YIELD CURVE ESTIMATION AND PREDICTION WITH VASIČEK MODEL

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YIELD CURVE ESTIMATION AND PREDICTION WITH VASIČEK MODEL

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Abstract

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The scope of this study is to estimate the zero-coupon bond yield curve of *tomorrow* by using Vasiček yield curve model with the zero-coupon bond yield data of *today*. The raw data of this study is the yearly simple spot rates of the Turkish zero-coupon bonds with different maturities of each day from July 1, 1999 to March 17, 2004. We completed the missing data by using Nelson-Siegel yield curve model and we estimated tomorrow yield curve with the discretized Vasiček yield curve model.

Keywords: One factor short rate models, Vasiček yield curve, Nelson-Siegel yield curve, Monte-Carlo method

Öz

VASIČEK MODELİ İLE VERİM EĞRİSİNİN BUGÜN VE ERTESİ GÜN İÇİN TAHMİNİ

Bayazıt, Derviş

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Bu çalışmanın amacı *bugünün* kuponsuz verim(tahvil, bono) verisini kullanarak, yarının kuponsuz verim eğrisini tahmin etmektir. Bu çalişmada ham veri olarak, 1-Temmuz-1999 ve 17-Mart-2004 tarihleri arasında günlük olarak elde edilmiş farklı vadeli kuponsuz tahvil ve bonoların verimleri kullanılmıştır. Eksik veriler Nelson-Siegel verim eğrisi modeli kullanılarak tamamlanmış ve Vasiček modeli kullanılarak *ertesi günün* verim eğrisi tahmin edilmiştir.

Anahtar Kelimeler: Tek faktörlü kısa dönem faiz haddi modelleri, Vasiček verim eğrisi, Nelson-Siegel verim eğrisi, Monte-Carlo metodu

To my family

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Chapter 1

Introduction

In our century the interest rate has an important place in all transactions that include simply lending and borrowing. However, its importance has risen with developing and slight stationary economic conditions of the world. The desire to have foresight for the level of future interest rate has become crucial in the sense that to know the lending and borrowing rate and not to missprice the interest rate instruments. As a result, the modelling of interest rate has risen as a problem. As solution to this problem there have been many models proposed. Their common feature is that they all model a stochastic problem, i.e. they deal with uncertainty. The first model in 1973 was proposed by Merton. However, the pioneering one was suggested by Vasiček in 1977 and in the following years many other models that are much more analytically untractable has come out. These stochastic models are mainly classified with respect to the number of factors which are assumed to have a stochastic evolution in the model.

In this study we will use Vasiček short-rate model among some of the most fundamental one factor interest-rate models to predict the yield curve of *tomorrow* by using *today*'s observed yield data. The models taken into account are presented in Table 1.1. In the second chapter of this study we will present these one factor short rate models with their solutions. In the following chapter, we will discuss two different yield curve models: Nelson-Siegel and Vasiček models. The explicit solution of the Vasiěk model will be presented in Section 3.2. In the last chapter, the raw data of this study which is the yearly simple spot rates of the Turkish zero-coupon bonds with different maturities of each day from July 1, 1999 to

Merton(1973)	$dr_t = \alpha\, dt + \sigma\, dW_t$
Vasiěk(1977)	$dr_t = \alpha(\beta - r_t)\, dt + \sigma\, dW_t$
Dothan(1978)	$dr_t = \sigma\, r_t\, dW_t^0$
Brennan-Schwartz(1980)	$dr_t = (\beta + \alpha\, r_t)\, dt + \sigma\, r_t\, dW_t$
Cox-Ingersoll-Ross(CIR)(1985)	$dr_t = \alpha(\beta - r_t)dt + \sigma\sqrt{r_t}dW_t$
Ho-Lee (1986)	$dr_t = \theta_t\, dt + \sigma\, dW_t$
Exponential Vasiček(EV)	$dr_t = r_t[\eta_t - a\log r_t]\, dt + \sigma\, r_t\, dW_t$
Black-Derman-Toy(1990)	$d(\log r_t) = [\theta_t + \frac{\sigma_t'}{\sigma_t}\log r_t]dt + \sigma_t dW_t$
Hull-White- *Extended Vasiček*(1990)	$dr_t = (\beta_t - \alpha_t\, r_t)\, dt + \sigma_t\, dW_t$
Hull-White Extended CIR(1990)	$dr_t = [\beta_t - \alpha_t r_t]dt + \sigma_t\sqrt{r_t}dW_t$
Black-Karazinsky(1991)	$d(\log r_t) = \phi_t[\log\mu_t - \log r_t]dt + \sigma_t dW_t$
Geometric Brownian Motion(GBM)	$dr_t = \beta\, r_t\, dt + \sigma\, r_t\, dW_t$
Marsh-Rosenfeld (1983)	$dr_t = [\beta\, r_t^{-(1-\gamma)} + \alpha\, r_t]\, dt + \sigma\, r_t^{\gamma/2}\, dW_t$

Table 1.1: One Factor Short-Rate Models

March 17, 2004 will be analyzed. Then we will fit a Nelson-Siegel curve to each of the day in the data set we choose. At the end of the last chapter we will predict the yield curve of March 17, 2004 by using Vasiček yield curve model in Monte Carlo method. Before working on these models it is necessary to introduce main mathematical notions used in these parts of this study.
We will begin with the following assumption.

Assumption 1.1. *All the random variables and the stochastic processes are defined on a given complete probability space* $(\Omega, \mathcal{A}, \boldsymbol{P})$.

Definition 1.1. A filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is an increasing sequence of sub-sigma algebras of $\mathcal{A}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$, for s<t.

$\mathbb{F}$ is used to model a flow of information. As time passes, an observer knows more and more detailed information. In mathematical sense, as t increases $\mathcal{F}_t$ as partition of Ω becomes finer.

Definition 1.2. A Brownian motion $(W_t)_{t\geq 0}$ is a real-valued, continuous stochastic process with the following defining properties:

(1) Independent increments: $\forall\, s \leq t$, $W_t - W_s$ is independent of the sub-σ algebra generated by $\{W_s,\ s \leq\ t\}$, that is denoted by $\mathcal{F}_s = \sigma\,(W_u, u \leq\ s)\,.$

(2) Stationary increments: $\forall\, s \leq t$, $W_t - W_s$ and $W_{t-s} - W_0$ have the same probability law.

(3) Continuity of paths: $W_t(\omega)$ is a continuous function of t.[1]

Remark 1.3. Since the mathematical foundation of Brownian motion as a stochastic process was discussed by N.Wiener in 1931, this process is also called Wiener process.

Theorem 1.4. *If $(W_t)_{t\geq 0}$ is a Brownian motion, then $W_t - W_0$ is a normal random variable with mean $c\,t$ and variance $\sigma^2\, t$, where c and σ are constant real numbers [1].*

Remark 1.5. A Wiener process is standard if

- $W_0 = 0$,
- $E(W_t) = 0$,
- $E[W_t^2] = t$.

From now on,we will surly consider standard Brownian motion without especially mentioning the word "standard".

Definition 1.6. The collection $\mathbb{F}$=$\{\mathcal{F}_t,\, t \in [0, T]\}$ with $\mathcal{F}_t$=$\sigma\{W_s,\, s \leq t, \mathcal{N}\}$, where $\mathcal{N}$ is the set of all P-negligible sets of ω is called the natural filtration of W or the filtration generated by W.

Definition 1.7. A real-valued continuous stochastic process is an $(\mathcal{F}_t)$-Brownian motion if it satisfies :

- $\forall\, t \geq 0$, W_t is $\mathcal{F}_t$-measurable.
- $\forall\, s \leq t$, $W_t - W_s$ is independent of the σ-algebra $\mathcal{F}_s$.
- $\forall\, s \leq t$, $W_t - W_s$ and $W_{t-s} - W_0$ have the probability law.

Remark 1.8. It is obvious that an $\mathcal{F}_t$ - Brownian motion is a Brownian motion with respect to its natural filtration.

The following notion is crucial for the characterization of arbitrage-free market, option pricing and hedging.

Definition 1.9. A stochastic process $(M_t)_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ is martingale if for any t it is integrable, i.e. $E(|M_t|) < +\infty$ and for any $s \leq t$ $E(M_t|\mathcal{F}_s) = M_s$ a.s.

Remark 1.10. Definition 1.9 implies $E(M_t) = E(M_0)$ for any t.

Definition 1.11. A zero coupon bond(T-bond) with maturity date T is a contract which guarantees the holder 1 unit of money to be paid on the date T. The price at time t of a bond with maturity T is denoted by $P(t, T)$.

Assumption 1.2. *We assume the following.*

- *There exists a (frictionless) market for T-bonds for every $T > 0$.*
- *The relation $P(t, t) = 1$ holds for all t.*
- *For each fixed t, the bond price $P(t, T)$ is differentiable with respect to maturity time T.*
- *For a fixed value of t, $P(t, T)$ is a smooth function of T.*
- *For a fixed maturity T, $P(t, T)$ is a stochastic process. This process gives the prices, at different times, of the bond for a fixed maturity T.*

Note that written payment value, equal to 1 unit, on the contract is known as the *principal value* or *face value.*

Suppose we are standing at time t, and let us fix two other points, S and T, with $t < S < T$. Now, let us write a contract at time t which will allow us to have a deterministic rate of return on the interval $[S, T]$ determined at the contract time t. This is achieved as follows [2].

1. At time t we sell one S-bond. This will give us $P(t, S)$ unit of money.
2. We buy with this income $\frac{P(t,S)}{P(t,T)}$ T-bonds resulting a net investment zero at time t.

3. At time S the S-bond matures, therefore we pay out one unit of money.

4. At time T the T-bonds mature at one unit of money a piece, thus we receive $\frac{P(t,S)}{P(t,T)}$ units of money.

The above transactions can be summarized as follows: We contracted at time t, to make an investment of one unit of money at time S, that is guaranteing a yield of $\frac{P(t,S)}{P(t,T)}$ at time T. Therefore, we contracted a riskless rate at time t,which is valid on the future period $[S, T]$. This rate is called as a **forward rate**.

Definition 1.12. The following definitions are the implications of above construction.

1. The simple forward rate for $[S, T]$ contracted at t is defined as

$$L(t; S, T) = -\frac{P(t, T) - P(t, S)}{(T - S)P(t, T)}.$$

2. The simple spot rate for $[S, T]$, is defined as

$$L(S, T) = -\frac{P(S, T) - 1}{(T - S)P(S, T)}.$$

3. The continuously compounded forward rate for $[S, T]$ contracted at t is defined as

$$R(t; S, T) = -\frac{\log P(t, T) \log P(t, S)}{T - S}.$$

4. The continuously compounded spot rate, $R(S, T)$, for the period $[S, T]$ is defined as

$$R(S, T) = -\frac{\log P(S, T)}{T - S}.$$

5. The instantaneous forward rate with maturity T, contracted at t, is defined as

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.$$

6. The instantaneous short rate at time t is defined by

$$r(t) = f(t, t).$$

Remark 1.13. The spot rates are forward rates where the time of contracting coincides with the start of the interval over which the interest rate is effective,i.e. $t = S$.

The first hypothesis of this study is that all the work being done is in a filtered probability space $(\Omega,\, \mathcal{A},\, \mathbf{P},\, (\mathcal{F}_t)_{0\leq t\leq T})$, where $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration of a standard Brownian motion $(W_t)_{0\leq t\leq T}$ and that $\mathcal{F}_T = \mathcal{A}$.

Definition 1.14. $e^{\int_0^t r_s\, ds}$ is called as discount factor, where r_t is an adapted process of short rate satisfying $\int_0^T |r_t|\, dt < \infty$, almost surely. r_t, itself as being short rate, provides a return equal to $r_t dt$ on the period $(t,\, t + dt]$.

In order to guarantee that the bond market is arbitrage-free the following **fundamental hypothesis** is made:
There is a probability $\mathbf{P*}$ equivalent to $\mathbf{P}$, under which, for all real valued $u \in [0,\, T]$, the process $\left(\tilde{P}(t,\, u)\right)_{0\leq t\leq u}$ which is defined by

$$\tilde{P}(t,\, u) = e^{-\int_0^t r_s\, ds}\, P(t,\, u)$$

is a martingale.

Definition 1.15. $\mathbf{P*}$ called as risk neutral probability is defined by

$$d\mathbf{P*} = L_T\, d\mathbf{P},$$

with density $L_T = exp(\int_0^T q_s\, dW_s - \frac{1}{2}\int_0^t q_s^2\, ds)$ *a.s.*, where q_s is an adapted process such that $\int_0^T q_s^2\, ds < \infty$ a.s..[1]

The martingale property under $\mathbf{P*}$ allows to obtain,

$$\tilde{P}(t,\, u) = E^*\left(\tilde{P}(u,\, u)|\mathcal{F}_t\right) = E^*\left(e^{-\int_0^u r_s\, ds}|\mathcal{F}_t\right)$$

and when the discounting is eliminated, the following basic equation is obtained,

$$P(t,\, u) = E^*\left(e^{-\int_t^u r_s\, ds}|\mathcal{F}_t\right), \text{ where } \mathbf{E}^* \text{ denotes expectation under } \mathbf{P*}.$$

Chapter 2

One Factor Interest Rate Models

In this chapter we will present some of the most famous one factor short-rate models. The stochastic evolution of short-rate models is identified with the following general stochastic differential equation;

$$dr_t = \alpha_t(\beta_t - r_t)dt + \sigma_t\, r_t^{\gamma} dW_t, \tag{2.0.1}$$

where α_t, β_t, σ_t, are the deterministic functions of time, and W is a Brownian motion.

2.1 Merton Model(1973)

In his work, Merton offered the following stochastic evolution

$$dr_t = \alpha\, dt + \sigma\, dW_t\ ,$$

where α and σ are positive constants. A simple solution is obtained for r.

$$
\begin{aligned}
r_t &= r_0 + \int_0^t \alpha\, ds + \int_0^t \sigma\, dW_s \\
&= r_0 + \alpha t + \sigma W_t \\
\Longrightarrow r_t &= r_u + \alpha(t-u) + \sigma(W_t - W_u)
\end{aligned}
$$

In this model r follows a Gaussian distribution which implies that there is a positive probability that short rate r can take a negative value.

2.2 Vasiček Model(1977)

One of the earliest stochastic models of the term structure was developed by Vasiček in 1977. His model is based on the evolution of an unspecified short-term interest rate. He supposes that r satisfies the following stochastic differential equation

$$dr_t = \alpha(\beta - r_t)\, dt + \sigma\, dW_t \tag{2.2.2}$$

where β, α and σ are non-negative constants and r_t is the current level of interest rate [1]. The parameter β is the long run normal interest rate. The model exhibits mean reversion, which means that if the interest rate is above the long run mean ($r > \beta$), then the drift becomes negative so that the rate will be pushed to be closer to the level β on average. Likewise, if the rate is less than the long run mean, ($r < \beta$), then the drift remain positive so that the rate will be pushed to the level β. The coefficient $\alpha > 0$ determines the speed of pushing the interest rate towards its long run normal level. Such mean reversion assumption agrees with the economic phenomenon that interest rates appear over time to be pulled back to some long run average value.That is, when the interest rates increase, the economy slows down, and there is less demand for loans and a natural tendency for rates to fall([19], [14]). The opposite case can be argued in a similar way.

To obtain an explicit formulae for r_t let us define a new process X_t where $X_t = r_t - \beta$. Therefore, X_t is the solution of the following stochastic differential equation

$$dX_t = -\alpha\, X_t\, dt + \sigma\, dW_t \tag{2.2.3}$$

which implies that X_t is an Ornstein-Uhlenbeck process. To solve this process let $Y_t = X_t\, e^{\alpha\, t}$. By integration-by-parts formulae

$$Y_t = x_0 + \int_0^t X_s\, d(e^{\alpha\, s}) + \int_0^t e^{\alpha\, s}\, dX_s + < X, e^{\alpha\cdot} >_t$$

By differentiating both sides,

$$\begin{aligned} dY_t &= X_t \alpha\, e^{\alpha\, t}\, dt + e^{\alpha\, t}\, dX_t + \underbrace{\underbrace{d < X, e^{\alpha\cdot} >_t}_{0}}_{Since\, e^{\alpha\, t}\, is\, a\, deterministic\, function.} \\ &= a\, X_t\, e^{\alpha\, t} dt + e^{\alpha\, t}[-\alpha\, X_t\, dt + \sigma\, dW_t] \\ &= \sigma\, e^{\alpha\, t}\, dW_t \end{aligned}$$

By integrating and doing back substitution we get

$$\begin{aligned} Y_t &= x_0 + \int_0^t \sigma\, e^{\alpha\, s}\, dW_s \\ X_t &= x_0 e^{-\alpha\, t} + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s} dW_s \\ r_t - \beta &= (r_0 - \beta) e^{-\alpha\, t} + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s}\, dW_s \end{aligned}$$

Therefore,

$$r_t = r_0 e^{-\alpha\, t} + \beta(1 - e^{-\alpha\, t}) + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s} dW_s$$

For $u \leq t$,

$$\begin{aligned} r_t &= r_0 e^{-\alpha\, t} + \beta\alpha \int_0^t e^{-\alpha(t-s)} ds + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \\ &= r_0 e^{-\alpha\, t} + \beta\alpha \int_u^t e^{-\alpha(t-s)} ds + \beta\alpha \int_0^u e^{-\alpha(t-s)} ds \\ &\quad + \sigma \int_u^t e^{-\alpha(t-s)} dW_s + \sigma \int_0^u e^{-a(t-s)} dW_s \end{aligned} \tag{2.2.4}$$

Also,

$$r_u \;=\; r_0 e^{-\alpha\, u} + \beta\alpha \int_0^u e^{-\alpha(u-s)} ds + \sigma \int_0^u e^{-\alpha(u-s)} dW_s$$

That is,

$$\beta\alpha \int_0^u e^{-\alpha(t-s)} ds + \sigma \int_0^u e^{-\alpha(t-s)} dW_s = e^{\alpha(u-t)}(r_u - r_0 e^{-\alpha\, u}) \qquad (2.2.5)$$

By using equation (2.2.6) we get the t-u expression of r_t from the equation (2.2.4).

$$r_t = r_u\, e^{-\alpha(t-u)} + \alpha\beta \int_u^t e^{-\alpha(t-s)} ds + \sigma \int_u^t e^{-\alpha(t-s)} dW_s \qquad (2.2.6)$$

Vasiček model besides its advantages such as being analytically tractable, it has several shortcomings. Since the short rate is normally distributed, for every t there is a positive probability that r is negative and this is unreasonable from an economic point of view. Because the nominal interest rate can not fall below zero as long as people can hold cash; it can become stuck at zero for long periods, however as when prices fall persistently and substantially.

Another drawback of the Vasiček model is that it assumes $\gamma = 0$. This assumption implies the conditional volatility of changes in the interest rate to be constant, independent on the level of r.

