ON PRINCIPLES OF B-SMOOTH DISCONTINUOUS FLOWS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ON PRINCIPLES OF B-SMOOTH DISCONTINUOUS FLOWS

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Discontinuous dynamical system defined by impulsive autonomous differential equation is a field that has actually been considered rarely. Also, the properties of such systems have not been discussed thoroughly in the course of mathematical researches so far.
This thesis comprises two parts, elaborated with a number of examples. In the first part, some results of the previous studies on the classical dynamical system are exposed. In the second part, the definition of discontinuous dynamical system defined by impulsive autonomous differential equation is formulated, and its properties are investigated, in the view of the known results of the studies on the classical dynamical system and impulsive differential equations.

Keywords:Dynamical systems, discontinuous flows, smoothness.

ÖZ

B-DÜZGÜN SÜREKSİZ AKIŞLARIN İLKELERİ ÜZERİNE

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" Impulsive " otonom diferensiyel denklemler ile tanımlanan süreksiz dinamik sistemler alanı nadiren ele alınmıştır. Ayrıca bu sistemin özellikleri, şu ana kadarki matematiksel araştırmalarda tartışılmamıştır.

Bu tez, örneklerle detaylandırılmış, iki bölümden oluşmaktadır. Birinci bölümde, klasik dinamik sistemler üzerine daha önceki çalışmaların bazı sonuçları belirtilmiştir. İkinci bölümde impulsive otonom diferensiyel denklemler ile tanımlanan süreksiz dinamik sistemlerin tanımı, klasik dinamik sistemler ve impalsif difensiyel denklemlerin bilinen sonuçları yardımi ile, açık bir şekilde belirtilmiş ve özellikleri araştırılmıştır.

Anahtar Kelimeler: Dinamik sistemler, süreksiz dinamik akışlar, düzgünlük.

To my lovely family and husband

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LIST OF SYMBOLS

- $\mathbb{N}$: the set of all natural numbers.
- $\mathbb{Z}$: the set of all integer numbers.
- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{R}^+ = (0, \infty)$.
- $\mathbb{N}^+$: the set of all positive natural numbers.
- $\|.\|$: the Euclidean norm, $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, where $x \in \mathbb{R}^n$, and $n \in \mathbb{N}$.
- $\mathcal{A}$: a set of indices is in $\mathbb{Z}/\{0\}$.
- ∂A: the boundary of a set A.
- $B(x_0, \xi) : \{x \in \mathbb{R}^n | \quad ||x - x_0|| < \xi\}$, a ball with center $x_0 \in \mathbb{R}^n$ and radius $\xi \in \mathbb{R}$, $\xi > 0$.
- $A \backslash B$: difference of sets and set B.
- $\bar{G}$: the closure of the set G in $\mathbb{R}^n$.
- A_ϵ : an open set A_ϵ is called ϵ-neighborhood. If $A \subset \mathbb{R}^n$, then $A_\epsilon = \bigcup_{x \in A} B(x, \epsilon)$.
- $\|A\| = \sup \{\|Ax\| \,|\, \|x\| = 1\}$, a norm of matrix A.
- $C^1(X, Y)$: set of C^1 maps from X to Y.
- $(\alpha \hat{,} \beta]$: an oriented interval that is;

$$(\alpha \hat{,} \beta] = \begin{cases} (\alpha, \beta], & \text{if } \alpha \leq \beta \\ (\beta, \alpha], & \text{if } \alpha > \beta \end{cases} .$$

- $dist(A, B)$: a distance between two sets $A, B \subset \mathbb{R}^n$ as $dist(A, B) = \inf\{||a - b|| \,|\, a \in A, b \in B\}$.

- A^{-1} : the inverse of matrix A.
- $i\,[0,t]$: the number of discontinuity points of a solution $x(t,x_0)$ in the interval $[0,t]$.
- $\tilde{i}\,[t,0]$: the number of discontinuity points of a solution $x(t,x_0)$ in the interval $[t,0]$.
- $[.]$: a greatest integer function.

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CHAPTER 1

Description of dynamical systems (flows)

1.1 Introduction

A flow is a type of a dynamical system (DS) which can be used to describe solutions of mathematical models, where state at time t is completely specified by the values of n real variables $x_1, x_2, \ldots, x_n$.Accordingly, the system is such that changing the rate of these variables merely depends upon the value of the variables themselves, so that the law of motion can be expressed by means of n differential equations of the first order

$$\dot{x} = f(x), \tag{1.1.1}$$

where $x = \{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$ and $\dot{x} = \frac{dx}{dt}$.

For the classical DS, one can consider a system of differential equation of the form (1.1.1), where f is assumed to be continuous function of its arguments in a certain domain G, which might be a Euclidean space or an open subset of Euclidean space $\mathbb{R}^n$.

Let us begin describing mathematically a DS as a map $\Phi : \mathbb{R} \times G \to G$ defined by $(t, x) \to x_t$ which is continuously differentiable or at least continuous and continuously differentiable in t.
We formalize DS in the following definition [16], [32].

Definition 1.1.1. *We say that a dynamical system is a C^1-map $\phi : \mathbb{R} \times G \to G$, which satisfies the following properties:*

i) $\phi(0, x) = x$, *for all* $x \in G$;

ii) *$\phi(t+s,x) = \phi(t,(\phi(s,x)))$, is valid for all $t, s \in \mathbb{R}$ and $x \in G$.*

Remark 1.1.1. *It follows from definition that the map $\phi_t : G \to G$, where $\phi_t(x) = \phi(t,x)$, is C^1 for each t and has C^1 inverse ϕ_{-t}.*

Remark 1.1.2. *The map ϕ_t with $t \in \mathbb{R}$ is a one parameter group of transformation of the solution space G into itself. It is known that any one parameter group of transformations of the set G is called a phase flow with the phase space G [49].*

Remark 1.1.3. *A DS which satisfies properties **i)** and **ii)** of Definition 1.1.1 is called the smooth dynamical system [7].*

Every dynamical system gives rise to a differential equation. We give precise example as follows [32]. It is easy to see that if A is an $n \times n$ matrix, then the function

$$\phi(t,x) = e^{At}x$$

defines DS on $\mathbb{R}^n$, and also, for each $x_0 \in \mathbb{R}^n$, $\phi(t,x_0)$ is the solution of the initial value problem

$$\dot{x} = Ax$$
$$x(0) = x_0.$$

Now, let us consider the general case. If $\phi(t,x)$ is DS on G, then the function

$$f(x) = \frac{d\phi(t,x)}{dt}|_{t=0},$$

generally defines a $C^1(G)-$vector field on G, and for each $x_0 \in G$,

$$x(0) = x_0. \tag{1.1.2}$$

$\phi(t,x_0)$ is the solution of the initial value problem (1.1.1), (1.1.2).
Moreover, for each $x_0 \in G$, the maximal interval of existence of $\phi(t,x_0)$ is $I(x_0) = (-\infty,\infty)$. Thus, each DS gives rise to function $f \in C^1$, and DS describes the solution set of the differential equation defined by this function. Conversely,

given a differential equation (1.1.1) with $f \in C^1(G)$ and G, open subset of $\mathbb{R}^n$, the solution $\phi(t, x_0)$ of the initial value problem (1.1.1), (1.1.2) will be DS on G if and only if for all $x_0 \in G$, $\phi(t, x_0)$ is defined for all $t \in \mathbb{R}$; i.e., if and only if for all $x_0 \in G$, the maximal interval of existence $I(x_0)$ of $\phi(t, x_0)$ is $(-\infty, \infty)$. In this case, we say that $\phi(t, x_0)$ is the dynamical system on G defined by (1.1.1).

In the following sections, we deal with fundamental existence and uniqueness theorem, maximal interval of existence, continuation of solutions, flow of differential equation, continuous dependence on initial value and differential dependence on initial value to consider the properties of classical DS. These are contained in [16], [32],[35],[37].

1.1.1 Existence and uniqueness

Often, computing solution explicitly is not possible, and indeed, it is a priori obvious that system (1.1.1) has not unique solutions for all initial conditions. The following is the fundamental existence and uniqueness theorems which can be found particulary in [7], [37].

Theorem 1.1.1. *[37]: Consider a system of differential equation (1.1.1) where the function $f(x)$ is assumed to be continuous in a closed and bounded domain $\bar{G}$ containing x_0. Then there exists a solution of IVP (1.1.1), (1.1.2) which is defined in the interval*

$$\frac{-d}{M\sqrt{n}} \leq t - t_0 \leq \frac{d}{M\sqrt{n}},$$

where $d = dist(\partial G, x_0)$ is the distance of x_0 from the boundary of the domain $\bar{G}$, and $M = max_{\bar{G}} \|f(x)\|$ is an upper bound of $\|f(x)\|$ in the domain $\bar{G}$.

Theorem 1.1.2. *[7]: Let G be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $f \in C^1(G)$. Then there exist $\alpha, \beta \in \mathbb{R}$ and $\alpha < 0 < \beta$ such that the IVP (1.1.1), (1.1.2) has a unique solution $x(t)$ on the interval (α, β).*

Lipschitz condition may be seen in many places in our study. A function f is

Lipschitz on G if for all $x, y \in G$ the following inequality is valid:

$$\|f(x) - f(y)\| \leq L \|x - y\|,$$

for a nonnegative constant L.

Sometimes, instead of lipschitzian function, local lipschitzian function may be more useful in the theory [16].

DEFINITION 1.1.2. *We call f locally Lipschitz if each point of G, an open subset of $\mathbb{R}^n$, has a neighborhood G_0 in G such that the restriction $f_{|G_0}$ is Lipschitz.*

The following assertion which can be seen in [16] will be needed in the main part of our study.

LEMMA 1.1.1. *Let the function $f : G \to \mathbb{R}^n$ be C^1, then f is locally Lipschitz.*

DEFINITION 1.1.3. *A point x is called a singular point of (1.1.1) if $f(x) = 0$.*

The following definitions are important to have a geometrical approach for the solution. Now, suppose that IVP (1.1.1), (1.1.2) has a unique solution $x(t)$ defined for t on an interval I containing x_0. By the *motion through* (t_0, x_0), we mean the set

$$\{(t, x(t)) : t \in I\}.$$

This is, of course, the graph of the function x.
By the *trajectory,* we mean the set

$$T(x_0) = \{x(t) : t \in I\}.$$

The *positive trajectory* is defined as

$$T_+(x_0) = \{x(t) : t \in I, \quad t \geq t_0\}.$$

Also, the *negative trajectory* is defined as

$$T_-(x_0) = \{x(t) : t \in I, \quad t \leq t_0\}.$$

1.1.2 The maximal interval of existence

The interval of existence and uniqueness of the solution of system (1.1.1) is quite important to continue with DSs. Let us recall the maximal interval of existence of the solution .

DEFINITION 1.1.4. *Assume that* $x_1 : J_1 \to G$, $x_2 : J_2 \to G$ *are solutions of (1.1.1). We say that* x_2 *is continuation of* x_1 *if* $J_1 \subset J_2$ *and* $x_2(t) = x_1(t)$*on* J_1.

DEFINITION 1.1.5. *If* $y : J \to G$ *is a solution of (1.1.1) and there is no continuation of* $y(t)$*, then* J *is said to be a maximal interval of existence of* $y(t)$.

The following appears to be a good example of maximal interval of existence;

EXAMPLE 1.1.1. *[47] Let us consider that.*

$$\dot{x} = x^p$$
$$x(0) = 1.$$

This scalar field is autonomous, and the interval I *is therefore not relevant. For any integer* $p \geq 0$*, we have* $G = \mathbb{R}$*. If* p *is a negative integer, we have to execute the origin,*$G = \mathbb{R}\backslash 0$*. If* p *is non-integer, negative* x *values are not allowed, and then,* $G = [0, \infty)$ *for* $p > 0$*, and* $G = (0, \infty)$ *for* $p < 0$*. The solutions are directly found from separation of variables leading to the implicit equation*

$$\int_1^x y^{-p} dy = t.$$

Hence:

$$x(t) = \begin{cases} [1 + (1-p)t]^{\frac{1}{1-p}}, & p \neq 1 \\ e^t, & p = 1 \end{cases} . \tag{1.1.3}$$

The maximum existence interval $I_{\max}$ *depends on* p *in a remarkable way, as we can see from the following cases:*

For $p > 1$*,*$I_{\max} = (-\infty, \frac{1}{p-1})$.

For $p = 1$*,*$I_{\max} = \mathbb{R}$.

For $p < 1$*and* $p \neq \frac{n-1}{n}$ *for some integer* $n \neq 0$, $I_{\max} = [\frac{1}{p-1}, \infty)$.
For $p = \frac{n-1}{n}$ *for some positive integer* n, $I_{\max} = \mathbb{R}$.
For $p = \frac{n-1}{n}$ *for some negative integer* n, $I_{\max} = \mathbb{R} \backslash \{n\}$.

1.1.3 Continuation of Solutions

Continuation of solution on $\mathbb{R}$ is one of the significant properties of DS. Therefore, let us consider how the solution can be continued uniquely over a much larger interval and how it may exist globally. That is, for all $t \in \mathbb{R}$. The idea of continuation is rather straightforward [15], [16], [32], [37].

Let us study continuation for the case $t > 0$. The case $t < 0$ is completely similar. Theorem 1.1.1 ensures that a unique solution $x(t)$ exists on $I_{\alpha_0} = [-\alpha_0, \alpha_0]$, where $\alpha_0 > 0$. Therefore, $x(t_1)$ with $t_1 = \alpha_0$ exists and $x(t_1) \in G$. After applying existence theorem around this point we get a new interval $I_{\alpha_1} = [t_1 - \alpha_1, t_1 + \alpha_1]$ on which unique solution of IVP (1.1.1), (1.1.2) exists. On $I_{\alpha_0} \cap I_{\alpha_1}$, this solution coincides with the solution in I_{α_0}, and on $I_{\alpha_1} \backslash I_{\alpha_0}$, it is its continuation. Repeating this procedure while taking the I_{α_i} as large as possible, we get a series of overlapping intervals I_{α_i}, $i = 0, 1, \ldots$ and the solution exists uniquely on their union.

As a corollary of the existence theorem, one can obtain the following results [37], which are important for the theory of the dynamical systems.

THEOREM 1.1.3. *[37]: If as time increases (or decreases), a given trajectory (an integral curve) remains in a closed bounded region* M *imbedded in an open domain* G *for which the conditions of our existence theorem are fulfilled, then the motion (the solution) may be continued for the whole infinite interval* $[t_0, \infty)$ *(or* $(-\infty, t_0]$*).*

Theorem 1.1.3 is not good for applications. Therefore, we may write another theorem which involves sufficient conditions for such continuation.

THEOREM 1.1.4. *[37]: If the function $f(x)$ is continuous for $x \in \mathbb{R}^n$, and, moreover, if $\|f(x)\| < A\max(\|x\|, 1)$ for $\|x\| > D > 0$, where A is some positive constant and D is sufficiently large number, then the solution of system (1.1.1) is defined on $\mathbb{R}$.*

The following theorems are assertions about continuation of solutions on $\mathbb{R}$. [32], [47].

THEOREM 1.1.5. *Suppose that $f \in C^1(\mathbb{R}^n)$ and that $f(x)$ satisfies the global Lipschitz condition*

$$\|f(x) - f(y)\| \leq M \|x - y\|$$

for all $x, y \in \mathbb{R}^n$. Then, for $x_0 \in \mathbb{R}^n$, IVP (1.1.1), (1.1.2) has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$.

For manifolds, for which some brief information is provided in section 1.2, there exists a theorem which is more simple in formulation.

The following theorem, which can be seen in [13], is a significant theorem for compact manifolds.

THEOREM 1.1.6. *(Chillingworth). Let D be a compact manifold and let $f \in C^1(D)$. Then for $x_0 \in D$, IVP (1.1.1), (1.1.2) has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$.*

1.1.4 Continuous dependence on initial value

DEFINITION 1.1.6. *The solution $x(t) = x(t, 0, x_0)$ is defined for $-T \leq t \leq T$ is called continuously dependent on initial value, if for every $\epsilon > 0$ there is a $\delta > 0$ such that for $\|\bar{x_0} - x_0\| < \delta$ the solution $\bar{x} = x(t, 0, \bar{x_0})$ is also defined for $-T \leq t \leq T$ and for all values of t in this interval $\left\|x(\bar{t}) - x(t)\right\| < \epsilon$.*

We can formulate this property as follows: If initial points are chosen sufficiently close, then in the course of a given interval $-T < t < T$, the distance between simultaneous positions of the moving points will remain less than an assigned quantity ϵ. The following theorems, particularly discussed in [37], [32], give this opportunity. Now, we will give sufficient conditions that must be satisfied by system (1.1.1) so that the solution there of can depend continuously on initial value. By using this knowledge, we find an estimate for the change of the solution of system (1.1.1), corresponding to a change of the initial value. Let us continue with the following definition and the theorems. The following property, continuous dependence on initial value, is one of the most important case for DS.

THEOREM 1.1.7. *Let $G \in \mathbb{R}^n$ be open and suppose $f : G \to \mathbb{R}^n$ has a Lipschitz constant K. Let $y(t)$, $z(t)$ be solutions of (1.1.1) on the closed interval $[t_0, t_1]$. Then, for all $t \in [t_0, t_1]$:*

$$|y(t) - z(t)| \leq |y(t_0) - z(t_0)| \, e^{K(t-t_0)}.$$

The theorem on differentiability with respect to the initial value provides a quite efficient method of studying the influence exerted on the solution by a small perturbation of the initial value. The following theorem [32] is a consequence of Theorem 1.1.1 on rectification.

THEOREM 1.1.8. *Assume that $f \in C^1(G)$.*

$$x(0) = y. \tag{1.1.4}$$

Then there exists an $a > 0$ and a $\delta > 0$ such that for all $y \in B(x_0, \delta)$ the IVP (1.1.1), (1.1.4) has a unique solution $\phi(t, y)$ with $\phi \in C^1(G)$ where $G = [-a, a] \times B(x_0, \delta) \subset \mathbb{R}^{n+1}$.

1.1.5 The Group property

We may continue discussing one of the most important properties of DS. Assume that $f \in C^1$ and all solutions are continuable on $\mathbb{R}$. Then the following lemmas are valid. These lemmas are taken from [15], [16], [32], [37].

LEMMA 1.1.2. *If $x(t) : \mathbb{R} \to G$ is a solution of system (1.1.1), then the function $x(t + \theta)$ is a solution of the system (1.1.1).*

Theorem 1.1.9. *[32] If $x(t)$ is a solution of system (1.1.1), then*

$$x(t_2, x(t_1, x_0)) = x(t_1 + t_2, x_0),$$

for all $\{t_1, t_2\} \in \mathbb{R}$ is valid.

By taking $t_1 = t$ and $t_2 = -t$ in Theorem 1.1.9, we can write the following corollary.

