STUDIES ON THE GENERALIZED AND REVERSE GENERALIZED BESSEL POLYNOMIALS

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STUDIES ON THE GENERALIZED AND REVERSE GENERALIZED BESSEL POLYNOMIALS

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ABSTRACT

STUDIES ON THE GENERALIZED AND REVERSE GENERALIZED BESSEL POLYNOMIALS

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The special functions and, particularly, the classical orthogonal polynomials encountered in many branches of applied mathematics and mathematical physics satisfy a second order differential equation, which is known as the equation of the hypergeometric type. The variable coefficients in this equation of the hypergeometric type are of special structures. Depending on the coefficients the classical orthogonal polynomials associated with the names Jacobi, Laguerre and Hermite can be derived as solutions of this equation.

In this thesis, these well known classical polynomials as well as another class of polynomials, which receive less attention in the literature called Bessel polynomials have been studied.

Keywords: Differential Equations of the Hypergeometric Type, Functions of the Hypergeometric Type, Orthogonal Polynomials, Bessel, Generalized Bessel and Reverse Generalized Bessel Polynomials.

ÖZ

GENELLEŞTİRİLMİŞ VE TERS ÇEVRİLMİŞ BESSEL POLİNOMLARI

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Uygulamalı matematiğin ve matematiksel fiziğin pek çok alanında karşılaşılan özel fonksiyonlar ve özellikle klasik ortogonal polinomlar hipergeometrik tipi denklem olarak bilinen ikinci mertebeden lineer bir diferansiyel denklemi sağlarlar. Bu hipergeometrik tipi denklemin değişken katsayıları belirli özel bir yapıdadır. Bu katsayılara bağlı olarak, klasik ortogonal polinomlar diye bilinen Jacobi, Laguerre ve Hermite polinomları hipergeometrik tipi denklemin çözümleri şeklinde elde edilirler.

Bu tezde, çok iyi incelenmiş bu klasik polinomlarla, Bessel polinomları olarak adlandırılan ve literatürde daha az ilgi toplayan bir başka polinom sınıfı üzerinde çalışılmıştır.

Anahtar Kelimeler: Hipergeometrik Tipi Diferansiyel Denklemler, Hipergeometrik Tipi Fonksiyonlar, Ortogonal Polinomlar, Bessel, Genelleştirilmiş Bessel ve Genelleştirilmiş Çevrilmiş Bessel Polinomları

To my family

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Modern engineering and physics applications demand a more through knowledge of applied mathematics. In particular, it is important to have a good understanding to the basic properties of special functions. These functions commanly arise in such areas of applications in physical sciences. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis.

In this century, the discoveries of new special functions and applications of special functions to new areas of mathematics have initiated a resurgence of interest in this field. In recert years particular cases of long familiar special functions have been clearly defined and applied to orthogonal polynomials.

In this thesis, before the discussion of Bessel polynomials (BP) firstly we have studied the foundation of special functions, equations of the hypergeometric type (HG) and their solutions. Afterwords we review the classical orthogonal polynomials.

Bessel polynomials arise in a natural way in the study of various aspects of applied mathematics. Enough literature on the subject has accumulated in the recent years to make these polynomials a respectable sub-area of special functions. The polynomials occur in surprising ways in problems in diverse fields; number theory, partial differential equations, algebra and statistics.

The importance of these polynomials was seen in 1949 by Krall and Frink [18] who introduced the name Bessel polynomials because of their close relationship with the Bessel functions. They initiated a systematic investigation of what are now well known in the mathematical literature as the Bessel polynomials.

The main object of the thesis is to discuss Bessel polynomials and also generalized and reverse generalized Bessel polynomials. We shall introduce general properties of these polynomials systematically, which are accepted along with the classical orthogonal polynomials.

In Chapter 2 we review the foundation of special functions and differential equations of the hypergeometric type and functions of the hypergeometric type.

Chapter 3 is concerned with a study of orthogonal polynomials. We define firstly orthogonality and orthogonal polynomials. The classical sets of Jacobi, Laguerre and Hermite polynomials are discussed and a brief summary of their general properties is given.

Finally, Chapter 4 is devoted to Bessel polynomials. These polynomials form a set of orthogonal polynomials on the unit circle in the complex plane. Thus they may be viewed as an additional member of family of classical orthogonal polynomials. We study in some detail properties of these polynomials. In particular, we define generalized Bessel polynomials and reverse generalized Bessel polynomials and introduce their orthogonality properties, generating functions, recurrence relations and interrelations to the other orthogonal polynomials.

Chapter 2

Foundation of the Theory of Special Functions

2.1 Equations of the Hypergeometric Type

Almost all special functions can be introduced as solutions of a differential equation of the type

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0, (u' = \frac{du}{dz},\ z \in \mathbb{C}) \tag{2.1}$$

where, $\tilde{\tau}(z)$ is a polynomial of degree at most 1 and $\sigma(z),\ \tilde{\sigma}(z)$ are polynomials of degree at most 2. This equation may be reduced to a simpler form by means of the transformation

$$u = \phi(z)y.$$

Actually, substituting u , $u' = \phi'(z)y + \phi(z)y'$ and

$$u'' = \phi''(z)y + \phi'(z)y' + \phi'(z)y + \phi(z)y'' = \phi''(z)y + 2\phi'(z)y' + \phi(z)y''$$

into (2.1) we obtain

$$y'' + (2\frac{\phi'}{\phi} + \frac{\tilde{\tau}}{\sigma})y' + (\frac{\phi''}{\phi} + \frac{\tilde{\tau}}{\sigma}\frac{\phi''}{\phi} + \frac{\tilde{\sigma}}{\sigma^2})y = 0. \tag{2.2}$$

Require that the coefficient of y' is of the form $\frac{\tau(z)}{\sigma(z)}$, with $\tau(z)$ a polynomial of degree at most 1, we have

$$2\frac{\phi'(z)}{\phi(z)} = \frac{\tau(z) - \tilde{\tau}(z)}{\sigma(z)}$$

or

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}, \tag{2.3}$$

where

$$\pi(z) = \frac{1}{2}[\tau(z) - \tilde{\tau}(z)]. \tag{2.4}$$

Note that $\pi(z)$ is also a polynomial of degree at most 1. By using the identity,

$$\frac{\phi''}{\phi} = \left(\frac{\phi'}{\phi}\right)' + \left(\frac{\phi'}{\phi}\right)^2$$

(2.2) takes the form

$$y'' + \left[2\frac{\pi(z)}{\sigma(z)} + \frac{\tilde{\tau}(z)}{\sigma(z)}\right]y' + \left[\frac{\pi^2}{\sigma^2(z)} + \frac{\pi'(z)\sigma - \sigma'\pi(z)}{\sigma^2(z)} + \frac{\tilde{\sigma}(z)\pi(z)}{\sigma^2(z)} + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\right]y = 0$$

which can be written as

$$y'' + \frac{\tau(z)}{\sigma(z)}y' + \frac{\tilde{\tilde{\sigma}}(z)}{\sigma^2(z)}y = 0 \tag{2.5}$$

where

$$\tau(z) = 2\pi(z) + \tilde{\tau}(z), \tag{2.6}$$

and

$$\tilde{\tilde{\sigma}}(z) = \pi^2(z) + [\tilde{\tau}(z) - \sigma'(z)]\pi(z) + [\tilde{\sigma}(z) + \pi'(z)\sigma(z)]. \tag{2.7}$$

Here $\tilde{\tilde{\sigma}}(z)$ is a polynomial of degree at most 2. From (2.5) and (2.1), we see that we have derived a class of transformations induced by the substitution

$$u = \phi(z)y$$

that do not change the type of the differential equation under consideration. Now, for simplicity, we shall choose $\pi(z)$ so that $\tilde{\tilde{\sigma}}$ in (2.7) is divisible by $\sigma(z)$. That is,

$$\tilde{\tilde{\sigma}}(z) = \lambda\sigma(z), \quad \lambda : \text{ constant} \tag{2.8}$$

which makes it possible to write (2.5) in the form

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0. \tag{2.9}$$

Equation (2.9) is referred to as a differential equation of the hypergeometric type (HG) and its solutions are referred to as the functions of the hypergeometric type. The starting equation in (2.1) may be called a generalized differential equation of hypergeometric type. Now we have to determine what $\pi(z)$ is. We can write from (2.8) and (2.7)

$$\pi^2 + (\tilde{\tau} - \sigma')\pi + (\tilde{\sigma} - k\sigma) = 0 \tag{2.10}$$

where k is another constant,

$$k = \lambda - \pi'(z). \tag{2.11}$$

So, we find that

$$\pi(z) = \frac{1}{2}\Big[\sigma'(z) - \tilde{\tau}(z)\Big] \mp \sqrt{\left[\frac{\tilde{\tau}(z) - \sigma'(z)}{2}\right]^2 - [\tilde{\sigma}(z) - k\sigma(z)]}. \tag{2.12}$$

The expression under the square root term is a quadratic polynomial in z. Since $\pi(z)$ defined in (2.4) is a linear polynomial, this expression must be the square of a linear polynomial. That is, the discriminant of the quadratic polynomial under the square root sign should be zero, from which we determine k. After that equation (2.12) gives us really a linear polynomial for $\pi(z)$. Then from (2.3), (2.6) and (2.11) $\phi(z)$, $\tau(z)$ and λ can be determined. Therefore, the differential equation

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0$$

can be reduced to a differential equation of the hypergeometric type

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 \tag{2.13}$$

in several ways corresponding to the different selections of k and the sign in (2.12) for $\pi(z)$[29].

Without any loss of generality we assume that $\sigma(z)$ in (2.1) does not have a double root. If $\sigma(z)$ has a double root, i.e $\sigma(z) = (z-a)^2$ then (2.1) can be transformed into

$$\frac{d^2u}{ds^2} + \frac{2 + s\tilde{\tau}(a+1/s)}{s}\frac{du}{ds} + \frac{s^2\tilde{\sigma}(a+1/s)}{s^2}u = 0$$

by the substitution $z - a = \frac{1}{s}$, which can be treated in a different way. We shall discuss this situation in detail in Chapter 4.

Basic properties of solutions of the differential equations of the hypergeometric type are summarized in the following theorems.