2.3 Dothan Model (1978)

In 1978, in the original paper of Dothan a driftless geometric Brownian motion is proposed as short rate process under the objective probability measure $\mathbf{P}$:

$$dr_t \;=\; \sigma\, r_t\, dW_t\,, \quad r(0) = r_0\,, \qquad (2.3.7)$$

where r_0 and σ are positive constants [9]. Then the following solutions are obtained for r.

$$r_t \;=\; r_0 + \int_0^t \sigma\, r_s\, dW_s.$$

Let $f(x) = \log x$. Then $f'(x) = \frac{1}{x}$ and $f^{''}(x) = -\frac{1}{x^2}$. By Ito Lemma,

$$\begin{aligned}
\log r_t &= \log r_0 + \int_0^t \frac{1}{r_s}\, r_s\, \sigma\, dW_s - \frac{1}{2}\int_0^t \frac{1}{r_s^2}\, r_s^2\, \sigma^2\, ds \\
&= \log r_0 + \int_0^t \sigma\, dW_s - \frac{1}{2}\int_0^t \sigma^2\, ds \\
&= \log r_0 + \sigma\, W_t - \frac{1}{2}\sigma^2\, t \\
\Rightarrow r_t &= r_0 \exp\{-\frac{1}{2}\sigma^2\, t + \sigma W_t\}
\end{aligned}$$

For $u \leq t$,

$$\begin{aligned}
\log r_u &= \log r_0 + \int_0^u \sigma\, dW_s - \frac{1}{2}\int_0^u \sigma^2\, ds \\
\log r_t &= \log\, r_0 + \int_u^t \sigma\, dW_s + \int_0^t \sigma\, dW_s - \frac{1}{2}\int_u^t \sigma^2\, ds - \frac{1}{2}\int_0^u \sigma^2\, ds \\
&= \log r_u + \int_u^t \sigma\, dW_s - \frac{1}{2}\int_u^t \sigma^2\, ds \\
&= \log r_u + \sigma(W_t - W_u) - \frac{1}{2}\sigma^2(t-u) \\
\Rightarrow r_t &= r_u \exp\{-\frac{1}{2}\sigma^2\,(t-u) + \sigma(W_t - W_u)\}
\end{aligned}$$

2.4 Brennan-Schwartz Model (1980)

In their work they offered a model to analyze the convertible bonds [5]. They proposed following stochastic differential equation,

$$dr_t = (\beta + \alpha\, r_t)\, dt + \sigma\, r_t\, dW_t \tag{2.4.8}$$

where α,β and σ are positive constants. Equation 2.4.8 is a linear nonhomogeneous stochastic differential equation in the form of

$$dr_t = [\beta_t + \alpha_t\, r_t]\, dt + [\gamma_t + \delta_t\, r_t]\, dW_t$$

where $\beta_t = \beta$, $\alpha_t = \alpha$, $\delta_t = 0$, $\sigma_t = \sigma$. To solve this SDE first consider the homogenous case in which $\beta = 0$.
Let,

$$\begin{aligned} dr_t &= \alpha\, r_t\, dt + \sigma\, r_t\, dW_t \\ \Rightarrow r_t &= r_0 + \int_0^t \alpha\, r_s\, ds + \int_0^t \sigma\, r_s\, dW_s \end{aligned}$$

Let $f(x) = \log x$. Applying Ito formula

$$\begin{aligned} \log r_t &= \log r_0 + \int_0^t \alpha\, ds + \int_0^t \sigma\, dW_s - \frac{1}{2}\int_0^t \sigma^2\, ds \\ &= \log r_0 + \alpha\, t + \sigma\, W_t - \frac{1}{2}\sigma^2\, t \end{aligned}$$

Then the homogeneous solution of SDE (2.4.8) is

$$r_t = r_0\, \underbrace{e^{(\alpha - \frac{1}{2}\sigma^2)\, t + \sigma\, W_t}}_{\phi_t}.$$

Now, let $\eta_t = \frac{1}{\phi_t}$ and define $\zeta_t = \eta_t\, r_t$. Let $f(x) = e^x$ and define

$$X_t = -(\alpha - \frac{1}{2}\sigma^2)\, t - \sigma\, W_t$$

$$\Rightarrow dX_t = -(\alpha - \tfrac{1}{2}\sigma^2)\, dt - \sigma\, dW_t$$

By Ito Lemma,

$$\begin{aligned} d\eta_t &= \eta_t\, dX_t + \frac{1}{2}\eta_t \sigma_t^2\, dt \\ &= \eta_t[-(\alpha - \frac{1}{2}\sigma^2)\, dt - \sigma\, dW_t] + \frac{1}{2}\eta_t\sigma^2\, dt \\ &= \eta_t[(-\alpha + \sigma^2)\, ds] - \sigma\eta_t\, dW_t \end{aligned} \tag{2.4.9}$$

By integration-by-parts,

$$\begin{aligned} d\zeta_t &= \eta_t\, dr_t + r_t\, d\eta_t + d < \eta, r >_t \\ &= \eta_t[(\beta + \alpha\, r_t)\, dt + \sigma\, r_t\, dW_t] + r_t[\eta_t(-\alpha + \sigma^2)\, dt - \sigma\eta_t\, dW_t] \\ &\quad -[\sigma^2\, r_t\, \eta_t\, dt] \\ &= \eta_t\, \beta\, dt \end{aligned} \tag{2.4.10}$$

By integrating both sides of equation 2.4.10

$$\zeta_t = \zeta_0 + \beta \int_0^t \eta_s\, ds,$$

where $\zeta_0 = r_0$.Therefore, r_t is

$$\begin{aligned} r_t &= \phi_t\, r_0 + \phi_t\beta \int_0^t \eta_s\, ds \\ &= e^{(\alpha - \frac{1}{2}\sigma^2)\, t + \sigma\, W_t} r_0 + e^{(\alpha - \frac{1}{2}\sigma^2)\, t + \sigma\, W_t} \beta \int_0^t \eta_s\, ds \\ &= e^{(\alpha - \frac{1}{2}\sigma^2)\, t + \sigma\, W_t} r_0 + \int_0^t e^{(\alpha - \frac{1}{2}\sigma^2)\, (t-s) + \sigma\, (W_t - W_s)} \beta\, ds \end{aligned} \tag{2.4.11}$$

For $u \le t$,

$$r_t = e^{(\alpha - \frac{1}{2}\sigma^2)\, (t-u) + \sigma\, (W_t - W_u)} r_u + \int_u^t e^{(\alpha - \frac{1}{2}\sigma^2)\, (t-s) + \sigma\, (W_t - W_s)} \beta\, ds$$

2.5 Cox-Ingersoll-Ross (CIR) Model(1985)

An intriguing case of a non-linear stochastic differential equation can be defined as

$$dX_t = (\theta_t X_t + \gamma_t)dt + \upsilon_t \sqrt{X_t} dW_t \ , \quad X_0 = x_0. \tag{2.5.12}$$

A process following such dynamics is traditionally referred to as square-root process. In a common sense, square-root processes are naturally linked to non-central χ-square distributions.One of the major examples of models based on this dynamics are the Cox, Ingersoll and Ross instantaneous short rate model in which short rate process r satisfies a simplified version of equation (2.5.12) producing strictly positive instantaneous short rate process.

The model formulation is

$$dr_t = \alpha(\beta - r_t)dt + \sigma\sqrt{r_t} dW_t \ , \quad r(0) = r_0 \tag{2.5.13}$$

where α, β, σ and r_0 are positive constants. The condition $2\alpha\beta \geq \sigma^2$ provides the positivity of r. It can be shown that equation (2.5.13) admits a unique solution that is positive, but we do not have an explicit form for it [22].

2.6 Ho-Lee Model(1986)

Ho and Lee pioneered a new approach by showing how an interest rate model can be designed so that it is automatically consistent with any specified initial term structure [2].

$$dr_t = \theta_t \, dt + \sigma \, dW_t \tag{2.6.14}$$

$$\begin{aligned} r_t &= r_0 + \int_0^t \theta_s \, ds + \int_0^t \sigma \, dW_s \\ &= r_0 + \int_0^t \theta_s \, ds + \sigma \, W_t \end{aligned} \tag{2.6.15}$$

$$r_t = r_u + \int_u^t \theta_s \, ds + \sigma[W_t - W_u] \tag{2.6.16}$$

2.7 Exponential Vasiček Model(EV)

EV is a lognormal model in which interest rate r satisfies the following stochastic differential equation

$$dr_t = r_t[\eta_t - a\log r_t]\,dt + \sigma r_t\,dW_t.$$

A natural way of obtaining a lognormal model is to assume that the logarithm of r follows an Ornstein-Uhlenbeck process y under the the risk neutral measure **P***. Here, y is defined by the following stochastic differential equation

$$dy_t = [\Theta - ay_t]\,dt + \sigma\,dW_t \tag{2.7.17}$$

where Θ, a and σ are positive constants and y_0 is a real number [18]. Therefore, to solve the 2.7.17 , it is convenient to obtain an Ornstein-Uhlenbeck process at first hand. So, applying the Ito Lemma for $f(x) = \log x$ and putting $y_t = \log r_t$, we get

$$\begin{aligned}
\log r_t &= \log r_0 + \int_0^t \frac{1}{r_s} r_s[\eta - a\log r_s]\,ds + \sigma\,dW_s \\
&+ \frac{1}{2}\int_0^t (\frac{-1}{r_s})\sigma^2 r_s^2\,ds \\
&= \log r_0 + \int_0^t [(\eta - \frac{1}{2}\sigma^2) - a\log r_s]\,ds + \int_0^t \sigma\,dW_s
\end{aligned}$$

that is

$$y_t = y_0 + \int_0^t [(\eta - \frac{1}{2}\sigma^2) - ay_s]\,ds + \int_0^t \sigma\,dW_s$$

we get,

$$dy_t = [(\eta - \frac{1}{2}\sigma^2) - ay_t]\,dt + \sigma\,dW_t \tag{2.7.18}$$

We can express equation 2.7.18 as

$$dy_t = [\Theta - ay_t]\,dt + \sigma\,dW_t, \tag{2.7.19}$$

where $\Theta = \eta - \frac{1}{2}\sigma^2$. If we put $\beta = \frac{\Theta}{a}$, $a = \alpha$ and $X_t = y_t - \beta$ we get

$$dX_t = -\alpha\, X_t\, dt + \sigma\, dW_t.$$

The solution of this SDE is

$$X_t = x_0 e^{-\alpha\, t} + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s} dW_s.$$

That is,

$$\begin{aligned} y_t - \beta &= (y_0 - \beta) e^{-\alpha\, t} + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s}\, dW_s \\ y_t &= y_0 e^{-\alpha\, t} + (1 - e^{-\alpha\, t})\beta + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s}\, dW_s \end{aligned}$$

By replacing y_t with $\log r_t$ we get

$$r_t = exp\{\log r_0 e^{-a\, t} + \frac{\Theta}{a}(1 - e^{-a\, t}) + \sigma\, e^{-a\, t} \int_0^t e^{a\, s}\, dW_s\}$$

For $u \leq t$,

$$y_t = y_0 e^{-\alpha\, t} + \beta\alpha \int_0^t e^{-\alpha(t-s)} ds + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \qquad (2.7.20)$$

$$y_t = y_u\, e^{-\alpha(t-u)} + \alpha\beta \int_u^t e^{-\alpha(t-s)} ds + \sigma \int_u^t e^{-\alpha(t-s)} dW_s \qquad (2.7.21)$$

By replacing y_t with $\log r_t$ we get

$$r_t = exp\{\log r_u e^{-\alpha(t-u)} + \alpha\beta \int_u^t e^{-\alpha(t-s)} ds + \sigma \int_u^t e^{-\alpha(t-s)} dW_s\}$$

That is,

$$r_t = exp\{\log r_u e^{-a(t-u)} + \Theta \int_u^t e^{-a(t-s)} ds + \sigma \int_u^t e^{-a(t-s)} dW_s\} \qquad (2.7.22)$$

2.8 Black-Derman-Toy Model (1990)

In their article, Black, Derman and Toy proposed a discrete time approach of interest rate modelling [3]. The continuous time equivalent of their model was expressed as a stochastic differential equation which can be shown to be [10]

$$d(\log r_t) = [\theta_t + \frac{\sigma_t^{'}}{\sigma_t} \log r_t]dt + \sigma_t dW_t. \tag{2.8.23}$$

In this model $\log r_t$ is mean reverting. The function σ_t is chosen to make the model consistent with the term structure of spot rate volatilities and may not give reasonable values for the future short rate volatility. The model has the disadvantage that bond prices can not be determined analytically.

$$d\,\text{log}r_t = \theta_t\,dt + \sigma\,dW_t \tag{2.8.24}$$

When we integrate, we get the following result;

$$\begin{aligned} \text{log}r_t &= \text{log}r_0 + \int_0^t \theta_s\,ds + \int_0^t \sigma\,dW_s \\ &= \text{log}r_0 + \int_0^t \theta_s\,ds + \sigma\,W_t \\ \Rightarrow & \\ r_t &= r_0\exp\{\int_0^t \theta_s\,ds + \sigma\,W_t\} \end{aligned}$$

For $u \leq t$

$$r_t = r_u\ \exp\{\int_u^t \theta_s\,ds + \sigma(W_t - W_u)\}$$

2.9 Hull-White Extended Short Rate Models (1990)

A number of authors have proposed one factor models of the term structure in which the short rate, r, follows a mean reverting process of the form

$$dr_t = \alpha(\beta - r)dt + \sigma\,r^{\gamma}dW_t\ , \tag{2.9.25}$$

where α, β, σ and γ are positive constants and W is the Wiener Process. In these models, the interest rate r, is pulled toward to a long term interest rate level β with rate α.

The condition of $\gamma = 0$ is considered by Vasiček. As we explained in section 2.2, besides being analytically tractable, this model has a fundamental drawback that the short term interest rate, r, can become negative. On the other hand, CIR considered an alternative where $\gamma = 1/2$ which results nonnegative r. However, CIR case is not analytically tractable.

It is reasonable to anticipate that in some situations the market's expectations about future interest rates involve time dependent parameters. In other words, the drift and diffusion terms can be defined as the functions of time as well as being functions of r. The time dependence can arise from the cyclical nature of the economy, expectations concerning the future impact of monetary policies, and expected trends in other macroeconomic variables. In their article Hull and White extend the model in (2.9.25) to reflect this time dependence. They add a time dependent drift, θ_t to the process for r, and allow both the reversion rate, α, and the volatility factor, σ, to be functions of time. This leads to the following model for r:

$$dr = [\theta_t + \alpha_t(\beta - r_t)]dt + \sigma_t r_t^{\gamma} dW_t \tag{2.9.26}$$

This can be regarded as a model in which a drift rate, θ_t, is imposed on a variable that would otherwise tend to revert to a constant level β. Since (2.9.26) can be expressed as

$$dr = \alpha_t[\frac{\theta_t}{\alpha_t} + \beta - r_t]dt + \sigma_t r_t^{\gamma} dW_t \tag{2.9.27}$$

it can also be regarded as a model in which the reversion level is a function, $\frac{\theta_t}{\alpha_t} + \beta$ of time.

2.9.1 Hull-White- *Extended Vasiček* Model

For $\gamma = 0$ and $\beta_t = \theta_t + \beta\alpha_t$ from (2.9.26) Hull-White proposed the following model,

$$dr_t = (\beta_t - \alpha_t\, r_t)\, dt + \sigma_t\, dW_t\,,$$

where β_t, α_t and σ_t are time dependent deterministic functions [11]. For the solution of this stochastic differential equation we define $K_t = \int_0^t \alpha_u \, du$. That is, $K_t' = \alpha_t$. Then, we multiply both sides of the equation with e^{K_t} and take the Ito differential of both sides with respect to t.
That is,

$$\begin{aligned} d(e^{K_t} r_t) &= e^{K_t} K_t' r_t \, dt + e^{K_t} \, dr_t \\ &= e^{K_t} \alpha_t r_t \, dt + e^{K_t}[(\beta_t - \alpha_t r_t) \, dt + \sigma_t \, dW_t] \\ &= e^{K_t}(\beta_t \, dt + \sigma_t \, dW_t) \end{aligned} \tag{2.9.28}$$

when we integrate both sides of equation 2.9.28 we get

$$e^{K_t} r_t = r_0 + \int_0^t e^{K_s} \beta_s \, ds + \int_0^t e^{K_s} \sigma_s \, dW_s, \tag{2.9.29}$$

that is,

$$r_t = e^{-K_t} r_0 + \int_0^t e^{-(K_t - K_s)} \beta_s \, ds + \int_0^t e^{-(K_t - K_s)} \sigma_s \, dW_s$$

To generalize this result for any $u \leq t$ we can make the following computations:

$$\begin{aligned} r_u &= e^{-K_u} r_0 + \int_0^u e^{-(K_u - K_s)} \beta_s \, ds + \int_0^u e^{-(K_u - K_s)} \sigma_s \, dW_s \\ r_t &= e^{-K_t} r_0 + \int_0^t e^{-(K_t - K_s)} \beta_s \, ds + \int_0^u e^{-(K_t - K_s)} \beta_s \, ds \\ &+ \int_u^t e^{-(K_u - K_s)} \sigma_s \, dW_s + \int_0^u e^{-(K_t - K_s)} \sigma_s \, dW_s \\ r_t &= e^{-K_t} r_0 + \int_u^t e^{-K_t + K_s} \beta_s \, ds + \int_u^t e^{-K_t + K_s} \sigma_s \, dW_s + e^{-K_t + K_u}(r_u - e^{-K_u} r_0) \\ &= e^{-(K_t - K_u)} r_u + \int_u^t e^{-(K_t - K_s)} \beta_s \, ds + \int_u^t e^{-(K_t - K_s)} \sigma_s \, dW_s \end{aligned} \tag{2.9.30}$$

2.9.2 Hull-White Extended CIR Model

For $\gamma = 0.5$ and $\beta_t = \theta_t + \beta\alpha_t$ from (2.9.26), Hull and White in the same article proposed [11]the extension of the CIR model based on the same idea of considering time dependent coefficients of their Vasiček extension. The short rate dynamics are then given by the following stochastic differential equation

$$dr_t = [\beta_t - \alpha_t r_t]dt + \sigma_t \sqrt{r_t} dW_t \; , \tag{2.9.31}$$

where α, β, and σ are deterministic functions of time. Such an extension is not analytically tractable. However, a simple version of (2.9.31) that turns out to be analytically tractable has been proposed by Jamshidian in 1995. He assumed that, for each t, the ratio β_t/σ_t^2 is equal to a positive constant δ, which must be greater than 1/2 to ensure that the origin is inaccessible.

2.10 Black-Karazinsky Model (1991)

In their original work, Black and Karazinsky proposed a mean reverting log-normal short rate model,

$$d(\log r_t) = \phi_t[\log \mu_t - \log r_t]dt + \sigma_t dW_t \; , \tag{2.10.32}$$

where μ_t is the target rate, ϕ_t is the mean reversion and σ_t is the local volatility in the expression for the local change in $\log r_t$ [4]. They assumed these time dependent deterministic functions to be inputs while they looked for the yield curve, as the output of their model. While we are solving equation (2.10.32), for the computational and traditional purposes, we will assume that $\alpha_t = \phi_t$ and $\beta_t = \phi_t \log \mu_t$.

$$d\text{log}r_t = (\beta_t - \alpha_t \, \text{log}r_t) \, dt + \sigma_t \, dW_t \tag{2.10.33}$$

Let $Y_t = \text{log}r_t$ and define a new deterministic function K_t as

$$K_t = \int_0^t \alpha_s \, ds$$

$$and,$$

$$K_t' = \alpha_t.$$

Let $\zeta_t = e^{K_t} Y_t$.

$$\begin{aligned} d(\zeta_t) &= e^{K_t} K_t' Y_t \, dt + e^{K_t} \, dY_t \\ &= e^{K_t} \alpha_t Y_t \, dt + e^{K_t}((\beta_t - \alpha_t Y_t) \, dt + \sigma_t \, dW_t) \\ &= e^{K_t}(\beta_t + \sigma_t \, dW_t) \end{aligned} \tag{2.10.34}$$

Integrate both sides of (2.10.34)

$$\begin{aligned} e^{K_t} Y_t &= Y_0 + \int_0^t e^{K_s} \beta_s \, ds + \int_0^t e^{K_s} \sigma_t \, dW_s \\ \Rightarrow Y_t &= e^{-K_t} Y_0 + \int_0^t e^{-(K_t - K_s)} \beta_s \, ds + \int_0^t e^{-(K_t - K_s)} \sigma_t \, dW_s \end{aligned}$$

For $u \leq t$,

$$Y_t = e^{-(K_t - K_u)} Y_u + \int_u^t e^{-(K_t - K_s)} \beta_s \, ds + \int_u^t e^{-(K_t - K_s)} \sigma_t \, dW_s$$

If we replace $\log r_t$ with Y_t in the last equation

$$\log r_t = e^{-(K_t - K_u)} \log r_u + \int_u^t e^{-(K_t - K_s)} \beta_s \, ds + \int_u^t e^{-(K_t - K_s)} \sigma_t \, dW_s \tag{2.10.35}$$

2.11 Geometric Brownian Motion (GBM) Model

When the noise of the interest rate process r_t is introduced we enlarge the number of multiplicative terms in the random increment dW_t. That is, we scale dW_t with $\sigma \, r_t$, so that return of interest rate has a constant standard deviation [20]. This process is called as geometric Brownian motion and it is given with the following stochastic differential equation.

$$dr_t = \beta \, r_t \, dt + \sigma \, r_t \, dW_t \tag{2.11.36}$$

$$r_t = r_0 + \int_0^t \beta\, r_s\, ds + \int_0^t \sigma\, r_s\, dW_s \tag{2.11.37}$$

The explicit solution satisfying above SDE can easily be derived by applying Ito Lemma. Let $f(x) = \log x$.

$$\begin{aligned}
\log r_t &= \log r_0 + \int_0^t \frac{1}{r_s} dr_s - \frac{1}{2}\int_0^t \frac{1}{r_s^2}\, d < r, r >_s \\
&= \log r_0 + \int_0^t \frac{1}{r_s}\, \beta\, r_s\, ds + \int_0^t \frac{1}{r_s}\, \sigma\, r_s\, dW_s - \frac{1}{2}\int_0^t \frac{1}{r_s^2}\, r_s^2\, \sigma^2\, ds \\
&= \log r_0 + \int_0^t \beta\, ds + \int_0^t \sigma\, dW_s - \frac{1}{2}\int_0^t \sigma^2\, ds \\
&= \log r_0 + (\beta - \frac{1}{2}\sigma^2)t + \sigma W_t
\end{aligned}$$

$$\Rightarrow r_t = r_0 e^{(\beta - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

For $u \leq t$,

$$\log r_t = \log r_u + (\beta - \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)$$

$$\Rightarrow r_t = r_u e^{(\beta - \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)} \tag{2.11.38}$$

2.12 Marsh-Rosenfeld Model(1983)

In 1975 constant elasticity of variance (CEV)process was studied in the stock price context by Cox and in 1976 by Black, which can be expressed in the short rate context as

$$dr_t = \alpha\, r_t dt + \sigma\, r_t^\gamma dW_t\ ,$$

where $\gamma \geq 0$. The constant elasticity of variance process includes, in turn, the square root and normal processes, and as a limiting case, the lognormal process.

In their article Marsh and Rosenfeld used a generalized case of the CEV diffusion process as

$$dr_t = [\beta\, r_t^{-(1-\gamma)} + \alpha\, r_t]\, dt + \sigma\, r_t^{\gamma/2}\, dW_t \,, \tag{2.12.39}$$

where $\gamma \geq 0$. If $\gamma = 1$, (2.12.39) turns into square root process with mean reverting drift [13]. If $\gamma = 0$, it becomes,

$$dr_t = [\frac{\beta}{r_t} + \alpha\, r_t]\, dt + \sigma\, dW_t \,.$$

Equation (2.12.39) is reducible to affine form as

$$dy_t = [c + by_t]\, dt + \sqrt{a' y_t}\, dW_t \,,$$

where

$$\begin{aligned} y &= r^{2-\gamma} \\ a' &= 0.5\sigma(2-\gamma)^2 \\ c &= (2-\gamma)(\beta + 0.5\sigma(1-\gamma)) \\ b &= \alpha(2-\gamma)\,. \end{aligned}$$

Marsh and Rosenfeld estimate the model on a time series of T-bill data using maximum likelihood. They are unable to reach strong conclusions, but notice that the likelihood is higher when $\gamma = 2$ than when γ takes lower values [20].

Chapter 3

Yield Curve Modelling

There are many methods which have been used to model the zero-coupon yield curve. We can put them into three categories; spline based models, function based models, and lastly stochastic models. The idea of spline based models is interpolating a spline function from the pre-known points. These points can be given as a pair (time to maturity, yield of zero-coupon bond at that time to maturity). The most famous models of this type are McCulloch and FNZ(Fischer, Nychka and Zervos) [16].