Corollary 1.1.1. *For all $t \in \mathbb{R}$, it is clearly seen that*

$$x(-t, x(t, x_0)) = x_0$$

It is obvious that $x(0, x_0) = x_0$. Thus on the bases of Theorem 1.1.9, one can conclude that $x(t, x_0)$ defines a one parameter group of transformations of G into itself.

Now, to go into details of DS, we study the following example which has physical sense [16].

Example 1.1.2. *Suppose we have a mass on a frictionless surface attached to a wall by spring. The state of this system is determined by two variables: υ for its velocity to the right, and x for the distance of the block from its neutral position. When $x = 0$, we assume that the spring is neither extended nor compressed and exerts no force on the block. As the block is moved to the right $(x > 0)$ of this neutral position, the spring pulls it to the left. Conversely, if the block is to the left of the neutral position $(x < 0)$, the spring is compressed and pushes the block to the right. We can mathematically express this system in the following way;*

$$\dot{\upsilon} = -x, \tag{1.1.5}$$
$$\dot{x} = \upsilon. \tag{1.1.6}$$

If we combine equations (1.1.5) and (1.1.6) by denoting $y = (x, \upsilon)^T$, then we get:

$$\dot{y} = Ay, \tag{1.1.7}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
The solution of this equation is in the form:

$$y(t, 0, y_0) = e^{At} y_0 = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} y_0, \tag{1.1.8}$$

where $y_0 = (x_0, \upsilon_0)^T$. *So the solution can be contracted to:*

$$y(t, t_0, y_0) = K(t) y_0,$$

where

$$K(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Let us see the properties of classical dynamical system step by step.
I) It is easily seen that the maximal interval of existence of this solution is $\mathbb{R}$, *as functions* $\sin t$ *and* $\cos t$ *have the domain* $\mathbb{R}$, *so the solution of the equation (1.1.7) is continuable on* $\mathbb{R}$.
II)By using Definition 1.1.6, $\|y(t, t_0, y_0) - y(t, t_0, \tilde{y_0})\| = \|K(t)y_0 - K(t)\tilde{y_0}\| = \|K(t)(y_0 - \tilde{y_0})\| \le \|y_0 - \tilde{y_0}\|$, *for all* $y_0, \tilde{y_0} \in G$, *since* $\max_{[0,2\pi]} \|K(t)\| = 1$ *so* $\delta = \epsilon$. *Therefore, all the solutions of system (1.1.7) are continuously dependent on initial value.*
III)One can see that $\frac{\partial y(t,t_0,y_0)}{\partial y_0} = K(t)$, *that is, all solutions are differentially dependent on initial value.*
IV)For the group property, we should show that the following two equalities are fulfilled ;

$$y(0, y_0) = y_0,$$

and

$$y(t_1 + t_2, y_0) = y(t_2, y(t_1, y_0)), \quad for \quad all \quad t_1, t_2 \in \mathbb{R}.$$

First one is easy to see. For the second case,

$$y(t_2, y(t_1, y_0)) = \begin{bmatrix} \cos t_1 & -\sin t_1 \\ \sin t_1 & \cos t_1 \end{bmatrix} \begin{bmatrix} \cos t_2 & -\sin t_2 \\ \sin t_2 & \cos t_2 \end{bmatrix} y_0.$$

After applying elementary trigonometry formulas, one can check that the following evaluation is right

$$y(t_2, y(t_1, y_0)) = \begin{bmatrix} \cos(t_1 + t_2) & -\sin(t_1 + t_2) \\ \sin(t_1 + t_2) & \cos(t_1 + t_2) \end{bmatrix} y_0.$$

We have checked all conditions (I)-(IV), so the given system (1.1.7) defines DS.

The following example can be found in [38].

EXAMPLE 1.1.3. *Consider an ideal pendulum, the bob has mass* m *and is attached by a rigid pole of length* L *to a fixed pivot. The state of this DS can be described by two numbers:* θ*, the angle the pendulum makes with the vertical, and* ω*, the rate of rotation (measured in radians per second). By definition,* $\omega = \frac{d\theta}{dt}$*. Then we get following system;*

$$\dot{\theta}(t) = \omega(t), \tag{1.1.9}$$

and

$$\dot{\omega}(t) = -\frac{g}{L} \sin \theta(t). \tag{1.1.10}$$

(The minus sign in the equation (1.1.10) reflects the fact that when $\theta > 0$*, the force tends to send the pendulum back to the vertical.) Let* $\varphi = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$ *be the state vector; then the equations (1.1.10) and (1.1.9) can be expressed:*

$$\dot{\varphi} = g(\varphi), \tag{1.1.11}$$

where $g : \mathbb{R}^2 \to \mathbb{R}^2$ *is defined by*

$$g(\varphi) = \begin{bmatrix} \omega \\ \frac{-g}{L} \sin\theta \end{bmatrix}. \tag{1.1.12}$$

By using the Theorems 1.1.1 -1.1.9, one can see that this is a DS,

The following example,which can be seen in [16], is useful to understand Example 2.1.1 about harmonic oscillator with impulses in the main part of our study.

Example 1.1.4. *We consider a particle of mass* m *moving in one dimension, its position at time* t *given by a function* $t \to x(t)$, $x : \mathbb{R} \to \mathbb{R}$. *Suppose the force on the particle at a point* $x \in \mathbb{R}$ *is given by* $-mp^2x$, *where* p *is some real constant.This model is called the harmonic oscillator. Then according to the laws of physics the motion of the particle satisfies*

$$\ddot{x} + p^2x = 0. \tag{1.1.13}$$

(1.1.13) is the equation of the harmonic oscillator. After this point, we can show that also this equation satisfies the conditions of classical DS. The equation (1.1.13) can be rewritten by the following system;

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -p^2x_1. \end{aligned} \tag{1.1.14}$$

with initial conditions $x_1(0) = x(0)$ *and* $x_2(0) = \dot{x}(0)$. *The solution of the system (1.1.14) is*

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos pt & \frac{1}{p}\sin pt \\ -p\sin pt & \cos pt \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}. \tag{1.1.15}$$

Let us denote $y(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, *we can rewrite the solution (1.1.15) in the following contracted form:*

$$y(t, t_0, y_0) = K(t)y_0,$$

where

$$K(t) = \begin{bmatrix} \cos pt & \frac{1}{p}\sin pt \\ -p\sin pt & \cos pt \end{bmatrix}.$$

Again, we may examine the properties of the classical dynamical system step by step.

By repeating the same discussion as we did in Example 1.1.2, one can see that all the solutions of the system (1.1.14) are continuable on $\mathbb{R}$, *continuously and differentially dependent on initial value, and satisfies the first case of group property. Then we should show that the following equality is right;*

$$y(t_1 + t_2, y_0) = y(t_2, y(t_1, y_0)), \quad for \quad all \quad t_1, t_2 \in \mathbb{R}.$$

$$x(t_2, x(t_1, x_0)) = \begin{bmatrix} \cos pt_2 & \frac{1}{p}\sin pt_2 \\ -p\sin pt_2 & \cos pt_2 \end{bmatrix} \begin{bmatrix} \cos pt_1 & \frac{1}{p}\sin pt_1 \\ -p\sin pt_1 & \cos pt_1 \end{bmatrix} x_0$$

$$= \begin{bmatrix} \cos pt_1 \cos pt_2 - \sin pt_1 \sin pt_2 & \frac{1}{p}[\sin pt_1 \cos pt_2 + \sin pt_2 \cos pt_1] \\ -p[\sin pt_1 \cos pt_2 + \sin pt_2 \cos pt_1] & \cos pt_1 \cos pt_2 - \sin pt_1 \sin pt_2 \end{bmatrix} x_0$$

$$= \begin{bmatrix} \cos p(t_1 + t_2) & \frac{1}{p}\sin p(t_1 + t_2) \\ -p\sin p(t_1 + t_2) & \cos p(t_1 + t_2) \end{bmatrix} x_0 = x(t_1 + t_2, x_0).$$

All conditions of DS are fulfilled, so the given system (1.1.14) defines DS.

CONCLUSION 1.1.1. *If system (1.1.1) satisfies conditions of theorems 1.1.1-1.1.9, then solutions* $\phi(t, x)$ *of the system define a (smooth) DS.*

1.2 Manifolds

In our work, in order to create systems that are continuously dependent on initial value, we will prefer to use manifolds without boundary as the sets of discontinuity Γ and $\tilde{\Gamma}$. Thus, we will give some basic definitions about a manifold with boundary and without boundary. For further information, one can look in [43], [44], [46], [48], [50], [51], [52].

Let $U \in \mathbb{R}^n$ and let $V \in \mathbb{R}^m$ be open sets. A mapping f from U to V is written as $f : U \to V$.

Definition 1.2.1. *A mapping $f : U \to V$ is called smooth if all of the partial derivatives $\frac{\partial^n f}{\partial x_{i_1}}, \ldots, \frac{\partial^n f}{\partial x_{i_n}}$ exist and are continuous.*

Definition 1.2.2. *Let X, Y be two topological spaces, and suppose $f : X \to Y$ is bijection (one to one and onto). If f is continuous, and at the same time its inverse $f^{-1} : Y \to X$ is continuous, then f is called a homeomorphism.*

Definition 1.2.3. *A map $f : X \to Y$ is called a diffeomorphism if f carries X homeomorphically onto Y and if both f and f^{-1} are smooth.*

Definition 1.2.4. *A topological space M is called an $n-$dimensional manifold if it is locally homeomorphic to $\mathbb{R}^n$. That is, there is an open cover $\{U_i\}_{i \in \mathcal{A}}$ of M such that for each $i \in \mathcal{A}$ there is a map $\phi_i : U_i \to \mathbb{R}^n$ which maps U_i homeomorphically onto an open subset of $\mathbb{R}^n$.*

Definition 1.2.5. *A subset $M \subset \mathbb{R}^n$ is called a smooth manifold of dimension r if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset U of the Euclidean space $\mathbb{R}^r$.*

Remark 1.2.1. *Any particular diffeomorphism $g : U \to W \cap M$ is called a parametrization of the region $W \cap M$.(The inverse diffeomorphism $W \cap M \to U$ is called a system of coordinates on $W \cap M$.)*

Example 1.2.1. *The unit sphere S^2, consisting of all $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$, is a smooth manifold of dimension 2. In fact the diffeomorphism*

$$(x, y) \to (x, y, \sqrt{1 - x^2 - y^2}),$$

for $x^2+y^2<1$ *parameterizes the region* $z>0$ *of* S^2*. By interchanging the roles of* x,y,z *and changing the signs of the variables, we obtain similar parametrization of the regions* $x>0$, $y>0$, $x<0$, $y<0$, $z<0$*. Since these cover* S^2*, it follows that* S^2 *is a smooth manifold.*

It forms a manifold of dimension $n-1$, and itself has no boundary. We could regard any manifold (as defined in 1.2.4) as a *manifold-without boundary.* Now, we will mention about manifold with boundary to allow comparison with the manifold without boundary, thereby preventing confusion. Consider first the closed half space

$$H^n = \{(x_1,\ldots,x_n)\in\mathbb{R}^n \quad | \quad x_m \geq 0\}\,.$$

The boundary ∂H^n is defined to be the hyperplane

$$\mathbb{R}^{n-1}\times 0 = \{(x_1,x_2,\ldots,x_n)|x_n=0\}\subset\mathbb{R}^n.$$

Definition 1.2.6. *A subset* $X\subset\mathbb{R}^n$ *is called a smooth n-manifold with boundary if each* $x\in X$ *has a neighborhood* $U\cap X$ *diffeomorphic to an open subset* $V\cap H^n$ *of* H^n*. The boundary* ∂X *is the set of* ∂H^n *under such a diffeomorphism.*

It is not hard to show that ∂X is well defined smooth manifold of dimension $n-1$. The interior $X-\partial X$ is a smooth manifold of dimension n.
A hemispherical cap (including the equator) or a right circular cylinder (including the circles at the ends) are typical examples of manifolds with boundary. Except for the equator, or the end-circles, they are 2-manifolds and these boundary sets are themselves manifolds of dimension one less. In fact, they are homeomorphic to S^1 or to $S^1\cup S^1$ in these two cases.

CHAPTER 2

Discontinuous Dynamical Systems

2.1 Introduction

A book [7] edited by D.V. Anosov and V.I. Arnold considers two fundamentally different DSs : flows and cascades. Roughly speaking, flows are DSs with continuous time and cascades are DSs with discrete time. One of the most important theoretical problems is to consider *Discontinuous Dynamical Systems* ($DDSs$). That is, the systems whose trajectories are piecewise continuous curves. It is well-recognized (for example, see [26]) that the general notion of such systems was introduced by Th. Pavlidis [27]-[29], although particular examples (the mathematical model of clock [6], [22] and so on) had been discussed before. Some basic elements of the theory are given in [14]-[30]. Analysing the behavior of the trajectories, we can conclude that $DDSs$ combine features of vector fields and maps. They can not be reduced to flows or cascades, but are close to flows since time is continuous. That is why we propose to call them also *Discontinuous Flows* (DFs). Applications of $DDSs$ in mechanics, electronics, biology and medicine were considered in [23], [27]-[29], [33]. Chaotic behavior of discontinuous processes was investigated in [13, 54]. One must emphasize that DFs are not *differential equations with discontinuous right side* which often have been accepted as $DDSs$ [11]. However, theoretical problems of nonsmooth dynamics and discontinuous maps [31], [53] are also very close to the subject of our theses. One should also agree that *nonautonomous impulsive differential equations*, which were thoroughly described in [20] and [30], are not DFs.

Papers of T. Pavilidis and V. Rozhko [27]-[29], [25] contain interesting practical and theoretical ideas concerning the DFs. These authors formulated some important conditions on differential equations, but not all of them were used to prove basic properties of DFs. Some aspects of DFs on manifolds were considered in [24]. One must remark that the authors of the paper formulated conditions for the group property, but as demonstrated by Example 2.9.3 those conditions do not guarantee it. In that paper the *smooth impulsive flow* was claimed to be considered, but differential dependence, as well as continuous dependence, were not defined and investigated. Thus one can say that the complexity of DFs necessitates more careful investigation. And this theses can be considered as an attempt to give a rigorous description of DFs.

The theses embodies results that provide sufficient conditions for the existence of a *differentiable DF*. Since DFs have specific smoothness of solutions we call these systems *B-differentiable DFs.* Apparently, it is the first time when notions of $B-$ continuous and $B-$ differentiable dependence of solutions on initial values [1]-[3] are applied to described $DDSs$ and sufficient conditions for the continuation of solutions and the group property are obtained. A central auxiliary result of the theses is the construction of a new form of the general autonomous impulsive equation (system (2.1.4)). Effective methods of investigation of systems with variable time of impulsive actions were considered in [1]-[5], [20, 21, 30].

As expressed above, some examples of DDS have been given in many places but with no concrete theory of DDS. Now, we will see one of these example [30] and examine the conditions of classical DS on it.

EXAMPLE 2.1.1. *Let us study the motion of the following system*

$$\ddot{x} + \omega^2 x = 0, \quad x \neq x_0$$
$$\dot{x}^+ = k + \dot{x}, \quad x = x_0,$$

where k is a positive constant and $x_0 > 0$.

Denote $x_1 = x$ and $x_2 = \frac{1}{\omega}\dot{x}$. By using this substitution, this system can be *rewritten in the form:*

$$\dot{x_1} = \omega x_2, \quad x_1 \neq x_0 \qquad (2.1.1)$$
$$\dot{x_2} = -\omega x_1,$$

$$x_2^+ = k_1 + x_2^-, \quad x_1 = x_0, \qquad (2.1.2)$$

where $k_1 = \frac{k}{\omega}$. *The solution of the system (2.1.1) is* $x_1(t) = r \sin \omega t$, $x_2(t) = r \cos \omega t$, *where* r *is a fixed real number. The system (2.1.1), (2.1.2) was mentioned in [30] as a DDS. One can easily see on figure 2.1 that how the solution of the system (2.1.1),(2.1.2) behaves, when* t *is positive and* $x_0 = r$. *Let us denote* $x^0(t) = x(t, 0, (0, x_0))$, *a solution of system (2.1.1) and (2.1.2). The solution moves on circle* ***c*** *until it meets the line* $x_1 = x_0$, *then it jumps and continues its motion on arc of the circle* c_1 *and again it meets the line* $x_1 = x_0$ *and jumps. And also, how the solution of the system (2.1.1), (2.1.2) behaves when* t *is negative and* $x_0 = r$, *is shown on figure 2.2.*

One may examine the solution do not guarantee continuous dependence on initial value on figure 2.1. Let us take sufficiently close another solution $x^\epsilon(t) = x(t, 0, (0, x_0 - \epsilon))$ *of this system, which starts its motion at the point* $(0, x_0 - \epsilon)$, *where* ϵ *is a fixed positive real number. According to Definition 1.1.6, the distance between simultaneous positions of the moving points should remain less than an assigned quantity* ξ. *The solution* $x^0(t)$ *jumps at the point* $(x_0, 0)$ *and continues its motion on the arc of the circle as explained above, however, solution* $x^\epsilon(t)$ *may continue its motion on circle* c_ϵ *without any jump. So, as seen in figure 2.1, the distance between these two solutions cannot remain less than* ξ, *despite the distance between initial points of these two solutions can be done arbitrary small. This explanation demonstrates that the solution of the system (2.1.1), (2.1.2) cannot continuously depend on initial value. Thus, this system* ***is not DDS.***

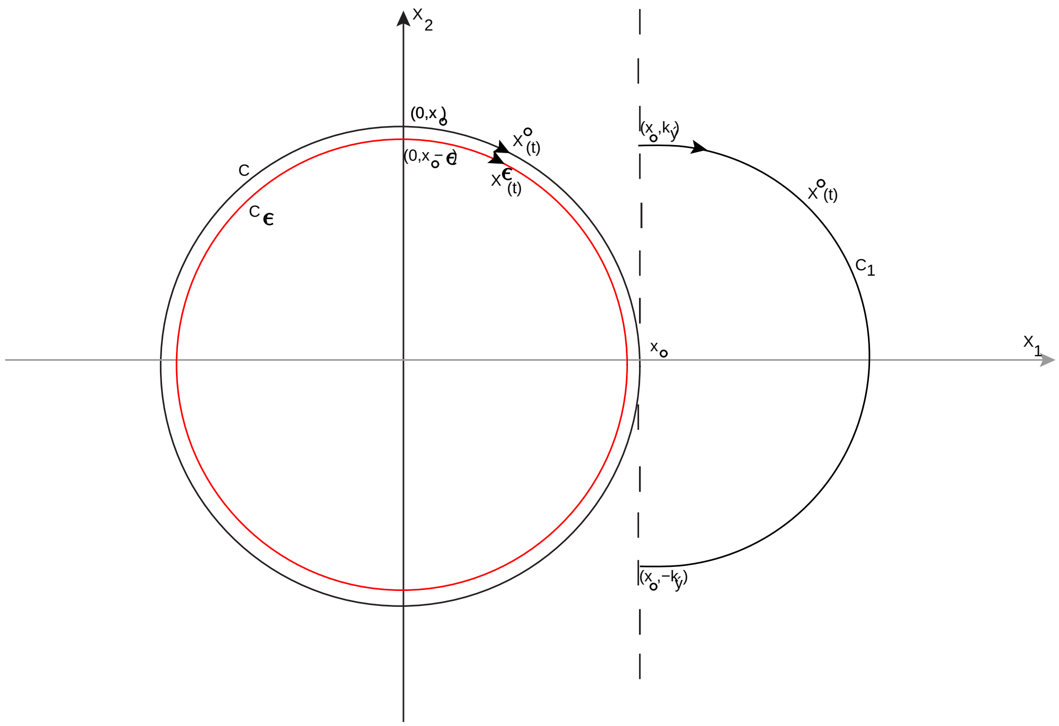

Figure 2.1:

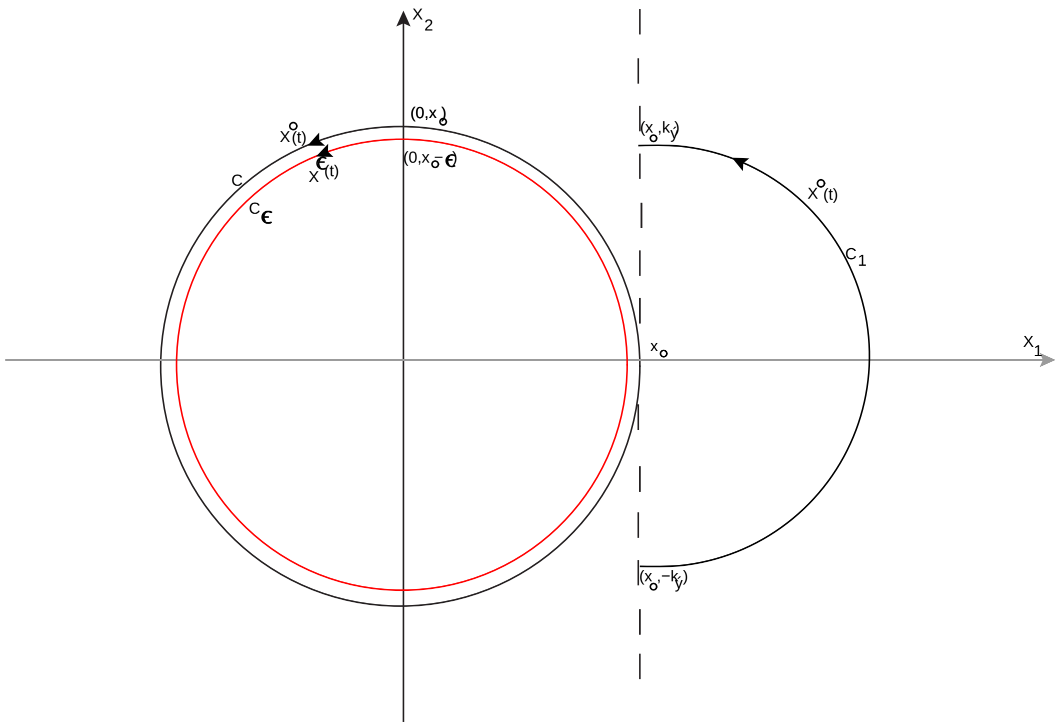

Figure 2.2:

We need to develop a simple examples of $DDSs$ which motivates the reader to B-smooth discontinuous flows with some of the basic ideas of classical DSs and the theory of impulsive differential equations. Later, these ideas will be put into a more systematic exposition. In particular examples themselves are important to follow the development of theoretical approach to B-smooth discontinuous flows. We consider the following examples:

EXAMPLE 2.1.2. *Consider the following impulsive autonomous system.*

$$\begin{array}{ll} \begin{array}{l} \dot{x_1} = -x_2 \\ \dot{x_2} = \quad x_1 \end{array}, & \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \left\{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\right\} \\ \begin{array}{l} x_1^+ = \cos\frac{\pi}{6}x_1^- - \sin\frac{\pi}{6}x_2^- \\ x_2^+ = \sin\frac{\pi}{6}x_1^- + \cos\frac{\pi}{6}x_2^- \end{array}, & \{x(t) \in \Gamma \wedge t \geq 0\} \\ \begin{array}{l} x_1^- = \cos\frac{\pi}{6}x_1^+ + \sin\frac{\pi}{6}x_2^+ \\ x_2^- = -\sin\frac{\pi}{6}x_1^+ + \cos\frac{\pi}{6}x_2^+ \end{array}, & \left\{x(t) \in \tilde{\Gamma} \wedge t \leq 0\right\} \end{array} \tag{2.1.3}$$

where

$$\Gamma = \left\{(x_1, x_2)| \quad x_1 = \sqrt{3}x_2, \quad x_1, x_2 \in \mathbb{R}^+\right\},$$

$$\tilde{\Gamma} = \left\{(x_1, x_2)| \quad \sqrt{3}x_1 = x_2, \quad x_1, x_2 \in \mathbb{R}^+\right\},$$

$D = \mathbb{R}^2 \backslash \left[\left\{(x_1, x_2)| \quad \frac{1}{\sqrt{3}}x_1 < x_2 < \sqrt{3}x_1, \quad x_1 > 0\right\} \cup (0,0)\right]$. *This system can be written in the form:*

$$\begin{array}{rcl} \dot{x} & = & Ax, \quad \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \left\{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\right\} \\ x^+ & = & Bx^-, \{x(t) \in \Gamma \wedge t \geq 0\} \\ x^- & = & B^{-1}x^+, \left\{x(t) \in \tilde{\Gamma} \wedge t \leq 0\right\} \end{array}$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$. *One can easily check that* $AB = BA$ *and* $AB^{-1} = B^{-1}A$, *then when* t *is positive and* $t_0 = 0$, *the solution* *of this system is verified as in the following form by [30]:*

$$x(t,x_0) = e^{At}B^{i[0,t]}x_0 = \begin{pmatrix} \cos(t+\frac{\pi}{6}i\,[0,t]) & -\sin(t+\frac{\pi}{6}i\,[0,t]) \\ \sin(t+\frac{\pi}{6}i\,[0,t]) & \cos(t+\frac{\pi}{6}i\,[0,t]) \end{pmatrix} x_0,$$

One can also write that

$$x(t,x_0) = K(t)x_0,$$

where $K(t) = \begin{pmatrix} \cos(t+\frac{\pi}{6}i\,[0,t]) & -\sin(t+\frac{\pi}{6}i\,[0,t]) \\ \sin(t+\frac{\pi}{6}i\,[0,t]) & \cos(t+\frac{\pi}{6}i\,[0,t]) \end{pmatrix}$.

When t *is negative and* $t_0 = 0$*, the solution becomes*

$$x(t,x_0) = e^{At}(B^{-1})^{\tilde{i}[t,0]}x_0 = \begin{pmatrix} \cos(t-\frac{\pi}{6}\tilde{i}\,[t,0]) & \sin(t-\frac{\pi}{6}\tilde{i}\,[t,0]) \\ -\sin(t-\frac{\pi}{6}\tilde{i}\,[t,0]) & \cos(t-\frac{\pi}{6}\tilde{i}\,[t,0]) \end{pmatrix} x_0.$$

Or

$$x(t,x_0) = \tilde{K}(t)x_0,$$

where $\tilde{K}(t) = \begin{pmatrix} \cos(t-\frac{\pi}{6}\tilde{i}\,[t,0]) & \sin(t-\frac{\pi}{6}\tilde{i}\,[t,0]) \\ -\sin(t-\frac{\pi}{6}\tilde{i}\,[t,0]) & \cos(t-\frac{\pi}{6}\tilde{i}\,[t,0]) \end{pmatrix}$.

For a geometrical approach to the solution of this system, one may study figure 2.3 and figure 2.4.

If t *is positive, as seen on figure 2.3, the solution continues its motion on circle* C *until it meets* Γ*. Immediately, it jumps on* $\tilde{\Gamma}$ *with the same radius and again moves on circle* C*. If* t *is negative, the solution continues its motion on circle* C *until it meets* $\tilde{\Gamma}$*. It jumps on* Γ *with the same radius and moves again on circle* C *as seen on figure 2.4. As we did in Example 1.1.2, we may examine the conditions of DS on impulsive autonomous system (2.1.3).*

I)The solution of the system $x(t,x_0)$ *is defined for all* $t \in \mathbb{R}$*, so it is continuable on* $\mathbb{R}$.

II)For all $x_0, \tilde{x_0} \in \mathbb{R}^2$*, and when* t *is positive:*

$$\|x(t,x_0) - x(t,\tilde{x_0})\| = \|K(t)x_0 - K(t)\tilde{x_0}\| = \|K(t)\,(x_0 - \tilde{x_0})\| \leq \|x_0 - \tilde{x_0}\|,$$

since $\max_{[0,2\pi]} \|K(t)\| = 1$ *so* $\delta = \epsilon$. *If* t *is negative, continuous dependence on initial value can be shown similarly. Therefore, all solutions of system (2.1.3) are continuously dependent on initial value.*

III)For the group property, we should show that the following equalities are fulfilled:

$$x(0, x_0) = x_0,$$

and

$$x(t_1 + t_2, x_0) = x(t_2, x(t_1, x_0)), \quad for \quad all \quad t_1, t_2 \in \mathbb{R}.$$

First one is easy to see, since $K(0) = \tilde{K}(0) = I$. *For the second case, consider only* $\theta_{i+1} > t_1 > \theta_i > 0$, *and* $\theta_{m+1} > t_1 + t_2 > \theta_m > 0$, *the other cases are very similar to this case.*

$$x(t_1, x(t_2, x_0)) = \begin{pmatrix} \cos(t_1 + \frac{\pi}{6} i\,[0, t_1]) & -\sin(t_1 + \frac{\pi}{6} i\,[0, t_1]) \\ \sin(t_1 + \frac{\pi}{6} i\,[0, t_1]) & \cos(t_1 + \frac{\pi}{6} i\,[0, t_1]) \end{pmatrix} \times$$

$$\begin{pmatrix} \cos(t_2 + \frac{\pi}{6} i\,[0, t_2]) & -\sin(t_2 + \frac{\pi}{6} i\,[0, t_2]) \\ \sin(t_2 + \frac{\pi}{6} i\,[0, t_2]) & \cos(t_2 + \frac{\pi}{6} i\,[0, t_2]) \end{pmatrix} x_0$$

$$= \begin{pmatrix} \cos(t_1 + t_2 + \frac{\pi}{6} i\,[0, t_1 + t_2]) & -\sin(t_1 + t_2 + \frac{\pi}{6} i\,[0, t_1 + t_2]) \\ \sin(t_1 + t_2 + \frac{\pi}{6} i\,[0, t_1 + t_2]) & \cos(t_1 + t_2 + \frac{\pi}{6} i\,[0, t_1 + t_2]) \end{pmatrix} x_0 = x(t_1 + t_2, x_0).$$

We can see that given impulsive autonomous system (2.1.3) satisfies some properties of DS. For example; the differential dependence on initial value is not clear.

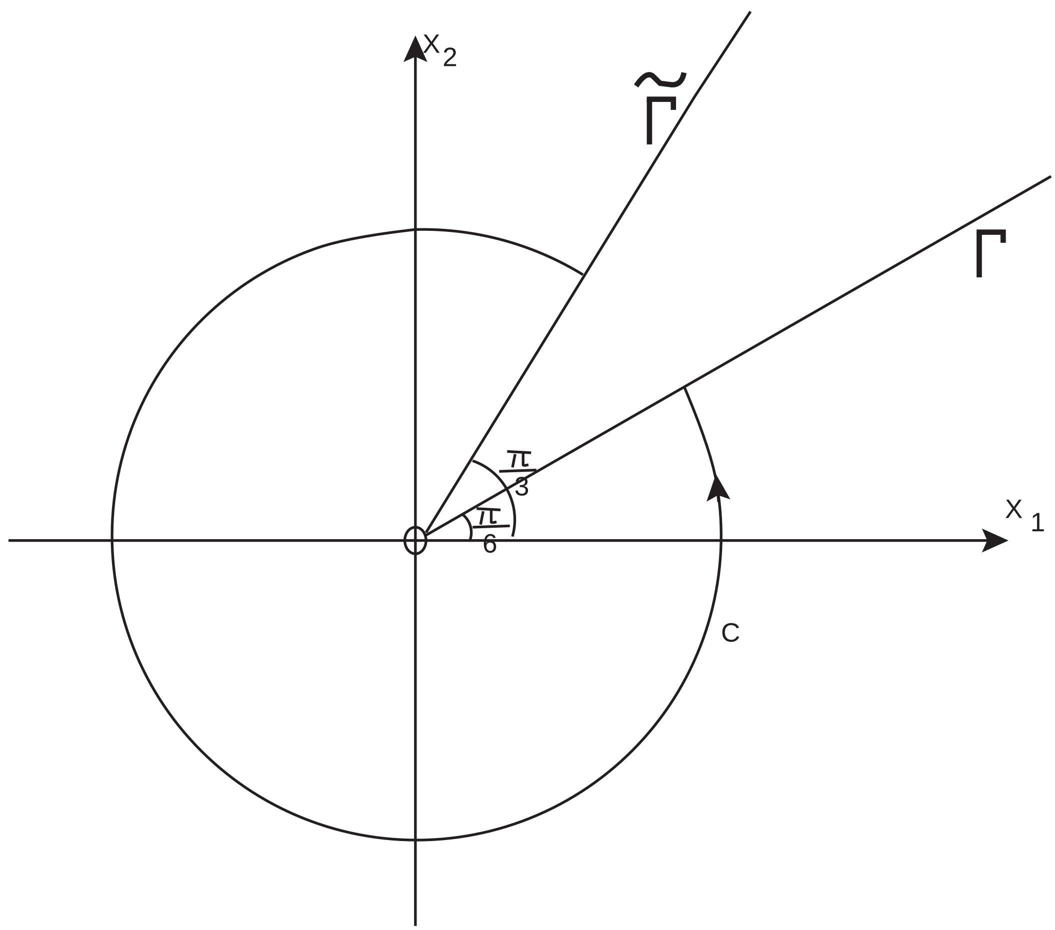

Figure 2.3:

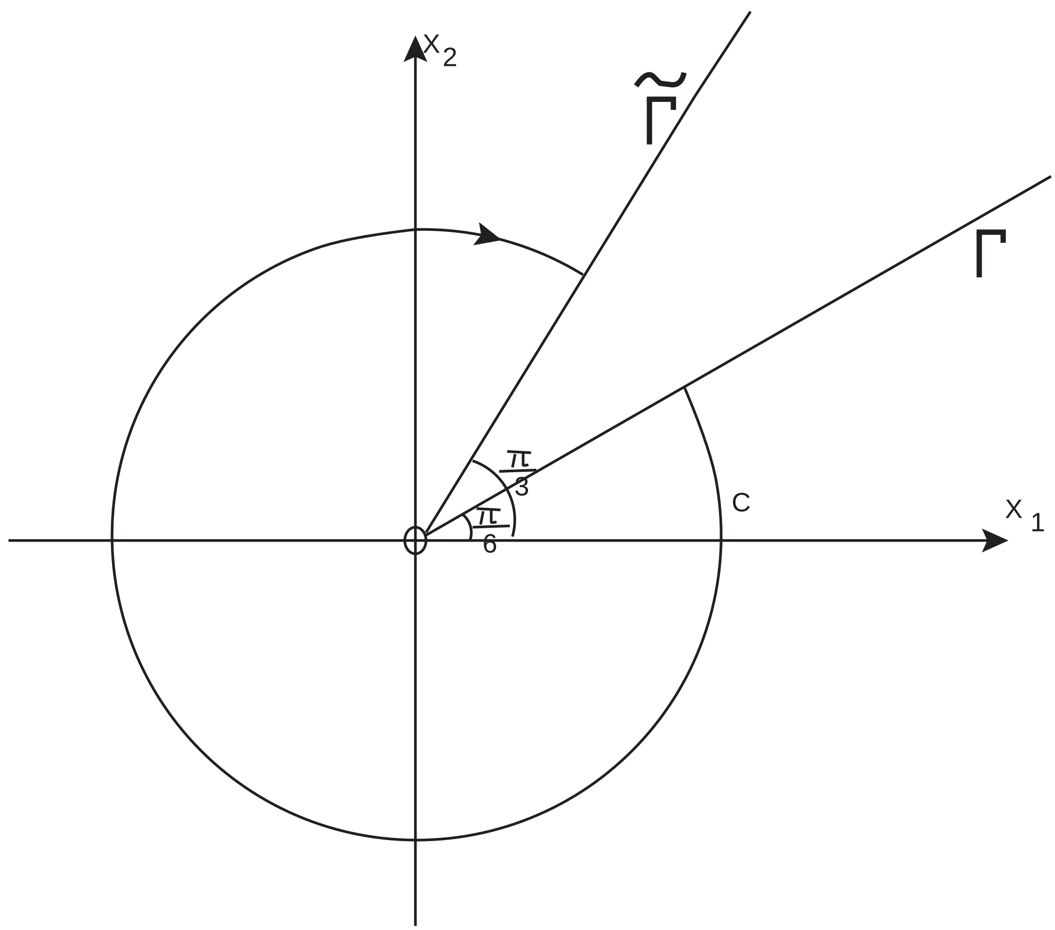

Figure 2.4:

Now, after these examples, we may start studyingn the theory of DDS. Consider a set of strictly ordered real numbers $\{\theta_i\}$, $i \in \mathcal{A}$, where $\mathcal{A}$ is an interval of indices from $\mathbb{Z}\backslash\{0\}$.

DEFINITION 2.1.1. *The set $\{\theta_i\}$ is said to be a sequence of $\beta - type$ if the product $i\theta_i \geq 0$ for all i and one of the following alternative cases holds:*

(a) $\{\theta_i\} = \emptyset$;

(b) $\{\theta_i\}$ *is a finite and nonempty set;*

(c) $\{\theta_i\}$ *is an infinite set such that* $|\theta_i| \to \infty$ *as* $|i| \to \infty$.

From the definition, it follows immediately that a sequence of $\beta - type$ does not have a finite accumulation point in $\mathbb{R}$.

DEFINITION 2.1.2. *A function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to belong to $\mathcal{PC}(\mathbb{R})$ if*

1. *$\varphi(t)$ is left continuous on $\mathbb{R}$;*
2. *there exists a sequence $\{\theta_i\}$ of $\beta-$ type such that φ is continuous if $t \neq \theta_i$ and φ has discontinuities of the first kind at the points θ_i.*

Particularly, $C(\mathbb{R}) \subset \mathcal{PC}(\mathbb{R})$.

EXAMPLE 2.1.3. *Consider the following function*

$$\phi\left(t\right) = t - \left[t\right],$$

for all $t \in \mathbb{R}$. Discontinuity set of function $\phi\left(t\right)$ is $\{n\}$, where $n \in \mathbb{Z}$ and, one can see easily that this set is β-type and $\phi\left(t\right)$ is continuous when $t \neq n$, $n \in \mathbb{Z}$ and $\phi\left(t\right)$ has first kind of discontinuity at t=n,

$$\lim_{t \to n^-} \phi(t) = 1,$$

and

$$\lim_{t \to n^+} \phi(t) = 0,$$

for all $t \in \mathbb{Z}$.
According to Definition 2.1.2, one can see that $\phi \in PC(\mathbb{R})$.