Theorem 2.1. *All derivatives of functions of the hypergeometric type are also functions of the hypergeometric type*

Proof. Differentiating (2.13) with respect to z, we obtain

$$\sigma(z)y''' + (\sigma' + \tau)(z)y'' + (\tau' + \lambda)(z)y' = 0. \tag{2.14}$$

Letting $y'(z) = v_1(z)$, we have

$$\sigma(z)v_1'' + \tau_1(z)v_1' + \mu_1 v_1 = 0$$

where, $\mu_1 = \tau' + \lambda$: constant and $\tau_1(z) = \sigma'(z) + \tau(z)$ is a polynomial of degree atmost 1. Clearly equation (2.14) is a differential equation of the hypergeometric type. By mathematical induction, we find that an equation, of the HG type for

$v_n(z) = y^{(n)}(z)$ which can be obtained as follows:

$$\sigma(z)v_n'' + \tau_n(z)v_n' + \mu_n v_n = 0, \quad v_0 \triangleq y \tag{2.15}$$

where,

$$\tau_n(z) = \sigma'(z) + \tau_{n-1}(z) = n\sigma'(z) + \tau(z), \quad \tau_0(z) \triangleq \tau(z) \tag{2.16}$$

$$\mu_n(z) = \mu_{n-1} + \tau_{n-1}' = \lambda + n\tau' + \frac{1}{2}n(n-1)\sigma'', \; \mu_0 \triangleq \lambda \tag{2.17}$$

for all $n = 1, 2, ...$ □

Theorem 2.2. *The differential equation (2.9) has polynomial solutions, say $y(z) \triangleq y_n(z)$, of degree n for particular values of λ so that,*

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''$$

Proof. Obviously, $\sigma(z)v_n''(z) + \tau_n(z)v_n'(z) + \mu_n v_n(z) = 0$ has a constant solution, that is

$$v_n = C_0, \quad \text{if} \;\; \mu_n = 0. \tag{2.18}$$

But we know

$$v_n = y^{(n)}(z) = C_0. \tag{2.19}$$

So if we integrate both sides of (2.19) successively, then we get

$$\begin{aligned}
y^{(n-1)}(z) &= C_0 z + C_1 \\
y^{(n-2)}(z) &= \frac{C_0}{2}z^2 + C_1 z + C_2 \\
&\;\;\vdots \\
y(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = y_n(z) \qquad (2.20)
\end{aligned}$$

which a polynomial of degree n. Indeed, $\mu_n = 0$ implies that

$$\lambda + n\tau' + \frac{1}{2}n(n-1)\sigma'' = 0 \Rightarrow \lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''$$

which completes the proof. $\square$

2.1.1 Rodrigues Formula

Let us consider the self-adjoint forms of

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 \tag{2.21}$$

and

$$\sigma(z)v_n'' + \tau_n(z)v_n' + \mu_n v_n = 0. \tag{2.22}$$

They can be written as

$$\rho(z)\sigma(z)y'' + \rho(z)\tau(z)y' + \rho(z)\lambda y = 0 \tag{2.23}$$

and

$$\rho_n(z)\sigma(z)v_n'' + \rho_n(z)\tau_n(z)v_n' + \rho_n(z)\mu_n v_n = 0 \tag{2.24}$$

where $\rho(z)$ and $\rho_n(z)$ are analytic functions. So, by using $(\sigma\rho)' = \tau\rho$ and $(\sigma\rho_n)' = \tau_n\rho_n$, the self adjoint forms of (2.21) and (2.22) can be obtain

$$(\sigma\rho y')' + \lambda\rho y = 0, \quad \lambda = \lambda_n \tag{2.25}$$

and

$$(\sigma\rho_n v_n')' + \mu_n\rho_n v_n = 0. \tag{2.26}$$

In fact, there is a connection between ρ_n and $\rho(z) \triangleq \rho_0(z)$.

From (2.24) and (2.25), we have

$$\frac{(\sigma\rho_n)'}{\rho_n} = \tau_n = n\sigma' + \tau' = n\sigma' + \frac{(\sigma\rho)'}{\rho}$$

which leads to

$$\sigma\frac{\rho_n'}{\rho_n} = -\sigma' + n\sigma' + \sigma' + \sigma\frac{\rho'}{\rho}.$$

By calculation we obtain

$$\rho_n(z) = \sigma^n(z)\rho(z), \quad n = 0, 1, 2, \cdots, \quad \rho_0(z) = \rho(z).$$

Then we see that, $\sigma\rho_n = \sigma^{n+1}\rho = \rho_{n+1}$. Since $v_n' = v_{n+1}$, (2.25) can be written as

$$(\rho_{n+1}v_{n+1})' + \mu_n\rho_n v_n = 0 \Rightarrow \rho_n v_n = \frac{-1}{\mu_n}(\rho_{n+1}v_{n+1})'.$$

Therefore for $m < n$, we get successively

$$\begin{aligned}
\rho_m v_m &= \frac{-1}{\mu_m}(\rho_{m+1}v_{m+1})' = \frac{(-1)'}{\eta_m}(\rho_{m+1}v_{m+1})' \\
&= (-1)^2\frac{1}{\mu_m \ \mu_{m+1}}(\rho_{m+2}v_{m+2})'' = (-1)^3\frac{1}{\mu_m \ \mu_{m+1} \ \mu_{m+2}} \ (\rho_{m+3}v_{m+3})'''
\end{aligned}$$

Finally

$$\rho_m v_m = (-1)^n\frac{1}{\mu_m \ \mu_{m+1} \ \mu_{m+2}\cdots\mu_{m+n-1}}(\rho_{m+n}v_{m+n})^{(n)}.$$

Replacing n by $n-m$, we find that

$$\begin{aligned}
\rho_m v_m &= \frac{(-1)^{n-m}}{\mu_m \ \mu_{m+1}\cdots\mu_{n-1}}(\rho_n v_n)^{(n-m)} \\
&= \frac{(-1)^m\mu_0 \ \mu_1\cdots\mu_{m-1}}{(-1)^n\mu_0 \ \mu_1\cdots\mu_m\mu_{m+1}\cdots\mu_{n-1}}(\rho_n v_n)^{(n-m)}. \qquad (2.27)
\end{aligned}$$

That gives

$$\rho_m v_m = \frac{A_m}{A_n}\frac{d^{n-m}}{dz^{n-m}}(\rho_n v_n)$$

where

$$A_j(\lambda) = (-1)^j \prod_{k=0}^{j-1} \mu_k(\lambda), \quad A_0 \triangleq 1 \quad \mu_0 = \lambda. \tag{2.28}$$

If $y(z)$ is a polynomial of degree n, say $y(z) = y_n(z)$ and $v_m(z) = y_n^{(m)}(z)$ we obtain a formula for the $v_m(z)$

$$v_m(z) = y_n^{(m)}(z) = B_n \frac{A_{mn}}{\rho_m(z)} \frac{d^{n-m}\rho_n}{dz^{n-m}} \tag{2.29}$$

where

$$A_{mn} = A_m(\lambda_n), \quad B_n = \frac{v_n(z)}{A_n(\lambda_n)} = \frac{y_n^{(n)}(z)}{A_{nn}} = \text{constant}. \tag{2.30}$$

In particular when $m = 0$, we derive an explicit representation polynomials of the HG type, ie

$$y_n(z) = B_n \frac{1}{\rho(z)} \frac{d^n}{dz^n} \left(\sigma^n(z)\rho(z)\right). \tag{2.31}$$

This formula is known as the Rodrigues formula for $n = 0, 1, \cdots$ with B_n a normalization constant which can be specified for historical reasons and these polynomials correspond to the values $\mu_n = 0$, that is

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma'' \tag{2.32}$$

2.1.2 Integral Representation of Functions of the Hypergeometric Type

Firstly let us remind Cauchy Integral Theorem. Let f be analytic in a simply connection region R and let φ be a simple closed contour in R. If z is a point inside φ then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\varphi} \frac{f(s)}{(s-z)^{n+1}} ds, \quad n \in \mathbb{N} \tag{2.33}$$

In particular, for $n = 0$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\varphi} \frac{f(s)}{s - z} dz. \tag{2.34}$$

So from Cauchy Integral Theorem, the integral representations of the polynomial solutions in

$$y_n(z) = B_n \frac{1}{\rho(z)} \frac{d^n}{dz^n} (\sigma^n(z)\rho(z)) \tag{2.35}$$

can be written as

$$y_n(z) = \frac{c_n}{\rho(z)} \int_{\varphi} \frac{\sigma^{(n)}(s)\rho(s)}{(s - z)^{n+1}} ds, \; c_n = \frac{n!}{2\pi i} B_n \tag{2.36}$$

where φ is a simple closed contour surrounding the point at $s = z$. This representation of a particular solution valid for $\lambda = \lambda_n$ makes it possible to propose a particular solution of the form

$$y(z) = y_v(z) = \frac{c_v}{\rho(z)} \int_{\varphi} \frac{\sigma^v(s)\rho(s)}{(s - z)^{v+1}} ds \tag{2.37}$$

for an arbitrary value of λ, where c_v is a normalization constant, and v is a parameter which will be in the form

$$\lambda = -v\tau' - \frac{1}{2} v(v - 1)\sigma''. \tag{2.38}$$

Theorem 2.3. *Consider the function*

$$u(z) = \int_{\varphi} \frac{\rho_v(s)}{(s - z)^{v+1}} ds \tag{2.39}$$

where v is a root of the equation, $\lambda + \tau' v + \frac{1}{2} v(v - 1)\sigma'' = 0$. Then the differential equation of the hypergeometric type

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0$$

has a particular solution of the form

$$y(z) \triangleq y_v(z) = \frac{c_v}{\rho(z)} u(z), \quad c_v : constant$$

provided that,

(i) Differentiation of $u(z)$ up to the second-order under integral sign with respect to z is valid, i.e.

$$u'(z) = (v+1) \int_{\varphi} \frac{\rho_v(s)}{(s-z)^{v+2}} ds, \tag{2.40}$$

$$u''(z) = (v+1)(v+2) \int_{\varphi} \frac{\rho_v(s)}{(s-z)^{v+3}} ds. \tag{2.41}$$

(ii) The contour φ is chosen in such a way that

$$\left. \frac{\sigma^{v+1}(s)\rho(s)}{(s-z)^{v+2}} \right|_{s=s_1}^{s=s_2} = 0 \tag{2.42}$$

where s_1 and s_2 are the boundary points of φ.