The most popular function based models are Nelson-Siegel [23], and Svensson models [15]. In fact, Svensson model is an extension of Nelson-Siegel model. In Nelson-Siegel model a relatively simple function is postulated for the instantaneous forward curve. Svensson extended this work by altering the functional form of the instantaneous forward curve suggested by Nelson-Siegel. In this study, however, we will be concentrated in Nelson-Siegel model, which is less complex in the sense of number of parameters.

The stochastic modelling of zero-coupon yield curve depends on interest rate modelling, especially on short-rate modelling. The idea is obtaining an explicit pricing formula for the zero-coupon bond and then extracting the yield from that formula by using the following fundamental equation:

$$P(t,T) = e^{-(T-t)R(t,T)}\,,$$

where $R(t,\,T)$ is the yield of zero-coupon bond on the given period $[t,\,T]$. In the following sections of the chapter we will obtain the explicit zero-coupon bond pricing and yield curve formulas for Vasiček short-rate model.

3.1 Nelson-Siegel Yield Curve Model

Nelson and Siegel proposed the instantaneous forward rate as a solution of a second order differential equations [15]. Thus, they give the following solution for the instantaneous forward rate:

$$f(\theta) = \beta_0 + \beta_1 e^{-\frac{\theta}{\tau}} + \beta_2 \left[\frac{\theta}{\tau} e^{-\frac{\theta}{\tau}}\right], \tag{3.1.1}$$

where $\theta = T - t$. Since yield on the period $[t, T]$ is the average of sum of the rates which are active on the period $[t, t + dt]$, by this simple intuition, yield $R(\theta)$ can be defined as the integral of 3.1.1. That is,

$$\begin{aligned}
R(t,T) &= \frac{1}{T-t}\int_t^T f(s,T)ds \qquad (3.1.2)\\
&= \frac{1}{T-t}\int_t^T \left(\beta_0 + \beta_1 e^{-\frac{(T-s)}{\tau}} + \beta_2 \left[\frac{(T-s)}{\tau} e^{-\frac{(T-s)}{\tau}}\right]\right) ds\\
&= \beta_0 + (\beta_1 + \beta_2)\frac{[1 - e^{-\frac{T-t}{\tau}}]}{\frac{T-t}{\tau}} - \beta_2 e^{-\frac{T-t}{\tau}}\\
&= \beta_0 + (\beta_1 + \beta_2)\frac{[1 - e^{-\frac{\theta}{\tau}}]}{\frac{\theta}{\tau}} - \beta_2 e^{-\frac{\theta}{\tau}}\\
&= \frac{1}{\theta}\int_0^\theta \left(\beta_0 + \beta_1 e^{-\frac{s}{\tau}} + \beta_2 \left[\frac{s}{\tau} e^{-\frac{s}{\tau}}\right]\right) ds\\
&= \frac{1}{\theta}\int_0^\theta f(s)ds\\
&= R(\theta) \qquad (3.1.3)
\end{aligned}$$

Although it is easy to see the last equality, we showed it to be consistent for the future notations of yield.

Now let us analyze the structure of the yield and forward curves. The limiting value of $R(\theta)$ as θ approaches to infinity is β_0 and as θ gets small values it is $(\beta_0 + \beta_1)$, which are necessarily the same as for the forward rate function since $R(\theta)$ is just an averaging of $f(\theta)$. Now, let us separate the forward rate function into three components: long-term, medium-term and lastly short-term. The long-term component is identified by the asymptotic value, β_0, of the function. The medium-term is identified by the functional component $\frac{\theta}{\tau} e^{-\frac{\theta}{\tau}}$ and for the designation of the short-term $e^{-\frac{\theta}{\tau}}$ is used.

The long term component is a constant larger than zero. Thus, it can not take zero value in the limit. The medium-term takes zero value at the starting point zero. This indicates that it is not short-term and since it decays to zero in the limit, therefore it is not long-term. As it is obvious that short-term curve has the largest negative slope resulting the fastest decay among all.It takes monotonically and asymptotically zero value.

In the model the contributions of these three components are given by β_0 for long-term, β_1 for short-term and β_2 for medium-term. Here, there are also some other features of the parameters that we should consider. If β_1 is negative the forward curve will have a positive slope and vice versa. Also, if β_2, as being the identifier of the magnitude and the direction of the hump, is positive, a hump will occur at τ whereas, if it is negative, a U-shaped value will occur at τ. Thus, we can conclude that parameter τ which is positive, specifies the position of the hump or U-shape on the entire curve. As a result, Nelson and Siegel have proposed that with appropriate choices of weights for these three components, it is possible to generate a variety of yield curves based on forward rate curves with monotonic and humped shapes [23].

3.2 Vasiček Yield Curve

In Vasiček model we assume that the process r_t satisfies the following stochastic differential equation:

$$dr_t = \alpha(\beta - r_t)\,dt + \sigma\,dW_t \tag{3.2.4}$$

Without repeating the solution for 3.2.4 that we followed in section 2.2 we will just change the probability measure to risk neutral one because of quite crucial reason. Unless we study with the risk neutral probability, it is not possible to make extinct the arbitrage possibilities in bond pricing and we know that the discounted bond prices have martingale property only under the risk neutral probability, which satisfies the non-arbitrage condition. With this aim, to change the measure we will use Girsanov Theorem. We assume a constant process $q(t) = -\lambda$, with $\lambda \in \mathbb{R}$. Therefore, $\tilde{W}_t = W_t + \int_0^t \lambda\, ds = W_t + \lambda\, t$ is a standard Brownian motion under $\mathbf{P^*}$. Thus, $dW_t = d\tilde{W}_t - \lambda\, dt$ and if we rewrite the equation 3.2.4 with respect to the new probability measure P^* we get the following result:

$$
\begin{aligned}
dr(t) &= \alpha\,(\beta - r(t))\, dt + \sigma \left[d\tilde{W}_t - \lambda\, dt \right] \\
&= (\alpha\,\beta - \alpha\, r(t) - \lambda\,\sigma)\, dt + \sigma\, d\tilde{W}_t \\
&= \alpha \left(\beta - \frac{\lambda\,\sigma}{\alpha} - r(t) \right) dt + \sigma\, d\tilde{W}_t \\
&= \alpha\,(\beta^* - r(t))\; dt + \sigma\, d\tilde{W}_t, \qquad (3.2.5)
\end{aligned}
$$

where $\beta^* = \beta - \frac{\lambda\,\sigma}{\alpha}$.

The main goal of this section is to obtain the Vasiček bond price formulae, therefore the yield curve. We will start with the fundamental equation of bond price, with the short rate r given by Equation 3.2.5.

$$\begin{aligned}
P(t,T) &= E^*\left(e^{-\int_t^T r_s\,ds}|\mathcal{F}_{\rm t}\right) \\
&= E^*\left(e^{-\int_t^T (r_s-\beta^*+\beta^*)\,ds}|\mathcal{F}_{\rm t}\right) \\
&= e^{-\int_t^T \beta^* ds}\,E^*\left(e^{-\int_t^T (r_s-\beta^*)\,ds}|\mathcal{F}_{\rm t}\right) \\
&= e^{-\beta^*(T-t)}\,E^*\left(e^{-\int_t^T X_s^*\,ds}|\mathcal{F}_{\rm t}\right) \qquad (3.2.6)
\end{aligned}$$

where $X_t^* = r_t - \beta^*$, and $\mathcal{F}_t$ is the natural filtration of W_t. Since (X_t^*) is a solution of the diffusion equation with constant coefficients

$$dX_t = -\alpha X_t + \sigma\, d\tilde{W}_t\,, \qquad (3.2.7)$$

by using the homogeneity of 3.2.7 and the Markov property of the solution $X_t^x = x\,e^{-\alpha\,t} + \sigma\,e^{-\alpha\,t}\int_0^t e^{\alpha\,s} d\tilde{W}_s$ we can write

$$\begin{aligned}
E^*\left(e^{-\int_t^T X_s^*\,ds}|\mathcal{F}_{\rm t}\right) &= F(T-t, X_t^*) \\
&= F(T-t, r_t - \beta^*) \qquad (3.2.8)
\end{aligned}$$

where F is the function defined by $F(\theta,x) = E^*\left(e^{-\int_0^\theta X_s^x\,ds}\right)$. Since (X_t^x) is Gaussian with continuous paths, we can calculate $F(\theta,x)$, explicitly [1].

For the complete calculation of $F(\theta,x)$, consider the random variable $Y = \int_0^\theta X_s^x\,ds \sim N(\mu,\,\sigma^2)$ with the mean μ, and the variance σ^2. The Laplace transform [17] of Y can be defined therefore,

$$\mathcal{L}_Y(t) := E\left(e^{-tY}\right) = \int_R e^{-tY} dF_Y(y)$$

The last integral turns into:

$$\begin{aligned}
\int_R e^{-tY} dF_Y(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_R e^{-ty}\, e^{\frac{-(y-\mu)^2}{2\sigma^2}}\, dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_R e^{\frac{-2ty\sigma^2 - y^2 + 2\mu y - \mu^2}{2\sigma^2}}\, dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_R e^{\frac{-\left[y^2 - (2\mu - 2\sigma^2 t)y + \mu^2\right]}{2\sigma^2}}\, dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_R e^{\frac{-\left[y^2 - (2\mu - 2\sigma^2 t)y + (\mu - \sigma^2 t)^2 - (\mu - \sigma^2 t)^2 + \mu^2\right]}{2\sigma^2}}\, dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_R e^{\frac{-(y-(\mu-\sigma^2 t))^2}{2\sigma_2} + \frac{\mu^2 - 2\mu\sigma^2 t + \sigma^4 t^2 - \mu^2}{2\sigma_2}}\, dy \\
&= e^{-\mu t + \frac{\sigma^2}{2} t^2} \int_R \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(y-(\mu-\sigma^2 t))^2}{2\sigma_2}}\, dy \\
&= e^{-\mu t + \frac{\sigma^2}{2} t^2}. \qquad (3.2.9)
\end{aligned}$$

where $\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(y-(\mu-\sigma^2 t))^2}{2\sigma_2}}$ is the density function of the random variable $\sim N(\mu - \sigma^2 t,\, \sigma^2)$. We need the case of $t = 1$.

Thus,

$$\begin{aligned}
E^*\left(e^{-\int_0^\theta X_s^x\, ds}\right) = E^*\left(e^{-Y}\right) &= e^{-\mu + \frac{\sigma^2}{2}} = e^{-E^*(Y) + \frac{1}{2}Var(Y)} \\
&= e^{-E^*\left(\int_0^\theta X_s^x\, ds\right) + \frac{1}{2}Var\left(\int_0^\theta X_s^x\, ds\right)} \qquad (3.2.10)
\end{aligned}$$

We have

$$\begin{aligned} E^*\left(\int_0^\theta X_s^x\,ds\right) &= \int_0^\theta E^*(X_s^x)\,ds \\ &= \int_0^\theta x\,e^{-\alpha s}\,ds \\ &= \frac{x}{\alpha}(1-e^{-\alpha\,\theta}) \end{aligned} \qquad (3.2.11)$$

Therefore, when we take the expectation of X_t^x we get zero for the stochastic part,since $\int_0^t e^{\alpha\,s}d\tilde{W}_s$ is a continuous $\mathcal{F}_t$-martingale.

$$Var\left(\int_0^\theta X_s^x\,ds\right) = Cov\left(\int_0^\theta X_s^x\,ds\ \int_0^\theta X_s^x\,ds\right)$$

For u = t = s

$$\begin{aligned} &= E^*\left(\int_0^\theta X_t\,dt\int_0^\theta X_u\,du\right) - E^*\left(\int_0^\theta X_t\,dt\right)E^*\left(\int_0^\theta X_u\,du\right) \\ &= \int_0^\theta\int_0^\theta E^*\left(X_t^x\,X_u^x\right)\,du\,dt - \int_0^\theta E^*\left(X_t^x\right)\,dt\int_0^\theta E^*\left(X_u^x\right)\,du \\ &= \int_0^\theta\int_0^\theta \left[E^*\left(X_t^x\,X_u^x\right) - E^*\left(X_t^x\right)\,E^*\left(X_u^x\right)\right]\,du\,dt \\ &= \int_0^\theta\int_0^\theta Cov\left(X_t^x\,X_u^x\right)\,du\,dt \end{aligned} \qquad (3.2.12)$$

For the calculation of $Cov\left(X_t^x\,X_u^x\right)$ we write [17],

$$
\begin{aligned}
Cov\left(X_t^x\, X_u^x\right) &= E^*\left[\left(X_t^x - E\left(X_t^x\right)\right)\left(X_u^x - E\left(X_u^x\right)\right)\right] \\
&= E^*\left[\sigma^2\, e^{-\alpha(t+u)} \int_0^t e^{\alpha\, s}\, d\tilde{W}_s \int_0^u e^{\alpha\, s}\, d\tilde{W}_s\right] \\
&= \sigma^2\, e^{-\alpha(t+u)}\, E^*\left[\int_0^t e^{\alpha\, s}\, d\tilde{W}_s \int_0^u e^{\alpha\, s}\, d\tilde{W}_s\right] \\
&= \sigma^2\, e^{-\alpha(t+u)} \int_0^{t\wedge\, u} e^{2\alpha\, s}\, ds \\
&= \sigma^2\, e^{-\alpha(t+u)}\, \frac{e^{2\alpha(t\wedge\, u)} - 1}{2\alpha} \qquad (3.2.13)
\end{aligned}
$$

For the complete calculation of $Var\left(\int_0^\theta X_s^x\, ds\right)$ let us insert the result we found in 3.2.13 in 3.2.12. Let us define $g(t,u) = Cov(X_t^x, X_u^x)$. We see that g is symmetric with respect to line $u = t$ on the region where the integral in 3.2.12 is taken.

Therefore it is enough to evaluate the above integral only for one of the regions to calculate $Var\left(\int_0^\theta X_s^x\, ds\right)$. Let us chose the region $\{(t\,,u) : 0 < u < t\}$.

$$\begin{aligned}
Var\left(\int_0^\theta X_s^x\, ds\right) &= \int_0^\theta \int_0^\theta Cov\,(X_t^x\, X_u^x)\; du\, dt \\
&= 2\int_0^\theta \int_0^t \sigma^2\, e^{-\alpha(t+u)}\, \frac{e^{2\alpha\, u} - 1}{2\alpha}\, du\, dt \\
&= 2\int_0^\theta \int_0^t \frac{\sigma^2 e^{-\alpha\, t}\, e^{\alpha\, u} - \sigma^2\, e^{-\alpha\, t}\, e^{-\alpha\, u}}{2\alpha}\, du\, dt \\
&= 2\int_0^\theta \left[\frac{\sigma^2 e^{-\alpha\, t}\, e^{\alpha\, u}}{2\alpha^2} + \frac{\sigma^2\, e^{-\alpha\, t}\, e^{-\alpha\, u}}{2\alpha^2}\right]_0^t\, dt \\
&= 2\int_0^\theta \left(\frac{\sigma^2}{2\alpha^2} + \frac{\sigma^2}{2\alpha^2}\, e^{-2\alpha\, t} - \frac{\sigma^2}{2\alpha^2}\, 2e^{-\alpha\, t}\right)\, dt \\
&= \frac{\sigma^2}{\alpha^2} \int_0^\theta \left(1 + e^{-2\alpha\, t} - 2e^{-\alpha\, t}\right)\, dt \\
&= \frac{\sigma^2}{\alpha^2} \left[t - \frac{e^{-2\alpha\, t}}{2\alpha} + \frac{2e^{-\alpha\, t}}{\alpha}\right]_0^\theta \\
&= \frac{\sigma^2}{\alpha^2} \left[\theta - \frac{e^{-2\alpha\, t}}{2\alpha} + \frac{2e^{-\alpha\, t}}{\alpha} - \frac{3}{2\alpha}\right] \\
&= \frac{\theta\sigma^2}{\alpha^2} - \frac{\sigma^2}{2\alpha^3}\, \left[e^{-2\alpha\theta} - 4\, e^{-\alpha\theta} + 3\right] \\
&= \frac{\theta\sigma^2}{\alpha^2} - \frac{\sigma^2}{2\alpha^3}\, \left[1 - 2e^{-\alpha\theta} + e^{-2\alpha\theta} + 2\left(1 - e^{-\alpha\theta}\right)\right] \\
&= \frac{\theta\sigma^2}{\alpha^2} - \frac{\sigma^2}{2\alpha^3}\, \left[\left(1 - e^{-\alpha\theta}\right)^2 + 2\left(1 - e^{-\alpha\theta}\right)\right] \\
&= \frac{\theta\sigma^2}{\alpha^2} - \frac{\sigma^2}{\alpha^3}\, \left(1 - e^{-\alpha\theta}\right) - \frac{\sigma^2}{2\alpha^3}\, \left(1 - e^{-\alpha\theta}\right)^2 \qquad (3.2.14)
\end{aligned}$$

By replacing 3.2.11 and 3.2.14 into 3.2.10, we get $F(\theta, x)$ explicitly. Thus,

$$
\begin{aligned}
F(\theta, x) &= E^* \left(e^{-\int_0^\theta X_s^x \, ds} \right) \\
&= e^{\frac{x}{\alpha}(1-e^{-\alpha\,\theta})+\frac{\theta\sigma^2}{\alpha^2}-\frac{\sigma^2}{\alpha^3}\left(1-e^{-\alpha\theta}\right)-\frac{\sigma^2}{2\alpha^3}\left(1-e^{-\alpha\theta}\right)^2} \qquad (3.2.15)
\end{aligned}
$$

Now, we can complete the calculation of bond price.

$$
\begin{aligned}
P(\theta) &= e^{-\beta^*\theta}\, e^{\frac{x}{\alpha}(1-e^{-\alpha\,\theta})+\frac{\theta\sigma^2}{\alpha^2}-\frac{\sigma^2}{\alpha^3}\left(1-e^{-\alpha\theta}\right)-\frac{\sigma^2}{2\alpha^3}\left(1-e^{-\alpha\theta}\right)^2} \\
&= e^{-\beta^*\theta+\frac{x}{\alpha}(1-e^{-\alpha\,\theta})+\frac{\theta\sigma^2}{\alpha^2}-\frac{\sigma^2}{\alpha^3}\left(1-e^{-\alpha\theta}\right)-\frac{\sigma^2}{2\alpha^3}\left(1-e^{-\alpha\theta}\right)^2} \\
&= e^{-\theta\, R(\theta, r)}, \qquad (3.2.16)
\end{aligned}
$$

where $R(\theta,\, r)$ can be seen as the average interest rate for the period $[t\,, T]$, i.e. $R(\theta,\, r)$ is the yield of the bond on the given period, and it can be calculated by inverting the price formula.

$$
\begin{aligned}
R(\theta\,, r) &= -\frac{-\beta^*\theta + \frac{x}{\alpha}(1-e^{-\alpha\,\theta}) + \frac{\theta\sigma^2}{\alpha^2} - \frac{\sigma^2}{\alpha^3}\,\left(1-e^{-\alpha\theta}\right) - \frac{\sigma^2}{2\alpha^3}\,\left(1-e^{-\alpha\theta}\right)^2}{\theta} \\
&= -\frac{-\beta^*\theta + \frac{\sigma^2\theta}{2\alpha^2} + \left(1-e^{-\alpha\,\theta}\right)\left[-\frac{x}{\alpha} - \frac{\sigma^2}{2\alpha^3} - \frac{\sigma^2}{4\alpha^3}\left(1-e^{-\alpha\,\theta}\right)\right]}{\theta} \\
&= -\left[-\beta^* + \frac{\sigma^2}{2\alpha^2} - \frac{1}{\alpha\theta}\left[\left(1-e^{-\alpha\theta}\right)\,\left(x + \frac{\sigma^2}{2\alpha^2} + \frac{\sigma^2}{4\alpha^2}\left(1-e^{-\alpha\theta}\right)\right)\right]\right] \\
&= -\left[-\beta^* + \frac{\sigma^2}{2\alpha^2} - \frac{1}{\alpha\theta}\left[\left(1-e^{-\alpha\theta}\right)\,\left((r-\beta^*) + \frac{\sigma^2}{2\alpha^2} + \frac{\sigma^2}{4\alpha^2}\left(1-e^{-\alpha\theta}\right)\right)\right]\right] \\
&= \beta^* - \frac{\sigma^2}{2\alpha^2} - \frac{1}{\alpha\theta}\left[\left(\beta^* - \frac{\sigma^2}{2\alpha^2} - r\right)\left(1-e^{-\alpha\theta}\right)\, - \frac{\sigma^2}{4\alpha^2}\left(1-e^{-\alpha\theta}\right)^2\right] \\
&= R_\infty - \frac{1}{\alpha\theta}\left[(R_\infty - r)\left(1-e^{-\alpha\theta}\right)\, - \frac{\sigma^2}{4\alpha^2}\left(1-e^{-\alpha\theta}\right)^2\right], \qquad (3.2.17)
\end{aligned}
$$

where

$$
R_\infty = \lim_{\theta\to\infty} R(\theta,\, r) = \beta^* - \frac{\sigma^2}{2\alpha^2}.
$$

Chapter 4

Applications

In the first section of this chapter, we are going to present and explain the data,representing the Turkish zero-coupon bond simple spot rate, that we use in the following sections. In the second section, we will use Nelson-Siegel yield curve model to fit the given data. To calibrate the Nelson-Siegel model we are going to construct a sum of squared errors. Then, we are going to minimize this function with appropriate constraints and an initial value. After the calibration, the performance of Nelson-Siegel yield curve will be measured by the value of the sum of squared errors. In the last section, we are going to use the Vasiček yield curve to predict the March 17, 2004 yield curve by using the March 16, 2004 data.

