrm{d}}{\mathrm{d}x} L_n^{(\alpha)}(0) = -\frac{n}{\alpha+1} L_n^{(\alpha)}(0) = -\frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+2)}. \tag{2.49}$$

Similarly differentiating (2.46) with respect to x and substituting $x = 0$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} L_n^{(\alpha)}(0) = -\frac{(n-1)}{\alpha+2} \frac{\mathrm{d}}{\mathrm{d}x} L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{(n-2)!\Gamma(\alpha+3)}. \tag{2.50}$$

By virtue of theorem (2.1) Laguerre polynomials are orthogonal on $(0, \infty)$ with respect to the weight $\rho(x) = x^\alpha e^{-x}$. That is,

$$\int_a^b L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} \mathrm{d}x = \mathcal{N}_n^2 \delta_{mn} \tag{2.51}$$

where

$$\mathcal{N}_n^2 = \frac{1}{n!}\Gamma(n+\alpha+1). \tag{2.52}$$

Then, the three term recurrence relation is given by [6]

$$(n+1) L_{n+1}^{(\alpha)}(x) = (\alpha + 2n + 1 - x) L_n^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x), \tag{2.53}$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$.

2.3.4 Hermite Polynomials

Finally, we introduce the Hermite polynomials. Let $\sigma(x) = 1$ and $\rho(x) = e^{-x^2}$ in the differential equation (2.9). Then, (2.11) gives $\tau(x) = -2x$ and $\lambda_n = 2n$. Corresponding polynomials are the Hermite polynomials defined by the Rodriguez formula in (2.11)

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}\right). \tag{2.54}$$

Hermite polynomials satisfy the differential equation

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0. \tag{2.55}$$

In the light of theorem (2.1), we see that Hermite polynomials are orthogonal on the real line $(-\infty, \infty)$ with respect to the weight $\rho(x) = e^{-x^2}$, i.e.,

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} \mathrm{d}x = \mathcal{N}_n^2 \delta_{mn} \tag{2.56}$$

where

$$\mathcal{N}_n^2 = 2^n n! \Gamma(\tfrac{1}{2}) = 2^n n! \sqrt{\pi}. \tag{2.57}$$

Thus, the recurrence relation reads as

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \tag{2.58}$$

with $H_0(x) = 1$ and $H_1(x) = 2x$. Finally, for example, from Rodriguez formula we can obtain another useful recurrence relation

$$H_n'(x) = 2n H_{n-1}(x). \tag{2.59}$$

CHAPTER 3

Asymptotic WKB Approximation

An introduction to asymptotic approximations has been given in the first chapter providing an example. Now, we study the fundamental concepts of the subject.

3.1 Introduction to asymptotic analysis

In order to describe the behaviour of $f(x)$ as $x \to x_0$ in terms of a known function $g(x)$ we shall often use certain notations called order symbols due to [11].

3.1.1 Order Symbols

Definition 3.1.1. (Big "O" symbol) If $|f(x)/g(x)|$ is bounded, we write

$$f(x) = O[g(x)] \quad (x \to x_0) \tag{3.1}$$

or $f = O(g)$; in words, f is of order not exceeding g.

Definition 3.1.2. (Small "o" symbol) If $f(x)/g(x) \to 0$, we write

$$f(x) = o[g(x)] \quad (x \to x_0) \tag{3.2}$$

or $f = o(g)$; again in words, f is of order less than g.

Special cases of thes definitions are $f = o(1)$ $(x \to x_0)$, meaning simply that f vanishes as $x \to x_0$, and $f = O(1)$ $(x \to x_0)$, meaning simply that f bounded as $x \to x_0$.

The statement (3.3) is of existential type: it asserts that there is a number M such that

$$|f(x)| \leq M|g(x)| \quad (x \geq a) \tag{3.3}$$

without giving information concerning the actual size of M. If $f(x)/g(x)$ tends to unity, we write

$$f(x) \sim g(x) \quad (x \to x_0) \tag{3.4}$$

or $f \sim g$; again in words, f is asymptotic to g or g is an asymptotic approximation of f [11].

3.1.2 Asymptotic sequence, asymptotic expansion and asymptotic power series

Definition 3.1.3. (Asymptotic sequence [5]) A sequence of functions

$$\{\phi_1(x), \phi_2(x), \phi_3(x), ..., \phi_n(x), ...\} = \{\phi_n(x)\} \tag{3.5}$$

for $n = 1, 2, 3, ...$ is an asymptotic sequence as $x \to x_0$, if

$$\phi_{n+1}(x) = o[\phi_n(x)] \tag{3.6}$$

for all n, i.e.

$$\lim_{x \to x_0} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0 \quad .$$

Definition 3.1.4. (Asymptotic expansion [3]) Let $\phi_n(x)$ be an asymptotic sequence as $x \to x_0$. $\sum a_n \phi_n(x)$ is said to be an asymptotic expansion for f as $x \to x_0$ if

$$f(x) = \sum_{n=1}^{N} a_n \phi_n(x) + o[\phi_N(x)] \tag{3.7}$$

where the a_n are constants. Here, we see that

$$a_N = \lim_{x \to x_0} \left[\frac{f(x) - \sum_{n=1}^{N-1} a_n \phi_n(x)}{\phi_N(x)} \right] \tag{3.8}$$

which implies that

$$f(x) = \sum_{n=1}^{N-1} a_n \phi_n(x) + O[\phi_N(x)].$$

A function f may have asymptotic expansions involving two different asymptotic sequences and two sequences need not be equivalent.

If $f \sim \sum a_n \phi_n$ and $g \sim \sum b_n \phi_n$ and if α,β are constants then

$$\alpha f(x) + \beta g(x) \sim \sum (\alpha a_n + \beta b_n) \phi_n.$$

Definition 3.1.5. (Asymptotic power series) Let $f(x)$ be defined and continuous on $\mathcal{D} \subset \mathbb{R}$. The formal power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to be an asymptotic power series expansion of f as $x \to x_0$ in $\mathcal{D}$ if the condition

$$f(x) = \sum_{n=0}^{m-1} a_n (x - x_0)^n + O[(x - x_0)^m]$$

is satisfied [4].