Proof. Consider the equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0.$$

In the self adjoint form we can write

$$(\sigma \rho y')' + \lambda \rho y = 0$$

where

$$[\sigma(s)\rho_v(s)]' = \tau_v(s)\rho_v(s),$$

$$\rho_v(s) = \sigma^v(s)\rho(s)$$

and

$$\tau_v(s) = \tau(s) + v\sigma'(s).$$

Let us multiply both sides of $[\sigma(s)\rho_v(s)]' = \tau_v(s)\rho_v(s)$, by $(s-z)^{-v-2}$ and integrate with respect to s along φ to get

$$\int_\varphi \frac{d}{ds}[\sigma(s)\rho_v(s)]\frac{ds}{(s-z)^{v+2}} = \int_\varphi \frac{\tau_v(s)\rho_v(s)}{(s-z)^{v+2}}ds. \tag{2.43}$$

Integrating the integral on the left hand side by parts, we obtain

$$(i) \quad \frac{\sigma(s)\rho_v(s)}{(s-z)^{v+2}}\Bigg|_{s_1}^{s_2} + (v+2)\int_\varphi \frac{\sigma(s)\rho_v(s)}{(s-z)^{v+3}}ds = \int_\varphi \frac{\tau_v(s)\rho_v(s)}{(s-z)^{v+2}}ds. \tag{2.44}$$

By hypothesis, boundary terms vanish since $\rho_v = \sigma^v\rho$. Furthermore, by Taylor's Theorem we can write the expansions of $\sigma(s)$ and $\tau_v(s)$ about $s = z$ which are

$$(ii) \quad \sigma(s) = \sigma(z) + \sigma(z)'(s-z) + \frac{\sigma''(z)}{2}(s-z)^2$$

and

$$\tau_v(s) = \tau_v(z) + \tau_v'(z)(s-z). \tag{2.45}$$

Using $u(z)$, $u'(z)$, $u''(z)$ and (ii), (i) can be written as

$$\sigma(z)u'' + [2\sigma'(z) - \tau(z)]u' - (v+1)[\frac{v-2}{2}\sigma''(z) + \tau'(z)]u = 0. \tag{2.46}$$

By using the identity, $[(\sigma\rho)y]' = (\sigma\rho)'y + \sigma\rho y'$ we may write

$$\sigma\rho y' \quad = \quad (\sigma\rho y)' - (\sigma\rho)'y = [\sigma(\rho y)]' - \tau(\rho y)$$

and by putting $\rho y = c_v u$, we find that

$$\sigma\rho y' \quad = \quad [\sigma c_v u]' - \tau c_v = c_v[(\sigma u)' - \tau u] \tag{2.47}$$

and on differentiation we have

$$
\begin{aligned}
(\sigma\rho y')' &= c_v[(\sigma u)'' - (\tau u)'] \\
&= c_v[(v+1)(\frac{v-2}{2}\sigma''(z) + \tau'(z))u + [\sigma''(z) - \tau'(z)]u]
\end{aligned} \tag{2.48}
$$

then it follows

$$
(\sigma\rho y')' = c_v\{\frac{1}{2}v(v-1)\sigma'' + v\tau'\}u = -\lambda c_v u = -\lambda\rho y \tag{2.49}
$$

and $(\sigma\rho y')' + \lambda\rho y = 0$ which is the self-adjoint form of (2.21) and that completes the proof.

The theorem is of fundemental importance in the particular of special functions. The theorem is valid for

$$
\left.\frac{\sigma_{v+1}(s)\rho(s)}{(s-z)^{v+2}}\right|_{s=s_1,s_2} \tag{2.50}
$$

at both end points of φ. Indeed, φ can be choosen so that (2.50) is satisfied at its end points at $s = s_1$ and $s = s_2$ [29]. $\square$

2.2 Hypergeometric Functions

2.2.1 Hypergeometric Equations

In part 1 we define differential equation of the HG type

$$
\sigma(z)y'' + \tau(z)y' + \lambda y = 0. \tag{2.51}
$$

There are 3 different cases corresponding to the degree of $\sigma(z)$.

CASE 1: Let $\sigma(z) = (z-a)(b-z)$ with $a \neq b$, under the substitution $z = a + (b-a)s$, (2.51) can be written as

$$
s(1-s)y'' + [\gamma - (\alpha+\beta+1)s]y' - \alpha\beta y = 0. \tag{2.52}
$$

Here α, β and γ are some constants. This equation is known as the Gauss HG differential equation. Actually,

$$s = \frac{z-a}{b-a} \Rightarrow 1 - s = \frac{b-z}{b-a}.$$

From Taylor Expansion Theorem, $\tau(z) = A + Bz$ can be written as

$$\tau(z) = \tau(a) + \tau'(a)(z-a) \tag{2.53}$$

so if we substitute $\frac{dy}{dz}$, $\sigma(z)$, $\frac{d^2y}{dz^2}$ and $\tau(z)$, then (2.51) becomes

$$s(1-s)\frac{d^2y}{ds^2} + [\frac{\tau(a)}{b-a} + \tau'(a)s]\frac{dy}{ds} + \lambda y = 0.$$

Let $\gamma = \frac{\tau(a)}{b-a}$ and choose α and β satisfying

$$\begin{aligned} \text{(i) } \alpha\beta &= -\lambda \qquad \text{and} \\ \text{(ii)}\alpha + \beta + 1 &= -\tau' \end{aligned} \tag{2.54}$$

so we get

$$s(1-s)y'' + [\gamma - (\alpha + \beta + 1)s]y' - \alpha\beta y = 0. \tag{2.55}$$

CASE 2: Let $\sigma(z) = z - a$. By the substitution $z = a + bs$, (2.51) is transformed into the differential equation

$$sy'' + (\gamma - s)y' - \alpha y = 0 \tag{2.56}$$

which is known as the Confluent hypergeometrik equation. Actually, since

$$s = \frac{1}{b}(z-a)$$

and

$$\tau(z) = \tau(a) + \tau'(a)(z-a)$$

we have

$$(z-a)\frac{d^2y}{dz^2}+[\tau(a)+\tau'(a)(z-a)]\frac{dy}{dz}+\lambda y=0,$$

which leads to

$$sy''+[\tau(a)+\tau'(a)bs]y'+\lambda by=0. \tag{2.57}$$

If $\tau'\neq 0$ then we set $\tau' b=-1$ and we obtain

$$sy''+[\tau(a)-s]y'-\frac{\lambda}{\tau'}y=0.$$

By putting $\tau(a)=\gamma$ and $\frac{\lambda}{\tau'}=\alpha$ we get (2.56). But if $\tau'=0$ the differential equations (2.51) leads to the so-called Lammel equation. And if $\sigma(z)$ is independent of z it can be taken as $\sigma(z)=1$, in this case if $\tau'=0$, we obtain

$$y''+\tau(a)y'+\lambda y=0 \tag{2.58}$$

a second-order differential equation with constant coefficients. So we assume that $\tau'\neq 0$.

CASE 3: Let $\sigma(z)=1$ and $\tau'\neq 0$. By the substitution $z=a+bs$, (2.51) can be written as

$$y''-2y's+2vy=0. \tag{2.59}$$

Actually by the transformation $z=a+bs$ and the equality

$$\tau(z)=\tau(a)+\tau'(a)(z-a)$$

we get

$$\frac{d^2y}{ds^2}+[b\tau(a)+b^2\tau'(a)s]\frac{dy}{ds}+b^2\lambda y=0. \tag{2.60}$$

If $\tau'\neq 0$ then we set $b^2\tau'(a)=-2$ and choose a so that $\tau(a)=0$, and then putting $\lambda=\frac{2v}{b^2}$, we obtain (2.59) [29].

2.2.2 Gamma and Beta Functions

One of the simplest but very important special functions is the gamma function, denoted by $\Gamma(z)$. It appears occasionally by itself in physical applications, but much of its importance stems from its usefulness in developing other functions such as hypergeometric functions.

The gamma function can be defined by Euler's integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-t} dt, \quad R(z) > 0. \tag{2.61}$$

The basic and important functional relation for the gamma function is

$$\Gamma(z+1) = z\Gamma(z)$$

and since $\Gamma(1) = \Gamma(2) = 1$,

$$\Gamma(n+1) = 1.2.\cdots.n = n!.$$

It is important to note that $\Gamma(z)$ is continued anallytical over the whole complex plane except the negative integers and zero.

A formula involving gamma functions that is somewhat comparable to the double-angle formulas for trigonometric functions is the Legendre duplication formula

$$2\Gamma(2z) = \frac{2^{2z}}{\Gamma(1/2)} \Gamma(z)\Gamma(z + \frac{1}{2}). \tag{2.62}$$

The Euler reflection formula is,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{2.63}$$

A particular combination of gamma function is given a name because it has a simple and useful integral representation. The beta function is defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt. \tag{2.64}$$

where $R(\alpha) > 0$ and $R(\beta) > 0$.

This function has symmetry property

$$B(\alpha,\beta) = B(\beta,\alpha)$$

and it can represented by

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2.65}$$

2.2.3 Pochhammer Symbol

In dealing with certain product forms, factorials and gamma functions, it is useful to introduce the abbreviation.

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1,2,3,\cdots$$

called the Pochhammer (Apell) symbol. By the properties of the gamma function, it follows that this symbol can also be defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1,2,3,\cdots$$

The Pochhammer Symbol $(a)_n$ satisfies the identities:

(i) $(1)_n = n!$

(ii) $(a+n)(a)_n = a(a+1)_n$

(iii) $(a)_{n+k} = (a)_k(a+k)_n = (a)_n(a+n)_k$

Another symbol $(a)^{(n)}$ is defined by

$$(a)^{(n)} = a(a-1)\cdots(a-n+1), \quad n = 1,2,3,\cdots$$

2.2.4 Hypergeometric Series

The series

$$1+\frac{\alpha_1\alpha_2\cdots\alpha_A}{\gamma_1\gamma_2\cdots\gamma_B}\frac{z}{1!}+\frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)\cdots\alpha_A(\alpha_A+1)}{\gamma_1(\gamma_1+1)\gamma_2(\gamma_2+1)\cdots\gamma_B(\gamma_B+1)}\frac{z^2}{2!}+\cdots$$
$$\equiv\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_A)_n}{(\gamma_1)_n(\gamma_2)_n\cdots(\gamma_B)_n}\frac{z^n}{n!} \qquad (2.66)$$

is called the generalized hypergeometric function. It has A numerator parameters $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_A$, B denominator parameters $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_B$ and one variable z. Any of these quantities may be real or complex but the γ parameters must not be negative integers, as in that case the series is not defined. The sum of this series, when it exists, is denoted by the symbol

$${}_AF_B[\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_A; \gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_B; z].$$

If any of the α parameters is a negative integer, the function reduces to a polynomial. These notations can be shortened still further to

$$\sum_{n=0}^{\infty}\frac{((\alpha)_A)_n}{((\gamma)_B)_n}\frac{z^n}{n!} = {}_AF_B[(\alpha);(\gamma);z].$$

It can be shown that such a series can converge only for $A \leq B+1$. If $A = B+1$ it is convergent only for $|z| < 1$.

