OSCILLATION OF SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

OSCILLATION OF SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES

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During the last decade, the use of time scales as a means of unifying and extending results about various types of dynamic equations has proven to be both prolific and fruitful. Many classical results from the theories of differential and difference equations have time scale analogues.

In this thesis we derive new oscillation criteria for second order dynamic equations on time scales.

Keywords: Differential equation, Time scale, First order Equation, Second order Equation, Oscillation, Nonoscillation.

ÖZ

ZAMAN SKALASI ÜZERİNDE İKİNCİ MERTEBEDEN DİNAMİK DENKLEMLERİN SALINIMI

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Son on yıl içinde değişik tiplerdeki dinamik denklemlerin sonuçlarını tekilleştirmek ve genişletmek için zaman dilimlerini kullanmanın üretken ve karlı olduğu farkedilmiştir. Differansiyel ve fark denklemleri teorilerinin birçok klasik sonuçlarının zaman dilimi benzerleri vardır.

Bu tezde zaman sıkalası üzerinde verilen ikinci dereceden dinamik denklemler için yeni salınım kriterleri çıkaracağız.

Anahtar Kelimeler: Diferensiyel denklem, Zaman skalası, Salınımlı çözüm, Salınımsız çözüm, Birinci Dereceden Denklem, İkinci Dereceden Denklem.

To my lovely uncle Sami

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CHAPTER 1

TIME SCALE CALCULUS

1.1 Introduction

In this thesis, we will be concerned with oscillating and nonoscillating solutions of equations on a time scale. By a time scale $\mathbb{T}$ we mean an arbitrary nonempty closed subset of the real numbers. For linear differential and difference equations, one could argue that the exponential function is one of the most essential one appearing in solutions. The exponential function on a time scale is therefore plays an important role in the investigation of oscillating and nonoscillating behavior of equations on time scale.

The exponential function, denoted by $e_p(t, t_0)$, is defined to be the unique solution of equation

$$y^{\Delta}(t) = p(t)y(t), \quad y(t_0) = 1,$$

where Δ is the *delta derivative* corresponding to the time scale $\mathbb{T}$ under certain conditions [1, 3]. For the time scale $\mathbb{T} = \mathbb{R}$, the operator is just the usual derivative $y^{\Delta}(t) = y'(t)$ and $e_p(t, t_0) = \exp[\int_{t_0}^{t} p(s)ds]$. If $\mathbb{T} = \mathbb{Z}$, $y^{\Delta}(t) = y(t+1) - y(t)$. We note that the exponential function can be oscillatory in an arbitrary time scale as opposed to the case when $\mathbb{T} = \mathbb{R}$.

We will present known definitions, theorems in Chapter 1 and present some known oscillation and nonoscillation theorems in Chapter 2. In Chapter 3 we will prove new oscillation and nonoscillation theorems for equations of the form

$$(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma} = f(t).$$

Examples are also presented to illustrate the results obtained.

1.2 Basic Definitions

A time scale (or measure chain) $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus

$$\mathbb{R}, \quad \mathbb{Z}, \quad \mathbb{N}, \quad \mathbb{N}_0, \quad [0,1] \cup [2,3], \quad \mathbb{Q}, \quad \text{Cantor set}$$

are examples of time scales.

To introduce a time scale calculus, the forward jump operator

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T}$$

and the backward jump operator

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T}$$

are defined for all $t \in \mathbb{T}$. In these definitions we put

$$\sigma(\emptyset) = \sup \mathbb{T}, \quad \rho(\emptyset) = \inf \mathbb{T}$$

If $\sigma(t) > t$, the point t is called right-scattered, while if $\rho(t) < t$, t is said to be left-scattered. If $\sigma(t) = t$, t is called right-dense, while if $\rho(t) = t$, t is left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

If $\mathbb{T} = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, then it follows that

$$\sigma(t) = \sqrt[3]{t^3 + 1} \text{ and } \rho(t) = \sqrt[3]{t^3 - 1},$$

and if $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$, then we have

$$\sigma(t) = 2t \text{ and } \rho(t) = \tfrac{t}{2}.$$

By an interval $[a, b]$ in $\mathbb{T}$, we mean the set

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}$$

Other types of intervals are defined similarly. The set $\mathbb{T}^k$ is derived from $\mathbb{T}$ as follows : If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

Below we state some well-known results concerning the differentiation and integration on an arbitrary time scale. The following theorems are extracted from [1].

Theorem 1.2.1. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then*

(a) if f is differentiable at t, then f is continuous at t,

(b) if f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(c) if f is differentiable and t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(d) if f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

Example 1.2.1. If $f(t) = t^2$, then $f^{\Delta}(t) = t + \sigma(t)$. In particular, If $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{2}} := \{\sqrt{n} : n \in \mathbb{N}_0\}$ and $f(t) = t^2$, then $f^{\Delta}(t) = \sqrt{t^2 + 1} + t$

It can be shown that if $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at a point $t \in \mathbb{T}^k$, then so are $f + g$, αf $(\alpha \in \mathbb{R})$, fg, f/g whenever $gg^{\sigma} \neq 0$. Indeed,

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t),$$

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t),$$

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta g(\sigma(t)),$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta g(t) - f(t)g^\Delta}{g(t)g(\sigma(t))}.$$

EXAMPLE 1.2.2. If $f(t) = 1/t$, then $f^\Delta(t) = -1/(t\sigma(t))$.

To define an antiderivative of a function we need the concept of regulated functions. A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

If f is regulated, then there exists a function F such that $F^\Delta(t) = f(t)$. In this case the Cauchy integral is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-continuous$ provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$$

It can be shown that every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by

$$F(t) := \int_{t_0}^t f(r)\Delta r \quad for \quad t \in \mathbb{T}$$

is an antiderivative of f. Furthermore,

$$\int_t^{\sigma(t)} f(r)\Delta r = \mu(t)f(t)$$

Let a, $b \in \mathbb{T}$ and $f \in C_{rd}$. The following results follow immediately:

(a) If $\mathbb{T} = \mathbb{R}$, then
$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$
where the integral on the right is the usual Riemann integral.

(b) If $[a, b]$ consists of only isolated points, then
$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b \end{cases}$$

(c) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then
$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & \text{if } a > b \end{cases}$$

(d) If $\mathbb{T} = \mathbb{Z}$, then
$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t=b}^{a-1} f(t) & \text{if } a > b \end{cases}$$

EXAMPLE 1.2.3. Let $f(t) = t$. Then
$$\int_0^t s\Delta s = \int_0^t sds = \frac{t^2}{2} \quad If \quad \mathbb{T} = \mathbb{R}$$
$$\int_0^t s\Delta s = \begin{cases} \frac{t(t-h)}{2}, & t > 0 \\ 0, & t = 0 \\ \frac{t(h-t)}{2}, & t < 0 \end{cases} \quad If \quad \mathbb{T} = h\mathbb{Z}$$
The next theorem summarizes the properties of the integral.

Theorem 1.2.2. *If a, b, $c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and f, $g \in \mathbb{C}_{rd}$, then*

(a) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$

(b) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,

(c) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,

(d) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

(e) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t$,

(f) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$,

(g) $\int_a^a f(t)\Delta t = 0$,

(h)

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t, \quad \textit{whenever } \ |f(t)| \leq g(t) \ \textit{on } [a,b),$$

(i)

$$\int_a^b f(t)\Delta t \geq 0, \quad \textit{whenever } \ f(t) \geq 0 \ \textit{on } [a,b)$$

For the existence and uniqueness of differential equations on a time scale, it is usually required that the involving functions are regressive. A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^k$. In a similar manner the regressiveness of differential equations are defined. For instance, $y^\Delta = p(t)y$ is said to be regressive if p is a regressive function. The set of all rd-continuous and regressive functions is denoted by $\Re$. We also say that $f \in \Re^+$ if $f \in \Re$ and $1 + \mu(t)f > 0$.

Definition 1.2.1. Suppose the that $p \in \Re$, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1,$$

is given by

$$y(t) = e_p(t, t_0),$$

and is called the exponential function.