DEFINITION 2.1.3. *A function $\varphi(t)$ is said to be from a space $\mathcal{PC}^1(\mathbb{R})$ if φ and $\varphi' \in \mathcal{PC}(\mathbb{R})$.*

Let T be an interval in $\mathbb{R}$.

DEFINITION 2.1.4. *We denote by $\mathcal{PC}(T)$ and $\mathcal{PC}^1(T)$, the sets of restrictions of all functions from $\mathcal{PC}(\mathbb{R})$ and $\mathcal{PC}^1(\mathbb{R})$ on T respectively.*

Let G be an open connected subset of $\mathbb{R}^n$, G_r be an $r-$ neighbourhood of G in $\mathbb{R}^n$ for a fixed $r > 0$ and $\hat{G} \subset G_r$ be an open subset of $\mathbb{R}^n$. Let $\Phi : \hat{G} \longrightarrow \mathbb{R}$ be a function from $C^1(\hat{G})$ and assume that a surface $\Gamma = \Phi^{-1}(0)$ is a connected subset of $\bar{G} \cap \hat{G}$. Moreover, define a function $J : \Gamma_r \to \bar{G}$, where Γ_r is an $r-$ neighbourhood of Γ. We shall need the following assumptions:

C1) $\nabla\Phi(x) \neq 0$, $\forall x \in \Gamma$;

C2) $J \in C^1(\Gamma_r)$ and $\det[\frac{\partial J(x)}{\partial x}] \neq 0$, for all $x \in \Gamma_r$.

One can see that the restriction $J|_\Gamma$ is a one-to-one function. Let also $\tilde{\Gamma} = J(\Gamma)$, $\tilde{\Gamma} \subset \bar{G}$. If $\tilde{\Phi}(x) = \Phi(J^{-1}(x))$, $x \in \tilde{\Gamma}$, then $\tilde{\Gamma} = \left\{x \in G |\, \tilde{\Phi}(x) = 0\right\}$. It is easy to verify that the following assertion is valid.

LEMMA 2.1.1. $\nabla\tilde{\Phi}(x) \neq 0$, $\forall x \in \tilde{\Gamma}$.

Proof: By using the definition of $\Phi(x)$, we can write that $\Delta\tilde{\Phi}(x) = \Delta\Phi\left(J^{-1}(x)\right)$. We can write from the calculus

$$\Delta\Phi\left(J^{-1}(x)\right) = \frac{\partial\Phi(y)}{\partial y}|_{y=J^{-1}(x)}\frac{\partial J^{-1}(x)}{\partial x}$$

we can obtain

$$\Delta\Phi\left(J^{-1}(x)\right) \neq 0$$

since by combining $C1)$ and $C2)$: $\det(\frac{\partial J^{-1}(x)}{\partial x}) \neq 0$, $\forall x \in \tilde{\Gamma_r}$ and $\frac{\partial\Phi(y)}{\partial y} \neq 0$
According to this discussion, lemma is proved.

Condition $C1)$ implies that for every $x_0 \in \Gamma$ there exists a number $j = \overline{1, n}$ and a function $\varphi_{x_0}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ such that in a neighborhood of x_0 the manifold Γ is the graph of the function $x_j = \varphi_{x_0}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. The same is true for every $x_0 \in \tilde{\Gamma}$.

Remark 2.1.1. *One can see from the description of* Γ *and* $\tilde{\Gamma}$ *that the surfaces are smooth* $n-1$ *dimensional manifolds without boundary [40].*

In order to simplify this discussion, we may give geometrical approach with the following examples.

EXAMPLE 2.1.4. *Consider the function* $\Phi(x) = x_2 - x_1$, *where* $x \in \mathbb{R}^2$. *When* $\Phi(x) = 0$ *this function can be written as* $x_2 = \varphi(x_1)$. *After calculating the gradient of* $\Phi(x)$, $\nabla\Phi(x) = (-1, 1) \neq 0$.

EXAMPLE 2.1.5. *Consider the set* $\Phi(x) = x_3 - x_1^2 - x_2^2$, $x \in \mathbb{R}^3$. *A map* Φ *defines a paraboloid when* $\Phi(x) = 0$. *Let us look at the gradient of* $\Phi(x)$, $\nabla\Phi(x) = (-2x_1, 2x_2, 1) \neq 0$. *We may say* $x_3 = x_1^2 + x_2^2$. *If we take* $\varphi(x_1, x_2) = x_1^2 + x_2^2$, *then we get a function* φ *which gives a graph of* x_3 *in terms of* x_1 *and* x_2.

EXAMPLE 2.1.6. *Consider the function* $\Phi(x) = x_1^2 - x_2^2 - 1$, *where* $x \in \mathbb{R}^2$ *when* $\Phi(x) = 0$ *this function defines a circle. More geometric way of understanding the* $C1)$ *on the partial derivative is in the terms of the tangent line. Let us take the tangent line at* $x_0 = (x_1^0, x_2^0)$, *given by:*

$$\nabla\Phi|_{x_0} \left(x_1 - x_1^0, x_2 - x_2^0\right) = 0$$

If $\frac{\partial\Phi}{\partial x_2}(x_0) \neq 0$, *then one can represent this line as a graph of* x_2 *in terms of* x_1, *i.e., we can solve for* x_2 *in terms of* x_1;

$$\left(x_1 - x_1^0\right) \frac{\partial\Phi}{\partial x_1}(x_0) + \left(x_2 - x_2^0\right) \frac{\partial\Phi}{\partial x_2}(x_0) = 0$$

$$\left(x_2 - x_2^0\right) = \frac{-\left(x_1 - x_1^0\right) \frac{\partial\Phi}{\partial x_1}(x_0)}{\frac{\partial\Phi}{\partial x_2}(x_0)}$$

If the tangent line at x_0 can be represented as a graph of x_2 in terms of x_1, then nearby the set $\Phi(x) = 0$ can be represented as a graph $x_2 = \varphi(x_1)$. The same ideas apply for $x \in \mathbb{R}^n$, $n \geq 1$.

EXAMPLE 2.1.7. *Let us take sphere in $\mathbb{R}^3$, consider the set*

$$\Phi(x) = x_1^2 + x_2^2 + x_3^2 - 1$$

after calculating the gradient of $\Phi(x)$, we get $\nabla\Phi(x) = (2x_1, 2x_2, 2x_3) \neq 0$ since the origin does not belong to the set Γ. Similar discussion, which is explained in Example 2.1.6 can be repeated for sphere to find a graph of x_2 in terms of x_1 and x_3.

Consider the following impulsive differential equation in the domain $D = \left[G \cup \Gamma \cup \tilde{\Gamma}\right] \setminus \left[(\bar{\Gamma} \backslash \Gamma) \cup \left(\bar{\tilde{\Gamma}} \backslash \tilde{\Gamma}\right)\right]$.

$$\begin{aligned} &x'(t) = f(x(t)), \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\}, \\ &x(t+)|_{x(t-) \in \Gamma \wedge t \geq 0} = J(x(t-)), \\ &x(t-)|_{x(t+) \in \tilde{\Gamma} \wedge t \leq 0} = J^{-1}(x(t+)), \end{aligned} \tag{2.1.4}$$

We make the following assumptions which will be needed throughout the paper.

C3) $f \in C^1(G_r)$.

C4) $\Gamma \cap \tilde{\Gamma} = \emptyset$, $\Gamma \cap \left(\bar{\tilde{\Gamma}} \backslash \tilde{\Gamma}\right) = \emptyset$, $\left(\bar{\Gamma} \backslash \Gamma\right) \cap \tilde{\Gamma} = \emptyset$.

C5) $\langle \nabla\Phi(x), f(x) \rangle \neq 0$ if $x \in \Gamma$.

C6) $\left\langle \nabla\tilde{\Phi}(x), f(x) \right\rangle \neq 0$ if $x \in \tilde{\Gamma}$.

EXAMPLE 2.1.8. *let us consider conditions $C1) - C6)$ on the following system:*

$$
\begin{cases}
\dot{x_1} = -\frac{1}{3}x_1 - 3x_2 \\
\dot{x_2} = 3x_1 - \frac{1}{3}x_2
\end{cases} , \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \left\{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\right\}
$$

$$
\begin{cases}
x_1^+ = 2\cos\frac{\pi}{6}x_1^- - 2\sin\frac{\pi}{6}x_2^- \\
x_2^+ = 2\sin\frac{\pi}{6}x_1^- + 2\cos\frac{\pi}{6}x_2^-
\end{cases} , \{x(t) \in \Gamma \wedge t \geq 0\} \tag{2.1.5}
$$

$$
\begin{cases}
x_1^- = \frac{1}{2}\cos\frac{\pi}{6}x_1^+ + \frac{1}{2}\sin\frac{\pi}{6}x_2^+ \\
x_2^- = -\frac{1}{2}\sin\frac{\pi}{6}x_1^+ + \frac{1}{2}\cos\frac{\pi}{6}x_2^+
\end{cases} , \left\{x(t) \in \tilde{\Gamma} \wedge t \leq 0\right\}
$$

where

$$
\Gamma = \left\{(x_1, x_2) | \quad x_1 = x_2, \quad x_1, x_2 \in \mathbb{R}^+\right\},
$$

$$
\tilde{\Gamma} = \left\{(x_1, x_2) | \quad \sqrt{3}x_1 = x_2, \quad x_1, x_2 \in \mathbb{R}^+\right\}.
$$

Let us assume that $D = \mathbb{R}^2 \setminus \left[\left\{(x_1, x_2) | \quad x_1 < x_2 < \sqrt{3}x_1, \quad x_1 > 0\right\} \cup (0,0)\right]$. *One can see that* $\Phi(x) = x_1 - x_2$, $\tilde{\Phi}(x) = \sqrt{3}x_1 - x_2$, $f(x) = (-\frac{1}{3}x_1 - 3x_2, 3x_1 - \frac{1}{3}x_2)$, $J(x) = (2\cos\frac{\pi}{6}x_1^- - 2\sin\frac{\pi}{6}x_2^-, 2\sin\frac{\pi}{6}x_1^- + 2\cos\frac{\pi}{6}x_2^-)$.
Let us start checking conditions $C1) - C6)$. $\nabla\Phi(x) = (1, -1) \neq 0$, *so, condition* $C1)$ *is satisfied. As can be seen easily,* J, f *are continuously differentiable functions and* $\det[\frac{\partial J(x)}{\partial x}] = det \begin{pmatrix} 2\cos\frac{\pi}{6} & 2\sin\frac{\pi}{6} \\ 2\sin\frac{\pi}{6} & 2\cos\frac{\pi}{6} \end{pmatrix} = 4(\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}) = 4 \neq 0$, *for all* x. *And also, it is obvious that* $\Gamma \cap \tilde{\Gamma} = \emptyset$.

$$
\langle \nabla\Phi(x), f(x) \rangle = \left\langle (1, -1), (-\frac{1}{3}x_1 - 3x_2, 3x_1 - \frac{1}{3}x_2) \right\rangle = \left(\frac{-10}{3}x_1 - \frac{8}{3}x_2 \right) \neq 0,
$$

for all $x \in \Gamma$. *One should know that* $x_1 = x_2$ *where* x=(x_1, x_2) *and* x *can not be singular point. The following inequality can be shown similarly.*

$$
\left\langle \nabla\tilde{\Phi}(x), f(x) \right\rangle \neq 0,
$$

for all $x \in \tilde{\Gamma}$. These two inequalities show that conditions $C5)$ and $C6)$ are valid. Hence, all conditions of $C1) - C6)$ are fulfilled for system (2.1.5).

EXAMPLE 2.1.9. *let us consider the conditions $C1) - C6)$ on the following system.*

$$\begin{cases} \begin{array}{lll} \dot{x_1} & = & -x_1 - 3x_2 \\ \dot{x_2} & = & 3x_1 - x_2 \end{array} & , \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \left\{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\right\} \\ \begin{array}{lll} x_1^+ & = & 2x_1^- \\ x_2^+ & = & 2x_2^- \end{array} & , \{x(t) \in \Gamma \wedge t \geq 0\} \\ \begin{array}{lll} x_1^- & = & \frac{1}{2}x_1^+ \\ x_2^- & = & \frac{1}{2}x_2^+ \end{array} & , \left\{x(t) \in \tilde{\Gamma} \wedge t \leq 0\right\} \end{cases} \tag{2.1.6}$$

where

$$\Gamma = \left\{(x_1, x_2) | \quad x_1^2 + x_2^2 = 1, \quad x_1, x_2 \in \mathbb{R}\right\},$$

$$\tilde{\Gamma} = \left\{(x_1, x_2) | \quad x_1^2 + x_2^2 = 4, \quad x_1, x_2 \in \mathbb{R}\right\}.$$

We assume that $G = \{(x_1, x_2) | \quad 1 < x_1^2 + x_2^2 < 4, \quad x_1, x_2 \in \mathbb{R}\}$. How the solution behaves for increasing and decreasing t can be seen on figure 2.5 and figure 2.6 respectively. One can see that $\Phi(x) = x_1^2 + x_2^2 - 1, \tilde{\Phi}(x) = x_1^2 + x_2^2 - 4, f(x) = (-x_1 - 3x_2, 3x_1 - x_2), J(x) = (2x_1, 2x_2)$. One can check conditions $C1) - C6)$. $\nabla\Phi(x) = (2x_1, 2x_2) \neq 0$ (the origin does not belong to Γ) so condition $C1)$ is satisfied. As can be seen easily, J, f are continuously differentiable functions and $\det[\frac{\partial J(x)}{\partial x}] = det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq 0$, for all x.

And, it is obvious that $\Gamma \bigcap \tilde{\Gamma} = \emptyset$.

$$\langle \nabla\Phi(x), f(x) \rangle = \langle (2x_1, 2x_2), (-x_1 - 3x_2, 3x_1 - x_2) \rangle = 2\left(-x_1^2 - x_2^2\right) = -2 \neq 0,$$

for all $x \in \Gamma$, *and*

$$\left\langle \nabla\tilde{\Phi}(x), f(x) \right\rangle = \langle (2x_1, 2x_2), (-x_1 - 3x_2, 3x_1 - x_2) \rangle = 2\left(-x_1^2 - x_2^2\right) = -8 \neq 0,$$

for all $x \in \tilde{\Gamma}$. $\bar{\Gamma} \backslash \Gamma$ *and* $\bar{\tilde{\Gamma}} \backslash \tilde{\Gamma}$ *are empty sets. One can say that, by these two inequalities, conditions* $C5)$ *and* $C6$ *are valid. As we have seen in the above discussion, for the system (2.1.6) all conditions* $C1) - C6)$ *are fulfilled.*

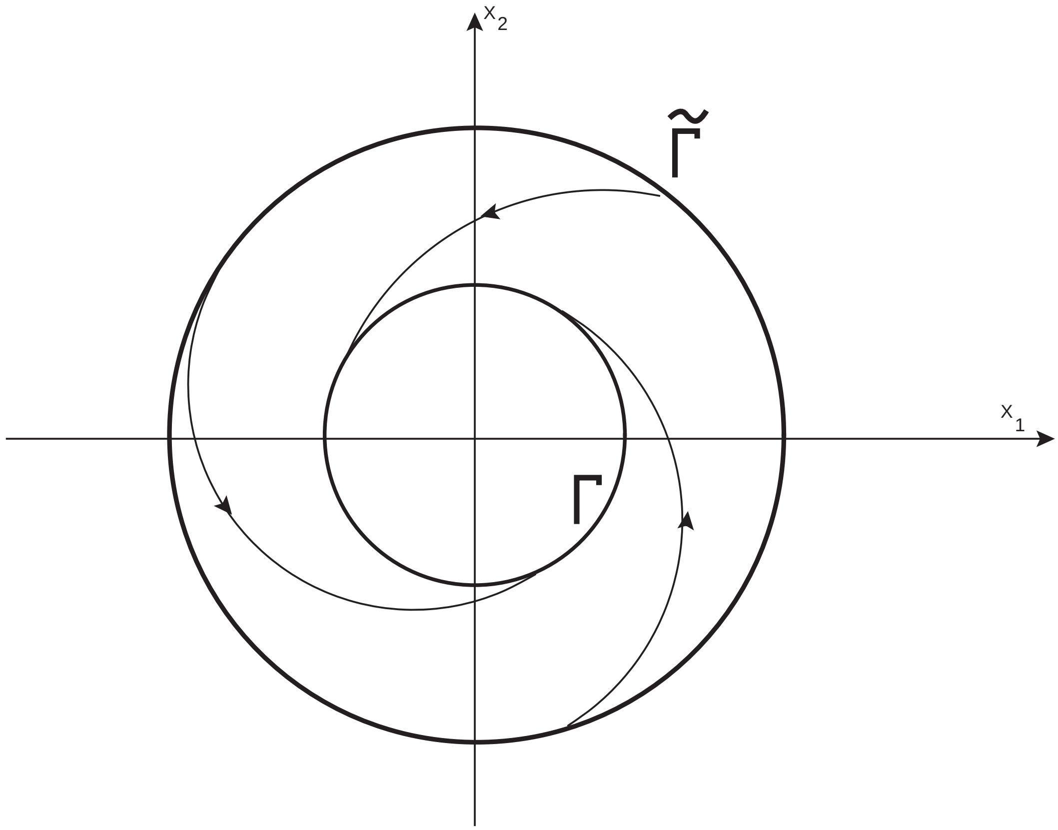

Figure 2.5:

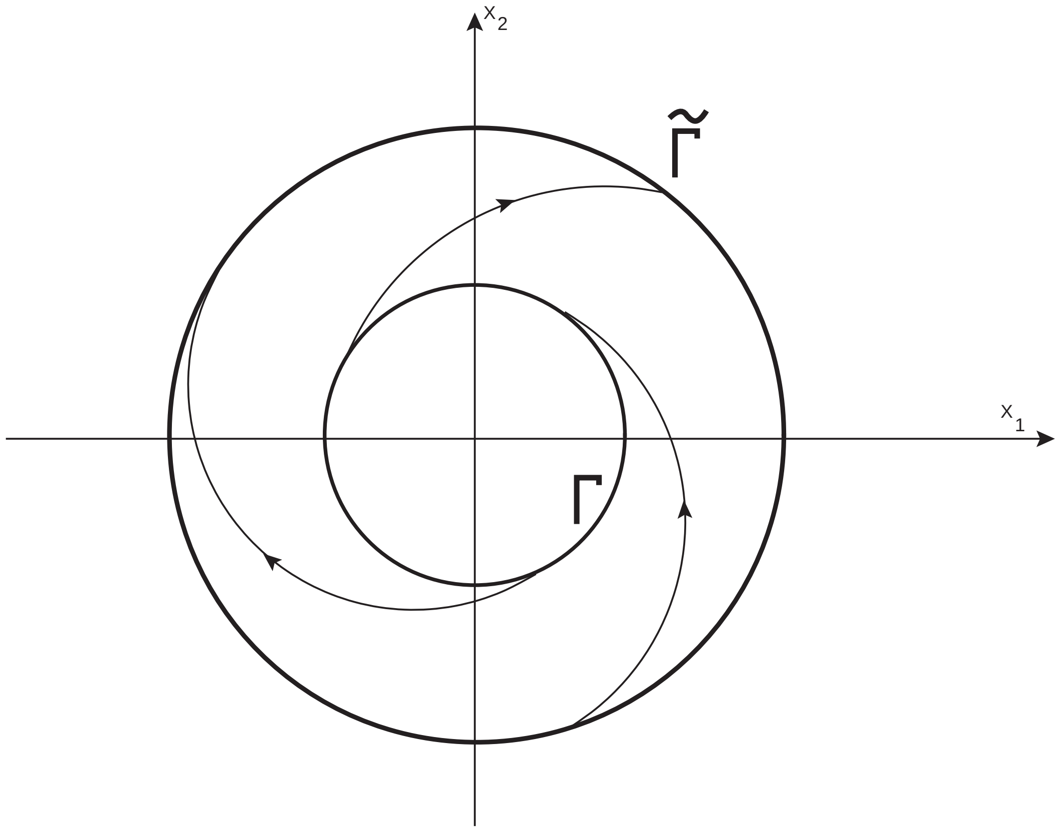

Figure 2.6:

2.2 Existence and Uniqueness

Definition 2.2.1. *A function $x(t) \in \mathcal{PC}^1(T)$ with a set of discontinuity points $\{\theta_i\} \subset T$ is said to be a solution of (2.1.4) on the interval $T \subset \mathbb{R}$ if it satisfies the following conditions:*

(i) equation (2.1.4) is satisfied at each point $t \in T \backslash \{\theta_i\}$ and $x'(\theta_i-) = f(x(\theta_i))$, $i \in \mathcal{A}$, where $x'(\theta_i-)$ is the left-sided derivative;

(ii) $x(\theta_i+) = J(x(\theta_i))$ for all θ_i.