It is often convenient to employ a contracted notation and write (2.66) in the abbreviated form

$${}_AF_B(\alpha_A;\gamma_B;z) = {}_AF_B\left(\begin{array}{c|c} \alpha_A & \\ & z \\ \gamma_B & \end{array}\right) = \sum_{n=0}^{\infty}\Big[(\alpha_A)_n z^n/(\gamma_B)_n n!\Big]. \qquad (2.67)$$

The derivatives of hypergeometric series are series of the same kind. The following relations are of this type

$$\frac{d}{dz}\ {}_AF_B\left(\begin{array}{c|c}\alpha_A & \\ & z\\ \gamma_B & \end{array}\right)=\frac{(\alpha_A)}{(\gamma_B)}\quad {}_AF_B\left(\begin{array}{c|c}\alpha_{A+1} & \\ & z\\ \gamma_{B+1} & \end{array}\right)$$

$$\frac{d^n}{dz^n}\ {}_AF_B\left(\begin{array}{c|c}\alpha_A & \\ & z\\ \gamma_B & \end{array}\right)=\frac{(\alpha_A)_n}{(\gamma_B)_n}\ {}_AF_B\left(\begin{array}{c|c}\alpha_{A+n} & \\ & z\\ \gamma_{B+n} & \end{array}\right). \tag{2.68}$$

Many of the elementary functions have representations as hypergeometric series. Here are some examples:

$$\frac{1}{z}\log(1+z) = \quad {}_2F_1(1,1;2;-z)$$

$$\frac{1}{z}\arctan z = \quad {}_2F_1(\frac{1}{2},1,\frac{3}{2};-z^2)$$

$${}_2F_1(\alpha,0;\gamma,z) = \quad {}_1F_1(0;\gamma;z)=1$$

$${}_1F_1(\alpha;\alpha;z)=e^z \quad {}_1F_1(0;\alpha;-z)=e^z.$$

We shall confine ourselves to the two seperate cases:

$A=B=1$, in which case the function will be called the Confluent hypergeometric function and $A=2$, $B=1$, we shall merely call it the hypergeometric function.

The integral representations can be found by Euler's formula which are,

$${}_2F_1(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}dt,$$
$$R(\gamma)>R(\alpha)>0, \quad |arg\ (1-z)|<\pi \tag{2.69}$$

$${}_1F_1(\alpha;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 e^{zt}t^{\alpha-\beta-1}dt, \quad R(\gamma)>R(\alpha)>0. \tag{2.70}$$

Gauss hypergeometric differential equation,

$$z(1-z)y'' + [\gamma - (\alpha+\beta+1)]y' - \alpha\beta y = 0$$

has two linearly independent solutions which can be denoted by generalized hypergeometric functions as

$$\begin{aligned} w_1 &= \ {}_2F_1(\alpha, \beta; \gamma; z) \\ w_2 &= z^{1-\gamma}\ {}_2F_1(1+\alpha-\beta, 1+\beta-\gamma; 2-\gamma; z) \end{aligned}$$

provided that γ is not an integer or zero.

The linearly independent solutions of confluent differential equations,

$$zy'' + (\gamma - z)y' - \alpha y = 0,$$

are proportional to

$$\begin{aligned} w_1 &= \ {}_1F_1(\alpha; \gamma; z) \\ w_2 &= z^{1-\gamma}\ {}_1F_1(1+\alpha-\gamma, 2-\gamma; z) \end{aligned}$$

provided γ is not an integer.

CHAPTER 3

Orthogonal Polynomials

It is important to remember that the classical orthogonal polynomials are the solutions of differential equatons of hypergeometric type. Firstly let us define the orthogonality.

3.1 Orthogonality

Definition 3.1. An orthonormal set of functions $\phi_0(x), \phi_1(x), \cdots, \phi_\ell(x)$, ℓ finite or infinite, is defined by the relation

$$(\phi_n, \phi_m) = \int_a^b \phi_n(x)\phi_m(x)d\alpha(x) = \delta_{mn}, \quad n, m = 0, 1, 2, \cdots, \ell$$

Here $\phi_n(x)$ is real-valued and belongs to the class L^2_α (a, b) and $\alpha(x)$ is a fixed non-decreasing function which is not constant in the interval $[a, b]$.

Functions of this kind are necessarily linearly independent. If $\alpha(x)$ has only a finite number N of points of increase (that is, points in the neighborhood of which $\alpha(x)$ is not constant), ℓ is necessarily finite and $\ell < N$ [28].

Theorem 3.1. *Let the real-valued functions*

$$f_0(x), f_1(x), f_2(x), \cdots, f_\ell(x), \tag{3.1}$$

ℓ finite or infinite, be of the class $L^2_\alpha(a, b)$ and linearly independent. Then an orthonormal set

$$\phi_0(x), \phi_1(x), \phi_2(x), \cdots, \phi_\ell(x) \tag{3.2}$$

exists such that, for $n = 0, 1, 2, \cdots, \ell$,

$$\phi_n(x) = \lambda_{n_0} f_0(x) + \lambda_{n_1} f_1(x) + \cdots + \lambda_{nn} f_n(x), \quad \lambda_{nn} > 0. \tag{3.3}$$

The set (3.2) is uniquely determined.

The procedure of derving (3.2) from (3.1) is called orthogonalization.

3.2 Orthogonal Polynomials

Definition 3.2. Let $\alpha(x)$ be a fixed non-decreasing function with infinitely many points of increase in the finite or infinite interval $[a, b]$ and let the moments

$$c_n = \int_a^b x^n d\alpha(x), \quad n = 0, 1, 2, \cdots \tag{3.4}$$

exist. If we orthogonalize the set of non-negative powers of x:

$$1, x, x^2, \cdots, x^n, \cdots, \tag{3.5}$$

we obtain a set of polynomials

$$p_0(x), p_1(x), p_2(x), \cdots, p_n(x) \tag{3.6}$$

uniquely determined by the following conditions:

(a) $p_n(x)$ is a polynomial of precise degree n in which the coefficient of x^n is positive;

(b) the system $\{p_n(x)\}$ is orthonormal, that is

$$\int_a^b p_n(x) p_m(x) d\alpha(x) = \delta_{mn} \quad n, m = 0, 1, 2, \cdots \tag{3.7}$$

The existence of the moments (3.4) is equivalent to the fact that the functions x^n are of the class $L_\alpha(a, b)$ [28].

Theorem 3.2. *If the $p_n(x)$ form a simple set of real polynomials and $w(x) > 0$*

on $a < x < b$, a necessary and sufficient condition that the set $p_n(x)$ be orthogonal with respect to $w(x)$ over the interval $a < x < b$ is that

$$\int_a^b w(x)x^k p_n(x)dx = 0, \quad k = 0, 1, 2, \cdots (n-1) \tag{3.8}$$

Proof. Suppose (3.8) is satisfied since x^k forms a simple set, there exist constants $b(k, m)$ such that

$$p_m(x) = \sum_{k=0}^{m} b(k, m)x^k \tag{3.9}$$

For the moment, let $m < n$. Then

$$\int_a^b w(x)p_n(x)p_m(x)dx = \int_{k=0}^{m} b(k, m) \int_a^b w(x)x^k p_n(x)dx = 0$$

since m, and therefore each k is less than n. If $m > n$, interchange m and n in the above argument. We have shown that if (3.8) is satisfied, it follows that

$$\int_a^b w(x)p_n(x)p_m(x)dx = 0, \quad m \neq n. \tag{3.10}$$

Now suppose (3.10) is satisfied. The $p_n(x)$ form a simple set, so there exist constans $a(m, k)$ such that

$$x^k = \sum_{m=0}^{k} a(m, k)p_m(x). \tag{3.11}$$

For any k in the range $0 \leq k < n$

$$\int_a^b w(x)x^k p_n(x)dx = \sum_{m=0}^{k} a(m, k) \int_a^b w(x)p_m(x)p_n(x)dx = 0,$$

since $m \leq k < n$ so that $m \neq n$. Therefore (3.8) follows (3.10) and the proof of theorem is complete [25]. $\square$

From Teorem 3.2 we obtain at once that orthogonal set p_n has the propety that

$$\int_a^b w(x)P(x)p_n(x)dx = 0,$$

for every polynomial $P(x)$ of degree $< n$. It is useful to note that since

$$\int_a^b w(x)p_n^2(x)dx \neq 0$$

it follows that also

$$\int_a^b w(x)x^n p_n(x)dx \neq 0.$$

3.3 Classical Orthogonal Polynomials

From Chapter 2 we know that, the differential equation of the HG type,

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 \tag{3.12}$$

has polynomial solutions which given by Rodrigues formula,

$$y_n(z) = \frac{B_n}{\rho(z)}\frac{d^n}{dz^n}[\sigma^n(z)\rho(z)] \tag{3.13}$$

for particular values of $\lambda = \lambda_n = -n\tau'(z) - \frac{1}{2}n(n-1)\sigma''(z),\ n = 0, 1, 2, \cdots$ and the function $\rho(z)$ saties the seperable differential equation

$$[\sigma(z)\rho(z)]' = \tau(z)\rho(z) \tag{3.14}$$

By some choice of $\sigma(z)$, $\tau(z)$ and λ in (3.12), we obtain classical orthogonal polynomials.

