MATRIX QUANTUM MECHANICS AND INTEGRABLE SYSTEMS

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ABSTRACT

MATRIX QUANTUM MECHANICS AND INTEGRABLE SYSTEMS

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In this thesis we improve and extend an algebraic technique pioneered by M. Gaudin. The technique is based on an infinite dimensional Lie algebra and a related family of mutually commuting Hamiltonians. In order to find energy eigenvalues of such Hamiltonians one has to solve the equations of Bethe ansatz. However, in most cases analytical solutions are not available. In this study we examine a special case for which analytical solutions of Bethe ansatz equations are not needed. Instead, some special properties of these equations are utilized to evaluate the energy eigenvalues. We use this method to find exact expressions for the energy eigenvalues of a class of interacting boson models.

In addition to that, we also introduce a q-deformation of the algebra of Gaudin. This deformation leads us to another family of mutually commuting Hamiltonians which we diagonalize using algebraic Bethe ansatz technique. The motivation for this deformation comes from a relationship between Gaudin algebra and a spin extension of the integrable model of F. Calogero. Observing this relation, we then consider a well known periodic version of Calogero's model which is due to B. Sutherland. The search for a Gaudin-like algebraic structure which is in a similar relationship with the spin extension of Sutherland's model naturally leads to the above mentioned q-deformation of Gaudin algebra. The deformation parameter q and the periodicity d of the Sutherland model are related by the formula $q = i\pi/d$.

Keywords: Gaudin algebras, Calogero-Sutherland model, pairing models, Yang-Baxter Equation, matrix mechanics.

ÖZ

MATRİS KUANTUM MEKANİĞİ VE ENTEGRE EDİLEBİLİR SİSTEMLER

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Bu tezde, ilk olarak M. Gaudin tarafından ortaya atılmış olan cebirsel bir tekniği iyileştirmekte ve genişletmekteyiz. Bu teknik sonsuz boyutlu bir Lie cebirine ve bununla ilişkili komütatif bir Hamiltonyenler ailesine dayanır. Bu Hamiltonyenlerin enerji özdeğerlerinin bulunabilmesi için Bethe ansatz denklemleri çözülmelidir. Bununla birlikte, çoğu kez bu denklemlerin analitik çözümleri bulunamaz. Biz bu tezde, özdeğerlerin bulunabilmesi için Bethe ansatz denklemlerinin analitik çözümlerinin gerekli olmadığı özel bir durumu inceledik. Analitik çözümler yerine bu denklemlerin bazı özeliklerini kullanarak bu özel durumda enerji özdeğerlerini tam olarak hesaplayabildik. Daha sonra bu tekniği etkileşimli bir boson modelleri ailesine uygulayarak bu modellere ait Hamiltoyenlerin enerji özdeğerlerini tam olarak bulduk.

Buna ek olarak bu tezde Gaudin cebirinin bir q-deformasyonunu da tanımladık. Bu deforme cebir, yukarıdakinden farklı yeni bir komütatif Hamiltonyenler ailesi üretti ki bunları da cebirsel Bethe ansatz yöntemiyle diagonalize ettik. Bu defomasyonu tanımlama nedenimiz Gaudin cebiriyle Calogero modelinin bir spin genellemesi arasında gözlemlediğimiz bir ilişkiydi. Bu ilişkiyi gözlemledikten sonra Calogero modelinin, Sutherland tarafından tanımlanmış meşhur bir periyodik versiyonunu ele aldık. Sutherland modelinin spin genellemesiyle benzeri bir ilişki içinde bulunan ve Gaudin'inkine benzeyen bir cebirsel yapı arayışı bizi yukarıda bahsedilen q-deforme Gaudin cebirine ulaştırdı. Deformasyon parametresi q ile Sutherland modelinin periyodu d arasındaki ilişkiyi de $q = i\pi/d$ şeklinde bulduk.

Anahtar Kelimeler: Gaudin cebirleri, Calogero-Sutherland modeli, eşleşme modelleri, Yang-Baxter denklemi, matris mekanigi.

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The author dedicates this thesis to her parents Neşe and Aydoğan Pehlivan and to her friends Cemsinan Deliduman, Ümit Akıncı and Ersun Işın with profound thanks for their good humor and joy of life.

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CHAPTER 1

INTRODUCTION

This thesis is based on an algebraic technique *a lá* M. Gaudin. In 1976, Gaudin introduced a set of mutually commuting operators and diagonalized them [1] using the algebraic technique of H. Bethe [2]. These operators are referred to as Gaudin magnet Hamiltonians. Gaudin has been inspired by the work of R.W. Richardson on the BCS theory of superconductivity. In 1957, J. Bardeen, L.N. Cooper and J.R. Schrieffer (BCS) proposed a pairing type model in order to explain the superconducting behavior of metals [3]. Soon after, in 1963, Richardson developed a method to diagonalize BCS Hamiltonian [4, 5]. He showed that exact eigenstates and energy eigenvalues of BCS model can be computed as soon as one solves some given systems of highly coupled nonlinear equations. In [1], Gaudin reproduced a special limit of Richardson's equations as the equations of Bethe ansatz that one has to solve to diagonalize the system of Gaudin magnet Hamiltonians. These Hamiltonians are identified as the constants of motion of BCS model in large coupling constant limit.

The approach by Richardson and Gaudin were brilliant but not very practical because of the necessity of solving highly coupled sets of equations. As a result, their work went unnoticed for a long time. As a matter of fact, in 1997, integrability of BCS Hamiltonian is proved once more [6] and in 2000 it is diagonalized again [7], this time using conformal field theory techniques.

Unlike Richardson, Gaudin approached the problem in an algebraic way and he used some step operators to diagonalize the Gaudin magnet Hamiltonians. Afterwards, these operators were shown to be elements of an infinite dimensional Lie algebra [8, 9] which is commonly known as the (rational) Gaudin algebra. The term "rational" is used when one needs to distinguish this algebra from trigonometric and elliptic Gaudin algebras. These are also infinite dimensional Lie algebras originating from Gaudin's work. All three named above are collectively known as Gaudin algebras.

Gaudin algebras played a central role in the study of integrable models and separable systems since the work of Gaudin [10, 11, 12]. They are known to correspond to the simplest solutions of classical Yang-Baxter equation [13, 14]. Study of integrable models in statistical mechanics and two dimensional quantum field theory has shown that classical and quantum Yang-Baxter equations have central roles in the theory of integrable systems. In particular, they guarantee the existence of a one-parameter family of mutually commuting operators $H(\lambda)$. The complex parameter λ is frequently called the "spectral parameter". These operators may be identified as the integrals of motion of a quantum system as well as the trace of transfer matrices of a vertex model. Main objective in both cases is to diagonalize them simultaneously. This is achieved through functional or algebraic Bethe ansatz techniques ([15, 16, 17] are classical reviews). But to do this one still has to solve the same Bethe ansatz equations that appeared in works of Richardson and Gaudin. As a matter of fact, Gaudin magnet Hamiltonians are the residues of $H(\lambda)$ at some singular points on the real line. Gaudin magnet Hamiltonians and $H(\lambda)$ for every $\lambda \in \mathbb{C}$ commute with each other and can be diagonalized simultaneously. Analytical solutions of these Bethe ansatz equations, however, still does not exist.

In this thesis, we examine a special case for which analytical solutions of Bethe ansatz equations are not needed to find the simultaneous eigenvalues of $H(\lambda)$. In this special case one is able to find exact expressions for the eigenvalues just by utilizing some special properties of Bethe ansatz equations. We apply this technique to a class of interacting boson models and compute the energy eigenvalues in an analytical way.

In addition to that, in this thesis, we also introduce a q-deformation of the algebra of Gaudin. Motivation comes from a relation between the system of Gaudin magnet Hamiltonians and a spin extension of the integrable model of Calogero. Calogero model was first introduced in 1969 as a many-body system in one dimension with inverse square two-body interactions [18, 19, 20, 21, 22]. It is a prime example of an exactly solvable model with many interesting features.

A variant of Calogero model was introduced by B. Sutherland in 1971. First, in [22], Sutherland considered a particle moving on a one dimensional lattice and interacting with lattice points though Calogero potential. This system is equivalent to a particle moving on a line under the influence of an external inverse sine square potential. Following this, he studied a model in which particles in one dimension interact among each other with two-body inverse sine square potentials [23, 24, 25, 26]. This is known as the Sutherland model in the literature and it is closely connected to Calogero model in various ways. For example, in [23], Sutherland introduced his model as a periodic version of the system of Calogero. Alternatively, one can view Sutherland model as a Calogero model on a circle, i.e. a system of particles which are constrained to move on the circle and interacting through Calogero potential along the cord distance between them. Sutherland model can also be obtained from Calogero model by mechanical reduction [27, 28, 29]. Reduction is a powerful technique which allows one to construct new integrable models from the existing ones. Reduced system has fewer degrees of freedom and is usually more complicated from the original system. As a matter of fact, it was shown in [30] that Calogero model itself is a reduction of a trivially integrable system.

Various generalizations of the models of Calogero and Sutherland include extensions to particles with $\mathfrak{su}(\mathfrak{n})$ internal degrees of freedom [30, 31, 32, 34, 35, 36, 37]. These models are known as the spin-Calogero and spin-Sutherland models in the literature and they enjoy increasing attention due to their intimate connection to two dimensional Yang-Mills theories. Spin-Calogero and spin-Sutherland models can be obtained from Hermitian and unitary matrix models, respectively, as was first shown in [30]. An alternative formulation with exchange matrices which is due to Polychronakos [38, 39] is also very interesting (see [40] for an excellent review on the relationships between exchange matrices, matrix models and the spin extensions of the models of Calogero and Sutherland).

In this thesis, we will consider the $\mathfrak{su}(2)$ spin extensions of the models. Firstly, we will show that one can express the $\mathfrak{su}(2)$ spin-Calogero model Hamiltonian in terms of the system of Gaudin magnet Hamiltonians. This naturally arises the question of whether or not one can find a set of mutually commuting operators, like Gaudin magnet Hamiltonians themselves, that can be related to $\mathfrak{su}(2)$ spin-Sutherland model. If the answer is yes, is it also possible to go further and find an algebra which is related to these operators? Relation between Calogero and Sutherland models is illuminating at that point. We were able to show that the answer to these questions is a q-deformation of rational Gaudin algebra and a related one-parameter family of mutually commuting operators. We also find a class of simultaneous eigenstates of this family and show that residues of these operators at some singular values of the complex parameter give us a q-deformation of the system of Gaudin magnet Hamiltonians. These operators can be related to spin-Sutherland Hamiltonian. As mentioned above one view the model of Sutherland as a periodic Calogero model with periodicity d or a Calogero model on a circle with circumference d. In this case the deformation parameter q is related to d by the formula $q = i\pi/d$. In $q \to 0$ limit q-deformed algebra approaches to ordinary Gaudin algebra and this is consistent with the fact that Sutherland model approaches to Calogero model in $d \to \infty$ limit.