Theorem 2.2.1. *Assume that conditions $C1) - C6)$ hold. Then for every $x_0 \in D$ there exists an interval $(a, b) \subset \mathbb{R}, a < 0 < b$, such that the solution $x(t) = x(t, 0, x_0)$ of (2.1.4) exists on the interval.*

Proof. To prove the theorem we consider the following several cases.

(a) Assume that $x_0 \notin \Gamma \cup \tilde{\Gamma}$. Then there exists a number $\epsilon > 0$ such that $B(x_0, \epsilon) \cap (\Gamma \cup \tilde{\Gamma}) = \emptyset$. Therefore, by the existence and uniqueness theorem [15], the solution exists and is unique on an interval (a, b) as a solution of the system

$$y' = f(y). \tag{2.2.7}$$

(b) If $x_0 \in \Gamma$, then $x(0+) \in \tilde{\Gamma}$. There exists a number $\epsilon > 0$ such that $B(x(0^+), \epsilon) \cap \Gamma \neq 0$ and $x(t)$ can be continued to the right continuously. Let us consider decreasing t now. By condition $C4)$ there exists a number $\epsilon > 0$ such that $B(x(0), \epsilon) \cap \tilde{\Gamma} \neq 0$ and $x(t)$ can be continued to the left continuously.

(c) We can discuss the case $x_0 \in \tilde{\Gamma}$ similarly to the previous one.

The uniqueness of the solution for all cases $(a) - (c)$ follows from the theorem on uniqueness of solutions of ordinary differential equations [15] and invertibility of the function J.

2.3 Continuation of solutions

In this section, we will give continuation theorems for system (2.1.4) and examples, where the solutions are continuable on $\mathbb{R}$ according to the these theorems.

DEFINITION 2.3.1. *A solution $x(t)$ of (2.1.4) is said to be continuable to ∞ if $x(t) : [a, \infty) \to \mathbb{R}^n$, $a \in \mathbb{R}$.*

DEFINITION 2.3.2. *A solution $x(t)$ of (2.1.4) is said to be continuable to $-\infty$ if $x(t) : (-\infty, b] \to \mathbb{R}^n$, $b \in \mathbb{R}$.*

DEFINITION 2.3.3. *A solution $x(t)$ of (2.1.4) is said to be continuable on $\mathbb{R}$ if it is continuable to ∞ and to $-\infty$.*

DEFINITION 2.3.4. *A solution $x(t) = x(t, 0, x_0)$ of (2.1.4) is said to be continuable to a set $S \subset \mathbb{R}^n$ as time decreases (increases) if there exists a moment $\xi \in \mathbb{R}$, such that $\xi \leq 0$ $(\xi \geq 0)$ and $x(\xi) \in S$.*

The following Theorem provides sufficient conditions for the continuation of solutions of (2.1.4).

THEOREM 2.3.1. *Assume that*

(a) every solution $y(t, 0, x_0), x_0 \in D,$ of (2.2.7) is either continuable to ∞ or continuable to Γ as time increases;

(b) for every $x \in \tilde{\Gamma}$ there exists a number ϵ_x such that $\bar{B}(x, \epsilon_x) \cap \Gamma = \emptyset$

(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times(0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)}\|f(x)\|} = \theta > 0$

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D,$ of (2.1.4) is continuable to ∞.

Proof. Let $x(\theta_i+) \in \tilde{\Gamma}$ for fixed i and denote $M_x = \sup_{B(x,\epsilon_x)} \|f(x)\|$. Assume that there exists a number $\xi > \theta_i$, such that $||x(\xi) - x(\theta_i+)|| = \epsilon_{x(\theta_i+)}$ (otherwise $x(t)$ is continuable to ∞). Then

$$x(\xi) = x(\theta_i+) + \int_{\theta_i}^{\xi} f(x(s))ds,$$

and $\epsilon_{x(\theta_i+)} \leq M_{x(\theta_i+)}(\xi - \theta_i) \leq M_{x(\theta_i+)}(\theta_{i+1} - \theta_i)$. The last inequality implies that $\theta_{i+1} - \theta_i \geq \theta$ for all i. That is θ_i is a sequence of $\beta-$ type if $\theta_i \geq 0$.
In a similar manner, one can prove that the following theorem is valid.

Theorem 2.3.2. *Assume that:*

(a) every solution $y(t,0,x_0)$, $x_0 \in D$, of (2.2.7) is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases;

(b) for every $x \in \Gamma$, there exists a number $\epsilon_x > 0$, such that $\bar{B}(x,\epsilon_x) \cap \tilde{\Gamma} = \emptyset$;

(c) $\inf_{(x,\epsilon_x)\in\Gamma\times(0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)}\|f(x)\|} = \theta > 0.$

Then, every solution $x(t) = x(t,0,x_0), x_0 \in D$, of (2.1.4) is continuable to $-\infty$.

Theorems 2.3.1 and 2.3.2 imply that the following assertion is valid.

Theorem 2.3.3. *Assume that*

(a) every solution $y(t,0,x_0), x_0 \in D$, of (2.2.7) satisfies the following conditions:

(a1) it is continuable either to ∞ or to Γ as time increases,

(a2) it is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases;

(b) for every $x \in \tilde{\Gamma}$, there exists a number $\epsilon_x > 0$, such that $\bar{B}(x,\epsilon_x) \cap \Gamma = \emptyset$;

b′) for every $x \in \Gamma$, there exists a number $\tilde{\epsilon}_x > 0$, such that $\bar{B}(x,\tilde{\epsilon}_x) \cap \tilde{\Gamma} = \emptyset$;

(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times\mathbb{R}} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)}\|f(x)\|} > 0;$

c′) $\inf_{(x,\tilde{\epsilon}_x)\in\Gamma\times\mathbb{R}} \frac{\tilde{\epsilon}_x}{\sup_{B(x,\tilde{\epsilon}_x)}\|f(x)\|} > 0.$

Then, every solution $x(t) = x(t,0,x_0)$, $x_0 \in D$, of (2.1.4) is continuable on $\mathbb{R}$.

Other sufficient conditions for the continuation of solutions of (2.1.4) are provided by the following theorems.

Theorem 2.3.4. *Assume that*

(a) every solution $y(t,0,x_0)$, $x_0 \in D$, of (2.2.7) satisfies the following conditions:

(a1) it is continuable either to ∞ or to Γ as t increases;

(a2) it is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as t decreases;

(b) $\sup_D \|f(x)\| < +\infty$*;*

(c) $\mathrm{dist}(\Gamma, \tilde{\Gamma}) > 0$.

Then a solution $x(t,0,x_0), x_0 \in D$, of (2.1.4) is continuable on $\mathbb{R}$.

Proof. Fix $x_0 \in D$ and let $x(t) = x(t,0,x_0)$ be the solution of (2.1.4). According to Definition 2.1.1, we shall consider the following three cases:

A) If x(t) is a continuous solution of (2.1.4), then it is a solution of (2.2.7) and, hence is continuable on $\mathbb{R}$.

B) Denote by $\theta_{\max}$ and $\theta_{\min}$ the maximal and minimal elements of the set $\{\theta_i\}$ respectively. Consider $t \geq \theta_{\max}$. By the condition on J the value $x(\theta_{\max}+) = J(x(\theta_{\max}-)) \in D$ and the solution $x(t) = y\left(t, \theta_{\max}, x(\theta_{\max}+)\right)$, where y is the solution of (2.2.7) and is continuable to ∞. For $t \leq \theta_{\min}$ one can apply the same arguments to show that x(t) is continuable to $-\infty$.

C) Three alternatives exist. Let us consider them in turn.

c_1) If the sequence $\{\theta_i\}$ has a maximal element $\theta_{\max} \in R$, then by using B), it is easy to prove that $x(t)$ is continuable to ∞. Let t be decreasing. We have that

$$x(\theta_i+) = x(\theta_{i+1}) + \int_{\theta_{i+1}}^{\theta_i} f(x(s))ds. \tag{2.3.8}$$

Denote $\sup_D \|f(x)\| = M$ and $\mathrm{dist}(\Gamma, \tilde{\Gamma}) = \alpha$. Then (2.3.8) implies that $\frac{\alpha}{M} \leq \left(\theta_{i+1} - \theta_i\right)$. Hence, $\frac{\alpha}{M}(i - i_0) \geq \left(\theta_i - \theta_{i_0}\right)$, where i_0 is fixed. The last inequality shows that $\theta_i \to -\infty$ as $i \to -\infty$. Thus, $x(t)$ is continuable to $-\infty$.

c_2) Assume that the sequence $\{\theta_i\}$ has a minimal element $\theta_{\min} \in R$. Then the arguments of B) indicate that $x(t)$ is continuable to $-\infty$. For increasing t we have that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))ds, \tag{2.3.9}$$

$\frac{\alpha}{M} \leq (\theta_{i+1} - \theta_i)$ or $\frac{\alpha}{M}(i - i_0) \leq (\theta_i - \theta_{i_0})$, where i_0 is fixed. Hence, $\theta_i \to \infty$ as $i \to \infty$. That is, $x(t)$ is continuable to ∞.

c_3) Assume that $\{\theta_i\}$ has neither a minimal nor a maximal element. The result for this case follows from c_1) and c_2). The proof is complete.

Theorem 2.3.5. *Assume that*

(a) every solution $y(t, 0, x_0), x_0 \in D,$ of (2.2.7) is continuable either to ∞ or to Γ as time increases;

(b) there exists a neighborhood S of Γ in D such that

(b1) $\mathrm{dist}(\Gamma, \partial \mathrm{S}) > 0$;

(b2) $\sup_S \|f(x)\| < \infty$;

(b3) $\tilde{\Gamma} \cap S = \emptyset$.

Then every solution $x(t) = x(t, 0, x_0), x_0 \in D,$ of (2.1.4) is continuable to ∞.

Proof. Denote $d = \mathrm{dist}(\Gamma, \partial \mathrm{S})$ and $M = \sup_S \|f(x)\|$. For fixed i one can see that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))ds.$$

Condition $b3$) implies that $d < \|x(\theta_{i+1}) - x(\theta_i+)\| \leq M(\theta_{i+1} - \theta_i)$. Thus $\theta_{i+1} - \theta_i \geq \frac{d}{M} > 0$ for all i. Further discussion is fully analogous to the proof of the last theorem.

Similarly, one can prove that the following assertion is valid.

Theorem 2.3.6. *Assume that:*

(a) every solution $y(t, 0, x_0), x_0 \in D,$ of (2.2.7) is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases,

(b) there exists a neighborhood $\tilde{S}$ of $\tilde{\Gamma}$ in D, such that:

(b1) $\mathrm{dist}(\tilde{\Gamma}, \partial\tilde{\mathrm{S}}) > 0$

(b2) $\sup_{\tilde{S}} \|f(x)\| < \infty$

(b3) $\Gamma \cap \tilde{S} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0), x_0 \in D$, of (2.1.4) is continuable to $-\infty$.

Using the conditions of both Theorems 2.3.5 and 2.3.6 one can formulate the following assertion.

THEOREM 2.3.7. *Assume that:*

(a) every solution $y(t, 0, x_0), x_0 \in D$, of (2.2.7) satisfies the following conditions:

(a1) it is continuable either to ∞ or to Γ as time increases;

(a2) it is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases;

(b) there exists a neighborhoods S and $\tilde{S}$ of Γ and $\tilde{\Gamma}$ in D, respectively, such that:

(b1) $\mathrm{dist}(\Gamma, \partial\mathrm{S}) > 0$, $\mathrm{dist}(\tilde{\Gamma}, \partial\tilde{\mathrm{S}}) > 0$;

(b2) $\sup_{S \cup \tilde{S}} \|f(x)\| < \infty$;

(b3) $\tilde{\Gamma} \cap S = \emptyset$, $\Gamma \cap \tilde{S} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0), x_0 \in D$, of (2.1.4) is continuable on $\mathbb{R}$.

EXAMPLE 2.3.1. *Let us consider system (2.1.5) in Example 2.1.8 and study the sufficient conditions to indicate continuation of the solution of this system.*
The differential equation (2.1.5) is a linear system and so the solution of this system is continuable to ∞ since maximal interval of existence is $\mathbb{R}$. The first condition is satisfied.

Let us choose the initial value $x_0 = (x_1^0, x_2^0) \in \tilde{\Gamma}$ *it means that* $\sqrt{3}x_1^0 = x_2^0$*, then, one can easily evaluate the distance between* Γ *and* x_0

$$\mathrm{dist}(\mathrm{x}_0, \Gamma) = \frac{|\mathrm{x}_1^0 - \mathrm{x}_2^0|}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}} \left|\mathrm{x}_1^0\right| = \frac{\sqrt{3}-1}{2\sqrt{2}} \|\mathrm{x}_0\|$$

since $|x_1^0| = \frac{1}{2}\|x_0\|$.
If we choose ;

$$\epsilon_{x_0} = \frac{\sqrt{3}-1}{2\sqrt{2}} \|x_0\| \tag{2.3.10}$$

and let us take any x *which belongs to* $B(x_0, \epsilon_{x_0})$, *then we can see that*

$$\|x - x_0\| < \epsilon_{x_0}.$$

It is clear that,

$$\|x\| < \epsilon_{x_0} + \|x_0\|. \tag{2.3.11}$$

Substituting ϵ_{x_0} *in (2.3.10) into (2.3.11), we have*

$$\|x\| < \left[\frac{\sqrt{3}-1+2\sqrt{2}}{2\sqrt{2}}\right] \|x_0\|.$$

After computing the norm of the function f *in this ball, we get*

$$\|f(x)\| = \sqrt{(\frac{1}{3})^2 + 3^2}\,\|x\| \leq \frac{\sqrt{82}}{3}\left[\frac{\sqrt{3}-1+2\sqrt{2}}{2\sqrt{2}}\right]\|x_0\|$$

$$= \frac{\sqrt{41}}{6}\left[\sqrt{3}-1+2\sqrt{2}\right]\|x_0\| = M_x.$$

Therefore

$$\inf \frac{\epsilon_x}{M_x} = \frac{\frac{\sqrt{3}-1}{2\sqrt{2}}\|x_0\|}{\frac{\sqrt{41}}{6}\left[\sqrt{3}-1+2\sqrt{2}\right]\|x_0\|} = \frac{3\left(\sqrt{3}-1\right)}{\sqrt{82}\left(\sqrt{3}-1+2\sqrt{2}\right)} > 0.$$

We can see from this calculation that $\inf \frac{\epsilon_x}{M_x}$ *is positive for all* x *which belongs to* $\tilde{\Gamma}$. *This demonstrates that all conditions of the theorem (2.3.1) are satisfied. So every solution of system (2.1.5) is continuable to* ∞, *for given* Γ *and* $\tilde{\Gamma}$. *The continuation of every solution of (2.1.5), when* t *is negative, can be shown similarly by using Theorem 2.3.2.*

EXAMPLE 2.3.2. *let us consider system (2.1.6) and examine sufficient conditions for continuation of the solution of this system. The differential equation (2.1.6) is a linear system and maximal interval of existence is* $\mathbb{R}$. *So the solution of the differential equation (2.1.6) is continuable. Thus, the first condition of Theorem 2.3.4 is satisfied.*
The domain of this system is $D = \{(x_1, x_2) | \quad 1 \leq x_1^2 + x_2^2 \leq 4, \quad x_1, x_2 \in \mathbb{R}\}$. *It means that the boundaries of the domain are circles with the radius* 1 *and* 2, *and hence one can easily evaluate the distance between* Γ *and* $\tilde{\Gamma}$ *by taking the difference between these radius. So*

$$\operatorname{dist}(\tilde{\Gamma}, \Gamma) = 1.$$

Since

$$\|f(x)\| = \sqrt{(-x_1 - 3x_2)^2 + (3x_1 - x_2)^2} = \sqrt{10}\sqrt{x_1^2 + x_2^2}, \qquad (2.3.12)$$

one can easily see that

$$\sup_D \|f(x)\| \leq 2\sqrt{10}.$$

This shows that all conditions of Theorem 2.3.4 are satisfied, so every solution of system (2.1.6) is continuable on $\mathbb{R}$ *for given* Γ *and* $\tilde{\Gamma}$.