3.3.1 Jacobi Polynomials

Let $\sigma(z) = 1 - z^2$ and $\rho(z) = (1-z)^\alpha(1+z)^\beta$. From (3.14)

$$[(1-z^2)(1-z)^\alpha(1+z)^\beta]' = \tau(z)(1-z)^\alpha(1+z)^\beta$$

one obtains

$$\tau(z) = -(\alpha+\beta+2)z + \beta - \alpha \tag{3.15}$$

and

$$\lambda = \lambda_n = -n\tau'(z) - \frac{1}{2}n(n-1)\sigma''(z)$$

which leads to

$$\lambda_n = n(n+\alpha+\beta+1). \tag{3.16}$$

From Rodrigues formula the corresponding polynomials are denoted by

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}[(1-z)^{n+\alpha}(1+z)^{n+\beta}] \tag{3.17}$$

where $B_n = \frac{(-1)^n}{n!}$ is chosen for historical reasons. By substituting $\sigma(z)$, $\tau(z)$ and λ we obtain

$$(1-z^2)y'' + [\beta - \alpha - (\alpha+\beta+2)z]y' + n(n+\alpha+\beta+1)y = 0 \tag{3.18}$$

that $P_n^{(\alpha,\beta)}$ satisfies. The differential equation (3.18) can be transformed into a Gauss hypergeometric equation by substitution of $z = 1-2s$. Actually, (3.18) takes the form

$$s(1-s)y'' + [\gamma' - (\alpha'+\beta'+1)s]y' - \alpha'\beta'y = 0 \tag{3.19}$$

with $\alpha' = n$, $\beta' = n+\alpha+\beta+1$ and $\gamma' = \alpha+1$. The equation (3.19) has polynomial solutions,

$$y(z) \quad = \quad {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}) \tag{3.20}$$

which leads to

$$P_n^{(\alpha,\beta)}(z) = C_n\ {}_2F_1(-n, n+\alpha+\beta+1;\ \alpha+1;\ \ \frac{1-z}{2}). \tag{3.21}$$

Let us remember the Leibnitz's rule for the derivatives of product

$$\frac{d^n}{dz^n}[u(z)v(z)] = \sum_{k=0}^{n} \binom{n}{k} [D^k u][D^{n-k} v], \quad D = \frac{d}{dz}, \quad D^k z^{n+\alpha} = \frac{(\alpha+1)_n}{(\alpha+1)_{\alpha-n}} z^{n+\alpha-k}.$$

Applying Leibnitz's rule for the derivatives of a product, it's seen from (3.17) that

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+1)_{n-k}(\beta+1)_k} (z-1)^{n-k}(z+1)^k \tag{3.22}$$

or equivalently

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)^n}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(z-1)^{n-k}(z+1)^k}{\Gamma(\alpha+1+n-k)\Gamma(\beta+1+k)} \tag{3.23}$$

Particularly we have

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)n}{n!}, \qquad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n}{n!}(\beta+1)_n. \tag{3.24}$$

Hence we can write the Jacobi polynomials in terms of the Gauss hypergeometric function

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{n!}(\alpha+1)_n \; {}_2F_1(-n, n+\alpha+\beta+1;\ \alpha+1; \frac{1-z}{2}) \tag{3.25}$$

By using the derivative of Gauss HG function in (3.25) we find

$$\begin{aligned}
\frac{d}{dz} P_n^{(\alpha,\beta)}(z) &= \frac{-1}{2} \frac{(\alpha+1)_n}{n!} \frac{(-n)(n+\alpha+\beta+1)}{\alpha+1} \; {}_2F_1(-n+1; n+\alpha+\beta+2; \alpha+2; \frac{1-z}{2}) \\
&= \frac{1}{2}(n+\alpha+\beta+1) \frac{(\alpha+1)_n}{(\alpha+1)(\alpha+2)_{n-1}} \frac{(\alpha+2)_{n-1}}{(n-1)!} \\
& \quad {}_2F_1(-(n-1), (n-1)+(\alpha+1)+(\beta+1)+1; (\alpha+1)+1; \frac{1-z}{2})
\end{aligned}$$

follows by

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z) \quad = \quad \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(z) \tag{3.26}$$

which is the differentiation formula for Jacobi Polynomials.

3.3.2 Laguerre Polynomials

Let $\sigma(z) = z$ and $\rho(z) = z^\alpha e^{-z}$, from (3.14) we find

$$\tau(z) = \alpha + 1 - z \quad \text{and} \quad \lambda_n = n. \tag{3.27}$$

These poynomials are the generalized Laguerre polynomials denoted by

$$L_n^\alpha(z) = \frac{1}{n!}e^z z^{-\alpha}\frac{d^n}{dz^n}[z^{n+\alpha}e^{-z}] \tag{3.28}$$

where $B_n = \frac{1}{n!}$. The differential equation that $L_n^\alpha(z)$ satisfies can be found by substituting $\sigma(z),\ \tau(z)$ and λ which is

$$zy'' + (\alpha+1-z)y' + ny = 0. \tag{3.29}$$

If we apply Leibnitz's rule for derivatives of a product, we get

$$L_n^\alpha(z) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{(-z)^{n-k}}{(\alpha+1)_{n-k}}. \tag{3.30}$$

So we have

$$L_n^\alpha(z) = \frac{(\alpha+1)}{n!}\ {}_1F_1(-n;\alpha+1;z). \tag{3.31}$$

Particularly

$$L_n^\alpha(0) \quad = \quad \frac{1}{n!}(\alpha+1)_n = P_n^{(\alpha,\beta)}(1). \tag{3.32}$$

$L_n^{(\alpha)}(z)$ form a simple set of polynomials as

$$
\begin{aligned}
L_0^{(\alpha)}(z) &= 1, \quad L_1^{(\alpha)}(z) = 1 + \alpha - z \\
L_2^{(\alpha)}(z) &= \frac{1}{2}(1+\alpha)(2+\alpha) - (2+\alpha)z + \frac{1}{2}z^2 \\
L_3^{(\alpha)}(z) &= \frac{1}{6}(1+\alpha)(2+\alpha)(3+\alpha) - \frac{1}{2}(2+\alpha)(3+\alpha)z + \frac{1}{2}(3+\alpha)z^2 - \frac{1}{6}z^3.
\end{aligned}
\tag{3.33}
$$

Differentiating both sides of (3.32) with respect to z we get,

$$\frac{d}{dz}L_n^{\alpha}(z) = \frac{(-n)}{\alpha+1}\frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n+1; \alpha+2; z)$$

or

$$\frac{d}{dz}L_n^{\alpha}(z) = \frac{-1}{(n+1)!}(\alpha+2)_{n-1}\ {}_1F_1(-(n-1); (\alpha+1)+1; z).$$

Then one obtains

$$\frac{d}{dz}L_n^{\alpha}(z) = -L_{n-1}^{\alpha+1}(z). \tag{3.34}$$

3.3.3 Hermite Polynomials

These are the last class of polynomials that we study on. Let $\sigma(z) = 1$, $\rho(z) = e^{-z^2}$. From the equality $[\sigma(z)\rho(z)]' = \tau(z)\rho(z)$ and (2.32) we obtain $\tau(z) = -2z$ and $\lambda = 2n$. Then the corresponding polynomials are defined by

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}[e^{-z^2}], \quad \text{where} \quad B_n = (-1)^n.$$

The derivation of the following properties of Hermite polynomials presents no difficulty:

$$H_{2m}(0) = (-1)^n\frac{(2m)!}{m!}, \quad H'_{2m+1}(0) = (-1)^m\frac{(2m+1)!}{(m+1)!}$$

We notice the following individual properties:

$$H_{n'}(z) = 2nH_{n-1}(z)$$
$$H_n(z) = 2zH_n(z) - H'_{n-1}(z)$$

The differential equation that Hermite polynomials satisfies,

$$y'' - 2zy' + 2vy = 0, \quad \text{with} \quad v = n = 0, 1, 2, \cdots \tag{3.35}$$

It's seen that Hermite polynomials are the solutions of Hermite differential equation.

3.4 Basic Properties of Polynomials of the Hypergeometric Type

Suppose $y_n(z)$ be a polynomial of order n and it is written in the form,

$$y_n(z) = a_n z^n + b_n z^{n-1} + \cdots, \quad y_0(z) = a_0, \quad a_n \neq 0 \tag{3.36}$$

where a_n and b_n are the coefficient of the highest order terms. Let us differentiate with respect to z both side of (3.36) k times,

$$y_n^{(k)}(z) = \frac{n!}{(n-k)!} a_n z^{n-k} + \frac{(n-1)!}{(n-k-1)!} b_n z^{n-k-1} + \cdots \tag{3.37}$$

we have

$$y_n^{(n-1)}(z) = n! a_n z + (n-1)! b_n. \tag{3.38}$$

For $k = n-1$, from (2.29) we get

$$y_n^{(n-1)}(z) = \frac{A_{n-1,n} B_n}{\sigma^{n-1}(z)\rho(z)} \frac{d}{dz}[\sigma^n(z)\rho(z)] \tag{3.39}$$

which leads to

$$y_n^{(n-1)}(z) = A_{n-1,n}B_n[(n-1)\sigma'(z) + \tau(z)] \tag{3.40}$$

By equation (3.38) and (3.40) we obtain

$$n!a_n z + (n-1)!b_n = A_{n-1,n}B_n\{[(n-1)\sigma''(z) + \tau'(z)]z + (n-1)\sigma'(0) + \tau(0),$$

$$a_n = \frac{1}{n!}A_{n-1,n}B_n[(n-1)\sigma''(z) + \tau'(z)] \tag{3.41}$$

and

$$\frac{b_n}{a_n} = n\frac{(n-1)\sigma'(0) + \tau(0)}{(n-1)\sigma''(z) + \tau'(z)} \tag{3.42}$$

where

$$A_{mn} = \frac{n!}{(n-m)!}\prod_{k=0}^{m-1}[\tau' + \frac{1}{2}(n+k-1)\sigma'']. \tag{3.43}$$

For Hermite polynomials, $\sigma(z) = 1$, $\tau(z) = -2z$ and $\lambda_n = 2n$ then by using (3.41) and (3.42) we get

$$A_{n-1,n} = \frac{n!}{(n-n+1)!}\prod_{k=0}^{n-2}[(-2z)' + \frac{1}{2}(n+k-1)0] = n!(-2)^{n-1}, \tag{3.44}$$

$$a_n = \frac{1}{n!}n!(-2)^{n-1}(-1)^n[0-2] = (-2)^n(-1)^n = 2^n \quad \text{where} \quad B_n = (-1)^n$$

and

$$\frac{b_n}{a_n} \quad = \quad n\frac{(n-1)\sigma'(0) + \tau(0)}{(n-1)\sigma'' + \tau'} = 0. \tag{3.45}$$