3.2 Asymptotic solutions of differential equations

A general form of a linear differential equation of the second order is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x) \frac{\mathrm{d}y}{\mathrm{d}x} + q(x) y = 0 \tag{3.9}$$

where $p(x)$ and $q(x)$ denote coefficient functions continuous on some open interval. If we change the dependent variable from y to w,

$$y = w e^{-\frac{1}{2} \int^x p(t) dt} \tag{3.10}$$

we transform the equation to a more appropriate form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + g(x)w = 0 \tag{3.11}$$

where

$$g(x) = \frac{1}{2}p'(x) + \frac{1}{2}[p(x)]^2 - q(x).$$

The function $g(x)$ may also depend on a parameter, say λ, so that $g = g(\lambda; x)$. Our objective is to find an asymptotic approximation for $w(x)$ as $x \to \infty$, i.e. a solution valid for large values of the argument x. To this end, we have to examine the "point at infinity", so the following cases may occur:

Case 1: If $x = \infty$ is an ordinary point of (3.11), then $w(x)$ consist of two linearly independent power series expansion in inverse power series of x which are convergent for $|x| > R$ for some R [2].

Case 2: If $x = \infty$ is a singular point, then there are two possibilities:

There is at least one solution of Frobenious series type, when the singularity is a regular singularity. If $x = \infty$ is an irregular singular point, we assume that $g(x)$ has an asymptotic form

$$g(x) \sim a_0 + \frac{a_1}{x^2} + ... = a_0 + O(x^{-2}) \quad \text{as} \quad x \to \infty, \quad a_0 \neq 0. \tag{3.12}$$

With this $g(x)$, we suggest an asymptotic solution of (3.11) of the form

$$w(x) = e^{\lambda x} x^{\sigma} f(x) \tag{3.13}$$

where

$$f(x) = \alpha_0 + \frac{\alpha_1}{x} + ... + \frac{\alpha_k}{x^k} + ... \tag{3.14}$$

and λ, σ, α_0, α_1,..., α_k are some constants. Substituting $w(x)$ in (3.11), after some calculations, we get the reccurence relation

$$[a_1 + 2(\sigma - n - 2)\lambda]\alpha_{n+2} + [a_2 + (n+1)(n+2) + \sigma^2 - \sigma(2n+3)]\alpha_{n+1} +$$

$$+\sum_{k=0}^{n} \alpha_{n-k} a_{k+3} = 0, \quad n = 0, 1, ... \tag{3.15}$$

to determine the constants. Then two asymptotic solutions for $w(x)$ as $x \to \infty$ may be written as

$$w_1(x) \sim Ae^{\lambda^{(1)}x} x^{\sigma^{(1)}} [1 + \frac{\alpha_1^{(1)}/\alpha_0}{x} + ... + \frac{\alpha_n^{(1)}/\alpha_0}{x^n} + ...]$$

and

$$w_2(x) \sim Be^{\lambda^{(2)}x} x^{\sigma^{(2)}} [1 + \frac{\alpha_1^{(2)}/\alpha_0}{x} + ... + \frac{\alpha_n^{(2)}/\alpha_0}{x^n} + ...]$$

where A and B are arbitrary constants. This procedure is closely related to procedure WKB approximation consisting of the first letters of the names of three mathematician Wentzel, Kramers and Brillauin, respectively [17].

3.2.1 WKB approximation with a large parameter

We consider again the differential equation

$$w'' + f(\lambda; x) w = 0 \tag{3.16}$$

with a large λ parameter. Now we try to find $w(\lambda; x)$ as $\lambda \to \infty$. The equation (3.16) is known as Liouville equation when $f(\lambda; x) = \lambda^2 \phi_0(x)$. More generally, WKB method uses originally

$$f(\lambda; x) = \lambda^2 \phi_0(x) + \lambda \phi_1(x) + \phi_2(x).$$

We study (3.16) with $f(\lambda; x)$ having the following asymptotic form

$$f(\lambda; x) \sim \lambda^2 \phi_0(x) + \lambda \phi_1(x) + \phi_2(x) + \lambda^{-1} \phi_3(x) + ... = \sum_{n=0}^{\infty} \lambda^{2-n} \phi_n(x) \tag{3.17}$$

as $\lambda \to \infty$ where the coefficient functions $\phi_n(x)$ are continuous and twice differentiable functions of x. We are looking for asymptotic solutions in the form

$$w(\lambda; x) \sim e^{g_0(\lambda)\psi_0(x)+g_1(\lambda)\psi_1(x)+\ldots} = e^{\sum_{n=0}^{\infty} g_n(\lambda)\psi_n(x)}$$

where the $g_n(\lambda)$ from an asymptotic sequence as $\lambda \to \infty$. Thus, we have to determine the sequence $\{g_n(\lambda)\}$ and the functions $\psi_n(x)$. Since

$$\ln w \sim \sum_{n=0}^{\infty} g_n(\lambda)\psi_n(x)$$

we have

$$\frac{w'}{w} \sim \sum_{n=0}^{\infty} g_n(\lambda)\psi_n'(x)$$

and

$$\frac{w''}{w} \sim \sum_{n=0}^{\infty} g_n(\lambda)\psi_n''(x) + \left[\sum_{n=0}^{\infty} g_n(\lambda)\psi_n'(x)\right]^2$$

so that the substitution of $\frac{w''}{w}$ into equation (3.16) gives

$$g_0(\lambda)\psi_0''(x) + g_1(\lambda)\psi_1''(x) + \ldots + \left[\sum_{n=0}^{\infty} g_n(\lambda)\psi_n(x)\right]^2 + \lambda^2\phi_0(x) + \lambda\phi_1(x) + \ldots \sim 0. \tag{3.18}$$

Equating similar asymptotic terms to zero, first we get

$$[g_0(\lambda)]^2[\psi_0'(x)]^2 + \lambda^2\phi(x) \sim 0 \tag{3.19}$$

for the dominant $O(\lambda^2)$ terms, which leads to

$$[g_0(\lambda)]^2 = \lambda^2 \Rightarrow g_0(\lambda) = \lambda \tag{3.20}$$

$$\lambda^2\{[\psi_0'(x)]^2 + \phi(x)\} \sim 0$$

$$[\psi_0'(x)]^2 = -\phi(x)$$

$$\psi_0(x) = \mp i \int^x \sqrt{\phi_0(\xi)}d\xi.$$

Notice that we assume $\phi_0(x) \neq 0$ in its domain. The next terms are of order λ $O(\lambda)$,

$$g_0(\lambda)\psi_0''(x) + \lambda\phi_1(x) + 2g_0(\lambda)\psi_0'(x)\psi_1'(x) \sim 0$$

$$\lambda\{\mp 2i[\phi_0(x)]^{-1/2}\phi_0'(x) + \phi_1(x) + 2g_1(\lambda)[\mp i\sqrt{\phi_0(x)}]\psi_1'(x)\} \sim 0 \tag{3.21}$$

which gives $g_1(\lambda) = \text{constant} = 1$ and

$$\psi_1'(x) = -\frac{1}{4}\frac{\phi_0'(x)}{\phi_0(x)} - \frac{(\mp)}{2i}\frac{\phi_1(x)}{\sqrt{\phi_0(x)}}$$

$$\psi_1(x) = \ln[\phi_0(x)]^{-1/4} \mp \frac{1}{2}i\int^x \frac{\phi_1(\xi)}{\sqrt{\phi_0(\xi)}}d\xi. \tag{3.22}$$