EXAMPLE 2.3.3. *Consider*

$$
\begin{cases}
\dot{x_1} = -2x_1 - 3x_2 \\
\dot{x_2} = 3x_1 - 2x_2
\end{cases} \quad , \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \left\{x(t) \notin \tilde{\Gamma} \wedge t \geq 0\right\}
$$

$$
\begin{aligned}
x_1^+ &= 2\cos\tfrac{\pi}{6}x_1^- - 2\sin\tfrac{\pi}{6}x_2^- \\
x_2^+ &= 2\sin\tfrac{\pi}{6}x_1^- + 2\cos\tfrac{\pi}{6}x_2^-
\end{aligned} \quad , \{x(t) \in \Gamma \wedge t \geq 0\}
$$

$$
\begin{aligned}
x_1^- &= \tfrac{1}{2}\cos\tfrac{\pi}{6}x_1^+ + \tfrac{1}{2}\sin\tfrac{\pi}{6}x_2^+ \\
x_2^- &= -\tfrac{1}{2}\sin\tfrac{\pi}{6}x_1^+ + \tfrac{1}{2}\cos\tfrac{\pi}{6}x_2^+
\end{aligned} \quad , \left\{x(t) \in \tilde{\Gamma} \wedge t \geq 0\right\} \tag{2.3.13}
$$

where

$$
\Gamma = \left\{(x_1, x_2)| \quad x_1 = \sqrt{3}x_2, \quad \frac{1}{2} < x_2 < \frac{3}{2}\right\},
$$

$$
\tilde{\Gamma} = \left\{(x_1, x_2)| \quad \sqrt{3}x_1 = x_2, \quad 1 < x_1 < 3\right\}.
$$

Assume that $D = \mathbb{R}^2 \setminus \left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right), \left(1, \sqrt{3}\right), \left(3, 3\sqrt{3}\right)\right\}$.
Let us check the sufficient conditions of Theorem 2.3.5. The differential equation (2.3.13) is a linear system and maximal interval of existence is $\mathbb{R}$*, so the solution of the differential equation (2.3.13) is continuable to* ∞ *as time increases. Hence, the first condition is satisfied.*

While dealing with other conditions, we prefer to use polar and cartesian coordinates together to have an easier calculation. Firstly, let us define S *in polar coordinates on figure 2.7. Clearly,*

$$
S = \left\{(\rho, \theta) \mid \quad \frac{9}{10} < \rho < \frac{21}{10}, \quad \frac{\pi}{12} < \theta < \frac{\pi}{4}\right\}.
$$

One can easily see that $\Gamma \subset S$ *and* $\tilde{\Gamma} \cap S = \emptyset$*. To calculate the distance between* Γ *and* ∂S*, it is enough to take minimum of the following two distances because of the* *symmetry; firstly, the distance between* Γ *and the circle* $\rho = \frac{9}{10}$ *and secondly the distance between* Γ *and the line* $\theta = \frac{\pi}{4}$. *Let us compute these distances* $dist(\Gamma, \rho = \frac{9}{10}) = \frac{1}{10}$. *Let us write the line* $\theta = \frac{\pi}{4}$ *in cartesian coordinates*

$$\ell = \left\{ (x_1, x_2) \mid \quad x_1 = x_2,\ x_1, x_2 \in \mathbb{R}^+ \right\}.$$

To find $dist(\Gamma, \ell)$, *it is enough to find out the distance between the line* ℓ *and the points* $A\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ *and* $B\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. *In fact,*

$$dist\,(\ell, A) = \frac{\left|\frac{\sqrt{3}}{2} - \frac{1}{2}\right|}{\sqrt{1+1}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{and} \quad dist\,(\ell, B) = \frac{\left|\frac{3\sqrt{3}}{2} - \frac{3}{2}\right|}{\sqrt{1+1}} = \frac{3\sqrt{3}-3}{2\sqrt{2}}$$

We also see that

$$dist(\Gamma, \partial S) = \frac{1}{10}.$$

One can get the distance between Γ *and the surface* ∂S. *Now, we take the norm of the function* $f(x)$, *where* x *belongs to* S

$$\|f(x)\| = \sqrt{4+9}\sqrt{x_1^2 + x_2^2}.$$

$$\sup_S \|f(x)\| = \frac{21\sqrt{13}}{10}, \quad \text{since} \quad \frac{9}{10} < \sqrt{x_1^2 + x_2^2} < \frac{21}{10}.$$

This demonstrates that all conditions of Theorem 2.3.5 are satisfied, so every solution of system (2.3.13) is continuable to ∞.

EXAMPLE 2.3.4. *let us show the continuation of the solution of system (2.3.13) for decreasing* t *by using Theorem 2.3.6. The solution of the differential equation (2.3.13) is continuable to* $-\infty$. *The differential equation (2.3.13) is a linear system. Hence, the first condition is satisfied. While dealing with other conditions, we prefer to use polar and cartesian coordinates together to have an easier calculation. Firstly, let us define* $\tilde{S}$ *in polar coordinates:*

$$\tilde{S} = \left\{ (\rho, \theta) \mid \quad \frac{11}{5} < \rho < \frac{31}{5}, \quad \frac{27\pi}{30} < \theta < \frac{33\pi}{30} \quad \right\}$$

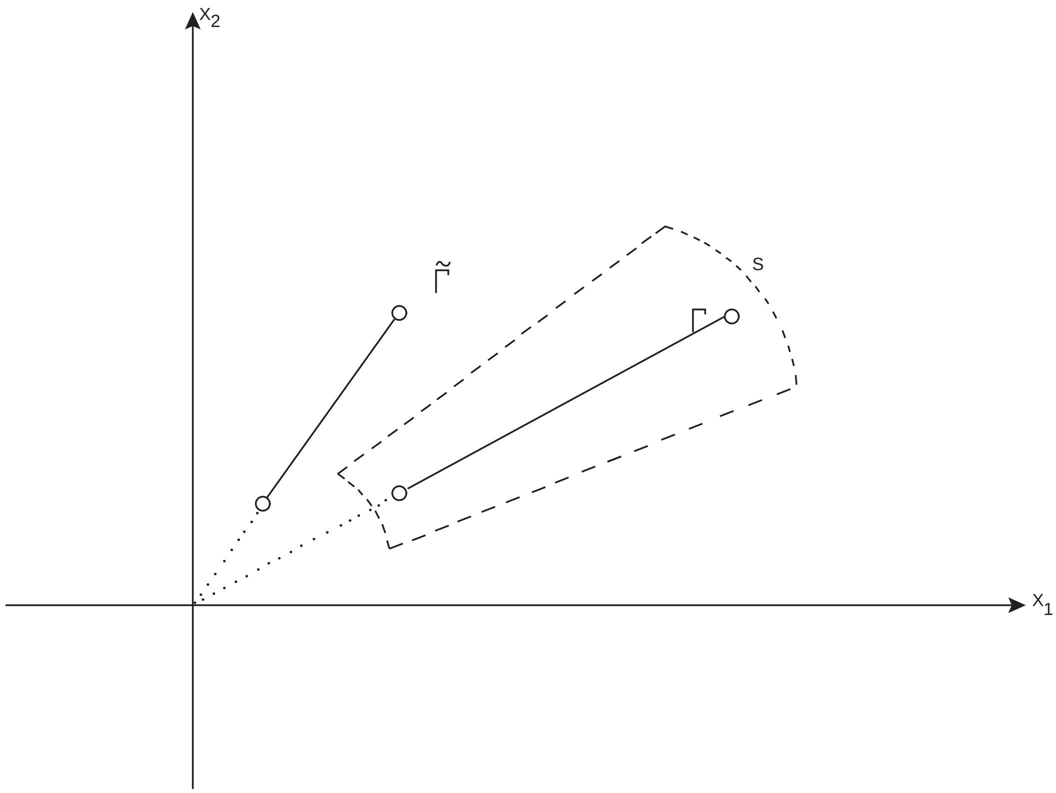

Figure 2.7:

One can easily see on figure 2.8 that $\tilde{\Gamma} \subset \tilde{S}$ *and* $\tilde{\Gamma} \cap \tilde{S} = \emptyset$. *To calculate the distance between* $\tilde{\Gamma}$ *and* $\partial\tilde{S}$, *it is enough to take minimum of the following two distances because of the symmetry; firstly, the distance between* $\tilde{\Gamma}$ *and the circle* $\rho = \frac{11}{5}$ *and the line* $\theta = \frac{27\pi}{30}$. $dist(\tilde{\Gamma}, \rho = \frac{11}{5}) = \frac{1}{5}$. *Let us write the line* $\theta = \frac{27\pi}{30}$ *in cartesian coordinates:*

$$\tilde{\ell} = \left\{ (x_1, x_2) \,|\, \tan \frac{27\pi}{30} x_1 = x_2, \quad , x_1, x_2 \in \mathbb{R}^+ \right\}.$$

To find $dist(\tilde{\Gamma}, \tilde{\ell})$,*it is enough to find out the distance between the line* $\tilde{\ell}$ *and the points* $\tilde{A}\left(1, \sqrt{3}\right)$ *and* $\tilde{B}\left(3, 3\sqrt{3}\right)$. *We easily see that*

$$dist\left(\tilde{\ell}, \tilde{A}\right) = \frac{\left|\tan \frac{27\pi}{30} - \sqrt{3}\right|}{(\sqrt{\tan \frac{27\pi}{30}})^2 + 1} = d_1$$

and

$$dist\left(\tilde{\ell}, \tilde{B}\right) = \frac{\left|3\tan \frac{27\pi}{30} - 3\sqrt{3}\right|}{(\sqrt{\tan \frac{27\pi}{30}})^2 + 1} = 3d_1.$$

The minimum of these distances becomes

$$dist(\tilde{\Gamma}, \partial\tilde{S}) = \frac{1}{5}.$$

Furthermore,

$$\sup_{\tilde{S}} |f(x)| = \frac{31\sqrt{13}}{5} \quad since \quad \frac{29}{5} < \sqrt{x_1^2 + x_2^2} < \frac{31}{5}.$$

So every solution of system (2.3.13) is continuable to $-\infty$ *when* $t < 0$. *By combining Examples 2.3.3 and 2.3.4, one can say that the solution of system (2.3.13) is continuable on* $\mathbb{R}$.

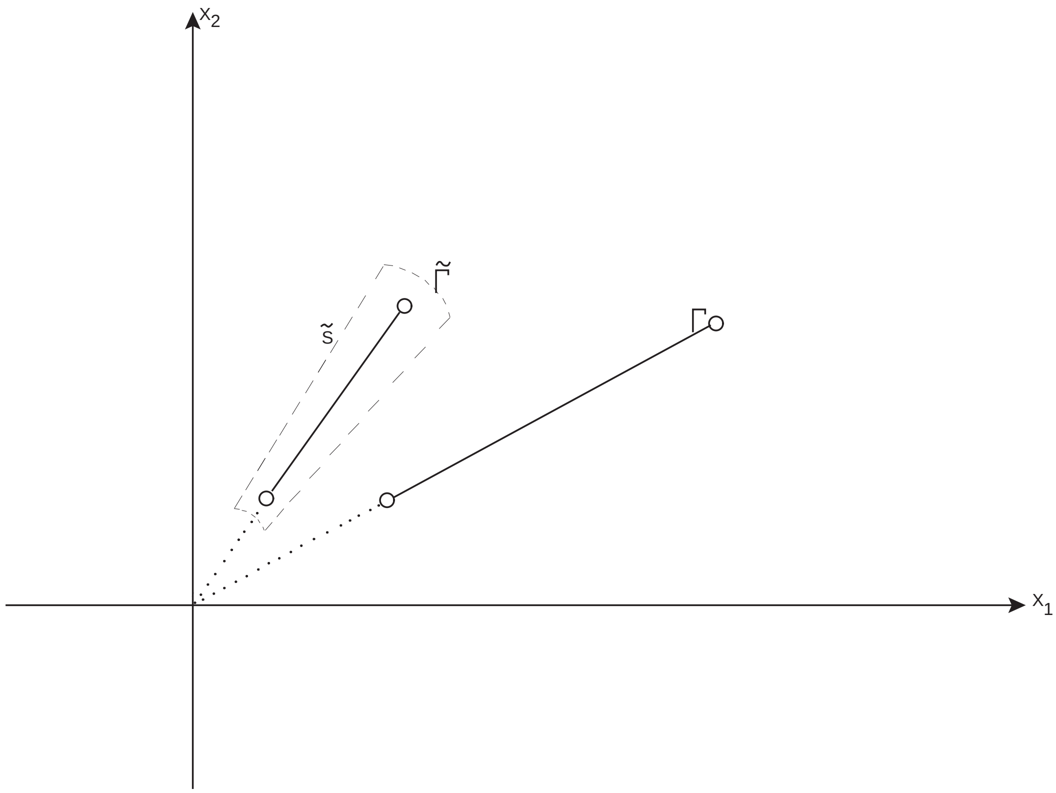

Figure 2.8:

2.4 The Group Property

In the above sections, we have dealt with existence and uniqueness of solutions of the system (2.1.4), smoothness of solutions of the system, and furthermore, we have given the conditions that are sufficient for all solutions of (2.1.4) to be continuable.

Now, we may discuss the group property, which is one of the most significant properties of DS, in this section.
Consider a solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ of (2.1.4). Let $\{\theta_i\}$ be the sequence of discontinuity points of $x(t)$. Fix $\theta \in \mathbb{R}$ and introduce a function $\psi(t) = x(t+\theta)$.

LEMMA 2.4.1. *The set* $\{\theta_i - \theta\}$ *is a set of all solutions of equation*

$$\Phi(\psi(t)) = 0. \tag{2.4.14}$$

Proof. We have $\Phi(\psi((\theta_i - \theta)) = \Phi(x((\theta_i - \theta) + \theta)) = \Phi(x(\theta_i)) = 0$. Assume that $t = \varphi$ is a solution of (2.4.14), then $\Phi(x(\varphi + \theta)) = \Phi(\psi(\varphi)) = 0$. That is, $\varphi + \theta$ is one of the numbers $\{\theta_i\}$. Let $\varphi + \theta = \theta_j$, then $\varphi = \theta_j - \theta$. The lemma is proved. The following condition is one of the main assumptions for DFs.

C7) a)for every $x \in \Gamma$ there exists $\epsilon_x > 0$ such that a function $sign\Phi(x)$ is constant in $[B(x, \epsilon_x) \cap G]\backslash\Gamma$;

b)for every $x \in \tilde{\Gamma}$ there exists $\epsilon_x > 0$ such that a function $sign\Phi(x)$ is constant in $[B(x, \epsilon_x) \cap G]\backslash\tilde{\Gamma}$.

LEMMA 2.4.2. *Assume that* $C1) - C7)$ *hold and* $x(t) : (-\alpha, \alpha) \to \mathbb{R}^n$, $\alpha > 0$, *is a solution of (1.1.1). Then* $x(0) \notin \Gamma$ *and* $x(0) \notin \tilde{\Gamma}$.

Proof. Assume, on the contrary, that $x(0) = x_0 \in \Gamma$. We have that

$$\Phi(x(t)) = \Phi(x(t)) - \Phi(x_0) = \langle \nabla\Phi(x_0), x(t) - x_0 \rangle + o(||x(t) - x_0||) =$$

$$\langle \nabla\Phi(x_0), f(x_0)t + o(|t|) \rangle + o(||f(x_0)||t + o(|t|)) = \langle \nabla\Phi(x(0)), f(x(0)) \rangle\, t + o(|t|).$$

By Condition $C7)$ function $sign\Phi(x(t))$ has a constant value for sufficiently small $|t|$. This contradiction proves our lemma for Γ. For $\tilde{\Gamma}$ the proof is similar.

LEMMA 2.4.3. *Assume that* $C1) - C7)$ *hold. Then* $x(-t, 0, x(t, 0, x_0)) = x_0$ *for all* $x_0 \in D, t \in \mathbb{R}$.

Proof. Consider only $t > 0$, as $t < 0$ is very similar to the first case and $t = 0$ is primitive. If the set $\{\theta_i\}$ is empty then proof follows immediately from the assertion for DS [7]. One can see that it remains to check the equality $x(-\theta_i, 0, x(\theta_i+)) = x(\theta_i)$ is valid for all i, and the condition $x(-\theta_1, 0, x(\theta_1, 0, x_0)) = x_0$ is fulfilled. The first one is obvious because of invertibility of J. Let us consider the second one. Denote $x(t) = x(t, 0, x_0)$, $\tilde{x}(t) = x(t, 0, x(\theta_1))$. Since $x(\theta_1) \in \Gamma$, then by $C4)$, the solution $\tilde{x}$ moves along the trajectory of (2.2.7) for decreasing t. And it could not meet $\tilde{\Gamma}$ if $t > -\theta_1$. Indeed, assume on the contrary that there exists θ, $-\theta_1 < \theta < 0$, moment where $\tilde{x}$ intersects $\tilde{\Gamma}$. Then $\tilde{x}(\theta+) = x(\theta + \theta_1)$. We have obtained a contradiction to Lemma 2.4.2 since $x(t)$ is the solution of (1.1.1) in a neighbourhood of $t = \theta + \theta_1$. The Lemma is proved. Beside the last lemma, let us show that condition $C7)$ and Lemma 2.4.3 are important for the group property. Particularly for the relation $x(-t, 0, x(t, 0, x_0)) = x_0$.

EXAMPLE 2.4.1. *Let us consider Example 2.1.9 where* $G = \{(x_1, x_2)| \quad x_1^2 + x_2^2 > 1, \quad x_1, x_2 \in \mathbb{R}\}$. *One can show that the phase portrait looks like in the figure 2.9. Consider a solution which starts at* x_0 *and comes to point* P *as* t *increases. Moving back, it could not return to* x_0, *for decreasing* t, *since of* $\tilde{\Gamma}$.

LEMMA 2.4.4. *If* $x(t) : \mathbb{R} \to \mathbb{R}^n$ *is a solution of (2.1.4), then* $x(t + \theta)$, $\theta \in \mathbb{R}$, *is also a solution of (2.1.4).*

Proof.

(a) From Lemma 2.4.1 it follows that $\psi = x(t+\theta)$ is continuous on the interval $(\theta_i - \theta, \theta_{i+1} - \theta], i \in \mathbb{Z}$. Fix $i \in \mathbb{Z}$, and consider $t \in (\theta_i - \theta, \theta_{i+1} - \theta]$. We have that $t + \theta \in (\theta_i, \theta_{i+1}]$ and in the same manner as for DSs, one can verify that $\psi'(t) = f(\psi(t))$. That is, the equation (2.1.4) is satisfied by $x(t + \theta)$ for all $t \neq \theta_i - \theta$, $i \in \mathbb{Z}$, if we mean the left sided derivatives.