This thesis is organized as follows: In the next chapter, we introduce rational Gaudin algebra together with the Hamiltonians $H(\lambda)$ and the related Bethe ansatz equations. Realizations of rational Gaudin algebra in terms of $\mathfrak{su}(2)$ generators are also given. We then compute the eigenvalues of $H(\lambda)$ without solving Bethe ansatz equations in a special realization. The last section of this chapter is devoted to an application of the method. We utilize the realizations of $\mathfrak{su}(2)$ in terms of the boson creation and annihilation operators to write $H(\lambda)$ in the form of a boson pairing Hamiltonian. Energy eigenvalues then immediately follow from the results of the chapter.

Chapter 3 begins with a modest review of Calogero and Sutherland models and their spin extensions. Then, Gaudin magnet Hamiltonians are introduced and the interaction term of spin-Calogero model is expressed in terms of these operators. Next, spin-Sutherland model is considered. The reduction process which links the Calogero model to Sutherland model is imitated on the system of Gaudin magnet Hamiltonians. In this way, another set of mutually commuting operators are found and the interaction term of spin-Sutherland model is then expressed in terms of these operators. The later set of operators can be considered as q-deformations of Gaudin magnet Hamiltonians. In the last section of Chapter 3 we introduce the related q-deformation of rational Gaudin algebra.

As emphasized above, Gaudin algebras correspond to some solutions of classical Yang-Baxter equation. A substantial part of Chapter 4 is devoted to review this fact. At the end of the chapter we examine the q-deformed rational Gaudin algebra in this context.

CHAPTER 2

GAUDIN ALGEBRA

2.1 Equations of Bethe Ansatz

We will denote the rational, trigonometric and elliptic Gaudin algebras by $\mathcal{G}^{(r)}$, $\mathcal{G}^{(t)}$ and $\mathcal{G}^{(e)}$, respectively. These are all infinite dimensional complex Lie algebras generated by three one parameter families of operators $J^+(\lambda)$, $J^-(\lambda)$ and $J^0(\lambda)$ parameterized by $\lambda \in \mathbb{C}$:

$$\mathcal{G}^{(r,t,e)} \equiv span_{\mathbb{C}}\{J^+(\lambda), J^-(\lambda), J^0(\lambda) | \lambda \in \mathbb{C}\}.$$

In this thesis, we are mainly involved with[1] $\mathcal{G}^{(r)}$ for which the Lie brackets are given by

$$[J^+(\lambda), J^-(\mu)] = 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu}, \tag{2.1}$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0$$

[1] Owing to this, the term Gaudin algebra will always refer to rational Gaudin algebra henceforth, unless otherwise is stated.

for $\lambda, \mu \in \mathbb{C}$ and $\lambda \neq \mu$. The Lie brackets of $J^+(\lambda), J^-(\lambda)$ and $J^0(\lambda)$ at the same value of the complex parameter are defined as the limits of the above equations as $\mu \to \lambda$ so that Lie brackets, as functions of two complex variables, are continuous.

The operator

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda) \tag{2.2}$$

looks like an $\mathfrak{su}(2)$ Casimir operator but it is not a Casimir of $\mathcal{G}^{(r)}$ because it does not commute with the generators $J^+(\lambda), J^-(\lambda)$ and $J^0(\lambda)$. On the other hand $H(\lambda)$ is still interesting because it forms a one-parameter family of mutually commuting operators:

$$[H(\lambda), H(\mu)] = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \tag{2.3}$$

One can diagonalize them simultaneously, starting from a lowest weight vector and using $J^+(\lambda)$ as step operators. Lowest weight vector $|0>$ by definition satisfies

$$J^-(\lambda)|0>= 0, \quad \text{and} \quad J^0(\lambda)|0>= W(\lambda)|0> \tag{2.4}$$

for every $\lambda \in \mathbb{C}$. Here $W(\lambda)$ is an arbitrary complex function of λ. It is easy to show that the lowest weight vector itself is an eigenvector of $H(\lambda)$:

$$H(\lambda)|0>= E_0(\lambda)|0> .$$

Its eigenvalue is given by

$$E_0(\lambda) = W(\lambda)^2 - W'(\lambda). \tag{2.5}$$

Suppose that we act on $|0>$ with $J^+(\xi)$ for an arbitrary $\xi \in \mathbb{C}$:

$$|\xi> \equiv J^+(\xi)|0> .$$

Commutator of $H(\lambda)$ and $J^+(\xi)$ is

$$[H(\lambda), J^+(\xi)] = \frac{2}{\lambda - \xi}\left(J^+(\lambda)J^0(\xi) - J^+(\xi)J^0(\lambda)\right).$$

We see that $J^+(\xi)$ behaves like a step operator if the first term in the parenthesis on right hand side vanishes when it acts on $|0>$. This is possible only if

$$W(\xi) = 0.$$

If this is satisfied then $J^+(\xi)|0>$ is an eigenstate of $H(\lambda)$ with the eigenvalue

$$E_1(\lambda) = E_0(\lambda) - 2\frac{W(\lambda)}{\lambda - \xi}.$$

In general, we fix a natural number n which is greater than one, and then write the Bethe ansatz state

$$|\xi_1, \xi_2, \ldots, \xi_n> \equiv J^+(\xi_1)J^+(\xi_2)\ldots J^+(\xi_n)|0> . \tag{2.6}$$

This is an eigenvector of $H(\lambda)$ if the numbers $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ satisfy the Bethe ansatz equations. These are a system of n coupled equations given by

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n. \tag{2.7}$$

If $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ is a solution of above equations then (2.6) is an eigenvector of $H(\lambda)$ with the following eigenvalue:

$$E_n(\lambda) = E_0(\lambda) - 2\sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}. \tag{2.8}$$

The existence or uniqueness of solutions of Bethe ansatz equations are not assumed. The only statement is that if $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ is a solution then (2.6) is a simultaneous eigenvector of the operators $H(\lambda)$. The form of the Bethe ansatz equations depend on the function $W(\lambda)$ which in turn depends on the chosen representation because of the definition (2.4). In most cases analytical solutions to Bethe ansatz equations cannot be given and one is compelled to use numerical techniques. However, there exists some representations in which eigenvalues of $H(\lambda)$ can be calculated analytically without the necessity of solving Bethe ansatz equations. These representations are induced by a special realization which will be presented in the next section.

2.2 An Alternative Method to Compute Eigenvalues

To begin with let us consider the $\mathfrak{su}(2)$ generators $t_i^{\pm,0}$ for $i = 1, 2, \ldots, N$ which obey

$$[t_i^+, t_j^-] = 2\delta_{ij} t_j^0, \quad \text{and} \quad [t_i^0, t_j^\pm] = \pm\delta_{ij} t_j^\pm$$

so that if $\mathfrak{su}(2)_i$ denotes the $\mathfrak{su}(2)$ algebra generated by $t_i^{\pm,0}$ then

$$span_{\mathbb{C}}\{t_i^{\pm,0} | i = 1, 2, \ldots, N\} = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \cdots \oplus \mathfrak{su}(2)_N.$$

Gaudin algebra has a well known realization in terms of those $\mathfrak{su}(2)$ generators which is given by

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{t_i^0}{x_i - \lambda} \quad \text{and} \quad J^{\pm}(\lambda) = \sum_{i=1}^{N} \frac{t_i^{\pm}}{x_i - \lambda}. \tag{2.9}$$

Here $x_1, x_2, \ldots, x_N$ are arbitrary real numbers which all differ from each other. As a matter of fact, even if $x_1, x_2, \ldots, x_N$ are not real or not all different from each other, (2.9) is a realization of Gaudin algebra. But in most physical applications one is interested in those realizations for which $x_1, x_2, \ldots, x_N$ are real and all different from each other[2]. Reality of these constants guarantees that

$$J^+(\lambda)^\dagger = J^-(\lambda^*) \quad \text{and} \quad J^0(\lambda)^\dagger = J^0(\lambda^*).$$

As far as the physicists are concerned, equation (2.9) means that the Hilbert space of a system of N spin sites carries an induced representation of the Gaudin algebra. If we label the sites by $j = 1, 2, \ldots, N$ and the corresponding spin values by $s_1, s_2, \ldots, s_N$ then the Hilbert space of the system is

$$V_1^{(s_1)} \otimes V_2^{(s_2)} \otimes \cdots \otimes V_N^{(s_N)}. \tag{2.10}$$

Here $V_j^{(s_j)}$ denotes the $2s_j + 1$ dimensional irreducible representation of $\mathfrak{su}(2)_j$. The lowest weight state $|0>$ defined in (2.4) corresponds to

$$|0>= |s_1,\ -s_1> \otimes |s_2,\ -s_2> \otimes \cdots \otimes |s_N,\ -s_N>$$

[2] We strongly emphasize that representations of Gaudin algebra induced by (2.9) are not faithful. Because of this, although the elements $J^{\pm,0}(\lambda)$ and $J^{\pm,0}(\mu)$ are linearly independent for $\lambda \neq \mu$ by definition, their images given by (2.9) do not have to be so. Therefore one should always keep in mind that the equality sing in equation (2.9) only tells us that the right hand side is the image of the left hand side under a not one-to-one mapping.

for which

$$W(\lambda) = \sum_{i=1}^{N} \frac{-s_i}{x_i - \lambda}.$$

Then Bethe ansatz equations in this Hilbert space take the following form:

$$\sum_{i=1}^{N} \frac{s_i}{x_i - \xi_\alpha} + \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{n} \frac{1}{\xi_\alpha - \xi_\beta} = 0 \quad \text{for} \quad \alpha = 1, 2, \ldots, n.$$

These are the equations that confronted Gaudin in [1], Richardson in [4] and Sierra in [7]. They look like the stability conditions of a system of $N + n$ charged particles on a line. n of these particles have unit charges and are free to move on the line whereas the remaining ones have charges s_i and are fixed at the points x_i. Observing this, one can write down the corresponding Hamiltonian and look for the time independent solutions. At any rate, exact solutions to these equations are not available at the time. In this study, however, we were able to show that for the special case of $N = 2$, an analytical expression for $H(\lambda)$ eigenvalues can be given without solving Bethe ansatz equations [41]. To present our results let us first note that the Bethe ansatz equations (2.7) imply

$$\sum_{\alpha=1}^{n} W(\xi_\alpha) = 0 \tag{2.11}$$

and

$$\sum_{\alpha=1}^{n} \xi_\alpha W(\xi_\alpha) = \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} \frac{\xi_\alpha}{\xi_\alpha - \xi_\beta} = \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \beta \neq \alpha}}^{n} \frac{\xi_\alpha - \xi_\beta}{\xi_\alpha - \xi_\beta} = \frac{n(n-1)}{2} \tag{2.12}$$

no matter what the form of $W(\lambda)$ is. For $N = 2$ we have

$$W(\lambda) = \frac{-s_1}{x_1 - \lambda} + \frac{-s_2}{x_2 - \lambda}. \tag{2.13}$$