Note that if $\phi_0(x)$ were equal to zero for some x in its domain, this would cause a singularity in $\psi_1(x)$ and hence the method would fail. The constant $O(1)$ terms give from (3.18), (3.20) and (3.22)

$$g_1(\lambda)\psi_1''(x) + g_1^2(\lambda)[\psi_1'(x)]^2 + 2g_0(\lambda)g_2(\lambda)\psi_0'(x)\psi_2'(x) + \phi_2(x) \sim 0 \tag{3.23}$$

$$\psi_1''(x) + [\psi_1'(x)]^2 + 2\psi_0'(x)\psi_2'(x) + \phi_2(x) = 0 \tag{3.24}$$

where $g_2(\lambda) = 1/\lambda$ and

$$\psi_2'(x) = \frac{-\psi_1''(x) + [\psi_1'(x)]^2 - \phi_2(x)}{2\psi_0'(x)}.$$

Notice that, we get the sequence $\{\lambda, 1, \lambda^{-1}, \lambda^{-2}, ...\}$ for $\{g_n(\lambda)\}$. Therefore an asymptotic solution of (3.16) may be taken as

$$w(x;\lambda) \sim e^{\mp i\lambda\int^x \sqrt{\phi_0(\xi)}d\xi + ln[\phi_0(x)]^{-1/4} \mp \frac{1}{2}i\int^x \phi_1(\xi)[\phi_0(\xi)]^{-1/2}d\xi + O(\lambda^{-1})}$$

$$\sim [\phi_0(x)]^{-1/4}e^{\mp i\int^x [\lambda\phi_0(\xi) + \frac{1}{2}\phi_1(\xi)][\phi_0(\xi)]^{-1/2}d\xi}[1 + O(\lambda^{-1})] \tag{3.25}$$

as $\lambda \to \infty$ valid for a domain of x where $\phi_0(x) \neq 0$.

Remark: This procedure is valid for a larger class of functions $f(\lambda; x)$. It is NOT valid only for $f(\lambda; x)$ given in (3.17). It is required that $f(\lambda; x)$ has an asymptotic expansion as $\lambda \to \infty$, or consists of a finite number of terms where the x and λ variations of $f(\lambda; x)$ are separable.

If in (3.17), $\phi_0(x) = 0$, $\phi_1(x) \neq 0$ in the range of x of interest, then

$$f(\lambda; x) = \lambda\phi_1(x) + \phi_2(x) + \lambda^{-1}\phi_3(x) + ... \tag{3.26}$$

which may be written in the form of (3.17)

$$f(\mu; x) = \mu^2\phi_1(x) + \phi_2(x) + \mu^{-2}\phi_3(x) + ... \tag{3.27}$$

on replacing λ by μ^2. In this case, the asymptotic solution of (3.16) can be reproduced directly from (3.25)

$$w(\lambda; x) \sim [\phi_1(x)]^{-1/4} e^{\mp i\sqrt{\lambda}\int^x \sqrt{\phi_1(t)}\mathrm{d}t}[1 + O(\lambda^{-1/2})] \tag{3.28}$$

on writing ϕ_1 instead of ϕ_0 and setting $\phi_1 = 0$.

3.2.2 WKB approximation of differential equations of hypergeometric type

Let us consider again the differential equation of hypergeometric type

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0 \tag{3.29}$$

which can be written in the self-adjoint form

$$[\rho(x)\sigma(x)y']' + \lambda\rho(x)y = 0 \tag{3.30}$$

where $\rho(x)$ satisfies the differential equation

$$[\rho(x)\sigma(x)]' = \tau(x)\rho(x). \tag{3.31}$$

Our objective is to reduce (3.30) to the canonical in (3.16) using the so-called Liouville transformations. To this end, first we make use of the substitution

$$y(x) = \phi(x)u(x)$$

$$y'(x) = \phi'(x)u(x) + \phi(x)u'(x)$$
$$y''(x) = \phi''(x)u(x) + 2\phi'(x)u'(x) + \phi(x)u''(x)$$

to obtain

$$\rho(x)\sigma(x)\phi(x)u''(x) + \{2\rho(x)\sigma(x)\phi'(x) + \left[\rho(x)\sigma(x)\right]'\phi(x)\}u'(x)+$$

$$\{\rho(x)\sigma(x)\phi''(x) + \left[\rho(x)\sigma(x)\right]'\phi'(x) + \lambda\rho(x)\phi(x)\}u(x) = 0. \tag{3.32}$$

Second, we introduce a new independent variable $s = s(x)$ with

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x} = s'(x)\frac{\mathrm{d}u}{\mathrm{d}s}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2u}{\mathrm{d}s^2}\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}^2s}{\mathrm{d}x^2} = [s'(x)]^2\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + s''(x)\frac{\mathrm{d}u}{\mathrm{d}s}$$

which transforms (3.32) to the form

$$u''(s) + f(s)u'(s) + [\lambda g(s) - q(s)]u(s) = 0 \tag{3.33}$$

where

$$f(s) = \frac{2\rho\sigma s'\phi' + (\rho\sigma s')'\phi}{\rho\sigma\phi[s']^2}$$

$$g(s) = \frac{1}{\sigma[s']^2} \quad \text{and} \quad q(s) = \frac{(\rho\sigma\phi')'}{\rho\sigma\phi[s']^2}.$$

Now, we may choose $s(x)$ and $\phi(x)$ so that $g(s) = 1$ and $f(s) = 0$, i.e

$$s(x) = \int_{x_0}^{x} [\sigma(t)]^{-1/2}\,\mathrm{d}t \tag{3.34}$$

and

$$\phi(x) = [\sigma(x)]^{-1/4}[\rho^2(x)]^{-1/2} \tag{3.35}$$

which leads to the canonical form

$$u'' + [\mu^2 - q(s)]u = 0 \quad , \quad \lambda = \mu^2 \tag{3.36}$$

where

$$q(s) = \frac{1}{4\sigma}\left[\left(\frac{(\rho\sigma)'}{\rho\sigma} + \frac{\rho'}{\rho}\right)' + \left(\frac{3}{4}\frac{(\rho\sigma)'}{\rho\sigma} - \frac{1}{4}\frac{\rho'}{\rho}\right)\left(\frac{(\rho\sigma)'}{\rho\sigma} + \frac{\rho'}{\rho}\right)\right]. \tag{3.37}$$