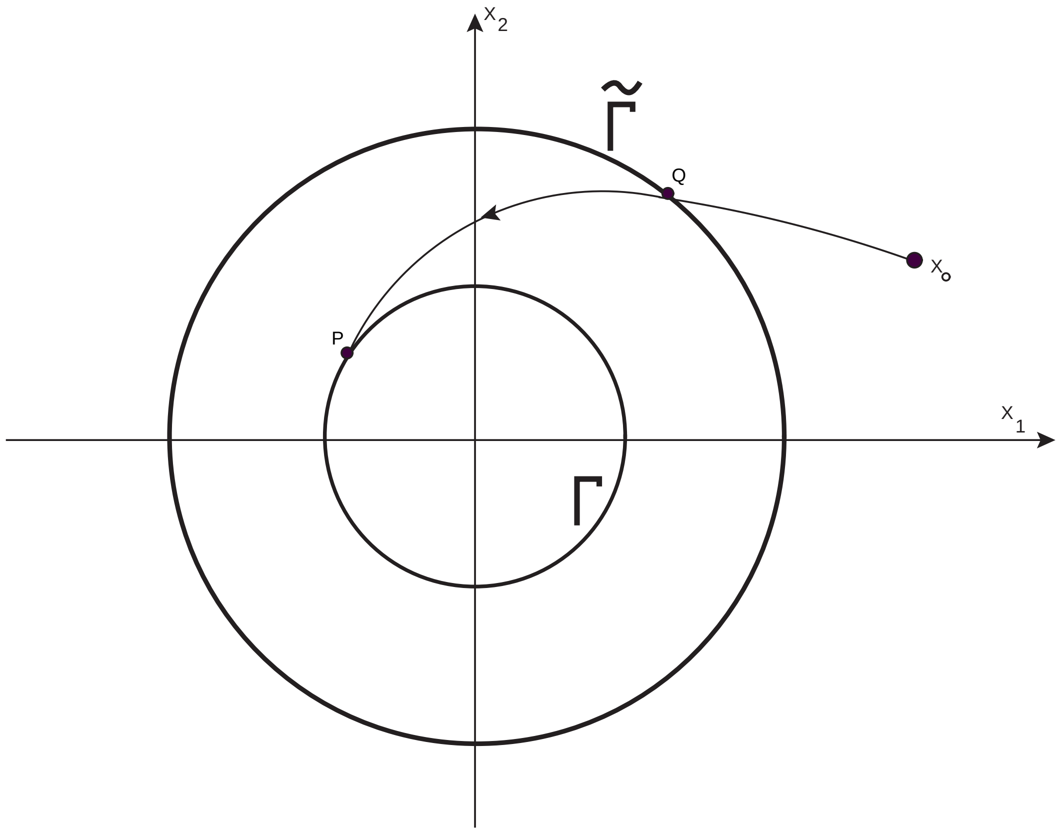

Figure 2.9:

(b) For fixed i we have that $\psi(\theta_i - \theta+) = x(\theta_i - \theta+) + \theta) = x(\theta_i+) = J(x(\theta_i)) = J(\psi(\theta_i - \theta))$. Thus, one can see that the impulsive equation in (2.1.4) is also satisfied by $x(t+\theta)$ and this completes the proof.

Lemmas 2.4.3-2.4.4 imply that the following theorem is valid.

THEOREM 2.4.1. *Assume that conditions $C1) - C7)$ are fulfilled. Then*

$$x(t_2, x(t_1, x_0)) = x(t_2 + t_1, x_0), \tag{2.4.15}$$

for all $t_1, t_2 \in \mathbb{R}$.

The proof of this theorem is similar to the continuous case [40].

Remark 2.4.1. *Since $x(0, x_0) = x_0$, one can conclude on the basis of Theorem 2.4.1 that $x(t, x_0)$ defines a one-parameter group of transformations of D into itself.*

2.5 Continuity of solutions in initial value

Dependence of solutions on an initial value is very effective method to investigate various problems of dynamical systems.

In this section, we will deal with the following topics: Continuous dependence of trajectories on initial value and B-equivalence. Also, we will give definitions and theorems about these subjects.

Assume that $x^0(t) : [a, b] \to \mathbb{R}^n, a \le 0 \le b$, is a continuous solution of (2.1.4), $x^0(t) = x(t, 0, x_0)$. Let $T = \{x \in G | x = x^0(t)$ for some $t \in [a, b]\}$. We shall show that the following lemma is valid.

LEMMA 2.5.1. *There exists an $\epsilon > 0$ such that $T_\epsilon \cap \Gamma = \emptyset$, $T_\epsilon \cap \tilde{\Gamma} = \emptyset$, if $T \cap \Gamma = \emptyset$, $T \cap \tilde{\Gamma} = \emptyset$.*

Proof. Assume, on the contrary, that there exists a sequence $x_n \in \Gamma, n \in \mathbb{N}$, such that $x_n \to T$, as $n \to \infty$. Since T is compact, there exists a subsequence, which we assume to be the sequence x_n itself, and a point $x_0 \in T$, such that

$x_n \to x_0$, as $n \to \infty$. As function Φ is continuous, either $x_0 \in \Gamma$ or $x_0 \in \bar{\Gamma} \backslash \Gamma$. But $x_0 \notin \Gamma$ by the assumption, and $x_0 \notin \bar{\Gamma} \backslash \Gamma$ by the definition of D. This contradiction proves our Lemma.

Now, we assume that solution $x^0(t)$ of (2.1.4) has an empty or nonempty set of discontinuity points, and all these points are interior in $[a, b]$. Denote by $x(t) = x(t, 0, \bar{x})$ another solution of (2.1.4).

Definition 2.5.1. *The solution $x(t) : [a, b] \to \mathbb{R}^n$ is said to be in an $\epsilon-$ neighborhood of $x^0(t)$ if*

1. *every point of discontinuity of $x(t)$ lies in an ϵ-neighborhood of a point of discontinuity of $x^0(t)$;*
2. *for each $t \in [a, b]$ outside ϵ-neighborhood of points of discontinuity of $x^0(t)$, the inequality $\|x^0(t) - x(t)\| < \epsilon$ holds.*

Definition 2.5.2. *A Hausdorff's topology, which is built on the basis of all $\epsilon-$ neighborhoods, $0 < \epsilon < \infty$, of piecewise continuous solutions is called $B_{[a,b]}-$ topology.*

Theorem 2.5.1. *Assume that conditions $C1) - C6)$ are satisfied. Then the solution $x^0(t)$ continuously depends on initial value in $B_{[a,b]}$ topology (B-continuous dependence).*
Moreover, if all $\theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are interior points of $[a, b]$, then for sufficiently small $||x_0 - \bar{x}||$, the solution $x(t) = x(t, 0, \bar{x}), x(t) : [a, b] \to \mathbb{R}^n$, meets the surface Γ exactly $m + k$ times.

Proof. We consider only the section $[0, b]$. The closeness of x(t) and $x^0(t)$ on $[a, 0]$ can be considered similarly. There are two cases: a) $x_0 \in \Gamma$ and b) $x_0 \notin \Gamma$.

Assume that $x^0(b) \notin \Gamma$. In other words, $t = b$ is not the discontinuity point of $x^0(t)$. For a positive number $\alpha \in R$ we shall construct a set G^α in the following way. Let
$F_\alpha = \{(t, x) | t \in [0, b], \|x - x^0(t)\| < \alpha\}, G_i(\alpha), i = \overline{0, m+1}$, be α-neighborhoods of points $(0, x_0)$, $(\theta_i, x(\theta_i)), i = \overline{1, m}$, $(b, x^0(b))$ in $\mathbb{R} \times \mathbb{R}^n$ respectively, and $\bar{G}_i(\alpha)$,

$i = \overline{1,m}$, be $\alpha-$ neighborhoods of points $(\theta_i, x^0(\theta_i+))$ respectively. Denote

$$G^\alpha = F_\alpha \cup \left(\cup_{i=0}^{m+1} G_i(\alpha)\right) \cup \left(\cup_{i=1}^{m} \bar{G}_i(\alpha)\right).$$

Take $\alpha = h$ sufficiently small so that $G^h \subset G_t \times G_x$, where G_t is an interval, such that $[0, b] \subset G_t$.
Fix $\epsilon \in \mathbb{R}$, $0 < \epsilon < h$.

1. In view of the theorem on continuous dependence on parameters [15], there exists $\bar{\delta}_m \in \mathbb{R}, 0 < \bar{\delta}_m < \epsilon$, such that $\bar{G_m}\left(\bar{\delta}_m\right) \cap \Gamma = \emptyset$ and every solution $x_m(t)$ of (2.2.7), which starts in $\bar{G_m}(\bar{\delta}_m)$, is continuable to $t = b$, does not intersect Γ, and
$$\left\|x_m(t) - x^0(t)\right\| < \epsilon,$$
for those t.

2. The continuity of J implies that there exists $\delta_m \in \mathbb{R}$, $0 < \delta_m < \epsilon$, such that $(\kappa, x) \in G_m(\alpha_m)$ implies $(\kappa, x + J(x)) \in \bar{G}_m(\bar{\alpha}_m) \cap D$.

3. Using corollary 2.6.1, continuous dependence of solutions on initial value, one can find $\bar{\delta}_{m-1}, 0 < \bar{\delta}_{m-1} < \epsilon$, such that a solution $x_{m-1}(t)$ of (2.2.7), which starts in $\bar{G}_{m-1}(\bar{\alpha}_{m-1}) \cap D$, $\bar{G}_{m-1}(\bar{\alpha}_{m-1}) \cap \Gamma \neq \emptyset$, intersects Γ in $G_m(\alpha_m)$ (we continue the solution $x_{m-1}(t)$ only to the moment of the intersection) and $\|x_{m-1}(t) - x^0(t)\| < \epsilon$ for all t from the common domain of $x_{m-1}(t)$ and $x^0(t)$.

Continuing the process for $m-2, m-3, \ldots, 1$, one can obtain a sequence of families of solutions of (2.2.7) $x_i(t), i = \overline{1,m}$, and a number $\delta \in \mathbb{R}, 0 < \delta < \epsilon$, such that a solution $x(t) = x(t, 0, \bar{x})$, which starts in $G_0(\delta) \cap D$, coincides over the first interval of continuity, except possibly, the δ_1-neighborhood of θ_1, with one of the solutions $x_1(t)$. Then on the interval $[\theta_1, \theta_2]$ it coincides with one of the solutions $x_2(t)$, except possibly, the δ_1-neighborhood of θ_1 and the δ_2-neighborhood of θ_2, etc. Finally, one can see that the integral curve of $x(t)$ belongs to G^ϵ, it has exactly k meeting points with Γ, θ_i^1, $i = \overline{1,m}$, $|\theta_i^1 - \theta_i| < \epsilon$ for all i and is continuable to $t = b$.

If $x^0(b) \in \Gamma$, then it is easy to see that $x(t)$ has either a discontinuity point $\theta_m^1 \leq \theta_m$ or only $m-1$ discontinuity points $\theta_i^1, i = \overline{1, m-1}$ in $[0, b]$.

Assume that $x_0 \in \Gamma$. In this case, $t = 0$ is a jump moment for $x^0(t)$ and $x(0+) \notin \Gamma$, that is $0 = \theta_1$. We assume that $x^0(t)$ has points of discontinuity $\theta_i, i = \overline{1, m}$. Similarly to the previous case, one can find the $\bar{\delta}_1$-neighborhood $\bar{G}(\bar{\delta}_1)$ of the point $(\theta_1, x(\theta_1^+))$ which serves the same role as $\bar{\delta}_1$ in the first case. That is, if $(\kappa, x) \in \bar{G}(\bar{\delta}_1) \cap D$, then the solution $x(t)$ belongs to the $\epsilon-$ neighborhood of $x^0(t)$ in $B_{[0,b]}-$ topology. Now, using condition $C5$) and continuity of f and J, it is easy to find $\delta, 0 < \delta < \epsilon$, such that every solution $x(t)$ of (2.1.4) which starts in $\delta-$ neighborhood of $(0, x_0)$ in D intersects Γ in $G_1(\delta_1) \cap D$.
For the case $a \leq t \leq 0$, we only should remark that similarly to $0 \leq t \leq b$ for a given $\epsilon > 0$, one can find δ', such that $(0, \bar{x}) \in G_0(\delta')$ implies that $x(t, 0, \bar{x})$ is in the ϵ-neighborhood of $x^0(t)$ in $B[a, 0]$-topology. Finally, if $\delta(\epsilon) = \min(\delta, \delta')$ and $(0, \bar{x}) \in G_0(\delta(\epsilon))$, then $x(t, 0, \bar{x})$ is in the $\epsilon-$neighborhood of $x^0(t)$ in $B[a, b]-$topology. The theorem is proved.

2.6 $B-$equivalence

Let us introduce the functions $\tau = \tau(x)$, $\Psi = \Psi(x)$, $\tilde{\tau} = \tilde{\tau}(x)$ and $\tilde{\Psi} = \tilde{\Psi}(x)$ which will be needed throughout the rest of the paper. Fix $\kappa \in \mathbb{R}$. Denote by $x(t) = x(t, \kappa, x)$ a solution of (2.2.7), $\tau = \tau(x)$ the moment of the first meeting of $x(t)$ with the surface Γ as t increases or decreases and $\tilde{\tau} = \tilde{\tau}(x)$ the moment of the first meeting of $x(t)$ with the surface $\tilde{\Gamma}$ as t increases or decreases.

Lemma 2.6.1. $\tau(x), \tilde{\tau}(x) \in C^1$.

Proof. Let us show $\tau \in C^1$. The proof of $\tilde{\tau} \in C^1$ is similar. Differentiating $\Phi(x(\tau, \kappa, x)) = 0$, and using $C5$) one can get that

$$\frac{\partial \Phi(x(\tau, \kappa, x))}{\partial \tau} = \frac{\partial \Phi(x(\tau, \kappa, x))}{\partial x} \frac{dx(t)}{dt}\bigg|_{t=\tau} = \frac{\partial \Phi(x(\tau, \kappa, x))}{\partial x} f(x(\tau, \kappa, x)) \neq 0$$

The proof of the lemma follows immediately from the implicit function theorem and conditions on (2.2.7).

Corollary 2.6.1. *$\tau(x), \tilde{\tau}(x)$ are continuous functions.*

Now let $x_1 = x(t,\tau,x(\tau)) + J(x(\tau)), \tilde{x}_1 = x(t,\tilde{\tau},x(\tilde{\tau})) + J^{-1}(x(\tilde{\tau}))$ be also solutions of (2.2.7). Define functions $\Psi(x) = x_1(\kappa), \tilde{\Psi}(x) = \tilde{x}_1(\kappa)$.
Similarly to Lemma 2.6.1, one can show that the following assertion is valid.

Lemma 2.6.2. $\Psi(x), \tilde{\Psi}(x) \in C^1$

Consider the solution $x^0(t) : [a,b] \to R^n, a \leq 0 \leq b$, of (2.1.4) again. This time we assume that all points of discontinuity $\{\theta_i\}$ are interior points of $[a,b]$. That is, $a < \theta_{-k}$ and $\theta_m < b$.

The following system of impulsive differential equations is very important in sequel

$$\begin{aligned} &\dot{y}(t) = f(y), \quad t \neq \theta_i, \\ &y(\theta_i+) = W_i(y(\theta_i)), \text{for } i > 0, \\ &y(\theta_i) = \tilde{W}_i(y(\theta_i+)), \text{for } i < 0, \end{aligned} \tag{2.6.16}$$

where the function f is the same as in (2.1.4) and the maps $W_i, \tilde{W}_i$ will be defined below.

Without loss of generality, assume that there exists $r_1 \in \mathbb{R}, 0 < r_1 < r$, such that the r_1- neighborhoods $G_i(r_1)$ of $(\theta_i, x^0(\theta_i))$ do not intersect each other. In view of $C5$), one can suppose that r_1 is sufficiently small so that every solution of (2.2.7) which starts in $G_i(r_1)$ intersects Γ in $G_i(r_1)$ exactly once as t increases or decreases.
Fix $i = 1,\ldots,m$ and let $\xi(t) = x(t,\theta_i,x), (\theta_i,x) \in G_i(r_1)$, be a solution of (2.2.7) and $\tau_i = \tau_i(x)$, $\tau_i \geq \theta_i$ or $\tau_i < \theta_i$, be a meeting time of $\xi(t)$ with Γ and $\psi(t) = x(t,\tau_i,\xi(\tau_i) + J(\xi(\tau_i))$ be another solution of (2.2.7). Denote $W_i(x) = \psi(\theta_i)$. One can see that

$$W_i(x) = \int_{\theta_i}^{\tau_i} f(\xi(s))ds + J(x + \int_{\theta_i}^{\tau_i} f(\xi(s))ds) + \int_{\tau_i}^{\theta_i} f(\psi(s))ds \tag{2.6.17}$$

is a map of an intersection of the plane $t = \theta_i$ with $G_i(r_1)$ into the plane $t = \theta_i$. Similarly for $i = -k,\ldots,-1$, if we denote by $\xi(t) = x(t,\theta_i,x)$ and $\psi(t) =$

$x(t, \tilde{\tau}_i, \xi(\tilde{\tau}_i) + J^{-1}(\xi(\tilde{\tau}_i))$ corresponding solutions of (2.2.7), then

$$\tilde{W}_i(x) = \int_{\theta_i}^{\tilde{\tau}_i} f(\xi(s))ds + J^{-1}(x + \int_{\theta_i}^{\tilde{\tau}_i} f(\xi(s))ds) + \int_{\tilde{\tau}_i}^{\theta_i} f(\psi(s))ds \quad (2.6.18)$$

The functions $W_i, \tilde{W}_i$ are the maps Ψ and $\tilde{\Psi}$ respectively defined in the beginning of this section with $\kappa = \theta_i$. Hence, Lemma 2.6.2 implies that all $W_i, \tilde{W}_i$ are continuously differentiable maps. It is obvious that for sufficiently small r_1, $W_i(x), \tilde{W}_i(x) \in G_r$. Furthermore, $(\alpha, \hat{\beta}], \{\alpha, \beta\} \subset R$, stands for an oriented interval. Let $x(t)$ be a solution of (2.1.4), $x(t) = x(t, a, x(a))$, and $x(t)$ be close to $x^0(t)$ in $B_{[a,b]}-$ topology so that x(t) has exactly $m - k$ points $\tau_i, i = -k, \ldots, -1, 1, 2, \ldots, m,$ of discontinuity in $[a, b]$. Denote by $G(h)$ an $h-$neighborhood of the point $x^0(0)$.

Definition 2.6.1. *The systems (2.1.4) and (2.6.16) are said to be $B-$equivalent in G^{r_1} if there exists $h \in R, 0 < h$, such that:*

1. *for every solution $x(t)$ of (2.1.4, such that $x(0) \in G(h)$, the integral curve of $x(t)$ belongs to G^{r_1} and there exists a solution $y(t) = y(t, 0, x(0))$ of (2.6.16) which satisfies*

$$x(t) = y(t), t \in [a, b] \setminus \cup_{i=-k}^{m} (\tau_i, \hat{\theta}_i]. \quad (2.6.19)$$

Particularly:

$$x(\theta_i) = \begin{cases} y(\theta_i), & \text{if } \theta_i \le \tau_i, \\ y(\theta_i^+), & \text{otherwise}, \end{cases} \qquad y(\tau_i) = \begin{cases} x(\tau_i), & \text{if } \theta_i \ge \tau_i, \\ x(\tau_i^+), & \text{otherwise}. \end{cases} \quad (2.6.20)$$

2. *Conversely, if (2.6.16) has a solution $y(t) = y(t, 0, x(0)), x(0) \in G(h)$, then there exists a solution $x(t) = x(t, 0, x(0))$ of (2.1.4) which has an integral curve in G^{r_1}, and (2.6.20) holds.*

Lemma 2.6.3. *$x^0(t)$ is a solution of (2.1.4) and (2.6.16) simultaneously.*

Proof. The proof follows immediately from (2.6.17) and (2.6.18).