4.1 Data Analysis

The data that is used in this study is the Turkish zero-coupon bond simple rate gathered for several maturities and days. Although the data includes yearly simple rate information of the zero-coupon bond between July 1, 1999 and March 17, 2004, we will use a set of data starts at May 1, 2001 and ends at March 17, 2004. The reason why we use this set is two fold. Firstly, we want to exclude the days which are effected by the harsh economic crises of the years,2000 and 2001. The most part of the data occurred during years includes daily disordered bond simple rate of returns. The second reason is that although the data is for a long period, it has the problem of having a very short number of bond prices for a given day. This stems from the fact that the bond market in Turkey is not

	MNO	MINO	MEAN
July 1, 1999-March 17, 2004	March 17, 2004(21)	March 9, 2001(2)	10.9933
July 1, 1999-April 30, 2001	July 9, 1999*(12)	March 9, 2001(2)	7.9542
May 1, 2001-March 17, 2004	March 17, 2004(21)	Jul 1, 1999*(7)	12.8784

Table 4.1: MNO: The maximum number of observations and its day for the given data set. MINO: The minimum number of observations and its day for the given data set. *: An example from several days.

liquid. The minimum number of observations occurs in March 9, 2001 with 2 observations.The maximum number 21 among all of the daily observations has occurred in March 17, 2004. The mean of observations for the whole data set is 10.9933. The mean of observations for the data set we have chosen is 12.8784 and the mean of the remaining data from Jul 1, 1999 to April 30, 2001 is 7.9542. These facts are summarized in the Table 4.1. Since we will fit a yield curve to the daily observations, we preferred to choose that data set:

· Excluding the day having the minimum number of daily observations.

· Including the day having the maximum number of daily observations.

· Having the highest mean of daily observations.

· Including the most number of days satisfying the above three conditions.

Since the data is a raw data by means of the interest rate information that it carries out, we worked on it to obtain an appropriate rate for the model based usage. To express the data for our purpose we used interest rate definitions given in definition set 1.12. Let us show our calculations, explicitly. First, we used simple spot rate definition and then the definition of continuously compounded spot rate.

$$L(S,\,T) \;=\; -\frac{P(S,\,T)-1}{(T-S)P(S,\,T)}$$

$$\Longleftrightarrow$$

$$P(S,\,T) \;=\; \frac{1}{1+L(S,\,T)(T-S)}$$

$$\Longleftrightarrow$$

$$R(S,\,T) \;=\; -\frac{\log P(S,\,T)}{T-S}$$

Since we are going to fit a yield curve, the data must be in the form of continuously compounded spot rate. For a numerical example, let us calculate the price of the zero coupon bond in May 1, 2001 having a yearly simple rate of return 0.7580 with time to maturity 113 days. Thus, $T-S = 113/365$ and $L(S,\,T) = 0.7580$. That is, $P(S,\,T) = 0.8010$ TL. This price coincide with the assumption $P(t,\,t) = 1$. However, the exact price will be 80100 TL. We multiplied the result with 100000 which is the face value of the Turkish zero-coupon bond used in this study. To evaluate $R(S,\,T)$ we use 0.8010. Therefore, $R(S,\,T) = 0.0018655$. This gives a continuously compounded rate of return of 186.55 TL over 100000 TL.

4.2 Yield Curve Fitting with Nelson-Siegel Model

In section 3.1 we studied the structure of the Nelson-Siegel yield curve model. In this section we are going to use this model to fit yield curves to the sets of zero-coupon bond yields for given days.

Let us recall the following yield defining equation 3.1.2:

$$R(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds$$

If we replace forward rate(in fact, instantaneous forward rate) with its definition given in 1.12 we get

$$\begin{aligned} R(t,T) &= \frac{1}{T-t}\int_t^T \left(-\frac{\partial \log P(t,\, s)}{\partial\, s}\right) ds \\ &= -\frac{1}{T-t}\left(\log P(t,\, T) - \log P(t,\, t)\right) \\ &= -\frac{\log P(t,\, T)}{T-t} \end{aligned} \tag{4.2.1}$$

Fitting implies calibration of the parameters of the given model. We will minimize the sum of squared errors(SSEs) to calibrate the parameters of the Nelson-Siegel model for a given day.
The Nelson-Siegel yield function obtained in Section 3.1 is as follows.

$$R(\theta) = \beta_0 + (\beta_1 + \beta_2)\,\frac{[1 - e^{-\frac{\theta}{\tau}}]}{\frac{\theta}{\tau}} - \beta_2\, e^{-\frac{\theta}{\tau}}$$

with parameters $\beta_0,\ \beta_1,\ \beta_2,\ \tau$. The minimization of objective function SSE is constructed as follows;

$$\mathbf{min}_B \sum_{\theta=1}^{m} \left(R(\theta) - \hat{R}(\theta)\right)^2 \tag{4.2.2}$$

where $B = [\beta_0, \beta_1, \beta_2, \tau]$, $R(\theta)$ is observed bond yield of time to maturity θ for a given day and $\hat{R}(\theta)$ is the theoretical bond yield of time to maturity θ for that day. To do this minimization we use five different sets of constraints and initial points. Now, let us introduce these sets. We denote by R_{long} the yield of the bond with longest maturity of the given day and by R_{short} the yield of the bond with shortest maturity of the given day.

(a) $B_{initial} = [R_{long}, R_{short} - R_{long}, 1, 50]$,
$\beta_0 > 0$,
$\beta_0 + \beta_1 = R_{short}$,

(b) $B_{initial} = [R_{long}, R_{short} - R_{long}, 0.9, 49.9]$,
$\beta_0 > 0$,

$\beta_0 + \beta_1 = R_{short}$,

(c) $B_{initial} = [R_{long}, R_{short} - R_{long}, 1, 50]$,
$\beta_0 > 0$,
$\beta_0 + \beta_1 < R_{short}$,

(d) $B_{initial} = [R_{long}, R_{short} - R_{long}, 0.9, 49.9]$,
$\beta_0 > 0$,
$\beta_0 + \beta_1 < R_{short}$,

(e) $B_{initial} = [R_{long}, R_{short} - R_{short}, -1, 50]$,
$\beta_0 > 0$,
$\beta_0 + \beta_1 < R_{short}$,
$\beta_2 < 0$,
$25 \leq \tau \leq 800$,

Note that in the initial sets first two initial values are default. However, the other two initials are perturbed on the axis $(0.9, 49.9)$. First, we perturbed 0.1 points in all directions and we reached to $(1, 50)$ and then we perturbed the axis with 0.01 points in all directions, but we remained on the same axis [24]. By using above constraint and initial point sets we calibrated the Nelson-Siegel yield function for all the days in our data set. That is we did 740x5 calibrations: For each of the day in our data set we repeated the minimization procedure for each of the constraint and initial point sets.

As a result, for each constraint-initial point set we estimated different parameters for a given day and we have chosen the parameters that minimize SSEs for that day. In the Figures 4.1, 4.2, 4.3 and 4.4 we presented fits of yield curves for some days. In most of the figures we resulted with a very fine fit. However, this does not mean that all the yield curve fits for our data set is fine. For example, in Figure 4.3 the days April 25 and April 26, 2002 fitted yield curves that are S-shaped and humped, respectively, are not fine.[1] Especially, the fit of April 25 is deficient which is mainly resulted from the difficulty of the data. The data

[1] We are describing *being fine* empirically since we are going to take into account all of the Nelson-Siegel fitted curves obtained from the above algorithm.

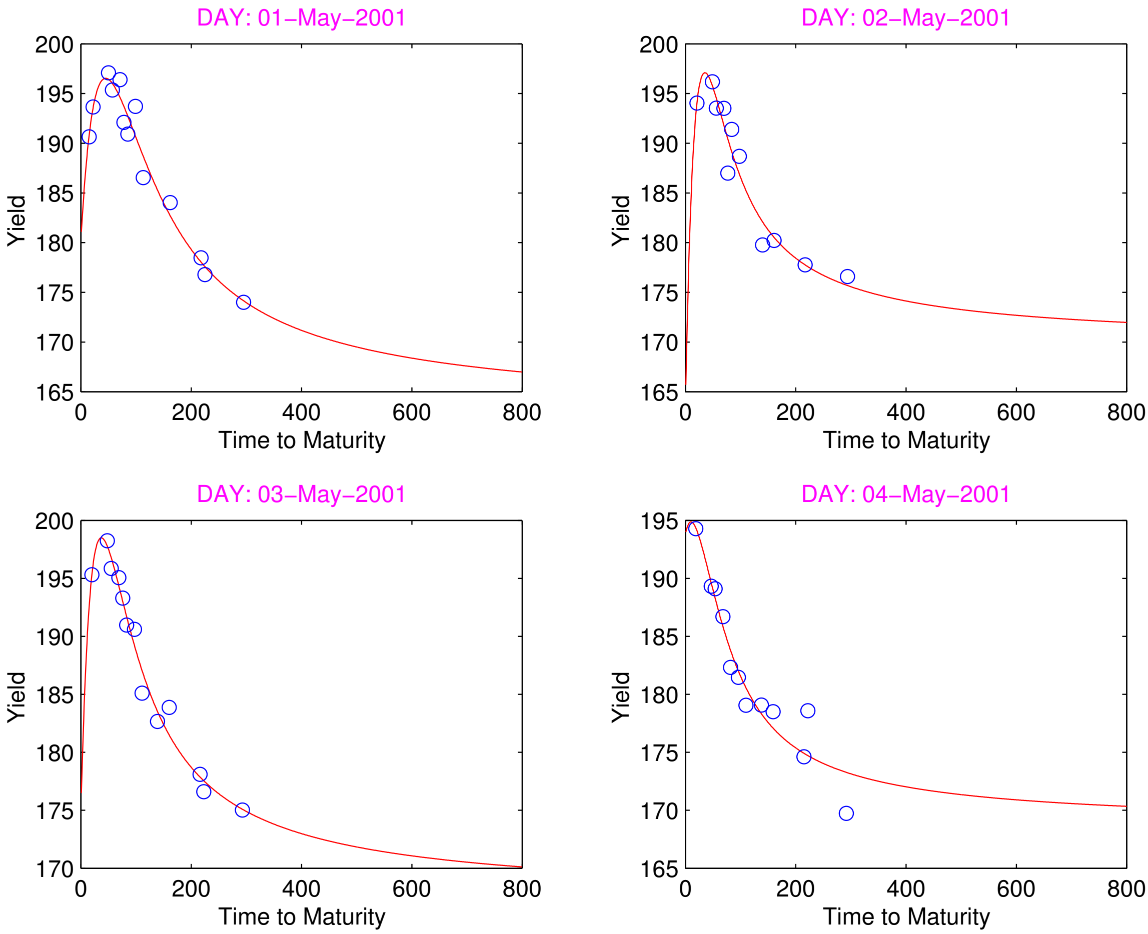

Figure 4.1: Fit of Yield Curves

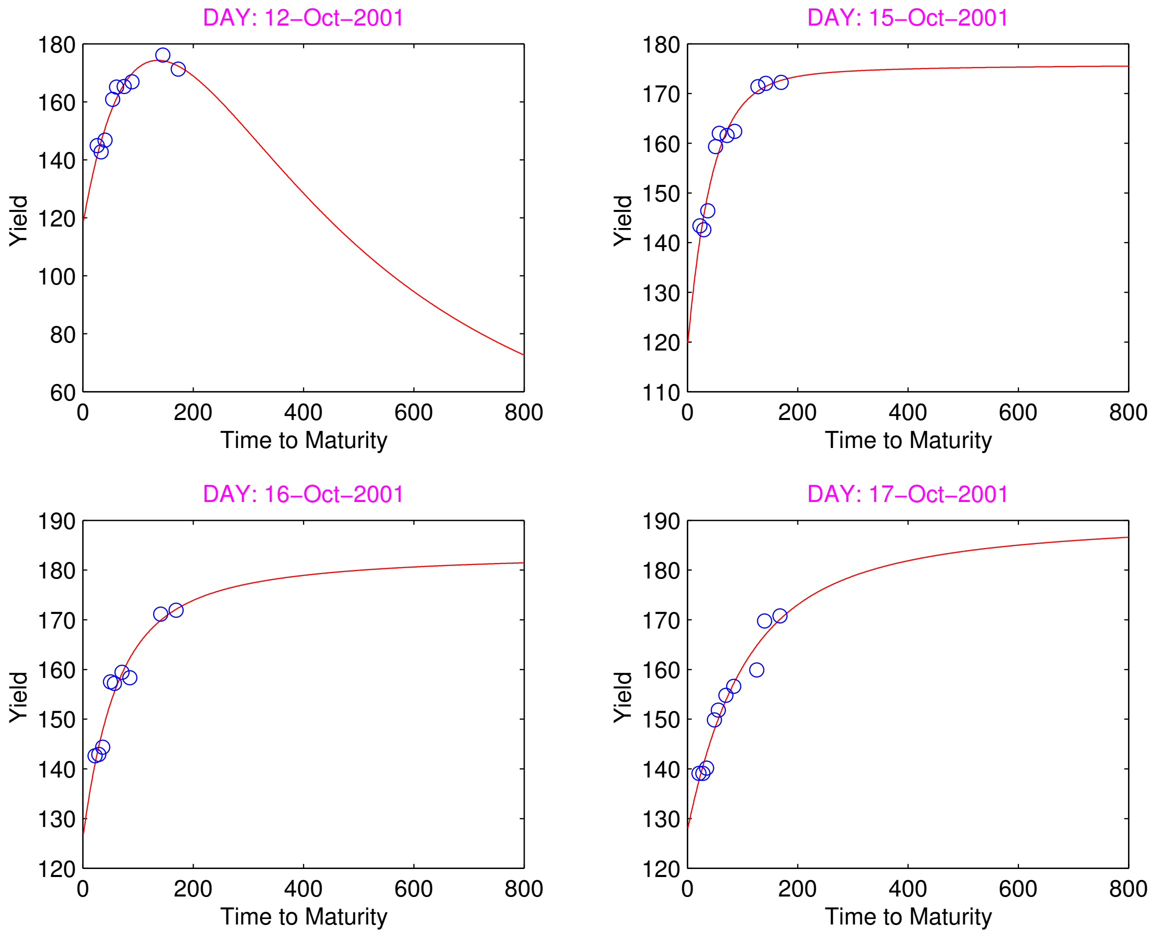

Figure 4.2: Fit of Yield Curves

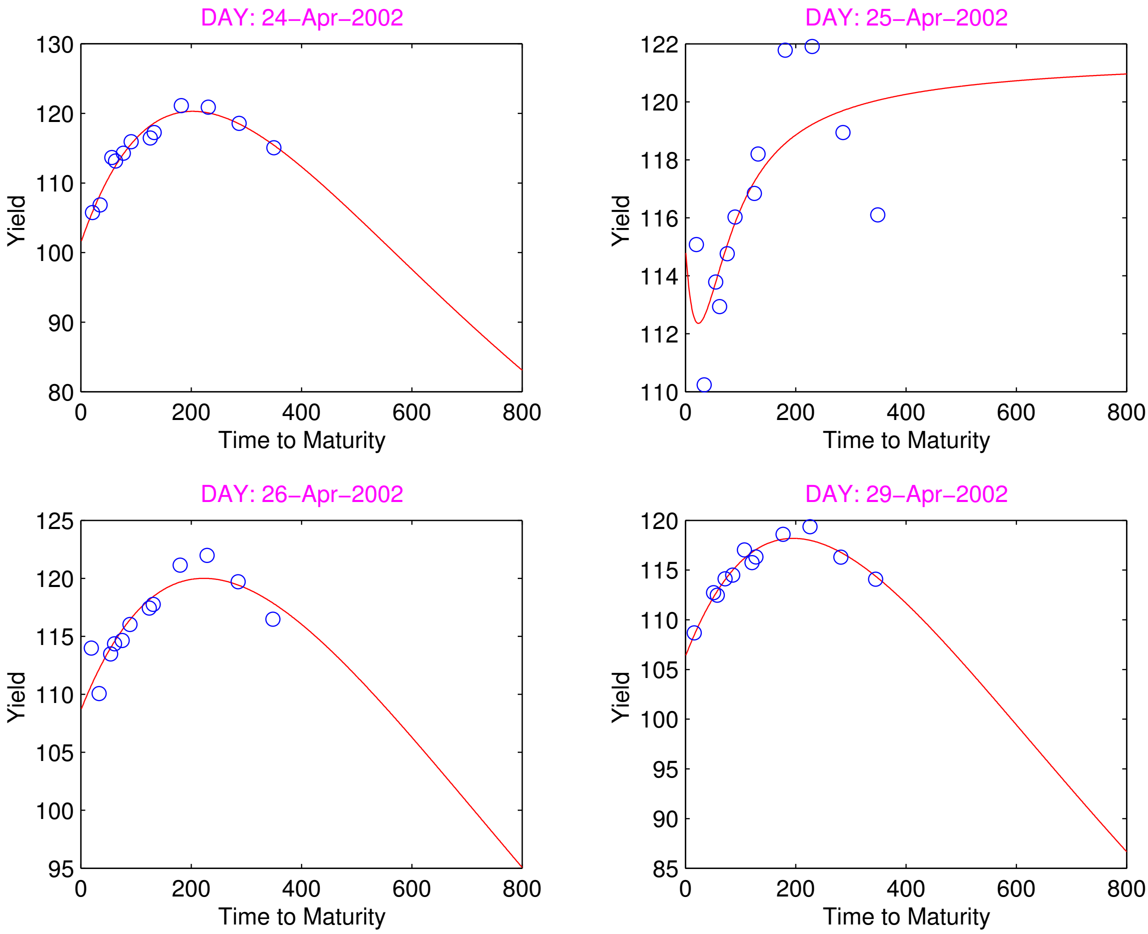

Figure 4.3: Fit of Yield Curves

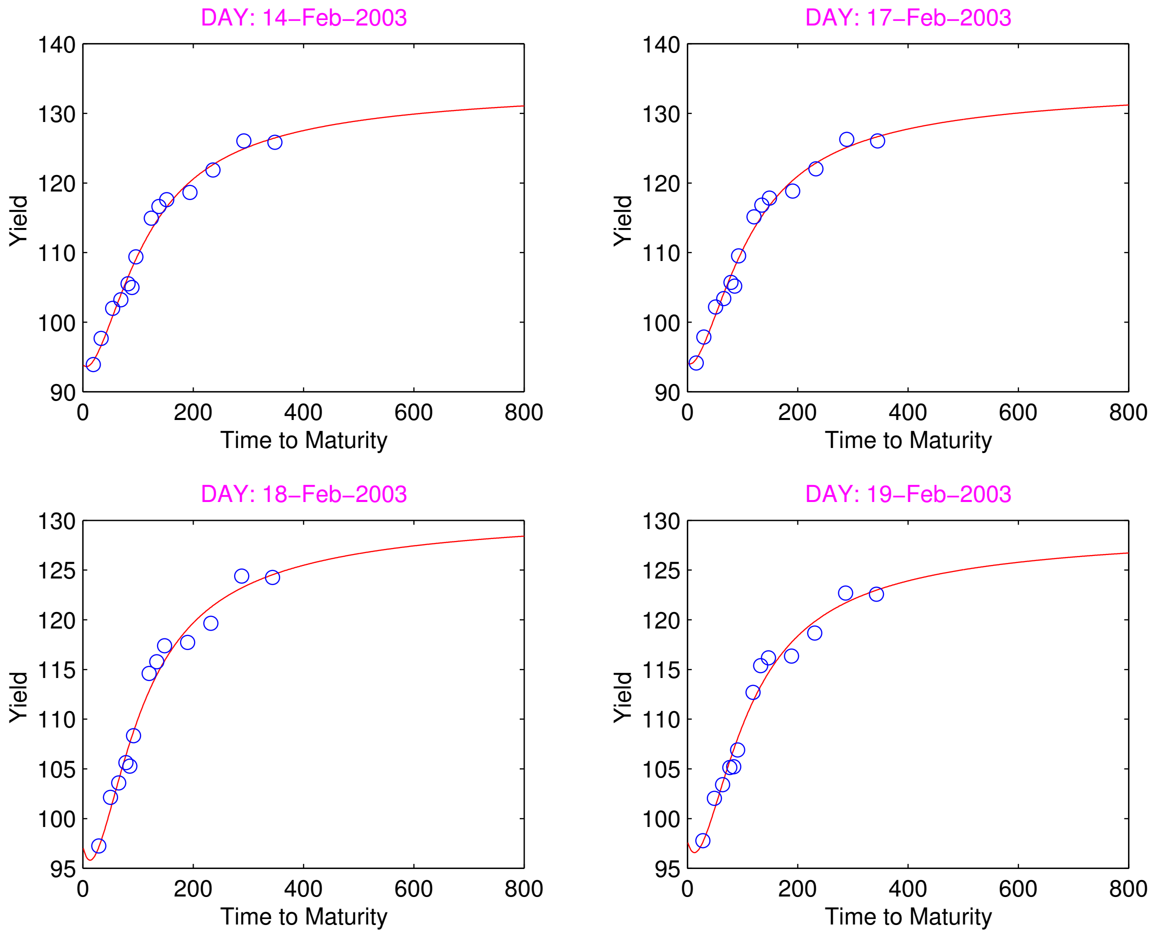

Figure 4.4: Fit of Yield Curves

with this structure, i.e. data having a U-shape and a hump, can be modelled by using Svenson model which has two place holder parameters τ_1 and τ_2, and two shape identifier parameters β_2 and β_3.