This equation is formally the same as the differential equation in (3.16)

$$u'' + f(\mu; s)u = 0$$

with

$$f(\mu; s) = \mu^2 - q(s).$$

Comparing $f(\mu; s)$ with (3.27) we see that

$$\phi_1 = 1 \quad , \quad \phi_2 = -q(s) \quad , \quad \phi_k = 0 \quad , \quad k = 3, 4, ...$$

Furthermore, from (3.28) we deduce that an asymptotic solution of (3.36) is given by

$$u(\mu; s) = e^{\mp i\mu \int^s \mathrm{d}t}[1 + O(\mu^{-1})] = e^{\mp i\mu s}[1 + O(\mu^{-1})]$$

as $\mu \to \infty$. Therefore, two real asymptotic solutions are expressible as

$$u_1(\mu; s) = \cos(\mu s) + O(\mu^{-1})$$

and

$$u_2(\mu; s) = \sin(\mu s) + O(\mu^{-1})$$

as $\mu \to \infty$, and, hence,

$$u(\mu; s) = c_1 \cos(\mu s) + c_2 \sin(\mu s) + O(\mu^{-1}) \tag{3.38}$$

for some arbitrary constants c_1 and c_2.

Returning back to the original variables, we obtain the general solution

$$y(\mu;x) = \phi(x)\{c_1 \cos[\mu s(x)] + c_2 \sin[\mu s(x)]\} + O(\mu^{-1})$$

or

$$y(\lambda;x) = \frac{1}{\sqrt{[\sigma(x)]^{1/2}\rho(x)}}\{c_1 \cos[\sqrt{\lambda}s(x)] + c_2 \sin[\sqrt{\lambda}s(x)]\} + O(\lambda^{-1/2}) \quad (3.39)$$

of the differential equation of the hypergeometric type in (3.29), where we have used (3.35) for $\phi(x)$. For convenience, we define new functions $\xi(x)$ and $p(x)$

$$\xi(x) = \sqrt{\lambda}s(x) = \int_{x_0}^{x} p(t)\mathrm{d}t \quad , \quad p(x) = \sqrt{\frac{\lambda}{\sigma(x)}} \quad (3.40)$$

from (3.34), and rewrite (3.39) in the form

$$y(\lambda;x) = \frac{1}{\sqrt{\sigma(x)\rho(x)p(x)}}[A\cos\xi(x) + B\sin\xi(x)] + O(\lambda^{-1/2}) \quad (3.41)$$

as $\lambda \to \infty$, where A and B are arbitrary constants.

Remark: The solution $u(s)$ in (3.38) of the equation (3.36) may be derived by an alternative method. Actually, writing (3.36) in the form

$$u'' + \mu^2 u = q(s)u$$

and assuming that the right hand side is known, we find the solution by the method of variation of parameters. To be specific, the general solution is given by

$$u(s) = u_c(s) + R_\mu(s) \quad (3.42)$$

where $u_c(s)$ is the complementary solution

$$u_c(s) = A\cos(\mu s) + B\sin(\mu s)$$

and $R_\mu(s)$ denotes a particular solution of the type

$$R_\mu(s) = \frac{1}{\mu}\int_0^s \sin[\mu(s-t)]q(t)u(t)dt. \tag{3.43}$$

Therefore, we have to prove that the neglected particular solution (3.43) in (3.38) is of order $O(\mu^{-1})$ as $\mu \to \infty$.

To prove this, suppose that the solution in (3.42) is continuous on some open interval. Then, $R_\mu(s)$ in (3.43) should be neglecible as $\mu \to \infty$, i.e.

$$\mu|R_\mu(s)| = O(1). \tag{3.44}$$

From (3.43), the magnitude of $R_\mu(s)$ can be written as

$$|R_\mu(s)| \leq \frac{1}{\mu}LM(\mu) \qquad (3.45).$$

where $L = \int_{c_1}^{d_1}|q(t)|dt,\ M(\mu) = \max|u(s)|$. Note also that since $\sin(\mu s)$ and $\cos(\mu s)$ are bounded, $M_c(\mu) = \max|u_c(s)|$ is of constant order. Taking the absolute values of both sides of (3.42), we have

$$|u(s)| \leq |u_c(s)| + |R_\mu(s)| \tag{3.46}$$

or

$$|u(s)| \leq M(\mu) \leq M_c(\mu) + \frac{1}{\mu}LM(\mu). \tag{3.47}$$

If we solve (3.47) for $M(\mu)$ and use (3.45) for $\mu > L$, we obtain

$$|R_\mu(s)| \leq \frac{L}{\mu - L}M_c(\mu)$$

which establishes (3.44) as $\mu \to \infty$. From this inequality we can see that the neglected term is of order $\mu^{-1} = \lambda^{-1/2}$.

3.2.3 Application to orthogonal polynomials

Now, we apply (3.41) to the special choices of $\sigma(x)$, $\rho(x)$ and λ in section (2.2), which lead to the Jacobi and Hermite polynomials.

Jacobi Polynomials: Let us obtain an approximate formula for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for large n when $\alpha, \beta \geq 0$ and $x \in (-1,1)$. The Jacobi differential equation in self adjoint is

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}y']' + n(n+\alpha+\beta+1)(1-x)^{\alpha}(1+x)^{\beta}y = 0 \qquad (3.48)$$

where $\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\sigma(x) = (1-x^2)$ and $\lambda = n(n+\alpha+\beta+1)$.