Making use of definition 1.2.1 of $e_p(t,t_0)$ one can easily derive some properties of the exponential function. For instance, if $p, q \in \Re$, then

(a) $e_0(t,s) = 1$ and $e_p(t,t) = 1$,

(b) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$,

(c) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$,

(d) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(e) $e_p(t,s).e_p(s,r) = e_p(t,r)$,

(f) $e_p(t,s).e_q(t,s) = e_{p\oplus q}(t,s)$,

(g) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$,

(h) $\left(\frac{1}{e_p(.,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(t,s)}$.

where

$$p \oplus q = p + q + \mu pq,$$

$$p \ominus q = \frac{p-q}{1+\mu q}$$

Note that, $e_p(t,t_0)$ may change sign. More specifically, if $p \in C_{rd}$, then $e_p(t,t_0)$ is positive on $\mathbb{T}$ when $1+\mu(t)p(t) > 0$ on $\mathbb{T}^k$; $e_p(r,t_0)e_p(\sigma(r),t_0)$ < 0 when $1+\mu(r)p(r) < 0$ for some $r \in \mathbb{T}^k$.

1.3 Self Adjoint Equation

Let p, $q \in C_{rd}$ and $p(t) \neq 0$ for all $t \in \mathbb{T}$. We consider the second order linear dynamic equation

$$Lx = 0, \quad Lx := (p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} \tag{1.1}$$

equation (1.1) is called self-adjoint equation. Let $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is continuous and $(px^{\Delta})^{\Delta} : \mathbb{T}^{k^2} \to \mathbb{R}$ is rd-continuous. A function $x \in \mathbb{D}$ is called a solution of equation (1.1) if it satisfies equation (1.1).

Theorem 1.3.1. *If $f \in C_{rd}$, $t_0 \in \mathbb{T}$, and x_0, x_0^Δ are given constants, then the initial value problem*

$$\begin{aligned} &Lx = f(t), \\ &x(t_0) = x_0, x^\Delta(t_0) = x_0^\Delta \end{aligned} \tag{1.2}$$

has a unique solution that exists on the whole time scale $\mathbb{T}$

The Wronskian of x and y of two solutions is defined by

$$W(x, y) = det \begin{pmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{pmatrix}.$$

for $t \in \mathbb{T}^k$ then The Lagrange bracket of x and y is defined as follows:

$$\{x; y\}(t) = p(t)W(x, y)(t)\text{fort} \in \mathbb{T}^k.$$

Theorem 1.3.2 (Lagrange Identity). *If x and y are in the domain of L, then*

$$x^\sigma(t)Ly(t) - y^\sigma(t)Lx(t) = \{x, y\}^\Delta(t) \quad for \quad t \in \mathbb{T}^{k^2}$$

Proof: By product rule, we have

$$\begin{aligned} \{x; y\}^\Delta &= (xpy^\Delta - px^\Delta y)^\Delta \\ &= x^\sigma(py^\Delta)^\Delta + x^\Delta py^\Delta - y^\sigma(px^\Delta)^\Delta - y^\Delta px^\Delta \\ &= x^\sigma(py^\Delta)^\Delta - y^\sigma(px^\Delta)^\Delta \\ &= x^\sigma\{(py^\Delta)^\Delta + qy^\sigma\} - y^\sigma\{(px^\Delta)^\Delta + qx^\sigma\} \\ &= x^\sigma Ly - y^\sigma Lx \end{aligned}$$

on $\mathbb{T}^{k^2}$.

Corollary 1.3.1 (Abel's Formula). *If x and y are solution of (1.1), then*

$$W(x, y) = \frac{c}{p(t)}$$

for all $t \in T^k$, where c is constant.

The Polya and Trench factorizations are often used in the oscillation theory. In particular, the theorem following these factorizations is mostly used. We will also employ that theorem in one of our main results in the last section. So, the theorem as well as its proof of the Polya factorization will be given.

DEFINITION 1.3.1. *A function $x(t)$ defined on a time scale $\mathbb{T}$ is said to have a generalized zero at t if $x(t) = 0$ or the generalized zero is contained in the real interval $(\rho(t), t)$ if tis a left-scattered and*

$$p(\rho(t))x(\rho(t))x(t) < 0$$

DEFINITION 1.3.2. *If the set of generalized zeros of $x(t)$ is unbounded from above, $x(t)$ is called oscillatory. Otherwise, it is called nonoscillatory.*

Theorem 1.3.3 (Polya Factorization). *If (1.1) has a solution u with no generalized zeros in $\mathbb{T}$ then for any $x \in \mathbb{D}$. We get the Polya factorization*

$$Lx(t) = \rho_1^{\sigma}(t)\{\rho_2(p_1 x)^{\Delta}\}^{\Delta}(t)$$

for all $t \in \mathbb{T}^{k^2}$, where

$$\rho_1 := \frac{1}{u} \quad and \quad \rho_2 := puu^{\sigma} > 0$$

Proof: Assume that u is a solution of (1.1) with no generalized zeros in $\mathbb{T}$. Then

$$\rho_2(t) = p(t)u(t)u^{\sigma}(t) > 0 \quad for \quad all \quad t \in \mathbb{T}^k$$

Let $x \in \mathbb{D}$. Then by the Lagrange identity (Theorem 1.3.2)

$$\begin{aligned} Lx(t) &= \frac{1}{u^\sigma(t)}\{u;x\}^\Delta(t) \\ &= \frac{1}{u^\sigma(t)}\{pW(u,x)\}^\Delta(t) \\ &= \frac{1}{u^\sigma(t)}\left\{puu^\sigma\left(\frac{x}{u}\right)^\Delta\right\}^\Delta(t) \\ &= \rho_1^\sigma(t)\{\rho_2(\rho_1 x)^\Delta\}^\Delta(t) \end{aligned}$$

for $t \in \mathbb{T}^{k^2}$,where ρ_2 is as in the statement of the theorem.

Theorem 1.3.4 (Trench Factorization). *Assume $a \in \mathbb{T}$, $p > 0$, and let $w := \sup \mathbb{T}$. If $w < \infty$, then assume $\rho(w) = w$. If (1.1) has a positive solution on $[a, w)$, then for any $x \in \mathbb{D}$ we get the Trench factorization*

$$Lx(t) = \gamma_1^\sigma(t)\{\gamma_2(\gamma_1 x)^\Delta\}^\Delta(t)$$

for all $t \in [a, w)$ where γ_1 and γ_2 are positive functions on $[a, w)$ and

$$\int_a^w \frac{1}{\gamma_2(t)}\Delta t = \infty \tag{1.3}$$

Proof: Since equation (1.1) has a positive solution on $[a, w)$, Lx has a polya factorization on $[a, w)$ by Theorem 1.3.3. Hence, there are positive functions ρ_1 and ρ_2 on $[a, w)$ such that

$$Lx(t) = \rho_1^\sigma(t)\{\rho_2(\rho_1 x)^\Delta\}^\Delta(t) = \frac{1}{\alpha_1^\sigma(t)}\left\{\frac{1}{\alpha_2}\left(\frac{x}{\alpha_1}\right)^\Delta\right\}^\Delta(t)$$

for $t \in [a, w)$, where

$$\alpha_i(t) := \frac{1}{\rho_i(t)} \quad for \quad i = 1, 2, \cdots$$

If

$$\int_a^w \alpha_2(t)\Delta t = \infty$$

then proof is completed. Assume

$$\int_a^w \alpha_2(t)\Delta t < \infty \tag{1.4}$$

In this case,

$$\beta_1(t) = \alpha_1(t)\int_t^w \alpha_2(s)\Delta s \quad and \quad \beta_2(t) = \frac{\alpha_2(t)}{\int_t^w \alpha_2(s)\Delta s \int_{\sigma(t)}^w \alpha_2(s)\Delta s}$$

for $t \in [a, w)$. Then by (1.4)

$$\begin{aligned}
\int_a^w \beta_2(t)\Delta t &= \lim_{b\to w, b\in\mathbb{T}} \int_a^b \frac{\alpha_2(t)}{\int_t^w \alpha_2(s)\Delta s \int_{\sigma(t)}^w \alpha_2(s)\Delta s}\Delta t \\
&= \lim_{b\to w, b\in\mathbb{T}} \int_a^b \left\{\frac{1}{\int_t^w \alpha_2(s)\Delta s}\right\}^\Delta \Delta t \\
&= \infty
\end{aligned}$$