4.3 Yield Curve Estimation with Vasiček Model

We proved in section 3.2 that for the risk neutral probability Vasiček short rate satisfies the below stochastic differential equation:

$$dr(t) = \alpha\left(\beta^* - r(t)\right)\, dt + \sigma\, d\tilde{W}_t \tag{4.3.3}$$

The solution of this SDE is going to be the adapted one to the solution of SDE 2.2.2 given under the objective probability in section 2.2 as:

$$r_t = r_0 e^{-\alpha\, t} + \beta^*(1 - e^{-\alpha\, t}) + \sigma\, e^{-\alpha\, t} \int_0^t e^{\alpha\, s} d\tilde{W}_s,$$

and for $u < t$, it is given as

$$r_t = r_u\, e^{-\alpha(t-u)} + \beta^*\left(1 - e^{-\alpha(t-u)}\right) + \sigma \int_u^t e^{-\alpha(t-s)} d\tilde{W}_s, \tag{4.3.4}$$

From the last solution we see that given r_u, the possible values of r_t are being normally distributed. The reason is that the distribution of $d\tilde{W}_s$ is normal with mean zero and variance one. Therefore, the integral itself is being normally distributed and since the remaining part of the equation other than the stochastic part is deterministic the current distribution of the stochastic part does not change except its mean and variance. As a result, r_t is normally distributed with mean $\mu_r(u,\, t) = r_u\, e^{-\alpha(t-u)} + \beta^*\left(1 - e^{-\alpha(t-u)}\right)$ and variance $\sigma_r^2(u,\, t) = E^*\left[\left(\sigma \int_u^t e^{-\alpha(t-s)} d\tilde{W}_s\right)^2\right] = \sigma^2 \int_u^t e^{-2\,\alpha\,(t-s)}\, ds = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\,\alpha\,(t-s)}\right)$.

The discretization of r_t can now be easily done [25]. That is,

$$\begin{aligned} r(t_{i+1}) &= \mu_r(t_i,\, t_{i+1}) + \sigma_r(t_i,\, t_{i+1}) Z_{i+1} \\ &= e^{-(t_{i+1}-t_i)}\, r(t_i) + \beta^* \left(1 - e^{-(t_{i+1}-t_i)}\right) + \sigma \sqrt{\frac{1}{2\alpha}(1 - e^{-2(t_{i+1}-t_i)})} Z_{i+1} \end{aligned} \tag{4.3.5}$$

where $0 = t_0 < t_1 < \ldots < t_n$ are the time points and Z_i's, $i = 1 \ldots n$ are the independent random variables with distribution $N(0, 1)$. Equation 4.3.5 is the exact discretization of the Vasiček short-rate process. However, a slightly simpler Euler scheme of the short-rate process can be given as

$$r(t_{i+1}) = r(t_i) + \alpha\,(\beta^* - r(t_i))(t_{i+1} - t_i) + \sigma\,\sqrt{t_{i+1} - t_i} Z_{i+1} \tag{4.3.6}$$

which entails some discretization error. In the following sections of the chapter we will measure the performance of both equations while we are simulating the Vasiček yield function.

In Section 3.2 we obtained the Vasiček yield function. We can rewrite it as follows.

$$R(\theta\,, r_t) \;=\; R_\infty - \frac{1}{\alpha\theta}\left[(R_\infty - r_t)\left(1 - e^{-\alpha\theta}\right) - \frac{\sigma^2}{4\alpha^2}\left(1 - e^{-\alpha\theta}\right)^2\right],$$

where $R_\infty = \lim_{\theta\to\infty} R(\theta,\, r) = \beta^* - \frac{\sigma^2}{2\alpha^2}$. As $R(\theta\,, r_t)$ is a smooth function of θ, it is a stochastic process of t and its stochastic evolution is determined by Vasiček short-rate process. Therefore, it is possible to use $R(\theta\,, r_t)$ for the future estimations of yield curve.

The discretization of $R(\theta\,, r_t)$ begins with the following stochastic differential equation.

$$dR(\theta\,, r_t) = \frac{\left(1 - e^{-\alpha\theta}\right)}{\alpha\theta} dr_t \tag{4.3.7}$$

and we write

$$R(\theta\,, r_{t_{i+1}}) - R(\theta\,, r_{t_i}) = \frac{\left(1 - e^{-\alpha\theta}\right)}{\alpha\theta}\left[r_{t_{i+1}} - r_{t_i}\right] \tag{4.3.8}$$

The following part of the discretization procedure has two folds:

(1) Replacing $r_{t_{i+1}} - r_{t_i}$ with

$$\alpha\,(\beta^* - r_{t_i})(t_{i+1} - t_i) + \sigma\,\sqrt{t_{i+1} - t_i} Z_{i+1}$$

which is obtained by the discretization of dr_t.

(2) Replacing $r_{t_{i+1}} - r_{t_i}$ with

$$\left(1 - e^{-\alpha(t_{i+1} - t_i)}\right)\,(\beta^* - r_{t_i})\,+\sigma\sqrt{\frac{1}{2\alpha}(1 - e^{-2\alpha(t_{i+1} - t_i)})} Z_{i+1}$$

which is obtained by the discretization of r_t, the solution of equation 4.3.3.

Assume we are in March 16, 2004. We will predict the yield curve of March 17, 2004 with the Vasiček yield curve model by using the information of and up to March 16, 2004. To make the prediction it is needed to determine the parameters of $R(\theta\,, r_t)$ of March 16, 2004. For this purpose we will fit $R(\theta\,, r_t)$ to the yield data of that day by using the same manner as we followed in Section 4.2 with a slight difference. In Section 4.2 we exactly fit the observed data, but now we will fit the data that is generated by Nelson-Siegel model for that day. Why we use this data has the reason that we will have a finer fit since the data is increased from 20 to 800 and to decrease the optimization difficulty of Vasiček model for small number of data. For the minimization procedure we proposed a set of parameters.

This set is formed by $P = [\alpha, R_\infty, \sigma, r]$. For this parameter set we will use two sets of constraints and for each sets of constraints we will use three different sets of initial points. After obtaining the estimated values of parameters $\hat{P} = [\hat{\alpha}, \hat{R}_\infty, \hat{\sigma}, \hat{r}]$ for each of the constraint-initial point tuple, we will use this estimated parameters to predict the yield curve of March 17, 2004 by using the Vasiček yield curve *process*.

Before all, let us introduce some notations used to express these constraint-initial point tuples.

	α	R_∞
a-i	3121926696.79853	8.89897355073613e-017
a-ii	3074582904.78327	57.5663641192539
a-iii	3122613471.94644	57.5663640805259
	σ	r
a-i	1.02702656015236e-015	-8.67535580071379e-026
a-ii	1.72842316498964e-014	1.66798115571358e-007
a-iii	1.75930947185959e-014	0

Table 4.2: Estimated values of parameters for constraint set **a**.

- *r_short*; the time series of yield of bonds with time to maturity with 1 day. We gathered this series in two steps:
  1) We fitted the Nelson-Siegel model to each daily data of our data set.
  2) We collected the yield of bond with time to maturity 1 day for each day of data set.

- *yield_ns*(#); the yield of bond with time to maturity # days and obtained from the Nelson-Siegel model for the day March 16, 2004.

- μ(*yield_ns*); the mean of the bond yield data obtained from the Nelson-Siegel model for the day March 16, 2004.

- σ(*r_short*); standard deviation of *r_short*.

- β_0; the parameter value estimated by the Nelson-Siegel model for March 16, 2004.

- β_1; the parameter value estimated by the Nelson-Siegel model for March 16, 2004.

- *yield_observed_long*; observed yield value of the bond with the longest time to maturity in March 16, 2004.

- *yield_observed_short*; observed yield value of the bond with the shortest time to maturity in March 16, 2004.

The followings are the constraint-initial point sets.

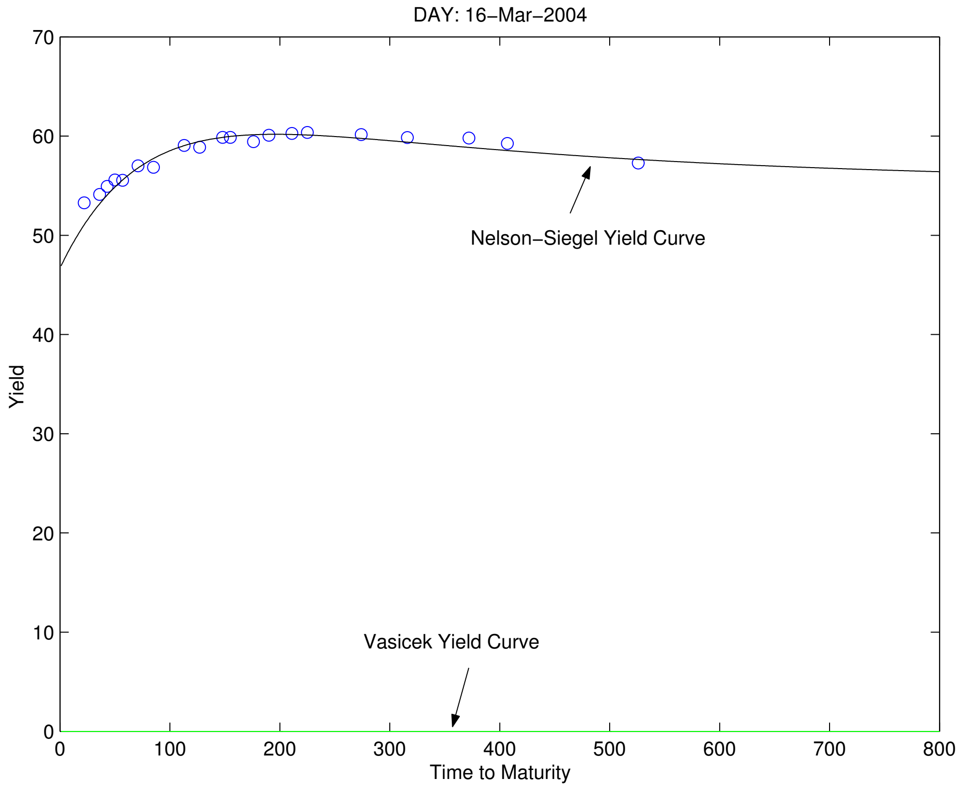

Figure 4.5: Yield Curve Fitting with the Vasiček and Nelson-Siegel Models by using constraint-initial point tuple set **a-i**.

(a) $0 < \alpha$; natural condition coming from the hypothesis of Vasiček short rate model,

$0 < R_\infty - r$;We expect long-run bond yield to be larger than short rate.

$0 < \sigma$;Positivity of σ comes from the assumptions of Vasiček model.

$0 < r < yield\_ns(1)$; it is logical to put this constraint in the sense that r is a short rate.

- **(i)** $P_{initial} = [0.2, \mu(yield\_ns), \sigma(r\_short), yield\_ns(1)]$. The result for March 16, 2004 is given in Figure 4.5.
- **(ii)** $P_{initial} = [0.2, \beta_0, \sigma(yield\_ns), \beta_0 + \beta_1]$. The result for March 16, 2004 is given in Figure 4.6.
- **(iii)** $P_{initial} = [0.2, yield\_observed\_long, \sigma(r\_short), yield\_observed\_short]$. The result for March 16, 2004 is given in Figure 4.7

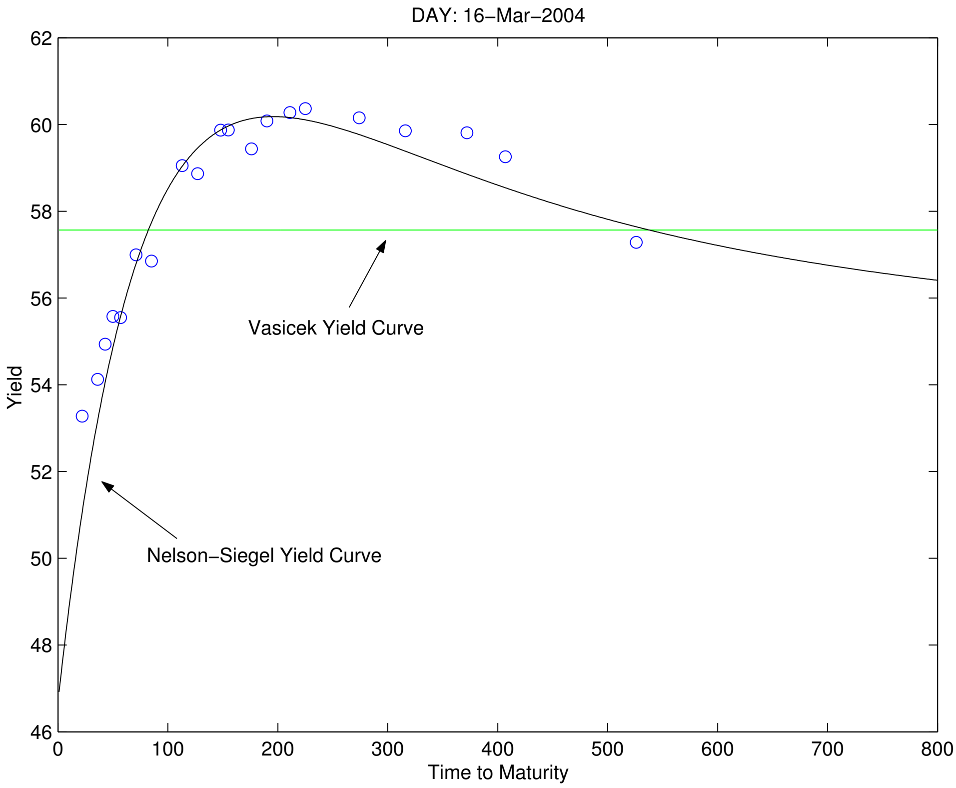

Figure 4.6: Yield Curve Fitting with the Vasiček Model by using constraint-initial point tuple set **a-ii**

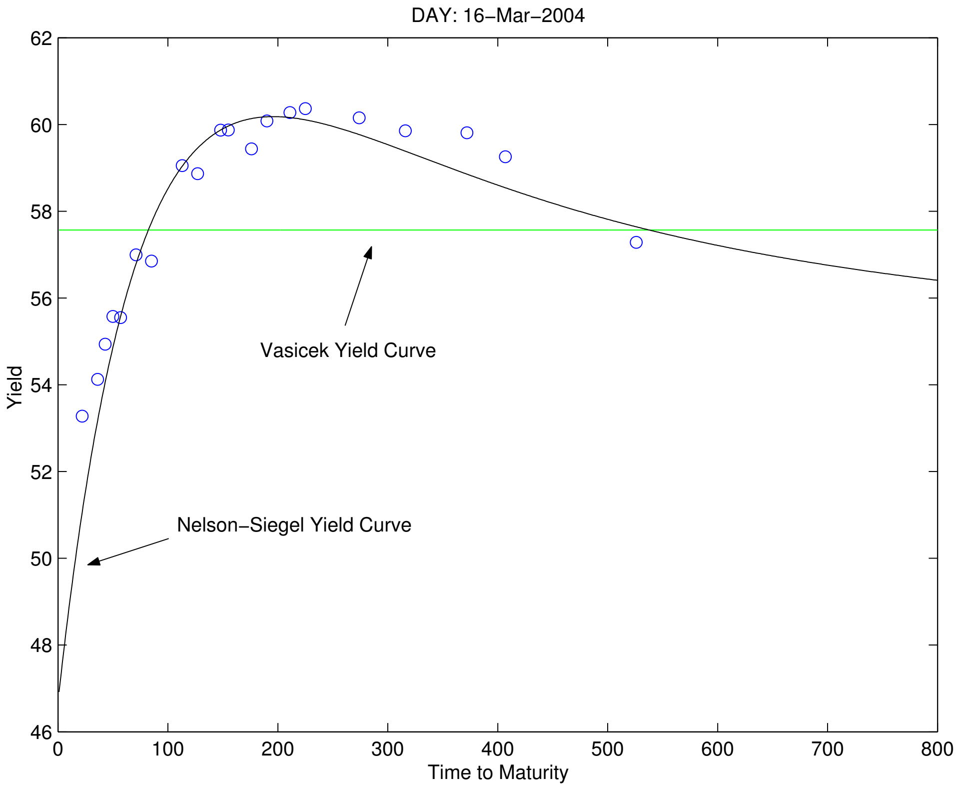

Figure 4.7: Yield Curve Fitting with the Vasiček Model by using constraint-initial point tuple set **a-iii**

	α	R_∞
b-i	0.12656452867452	58.0289089496322
b-ii	0.126565260211965	58.0289077616854
b-iii	0.126565348814999	58.0289100190384
	σ	r
b-i	3.9547173954881e-007	48.9046368960016
b-ii	2.91328054828062e-006	48.904580334028
b-iii	-1.34651426739607e-005	48.9045822601353

Table 4.3: Estimated values of parameters constraint set **b**.

Vasiček Yield Curve is constant in the Figures 4.5, 4.6, 4.7. The reason of this situation is that for each of the tuples **a-i, a-ii, a-iii** the estimated values of σ and r considerably small, and the estimated values of σ is considerably large. The Table 4.2 summarizes the estimated parameter values for constraint-initial point tuples, **a-i, a-ii, a-iii**.

(b) $0.01 < \alpha < 1$; We give an upper boundary 1 to α since α is the mean reversion rate.
$0.01 < R_\infty - r$;
$0.01 < \sigma$;
$0.01 < r < yield\_ns(1)$.
For this constraint set we also take the below boundary of the conditions 0.01 units perturbed, which provides a different initial direction vector for the minimization procedure.

- **(i)** $P_{initial} = [0.2, \mu(yield\_ns), \sigma(r\_short), yield\_ns(1)]$. The result for March 16, 2004 is given in Figure 4.8.
- **(ii)** $P_{initial} = [0.2, \beta_0, \sigma(yield\_ns), \beta_0 + \beta_1]$. The result for March 16, 2004 is given in Figure 4.9.
- **(iii)** $P_{initial} = [0.2, yield\_observed\_long, \sigma(r\_short), yield\_observed\_short]$. The result for March 16, 2004 is given in Figure 4.10.

After we changed the constraint-initial point tuples, we received better fits of Vasiček yield curve to the yield data set of March 16, 2004. The Table

	a-i	**b-i**
fold1	14.385093067632	17.6622167220191
fold2	14.385093067632	17.4400854139305

Table 4.4: Sum of Squared Errors-SSEs of Vasiček Yield Curve Estimations: given for each initial **i** point tuple and folds. Nelson-Siegel SSEs is 5.01802205688122.

	a-ii	**b-ii**
fold1	50.7273031357703	17.6622419124489
fold2	14.3850930738333	17.4401076514605

Table 4.5: Sum of Squared Errors-SSEs of Vasiček Yield Curve Estimations: given for each initial **ii** point tuple and folds. Nelson-Siegel SSEs is 5.01802205688122.

4.3 summarizes the estimated parameter values for constraint-initial point tuples, **b-i, b-ii, b-iii**.

To estimate the yield curve of *tomorrow* we will apply Monte-Carlo Method by using the discretized Vasiček yield function. We will produce a sample of 1000 throws. Then, we will take the mean of the sample. This is going to be our estimator for the yield of the bond with time to maturity θ of the day March 17, 2004. For this purpose we will use two types of discretized equation as we explained before;

$$R(\theta\,, r_{t_{i+1}}) = R(\theta\,, r_{t_i}) + \frac{\left(1 - e^{-\alpha\theta}\right)}{\alpha\theta}\left[\alpha\,(\beta^* - r_{t_i})(t_{i+1} - t_i) + \sigma\,\sqrt{t_{i+1} - t_i} Z_{i+1}\right] \tag{4.3.9}$$

	a-iii	**b-iii**
fold1	50.7273032551583	17.662240917152
fold2	14.3850930737449	17.4401091872071

Table 4.6: Sum of Squared Errors-SSEs of Vasiček Yield Curve Estimations: given for each initial **iii** point tuple and folds. Nelson-Siegel SSEs is 5.01802205688122.