Substituting this form of $W(\lambda)$ into (2.11) and (2.12) and employing the identity

$$\frac{\xi_\alpha}{x - \xi_\alpha} = \frac{x}{x - \xi_\alpha} - 1$$

one finds

$$s_1 \sum_{\alpha=1}^{n} \frac{1}{x_1 - \xi_\alpha} + s_2 \sum_{\alpha=1}^{n} \frac{1}{x_2 - \xi_\alpha} = 0 \tag{2.14}$$

and

$$s_1 \sum_{\alpha=1}^{n} \left(\frac{x_1}{x_1 - \xi_\alpha} - 1 \right) + s_2 \sum_{\alpha=1}^{n} \left(\frac{x_2}{x_2 - \xi_\alpha} - 1 \right) = -\frac{n(n-1)}{2}, \tag{2.15}$$

respectively. Using equation (2.14) in equation (2.15) one then obtains

$$s_2 (x_1 - x_2) \sum_{\alpha=1}^{n} \frac{1}{x_2 - \xi_\alpha} = \frac{n(n-1)}{2} - (s_1 + s_2) n. \tag{2.16}$$

Let us consider now the expression

$$2 \sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha} = 2 \sum_{\alpha=1}^{n} \left[\frac{-s_1}{(x_1 - \lambda)(x_1 - \xi_\alpha)} + \frac{-s_2}{(x_2 - \lambda)(x_2 - \xi_\alpha)} \right]$$

for $n > 1$. Using equation (2.14) we can write the right hand side of above expression as follows:

$$2 s_2 \left(\frac{1}{x_1 - \lambda} - \frac{1}{x_2 - \lambda} \right) \sum_{\alpha=1}^{n} \frac{1}{x_2 - \xi_\alpha}.$$

Then, equation (2.16) tells us that

$$2\sum_{\alpha=1}^{n}\frac{W(\lambda)-W(\xi_\alpha)}{\lambda-\xi_\alpha}=\frac{-2}{(x_1-\lambda)(x_2-\lambda)}\left[\frac{n(n-1)}{2}-(s_1+s_2)n\right].$$

In conclusion, one can write eigenvalues of $H(\lambda)$ given in (2.8) as follows:

$$E_n(\lambda)=E_0(\lambda)+\frac{2}{(x_1-\lambda)(x_2-\lambda)}\left[\frac{n(n-1)}{2}-(s_1+s_2)n\right]. \tag{2.17}$$

Here $E_0(\lambda)$ is the ground state eigenvalue defined in (2.5). Substituting $W(\lambda)$ we find it as

$$E_0(\lambda)=\frac{s_1(s_1-1)}{(x_1-\lambda)^2}+\frac{s_2(s_2-1)}{(x_2-\lambda)^2}+\frac{2s_1s_2}{(x_1-\lambda)(x_2-\lambda)}. \tag{2.18}$$

This expression is independent of the solutions of Bethe ansatz equations. The formula is trivially correct for $n=0$. In addition, although the derivation above is done for $n>1$, one can easily repeat it for $n=1$ and show that the formula produces the correct result for this case as well. Therefore (2.17) is effectively true for all n.

2.3 An Exactly Solvable Model of Interacting Bosons

It is a well known procedure to construct representations of compact and noncompact Lie groups in terms of boson creation and annihilation operators. By this means, Gaudin algebra can be naturally linked to interacting boson models [41].

Specific to this section only, we will switch to the so called $\mathfrak{su}(1,1)$ basis of rational Gaudin algebra. By this we mean that the Lie brackets of the rational Gaudin algebra introduced in (2.1) will be replaced by

$$[J^+(\lambda), J^-(\mu)] = -2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu},$$

$$[J^0(\lambda), J^\pm(\mu)] = \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu}, \tag{2.19}$$

$$[J^0(\lambda), J^0(\mu)] = [J^\pm(\lambda), J^\pm(\mu)] = 0.$$

These brackets differ from the ones in (2.1) only with the minus sign in the first commutator. Since Gaudin algebra is a complex algebra this is nothing but a change of basis given by $J^\pm(\lambda) \to iJ^\pm(\lambda)$ and $J^0(\lambda) \to J^0(\lambda)$.

Then the operator $H(\lambda)$ becomes

$$H(\lambda) = J^0(\lambda)J^0(\lambda) - \frac{1}{2}J^+(\lambda)J^-(\lambda) - \frac{1}{2}J^-(\lambda)J^+(\lambda). \tag{2.20}$$

Now it looks like an $\mathfrak{su}(1,1)$ Casimir operator. It also obeys (2.3) since all we are doing is a change of basis. The Bethe ansatz equations presented in Section 2.1 are also valid for (2.19) and (2.20). That is to say, one can start with (2.19), define the lowest weight vector as in (2.4) and apply $J^+(\xi_\alpha)$ operators to obtain the higher eigenstates. Bethe ansatz equations are exactly the same.

As one expects, there exists a realization of (2.19) in terms of $\mathfrak{su}(1,1)$ generators $t_i^{\pm,0}$ for $i = 1, 2, \ldots, N$ which obey

$$[t_i^+, t_j^-] = -2\delta_{ij}t_j^0, \quad \text{and} \quad [t_i^0, t_j^\pm] = \pm\delta_{ij}t_j^\pm.$$

This realization is the same as (2.9):

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{t_i^0}{x_i - \lambda} \quad \text{and} \quad J^\pm(\lambda) = \sum_{i=1}^{N} \frac{t_i^\pm}{x_i - \lambda} \tag{2.21}$$

with the exception that $t_i^{\pm,0}$ are now $\mathfrak{su}(1,1)$ operators. As before $x_1, x_2, \ldots, x_N$ are arbitrary real numbers which all differ from each other.

Considering that the exact eigenvalues of $H(\lambda)$ are found for $N = 2$, we will fix this value for N in this section. In addition to that, we will also realize the two $\mathfrak{su}(1,1)$ algebras that appear in (2.21) with the boson creation and annihilation operators. At least two different types of bosons are needed for this. But for the sake of generality here we will consider a number of bosons and then divide them into two groups. One group of bosons, called d, are used to realize one $\mathfrak{su}(1,1)$ whereas the other group, called s, are used the realize the other. That is to say, we will deal with the operators $d_\mu, d_\mu^\dagger$ and $s_\alpha, s_\alpha^\dagger$ for $\mu = 1, 2, \ldots, n_1$ and $\alpha = 1, 2, \ldots, n_2$ which obey

$$[d_\mu, d_\nu^\dagger] = \delta_{\mu\nu} \quad [s_\alpha, s_\beta^\dagger] = \delta_{\alpha\beta}$$

Then we realize the two $\mathfrak{su}(1,1)$ algebras as follows:

$$t_1^+ = \frac{1}{2}\sum_{\mu=1}^{n_1} \Lambda_\mu d_\mu^\dagger d_\mu^\dagger, \quad t_1^- = \frac{1}{2}\sum_{\mu=1}^{n_1} \Lambda_\mu d_\mu d_\mu, \quad t_1^0 = \frac{1}{4}\sum_{\mu=1}^{n_1} \left(d_\mu^\dagger d_\mu + d_\mu d_\mu^\dagger\right),$$

$$t_2^+ = \frac{1}{2}\sum_{\alpha=1}^{n_2} \Xi_\alpha s_\alpha^\dagger s_\alpha^\dagger, \quad t_2^- = \frac{1}{2}\sum_{\alpha=1}^{n_2} \Xi_\alpha s_\alpha s_\alpha, \quad t_2^0 = \frac{1}{4}\sum_{\alpha=1}^{n_2} \left(s_\alpha^\dagger s_\alpha + s_\alpha s_\alpha^\dagger\right).$$

Here Λ_μ and Ξ_α are just constants which can be ± 1. Substituting above equations into (2.21) we find the following realization of (2.19):

$$J^+(\lambda) = \frac{1}{2}\sum_{\mu=1}^{n_1} \Lambda_\mu \frac{d_\mu^\dagger d_\mu^\dagger}{x_1 - \lambda} + \frac{1}{2}\sum_{\alpha=1}^{n_2} \Xi_\alpha \frac{s_\alpha^\dagger s_\alpha^\dagger}{x_2 - \lambda}$$

$$J^-(\lambda) = \frac{1}{2}\sum_{\mu=1}^{n_1} \Lambda_\mu \frac{d_\mu d_\mu}{x_1 - \lambda} + \frac{1}{2}\sum_{\alpha=1}^{n_2} \Xi_\alpha \frac{s_\alpha s_\alpha}{x_2 - \lambda} \tag{2.22}$$

$$J^0(\lambda) = \frac{1}{4}\sum_{\mu=1}^{n_1} \frac{d_\mu^\dagger d_\mu + d_\mu d_\mu^\dagger}{x_1 - \lambda} + \frac{1}{4}\sum_{\alpha=1}^{n_2} \frac{s_\alpha^\dagger s_\alpha + s_\alpha s_\alpha^\dagger}{x_2 - \lambda}$$

This realization of Gaudin algebra is very interesting in that it looks very similar to an $\mathfrak{su}(1,1)$ realization which is commonly used in nuclear physics. It appears in the context of the so called "*sd*-boson" model. In this model one deals with a system of bosonic particles which are able to occupy two energy levels. These levels have orbital angular momenta $L = 0$ and $L = 2$. They are traditionally called s and d levels, respectively. Introducing the boson operators $s, s^\dagger$ and $d_\mu, d_\mu^\dagger$ for $\mu = 0, \pm 1, \pm 2$, one can represent the one particle states by $s^\dagger|0>$ and $d_\mu^\dagger|0>$. Following realization of $\mathfrak{su}(1,1)$ in terms of $s, s^\dagger$ and $d_\mu, d_\mu^\dagger$ operators is commonly used in sd model:

$$S^+ = \frac{1}{2}\sum_{\mu=-2}^{2} (-1)^\mu d_\mu^\dagger d_\mu^\dagger - \frac{1}{2} s^\dagger s^\dagger,$$

$$S^+ = \frac{1}{2}\sum_{\mu=-2}^{2} (-1)^\mu d_\mu d_\mu - \frac{1}{2} ss \tag{2.23}$$

$$S^0 = \frac{1}{4}\sum_{\mu=-2}^{2}(d_\mu^\dagger d_\mu + d_\mu d_\mu^\dagger) + \frac{1}{4}(s^\dagger s + s s^\dagger).$$

Comparing above equations with (2.22) we see that choosing $n_1 = 5$ and $n_2 = 1$ equation (2.22) is a much more general mixing of s and d bosons than (2.23). Because (2.23) is just a special case of (2.22) for $x_1 = 1$, $x_2 = 0$ and $\lambda = 1/2$.