For $x \in \mathbb{D}$, note that

$$\left(\frac{x}{\beta_1}\right)^\Delta (t) = \left\{\frac{\frac{x(t)}{\alpha_1(t)}}{\int_t^w \alpha_2(s)\Delta s}\right\}^\Delta = \frac{\int_t^w \alpha_2(s)\Delta s \left(\frac{x}{\alpha_1}\right)^\Delta (t) - \frac{x(t)}{\alpha_1(t)}(-\alpha_2(t)}{\int_t^w \alpha_2(s)\Delta s \int_{\sigma(t)}^w \alpha_2(s)\Delta s}$$

for $t \in [a, w)$. Hence,

$$\frac{1}{\beta_2(t)}\left(\frac{x}{\beta_1}\right)^\Delta (t) = \left[\frac{1}{\alpha_2(t)}\left(\frac{x}{\alpha_1}\right)^\Delta (t)\right]\int_t^w \alpha_2(s)\Delta s + \frac{x(t)}{\alpha_1(t)}$$

for $t \in [a, w)$. Taking derivative both sides we get

$$\left\{\frac{1}{\beta_2}\left(\frac{x}{\beta_1}\right)^\Delta\right\}^\Delta (t) = \left\{\frac{1}{\alpha_2}\left(\frac{x}{\alpha_1}\right)^\Delta\right\}^\Delta \int_{\sigma(t)}^w \alpha_2(s)\Delta s$$

for $t \in [a, w)$. It follows that

$$\frac{1}{\beta_1^\sigma(t)}\left\{\frac{1}{\beta_2}\left(\frac{x}{\beta_1}\right)^\Delta\right\}^\Delta(t) = \frac{1}{\alpha_1^\sigma(t)}\left\{\frac{1}{\alpha_2}\left(\frac{x}{\alpha_1}\right)^\Delta\right\}^\Delta = Lx(t)$$

for $t \in [a, w)$. If

$$\gamma_i(t) := \frac{1}{\beta_i(t)} \quad for \quad i = 1, 2, \cdots \quad t \in [a, w),$$

then

$$Lx(t) = \gamma_1^\sigma(t)\{\gamma_2(\gamma_1 x)^\Delta\}^\Delta(t)$$

for $t \in [a, w)$, where (1.3) is satisfied.

Theorem 1.3.5. *[1] Assume $a \in \mathbb{T}, p > 0$, let $w := \sup \mathbb{T}$. If $w < \infty$, then we assume $\rho(w) = w$. If (1.1) has a positive solution on $[a, w)$, then there is a positive solution u, called recessive solution at w, such that for any second linearly independent solution v, called a dominant solution at w,*

$$\lim_{t \to \overline{x}} \frac{u(t)}{v(t)} = 0$$

,

$$\int_a^w \frac{1}{p(t)u(t)u^\sigma(t)}\Delta t = \infty \quad and \quad \int_b^w \frac{1}{p(t)v(t)v^\sigma(t)}\Delta t < \infty$$

where $b < w$ sufficiently close . Furthermore

$$\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)} \tag{1.5}$$

for t sufficiently close to w ($t < w$).

1.4 Riccati Equation

It is well known that the Riccati equation is the most powerful tool in the oscillation theory in both continuous and discrete cases. The corresponding equation on an arbitrary time scale is therefore a must.

To derive a Riccati type equation on a time scale, we begin with considering a solution x of (1.1) with no generalized zeros. Define

$$z = \frac{px^{\Delta}}{x} \quad for \quad t \in \mathbb{T}^{k^2}. \tag{1.6}$$

It follows that

$$p + \mu z \;=\; p + \mu\frac{px^{\Delta}}{x} = \frac{p(x + \mu x^{\Delta})}{x} = \frac{px^{\sigma}}{x} > 0 \quad on \quad \mathbb{T}^{k^2}$$

and

$$z^{\Delta} \;=\; \left(\frac{px^{\Delta}}{x}\right)^{\Delta} = -q - \frac{z^2}{p + \mu z} \quad for \quad t \in \mathbb{T}^{k^2}.$$

The equation

$$Rz = 0 \tag{1.7}$$

for $t \in \mathbb{T}^k$, where

$$Rz = z^{\Delta} + q(t) + \frac{z^2}{p(t) + \mu(t)z}, \tag{1.8}$$

$$p(t) + \mu(t)z(t) > 0. \tag{1.9}$$

is called a Riccati equation.

The following results are classical.

Theorem 1.4.1. *If $x(t)$ is a solution of (1.1) having no generalized zeros, then*

$$z = \frac{px^{\Delta}}{x}$$

is a solution of the Riccati equation (1.7) and (1.9) holds for all $t \in \mathbb{T}^k$.

Theorem 1.4.2. *If $p(t) > 0$, then (1.1) has a positive solution on $\mathbb{T}$ iff the*

Riccati equation (1.7) has a solution z on $\mathbb{T}^k$ satisfying (1.9).

Proof: First assume that (1.1) has a positive on $\mathbb{T}$. Then by 1.4.1, the Riccati equation (1.7) has a solution z on $\mathbb{T}^k$ satisfying (1.9) on $\mathbb{T}^k$.
Conversely assume that z is a solution of (1.7) on T^k such that (1.9) holds on $\mathbb{T}^k$. Since by (1.9)

$$1 + \mu(t)\frac{z(t)}{p(t)} = \frac{p(t) + \mu(t)z(t)}{p(t)} > 0 \quad for \quad all \quad t \in \mathbb{T}^k$$

we have $z/p \in R^+$, and hence

$$x := e_{\frac{z}{p}}(t, t_0) > 0$$

where $t_0 \in \mathbb{T}$. Now $x^\Delta(t) = (z/p)x$ implies that

$$px^\Delta = zx \quad on \quad \mathbb{T}^k.$$

Taking the derivative of both sides we get that

$$\begin{aligned}
(px^\Delta)^\Delta &= (xz)^\Delta \\
&= x^\sigma z^\Delta + zx^\Delta \\
&= x^\sigma\left(-q - \frac{z^2}{p + \mu z}\right) + \frac{z^2 x}{p} \\
&= -qx^\sigma - (x + \mu x^\Delta)\frac{z^2}{p + \mu z} + \frac{z^2 x}{p} \\
&= -qx^\sigma.
\end{aligned}$$

Hence $x = e_{\frac{z}{p}}(., t_0)$ is a positive solution of (1.1).

CHAPTER 2

SOME KNOWN OSCILLATION CRITEIA FOR SECOND ORDER DYNAMIC EQUATIONS

2.1 Introduction

In this chapter we provide some known oscillation criteria for second order dynamic differential equations. We should point out that there is not enough literature on the oscillation theory of dynamic equations. For some additional results we refer, in particular, to [3, 5, 7].

In the next section we will consider the second order dynamic equations of the form

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)[(f \circ x^{\sigma}(t))]^{\gamma} = 0, \tag{2.1}$$

where p, $q \in C_{rd}$ are positive, real-valued functions, and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Function $x(t)$ of (2.1) satisfying $\sup\{|x(t)| : t > t_0\} > 0$ is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is said to be nonoscillatory. equation (2.1) is said to be oscillatory if all solutions are oscillatory.

It is clear that $\mathbb{T}$ is necessarily assumed to unbounded from above.

2.2 The oscillation criteria

We first consider the case $\gamma = 1$. Then equation (2.1) becomes

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)(f \circ x^{\sigma}(t)) = 0 \tag{2.2}$$

The following lemma is obtained by Saker in [4].