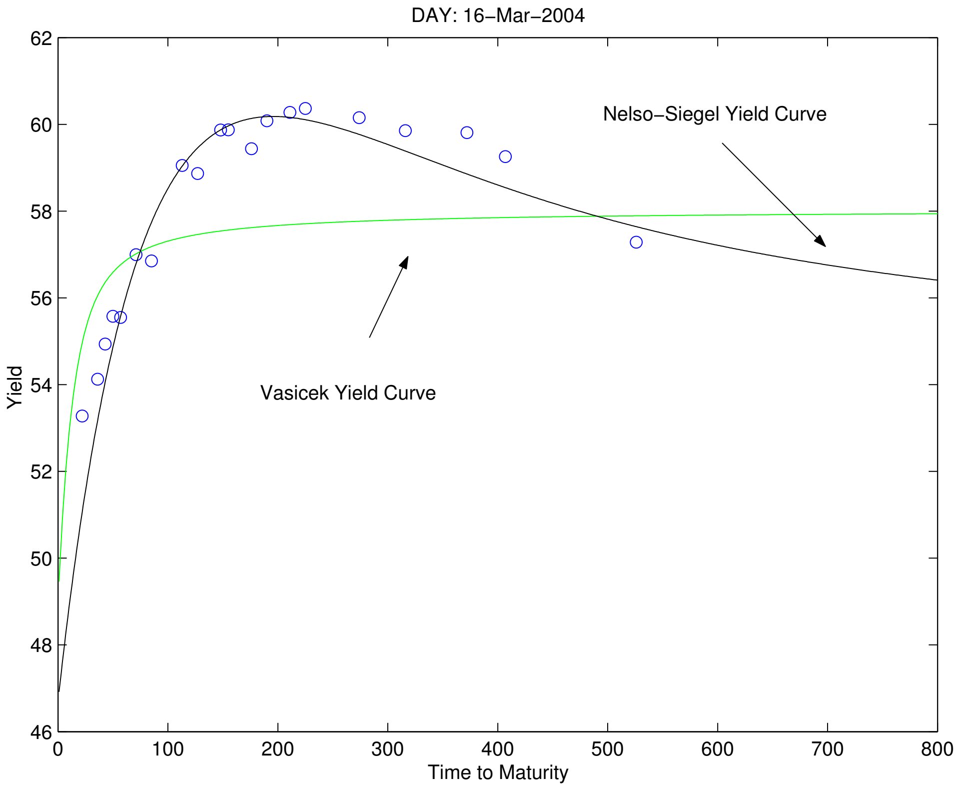

Figure 4.8: Yield Curve Fitting with the Vasiček Model by using constraint-initial point tuple **b-i**

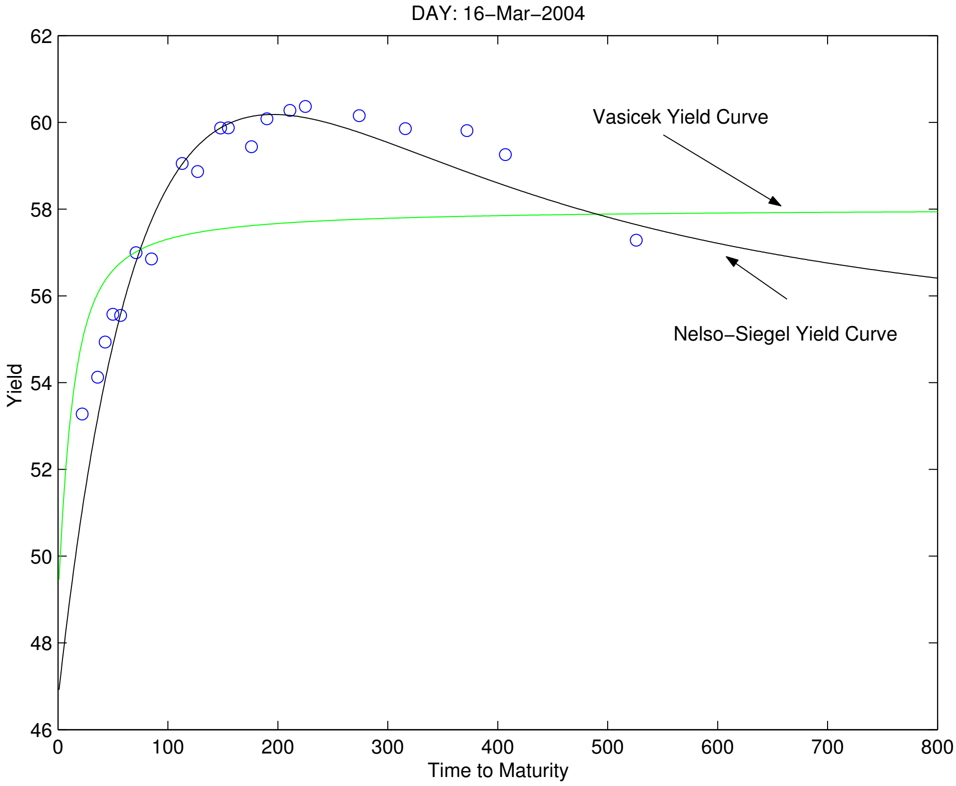

Figure 4.9: Yield Curve Fitting with the Vasiček Model by using constraint-initial point tuple **b-ii**

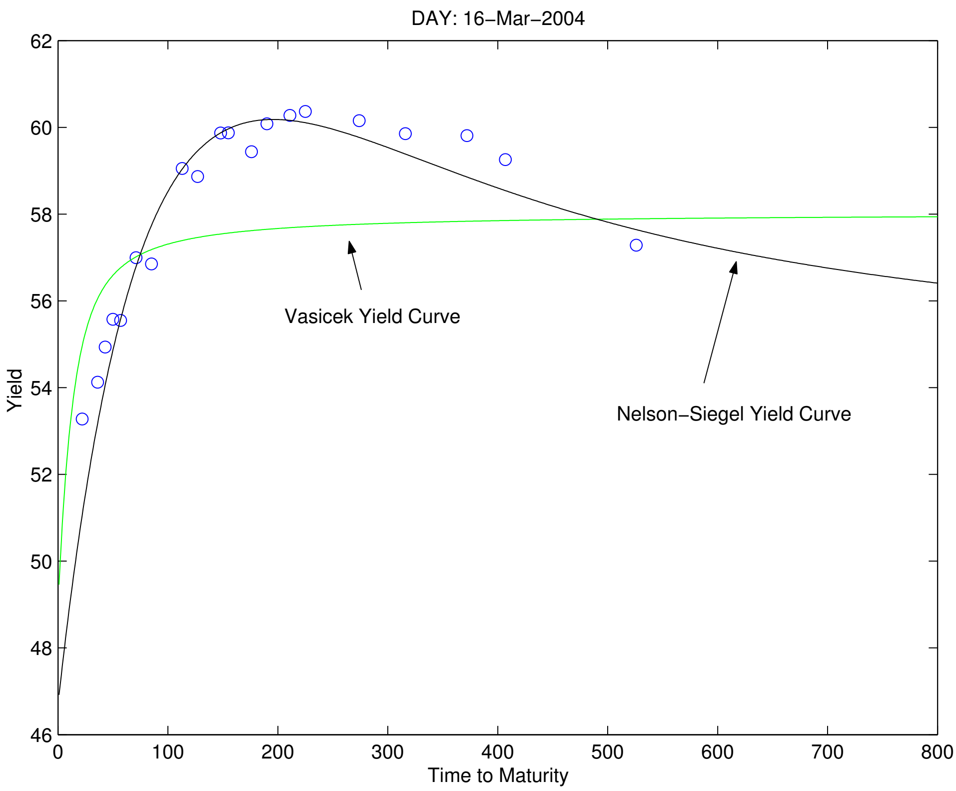

Figure 4.10: Yield Curve Fitting with the Vasiček Model by using constraint-initial point tuple **b-iii**

$$
\begin{aligned}
R(\theta\,, r_{t_{i+1}}) \;&=\; R(\theta\,, r_{t_i}) + \frac{\left(1 - e^{-\alpha\theta}\right)}{\alpha\theta} *\\
&\left[\left(1 - e^{-\alpha(t_{i+1}-t_i)}\right)\,(\beta^* - r_{t_i})\, + \sigma\sqrt{\frac{1}{2\alpha}(1 - e^{-2\alpha(t_{i+1}-t_i)})}Z_{i+1}\right]
\end{aligned}
\tag{4.3.10}
$$

When we compare the Figures 4.5 and 4.8, we see a non-zero Vasiček yield curve fitted to the data of March 16, 2004. The reason of this improve is better estimated parameter values. While we are estimating the parameter values with **b-i**, the constraint $0.01 < \alpha < 1$ gives a better estimation(Table 4.3). As the estimated value of α decreases the effect of exponential terms increase and the effect of $\frac{1}{\alpha\theta}$ decreases in amount in $R(\theta, r)$. As the effect of exponential term increases we obtain a hump in the figure. Also, we obtained the difference $R_\infty - r$ getting faraway from 0 in positive way with **b-i**. The Figures 4.9 and 4.10 can be explained in a way similar to 4.8.

Moreover, when we compare the Figures 4.6 and 4.5 obtained from **a-ii** and **a-i**, respectively, we see that the Vasiček yield curve in Figure 4.6 is a better fit for March 16, 2004 in the sense that it is not zero. But, again, it is linear with an approximately zero slope. When we look at the Table 4.2 we see that this is resulted from the fact that the large value of α decreases the effect of exponential form in $R(\theta,\, r)$. In the graph, we observe that initial value is almost equal to its asymptotic value R_∞. Although this is theoretically not true ($since\ \lim_{\theta\to 0} R(\theta,\, r) = r$), it is possible in machine size calculation: That value on the graph is calculated by taking $\theta = 1$. When this equality holds, the asymptotic value of $R(\theta, r)$ is R_∞ and this result is consistent with the zero slope value. A similar result to **a-ii** can be obtained for the Vasiček yield curve fit of **a-iii**.

The Figures 4.14 and 4.15 represent the Vasiček yield curve estimations for **a-i fold1** and **a-i fold2**. Although both of the graphs are obtained from the same constraint- initial point **a-i**, the **folds**, .i.e. discretization methods of $R(\theta, r_t)$, are different. However, the sums of squared errors for each fold are same as it is presented in Table 4.4. This result is possible for very high values of α (see Table 4.2). When we examine Figure 4.16 we see that it is a quite similar one to the graph presented in Figure 4.14 and 4.15.[2] However, the sum of squared errors of the estimation with **b-i fold2** is 17.4400854139305 which is 3.0549923462985 larger than **b-i** of both folds.

The Figures 4.17 and 4.18 represents the yield curve estimation of March 17, 2004 with **a-ii fold1** and **a-ii fold2**, respectively. Figure 4.17 represents an estimated yield curve which has a better fit for θ nearly larger than 108. However, its performance is not good for the short period. It overestimates the yield of bonds having short-term maturity. This results from the estimated parameter set of **a-ii** and discretization method **fold1**. For the small values of $\theta \geq 1$, **a-ii fold1** gives a result such that

$$R(\theta, r_{t_{i+1}}) \approx R(\theta, r_{t_i}) + R_\infty.$$

This result is not surprising since α is quite large(see Table 4.2). On the other hand in Figure 4.18, i.e. for **a-ii fold2** this approximation can not be seen because of the term $\frac{1-e^{\alpha\,\theta}}{\alpha\,\theta}$. This term decreases the value for small $\theta \geq 1$ with the help of the negative draws. The performance of the estimation with **a-ii fold1** is quite below the performance of the estimation with **a-ii fold2** by means of SSEs. This huge difference is represented in Table 4.5. Also, for not to be confused by the Figures 4.17 and 4.18, we measured the performance of the estimation of both **a-ii fold1** and **a-ii fold2** for $\theta \geq 112$ by means of SSEs. We observed that the estimation with **a-ii fold2** is again better than the estimation with **a-ii fold1** in the long-run. The measured value for **fold2** is 1.87909334435726 and for **fold1**, it is 2.84815794190374.

The collected information of performance of all the tuples is given in Tables 4.4, 4.5 and 4.6. We can conclude that constraint set **b** is much more consistent with the all three initial points and it gives an estimated yield curve similar to Nelson-Siegel fitted yield curve. However, when tomorrow comes out we see that Nelson-Sigel gives a better fit of the day with 5.01802205688122.

The most important problem here is the distributions of data occurred in both March 16, 2004 and March 17, 2004. If the both distributions are Gaussian, we

[2]The figures of constraint set **b** with **fold1** are not given since they are very similar to the figures obtained from the constraint set **b** with **fold2**.This similarity can also be seen in Tables 4.4, 4.5, and 4.6.

expect a very well fit. In our case both of our data are not Gaussian distributed. In Figures 4.11 and 4.12 the quantile-quantile plots of March 16 and March 17 are given. As it can be seen from these plots that the distributions of each of the days are not Gaussian. However, for each of the days there is are parts of the data drawn from a distribution closed to a Gaussian one. If the estimation of yield curve for March 16 is done with this part we obtain a better fit in the prediction of the yield curve of March 17. We will show this result for March 16 [3]. Let us exclude the last three yield data of bonds with the longest maturities from the whole data set of March 16, 2004 yield data. That is we exclude the yields of bonds with maturities 372, 407 and 526 days since these data points are the outliers of the data. When we apply Kolmogorov-Simirnov test to the remaining part of the data we can not reject that the data is Gaussian distributed. In the next step, we repeated all the procedures that we have followed before(estimation and prediction procedures). After obtaining the estimated parameters of Vasiček model we predict the yield curve of March 17 again with the Vasiček yield curve model. As a result, we improved our prediction by means of fitting the observed data of March 17, i.e. we decreased the SSEs approximately from 17.4401 to 16.3376(see also Figure 4.13).

[3]We repeated this procedure just for constraint-initial point tuple b-ii and for fold2 as an example of above argument.

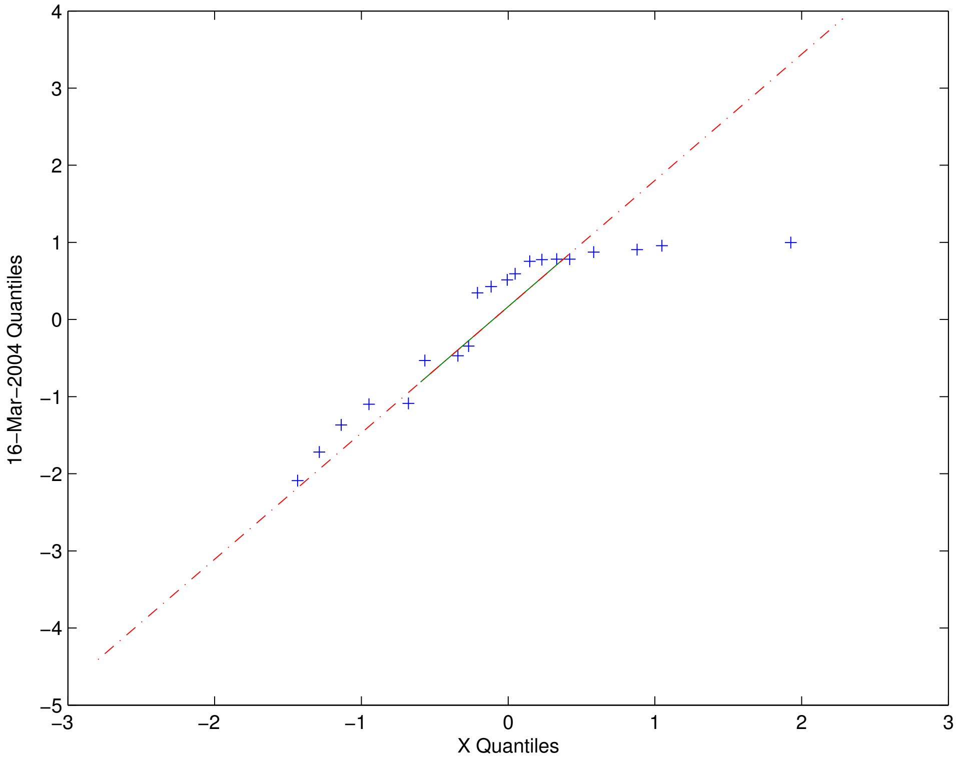

Figure 4.11: Quantile-quantile Plot of X and March 16, 2004 yield data.

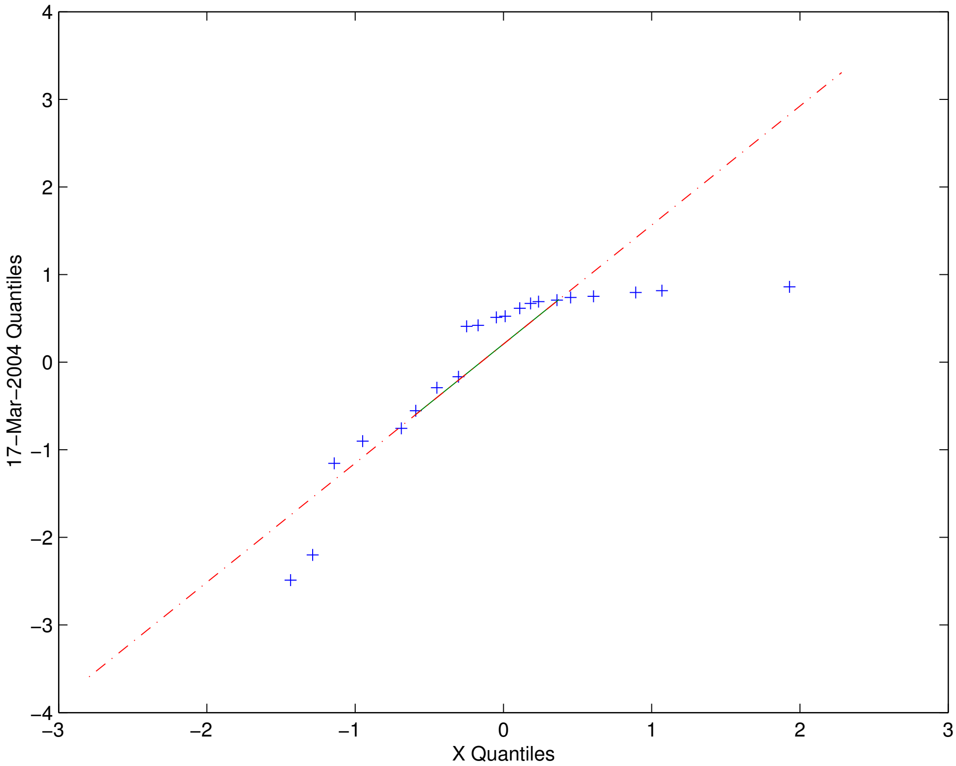

Figure 4.12: Quantile-quantile Plot of X and March 17, 2004 yield data.

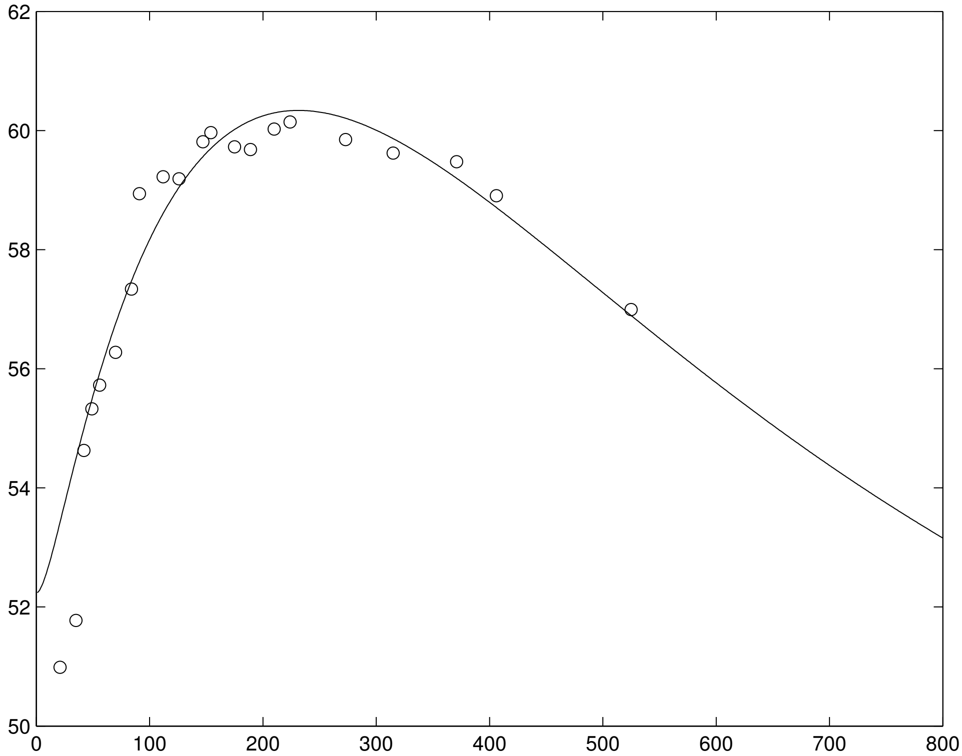

Figure 4.13: Constraint initial point tuple **b-ii** and discretization **fold2**.