The asymptotic solution of (3.48) as $n \to \infty$ or $\lambda \to \infty$, is found from equation (3.41) as

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}}\{A\cos[\xi(x)] + B\sin[\xi(x)]\} + O(n^{-1}) \quad (3.49)$$

where $\xi(x) = \sqrt{n(n+\alpha+\beta+1)}\int_{x_0}^{x}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \sqrt{n(n+\alpha+\beta+1)}\arcsin(x)$ for $x_0 = 0$, $x \in (-1,1)$. This is an approximation to Jacobi polynomials, for a suitable choice of the coefficients A and B. Indeed, if we equate (3.49) and its derivative at $x=0$ to $P_n^{(\alpha,\beta)}(0)$ and $[P_n^{(\alpha,\beta)}]'(0) = 0$, we find that

$$A = P_n^{(\alpha,\beta)}(0)$$

and

$$B = [P_n^{(\alpha,\beta)}]'(0) - P_n^{(\alpha,\beta)}(0).$$

Legendre Polynomials: When $\alpha = \beta = 0$, (3.49) becomes

$$P_n(x) = \frac{1}{(1-x)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}}\{A\cos[\xi(x)] + B\sin[\xi(x)]\} + O(n^{-1}) \qquad (3.50)$$

where $\xi(x) = \sqrt{n(n+1)} \int_{x_0}^{x} \frac{1}{\sqrt{1-t^2}} \mathrm{d}t = \sqrt{n(n+1)} \arcsin(x)$ for $x_0 = 0$, $x \in (-1, 1)$ which is an asymptotic formula for Legendre polynomials. In this case, it is possible to show that

$$P_{2n}(x) = P_{2n}(0) \cos[\xi(x)] + O(n^{-1})$$

and

$$P_{2n+1}(x) = P'_{2n+1}(0) \sin[\xi(x)] + O(n^{-1})$$

respectively.

Hermite Polynomials: As a last application, we will obtain an approximate formula for the Hermite polynomials $H_n(x)$ for large n when $x \in (-\infty, \infty)$. The Hermite differential equation in self adjoint form is

$$[e^{-x^2} y']' + 2ny = 0 \tag{3.53}$$

where $\rho(x) = 1$, $\sigma(x) = e^{-x^2}$ and $\lambda = 2n$. The approximate solution of (3.53) as $n \to \infty$, is found from equation (3.41)

$$y_n(x) = e^{x^2} \{A \cos[\sqrt{2n}x] + B \sin[\sqrt{2n}x]\} + O(n^{-1/2}) \tag{3.54}$$

which implies that

$$H_{2n}(x) = H_{2n}(0) e^{x^2} \cos[\sqrt{4n}x] + O(n^{-1/2})$$

and

$$H_{2n+1}(x) = \frac{1}{2}\left(n + \frac{1}{2}\right)^{-1/2} [H_{2n+1}]'(0) e^{x^2} \cos[\sqrt{4n+2}x] + O(n^{-1/2})$$

for appropriate selection of constants A and B.

As is shown, apart from that of Hermite polynomials, the asymptotic solutions have singularities at the end points of their respective intervals. In other words, the approximation in (3.49) for the Jacobi polynomials is not valid at the end points. Now, we should look for a different approach to obtain asymptotic solutions valid at the end points as well.

3.2.4 Approximate solution of differential equation of hypergeometric type valid at the end points of domain

We try to determine the behaviour of the classical orthogonal polynomials about the singular points of their differential equations.

Consider, the self-adjoint form of (3.30) for $a \le x < b$. We look for the case when one of the $\sigma(x)\rho(x)$ and $\rho(x)$ is zero at $x = a$. We can write $\sigma(x)\rho(x) = (x-a)^m\bar{\sigma}(x)$, $\rho(x) = (x-a)^l\bar{\rho}(x)$ where $\bar{\sigma}(a) > 0$, $\bar{\rho}(a) > 0$ and $\bar{\sigma}(x)$ and $\bar{\rho}(x)$ have continuous second derivatives for $a \le x < b$. We assume that $l - m > -2$ so that $s(x)$ in (3.34) is finite at $x = a$ [10]. Substituting $\sigma(x)\rho(x)$ and $\rho(x)$ in $s(x)$ and $q(x)$ defined at (3.34) and (3.35) respectively, we get

$$s(x) = \int_a^x \sqrt{\frac{\bar{\rho}(t)}{\bar{\sigma}(t)}}(t-a)^{\frac{l-m}{2}}dt \tag{3.55}$$

and

$$q(s) = \frac{1}{16}(x-a)^{m-l-2}\frac{\bar{\sigma}(x)}{\bar{\rho}(x)}\left\{(l+m)(3m-l-4) + 2(x-a)\mathcal{A}(x) + 4(x-a)^2\mathcal{B}(x)\right\} \tag{3.56}$$

where

$$\mathcal{A}(x) = (3m+l)\frac{\bar{\sigma}'(x)}{\bar{\sigma}(x)} + \frac{\bar{\rho}'(x)}{\bar{\rho}(x)},$$

$$\mathcal{B}(x) = \left[\frac{\bar{\sigma}'(x)}{\bar{\sigma}(x)} + \frac{\bar{\rho}'(x)}{\bar{\rho}(x)}\right]' + \left[\frac{3}{4}\frac{\bar{\sigma}'(x)}{\bar{\sigma}(x)} - \frac{1}{4}\frac{\bar{\rho}'(x)}{\bar{\rho}(x)}\right]\left[\frac{\bar{\sigma}'(x)}{\bar{\sigma}(x)} + \frac{\bar{\rho}'(x)}{\bar{\rho}(x)}\right].$$

Now, let us find an approximation for $s(x)$ as $x \to a$. By using the mean value theorem for integrals, (3.55) can be taken as

$$s(x) = \sqrt{\frac{\bar{\rho}_0(\xi)}{\bar{\sigma}_0(\xi)}}\int_a^x (t-a)^{\frac{l-m}{2}}dt = \sqrt{\frac{\bar{\rho}_0(\xi)}{\bar{\sigma}_0(\xi)}}\frac{(x-a)^{\frac{l-m+2}{2}}}{(l-m+2)/2}$$

where $a < \xi < x$, so we get

$$s(x) \sim \sqrt{\frac{\bar{\rho}(a)}{\bar{\sigma}(a)}} \frac{(x-a)^{\frac{l-m+2}{2}}}{(l-m+2)/2}$$

as $x \to a$. Now, $q(s)$ in (3.56) reduces to

$$q(s) = \frac{\nu^2 - \frac{1}{4}}{s^2} + s^{\gamma-2} f(s)$$

where $\gamma = \frac{2}{l-m+2} > 0$, $\nu = \frac{|m-1|}{l-m+2}$ and $f(s)$ stands for a continuous function on $0 \leq s < s(b)$. In this case, the differential equation in (3.36) takes the form

$$u'' + \left(\mu^2 - \frac{\nu^2 - \frac{1}{4}}{s^2}\right) u = s^{\gamma-2} f(s) u \qquad (3.57)$$

which has a singular point at $s = 0$. As is made in the remark of the previous section, we may solve this equation by the method of variation of parameters assuming that the right hand side is known. The homogenous part

$$u'' + \left(\mu^2 - \frac{\nu^2 - \frac{1}{4}}{s^2}\right) u = 0 \qquad (3.58)$$

is the so-called Lommel equation which may be transformed into a Bessel equation. Actually, by the change of the variable form s to t, $t = \mu s$, we have

$$\frac{d^2u}{dt^2} + \left(1 - \frac{\nu^2 - \frac{1}{4}}{t^2}\right) u = 0. \qquad (3.59)$$