THEOREM 2.6.1. *Assume that conditions $C1) - C6)$ are fulfilled. Then systems (2.1.4) and (2.6.16) are B-equivalent in G^{r_1} if r_1 is sufficiently small.*

*Proof.*Assume that $r_1 > 0$ is sufficiently small so that W_i, $i = 1, \ldots, m$, and $\tilde{W}_i$, $i = -k, \ldots, -1$ are defined. Let us check only the first part of Definition 2.6.1,because for the second part, the proof is analogous. Theorem 2.5.1 implies that there exists a small $h, 0 < h < r_1$, such that if $\|\bar{x} - x_0\| < h$ and $\bar{x} \in D$, then the solution $x(t) = x(t, 0, \bar{x})$ belongs to $G^{r_1} \cap G_t \times D$, where $r_1 > 0$ has been chosen for W_i above. Assume that h is sufficiently small so that $x(t)$ has exactly $m + k - 1$ moments of discontinuity $t = \tau_i, i = -k, \ldots, -1, 1, \ldots, m$. Without loss of generality, we suppose that $\theta_i > \tau_i$ for all i and $x(0)$ is not the point of discontinuity. It is obvious that we need only to prove the theorem for $[0, b]$, because for $[a, 0]$, the proof is similar. Consider the solution $y(t) = x(t, 0, x(0))$ of (2.6.16). By the theorem on existence and uniqueness [15] the equality

$$x(t) = y(t) \tag{2.6.21}$$

on $[0, \tau_1]$ is valid. Since $(\tau_1, x(\tau_1)) \in G^{r_1}$ we have

$$y(\theta_1+) = \int_{\tau_1}^{\theta_1} f(y(s))ds + W_i(y(\theta_1)). \tag{2.6.22}$$

Moreover,

$$x(\theta_1) = x(\tau_1) + J(x(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))ds. \tag{2.6.23}$$

Using (2.6.21)-(2.6.23) one can obtain that

$$y(\theta_1+) = x(\tau_1) + \int_{\tau_1}^{\theta_1} f(y(s))ds + \int_{\theta_1}^{\tau_1} f(y(s))ds$$

$$+J(y(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))ds = x(\theta_1).$$

Now, defining $x(t)$ and $y(t)$ as solutions of (2.2.7) with a common initial value $x(\theta_1)$, one can see that $x(t) = y(t), t \in (\theta_1, \tau_2]$. Continuing in the same manner for all $t \in [0, b]$ one can show that $y(t)$ is continuable to $t = b$ and (2.6.19) holds. Moreover, it is easily seen that for sufficiently small r_1 the integral curve of $y(t)$ belongs to G_r. The theorem is proved.

2.7 Differentiability of solutions in initial value

Let us define derivatives of functions $\tau_i(x)$, $W_i(x)$, $i = 1 \ldots, m$, and $\tilde{\tau}_i(x)$, $\tilde{W}_i(x)$, $i = -k, \ldots, -1$, which were described in Section 2.6, at the points $(x^0(\theta_i))$ and $(x^0(\theta_i+))$ respectively. We start with derivatives of $\tau_i(x)$ and $\tilde{\tau}_i(x)$. One should emphasize that $\tau_i, \tilde{\tau}_i$ are maps $\tau, \tilde{\tau}$ defined in Section 2.6 with $\kappa = \theta_i$. The equalities $\Phi(x(\tau_i(x))) = 0$ and $\tilde{\Phi}(x(\tilde{\tau}_i(x))) = 0$ imply that

$$\Phi_x(x^0(\theta_i))f(x^0(\theta_i))d\tau_i + \sum_{j=1}^{n} \Phi_x(x^0(\theta_i))\frac{\partial x^0(\theta_i)}{\partial x_j}dx_j = 0$$

$$\tilde{\Phi}_x(x^0(\theta_i+))f(x^0(\theta_i+))d\tau_i + \sum_{j=1}^{n} \tilde{\Phi}_x(x^0(\theta_i+))\frac{\partial x^0(\theta_i+)}{\partial x_j}dx_j = 0.$$

Using the last expression, one can obtain that

$$\frac{\partial \tau_i(x^0(\theta_i))}{\partial x_j} = -\frac{\Phi_x(x^0(\theta_i))\frac{\partial x^0(\theta_i)}{\partial x_j}}{\Phi_x(x^0(\theta_i))f(x^0(\theta_i))},$$

and

$$\frac{\partial \tilde{\tau}_i(x^0(\theta_i+))}{\partial x_j} = -\frac{\partial \Phi_x(x^0(\theta_i+))\frac{\partial x^0(\theta_i+)}{\partial x_j}}{\tilde{\Phi}_x(x^0(\theta_i+))f(x^0(\theta_i+))}. \qquad (2.7.24)$$

Similarly, the following expressions are valid:

$$\frac{\partial W_i(x^0(\theta_i))}{\partial x_j} = f\frac{\partial \tau_i}{\partial x_j} + \frac{\partial J}{\partial x}(e_j + f\frac{\partial \tau_i}{\partial x_j}) - f^+\frac{\partial \tau_i}{\partial x_j},$$
$$\frac{\partial \tilde{W}_i(x^0(\theta_i+))}{\partial x_j} = f^+\frac{\partial \tilde{\tau}_i}{\partial x_j} + \frac{\partial J^{-1}}{\partial x}(e_j + f^+\frac{\partial \tilde{\tau}_i}{\partial x_j}) - f\frac{\partial \tilde{\tau}_i}{\partial x_j}, \qquad (2.7.25)$$

where $e_j = (0, \ldots, 1, \ldots, 0)^T$. Assume that $x^0(t) : [a, b] \to \mathbb{R}^n$ is the solution of (2.1.4) and (2.6.16). Moreover, systems (2.1.4) and (2.6.16) are $B-$equivalent in G^r and there exists $\delta \in R, \delta > 0$, such that every solution which starts in $G_0(\delta)$ is continuable to $t = b$. Without loss of generality, assume that all points of discontinuity of $x^0(t)$ are interior. Denote by $x^j(t), j = \overline{1, n}$, a solution of (2.1.4) such that $x^j(t_0) = x_0 + \xi e_j = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0), \xi \in \mathbb{R}$, $(t_0, x_0 + \xi e_j, \mu_0) \in C_0(\delta)$ and let θ_i^j be the moments of discontinuity of $x^j(t)$. By Theorem 2.5.1, for sufficiently small $|\xi|$ the solution $x^j(t)$ is defined on $[a, b]$.

Definition 2.7.1. *The solution $x^0(t)$ is said to be differentiable in $x_j^0, j = \overline{1, n}$, if*

A) there exist constants $\nu_{ij}, i = -k, \ldots, -1, 1, \ldots, m$, such that

$$\theta_i^j - \theta_i = \nu_{ij}\xi + o(|\xi|), \quad (\xi_{ij} \to 0) \tag{2.7.26}$$

B) for all $t \in [a, b] \setminus \cup_{i=-k}^{m} (\theta_i, \hat{\theta_i^j}]$, the following equality is satisfied

$$x^j(t) - x^0(t) = u_j(t)\xi + o(|\xi|), \quad (\xi_{ij} \to 0) \tag{2.7.27}$$

where $u_j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i, i = -k, \ldots, -1, 1, \ldots, m$.

The pair $\{u_j, \{\nu_{ij}\}_i\}$ is said to be a $B-$ derivative of $x^0(t)$ in initial value x_0^j on $[a, b]$.

Lemma 2.7.1. *Assume that conditions $C1) - C6)$ Then the solution $x^0(t)$ of (2.6.16) has $B-$ derivatives in the initial value on $[a, b]$. Moreover*

1) $u_j, j = \overline{1, n}$, are solutions of the linear system

$$\begin{aligned} &\frac{du}{dt} = f_x(x^0(t))u, \quad t \neq \theta_i, \\ &u(\theta_i+) = W_{ix}(x^0(\theta_i))u(\theta_i), if i > 0, \\ &u(\theta_i) = \tilde{W}_{ix}(x^0(\theta_i+))u(\theta_i+), if i < 0, \end{aligned} \tag{2.7.28}$$

with the initial conditions $u(t_0) = e_j, j = \overline{1, n}$, respectively and constants $\nu_{ij} =$ 0, for all i, j.

Proof. Fix $p = \overline{1,n}$. We shall prove the Lemma only for the derivative in x_0^p and for $t \geq 0$. Let $y_p(t) = y(t, t_0, x_0 + \xi e_p, \mu_0)$. By the theorem on differentiability with respect to parameters [15] we have that $y_p(t) - x^0(t) = u_p(t)\xi + \rho(\xi), \quad \rho(\xi) = o(|\xi|)$, for all $t \in [0, \theta_1]$. Particularly, $y_p(\theta_1) - x^0(\theta_1) = u_p(\theta_1)\xi + \rho(|\xi|)$. Then $y_p(\theta_1+) - x^0(\theta_1+) = W_1(y_p(\theta_1)) - W_1(x^0(\theta_1)) = W_{1x}(x^0(\theta_1))[u_p(\theta_1)\xi + \rho(\xi)] + \bar{\rho}_1(\xi)$. Since $\bar{\rho}_1 = o(|\xi|)$, we have that $y_p(\theta_1+) - x^0(\theta_1+) = u_p(\theta_1+)\xi + \tilde{\rho}_1(\xi)$, where $\tilde{\rho}_1 = o(|\xi|)$. Denote by $U(t), U(\theta_1) = I$, the fundamental matrix of solutions of the system $u'(t) = f_x(x^0(t))$. Using the theorem from [15] again one can obtain that for all $t \in (\theta_1, \theta_2]$ the following relation is true $y_p(t) - x^0(t) = U(t)(y_p(\theta_1+) - x^0(\theta_1+)) + \rho(y_p(\theta_1+) - x^0(\theta_1+)) = U(t)u_p(\theta_m+)\xi + \rho_2(\xi) = u_p(t)\xi + \rho_2(\xi)$, where $\rho_2 = o(|\xi|)$. Continuing the process we can prove that (2.7.27) is valid. Formula (2.7.26) involving constants ν_i^j is trivial. The Lemma is proved.

THEOREM 2.7.1. *Assume that conditions $C1) - C6)$ are satisfied. Then the solution $x^0(t)$ of (2.1.4) has $B-$ derivatives in the initial value on $[a, b]$. Moreover:*

$u_j(t), j = \overline{1,n}$, are respectively solutions of equation (2.7.28) with the initial conditions $u(t_0) = e_j, j = \overline{1,n}$, and

$$\nu_{ij} = -\frac{\Phi_x u_j(\theta_i)}{\Phi_x f}, j = \overline{1,n}, i = \overline{1,m}, \nu_{ij} = -\frac{\tilde{\Phi}_x(x^0(\theta_i+))u_j(\theta_i+)}{\tilde{\Phi}_x(x^0(\theta_i+))f(x^0(\theta_i+))},$$

$j = \overline{1,n}$, $i = \overline{-k,-1}$.

The proof of the theorem follows immediately from Theorem 2.6.1, Lemma 2.7.1 and formulas (2.7.24), (2.7.25).

Remark 2.7.1. *Higher order smoothness of DDS is considered in [3].*

2.8 Conclusion

Let $D \subset \mathbb{R}^n$ be as in Section 2.1.

DEFINITION 2.8.1. *We say that a $B-$ smooth DF is a map $\phi : \mathbb{R} \times D \to D$, which satisfies the following properties:*

I) The group property:

(i) $\phi(0, x) : D \to D$ *is the identity;*

(ii) $\phi(t, \phi(s, x)) = \phi(t + s, x)$, *is valid for all* $t, s \in \mathbb{R}$ *and* $x \in D$.

II) If $x \in D$ *is fixed, then* $\phi(t, x) \in \mathcal{PC}^1(\mathbb{R})$, *and* $\phi(\theta_i, x) \in \Gamma, \phi(\theta_i+, x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t, x)$.

III) The function $\phi(t, x)$ *is* $B-$ *differentiable in* $x \in D$ *on* $[a, b] \subset \mathbb{R}$ *for every* $\{a, b\} \subset \mathbb{R}$, *assuming that all discontinuity points of* $\phi(t, x)$ *are interior points of* $[a, b]$.

Remark 2.8.1. *One can see that system (2.1.4) defines* $B-$ *smooth DF provided conditions* $C1) - C7)$ *and the conditions of one of the continuation theorems are fulfilled.*

DEFINITION 2.8.2. *We say that a DF is a map* $\phi : \mathbb{R} \times D \to D$, *which satisfies the property I) of Definition 2.8.1 and the following conditions:*

IV) If $x \in D$ *is fixed, then* $\phi(t, x) \in \mathcal{PC}(\mathbb{R})$, *and* $\phi(\theta_i, x) \in \Gamma$,$\phi(\theta_i+, x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t, x)$.

V) The function $\phi(t, x)$ *is* $B-$ *continuous in* $x \in D$ *on* $[a, b] \subset \mathbb{R}$ *for every* $\{a, b\} \subset \mathbb{R}$.

Remark 2.8.2. *Comparing definitions of the* $B-$ *differentiability and the* $B-$ *continuity one can conclude that every* $B-$ *smooth DF is a DF.*

2.9 Examples

EXAMPLE 2.9.1. *Consider the impulsive differential equation*

$$\begin{cases} \dot{x_1} = \alpha x_1 - \beta x_2 \\ \dot{x_2} = \beta x_1 + \alpha x_2 \end{cases} , (x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0),$$

$$\begin{cases} x_1(t+) = \sqrt{3}x_1(t-) - x_2(t-) \\ x_2(t+) = x_1(t-) + \sqrt{3}x_2(t-) \end{cases} , (x(t) \in \Gamma \wedge t \geq 0)$$

$$\begin{cases} x_1(t-) = \frac{\sqrt{3}}{4}x_1(t+) + \frac{1}{4}x_2(t+) \\ x_2(t-) = -\frac{1}{4}x_1(t+) + \frac{\sqrt{3}}{4}x_2(t+) \end{cases} , (x(t) \in \tilde{\Gamma} \wedge t \leq 0) \tag{2.9.29}$$

where $\Gamma = \{(x_1, x_2)|\, x_2 = \frac{1}{2}x_1, x_1 > 0\}$, $\tilde{\Gamma} = \{(x_1, x_2)| \quad x_2 = \frac{\sqrt{3}}{2}x_1, \quad x_1 > 0\}$, *constants* α, β *are positive. One can see that* $\Phi(x) = x_2 - \frac{1}{2}x_1$, $f(x) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2)$, $J(x) = (\sqrt{3}x_1 - x_2, x_1 + \sqrt{3}x_2)$. *We assume that*

$$D = \mathbb{R}^2 \backslash \left[\left\{ (x_1, x_2)| \quad \frac{1}{2}x_1 < x_2 < \frac{\sqrt{3}}{2}x_1, \quad x_1 > 0 \right\} \cup (0,0) \right].$$

One can verify that $C1) - C7)$ *are valid. Let us check if conditions of Theorem 2.3.3 hold or not. Fix* $x \in \tilde{\Gamma}$. *Then* $\mathrm{dist(x, \Gamma)} = \frac{1}{2}||\mathrm{x}||$ *and*

$$||f(x)|| = \sqrt{(\alpha x_1 - \beta x_2)^2 + (\beta x_1 + \alpha x_2)^2} = \sqrt{\alpha^2 + \beta^2}||x||.$$

Thus

$$\sup_{B(x,\epsilon_x)} ||f|| = \sqrt{\alpha^2 + \beta^2}(||x|| + \frac{1}{2}||x||) = \frac{3}{2}\sqrt{\alpha^2 + \beta^2}||x||,$$

and

$$\inf_{\tilde{\Gamma} \times (0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)} ||f||} = \frac{2}{3\sqrt{\alpha^2 + \beta^2}} > 0.$$

Hence, all conditions of a DF for the system are fulfilled.

Example 2.9.2. *Consider the following model for simple neural nets from [27]. We have modified it according to the proposed equation (2.1.4).*

$$
\begin{cases}
\dot{x_1} = x_2 \\
\dot{x_2} = -\beta^2 x_1 & ,(x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0), \\
p' = -\gamma p + x_1 + B_0 \\
\\
x_1(t+) = x_1(t-) \\
x_2(t+) = x_2(t-) & ,(x(t) \in \Gamma \wedge t \geq 0) \\
p(t+) = 0 \\
\\
x_1(t-) = x_1(t+) \\
x_2(t-) = x_2(t+) & ,(x(t) \in \tilde{\Gamma} \wedge t \leq 0) \\
p(t-) = r
\end{cases}
\tag{2.9.30}
$$

where $\Gamma = \{(x_1, x_2, p) | \, p = r, \, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$, $\tilde{\Gamma} = \{(x_1, x_2, p)| \quad p = 0, \, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$, $\Phi(x) = p - r, f(x) = (x_2, -\beta^2 x_1, -\gamma p + x_1 + B_0)$, $J(x) = (x_1, x_2, r)$, $\beta, \gamma, r > 0$, *are constants and* $B_0 > \gamma r + 1$. *We assume that* $D = \{(x_1, x_2, p)| 0 \leq p \leq r, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$. *In the system the variable* $p(t)$ *is a scalar input of a neural trigger and* x_1, x_2, *are other variables. The value of* r *is the threshold. One can verify that the functions and the sets satisfy* $C1) - C7)$ *and the conditions of Theorem 2.3.4. That is, the system defines a DF.*

Example 2.9.3. *Let us consider the discontinuous system*

$$
\begin{cases}
\dot{x_1} = = \alpha x_1 - \beta x_2 \\
\dot{x_2} = \beta x_1 + \alpha x_2 & ,(x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0), \\
\\
x_1(t+) = k x_1(t-) \\
x_2(t+) = k x_2(t-) & ,(x(t) \in \Gamma \wedge t \geq 0) \\
\\
x_1(t-) = \frac{1}{k} x_1(t+) \\
x_2(t-) = -\frac{1}{k} x_2(t+) & ,(x(t) \in \tilde{\Gamma} \wedge t \leq 0)
\end{cases}
\tag{2.9.31}
$$

where $\Gamma = \{(x_1, x_2) |\, x_1^2 + x_2^2 = r_1\}, \tilde{\Gamma} = \{(x_1, x_2) |\, x_1^2 + x_2^2 = kr_1\},\ \alpha, \beta, k$ *are constants such that* $\alpha,\ \beta < 0, 1 < k$. *Assume that* $D = \mathbb{R}^2$.

One can see that all conditions $C1) - C6)$ *are valid for the system, and so are conditions of Theorem 2.3.4. But* $C7)$ *is not fulfilled, and it is easy to see that a solution* $x(t, 0, x_0)$ *of (2.9.31), which starts outside of* $\tilde{\Gamma}$, *does not satisfy the condition* $x(-t, 0, x(t, 0, x_0)) = x_0$ *for all* t. *Thus (2.9.31) does not define a DF.*

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