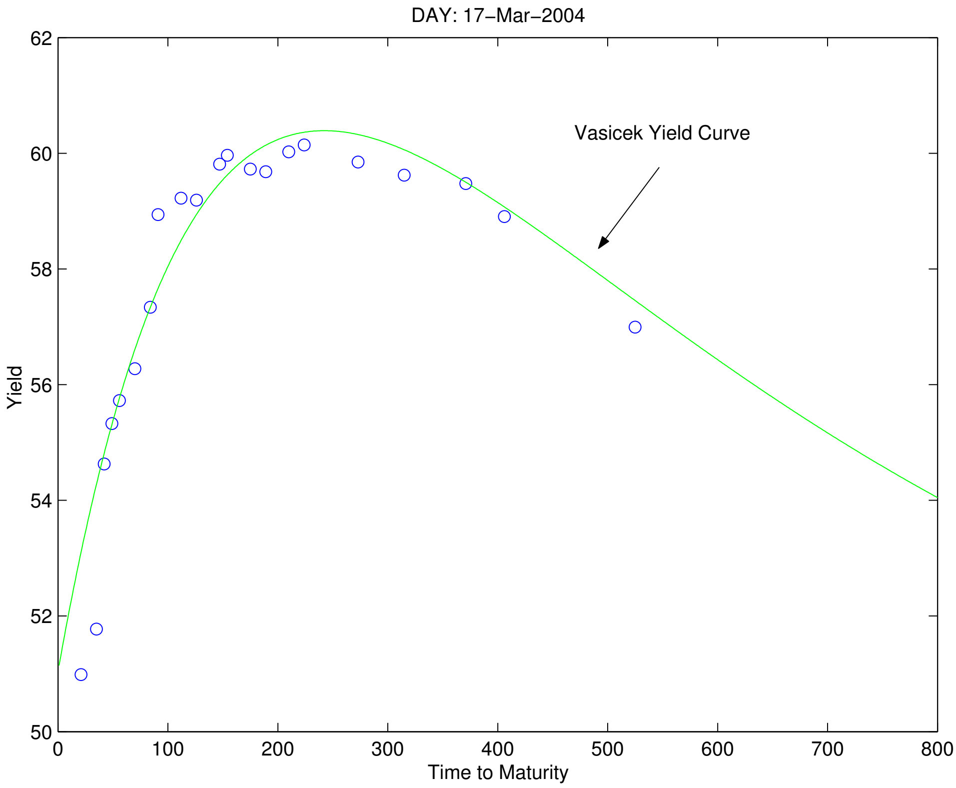

Figure 4.14: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-i** and discretization **fold-1**

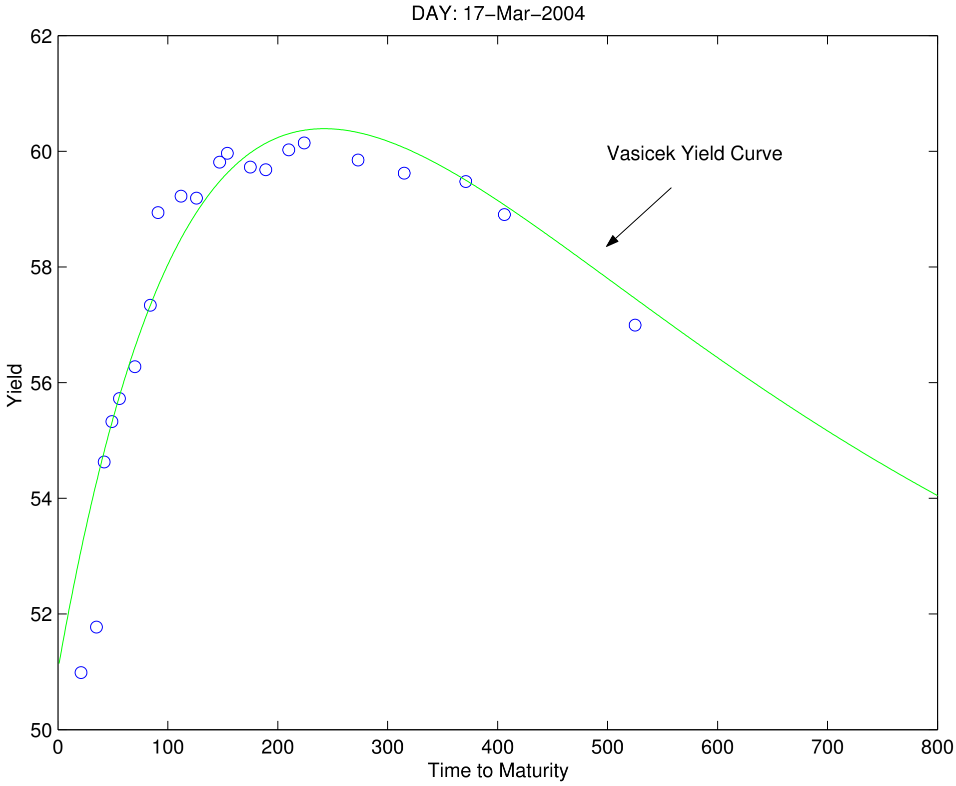

Figure 4.15: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-i** and discretization **fold-2**

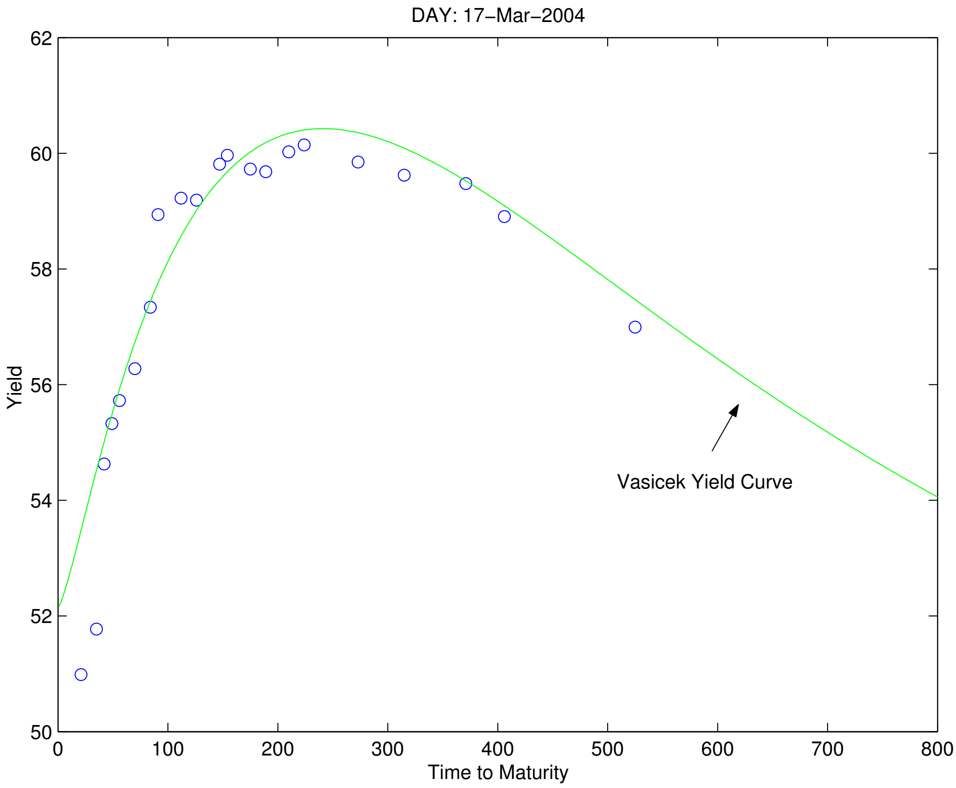

Figure 4.16: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **b-i** and discretization **fold-2**

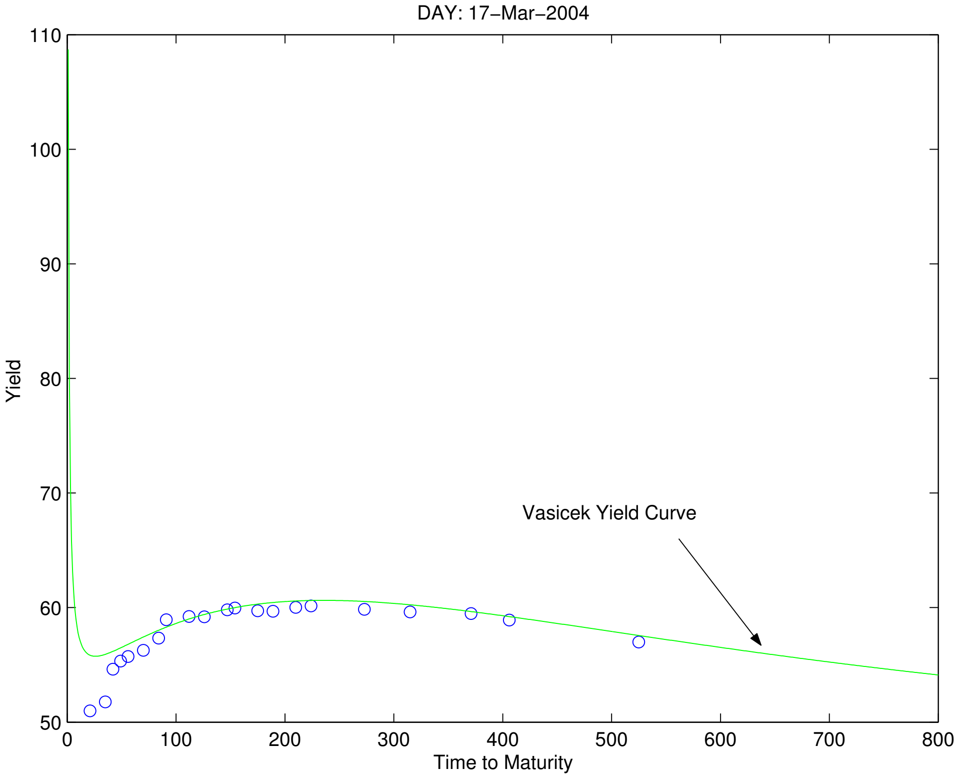

Figure 4.17: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-ii** and discretization **fold-1**

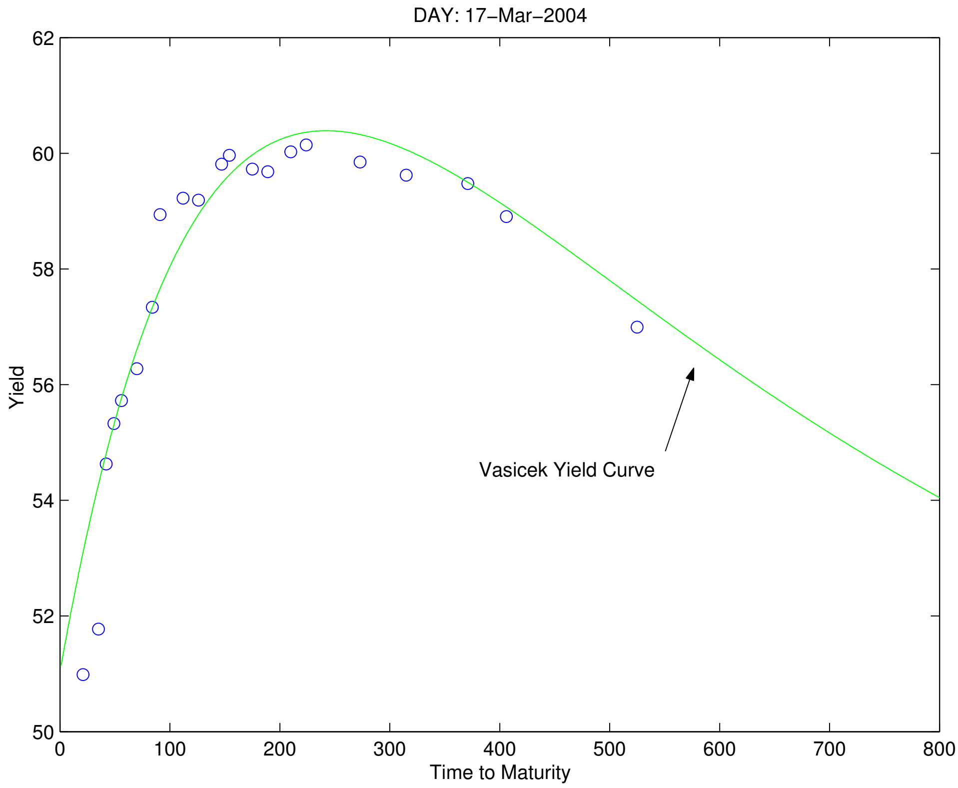

Figure 4.18: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-ii** and discretization **fold-2**

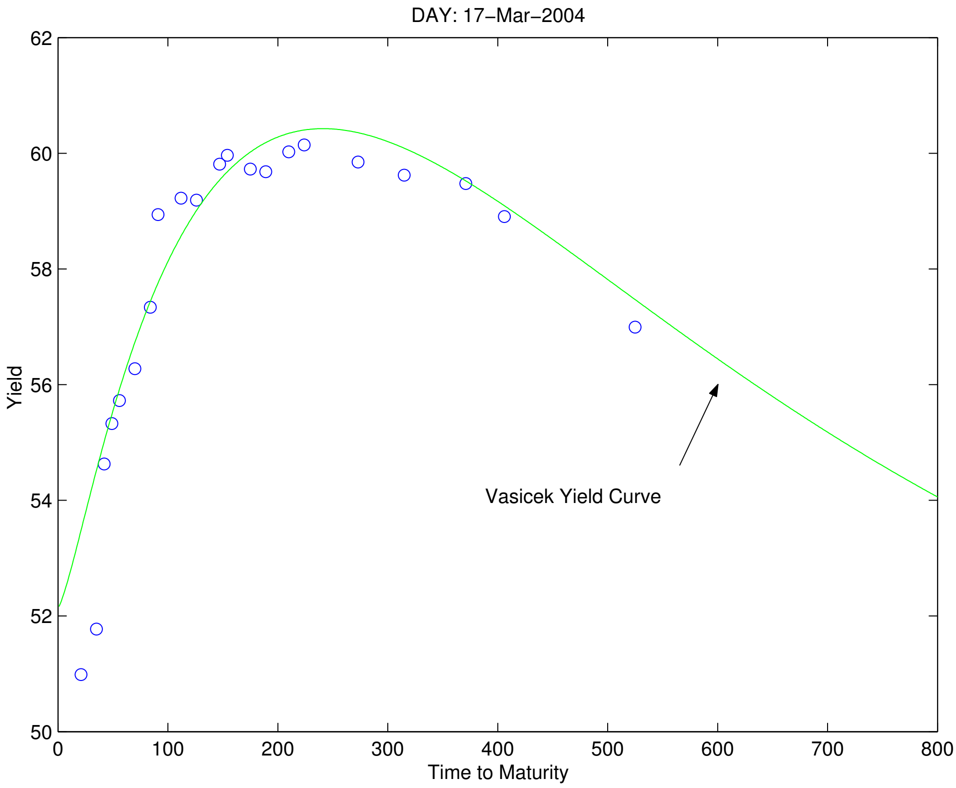

Figure 4.19: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **b-ii** and discretization **fold-2**

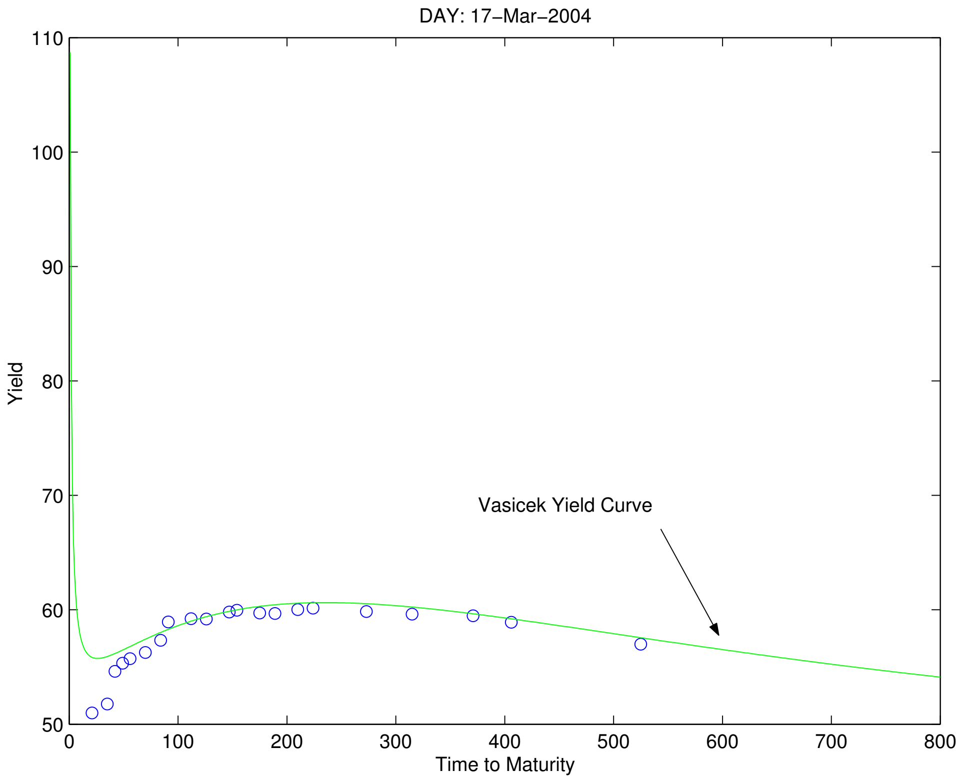

Figure 4.20: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-iii** and discretization **fold-1**

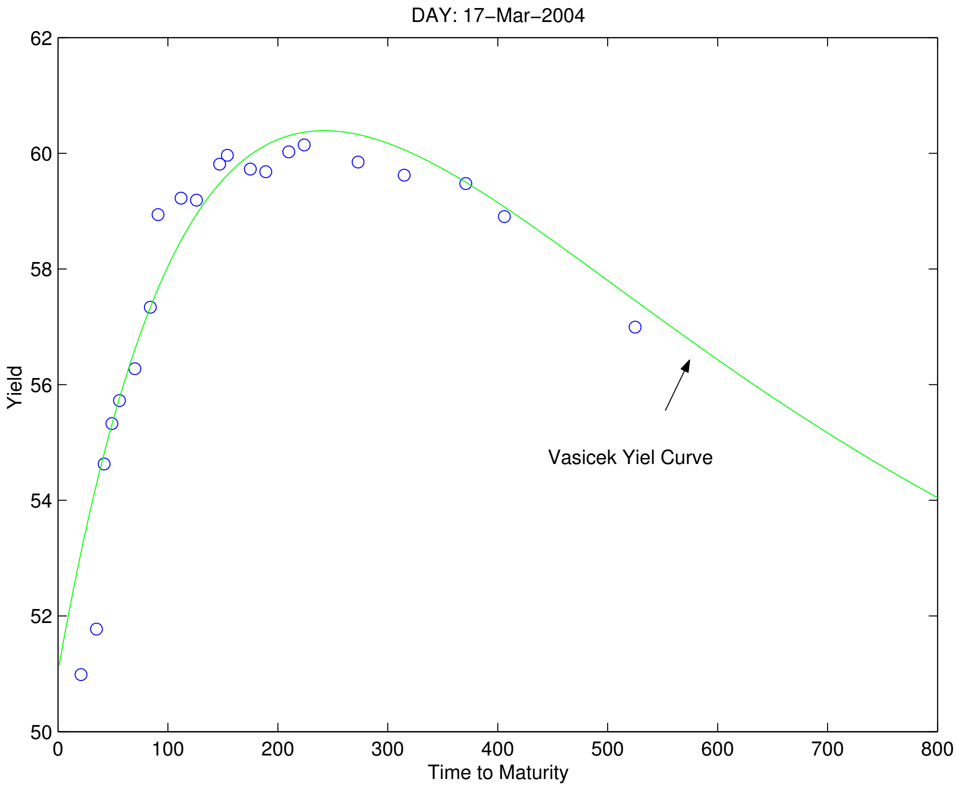

Figure 4.21: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-iii** and discretization **fold-2**

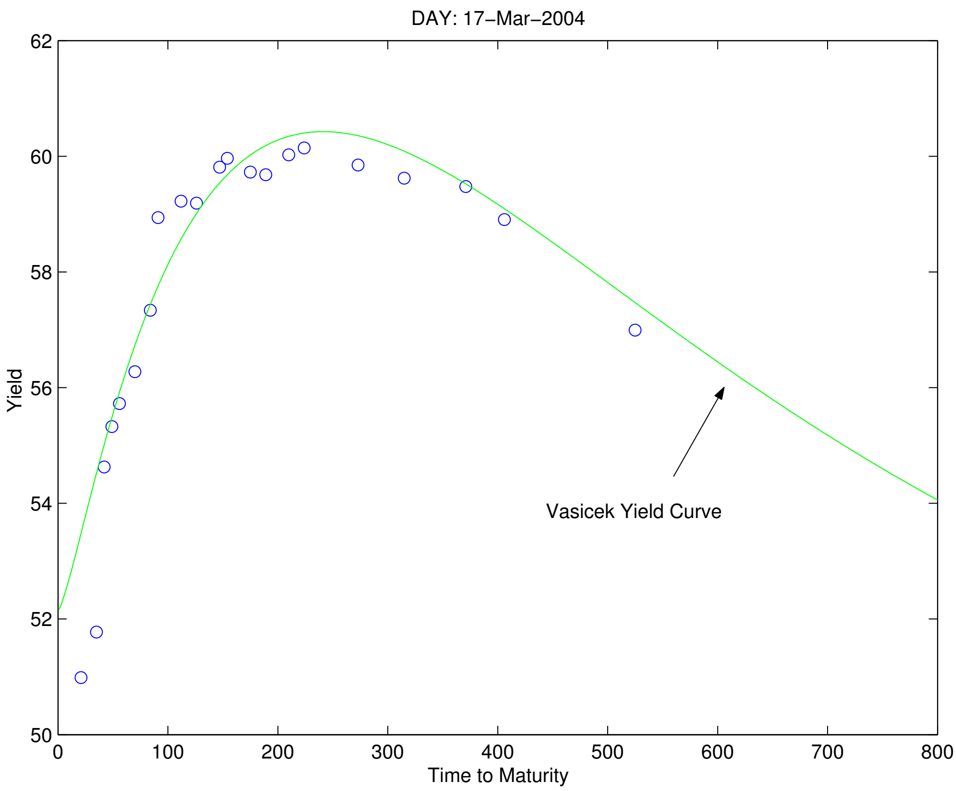

Figure 4.22: Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **b-iii** and discretization **fold-2**

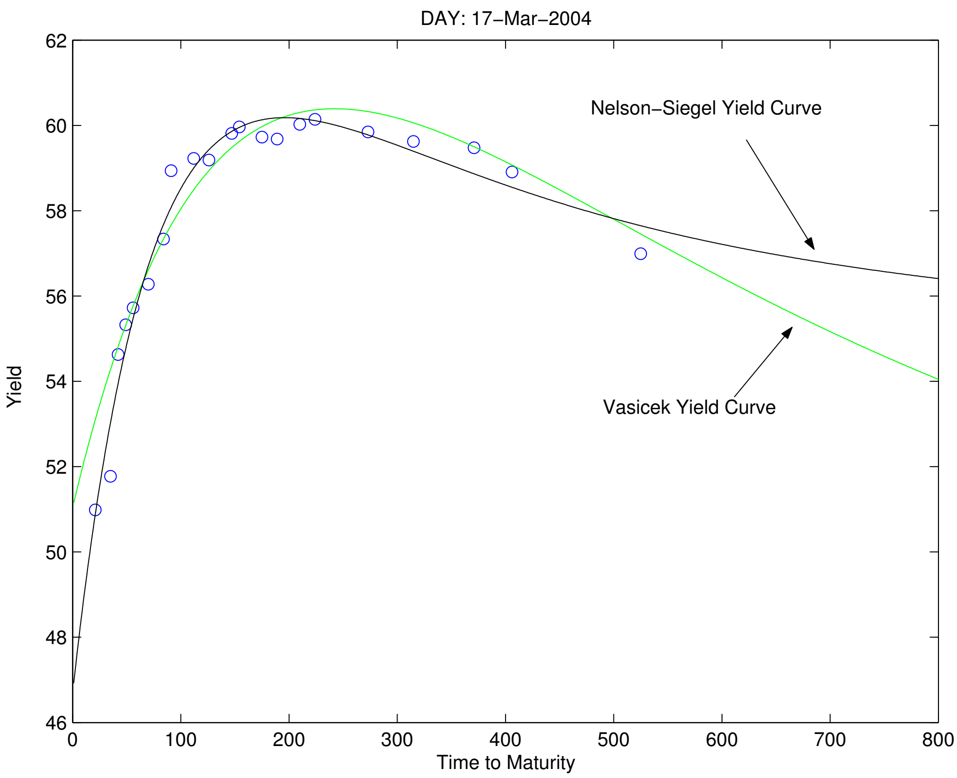

Figure 4.23: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-i** and discretization **fold-1**