Lemma 2.2.1 ([4]). *If $x(t) > 0$ for all $t > t_0$ is a solution of (2.2) and if*

$$\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} = \infty, \tag{2.3}$$

then for $y = px^{\Delta}$ we have

$$y^{\Delta}(t) < 0, \quad 0 \leq y(t) \leq \frac{x(t)}{\int_{t_0}^{t} \frac{\Delta s}{p(s)}}, \quad t > t_0$$

and

$$0 \leq \frac{x^{\Delta}(t)}{x(t)} \leq \frac{1}{p(t)\int_{t_0}^{t} \frac{\Delta s}{p(s)}}, \quad t > t_0.$$

Proof: Clearly $x(\sigma(t)) > 0$ for $t > t_0$ and(2.2) implies

$$y^{\Delta}(t) = (p(t)x^{\Delta}(t))^{\Delta} = -q(t)f(x^{\sigma}(t)) < 0, \quad t > t_0.$$

Thus $y(t)$ is decreasing for $t > t_0$. Assume that there exists a $t_1 > t_0$ such that $y(t_1) = c < 0$. Then

$$p(s)x^{\Delta}(s) = y(s) \leq y(t_1) = c, \quad s \geq t_1$$

and therefore

$$x^{\Delta(s)} \leq \frac{c}{p(s)}, \quad s \geq t_1$$

Taking the integral from t_1 to $t > t_1$

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(s)\Delta s \leq x(t_1) + c\int_{t_1}^{t} \frac{\Delta s}{p(s)} \to -\infty \quad as \quad t \to \infty$$

yields a contradiction. Hence $y(t) = p(t)x^{\Delta}(t) \geq 0$ for all$t > t_0$.

$$x(t) \geq x(t) - x(t_0) = \int_{t_0}^{t} \frac{y(s)\Delta s}{p(s)} \geq y(t) \left\{ \int_{t_0}^{t} \frac{\Delta s}{p(s)} \right\}$$

for $t > t_0$. Since p is positive, the proof is complete.

Let $r \in \Re$ and $p\,r$ be a differentiable function. Define the auxiliary functions:

$$C(t) = C(t, t_0) := 1 + \frac{\mu(t)}{p(t) \int_{t_0}^{t} \frac{\Delta s}{p(s)}}$$

$$Q_1(t) = Q_1(t, t_0) := \frac{1 + \mu(t)r(t)}{p(t)e_r(t, t_0)}$$

$$\psi(t) = \psi(t, t_0) := e_r(\sigma(t), t_0) \left[Kq(t) + \frac{1}{2}(p(t)r(t))^{\Delta} + \frac{r^2 8t)p(t)}{4C(t)} \right]$$

$$Q(t) = Q(t, t_0) := -\frac{r(t)(1 + \mu(t)r(t))}{C(t)} + r(t)$$

for $t > t_0$. We shall make use of the following conditions:

(C1) There exists an $M > 0$ such that $r(t)e_r(t, t_0)p(t) \leq M$ for all large t.

(C2) $xf(x) > 0$ and $|f(x)| \geq K|x|$ for $x \neq 0$ for some $K > 0$

Theorem 2.2.1 ([4]). *Assume that (C1), (C2) and (2.3) hold. Furthermore, assume that there exists an $r \in \Re^+$ such that $p.r$ is differentiable and such that for any $t_0 \geq a$ there exists a $t_1 > t_0$ such that*

$$\limsup_{t \to \infty} \int_{t_1}^{t} H(s)\Delta s = \infty, \tag{2.4}$$

where

$$H(t) = \psi(t) - \frac{Q^2(t)C(t)}{4Q_1(t)}$$

for $t > t_0$. Then equation (2.2) is oscillatory.

Corollary 2.2.1. *(Leighton-Wintner theorem) Assume that (C2) and (2.3) hold.*

If

$$\int_a^\infty q(t)\Delta s = \infty, \tag{2.5}$$

then equation (2.2) is oscillatory.

Corollary 2.2.2 ([3]). *Assume that p is bounded above that (C2) and (2.3) hold, and that for any $t_0 \geq t_0$ such that*

$$\lim_{t\to\infty} sup \int_{t_1}^t \left[\sigma(s)\left[Kq(s) + \left(\frac{p(s)}{2s}\right)^\Delta + \frac{p(s)}{4s^2C(s)}\right] - \frac{A^2(s)C(s)}{4B(s)}\right]\Delta s = \infty \tag{2.6}$$

where

$$A(s) := \frac{-1}{sC(s)}\left(1 + \frac{1}{s}\mu(s) - C(s)\right), \quad B(s) := \frac{s+\mu(s)}{s^2p(s)}$$

Then equation (2.2) is oscillatory.

Theorem 2.2.2 ([4]). *Assume that (C2) and (2.3) hold. Furthermore, assume that there exists a function $r \in \Re^+$ such that $p\,r$ is differentiable and given any $t_0 \geq a$ there is a $t_1 > t_0$ such that*

$$\lim_{t\to\infty} sup \frac{1}{t^m}\int_{t_1}^t (t-s)^m \left[\psi(s) - \frac{Q^2(s)C(s)}{4Q_1(s)}\right]\Delta s = \infty \tag{2.7}$$

where m is a positive integer. Assume further that

$$\left(\frac{1}{t^m}\right)\int_{t_1}^t (e_r)^\sigma(s,t_0)p^\sigma(s)r^\sigma(s)\sum_{v=0}^{m-1}(\sigma(s)-t)^v(s-t)^{m-v-1}\Delta s$$

is bounded above. Then equation (2.2) is oscillatory.

Theorem 2.2.3. *[[4]] Assume that (C2) and*

$$\int_{t_0}^\infty \frac{\Delta t}{p(t)} < \infty \tag{2.8}$$

hold and there exists an $r \in \Re^+$ such that pr is differentiable such that (2.4) *holds. Furthermore, assume*

$$\int_a^\infty \frac{1}{p(s)} \int_a^t q(s)\Delta s \Delta t = \infty \tag{2.9}$$

and let (C1) hold. Then every solution of equation (2.2) is either oscillatory or converges to zero as $t \to \infty$.

Theorem 2.2.4 ([4]). *Let all conditions of Theorem 2.2.3 hold with an exception that (2.7) is replaced by (2.4). If*

$$\liminf_{t\to\infty} \int_T^t q(s)\Delta s \geq 0 \quad and \quad \not\equiv 0 \tag{2.10}$$

for all large T, *then every solution of equation (2.2) is oscillatory or converges to zero as* $t \to \infty$.

Next, we turn our attention to equation (2.1) when $f(x) = x$ and $p(t) \equiv 1$. That is, we consider

$$x^{\Delta\Delta}(t) + q(t)[x^\sigma]^\gamma = 0 \tag{2.11}$$

The next lemma is contained in [7].

Lemma 2.2.2. *Let* x *be a nonoscillatory solution of equation (2.1) and assume conditions (2.3) and (2.10) hold. Then, there exists* $T_1 \geq T$ *such that*

$$x(t)x^\Delta(t) > 0 \quad for \quad t \geq T_1.$$

equation (2.11) was studied when the time scale $\mathbb{T}$ contained only isolated points. The following lemma can be found [3].