Now, transforming the dependent variable, we obtain

$$v'' + \frac{1}{t} v' + \left(1 - \frac{\nu^2}{t^2}\right) v = 0 \qquad (3.60)$$

where $v(t) = t^{-1/2} u(t)$, which is the Bessel equation . When ν is not an integer, it is known that the general solution of the Bessel equation is given by

$$y(t) = A J_\nu(t) + B J_{-\nu}(t) \qquad (3.61)$$

where $J_\nu(t)$ is the Bessel function of the first kind of order ν [8]. If ν is an integer, we introduce the Bessel function of the second kind denoted by $Y_\nu(t)$ and write the general solution in the form

$$y(t) = AJ_\nu(t) + BY_\nu(t). \tag{3.62}$$

The solution of the homogenous equation in (3.58) is now written as

$$u_c(s) = \sqrt{\mu s}[AJ_\nu(\mu s) + BJ_{-\nu}(\mu s)]$$

for a non-integer ν. Hence the general solution of (3.57) is

$$u(s) = u_c(s) + R_\mu(s)$$

where

$$R_\mu(s) = \int_{s_0}^{s} K_\mu(t; s)t^{\gamma-2} f(t)\mathrm{d}t$$

in which

$$K_\mu(t; s) = \frac{\pi}{2\mu\sin(\pi\nu)}\left\{\sqrt{\mu s}\sqrt{\mu t}[J_\nu(\mu s)J_{-\nu}(\mu t) + J_\nu(\mu t)J_{-\nu}(\mu s)]\right\}.$$

If we keep in mind the inequality

$$|(\mu s)^{\frac{1}{2}} J_{\pm\nu}(\mu s)| \leq \begin{cases} C & \text{for} \quad \mu s > 1 \\ C(\mu s)^{\pm\nu+1/2} & \text{for} \quad \mu s \leq 1 \end{cases} \tag{3.64}$$

valid for the Bessel functions, $R_\mu(s)$ can be neglected [10]. More specifically, returning back to the original variables, we find an asymptotic solution of (3.30) of the form

$$y(x) = \sqrt{\frac{\xi(x)}{\rho(x)\sigma(x)p(x)}}\left\{AJ_\nu[\xi(x)] + BJ_{-\nu}[\xi(x)]\right\} + \\ + \begin{cases} O(\mu^{-1/2}) & , \quad \xi > 1 \\ O(\mu^{\nu}) & , \quad \xi \leq 1 \end{cases} \tag{3.65}$$

for $a \leq x < b$ as $\mu \to \infty$, where $p(x) = \mu\,[\sigma(x)]^{-1/2}$, $\xi(x) = \int_a^x p(t)dt$, and $\lambda = \mu^2$. When ν is an integer $J_{-\nu}(\xi)$ is replaced by $Y_\nu(\xi)$. Notice that, this asymptotic solution is valid at the singular point of the differential equation at $x = a$. Similar procedure can be adopted for the other end point, $x = b$, of the interval $a < x \leq b$ if there is a singularity there.

3.2.5 Approximate solution of differential equations of orthogonal polynomials valid on their closed orthogonality intervals

Jacobi Polynomials: Consider the Jacobi differential equation

$$[(1-x)^{\alpha+1}(1+x)^{\beta+1}y']' + n(n+\alpha+\beta+1)(1-x)^{\alpha}(1+x)^{\beta}y = 0$$

generating the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. It is known that these polynomials are orthogonal on $x \in [-1,1]$ when $\alpha, \beta \geq -1$. For large values of n, we may now derive an asymptotic solution from (3.65) which is valid for $-1 \leq x < 1-\delta$. We see that, in Jacobi differential equation, $\sigma(x)\rho(x) = (1+x)^{\beta+1}\bar{\sigma}(x)$ and $\rho(x) = (1+x)^{\beta}\bar{\rho}(x)$ with

$$\bar{\sigma}(x) = (1-x)^{\alpha+1}, \quad \bar{\rho}(x) = (1-x)^{\alpha}.$$

Therefore the parameters a, l and m in (3.55) are $a = -1$, $l = \beta$ and $m = \beta + 1$, respectively. Furthermore, $\nu = \beta$ and $\mu = \sqrt{n(n+\alpha+\beta+1)}$. Then, the asymptotic solution in (3.65) reads as

$$y(x) = \frac{\sqrt{\xi(x)}}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}}\{AJ_\beta[\xi(x)] + BJ_{-\beta}[\xi(x)]\} + \begin{cases} O(n^{-1/2}) & , \quad \xi > 1 \\ O(n^{\beta}) & , \quad \xi \leq 1 \end{cases} \tag{3.66}$$

where

$$\xi(x) = \mu \int_{-1}^{x} \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \mu \arccos(-x)$$

and $\mu \sim n$ as $n \to \infty$. With the suitable choice of the coefficients A and B, we get an approximation to the Jacobi polynomials. To this end, we must have

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)} n! = \lim_{x \to -1} y(x).$$

In order for the limit on the right hand side to exist, we choose $B = 0$ in (3.66) since $J_{-\beta}(\xi)$ tends to the infinity as $x \to -1$. Also the other constant A is given by

$$A = P_n^{(\alpha,\beta)}(-1) \lim_{x \to -1} \frac{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}}{\sqrt{\xi(x)} J_\beta[\xi(x)]}$$

which may be written as

$$A = 2^{\frac{\alpha}{2}+\frac{1}{4}} P_n^{(\alpha,\beta)}(-1) \lim_{x \to -1} \left[\frac{\sqrt{1+x}}{\xi(x)} \right]^{\beta+\frac{1}{2}} \lim_{\xi \to 0} \frac{\xi^\beta}{J_\beta(\xi)}.$$