If we substitute (2.22) in (2.20) we see that $H(\lambda)$ is a pairing type Hamiltonian of s and d bosons. For simplicity we also subtract the ground state energy:

$$\begin{aligned}\tilde{H}(\lambda) \;&= \tfrac{1}{4}\left[\tfrac{\hat{N}_d}{x_1-\lambda} + \tfrac{\hat{N}_s}{x_2-\lambda}\right]^2 - \tfrac{1}{2}\left[\tfrac{\hat{N}_d}{(x_1-\lambda)^2} - \tfrac{\hat{N}_s}{(x_2-\lambda)^2}\right] \\ &+W(\lambda)\left[\tfrac{\hat{N}_d}{x_1-\lambda} + \tfrac{\hat{N}_s}{x_2-\lambda}\right] - J^+(\lambda)J^-(\lambda). \end{aligned} \tag{2.24}$$

Here $\tilde{H}(\lambda) = H(\lambda) - E_0(\lambda)$ and $\hat{N}_d$ and $\hat{N}_s$ are total number operators for the s and d type bosons, i.e.,

$$\hat{N}_d = \sum_{\mu=1}^{n_1} d_\mu^\dagger d_\mu \quad \text{and} \quad \hat{N}_s = \sum_{\alpha=1}^{n_2} s_\alpha^\dagger s_\alpha.$$

The last term in (2.24) is a pairing operator. It involves the terms $d_\mu^\dagger d_\mu^\dagger d_\nu d_\nu$ and $s_\alpha^\dagger s_\alpha^\dagger s_\beta s_\beta$ as well as the cross terms $d_\mu^\dagger d_\mu^\dagger s_\alpha s_\alpha$ and $s_\alpha^\dagger s_\alpha^\dagger d_\mu d_\mu$. Therefore $\tilde{H}(\lambda)$ allows s-pairs to annihilate each other to form s-pairs or d-pairs. Similarly d-pairs can also annihilate each other to produce s or d pairs.

For the realization given by (2.22), the lowest weight state $|0>$ corresponds to the boson vacuum for which the function $W(\lambda)$ defined in (2.4) is given by

$$W(\lambda) = \frac{1}{4}\left(\frac{n_1}{x_1-\lambda} + \frac{n_2}{x_2-\lambda}\right) \tag{2.25}$$

A comparison between the above form of $W(\lambda)$ and (2.13) shows us that eigenvalues of $H(\lambda)$ are given by (2.17) with $-s_1$ and $-s_2$ replaced by $n_1/4$ and $n_2/4$, respectively. If we also subtract the ground state energy from (2.17) we read the eigenvalues of (2.24) as

$$\tilde{E}_n(\lambda) = \frac{2}{(x_1-\lambda)(x_2-\lambda)}\left[\frac{n(n-1)}{2} + (n_1+n_2)\frac{n}{4}\right].$$

Note that for the complex values of λ, the Hamiltonians $H(\lambda)$ and $\tilde{H}(\lambda)$ are not Hermitian. For real values of λ, however, they are Hermitian and the eigenvalues $E_n(\lambda)$ and $\tilde{E}_n(\lambda)$ are real.

CHAPTER 3

CALOGERO-SUTHERLAND MODEL AND A q-DEFORMED GAUDIN ALGEBRA

3.1 Integrable Models of Calogero and Sutherland

Calogero model is an exactly solvable model which was first introduced in 1969 [18, 19, 20, 21, 22]. It is a system of N identical, non-relativistic particles moving on a line and interacting among each other with a two body potential which is inversely proportional to the square of the distance between them. If we label the particles by $i = 1, 2, \ldots, N$ then the Hamiltonian is given by

$$H^{(0)} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{g}{(x_i - x_j)^2}. \tag{3.1}$$

Here x_i and p_i denote the positions and momenta. Masses are scaled to unity and g is the interaction strength. Classically it is usually assumed to be positive in other to prevent the system from collapsing[1]. But quantum mechanically, Pauli exclusion principle allows small negative values of g.

[1] In most cases an harmonic oscillator term is also added to confine the system into a finite region of the line. The model with the external harmonic oscillator potential is called harmonic Calogero model whereas the one without any external potentials is called the free Calogero model. Here we will consider the free model because external potentials are not relevant in this discussion.

A variant of this model was introduced by B. Sutherland in 1971. First, in [22] he considered a particle moving on a line and interacting with the the points of a one-dimensional lattice though Calogero potentials. He showed that his system is equivalent to a particle moving on a line under the influence of an external inverse sine square potential. Following this, he studied a model in which particles in one-dimension interact with each other through two-body inverse sine square potentials [23, 24, 25, 26]. This is known as the Sutherland model in literature. Its Hamiltonian is given by

$$H^{(d)} = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \left(\frac{\pi}{d}\right)^2 \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{g}{\sin^2\left[\pi(x_i - x_j)/d\right]}. \tag{3.2}$$

For future convenience we define $q = i\pi/d$. Then using $\sin^2(ix) = -\sinh^2(x)$ we write (3.2) in the following form:

$$H^{(q)} = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + q^2 \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{g}{\sinh^2\left[q(x_i - x_j)\right]}. \tag{3.3}$$

The reason for superscripts (0) and (d) (or (q)) on the Hamiltonians is the following: If we consider a circle with radius r and circumference $d = 2\pi r$, the cord distance between two points on the circle is $|(d/\pi)\sin\left[\pi(x_1 - x_2)/d\right]|$. Here $x_i = r\theta_i$ and θ_i for $i = 1, 2$ are the angular coordinates of the points. As a result, one can consider Sutherland model as a Calogero model in two dimensions in which particles are constrained to move on a circle but nevertheless they interact though the cord distance between them. In $d \to \infty$ or equivalently $q \to 0$ limit $H^{(q)}$ approaches to $H^{(0)}$.

Alternatively, one can obtain Sutherland's model by reducing the Calogero model through some of its discrete symmetries [29]. In the reduction process, we start with a system whose phase space variables are $\phi = \{\phi_i\}$. We take a symmetry transformation $\mathcal{D}$ of the system and consider the subspace of the phase space for which $\phi = \mathcal{D}\phi$ is satisfied. When the system starts its motion in this subspace it always remains in it. Equations of motion cannot take the system out of this subspace. In that case, if we impose this condition into the Hamiltonian, we obtain a new system which has less degrees of freedom than the original system. If the original system is integrable, the new system is also integrable, though in most cases it is slightly more complicated than the original one. In the case of the Calogero and Sutherland models, the symmetry transformation under consideration involves a permutation of the particles and a translation. We start with a Calogero system with infinitely many particles[2]. We then fix a natural number N and label the particles with an index which is of the form $i + nN$ where $i = 1, 2, \ldots, N$ and $n \in \mathbb{Z}$. This indexing divides the system into small parts each of which includes N particles. We can write the Hamiltonian of the corresponding Calogero model as

$$\frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{i=1}^{N} p_{i+nN}^2 + \frac{1}{2}\left(\sum_{n,m\in\mathbb{Z}}\sum_{i,j=1}^{N}\right)' \frac{g}{(x_{i+nN} - x_{j+mN})^2}. \tag{3.4}$$

The prime sign on the sums in second term above means that we are excluding those terms for which $i + nN = j + mN$. We are excluding those terms because

[2] Here we closely follow [29].

they correspond to self interactions. Let $\mathcal{D}$ be a symmetry transformation which takes i^{th} particle to $(i+N)^{th}$ particle and then translates the whole system on the line by a distance of d. That the system is invariant under this transformation requires $x_{i+nN} = x_i + nd$ and $p_{i+nN} = p_i$ to be satisfied. When these conditions are satisfied by the initial data they will be satisfied at all times. If this is the case then we can write (3.4) in the following way:

$$\frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{n,m\in\mathbb{Z}}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{g}{(x_i - x_j + (n-m)d)^2} + \frac{1}{2}\sum_{\substack{n,m\in\mathbb{Z}\\ n\neq m}}\sum_{i=1}^{N}\frac{g}{(n-m)^2 d^2}. \tag{3.5}$$

Last term is just a constant and can be omitted. Using the formula

$$\frac{q^2}{\sinh^2(qx)} = \sum_{m\in\mathbb{Z}}\frac{1}{(x-md)^2}\ , \qquad q = \frac{i\pi}{d} \tag{3.6}$$

which is valid for $0 < x < d$, we can write (3.5) as

$$\sum_{n\in\mathbb{Z}}\left\{\frac{1}{2}\sum_{i=1}^{N} p_i^2 + q^2\sum_{\substack{i,j=1\\ i<j}}^{N}\frac{g}{\sinh^2[q(x_i - x_j)]}\right\} \tag{3.7}$$

which is a set of infinitely many decoupled Sutherland models.

Spin-Calogero and spin-Sutherland models [30, 31, 32, 34, 35, 36, 37] are generic names for the extensions of these models to particles with $\mathfrak{su}(q)$ internal degrees of freedom. These models are intimately connected to Hermitian and unitary Matrix models as was first shown in [30] (see Appendix C). The word "spin" is generally used in quotation marks to indicate that it does not necessarily refer to the $\mathfrak{su}(2)$ spin as in the traditional use of the term. But here we do not use quotation marks because we are actually interested in $\mathfrak{su}(2)$ case. The Hamiltonians in this case are given by

$$H^{(0)} = \sum_{i=1}^{N} \frac{1}{2} p_i + \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{g \vec{t}_i \cdot \vec{t}_j}{(x_i - x_j)^2} \tag{3.8}$$

and

$$H^{(q)} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + q^2 \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{g \vec{t}_i \cdot \vec{t}_j}{\sinh^2[q(x_i - x_j)]} \tag{3.9}$$

for the spin-Calogero and spin-Sutherland models, respectively. Here $\vec{t}_i$ refers to the spin of the i^{th} particle. The particle pairs interact with anti-ferromagnetic spin-spin coupling. This type of coupling is traditionally called the "exchange interaction."

3.2 System of Gaudin Magnet Hamiltonians

Returning to the realization of Gaudin algebra in terms of $\mathfrak{su}(2)$ generators given in (2.9) we first observe that $H(\lambda)$ in this realization is not well defined everywhere on the complex plane. In the realization (2.9), $H(\lambda)$ becomes[3]

$$H(\lambda) = \sum_{i,j=1}^{N} \frac{\vec{t}_i \cdot \vec{t}_j}{(x_i - \lambda)(x_j - \lambda)} \tag{3.10}$$

[3] The operators $H(\lambda)$ and $H(\mu)$ are, by definition, linearly independent from each other if $\lambda \neq \mu$. But the image given by (3.10) is written in terms of only N^2 linearly independent operators. The reason is the same as in footnote (2) of Chapter 2, i.e., representations induced by (2.9) are not faithful

where

$$\overrightarrow{t}_i \cdot \overrightarrow{t}_j = t_i^0 t_j^0 + \frac{1}{2}(t_i^+ t_j^- + t_i^- t_j^+). \tag{3.11}$$

We see that $H(\lambda)$ has simple poles on the real axis at $\lambda = x_i$. Although $H(\lambda)$ itself is not defined at these points its residues given by

$$h_i = -2 \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \frac{\overrightarrow{t}_i . \overrightarrow{t}_j}{x_i - x_j} \tag{3.12}$$

are well defined. These residues are called "Gaudin magnet Hamiltonians". The reason for the name is that if we are concerned with a system of N spin sites on a line with positions x_i, coupled to each other trough a pairwise ferromagnetic interaction inversely proportional to the distance, then h_i is the Hamiltonian of the i^{th} spin. Note that equation (2.3) implies

$$[H(\lambda), h_i] = 0 \quad \text{and} \quad [h_i, h_j] = 0 \tag{3.13}$$

for every $i, j = 1, 2, \ldots, N$, and $\lambda \in \mathbb{C}$.

Gaudin magnet Hamiltonians were first introduced and diagonalized in [1]. One encounters them in many different problems of theoretical physics: They are the constants of motion of the BCS Hamiltonian in the large coupling constant limit [6, 7]. They appear in the problem of separation of variables for some classes of differential equations [10, 11]. They can also be identified as the the constant of motion of Euler-Manakov top [10, 12].