Lemma 2.2.3. *Assume that* $q(t) \geq 0$ *for all* $t \in \mathbb{T}$, *and for every* $a \in \mathbb{T}$, $q(t) > 0$ *for some* $t \in [\sigma(a), \infty)$. *If* $x(t)$ *is a positive solution of equation (2.11) such that*

$$x(t) > 0$$

for all $t \in [a, \infty)$*, then*

$$x(\sigma(t)) > x(t), \tag{2.12}$$

$$0 < x^{\Delta}(\sigma(t)) \leq x^{\Delta}(t), \tag{2.13}$$

for all $t \in [a, \infty)$*.*

Theorem 2.2.5 ([5]). *Let* $q(t)$ *be as in Lemma 2.2.3,* $a \in \mathbb{T}, a \geq 0$*, and* $\gamma > 1$*. Then equation (2.11) is oscillatory if and only if*

$$\int_a^{\infty} \sigma(\ell)q(\ell)\Delta\ell = \infty.$$

Theorem 2.2.6 ([5]). *Let* $q(t)$ *be as in Lemma 2.2.3. Then equation (2.11) has a bounded nonoscillatory solution if and only if*

$$\int_a^{\infty} \sigma(\ell)q(\ell)\Delta\ell < \infty,$$

where $a \in \mathbb{T}, \quad a \geq 0$*.*

Theorem 2.2.7 ([5]). *Assume* $q(t)$ *is as in Lemma 2.2.3 and* $0 < \gamma < 1$*. Then equation (2.11) is oscillatory if and only if*

$$\int_a^{\infty} (\sigma(\ell))^{\gamma} q(\ell)\Delta\ell = \infty$$

where $a \in \mathbb{T}$*,* $a \geq 0$*.*

Proof: Let $x(t)$ be a solution of equation (2.11) such that $x(t) > 0$ for all $t \in [a, \infty)$ where $a \in \mathbb{T}, a \geq 0$. By Lemma 2.2.3, $x(t)$ is increasing and $x^{\Delta}(t)$ is positive and nonincreasing for all $t \in [a, \infty)$. Fix $j \in \mathbb{T}$ such that $j > 2a$. Then for all $t \in [j, \infty)$, we have

$$\begin{aligned}
x(t) &= x(a) + \int_a^t x^{\Delta}(\ell)\Delta\ell \\
&> \int_a^t x^{\Delta}(t)\Delta\ell \\
&= (t-a)x^{\Delta}(t) \qquad (2.14) \\
&> \frac{t}{2}x^{\Delta}(t) \qquad (2.15)
\end{aligned}$$

That is, $\frac{x(\sigma(t))}{x^{\Delta}(\sigma(t))} < \frac{\sigma(t)}{2}$. Dividing equation (2.11) by $(x^{\Delta}(\sigma(t)))^{\gamma}$, using inequality (2.15), and integrating from j to t, we obtain

$$\int_j^t \frac{x^{\Delta\Delta}(\ell)}{(x^{\Delta}(\sigma(\ell)))^{\gamma}}\Delta\ell + \frac{1}{2^{\gamma}}\int_j^t q(\ell)(\sigma(\ell))^{\gamma}\Delta\ell < 0 \qquad (2.16)$$

for $t \in [j, \infty)$. By hypothesis, the second integral in (2.16) approaches ∞ as $t \to \infty$, so the first term approaches $-\infty$. However, we will show that this is impossible. Let

$$r(k) = x(\ell) + (k-\ell)x^{\Delta}(\ell),$$

$\ell \leq k \leq \sigma(\ell), \ell \geq a$ so that r is positive, continuous, and increasing. Furthermore, let

$$s(k) = \frac{r(k+\mu(\ell)) - r(k)}{\mu(\ell)},$$

$k \geq a$ so that s is positive and continuous. Since $\ell \leq k \leq \sigma(\ell)$, we have $\sigma(\ell) \leq k + \mu(\ell) \leq 2\sigma(\ell) - \ell$. Therefore,

$$\begin{aligned}
r(k+\mu(\ell)) &= x(\sigma(\ell)) + (k+\mu(\ell)-\sigma(\ell))x^{\Delta}(\sigma(\ell)) \\
&= x(\sigma(\ell)) + (k-\ell)x^{\Delta}(\sigma(\ell)).
\end{aligned}$$

This implies that

$$\begin{aligned}
s(k) &= \frac{x(\sigma(\ell)) +^(k-\ell)x^{\Delta}(\sigma(\ell)) - x(\ell) - (k-l)x^{\Delta}(\ell)}{\mu(\ell)} \\
&= \frac{x(\sigma(\ell)) - x(\ell)}{\mu(\ell)} + (k-\ell)\left[\frac{x^{\Delta}(\sigma(\ell)) - x^{\Delta}(\ell)}{\mu(\ell)}\right] \\
&= x^{\Delta}(\ell) + (k-\ell)x^{\Delta\Delta}(\ell).
\end{aligned}$$

Hence, $s'(k) = x^{\Delta\Delta}(\ell) \leq 0$ for$\ell < k < \sigma(\ell)$ implies that s is nonincreasing and $0 < s(k) \leq s(\ell) = x^{\Delta}(\ell)$. Then, for $\ell < k < \sigma(\ell)$ we have

$$\begin{aligned}
\frac{x^{\Delta\Delta}(\ell)}{(x^{\Delta}(\sigma(\ell)))^{\gamma}} &= \frac{1}{\mu(\ell)}\int_{\ell}^{\sigma(\ell)} \frac{x^{\Delta^2}(\ell)}{(x^{\Delta}(\sigma(\ell)))^{\gamma}} dk \\
&\geq \frac{1}{\mu(\ell)}\int_{\ell}^{\sigma(\ell)} \frac{s'(k)}{s^{\gamma}(\sigma(k))} dk \\
&= \frac{1}{\mu(\ell)}\int_{\ell}^{\sigma(\ell)} \frac{s'(k)}{s^{\gamma}(k)} dk \\
&= \frac{1}{\mu(\ell)}\frac{1}{1-\gamma}[s^{1-\gamma}(\sigma(\ell)) - s^{1-\gamma}(\ell)] \\
&= \frac{1}{1-\gamma}(s^{1-\gamma})^{\Delta}(\ell).
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{j}^{t} \frac{x^{\Delta\Delta}(\ell)}{(x^{\Delta}(\sigma(\ell)))^{\gamma}} &\geq \frac{1}{1-\gamma}\int_{j}^{t} (s^{1-\gamma})^{\Delta}(\ell)\Delta\ell \\
&= \frac{1}{1-\gamma}[s^{1-\gamma}(t) - s^{1-\gamma}(j)].
\end{aligned}$$

But, $s^{1-\gamma}(t) > 0$ and $0 < \gamma < 1$ for all $t \geq a$, so $\int_{j}^{t} \frac{x^{\Delta\Delta}(\ell)}{(x^{\Delta}(\sigma(\ell)))^{\gamma}}\Delta\ell$ is bounded, and hence is a contradiction.

Let us now consider the dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\sigma}(t) = 0. \tag{2.17}$$

The following results are contained in [1].

Theorem 2.2.8 ([1]). *Assume* $\sup \mathbb{T} = \infty$, $p > 0$ *and let* $a \in \mathbb{T}$. *Suppose that for each* $t_0 \in [a, \infty)$ *there exists* $a_0 \in [t_0, \infty)$ *and* $b_0 \in (a_0, \infty)$, *such that* $\mu(a_0) > 0, \mu(b_0) > 0$ *and*

$$\int_{a_0}^{b_0} q(s)\Delta s \geq \frac{p(a_0)}{\mu(a_0)} + \frac{p(b_0)}{\mu(b_0)}. \tag{2.18}$$

Then equation (2.17) is oscillatory on $[a, \infty)$

Theorem 2.2.9 (Leighton-Wintner Theorem[4]). *Assume* $a \in \mathbb{T}$, $p > 0$, $\sup \mathbb{T} = \infty$ *and*

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \int_a^\infty q(t)\Delta t = \infty.$$

Then equation (2.17) is oscillatory on $[a, \infty)$

Theorem 2.2.10 ([4]). *Assume that equation (2.17) is oscillatory for all* $\lambda > 0$ *and suppose that (2.8), and (2.10) hold. In addition, assume that*

$$\int_T^\infty \frac{1}{p(s)} \int_T^s q(\eta)\Delta\eta\Delta s = \infty.$$

Then every solution of equation (2.2) is either oscillatory or converges to zero on $[a, \infty)$

If $p(t) = 1$,$\gamma = 1$ and $f(x) = x$, then equation (2.2) reduces to

$$x^{\Delta\Delta}(t) + q(t)x^\sigma = 0 \tag{2.19}$$

for $t \in [a, \infty)$.

The following result can also be found in [1].