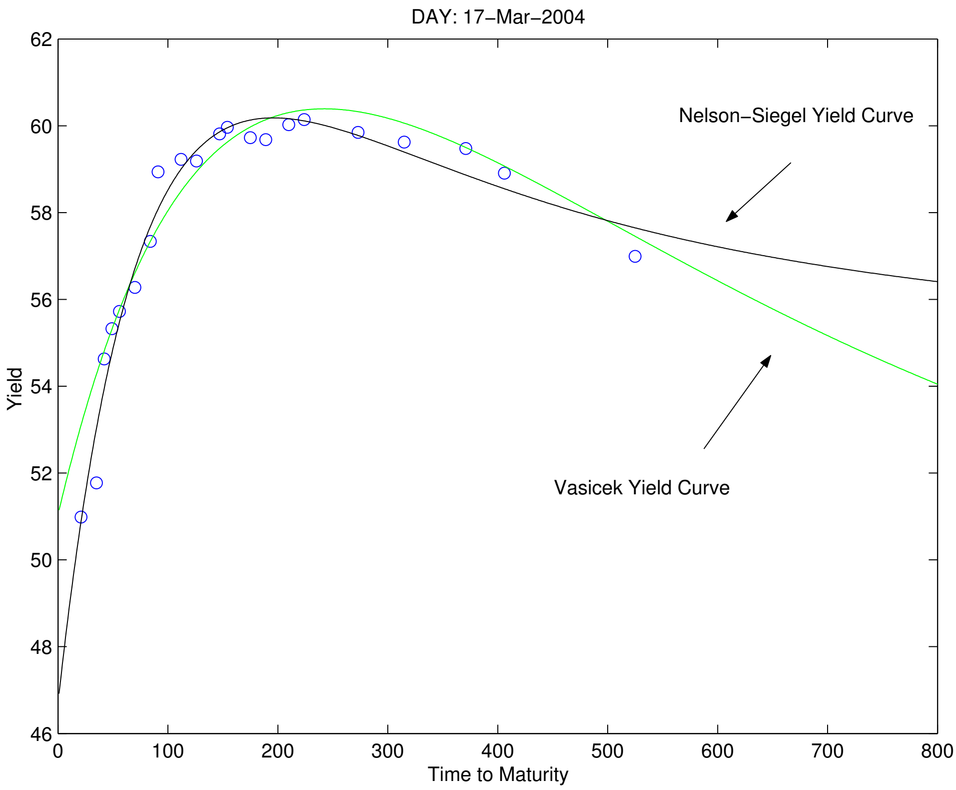

Figure 4.24: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-i** and discretization **fold-2**

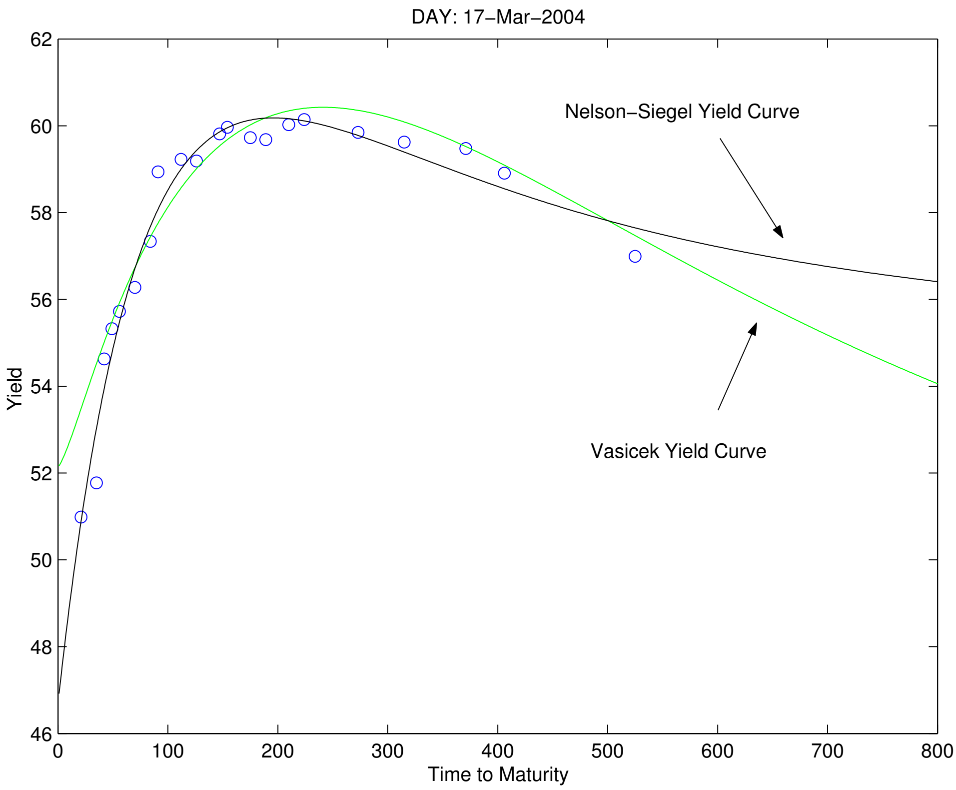

Figure 4.25: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **b-i** and discretization **fold-2**

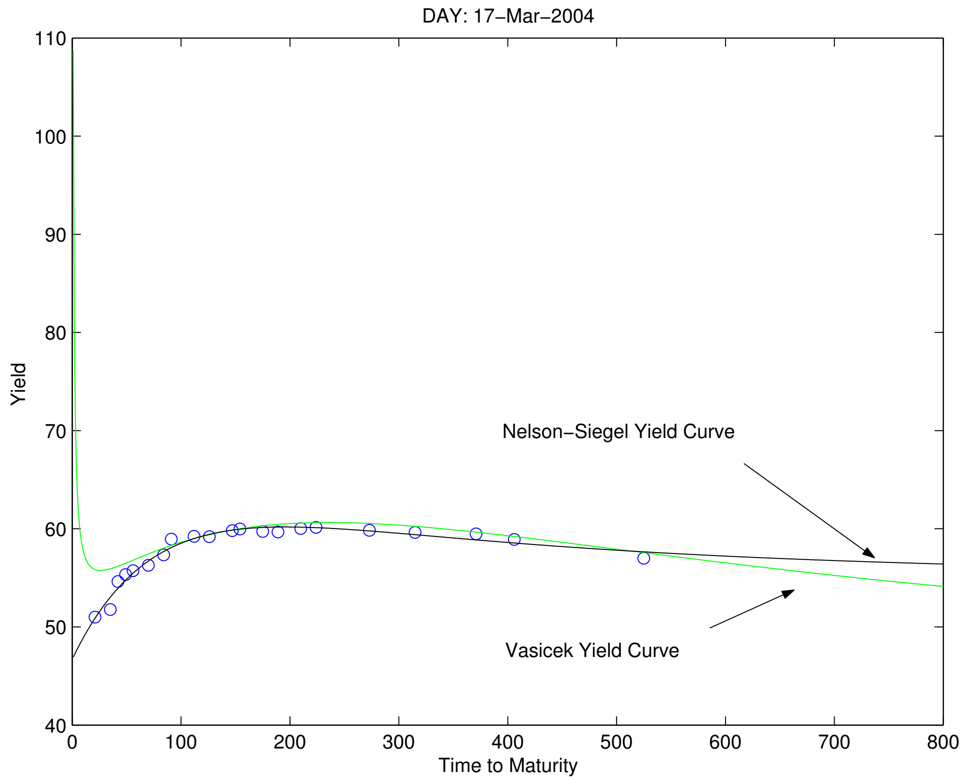

Figure 4.26: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-ii** and discretization **fold-1**

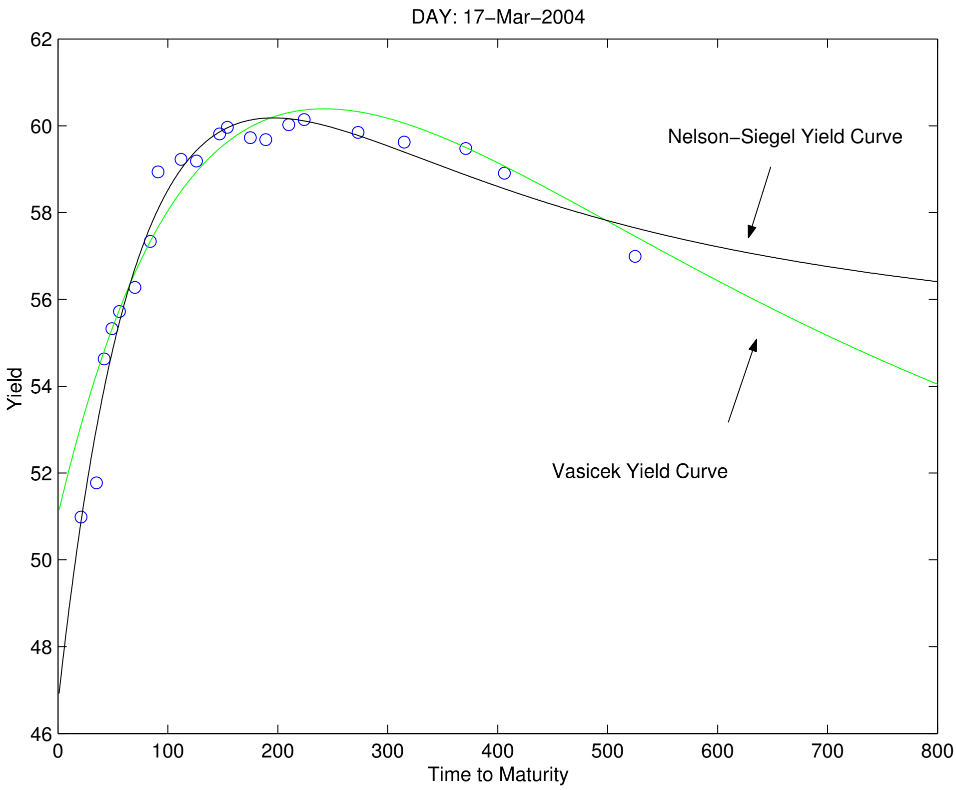

Figure 4.27: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-ii** and discretization **fold-2**

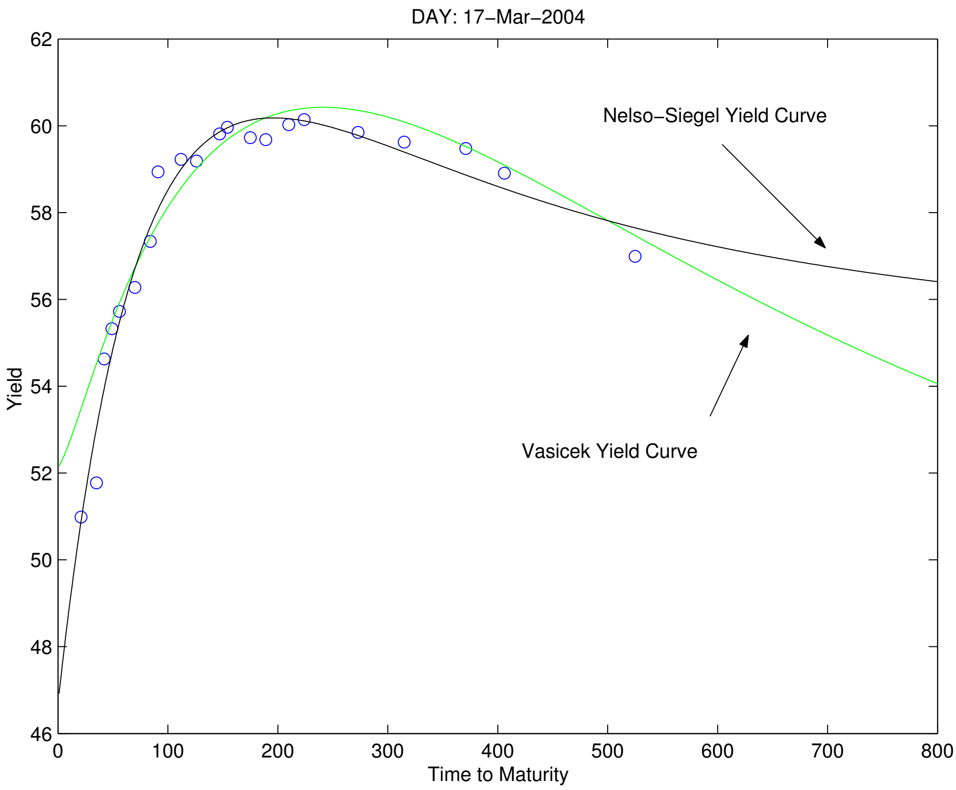

Figure 4.28: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **b-ii** and discretization **fold-2**

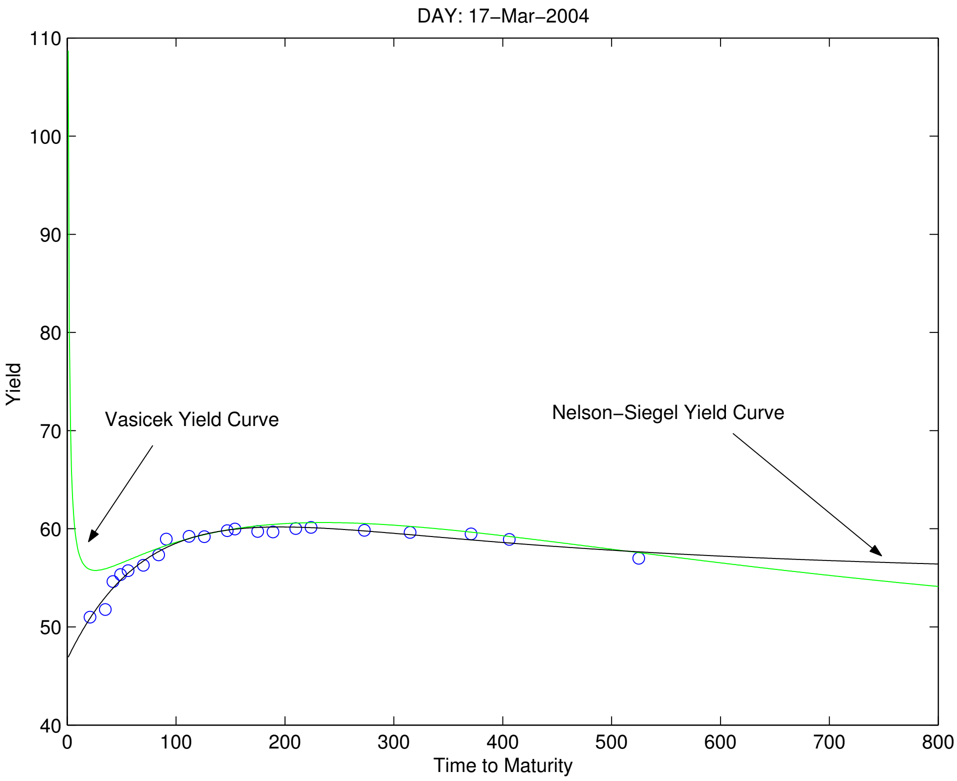

Figure 4.29: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-iii** and discretization **fold-1**

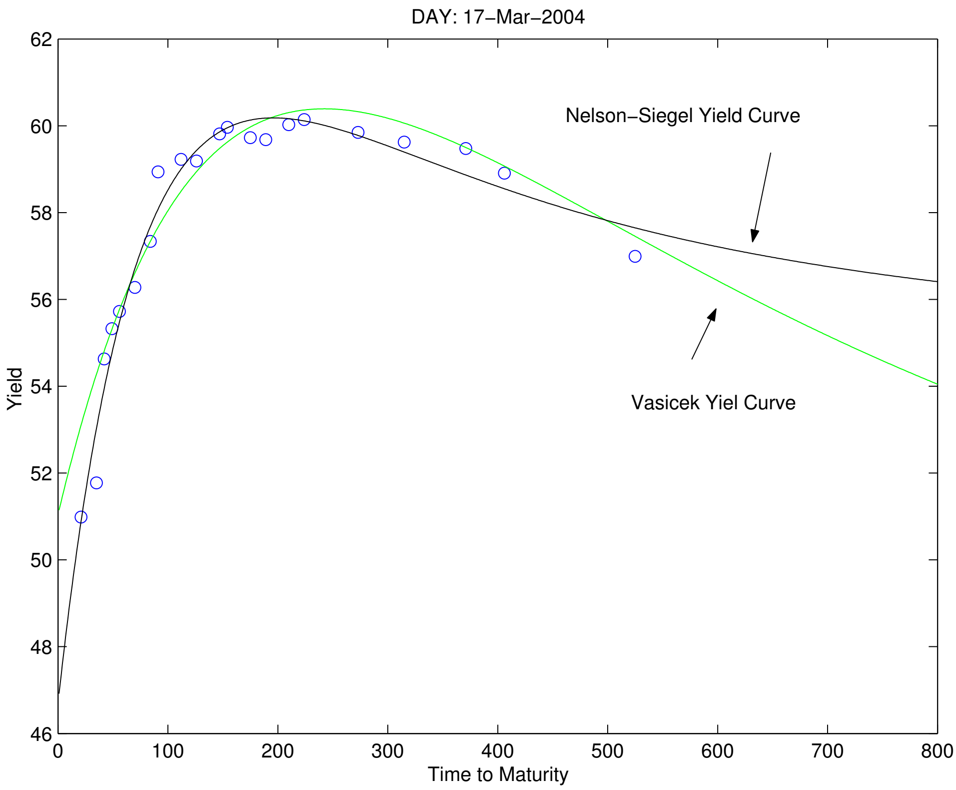

Figure 4.30: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **a-iii** and discretization **fold-2**

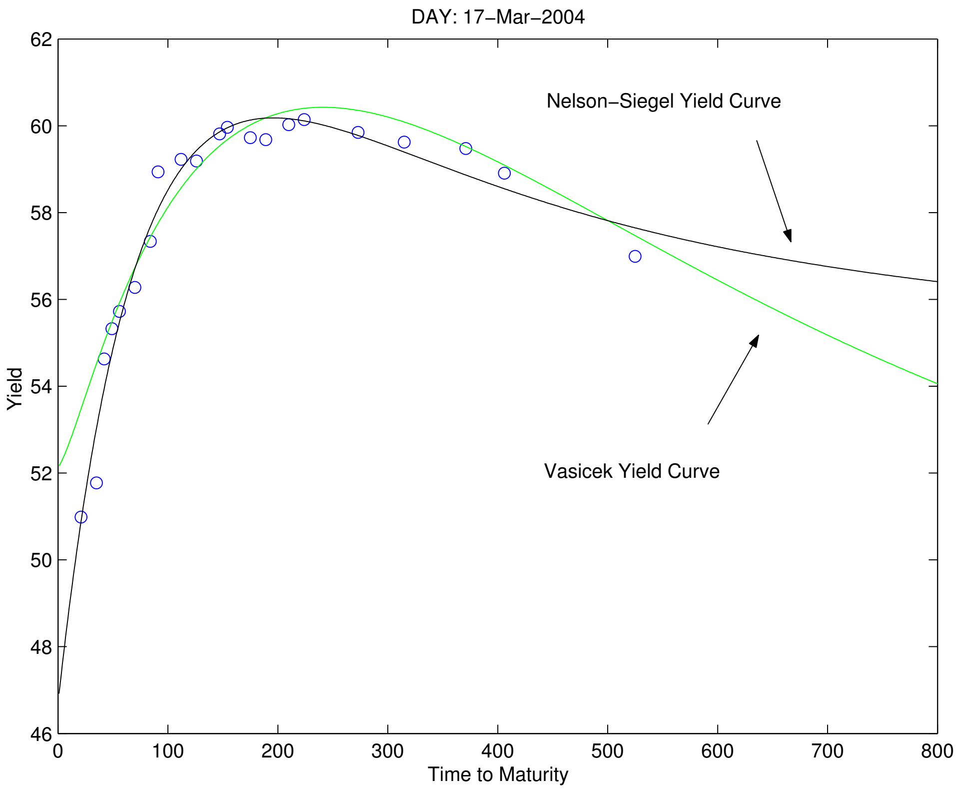

Figure 4.31: Nelson-Siegel Yield Curve Fitting, and Yield Curve Estimation with the Vasiček Model by using constraint-initial point tuple **b-iii** and discretization **fold-2**

Chapter 5

Conclusion

In this study, we presented some of the most fundamental one factor interest rate models with their solutions. Among them, Vasiček model is used to reach the main scope of this study: prediction of the yield curve of *tomorrow*. Nelson-Siegel model which is a function base yield curve model is used to complete the missing data which is Turkish zero coupon bond simple spot rate gathered for several maturities from May 1, 2001 to March 17, 2004. We discretized Vasiček short rate model, that is used in Monte Carlo method to predict the yield curve of *tomorrow*. We used two different discretization methods: discretization of 4.3.3(fold1) and discretization of 4.3.4(fold2). It is seen that discretization method fold2 results with a better prediction of yield curve of March 17 by means of smaller SSEs. In the last section, it is also shown that if the distribution of the data(*today*'s data) used to predict the yield curve of *tomorrow* is Gaussian, then we obtain a better prediction.

Although this exploratory method followed in this study can be extended to more elaborated methods, we conclude that by taking the Nelson-Siegel model as a benchmark we measured the performance of Vasiček model as a predictor and we found a considerable difference. The performance might be improved by using different constraint initial point tuples and by developing other methods in the optimization of SSEs. We also conclude that all the methods followed in this study are applicable to affine models in which the solution of bond price is explicitly obtained.

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