By L'Hospital rule, we have

$$\lim_{x \to -1} \frac{\sqrt{1+x}}{\xi(x)} = \frac{1}{\sqrt{2n(n+\alpha+\beta+1)}}$$

and, therefore

$$A = \frac{(-1)^n 2^{\alpha+\frac{\beta}{2}} \Gamma(n+\beta+1)}{n! [n(n+\alpha+\beta+1)]^{\frac{\beta}{2}+\frac{1}{4}}} \tag{3.67}$$

where we have used the fact that

$$\lim_{\xi \to 0} \frac{J_\beta(\xi)}{\xi^\beta} = \frac{1}{2^\beta \Gamma(\beta+1)}$$

known for the Bessel functions [14]. Substituting A into (3.66), we get

$$P_n^{(\alpha,\beta)}(-\cos\theta) = \frac{(-1)^n \Gamma(n+\beta+1)\sqrt{\theta/2}}{n! \mu^\beta [\cos(\theta/2)]^{\alpha+\frac{1}{2}} [\sin(\theta/2)]^{\beta+\frac{1}{2}}} J_\beta(\mu\theta) + \begin{cases} O(n^{-1/2}), & n\theta > 1 \\ O(n^\beta), & n\theta \le 1 \end{cases} \tag{3.68}$$

for $0 \le \theta \le \pi - \epsilon$ or $-1 \le x < 1$ on setting $x = -\cos\theta$, where ϵ is a small parameter.

Using the relationship

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

we can easily find an approximate formula for $P_n^{(\alpha,\beta)}(x)$ from (3.68) of the form

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\beta+1)\sqrt{\theta/2}}{n!\mu^{\beta}[\cos(\theta/2)]^{\beta+\frac{1}{2}}[\sin(\theta/2)]^{\alpha+\frac{1}{2}}} J_\alpha(\mu\theta) + \begin{cases} O(n^{-1/2}), & n\theta > 1 \\ O(n^{\alpha}), & n\theta \leq 1 \end{cases}. \tag{3.69}$$

valid for $0+\epsilon \leq \theta \leq \pi$ or $-1 < x \leq 1$.

In particular, when $\alpha = \beta = 0$ (3.68) and (3.69)

$$P_n(-\cos\theta) = \frac{(-1)^n\Gamma(n+1)\sqrt{\theta}}{n!\sqrt{\sin\theta}} J_0(\mu\theta) + \begin{cases} O(n^{-1/2}) & , \ n\theta > 1 \\ O(1) & , \ n\theta \leq 1 \end{cases} \tag{3.70}$$

and

$$P_n(\cos\theta) = \frac{(-1)^n\Gamma(n+1)\sqrt{\theta}}{n!\sqrt{\sin\theta}} J_0(\mu\theta) + \begin{cases} O(n^{-1/2}), & n\theta > 1 \\ O(1), & n\theta \leq 1 \end{cases} \tag{3.71}$$

reduce to asymptotic formulas for the Legendre polynomials, respectively.

Laguerre polynomials: Now, consider the Laguerre differential equation

$$[x^{\alpha+1}e^{-x}y']' + nx^{\alpha}e^{-x}y = 0$$

generating the Laguerre polynomials $L_n^{\alpha}(x)$. For large values of n, we may now derive an asymptotic solution from (3.65) which is valid for $0 \leq x < \infty$. We can see that, in Laguerre differential equation, $\sigma(x)\rho(x) = x^{\alpha+1}\bar{\sigma}(x)$ and $\rho(x) = x^{\alpha}\bar{\rho}(x)$ with

$$\bar{\sigma}(x) = e^{-x}, \quad \bar{\rho}(x) = e^{-x}.$$

Therefore the consants a, l and m in (3.55) are $a = 0$, $l = \alpha$ and $m = \alpha + 1$, respectively. The parameter ν and μ are now $\nu = \alpha$ ad $\mu = \sqrt{n}$. Then, the asymptotic solution in (3.65) takes the form

$$y(x) = \frac{\sqrt{2}}{x^{\alpha/2}e^{-x/2}}\{AJ_\alpha(2\sqrt{nx}) + BJ_{-\alpha}(2\sqrt{nx})\} + \begin{cases} O(n^{-1/4}) & , \quad 2\sqrt{nx} > 1 \\ O(n^{\alpha/2}) & , \quad 2\sqrt{nx} \leq 1 \end{cases}. \tag{3.72}$$

By choosing A and B appropriately, we can write an approximation to Laguerre polynomials. To this end, we must have

$$L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} = \lim_{x\to 0} y(x). \tag{3.73}$$

In order for the limit on the right hand side to exist, we choose $B = 0$ in (3.72) since $J_{-\alpha}(2\sqrt{nx})$ tends to the infinity as $x \to 0$. Also the other constant A is given by

$$A = L_n^\alpha(0) \lim_{x\to 0} \frac{x^{\alpha/2}e^{-x/2}}{\sqrt{2}J_\alpha(2\sqrt{nx})}$$

which may be written as

$$A = \frac{1}{2^{\alpha+1/2}n^{\alpha/2}} L_n^\alpha(0) \lim_{x\to 0} \frac{(2\sqrt{nx})^\alpha}{J_\alpha(2\sqrt{nx})}.$$

Therefore

$$A = \frac{\Gamma(n+\alpha+1)}{\sqrt{2}n^{\alpha/2}n!}.$$

where we have used the fact to find (3.67) which is about Bessel functions. Substituting A in (3.72), we get

$$L_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)e^{x/2}}{n!(nx)^{\alpha/2}} J_\alpha(2\sqrt{nx}) + \begin{cases} O(n^{-1/4}) & , \quad 2\sqrt{nx} > 1 \\ O(n^{\alpha/2}) & , \quad 2\sqrt{nx} \leq 1 \end{cases}.$$

valid for $0 \leq x < \infty$.