Theorem 2.2.11 ([1]). *Assume* $a \in \mathbb{T}$ *and* $\sup \mathbb{T} = \infty$ *for any* $t_0 \in [a, \infty)$ *that there exists a strictly increasing sequence* $\{t_k\}_{k=1} \subset [t_0, \infty)$ *with* $\lim_{k\to\infty} t_k = \infty$, *and that there are constants* k_1 *and* k_2 *such that* $0 < k_1 \leq \mu(t_k) \leq k_2$ *for* $k \in \mathbb{N}$ *and*

$$\lim_{k\to\infty} \int_{t_1}^{t_k} q(t)\Delta t \geq \frac{1}{\mu(t_1)} \tag{2.20}$$

Then equation (2.19) is oscillatory on $[a, \infty)$

Finally, we provide some oscillation criteria obtained in [4] for dynamic equations of the form

$$x^{\Delta\Delta} + a(t)x^{\Delta^\sigma} + b(t)x^\sigma = 0. \tag{2.21}$$

Define

$$\alpha(t) = \frac{a(t) - \mu(t)b(t)}{1 - a(t)\mu(t) + b(t)\mu^2(t)}.$$

If $a \in \Re$ then equation (2.21) can be written self adjoint form (1.1), where

$$p = e_\alpha(., t_0) \quad and \quad q = bp \tag{2.22}$$

It is easy to see that an equation of the form

$$x^{\Delta\Delta} + p(t)x^\Delta + q(t)x = 0 \tag{2.23}$$

can be transformed into an of the form equation (1.1), if $1 - \mu p + \mu^2 q \neq 0$.

Theorem 2.2.12 ([4]). *Let* p, q *be defined as in (2.22) and assume that (2.3) holds. Furthermore, assume that there exists an* $r \in \Re$ *with differentiable such that (2.4) holds with*

$$\psi(t) = e_r(\sigma(t), t_0)\left[q(t) + \frac{1}{2}(p(t)r(t))^\Delta + \frac{r^2(t)p(t)}{4C(t)}\right] \tag{2.24}$$

Then equation (2.23) is oscillatory.

Corollary 2.2.3 ([4]). *Assume that (2.3) and (2.5) hold, where* p *and* q *are defined in (2.22). Then equation (2.23) is oscillatory.*

Corollary 2.2.4 ([4]). *Assume that (2.3) and (2.6) hold except the term* $K_q(t)$ *is replaced by* $q(t)$*, where* p *and* q *are as defined in (2.22). Then the equation (2.23) is oscillatory.*

CHAPTER 3

NEW OSCILLATION CRITERIA FOR SECOND ORDER DYNAMIC EQUATIONS

3.1 Introduction

In this chapter we derive new oscillation criteria for certain type of dynamic equations. Specifically, we are concerned with the oscillatory and nonoscillatory behavior of second order dynamic equations of the form

$$(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma} = f(t) \tag{3.1}$$

where it is tacitly assumed that p, q, $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $p > 0$, $t \in \mathbb{T}$

Definition 3.1.1. *By a solution of equation (3.1) on $[t_0, \infty)$ we mean a function y defined on $[t_0, \infty)$ such that y, $p(t)y^{\Delta}$ are differentiable, and satisfies the equation (3.1) for $t \geq t_0$.*

For simplicity, let us define the operator L by

$$Ly = (p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma}.$$

Clearly, (3.1) can be written as

$$Ly = f. \tag{3.2}$$

The equation

$$Ly = 0 \tag{3.3}$$

is called the homogeneous equation associated with (3.2).

It is well-known that, see Theorem 1.3.5, if equation (3.3) is nonoscillatory; i.e.,If every solution of equation (3.3) is nonoscillatory, then there exist two linearly independent solutions $z(t)$ and $u(t)$ of equation (3.3) such that

$$\int^{\infty} \frac{1}{p(t)z(t)z(\sigma(t))} \Delta t < \infty \tag{3.4}$$

and

$$\int^{\infty} \frac{1}{p(t)u(t)u(\sigma(t))} \Delta t = \infty$$

The solution $z(t)$, which is referred to as a dominant solution, plays an important role in investigating oscillation behavior of solutions of equation (3.2).

It is easy to see that an equation of the form

$$y^{\Delta\Delta} + p_1(t)y^{\Delta} + q_1(t)y = 0 \tag{3.5}$$

can be transformed into an of the form (3.3), if $1 - \mu p_1 + \mu^2 q_1 \neq 0$. Therefore, oscillation of equation (3.5) can be deduced from that of equation (3.3), which will be illustrated in this Chapter.

Finally, we will consider

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y^{\sigma} = 0 \tag{3.6}$$

and obtain a nonoscillation theorem by employing a Riccati substitution.

In recently there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales [11, 12, 13, 14].

In [9], Wong considered the equation (3.1) when $\mathbb{T} = \mathbb{R}$ and gave a necessary and sufficient condition for oscillation, where it was assumed that $p > 0$, q, f are continuous functions.

In [15], Dosly and Hilger considered the equation (3.3) and gave necessary and sufficient conditions for oscillation of all solutions. Often, however, the oscillation criteria require additional assumptions on the unknown solutions, which may not be easy to check. In Erbe and Peterson [16], the authors consider the same equation and assume that there exists $t_0 \in \mathbb{T}$, such that $p(t)$ bounded above on $[t_0, \infty)$, and

$$h_0 = \inf\{\mu(t) : t \in [t_0, \infty)\} > 0$$

and showed via Riccati techniques that

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty$$

implies that every solution is oscillatory. It is clear that the results given by Erbe and Peterson [16] cannot be applied when p is unbounded, $\mu(t) = 0$ and $q(t) = t^{-\alpha}$ when $\alpha > 1$. We refer also to papers by Erbe and Peterson [16] and Erbe [17] for additional linear oscillation criteria, which also treat more general situations. For some sufficient conditions for nonoscillation of the solutions of equation (3.5) we also refer to Guseinov and Kaymakçalan [14].

3.2 Main Results

We will first consider self-adjoint dynamic equation (3.2). Define

$$H(t) = \int^t \frac{1}{pzz^\sigma} \left(\int^s fz^\sigma \Delta\tau \right) \Delta s$$

Theorem 3.2.1. *Suppose (3.3) is nonoscillatory and that $z(t)$ is a dominant solution. If*

$$\overline{\lim_{t\to\infty}} H(t) = -\underline{\lim_{t\to\infty}} H(t) = \infty \tag{3.7}$$

then equation (3.2) is oscillatory.

Proof: Let z be a dominant solution of (3.3). In Theorem 1.3, we set $\gamma_1 = \frac{1}{z}$, $\gamma_2 = pzz^\sigma$. Then, we have

$$z^\sigma Ly - y^\sigma Lz \quad = \quad \left[pzz^\sigma \left(\frac{y}{z} \right)^\Delta \right]^\Delta$$

for any y and z satisfying $Ly = f$ and $Lz = 0$, we use $y(t) = z(t)w(t)$ and obtain

$$[pzz^\sigma w^\Delta]^\Delta = z^\sigma f \tag{3.8}$$

Integrating (3.8) twice from t_0 to t, we obtain

$$\begin{aligned} w^\Delta(t) &= \frac{w^\Delta(t_0)}{pzz^\sigma} + \frac{1}{pzz^\sigma}\int_{t_0}^t z^\sigma f \Delta\tau, \\ w(t) &= w(t_0) + w^\Delta(t_0)\int_{t_0}^t \frac{\Delta s}{pzz^\sigma} + \int_{t_0}^t \frac{1}{pzz^\sigma}\left(\int^s fz^\sigma\Delta\tau\right)\Delta s. \end{aligned}$$

In view of (3.4) and (3.7), we see that

$$\overline{\lim_{t\to\infty}} w(t) = -\underline{\lim_{t\to\infty}}\, w(t) = \infty.$$

This means that $w(t)$ and hence $y(t) = z(t)w(t)$ is oscillatory.

Remark: If $\mathbb{T} = \mathbb{R}$ then we recover a theorem proved by Wong in [9]. The discrete case, i.e, $\mathbb{T} = \mathbb{Z}$, has not been considered in the literature.

Example 3.2.1. Let $\mathbb{T} = \mathbb{Z}$ and consider

$$\Delta^2 y + (4 - 2\sqrt{2})\Delta y = \lambda^n(\sqrt{2}+1)^n[\lambda(3+2\sqrt{2}) - 1] \tag{3.9}$$

where $\lambda < 0$.