In this thesis, we also point out a simple relation between the system of Gaudin magnet Hamiltonians and the spin-Calogero model introduced in the previous section. First, let us write the spin-Calogero model Hamiltonian (3.8) in the following form:

$$H^{(0)} = \sum_{i=1}^{N} \left(\frac{1}{2} p_i + V(x_i) \right)$$

Here $V(x_i)$ contains the interaction of the i^{th} particle with all other particles, i.e.

$$V(x_i) = \frac{g}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\vec{t}_i \cdot \vec{t}_j}{(x_i - x_j)^2}$$

One can write this effective potential in terms of Gaudin magnet Hamiltonians in the following way:

$$V(x_i) = \frac{g}{4} \frac{\partial h_i}{\partial x_i} = \frac{ig}{4} [p_i, h_i].$$

Here we assumed $\hbar = 1$. The spin-Calogero model Hamiltonian then becomes

$$H^{(0)} = \frac{1}{2} \sum_{i=1}^{N} \left(p_i^2 + i \frac{g}{2} [p_i, h_i] \right). \tag{3.14}$$

Looking at (3.14) and considering that Gaudin magnet Hamiltonians mutually commute, one wonders if it is also possible to find a set of mutually commuting operators $h_i^{(q)}$ which give us the spin-Sutherland model (3.9) when substituted in the similar formula

$$H^{(q)} = \frac{1}{2} \sum_{i=1}^{N} \left(p_i^2 + i \frac{g}{2} [p_i, h_i^{(q)}] \right). \tag{3.15}$$

Note that we also require

$$\lim_{q\to 0} h_i^{(q)} = h_i \tag{3.16}$$

since the Sutherland model approaches to Calogero model in $q \to 0$ limit. We know that Sutherland model can be obtained from Calogero model through mechanical reduction. Therefore the first guess would be to consider the system of Gaudin magnet Hamiltonians and apply to it the same reduction process. The key point in the reduction of Calogero model is the formula (3.6). But the spins in the Gaudin system interact with a potential which is inversely proportional to distance between them whereas the particles in spin-Calogero model interact through the square of this distance. Because of this, instead of (3.6) we need to use the formula

$$q \coth(qx) = \sum_{n\in\mathbb{Z}} \frac{1}{x+nd}\,, \qquad q = \frac{i\pi}{d} \tag{3.17}$$

which is valid for $0 < x < d$. This gives us the following operators:

$$h_i^{(q)} = -2\sum_{\substack{j=1\\ j\neq i}}^{N} q\coth\left[q(x_i - x_j)\right] \overrightarrow{t_i} \cdot \overrightarrow{t_j}\,. \tag{3.18}$$

It is quite straightforward to show that these operators do produce the Sutherland Hamiltonian when they are substituted in (3.15) and they also obey the condition

(3.16). But they do not mutually commute:

$$[h_i^{(q)}, h_j^{(q)}] = \frac{1}{2} \sum_{\substack{k=1 \\ k \neq i,j}}^{N} q^2 \left(t_i^+ t_j^- t_k^0 + t_k^+ t_i^- t_j^0 + t_j^+ t_k^- t_i^0 - t_i^+ t_k^- t_j^0 - t_k^+ t_j^- t_i^0 - t_j^+ t_i^- t_k^0 \right). \tag{3.19}$$

None the less, if we add the term which is shown below

$$h_i^{(q)} \longrightarrow h_i^{(q)} + 2q \sum_{j=1}^{N} \overrightarrow{t_i} \cdot \overrightarrow{t_j} \tag{3.20}$$

then the resulting set of operators

$$h_i^{(q)} = -2 \sum_{\substack{j=1 \\ j \neq i}}^{N} q \coth\left[q(x_i - x_j)\right] \overrightarrow{t_i} \cdot \overrightarrow{t_j} + 2q \sum_{j=1}^{N} \overrightarrow{t_i} \cdot \overrightarrow{t_j} \tag{3.21}$$

do form a commutative set and yet they continue to obey (3.15) and (3.16). Because the term we added commutes with p_i and its limit as $q \to 0$ is zero.

The operators $h_i^{(q)}$ given in (3.21) can be considered as q deformations of Gaudin magnet Hamiltonians [42]. Inspired by this observation, we wonder if we can find a q deformation of Gaudin algebra which produces the operators $h_i^{(q)}$ as the residues of a one-parameter family of mutually commuting operators. Considering the resemblance between the realization (2.9) and the Gaudin magnet Hamiltonians (3.12), we introduce the operators

$$J^{\pm,0}(\lambda) = \sum_{i=1}^{N} q \left(\coth\left[q(x_i - \lambda)\right] + 1 \right) t_i^{\pm,0}. \tag{3.22}$$

The $+1$ term is supposed to produce the term that we added in (3.20). It is easy to show that these operators obey the commutation relations which are given in the equation (3.24) of the next section. But in the next section we will forget about (3.22) and consider the above commutators as the Lie brackets of an infinite dimensional abstract Lie algebra.

3.3 A q-Deformed Gaudin Algebra

Let us consider the Lie algebra

$$\mathcal{G}_q^{(r)} = span_{\mathbb{C}}\{J^+(\lambda), J^-(\lambda), J^0(\lambda) | \lambda \in \mathbb{C}\} \tag{3.23}$$

generated three one-parameter families of operators $J^+(\lambda)$, $J^-(\lambda)$ and $J^0(\lambda)$ which obey

$$\begin{aligned}
[J^+(\lambda), J^-(\mu)] &= 2q\frac{J^0(\lambda) - J^0(\mu)}{\tanh\left[q(\lambda-\mu)\right]} + 2q\left(J^0(\lambda) + J^0(\mu)\right), \\
[J^0(\lambda), J^\pm(\mu)] &= \pm q\frac{J^\pm(\lambda) - J^\pm(\mu)}{\tanh\left[q(\lambda-\mu)\right]} \pm q\left(J^\pm(\lambda) + J^\pm(\mu)\right), \\
[J^0(\lambda), J^0(\mu)] &= [J^\pm(\lambda), J^\pm(\mu)] = 0.
\end{aligned} \tag{3.24}$$

That the Jabobi identity is satisfied is a result of the fact that the function

$$f(x) = q\coth(qx) + q \tag{3.25}$$

is a solution of

$$f(\sigma-\mu)f(\lambda-\sigma) + f(\mu-\sigma)f(\lambda-\mu) - f(\lambda-\sigma)f(\lambda-\mu) = 0. \tag{3.26}$$

$\mathcal{G}_q^{(r)}$ can be considered as a q-deformation of rational Gaudin algebra [42]. Because in $q \to 0$ limit, Lie brackets of $\mathcal{G}_q^{(r)}$ approach to Lie brackets of $\mathcal{G}^{(r)}$. Equation (3.22) is a realization of $\mathcal{G}_q^{(r)}$ in terms of $\mathfrak{su}(2)$ generators. In $q \to 0$ limit, it also approaches to the $\mathfrak{su}(2)$ realization of rational Gaudin algebra given by (2.9) [4].

Let us define

$$H^{(q)}(\lambda) = J^0(\lambda)J^0(\lambda) + \frac{1}{2}J^+(\lambda)J^-(\lambda) + \frac{1}{2}J^-(\lambda)J^+(\lambda). \tag{3.27}$$

These operators form a one-parameter family of mutually commuting operators:

$$[H^{(q)}(\lambda), H^{(q)}(\mu)] = 0 \tag{3.28}$$

We can simultaneously diagonalize them starting from a lowest weight vector which satisfies by definition

$$J^-(\lambda)|0>= 0, \quad J^0(\lambda)|0>= W(\lambda)|0> . \tag{3.29}$$

Here $W(\lambda)$ is a complex valued function of λ. $|0>$ is an eigenvector of $H^{(q)}(\lambda)$ with the eigenvalue[5]

$$E_0(\lambda) = W(\lambda)^2 - W'(\lambda) - 2qW(\lambda) \tag{3.30}$$

---

[4] $\mathcal{G}_q^{(r)}$ admits one more realization in terms of $\mathfrak{su}(2)$ generators which is given by

$$J^{\pm,0}(\lambda) = \sum_{i=1}^{N} q\left(\tanh\left[q(x_i - \lambda)\right] + 1\right) t_i^{\pm,0}.$$

But in $q \to 0$ limit this realization approaches to zero which is only a trivial realization of $\mathcal{G}^{(r)}$.

[5] See the appendix A for the details.

A vector of the form

$$|\xi >\equiv J^{+}(\xi)|0>$$

is also an eigenvector of $H^{(q)}(\lambda)$ if $\xi \in \mathbb{C}$ is a root of $W(\lambda)$:

$$W(\xi) = 0.$$

If this is the case then the eigenvalue of $|\xi >$ is given by

$$E_1(\lambda) = E_0(\lambda) - 2q\left(\coth\left[q(\lambda - \xi)\right] - 1\right) W(\lambda). \tag{3.31}$$

In general, for $n > 1$, one can write a Bethe ansatz:

$$|\xi_1, \xi_2, \ldots, \xi_n >\equiv J^{+}(\xi_1) J^{+}(\xi_2) \ldots J^{+}(\xi_n)|0> . \tag{3.32}$$

For this to be an eigenvector of $H^{(q)}(\lambda)$, the numbers $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ must satisfy the following system of equations:

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{n} q\left(\coth\left[q(\xi_\alpha - \xi_\beta)\right] - 1\right), \quad \text{for} \quad \alpha = 1, 2, \ldots, n. \tag{3.33}$$

If these equations are satisfied then the eigenvalue of $|\xi_1, \xi_2, \ldots, \xi_n >$ is given by

$$E_n(\lambda) = E_0(\lambda) - 2\sum_{\alpha=1}^{n} q\left(\coth\left[q(\lambda - \xi_\alpha)\right] - 1\right)\left(W(\lambda) - W(\xi_\alpha)\right). \tag{3.34}$$

In the representations of $\mathcal{G}_q^{(r)}$ induced by the realization (3.22), $H^{(q)}(\lambda)$ can be written as

$$H^{(q)}(\lambda) = \sum_{i,j=1}^{N} q^2\left(\coth\left[q(x_i - \lambda)\right] + 1\right)\left(\coth\left[q(x_j - \lambda)\right] + 1\right) \overrightarrow{t_i} \cdot \overrightarrow{t_j}.$$

We see that $H^{(q)}(\lambda)$ has simple poles on real axis at the points $\lambda = x_i$. It is straightforward to show that $h_i^{(q)}$ defined in (3.21) is the residue of $H^{(q)}(\lambda)$ at $\lambda = x_i$.