CHAPTER 4

Hilb's Type Asymptotic Approximation of Jacobi Polynomials

Let us take the differential equation [1]

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left[\frac{1/4-\alpha^2}{4\sin^2(\theta/2)} + \frac{1/4-\beta^2}{4\cos^2(\theta/2)} + N^2\right]u = 0 \tag{4.1}$$

whose solution is written as

$$u = u(\theta) = \sin^{\alpha+1/2}(\theta/2)\cos^{\beta+1/2}(\theta/2)P_n^{(\alpha,\beta)}(\cos\theta), \quad 0 \le \theta < \pi-\delta, \quad \delta > 0$$

in terms of Jacobi polynomials where $N = n+(\alpha+\beta+1)/2$ and $\alpha, \beta > -1$. Now, we want to find an asymptotic approximations of solutions of this equation as $N \to \infty$, by the method given in part 3.2.4. The equation (4.1) can be rewritten in form (3.57) as

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left[N^2 + \frac{1/4-\alpha^2}{\theta^2}\right]u = \left[\frac{\beta^2-1/4}{4\cos^2(\theta/2)} + \left(\frac{1}{4}-\alpha^2\right)\left(\frac{1}{\theta^2} - \frac{1}{4\sin^2(\theta/2)}\right)\right]u. \tag{4.2}$$

We can solve this equation by the method of variation of parameters, assuming the right-hand side is known. Since the homogenous part of (4.2)

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \left[N^2 + \frac{1/4-\alpha^2}{\theta^2}\right]u = 0$$

is the Lommel equation, its solution is written as

$$u_c(\theta) = \sqrt{\theta}[AJ_\alpha(N\theta) + BJ_{-\alpha}(N\theta)]$$

where A and B are constants. We obtain the general solution of (4.2) in the form

$$u(\theta) = \sin^{\alpha+1/2}(\theta/2)\cos^{\beta+1/2}(\theta/2)P_n^{(\alpha,\beta)}(cos\theta) = u_c(\theta) + \sqrt{\theta}R_N(\theta). \tag{4.3}$$

where

$$R_N(\theta) = \int_{\theta_0}^{\theta} \sqrt{t}K(t;\theta)f(t)u(\theta)dt,$$

in which

$$K(t;\theta) = [J_\alpha(N\theta)J_{-\alpha}(Nt) - J_{-\alpha}(N\theta)J_\alpha(Nt)]$$

and

$$f(t) = \frac{\pi}{2\sin\alpha\pi}\left[\frac{\beta^2 - 1/4}{4\cos^2(t/2)} + \left(\frac{1}{4} - \alpha^2\right)\left(\frac{1}{t^2} - \frac{1}{4\sin^2(t/2)}\right)\right]$$

where $f(t)$ is independent of n. It can be shown that $R_N(\theta)$ can be neglected in (4.3) as $N \to \infty$. If $N \to \infty$, we have $n \to \infty$, by writing ν and μs as α and $n\theta$, respectively, the inequality in (3.64) reads as

$$|J_{\pm\alpha}(n\theta)| \leq \begin{cases} Cn^{-1/2} & , \quad n\theta > 1 \\ Cn^{\alpha} & , \quad n\theta \leq 1 \end{cases} .$$

Then, $R_N(\theta)$ can be neglected by the method in remark in part 3.2.4. More specifically, (4.3) can be represented in the form

$$\theta^{-1/2}\sin^{\alpha+1/2}(\theta/2)\cos^{\beta+1/2}(\theta/2)P_n^{(\alpha,\beta)}(cos\theta) = AJ_\alpha(N\theta) + BJ_{-\alpha}(N\theta) +$$

$$+ \begin{cases} O(n^{-3/2}) & , \quad n\theta > 1 \\ O(n^{\alpha}) & , \quad n\theta \leq 1 \end{cases} .$$

Now, we will look for the suitable coefficients A and B to get the asymptotic approximation of Jacobi polynomials at $[-1, 1]$. To this end, we must have

$$\lim_{\theta\to 0} P_n^{(\alpha,\beta)}(\cos\theta) = P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}.$$

In order for the limit on the right hand side to exist, we choose $B = 0$ in (3.66) since $J_{-\alpha}(N\theta)$ tends to the infinity as $\theta \to 0$. Also the other constant A is given by

$$A = \lim_{\theta\to 0} \frac{\theta^{-1/2}\sin^{\alpha+1/2}(\theta/2)\cos^{\beta+1/2}(\theta/2)P_n^{(\alpha,\beta)}(\cos\theta)}{J_\alpha(N\theta)}$$

which may be written as

$$A = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{N^\alpha} \lim_{\theta\to 0}\left[\frac{\sin(\theta/2)}{\theta}\right]^{\alpha+1/2} \lim_{\theta\to 0}\frac{(N\theta)^\alpha}{J_\alpha(N\theta)}.$$

By L'Hospital rule, we have

$$\lim_{\theta\to 0}\left[\frac{\sin(\theta/2)}{\theta}\right]^{\alpha+1/2} = 2^{-\alpha-1/2}.$$

and therefore

$$A = \frac{\Gamma(n+\alpha+1)}{\sqrt{2}N^\alpha n!}.$$

where we have used (3.67). Thus, the asymptotic solution of (4.1) in terms of Jacobi polynomials that is valid on $[-1, 1]$ as $N \to \infty$, $\alpha, \beta > -1$ is found in the form

$$\sin^\alpha(\theta/2)\cos^\beta(\theta/2)P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+1)}{\sqrt{2}N^\alpha n!}\sqrt{\frac{\theta}{\sin\theta}}J_\alpha(N\theta) +$$

$$+\begin{cases} O(n^{-3/2}) & , \quad n\theta > 1 \\ O(n^\alpha) & , \quad n\theta \le 1 \end{cases}. \tag{4.4}$$

This solution is reffered to as Hilb's type asymptotic solution of (4.1) [15]. It follows from $u(\theta)$ that

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+1)\sqrt{\theta/2}}{\sqrt{2}N^\alpha n! \sin^{\alpha+1/2}(\theta/2)\cos^{\beta+1/2}(\theta/2)} J_\alpha(N\theta) + \\ + \begin{cases} O(n^{-3/2}) & , \quad n\theta > 1 \\ O(n^\alpha) & , \quad n\theta \leq 1 \end{cases} . \tag{4.5}$$

which is an alternative asymptotic form of the Jacobi polynomials. Clearly, comparing with (3.69), we see that it is more accurate since the error term is of order $O(n^{-3/2})$.

CHAPTER 5

CONCLUSION

In this thesis we deal with the asypmtotic WKB method for differential equations of the hypegeometric type. It is shown that the asymptotic solutions in (3.41) of section 3.2.2, obtained by the standart WKB method are not valid at the singular points of the differential equations under consideration. In section 3.2.4, we extend our analysis to find aymptotic approximations reflecting the behaviour of solutions at singular points as well. Thus, we present formulas for $P_n^{(\alpha,\beta)}(x)$ and $L_n^{(\alpha)}(x)$ as $n \to \infty$ which are defined in their orthogonality intervals $[-1, 1]$ and $[0, \infty)$, respectively.

Furthermore, we consider an alternative differential equation in chapter 4 whose solutions are involved again in the Jacobi polynomials. Then, we have verified that such a differential equation has asymptotic solutions leading to a more accurate approximation for the Jacobi polynomials. Therefore, we observe that several asymptotic approximations can be derived for a function in this way.

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