For Laguerre polynomial $\sigma(z) = z$ and $\tau(z) = \alpha + 1 - z$. Then one obtains

$$\begin{aligned} A_{n-1,n} &= \frac{n!}{1!}\prod_{k=0}^{n-2}[(\alpha+1-z)' + \frac{1}{2}(n+k-1)0] \\ &= n!\prod_{k=0}^{n-2}(-1) = n!(-1)^{n-1} \end{aligned} \tag{3.46}$$

and

$$a_n = \frac{1}{n!}A_{n-1,n}B_n[(n-1)\sigma'' + \tau'] = \frac{(-1)^n}{n!}, \quad B_n = 1/n!, \tag{3.47}$$

$$\frac{b_n}{a_n} = n\frac{(n-1)\sigma'(0) + \tau(0)}{(n-1)\sigma'' + \tau'} = -n(n+\alpha). \tag{3.48}$$

For Jacobi polynomials if we substitute $\sigma(z) = 1 - z^2$ and $\tau(z) = \beta - \alpha - (\alpha + \beta + 2)z$ into (3.41) and (3.42) then we obtain

$$\begin{aligned} A_{n-1,n} &= n!\prod_{k=0}^{n-2}[-(\alpha+\beta+2) + \frac{1}{2}(n+k-1)(-2)] \\ &= n!(-1)^{n-1}(\alpha+\beta+n+1)_{n-1} \end{aligned} \tag{3.49}$$

and

$$a_n = \frac{1}{n!}A_{n-1,n}B_n[(n-1)\sigma'' + \tau'] = \frac{1}{2^n n!}(\alpha+\beta+n+1)_n \tag{3.50}$$

where $B_n = \frac{(-1)^n}{2^n n!}$ and

$$\frac{b_n}{a_n} = n\frac{(n-1)\sigma'(0) + \tau(0)}{(n-1)\sigma'' + \tau'} = -n\frac{\beta - \alpha}{\alpha + \beta + 2n} \tag{3.51}$$

3.4.1 Generating Function

A function $\Phi(z, t)$ is called generating function for each polynomial of the hypergeometric type, such that

$$\Phi(z,t)=\sum_{n=0}^{\infty}\tilde{y}_n(t)\frac{t^n}{n!},\quad y_n(z)=B_n\tilde{y}_n(z) \tag{3.52}$$

where

$$\tilde{y}_n(z)=\frac{1}{\rho(z)}D^n[\sigma^n(z)\rho(z)]. \tag{3.53}$$

It can be shown that the generating function so defined is a function of two variables whose coefficients of its expansion in powers of t consist of polynomials of the HG type. Such an expansion is valid at least for sufficiently small $|t|$.

Let us recall the integral form of polynomial solutions

$$\tilde{y}_n(s)=\frac{1}{\rho(z)}\frac{n!}{2\pi i}\int_{\varphi}\frac{\sigma^n(s)\rho(s)}{(s-z)^{n+1}}ds \tag{3.54}$$

where φ is a closed contour surrounding the point at $s=z$ then we can write,

$$\Phi(z,t)=\sum_{n=0}^{\infty}\frac{1}{\rho(z)}\frac{n!}{2\pi i}\left[\int_{\varphi}\frac{\sigma^n(s)\rho(s)}{(s-z)^{n+1}}ds\right]\frac{t^n}{n!}. \tag{3.55}$$

For some z fixed and sufficiently small $|t|$, we can justify the interchange of summation and integration which follows that

$$\begin{aligned}\Phi(z,t) &= \frac{1}{\rho(z)2\pi i}\int_{\varphi}\frac{\rho(s)}{s-z}\left\{\sum_{n=0}^{\infty}[\frac{\sigma(s)t}{s-z}]^n\right\}ds\\ &= \frac{1}{2\pi i\rho(s)}\int_{\varphi}\frac{\rho(s)}{s-z-\sigma(s)t}dt\end{aligned} \tag{3.56}$$

since

$$\sum_{n=0}^{\infty}[\frac{\sigma(s)t}{s-z}]^n=\frac{1}{1-[\frac{\sigma(s)t}{s-z}]}=\frac{s-z}{s-z-\sigma(s)t}. \tag{3.57}$$

We suppose that there is only one zero of the denominator of the integrand located at $s = \xi_0(z,t)$ which is in the near vicinity of $s = z$ and we assume that such a single pole of the integrand of (3.54) lies inside φ so that

$$\Phi(z,t) = \frac{1}{\rho(z)} Res_{\xi_0} \left[\frac{\rho(s)}{s - z - \sigma(s)t} \right] \tag{3.58}$$

where, $s - z - \sigma(s)t \Big|_{s=\xi_0} = 0$, hence $Res_{\xi_0} f(s) = \lim_{s \to \xi_0} (s - \xi_0) f(s)$.

$$\begin{aligned} Res_{\xi_0} \left[\frac{\rho(s)}{s - z - \sigma(s)t} \right] &= \lim_{s \to \xi_0} (s - \xi_0) \frac{\rho(s)}{s - z - \sigma(s)t} \\ &= \lim_{s \to \xi_0} \frac{\rho(s) + (s - \xi_0)\rho'(s)}{1 - \sigma'(s)t} = \frac{\rho(s)}{1 - \sigma'(s)t} \Big|_{s=\xi_0(z,t)} \end{aligned}$$

which leads to

$$\Phi(z,t) = \frac{\rho(s)}{\rho(z)} \frac{1}{1 - \sigma'(s)t} \Big|_{s=\xi_0(z,t)} \tag{3.59}$$

for sufficiently small $|t|$. By the principle of analytic continuation, this is valid for $|t| < R$. For a fixed z where R is the distance from the origin to the nearest singular point of $\Phi(z,t)$ with respect to t.

Generating Function for Laguerre Polynomials

If we substitute $\sigma(z) = z$ and $\rho(z) = z^\alpha e^{-z}$ into (3.59) then we have

$$\Phi(z,t) = \frac{\rho(s)}{\rho(z)} \frac{1}{1 - \sigma'(s)t} \Big|_{s=\xi_0} = (\frac{s}{z})^\alpha e^{z-s} \frac{1}{1-t} \Big|_{s=\xi_0} \tag{3.60}$$

ξ_0 is the root of $s - z - zt$, so it can be found $\xi_0 = z + zt$. If we insert ξ_0 into (3.60) then we get

$$\Phi(z,t) = (1+t)^\alpha e^{-zt} (1-t)^{-1} = \sum_{n=0}^{\infty} L_n^\alpha(z) \tag{3.61}$$

which is the generating function for Laguerre Polynomials.

Generating Function for Hermite Polynomials

By substituting $\sigma(z) = 1$ and $\rho(z) = e^{-z^2}$ into (3.60) we get

$$\Phi(z,t) = e^{z^2 - s^2}\Big|_{s=\xi_0(z,t)} = \sum_{n=0}^{\infty}(-1)^n H_n(z)\frac{t^n}{n!}$$

Here ξ_0 is the root of $s - z - \sigma(s)t$. So $s = \xi_0 = z + t$. From (3.60) we have

$$e^{z^2-(z+t)^2} = \sum_{n=0}^{\infty}(-1)^n H_n(z)\frac{t^n}{n!}$$

or

$$e^{-2zt-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{(-t)^n}{n!}.$$

Substituting $\ell = -t$ we obtain

$$e^{-2z\ell-\ell^2} = \sum_{n=0}^{\infty} H_n(z)\frac{s^n}{n!}$$

and instead of s putting t one obtains

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}. \tag{3.62}$$

Generating Function for Jacobi Polynomials

Generating function for Jacobi Polynomials can be found by using (3.60) as

$$2^{\alpha+\beta}(1-2zt+t^2)^{-1}(1-t+\sqrt{1-2zt+t^2})^{-\alpha}(1+t+\sqrt{1-2zt+t^2})^{-\beta} = \sum_{n=0}^{\infty} p_n^{(\alpha,\beta)}(z)t^n \tag{3.63}$$

3.4.2 Orthogonality of Polynomial of the Hypergeometric Type

Let us consider the orthogonality properties of classical polynomials by the following theorem

Theorem 3.3. *Let the coefficients in the differential equation*

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0$$

be such that

$$\sigma(x)\rho(x)x^k\Big|_{x=a,b} = 0 \quad \textit{for} \quad k = 0, 1, \cdots \tag{3.64}$$

at the boundaries of an x-interval (a, b). Then the polynomials of the HG type which constitute a sequence of real functions of the real argument x

$$\{y_0(x), y_1(x), \cdots, y_m(x), \cdots, y_n(x), \cdots\} \tag{3.65}$$

corresponding to different values of $\lambda = \lambda_n$, i.e.

$$\lambda_0, \lambda_1, \cdots, \lambda_m, \cdots, \lambda_n, \cdots \tag{3.66}$$

are orthogonal on (a, b) in the sense that

$$\int_a^b \rho(x) y_m(x) y_n(x) dx = 0 \tag{3.66}$$

for $m \neq n$ where $\rho(x)$ is now called the weighting function.