Let us consider a mapping on the complex plane which is of the form

$$\lambda \to f(\lambda) = \frac{\tanh(q\lambda)}{q}.$$

Defining $x = f(\lambda)$ and $y = f(\mu)$ we can then write the algebra (3.24) as following way [43]:

$$[K^+(x), K^-(y)] = 2\frac{K^0(x) - K^0(y)}{x-y} + 2q(K^0(x) + K^0(y)) + 2q^2\frac{K^0(x) - K^0(y)}{1/x - 1/y},$$

$$[K^0(x), K^\pm(y)] = \pm\frac{K^\pm(x) - K^\pm(y)}{x-y} \pm q(K^\pm(x) + K^\pm(y)) \pm q^2\frac{K^\pm(x) - K^\pm(y)}{1/x - 1/y},$$

$$[K^0(x), K^0(y)] = [K^+(x), K^+(y)] = [K^-(x), K^-(y)] = 0. \qquad (3.35)$$

Here the elements $K^{\pm,0}(x)$ are just $J^{\pm,0}(\lambda)$ expressed in terms of the new parameter x. In $q \to 0$ limit we have $f(\lambda) \to \lambda$. That the rational Gaudin algebra is the $q \to 0$ limit of $\mathcal{G}_q^{(r)}$ is more apparent in the above form. Terms proportional to q and q^2 disappear and we are left with Gaudin algebra. When $q \neq 0$, however, we have two more terms. Each term is symmetric under the exchange of the complex parameters x and y.

An interesting observation about the Lie brackets in (3.35) is that they remain invariant under the mapping $x \to 1/q^2x$.

CHAPTER 4

CLASSICAL YANG-BAXTER EQUATION

As mentioned in Introduction, Gaudin algebras correspond to some simple solutions of classical Yang-Baxter equation. Here we do not drive this result but just explain what is meant by this. We also examine the q-deformed Gaudin algebra introduced above in this context.

Let us start with a complex algebra $\mathcal{A}$ together with an associative product. Commutator with respect to this product is a Lie bracket which is denoted by $[,]$. We denote the set of $D \times D$ matrices with entries from $\mathcal{A}$ by $Mat_D(\mathcal{A})$. If $L(\lambda)$ is a $Mat_D(\mathcal{A})$ valued function of a complex variable λ, then it can be written as

$$L(\lambda) = \sum_{i,j=1}^{D} L_{ij}(\lambda) E_{ij}, \quad \lambda \in \mathbb{C}.$$

Here $L_{ij}(\lambda)$ are elements of $\mathcal{A}$ and $E_{ij} \in Mat_D(\mathbb{C})$ is a $D \times D$ matrix whose only nonzero entry is the $(ij)^{th}$ one which is equivalent to unity. We denote the identity element of $\mathcal{A}$ by $\mathcal{I}$. Then

$$I = \sum_{i,j=1}^{D} \delta_{ij} \mathcal{I} E_{ij}$$

is the $\mathcal{I}$ times the $D \times D$ identity matrix. Given a $Mat_D(\mathcal{A})$ valued function $L(\lambda)$

we define

$$[L(\lambda) \otimes_, L(\mu)] = [L(\lambda) \otimes I, I \otimes L(\mu)]$$

Here the brackets [,] denote the matrix commutator which respects the non-commutativity of the matrix elements since the matrices are $\mathcal{A}$ valued. We use the same symbol for matrix commutator and the Lie bracket of $\mathcal{A}$ but which operation is meant is clear from the operands.

It takes some algebra to show that

$$[L(\lambda) \otimes_, L(\mu)] = \sum_{i,j,k,l=1}^{D} [L_{ij}(\lambda), L_{kl}(\mu)] E_{ij} \otimes E_{kl}.$$

We see that $[L(\lambda) \otimes_, L(\mu)]$ is an element of $Mat_D(\mathcal{A}) \otimes Mat_D(\mathcal{A})$ whose components are of the form $[L_{ij}(\lambda), L_{kl}(\mu)]$. This means that $[L(\lambda) \otimes_, L(\mu)]$ is equal to an expression involving the structure constants of the algebra $\mathcal{A}$ and terms linear in $L_{ij}(\lambda)$ and $L_{kl}(\mu)$. Suppose that we can find a $Mat_D(\mathbb{C}) \otimes Mat_D(\mathbb{C})$ valued function $r(z)$ of a complex variable z which satisfies

$$[L(\lambda) \otimes_, L(\mu)] + [r(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)] = 0. \qquad (4.1)$$

If such an r-matrix exists[1] it contains information about structure constants of $\mathcal{A}$. Consequently, $r(z)$ has to obey an equation imposed by Jacobi identity. Before writing down this equation let us introduce the notation. We define three embeddings of $Mat_D(\mathbb{C}) \otimes Mat_D(\mathbb{C})$ into $Mat_D(\mathbb{C}) \otimes Mat_D(\mathbb{C}) \otimes Mat_D(\mathbb{C})$ as

[1] It is important to note that it may not be found for every $\mathcal{A}$.

follows:

$$\varphi_{12} : a \otimes b \longrightarrow a \otimes b \otimes I,$$

$$\varphi_{13} : a \otimes b \longrightarrow a \otimes I \otimes b,$$

$$\varphi_{23} : a \otimes b \longrightarrow I \otimes a \otimes b, \tag{4.2}$$

where $a, b \in Mat_D(\mathbb{C})$. We denote the images of $r(z)$ under these mappings by

$$\varphi_{12} r(z) = r^{12}(z), \quad \varphi_{13} r(z) = r^{13}(z), \quad \varphi_{23} r(z) = r^{23}(z).$$

Then condition imposed by Jacobi identity on $r(z)$ is

$$[r^{12}(\lambda-\mu), r^{13}(\lambda-\sigma)] + [r^{12}(\lambda-\mu), r^{23}(\mu-\sigma)] + [r^{13}(\lambda-\sigma), r^{23}(\mu-\sigma)] = 0. \tag{4.3}$$

This is the classical Yang-Baxter equation (Although the name "Yang-Baxter equation" and the notion of r-matrix goes back to late 1970's , their connection with the algebraic structures as described above was first considered in [44]. See [45] for a review on the history of the subject.)

To give some simple solutions to this equation let us assume that $D = 2$. In this case $r(z)$ is a 4×4 matrix. If one assumes a solution of classical Yang-Baxter equation which is of the form

$$r(z) = \begin{pmatrix} f(z) & 0 & 0 & 0 \\ 0 & 0 & g(z) & 0 \\ 0 & g(z) & 0 & 0 \\ 0 & 0 & 0 & f(z) \end{pmatrix} \tag{4.4}$$

such that $f(z)$ and $g(z)$ are arbitrary complex valued functions, then (4.3) tells us that these functions have to obey [2,3]

$$g(z)g(w) = g(z+w)(f(z)+f(w))$$

$$f(z)g(w) = g(z+w)g(z) + f(z+w)g(w). \tag{4.5}$$

One solution to these equations is

$$f(z) = g(z) = \frac{1}{z}. \tag{4.6}$$

This is the solution of classical Yang-Baxter equation which is related to rational Gaudin algebra [14]. To see this let us take $\mathcal{A} = \mathcal{G}^{(r)}$ and

$$L(\lambda) = \begin{pmatrix} J^0(\lambda) & J^+(\lambda) \\ J^+(\lambda) & -J^0(\lambda) \end{pmatrix}. \tag{4.7}$$

Then it is fairly straightforward to show that (4.1) together with the r-matrix which is given by (4.4) and (4.6) is equivalent to (2.1). Note also that $H(\lambda)$ is equivalent to $Tr(L(\lambda)^2)$. That the trace of the powers of $L(\lambda)$ commute is typical

[2] To find $r^{12}(z)$, $r^{23}(z)$ and $r^{13}(z)$ one should write $r(z)$ in a form which is suitable for (4.2). One suggestion is

$$r(z) = f(z)(2\sigma^0 \otimes \sigma^0 + \frac{1}{2} I \otimes I) + g(z)(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+)$$

where $\sigma^{\pm,0}$ are Pauli spin matrices and I is 2×2 identity matrix.

[3] Note that if we assume f(z)=g(z), these equations tell us that

$$f(z+w)f(z) + f(z+w)f(w) - f(z)f(w) = 0.$$

This is similar to, but not the same as (3.26). Equation (3.26) is the same as the above equation only when the function $f(x)$ in (3.26) is an odd function which is not the case for $f(x)$ given in (3.25).

for the algebras for which an r-matrix satisfying classical Yang-Baxter equation can be found.

Another solution of equations (4.5) which is given by

$$f(z) = p \coth pz, \quad g(z) = \frac{p}{\sinh pz} \tag{4.8}$$

corresponds to trigonometric Gaudin algebra which is introduced in Appendix B. If we take $\mathcal{A} = \mathcal{G}^{(t)}$ then (4.1) together with the r-matrix given by (4.4) and (4.8) is equivalent to the Lie brackets of trigonometric Gaudin algebra given in (B.1).

As far as $\mathcal{G}_q^{(r)}$ is concerned, the operator $H^{(q)}(\lambda)$ is also equivalent to $Tr(L(\lambda)^2)$. But the Lie brackets given in (3.24) or in (3.35) cannot be written in the form of (4.1). In other words there is no r-matrix[4] for $\mathcal{G}_q^{(r)}$. This is simply because the right hand sides of (3.24) and (3.35) involve terms which are proportional to both the sum and the difference of the generators. Rational and trigonometric Gaudin algebras involve only the difference and consequently they can be put in the form of equation (4.1).

Nevertheless, one can easily show that Lie brackets of $\mathcal{G}_q^{(r)}$ can be written in the following form:

$$\begin{aligned} &[L(\lambda) \otimes I, I \otimes L(\mu)] + [r(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)] \\ &+[s(\lambda + \mu), L(\lambda) \otimes I - I \otimes L(\mu)] = 0. \end{aligned} \tag{4.9}$$

[4] At least with $L(\lambda)$ given by (4.7).

Here $L(\lambda)$ is again given by (4.7) and the matrices $r(z)$ and $s(z)$ are as follows:

$$r(z) = \frac{q}{\tanh(qz)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s(z) = q \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

CHAPTER 5

CONCLUSION

Using the algebraic technique of M. Gaudin one can evaluate eigenvalues and eigenstates of the Hamiltonians $H(\lambda)$. Although in this thesis we presented only the rational Gaudin algebra in detail, trigonometric and elliptic Gaudin algebras also have associated one-parameter families of commuting operators $H(\lambda)$ and diagonalization of these operators follow very similar lines. It is clear that what lies under the success of the formalism is its connection to classical Yang-Baxter equation. We must emphasize, however, that Gaudin's work is much earlier than the Yang-Baxter equation. Generalizations of Gaudin formalism to semi-simple Lie algebras other than $\mathfrak{su}(2)$ are well studied. They usually go with the name of Yangians in the literature.

In different realizations of Gaudin algebras $H(\lambda)$ appear in different forms and may have different physical meanings. As a result, Gaudin's work has a wide range of physical applications. Eigenvalues and eigenstates of such Hamiltonians just follow from the beautiful algebraic structure of Gaudin's formalism. Solving Bethe ansatz equations, on the other hand, is the most difficult part of this technique and practically makes it much less useful. In this thesis, we showed that solutions of these equations are not necessary in a special realization involving only two $\mathfrak{su}(2)$ algebras. One can simply utilize some properties of the equations of Bethe ansatz to compute eigenvalues in an analytical way.