From $Lz = 0$ we easily obtain $z_1(n) = (\sqrt{2}+1)^n$ (dominant solution) and $z_2(t) = (\sqrt{2}-1)^n$ as nonoscillatory solutions.

Note that since

$$
\begin{aligned}
H(n) &= \int_0^n \frac{1}{(\sqrt{2}+1)^{2s}} \left(\int_0^s \lambda^\tau (\sqrt{2}+1)^{2\tau} [\lambda(3+2\sqrt{2})-1]\Delta\tau \right) \Delta s, \\
&= \int_0^n \frac{\lambda^s(\sqrt{2}+1)^{2s}-1}{(\sqrt{2}+1)^{2s}} \Delta s \\
&= \int_0^n \lambda^s \Delta s - \int_0^n \frac{1}{(1+\sqrt{2})^{2s}} \Delta s \\
&= \sum_{k=0}^{n-1} \lambda^k - \sum_{k_0}^{n-1} \frac{1}{(3+2\sqrt{2})^s} \\
&= \frac{1-\lambda^n}{1-\lambda} - \frac{1-\frac{1}{(3+2\sqrt{2})^n}}{1-\frac{1}{3+2\sqrt{2}}}.
\end{aligned}
$$

We have

$$\overline{\lim_{n\to\infty}} H(n) = -\underline{\lim_{n\to\infty}} H(n) = \infty.$$

Therefore by Theorem 3.2.1, equation (3.9) is oscillatory.

EXAMPLE 3.2.2. Let $\mathbb{T} = \mathbb{R}$ and consider

$$y'' - y = (2\cos t + 2\sin t - t\sin t + 2t\cos t)e^t. \tag{3.10}$$

In this case we see that $z_1(t) = e^t$(dominant solution), $z_2(t) = e^{-t}$ are nonoscillatory solutions of the corresponding homogeneous equation. Furthermore, from

$$
\begin{aligned}
H(t) &= \int^t \frac{1}{e^{2s}} \int^s (2\cos\tau + 2\sin\tau - \tau\sin\tau + 2\tau\cos\tau)e^{2\tau} d\tau ds \\
&= \int^t \frac{1}{e^{2s}} (\sin s + s\cos s)e^{2s} ds \\
&= \int^t (\sin s + s\cos s) ds \\
&= t\sin t
\end{aligned}
$$

it follows that

$$\overline{\lim_{t\to\infty}} H(t) = -\underline{\lim_{t\to\infty}} H(t) = \infty.$$

Since the conditions of Theorem 3.2.1 are satisfied, we may conclude that equation

(3.10) is oscillatory.

If p_1, $q_1 \in C_{rd}$ and

$$1 - p_1(t)\mu(t) + q_1(t)\mu^2(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^{k^2},$$

then the second order dynamic equation

$$y^{\Delta\Delta} + p_1(t)y^{\Delta} + q_1(t)y = 0. \tag{3.11}$$

can be written in a self-adjoint form (1.1), where

$$p(t) = e_\alpha(t, t_0), \quad q(t) = e_\alpha^\sigma(t, t_0)q_1(t) = (1 + \mu(t)\alpha(t))p(t)q_1(t) \tag{3.12}$$

and

$$\alpha(t) = \frac{p_1(t) - \mu(t)q_1(t)}{1 - p_1(t)\mu(t) + q_1(t)\mu^2(t)}$$

with $t_0 \in \mathbb{T}^k$.

By transforming (3.5) into a self-adjoint form, we are able to obtain the following theorem.

Theorem 3.2.2. *Let $p_1, q_1 \in C_{rd}$ be such that*

$$1 - \mu(t)p_1(t) + \mu^2(t)q_1(t) > 0$$

and

$$\int^\infty e_{\ominus\alpha}(t, t_0)\Delta t = \infty, \tag{3.13}$$

where

$$\alpha(t) = \frac{p_1(t) - \mu(t)q_1(t)}{1 - \mu(t)p_1(t) + \mu^2(t)q_1(t)}.$$

If

$$\int^\infty e_\alpha^\sigma(t, t_0)q_1(t)\Delta t = \infty \tag{3.14}$$

then, equation (3.5) is oscillatory.

Proof: equation (3.5) can be written in self-adjoint form

$$(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma} = 0, \tag{3.15}$$

where

$$p(t) = e_{\alpha}(t, t_0) \quad \textit{and} \quad q(t) = e^{\sigma}_{\alpha}(t, t_0)q_1(t).$$

Since $1 - \mu(t)p_1(t) + \mu^2(t)q_1(t) > 0$, we see that

$$1 + \mu\alpha = 1 + \mu\frac{p_1 - \mu q_1}{1 - \mu p_1 + \mu^2 q_1} = \frac{1}{1 - \mu p_1 + \mu^2 q_1} > 0,$$

and hence

$$1 + \mu(\ominus\alpha) = 1 + \mu\frac{-\alpha}{1 + \mu\alpha} > 0.$$

Thus, $\alpha, \ominus\alpha \in \Re^{+}$ and $p(t) > 0$. Apply Theorem 2.2.9 to equation (3.15), it follows that equation (3.5) must be oscillatory.

Remark: The case $\mathbb{T} = \mathbb{R}$ was proved in [8].

EXAMPLE 3.2.3. Consider second order linear dynamic equation

$$y^{\Delta\Delta} + 4\mu(t)y^{\Delta} + 4y = 0 \tag{3.16}$$

so that

$$p(t) = 4\mu(t), \quad q(t) = 4, \quad \alpha(t) = 0, \quad \ominus\alpha(t) = 0.$$

Clearly,

$$\int_a^{\infty} e_{\alpha}(t, t_0)\Delta t = \int_a^{\infty} e_0(t, t_0)\Delta t = \lim_{t\to\infty} t = \infty,$$
$$\int_a^{\infty} e^{\sigma}_{\alpha}(t, t_0)q(t)\Delta, t = 4\int_a^{\infty} e^{\sigma}_0(t, t_0)\Delta t = \lim_{t\to\infty} 4t = \infty.$$

Because all the conditions of Theorem 3.2.2 are satisfied, equation (3.16) is oscillatory.

Note that if $\mathbb{T} = \mathbb{R}$ then equation (3.16) becomes

$$y'' + 4y = 0,$$

whose general solution $y(t) = c_1 \sin 2t + c_2 \cos 2t$ is oscillatory, and If $\mathbb{T} = \mathbb{Z}$ then equation (3.16) gives

$$\Delta^2 y + 4\Delta y + 4y = 0$$

It is easy to verify that $y(t) = (c_1 + c_2 t)(-1)^t$ is oscillatory.

In case $\mathbb{T} = q^{\overline{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, equation (3.16) reads as

$$y^{\Delta\Delta} + 4(q-1)y^{\Delta} + 4y = 0,$$

and may be concluded that it is oscillatory.

$$(p_0(t)y^{\Delta})^{\Delta} + q_0(t)y^{\sigma} = 0 \tag{3.17}$$

where p_0 and q_0 satisfy the same assumptions as p and q

Theorem 3.2.3. *(Sturm's Comparison Theorem) Suppose we have for all $t \in \mathbb{T}$*

$$p_0(t) \leq p(t) \quad q_0(t) \geq q(t)$$

If equation (3.3) has a solution with two consecutive generalized zeros t_1, t_2, then every solution of equation (3.17) has a generalized zero between t_1 and t_2.

The next result provides a necessary and sufficient condition for the nonoscillation of equation (3.6). The theorem is well-known in both continuous and discrete cases.

Theorem 3.2.4. *Assume $-p \in \Re^+$ that equation (3.6) is nonoscillatory if and* *only if there exist* $t_0 \in [0, \infty)$ *and a differentiable function* $r(t) \in C_{rd}$ *such that*

$$r^{\Delta}(t) \geq \frac{r^2(t)}{1-\mu(t)r(t)} - \frac{p(t)r(t)}{1-\mu(t)r(t)} + q(t), \tag{3.18}$$

$$1-\mu(t)r(t) > 0 \tag{3.19}$$

for all $t \geq t_0$.