Proof. The elements y_n and y_m of sequence satisfy the differential equations

$$y_m/[\sigma(x)\rho(x)y_n']' + \lambda_n \rho(x) y_n = 0 \tag{3.67}$$

$$y_n/[\sigma(x)\rho(x)y'_m]' + \lambda_m\rho(x)y_m = 0 \tag{3.68}$$

multiplying the first by y_n and second by y_m and subtracting we obtain

$$y_m(\sigma\rho y'_n)' - y_n(\sigma\rho y'_m)' = (\lambda_m - \lambda_n)\rho y_m y_n$$

which is equal to

$$\frac{d}{dx}[\sigma\rho W(y_m, y_n)] = (\lambda_m - \lambda_n)\rho y_m y_n \tag{3.69}$$

where $W(y_m, y_n)$ is the wronskiyen. Integrating both sides from a to b, we get

$$(\lambda_m - \lambda_n)\int_a^b \rho(x)y_m(x)y_n(x)dx = \sigma(x)\rho(x)W(y_m, y_n)\Big|_a^b = 0. \tag{3.70}$$

From the hypothesis the left hand side is equal to zero. Hence, for $m \neq n$ ($\lambda_m \neq \lambda_n$) we must have

$$\int_a^b \rho(x)y_m(x)y_n(x)dx = 0 \tag{3.71}$$

and more specifically we may write

$$\int_a^b \rho(x)y_m(x)y_n(x)dx = \mathcal{N}_n^2\delta_{mn} = \begin{cases} 0 & , \quad m \neq n \\ \mathcal{N}_n^2 & , \quad m = n \end{cases} \tag{3.72}$$

where $\mathcal{N}_n$ is a normalization constant [29]. $\square$

Orthogonality of Laguerre Polynomials

We know for Laguerre polynomials $\sigma(x) = x$ and $\rho(x) = e^{-x}x^{\alpha}$ so by using

Theorem 3.3 one obtains

$$\sigma(x)\rho(x)x^k\Big|_{a,b} = e^{-x}x^{\alpha+1+k}\Big|_{x=0,\infty} = 0$$

where $(a,b) = (0,\infty)$, then these polynomials are orthogonal provided that $\alpha > -1$

Orthogonality of Hermite Polynomials

For Hermite Polynomials $\sigma(x) = 1$ and $\rho(x) = e^{-x^2}$, so

$$\sigma(x)\rho(x)x^k\Big|_{a,b} = e^{-x^2}x^k\Big|_{\mp\infty} = 0$$

where $(a,b) = (-\infty,+\infty)$.

Orthogonality of Jacobi Polynomials

For Jacobi Polynomial $\sigma(x) = 1 - x^2$ and $\rho(x) = (1+x)^\beta(1-x)^\alpha$, then

$$\sigma(x)\rho(x)x^k\Big|_{a,b} = (1-x)^{\alpha+1}(1+x)^{\beta+1}x^k\Big|_{x=\mp1} = 0$$

where $(a,b) = (-1,1)$. So Jacobi polynomials are orthogonal provided that $\alpha > -1$ and $\beta > -1$.

CHAPTER 4

Bessel Polynomials

4.1 Generalized and Reversed Generalized Bessel Polynomials

We have mentioned in Chapter 2 that the case in which $\sigma(z)$ has a double root, i.e. $\sigma(z) = (z - z_0)^2$, can be treated in a different way.

Clearly the differential equation of the HG type

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 \tag{4.1}$$

with $\sigma(z) = (z - z_0)^2$ has polynomial solutions $y_n(z)$ of the HG type whenever

$$\lambda = \lambda_n = -n\tau'(z) - \frac{1}{2}n(n-1)\sigma''(z) = -n[\tau'(z) + n - 1]. \tag{4.2}$$

Taking $\tau'(z) = a$ and writing the Taylor polynomial of $\tau(z)$ about $z = z_0$, we have

$$\tau(z) = \tau(z_0) + a(z - z_0) \tag{4.3}$$

so that the differential equation (4.1) can be written as

$$y'' + \left[\frac{a}{z - z_0} + \frac{\tau(z_0)}{(z - z_0)^2}\right] y' + \frac{\lambda_n}{(z - z_0)^2}\, y = 0. \tag{4.4}$$

If we introduce a new variable s such that

$$s = \frac{1}{z - z_0} \quad \text{or} \quad z - z_0 = \frac{1}{s},$$

$$\frac{ds}{dz} = -s^2, \quad \frac{d^2s}{dz^2} = 2s^3$$

$$\frac{dy}{dz} = -s^2\frac{dy}{ds}, \quad \frac{d^2y}{dz^2} = s^4\frac{d^2y}{ds^2} + 2s^3\frac{dy}{ds},$$

then, the differential equation in (4.4) is transformed to

$$s^2\frac{d^2y}{ds^2} + s[2 - a - \tau(z_0)s]\frac{dy}{ds} + \lambda_n y = 0. \tag{4.5}$$

This is not an equation of the HG type but it is a generalized differential equation. Therefore, by means of the substitution

$$y = \phi(s)u \tag{4.6}$$

and the identities

$$\frac{y'}{y} = \frac{\phi'}{\phi} + \frac{u'}{u}, \qquad \frac{y''}{y} = \frac{u''}{u} + 2\frac{\phi'}{\phi}u' + \frac{\phi''}{\phi}$$

we transform the last differential equation into the HG type. Actually we obtain,

$$\left[\frac{u''}{u} + \frac{2\phi'}{\phi}\frac{u'}{u} + \frac{\phi''}{\phi}\right]s^2 + \left[(\frac{\phi'}{\phi} + \frac{u'}{u})\frac{-f(s)}{s}\right] + \lambda_n = 0$$

leading to

$$u'' + \left[2\frac{\phi'}{\phi} - \frac{f(s)}{s}\right]u' + \left[\frac{\phi''}{\phi} - \frac{f(s)}{s}\frac{\phi'}{\phi} + \frac{\lambda_n}{s^2}\right]u = 0 \tag{4.7}$$

where $f(s) = \tau(z_0)s + a - 2$ is a polynomial in s of degree 1. A suitable form for $\phi(s)$ is s^{-n}. Actually, since

$$\phi(s) = s^{-n}, \quad \frac{\phi'}{\phi} = -\frac{n}{s}, \quad \frac{\phi''}{\phi} = \frac{n^2+n}{s^2}, \quad n = 0, 1, \cdots .$$

we obtain the differential equation

$$su'' - [2n + f(s)]u' + n\tau(z_0)u = 0 \tag{4.8}$$

which is an equation of the HG type with

$$\tilde{\sigma}(s) = s, \quad \tilde{\tau}(s) = -[2n + a - 2 + \tau(z_0)s], \quad \tilde{\lambda} = n\tau(z_0).$$

This equation has also polynomial solutions because,

$$\tilde{\lambda} = \tilde{\lambda}_n = -n\tilde{\tau}' - \frac{1}{2}n(n-1)\tilde{\sigma}'' = -n[-\tau(z_0)] + 0 = n\tau(z_0). \tag{4.9}$$

The self-adjoint form of (4.8) implies that

$$[s\rho(s)]' = -[2n + f(s)]\rho(s) = -[2(n-1) + a + \tau(z_0)s]\rho(s)$$

which leads to

$$\rho(s) = s^{1-2n-a}e^{-\tau(z_0)s}.$$

Rodrigues formula defining polynomial solutions, say $u_n(s)$, of (4.8) is

$$u_n(s) = B_n s^{a+2n-1} e^{\tau(z_0)s} \frac{d^n}{ds^n}[s^{1-a-n}e^{-\tau(z_0)s}] \tag{4.10}$$

where B_n is a constant. In fact these polynomials are closely related to the Laguerre polynomials L_n^{α}. With $t = \tau(z_0)s$, $\alpha = 1 - a - 2n$, the differential equation (4.8) takes the form

$$tu''(t) + (\alpha + 1 - t)u'(t) + nu(t) = 0 \tag{4.11}$$

which is the Laguerre differential equation. Thus the polynomial solutions are given by

$$u(t) = L_n^{\alpha}(t)$$

so the polynomials in (4.10) are expressible as

$$u_n(s) = C_n L_n^{\alpha}(\tau(z_0)s) \tag{4.12}$$

in terms of the generalized Laguerre polynomials. It follows then that,

$$y_n(s) = \frac{1}{s^n} u_n(s) = s^{-n} L_n^{\alpha}(\tau(z_0)s)$$

and

$$y_n(z) = C_n (z - z_0)^n L_n^{1-a-2n}\left(\frac{\tau(z_0)}{z - z_0}\right) \tag{4.13}$$

standing for the polynomial solutions of the original equation in (4.1), where C_n is a constant. It is important to note that we have still polynomials of degree n even if $\tau(z_0) = 0$, in which case the differential equation (4.1) reduces to a Cauchy-Euler equation. The best known polynomials of the HG type falling into this catagory, where $\sigma(z) = (z - z_0)^2$ are the generalized Bessel polynomials denoted by $y_n(z; a, b)$. More specifically the generalized Bessel polynomials are the solutions of the equation

$$z^2 y'' + (b + az)y' - n(a + n - 1)y = 0. \tag{4.14}$$

in which $z_0 = 0$, $\tau(z_0) = b$ and $\sigma(z) = z^2, \quad \tau(z) = b + az, \quad \lambda_n = -n(a + n - 1)$. The connection with the Laguerre polynomials is written as

$$y_n(z; a, b) = C_n z^n L_n^{1-a-2n}\left(\frac{b}{z}\right), \qquad C_n = \left(\frac{-1}{b}\right)^n n! \tag{4.15}$$

and the weighting function $\rho(z)$ satisfies

$$[z^2 \rho(z)]' = (b + az)\rho(z)$$

from which

$$\rho(z) = e^{-b/z} z^{a-2} \tag{4.16}$$

is obtained. Therefore

$$y_n(z;a,b) = B_n z^{2-a} e^{b/z} D^n[z^{2n+a-2} e^{-b/z}], \quad B_n = (\frac{1}{b})^n \tag{4.17}$$

is the Rodrigues formula for the generalized Bessel polynomials [29].

In fact the value of the constant b is not important and a linear transformation for z transforms (4.15) into

$$z^2 y'' + (2 + az)y' - n(a + n - 1)y = 0. \tag{4.18}$$

In this case instead of $y_n(z;a,b)$ we can consider a one parameter family of functions denoted by $y_n(z,a)$. All formulas above can be reproduced for $y_n(z,a)$ if b is taken 2. Notice that we get polynomial solutions for each n, $\quad n = 0, 1, 2, \cdots$ provided that a is not negative integer or zero.

As a more special case, if $a = b = 2$ then $y_n(z,a)$ reduces to Bessel polynomials $y_n(z)$ which satisfy the differential equation

$$z^2 y'' + (2 + 2z)y' - n(2 + n - 1)y = 0. \tag{4.19}$$

The Rodrigues formula can be found as

$$y_n(z) := y_n(z,2) = B_n e^{2/z} D^n[z^{2n} e^{-2/z}], \text{ where } \quad B_n = (\frac{1}{2})^n \tag{4.20}$$

and in terms of the Laguerre polynomials, it can be written as

$$y_n(z) = n!(-z/2)^n L_n^{-2n-1}(2/z), \qquad C_n = (-1/2)^n n! \tag{4.21}$$

Thus normalization is obvious when we introduce the coefficients of the leading order terms of classical polynomials.