In this thesis, a q-deformation of rational Gaudin algebra is also introduced together with an associated one-parameter family of mutually commuting operators. Our motivation came from spin-extensions of the models of Calogero and Sutherland. These are one-dimensional integrable systems directly related to Hermitian and unitary matrix models, respectively. Unitary matrices are exponentials of Hermitian ones. It is, therefore, not surprising to see that spin-Sutherland model is connected to a q-deformation of rational Gaudin algebra involving hyperbolic functions.

Although the q-deformed rational Gaudin algebra introduced here and the trigonometric Gaudin algebra which was introduced by Gaudin himself in [8] (see Appendix B) both involve hyperbolic functions and a complex deformation parameter, they are completely different from each other. Perhaps the most intuitive argument in favor of this is given in Appendix B.

Associated to rational, trigonometric and elliptic Gaudin algebras are one-parameter families of mutually commuting Hamiltonians $H(\lambda)$. They all have the form of $Tr(L(\lambda)^2)$ where $L(\lambda)$ is given in Chapter 4 by equation (4.7). In that, $\mathcal{G}_q^{(r)}$ can be treated in the same footing with Gaudin algebras because the operators $H^{(q)}(\lambda)$ associated with $\mathcal{G}_q^{(r)}$ also commute with each other and they have the form of $Tr(L(\lambda)^2)$. As was mentioned in Chapter 4, Gaudin algebras are related to some solutions of classical Yang-Baxter equation. That the above operators $H(\lambda)$ commute is a result of this. Nevertheless, the equation (4.9) at the end of Chapter 4 tells us that $\mathcal{G}_q^{(r)}$ is not likely to be related to classical Yang-Baxter equation in its present form. Therefore it looks mysterious that the operators $H^{(q)}(\lambda)$ form a commutative set.

One of the possible directions to extend this study is to consider the equation (4.9) and to find the condition that is imposed on the r and s-matrices by Jacobi identity. This will certainly be an extension of classical Yang-Baxter equation. Next, it should be checked whether or not this extended classical Yang-Baxter equation also allows the trace of the powers of $L(\lambda)$ to commute as in the case of the q-deformed rational Gaudin algebra.

Studies in different branches of mathematical physics such as the matrix quantum mechanics, the inverse scattering problem and the two-dimensional statistical mechanics and quantum field theory, seem to point out beautiful and powerful algebraic structures behind the concept of integrability. We hope that this study proves to be a useful and fruitful contribution to this field.

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APPENDIX A

COMPUTING THE EIGENVALUES

To see that the lowest weight state given by (3.29) is an eigenstate of $H^{(q)}(\lambda)$ we first use the Lie brackets (3.24) to write $H^{(q)}(\lambda)$ in the following form:

$$
\begin{aligned}
H^{(q)}(\lambda) &= J^0(\lambda)J^0(\lambda) + J^+(\lambda)J^-(\lambda) + \frac{1}{2}[J^-(\lambda), J^+(\lambda)] \\
&= J^0(\lambda)J^0(\lambda) + J^+(\lambda)J^-(\lambda) \qquad \text{(A.1)} \\
&\quad -\frac{1}{2}\lim_{\mu\to\lambda} 2q\left(\frac{J^0(\lambda) - J^0(\mu)}{\tanh\left[q(\lambda-\mu)\right]} + J^0(\lambda) + J^0(\mu)\right).
\end{aligned}
$$

Then we see that the action of $H^{(q)}(\lambda)$ on the lowest weight state is

$$
H^{(q)}(\lambda)|0>= \big(W(\lambda)^2 - W'(\lambda) - 2qW(\lambda)\big)\,|0>. \qquad \text{(A.2)}
$$

We see that the lowest weight state is an eigenstate of $H^{(q)}(\lambda)$ with the eigenvalue $E_0(\lambda)$ given by (3.30).

Let us consider the state $|\xi>= J^+(\xi)|0>$. Action of $H^{(q)}(\lambda)$ on $J^+(\xi)|0>$ is

$$
\begin{aligned}
H^{(q)}(\lambda)J^+(\xi)|0> &= 2q\left(\coth\left[q(\lambda-\xi)\right] - 1\right)W(\xi)J^+(\lambda)|0> \qquad \text{(A.3)} \\
&\quad + \left(E_0(\lambda) - 2qW(\lambda)\coth\left[q(\lambda-\xi)\right] + 2qW(\lambda)\right)J^+(\xi)|0>.
\end{aligned}
$$

We see that $H^{(q)}(\lambda)J^+(\xi)|0>$ is a superposition of the states $J^+(\lambda)|0>$ and $J^+(\xi)|0>$. Therefore, for a generic ξ, the vector $J^+(\xi)|0>$ is not an eigenstate of

$H^{(q)}(\lambda)$. But if ξ is a root of the $W(\lambda)$ then the coefficient of $J^+(\lambda)|0>$ vanishes. In this case $J^+(\xi)|0>$ is an eigenstate of $H^{(q)}(\lambda)$ with the eigenvalue $E_1(\lambda)$ given by (3.31). For $n>1$, action of $H^{(q)}(\lambda)$ on the state $J^+(\xi_1)J^+(\xi_2)\dots J^+(\xi_n)|0>$ is given by

$$\begin{aligned}
&H^{(q)}(\lambda)J^+(\xi_1)J^+(\xi_2)\dots J^+(\xi_n)|0>= \\
&-\sum_{\alpha=1}^{n} 2q\left(\coth\left[q(\lambda-\xi_\alpha)\right]-1\right)\left(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} q\left(\coth\left[q(\xi_\alpha-\xi_\beta)\right]-1\right)-W(\xi_\alpha)\right) \\
&J^+(\xi_1)\dots J^+(\xi_{\alpha-1})J^+(\lambda)J^+(\xi_{\alpha+1})\dots J^+(\xi_n)|0> \\
&+\sum_{\alpha=1}^{n} 2q\left(\coth\left[q(\lambda-\xi_\alpha)\right]-1\right)\left(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} q\left(\coth\left[q(\xi_\alpha-\xi_\beta)\right]-1\right)-W(\lambda)\right) \\
&J^+(\xi_1)J^+(\xi_2)\dots J^+(\xi_n)|0>\,. \qquad \text{(A.4)}
\end{aligned}$$

Here we see that $H^{(q)}(\lambda)J^+(\xi_1)J^+(\xi_2)\dots J^+(\xi_n)|0>$ is a superposition of the states

$$J^+(\xi_1)\dots J^+(\xi_{\alpha-1})J^+(\lambda)J^+(\xi_{\alpha+1})\dots J^+(\xi_n)|0> \qquad \text{(A.5)}$$

for $\alpha=1,2,\dots,n$ and the state

$$J^+(\xi_1)J^+(\xi_2)\dots J^+(\xi_n)|0>\,. \qquad \text{(A.6)}$$

Coefficients of the states $J^+(\xi_1)\dots J^+(\lambda)\dots J^+(\xi_n)|0>$ vanish when the conditions (3.33) are satisfied. Consequently, $J^+(\xi_1)J^+(\xi_2)\dots J^+(\xi_n)|0>$ become an eigenstate of $H^{(q)}(\lambda)$ and its eigenvalue is given by (3.34).

APPENDIX B

TRIGONOMETRIC GAUDIN ALGEBRA

Like the rational Gaudin algebra $\mathcal{G}^{(r)}$, trigonometric Gaudin algebra $\mathcal{G}^{(t)}$ is also also an infinite dimensional complex Lie algebra spanned by $J^{\pm,0}(\lambda)$. Its Lie brackets are given by

$$[J^{+}(\lambda), J^{-}(\mu)] = 2p\frac{J^{0}(\lambda) - J^{0}(\mu)}{\sinh{[p(\lambda-\mu)]}},$$

$$[J^{0}(\lambda), J^{\pm}(\mu)] = \pm p\frac{J^{\pm}(\lambda) - \cosh{[p(\lambda-\mu)]}J^{\pm}(\mu)}{\sinh{[p(\lambda-\mu)]}}, \tag{B.1}$$

$$[J^{0}(\lambda), J^{0}(\mu)] = [J^{\pm}(\lambda), J^{\pm}(\mu)] = 0$$

Here p is a complex deformation parameter and $\mathcal{G}^{(t)}$ approaches to rational Gaudin algebra in $p \to 0$ limit.

A realization of $\mathcal{G}^{(t)}$ in terms of $\mathfrak{su}(2)$ generators $t_i^{\pm,0}$ for $i = 1, 2, \ldots, N$ is given by

$$J^{0}(\lambda) = \sum_{i=1}^{N} p\coth{[p(x_i - \lambda)]}t_i^{0} \quad \text{and} \quad J^{\pm}(\lambda) = \sum_{i=1}^{N}\frac{p}{\sinh{[p(x_i-\lambda)]}}t_i^{\pm}. \tag{B.2}$$

This realization also approaches to (2.9) as $p \to 0$ because both $p\coth{(px)}$ and $p/\sinh{(px)}$ approach to $1/x$ in this limit.

We see that realization of trigonometric Gaudin algebra involve different functions for $J^{\pm}(\lambda)$ and $J^0(\lambda)$ which is not the case for the rational Gaudin algebra and its q-deformation introduced in Chapter 3. This, intuitively tells us that the q-deformed rational Gaudin algebra introduced in Section 3.3 is not the same as trigonometric Gaudin algebra, although it involves trigonometric functions as well. That is to say, $\mathcal{G}_q^{(r)}$ cannot be obtained from trigonometric Gaudin algebra using a transformation $\lambda \to f(\lambda)$. Obviously no such transformation will turn both of the functions $p \coth(p\lambda)$ and $p/\sinh(p\lambda)$ into $q \coth(q\lambda) + q$.

Owing to the fact that two different functions, $p \coth(px)$ and $p/\sinh(px)$, are used in the $\mathfrak{su}(2)$ realization of trigonometric Gaudin algebra, operators in (B.2) are not $\mathfrak{su}(2)$ tensor operators. But realizations of rational Gaudin algebra and its q-deformation are$\mathfrak{su}(2)$ tensor operators. To see that let us introduce

$$Q^0 = \sum_{i=1}^{N} t_i^0 \quad \text{and} \quad Q^{\pm} = \sum_{i=1}^{N} t_i^{\pm}. \tag{B.3}$$

Obviously $Q^{\pm,0}$ span an $\mathfrak{su}(2)$ algebra that we will denote by $\mathfrak{su}(2)_Q$. One can show that realizations of $\mathcal{G}^{(r)}$ and $\mathcal{G}_q^{(r)}$ obey

$$[Q^0, J^{\pm}(\lambda)] = \pm J^{\pm}(\lambda), \quad [Q^{\pm}, J^0(\lambda)] = \mp J^{\pm}(\lambda), \quad [Q^{\pm}, J^{\mp}(\lambda)] = \pm 2J^0(\lambda) \tag{B.4}$$

which tells us that realizations of $\mathcal{G}^{(r)}$ and $\mathcal{G}_q^{(r)}$ in terms of $\mathfrak{su}(2)$ generators are rank one tensor operators of $\mathfrak{su}(2)_Q$. However, (B.4) is not satisfied by the realizations of $\mathcal{G}^{(t)}$. Therefore these realizations are not $\mathfrak{su}(2)_Q$ tensor operators.