Proof: $\Rightarrow$:)Suppose that equation (3.6) is nonoscillatory. Without loss of generality we may assume that there exists a positive solution $y(t)$ on the semi-definite interval $[t_0, \infty)$, where $t_0 \geq 0$ depends on the solution $y(t)$. Let

$$r(t) = -\frac{y^{\Delta}(t)}{y(t)}, \quad t \geq t_0.$$

We first note that

$$1-\mu r = 1+\mu\frac{y^{\Delta}}{y} = \frac{y+\mu y^{\Delta}}{y} = \frac{y^{\sigma}}{y} > 0.$$

Next, by taking the derivative of $r(t)$ and using (3.3) in the process, we obtain

$$r^{\Delta} \quad = \quad \frac{r^2}{1-\mu r} - \frac{pr}{1-\mu r} + q \tag{3.20}$$

This means that $r(t)$ satisfies (3.18) and (3.19) for all $t \geq t_0$.

$\Leftarrow$:) Suppose that there exists a differentiable function $r \in C_{rd}$ that satisfies (3.18) and (3.19). Let $q_0(t) \geq 0$ be such that

$$r^{\Delta}(t) = \frac{r^2(t)}{1-\mu(t)r(t)} - \frac{p(t)r(t)}{1-\mu(t)r(t)} + (q(t)+q_0(t)). \tag{3.21}$$

Let us consider the inital value problem

$$y^{\Delta} = -r(t)y, \quad y(t_0) = 1 \tag{3.22}$$

By the existence and uniqueness theorem, this problem has a unique solution $y = e_{-r}(t, t_0)$. Since $1-\mu r > 0$, we see that $y(t) > 0$ for all $t \geq t_0$. Differentiating

(3.22) we arrive at

$$y^{\Delta\Delta} + p(t)y^{\Delta} + (q(t) + q_0(t))y^{\sigma} = 0. \tag{3.23}$$

Equation (3.23) can be written as

$$y^{\Delta\Delta} + (p(t) + (q(t) + q_0(t))\mu(t))\, y^{\Delta} + (q(t) + q_0(t))y = 0$$

or

$$(p_1(t)y^{\Delta})^{\Delta} + q_1(t)y^{\sigma} = 0,$$

where

$$p_1(t) = e_{\alpha_1}(t, t_0), \quad q_1(t) = e^{\sigma}_{\alpha_1}(t, t_0)(q(t) + q_0(t)),$$

and

$$\alpha_1 = \frac{p + (q + q_0)\mu - (q + q_0)\mu}{1 - \mu(p + (q + q_0)\mu) + (q + q_0)\mu^2} = \frac{p}{1 - \mu p}$$

Since $-p \in \Re^+$, we see that

$$1 + \mu\alpha_1 = 1 + \mu\frac{p}{1 - \mu p} = \frac{1}{1 - \mu p} > 0$$

and hence

$$1 + \mu(\ominus\alpha_1) = 1 + \mu\frac{-\alpha_1}{1 + \mu\alpha_1} = \frac{1}{1 + \mu\alpha_1} > 0.$$

Thus $\alpha, \ominus\alpha \in \Re^+$ and $p_1(t) > 0$.

Clearly, Riccati equation (1.8) corresponds with the nonoscillatory equation

$$y^{\Delta\Delta} + py^{\Delta} + qy^{\sigma} = 0. \tag{3.24}$$

Equation (3.24) can be written as

$$y^{\Delta\Delta} + (p(t) + q(t)\mu(t))\, y^{\Delta} + q(t)y = 0$$

or

$$(p_2(t)y^{\Delta})^{\Delta} + q_2(t)y^{\sigma} = 0,$$

where

$$p_2(t) = e_{\alpha_2}(t, t_0), \quad q_2(t) = e^{\sigma}_{\alpha_2}(t, t_0)q(t)$$

and

$$\alpha_2 = \frac{p + q\mu - q\mu}{1 - \mu(p + q\mu) + q\mu^2} = \frac{p}{1 - \mu p}.$$

We see that $\alpha_1 = \alpha_2$ and so $p_1 = p_2$. Since $q_1 \geq q_2$ by Theorem 3.2.3, we conclude that equation (3.6) cannot be oscillatory.

Remark: If $\mathbb{T} = \mathbb{R}$ then the above theorem can be found in [10].

As an application of the above theorem we obtain a nonoscillation theorem for equation (3.6):

Theorem 3.2.5. *Let there exits $Q(t)$ satisfying $Q^{\Delta}(t) = q(t)$. If*

$$(p(t) - Q(t))Q(t) \geq 0, \quad \textit{and} \quad 1 - \mu Q > 0,$$

then equation (3.6) is nonoscillatory.

Proof: Multiplying the inequality

$$Q^2(t) - p(t)Q(t) \leq 0$$

by $1 - \mu Q$ leads to

$$\frac{Q^2}{1 - \mu Q} - \frac{p\,Q}{1 - \mu Q} \leq 0. \tag{3.25}$$

Adding $Q^{\Delta}(t)$ on both sides we see that

$$q + \frac{Q^2}{1 - \mu Q} - \frac{p\,Q}{1 - \mu Q} \leq Q^{\Delta} \tag{3.26}$$

In view of Theorem 3.2.4, we may deduce from (3.26) that equation (3.6) is nonoscillatory.

Remark: If $\mathbb{T} = \mathbb{R}$ then the above theorem was proved by Wong [8]. Note that in this case $1 - \mu Q = 1 > 0$ is automatically satisfied.

Example 3.2.4. Let $\mathbb{T} = \mathbb{Z}$ and consider

$$\Delta^2 y + \left(\frac{2t}{t-1}\right)\Delta y - \frac{1}{t(t-1)} y(t+1) = 0. \tag{3.27}$$

Here, we have

$$p(t) = \frac{2t}{t-1}, \quad q(t) = -\frac{1}{t(t-1)}.$$

We observe that

$$Q(t) = -\int^t \frac{1}{s(s-1)} \Delta s = \frac{t}{t-1},$$

$$1 - \mu(t)Q(t) = -\frac{1}{t-1} > 0,$$

and

$$(p(t) - Q(t))Q(t) = \frac{t^2}{(t-1)^2} \geq 0.$$

Since the conditions of Theorem 3.2.5 are satisfied, equation (3.27) is nonoscillatory.

Example 3.2.5. Let $q > 1$ and $q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\}$ and $q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$. Here, we consider the time scale $\mathbb{T} = q^{\overline{\mathbb{Z}}}$ in which $\sigma(t) = qt$, $\mu(t) = (q-1)t$. Consider

$$y^{\Delta\Delta} + \frac{2}{qt} y^{\Delta} - \frac{1}{q^2 t^2} y^{\sigma} = 0 \tag{3.28}$$

so that

$$p(t) = \frac{2}{qt}, \quad q(t) = -\frac{1}{q^2 t^2}.$$

We see that

$$Q(t) = -\int^t \frac{1}{q^2 s^2}\Delta s = \frac{1}{qt},$$

$$1 - \mu(t)Q(t) = \frac{1}{q} > 0,$$

and

$$(p(t) - Q(t))Q(t) = \frac{1}{q^2 t^2} \geq 0.$$

Since the conditions of Theorem 3.2.5 are satisfied, equation (3.28) is nonoscillatory.

Example 3.2.6. Let $\sigma(t) < \frac{1+t^3}{t^2}$ and consider

$$y^{\Delta\Delta} + 2t^2 y^{\Delta} + (t + \sigma(t))y^{\sigma} = 0. \tag{3.29}$$

In this case, we have

$$p(t) = 2t^2, \quad q(t) = t + \sigma(t).$$

Since

$$Q(t) = \int^t (s + \sigma(s))\Delta s = t^2$$

Since $\sigma(t) < \frac{1+t^3}{t^2}$, we clearly see that

$$1 - \mu(t)Q(t) = 1 - (\sigma(t) - t)t^2 > 0,$$

and

$$(p(t) - Q(t))Q(t) = t^4 \geq 0.$$

Since the conditions of Theorem 3.2.5 are satisfied, equation (3.29) is nonoscillatory.

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