APPENDIX C

HERMITIAN AND UNITARY MATRIX MODELS

This Appendix is a modest review of the connection between Hermitian and unitary matrix models and the integrable systems of Calogero and Sutherland. We closely follow the references [30, 36, 39, 40].

When we consider a system of N identical particles, formulation must be invariant under the permutations of particle labels. This inspires us to view the phase space variables of the system as eigenvalues of $N \times N$ matrices. Because eigenvalues of matrices are just permuted and otherwise left invariant under unitary transformations.

Let us consider a time dependent $N \times N$ Hermitian matrix M together with the Lagrangian

$$L = \frac{1}{2} tr \dot{M}^2. \tag{C.1}$$

The action is invariant under the time independent unitary transformations

$$M \to UMU^{-1}.$$

Conserved quantity corresponding to this symmetry is the following Hermitian traceless matrix:

$$J = i[M, \dot{M}].$$

Eigenvalues $x_1(t), x_2(t), \ldots, x_N(t)$ of $M(t)$ are real numbers and the matrix

$$\Lambda(t) = diag\{x_1(t), x_2(t), \ldots, x_N(t)\}$$

is just $M(t)$ in the basis where it is diagonal. One can always change into this basis using a unitary matrix, i.e. there exists a time dependent unitary matrix $U(t)$ satisfying

$$M(t) = U(t)\Lambda(t)U^{-1}(t).$$

In this parameterization, $\dot{M}$ and J can be written as

$$\dot{M} = U\left(\dot{\Lambda} + [\Lambda, A]\right)U^{-1} \tag{C.2}$$

$$J = iUKU^{-1} \tag{C.3}$$

where A and K are defined as follows:

$$A = -U^{-1}\dot{U} \quad \text{and} \quad K = [\Lambda, [\Lambda, A]]. \tag{C.4}$$

If we substitute (C.2) into the Lagrangian (C.1) we find

$$\begin{aligned} L &= \frac{1}{2} tr\left(\dot{\Lambda}^2 + [\Lambda, A]^2\right) \\ &= \sum_{i=1}^{N} \frac{1}{2}\dot{x}_i - \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} (x_i - x_j)^2 A_{ij}A_{ji}. \end{aligned} \tag{C.5}$$

One can easily show that components of A and K are related by the formula

$$K_{ij} = A_{ij} \left(x_i - x_j\right)^2$$

If we substitute this into (C.5), we find

$$L = \sum_{i=1}^{N} \frac{1}{2} \dot{x}_i - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{K_{ij} K_{ji}}{\left(x_i - x_j\right)^2}. \tag{C.6}$$

Since the conserved charge J is not dynamical one can choose it at will and then calculate corresponding K from (C.3). The only requirement is that the diagonal elements of K must be zero. Because otherwise diagonal elements of A blow up as can be seen from equation (C.3). If we make a choice which gives $K = 0$, then L is the Lagrangian of a system of ordinary non-interacting particles. A less trivial alternative is to choose J in such a way that $K_{ij} K_{ji} = g$ is satisfied for $i \neq j$. This choice leads us to

$$L = \sum_{i=1}^{N} \frac{1}{2} \dot{x}_i - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{g}{\left(x_i - x_j\right)^2}$$

which is the Lagrangian of Calogero model given in (3.1).

As mentioned in Section 3.1, Sutherland's model can be considered as a Calogero model on a circle. Natural coordinates for the particles on a circle are unimodular complex numbers which are known to be the eigenvalues of unitary matrices. Consequently, one may expect to relate Sutherland's model to a unitary matrix model. Consider the following Lagrangian:

$$L = -\frac{1}{2} tr \left(W^{-1} \dot{W}\right)^2 \tag{C.7}$$

where W is a unitary matrix. This Lagrangian is invariant under the left and right multiplication of W with time independent unitary matrices U and V^{-1}:

$$W \to UW \quad \text{and} \quad W \to WV^{-1}.$$

Conserved charges corresponding to these symmetries are

$$L = i\dot{W}W^{-1} \quad \text{and} \quad R = -iW^{-1}\dot{W},$$

respectively. In particular $W \to UWU^{-1}$ is a symmetry of the system because this is equivalent to above transformations with $U = V$ applied successively. The conserved charge for this symmetry is

$$J = L + R = i[\dot{W}, W^{-1}].$$

This transformation is the unitary conjugation which permutes the eigenvalues of W. Since the eigenvalues of W are unimodular complex numbers we will denote them by $e^{ix_1(t)}, e^{ix_2(t)}, \ldots, e^{ix_N(t)}$. One can always find a basis where W is diagonal and perform a unitary transformation into this basis. In other words, there exists a time dependent unitary matrix $U(t)$ such that

$$W(t) = U(t)\Lambda(t)U(t)^{-1}.$$

Here $\Lambda(t) = diag\{e^{ix_1(t)}, e^{ix_2(t)}, \ldots, e^{ix_N(t)}\}$ is the diagonalized form of $W(t)$. In this parameterization we can write $\dot{W}$ and J in the following way:

$$\dot{W} = U\left(\dot{\Lambda} + [\Lambda, A]\right)U^{-1}$$

$$J = iUKU^{-1}.$$

Here A and K are defined as follows:

$$A = -U^{-1}\dot{U} \quad \text{and} \quad K = [\Lambda^{-1}, [\Lambda, A]].$$

Using

$$([\Lambda, A])_{ij} = A_{ij}\left(e^{ix_i} - e^{ix_j}\right)$$

we can write the Lagrangian (C.7) as

$$L = \frac{1}{2}\sum_{i=1}^{N}\dot{x}_i^2 - 2\sum_{\substack{i,j=1 \\ i\neq j}}^{N} A_{ij}A_{ji}\sin^2\frac{x_i - x_j}{2}. \tag{C.8}$$

As for the Hermitian matrix model, it is advantageous to use K_{ij} instead of A_{ij}, since K is related to a conserved quantity whereas A is dynamical. Using the definition of K we find

$$K_{ij} = 4A_{ij}\sin^2\frac{x_i - x_j}{2}.$$

This leads us to the following Lagrangian upon substitution in (C.8):

$$L = \frac{1}{2}\sum_{i=1}^{N}\dot{x}_i^2 - \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N}\frac{K_{ij}K_{ji}}{4\sin^2[(x_i - x_j)/2]}. \tag{C.9}$$

Conserved charge J can be chosen at will. If we choose it to be zero then K is also zero and we obtain a free model as before. The choice of J leading to $K_{ij}K_{ji} = g$ gives us

$$L = \frac{1}{2}\sum_{i=1}^{N}\dot{x}_i^2 - \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N}\frac{g}{4\sin^2[(x_i - x_j)/2]}$$

which is the Lagrangian of Sutherland's model given in (3.2) for $d = 2\pi$.

Quantization of Hermitian and unitary matrix models is a tedious but straightforward calculation which will not be reproduced here in its full content. In their classical versions these matrix models are shown to be equivalent to some Calogero-Sutherland like models but unlike the standard Calogero-Sutherland model, there is an additional degree of freedom in matrix models. This degree of freedom is non-dynamical and it seems to control the way the particles interact. This additional degree of freedom can be viewed as an an internal $\mathfrak{su}(q)$ degree of freedom of individual particles when the models are quantized. To see this let us first note that in the classical models, Poisson brackets of the components of K among each other are

$$\{K_{ij}, K_{kl}\} = i\left(\delta_{jk}K_{il} - \delta_{il}K_{kj}\right). \tag{C.10}$$

We also know that K is constrained with $K_{ii} = 0$ (there is no summation in this formula, what is meant is that the diagonal elements of K are zero). These are true for both Hermitian and unitary matrix models. As a result, when we quantize these models, components of K are mapped to operators $\hat{K}_{ij}$ which obey

$$[\hat{K}_{ij}, \hat{K}_{kl}] = \delta_{il}\hat{K}_{kj} - \delta_{jk}\hat{K}_{il}. \tag{C.11}$$

The constraint is implemented by requiring that $\hat{K}_{ii}$ gives zero on physical states. It was shown in [36] that considering the boson operators $a_{mi}, a^{\dagger}_{mi}$ for $m =$

$1, 2, \ldots, q$ and $i = 1, 2, \ldots, N$ which obey

$$[a_{mi}, a_{nj}^{\dagger}] = \delta_{mn}\delta_{ij}$$

one can realize the operators $\hat{K}_{ij}$ as follows:

$$\hat{K}_{ij} = \sum_{m=1}^{q} a_{mi}^{\dagger} a_{mj} - \delta_{ij}\alpha \sum_{m=1}^{q}\sum_{k=1}^{N} a_{mk}^{\dagger} a_{mk}$$

Here the constant α serves to fulfill the constraint because the second term above is proportional to the total number operator

$$\hat{N} = \sum_{m=1}^{q}\sum_{k=1}^{N} a_{mk}^{\dagger} a_{mk}$$

and owing to this term the constraint $\hat{K}_{ii} = 0$ is transformed into

$$\sum_{m=1}^{q} a_{mi}^{\dagger} a_{mi} = \alpha \hat{N},$$

i.e. the total number of i type bosons are always a certain fraction of the total number of particles. Since this is equivalently valid for every $i = 1, 2, \ldots, N$ we must choose $\alpha = 1/N$.

Second step is to define

$$\hat{S}_{i,nm} = a_{mi}^{\dagger} a_{ni} - \frac{1}{q}\sum_{s=1}^{q} a_{si}^{\dagger} a_{si}.$$

For every $i = 1, 2, \ldots, N$ the operators $\hat{S}_{i,mn}$ span an $\mathfrak{su}(q)$ algebra and $\hat{K}_{ij}\hat{K}_{ji}$ can be written as

$$\hat{K}_{ij}\hat{K}_{ji} = \sum_{m,n=1}^{q} \hat{S}_{i,mn}\hat{S}_{j,nm} + \frac{\ell(\ell+q)}{q} \tag{C.12}$$

where ℓ is a positive integer or half integer. In short, it is possible to distribute the additional degree of freedom $K_{ij}K_{ji}$ to the individual particles as an internal $\mathfrak{su}(q)$ degree of freedom. Here q is an arbitrary positive integer can be chosen at will. Substituting (C.12) into (C.6) and (C.9) we find

$$L = \sum_{i=1}^{N} \frac{1}{2} \dot{x}_i - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\overrightarrow{S}_i \cdot \overrightarrow{S}_j + \frac{\ell(\ell+q)}{q}}{(x_i - x_j)^2}$$

and

$$L = \frac{1}{2} \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\overrightarrow{S}_i \cdot \overrightarrow{S}_j + \frac{\ell(\ell+q)}{q}}{4\sin^2[(x_i - x_j)/2]}$$

respectively, where $\overrightarrow{S}_i \cdot \overrightarrow{S}_j = Tr(\hat{S}_i \hat{S}_j)$. We see that quantization of Hermitian and unitary matrix models lead to Calogero-Sutherland like models involving particles with $\mathfrak{su}(q)$ internal degrees of freedom. These particles interact via the anti-ferromagnetic coupling of these internal degrees of freedom as well as the ordinary Calogero-Sutherland interaction term with $g = \ell(\ell + q)/q$.

VITA

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