If we replace s by z and set $\tau(z_0) = b$ in (4.8), (4.10), (4.12) the polynomial solutions $u_n(z)$ denoted and defined by

$$\begin{aligned}\theta_n(z;a) &= z^n y_n(z^{-1};a)\\ \theta_n(z;a,b) &= z^n y_n(z^{-1};a,b)\end{aligned} \tag{4.22}$$

are called the reverse generalized Bessel polynomials. These polynomials satisfy the differential equation

$$z\theta'' - (2n-2+a+bz)\theta' + bn\theta = 0. \tag{4.23}$$

The phase reverse can be justified because, if

$$y_n(z;a) = b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n$$

then

$$\begin{aligned}\theta_n(z;a) &= z^n[b_0(\frac{1}{z})^n + b_1(\frac{1}{z})^{n-1} + \cdots + b_{n-1}(\frac{1}{z}) + b_n]\\ &= b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} + b_n z^n.\end{aligned} \tag{4.24}$$

Clearly it is seen that reverse generalized Bessel polynomial is a polynomial with the same coefficients but in reverse order. The differential equation for the generalized and reverse generalized Bessel polynomials in (6) and (8) respectively have a basic difference. In (4.17) $z = 0$ is irregular singular point and $z = \infty$ is regular singular point. However in (4.23) the point at the origin $z = 0$ is a regular singular point while the point at infinity represents an irregular singularity which is more preferable. If we use (4.22) we obtain, by routine computations, the differential equation satisfied by $\theta_n = \theta_n(z;a)$ namely

$$z\theta_n'' - (2n-2+a+2z)\theta n' + 2n\theta_n = 0. \tag{4.25}$$

For $a = b = 2$, (4.25) reduces to

$$z\theta_n'' - 2(z+n)\theta n' + 2n\theta_n = 0. \tag{4.26}$$

4.2 Recurrence Relation

We shall show that there is a connection between the Bessel function and the Bessel polynomials. It is well known that the general solution of Bessel differential equation

$$z^2y'' + zy' + (\alpha^2 z^2 - \nu^2)z = 0 \quad \text{for fixed } \nu, \quad \text{Reel } \nu \geq 0 \tag{4.27}$$

is given by

$$y = C_1 J_\nu(bz) + C_2 Y_\nu(bz) \tag{4.28}$$

where C_1 and C_2 are arbitrary constants. J_ν and Y_ν denote Bessel functions of the first and second kinds, respectively. The related differential equation

$$z^2y'' + zy' - (z^2 + \nu^2)y = 0, \quad \textit{Reel } \nu \geq 0 \tag{4.29}$$

which bears great resemblance to Bessel's equation, is modified Bessel equation. It is of the form (4.27) with $\alpha^2 = -1$. The function $I_\nu(z)$, $K_\nu(z)$ are independent solutions of the modified Bessel equation for all values of ν and its general solution can be taken to be

$$y(z) = A_1 I_\nu(z) + A_2 K_\nu(z). \tag{4.30}$$

Let y be a solution of (4.29). Under the transformation $y = z^{-\nu}e^{-z}\theta$, by substituting y' and y'' into (4.29), we obtain

$$[z\theta'' - (2z + 2\nu - 1)\theta' - (2\nu - 1)\theta]zy = 0. \tag{4.31}$$

As $zy(z) \neq 0$, it follows that $\theta(z)$ satisfies the differential equation

$$z\theta'' - (2z + 2\nu - 1)\theta' + (2\nu - 1)\theta = 0 \tag{4.32}$$

which is nothing but the differential equations in (4.26) for reverse Bessel polynomials whenever $2\nu - 1 = 2n$. This is why the polynomial solutions of (4.14) and (4.23) are called Bessel polynomials.

By the general theory of linear differential equations, $z = 0$ is a regular singular point and in general there exist two independent particular solutions of the form $\theta = z^{\alpha} \sum_{m=0}^{\infty} c_m z^m$. By Frobenius' method one finds that the indicial equation is,

$$\alpha(\alpha - 2\nu) = 0. \tag{4.33}$$

The solution of (4.33) are indeed distinct, except for $\nu = 0$. For $\nu \neq 0$, set $\theta = \theta(z,\nu) = \sum_{m=0}^{\infty} c_m z^m$ and $\tilde{\theta} = \tilde{\theta}(z,\nu) = z^{2\nu} \sum_{m=0}^{\infty} c'_m z^m$. After differentiation and substituting them into (4.32) then we obtain

$$\begin{aligned} \theta(z,\nu) &= c_0 \Bigg[1 + z + \frac{(2\nu-1)(2\nu-3)}{(2\nu-1)(2\nu-2)} \frac{z^2}{2!} + \cdots + \\ &\qquad \frac{(2\nu-1)(2\nu-3)\cdots(2\nu-2m+1)}{(2\nu-1)(2\nu-2)\cdots(2\nu-m)} \frac{z^m}{m!} + \cdots \Bigg] \end{aligned} \tag{4.34}$$

similarly,

$$\begin{aligned} \tilde{\theta}(z,\nu) &= c'_0 z^{2\nu} \Bigg[1 + z + \frac{(2\nu+1)(2\nu+3)}{(2\nu+1)(2\nu+2)} \frac{z^2}{2!} + \\ &\qquad + \cdots + \frac{(2\nu+1)(2\nu+3)\cdots(2\nu+2m-1)}{(2\nu+1)(2\nu+2)\cdots(2\nu+m)} \frac{z^m}{m!} + \cdots \Bigg]. \end{aligned} \tag{4.35}$$

It is obvious that $\theta(z,\nu)$ reduces to a polynomial of exact degree n if $2\nu = 2n + 1$, $n = 0, 1, \cdots$ and $z^{-2\nu}\tilde{\theta}(z,\nu)$ reduces to a polynomial of exact degree n if $2\nu = -2n - 1$, $n = 0, 1, \cdots$ In general, it will be sufficient to consider only non negative values for the half integral second parameter so, for $\nu = n + \frac{1}{2}$,

$$\begin{aligned}\theta(z, n+\frac{1}{2}) \;&=\; c_0\Bigg[1 + z + \frac{2n(2n-2)z^2}{2n(2n-1)2!} + \\ &+ \cdots + \frac{2n(2n-2)(2n-4)\cdots(2n-2(m-1))}{2n(2n-1)(2n-2)\cdots(2n-(m-1))}\frac{z^m}{m!} \\ &+ \cdots + \frac{2n(2n-2)(2n-4)\cdots 4.2}{2n(2n-1)(2n-2)\cdots(n+2)(n+1)}\frac{z^n}{n!}\Bigg] \quad (4.36)\end{aligned}$$

The polynomial defined in (4.36) is the solution of (4.32) while $\nu = n + \frac{1}{2}$. The coefficient of z^n in $\theta(z, n+\frac{1}{2})$ equals $\frac{c_0 2^n \cdot n!}{(2n)!}$. If we select $c_0 = \frac{(2n)!}{2^n \cdot n!}$, the leading coefficients becomes unity. With this normalization we shall denote $\theta(z, n+\frac{1}{2})$ simply by $\theta_n(z)$. One now easily verifies that

$$\theta_n(z) = \sum_{m=0}^{n} a_{n-m} z^m = \sum_{m=0}^{n} a_m z^{n-m}.$$

In (4.36) if we substitute c_0 then we find

$$a_{n-m} = \frac{2^{m-n}(2n-m)!}{(n-m)!m!}, \qquad (4.37)$$

and if we replace m by $n-m$ we get

$$a_m = \frac{(n+m)!}{2^m(n-m)!m!}, \quad m = 0, 1, 2, \cdots n. \qquad (4.38)$$

The coefficients of generalized Bessel polynomial, $y_n(z; a, b)$ could be found by using the some procedure. The solution of (4.14) is of the form

$$y = y_n(z; a, b) = \sum_{k=0}^{n} f_k^{(n)} z^k$$

where

$$f_k^{(n)} = \frac{n!(n+k+a-2)^{(k)}}{k!(n-k)!b^k}. \qquad (4.39)$$

One verifies that for $a = b = 2$ [17],

$$f_k^{(n)} = \frac{(n+k)!}{k!(n-k)!2^k}. \tag{4.40}$$

The corresponding generalization of $\theta_n(z; a, b)$ is obtained most conveniently by setting

$$y_n(z; a, b) = z^n \theta_n(z^{-1}; a, b)$$

The functions $K_\nu(z)$ and $I_\nu(z)$ satisfy some recurrence relations [26], [31] such as

$$I'_\nu(z) + \frac{\nu}{z} I_\nu(z) = I_{\nu-1}(z), \tag{4.41}$$

$$I'_\nu(z) - \frac{\nu}{z} I_\nu(z) = I_{\nu+1}(z), \tag{4.42}$$

$$I_{\nu-1}(z) + I_{\nu+1}(z) = 2I'_\nu(z), \tag{4.43}$$

and

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z). \tag{4.44}$$

By the remarkable identity

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \tag{4.45}$$

and above recurrence formulas for $I_\nu(z)$ one obtains

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -(2\nu/z) K_\nu(z), \tag{4.46}$$

$$K'_\nu(z) = -K_{\nu-1}(z) - (\nu/z)K_\nu(z), \tag{4.47}$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z), \tag{4.48}$$

$$K'_\nu(z) = -K_{\nu+1}(z) + (\nu/z)K_\nu(z). \tag{4.49}$$

In general the modified Bessel function of half integral order written as

$$I_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}}\left[e^z \sum_{r=0}^{n} \frac{(-1)^r (n+r)!}{(n-r)!r!(2z)^r} + (-1)^{n+1} e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!(2z)^r}\right] \tag{4.50}$$

$$I_{-(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}}\left[e^z \sum_{r=0}^{n} \frac{(-1)^r (n+r)!}{(n-r)!r!(2z)^r} + (-1)^{n} e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!(2z)^r}\right] \tag{4.51}$$

thus, we have

$$I_{-(n+\frac{1}{2})} - I_{(n+\frac{1}{2})} = \frac{1}{\sqrt{2\pi z}} 2(-1)^n e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!(2z)^r}. \tag{4.52}$$

Substituting this identity into (4.45) by taking $\nu = n + \frac{1}{2}$, we get

$$K_{n+\frac{1}{2}}(z) = \frac{\pi}{2} \frac{I_{(n+1/2)} - I_{-(n+\frac{1}{2})}}{\sin(n+\frac{1}{2})\pi} = \frac{\pi}{2} \frac{2(-1)^n e^{-z} \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!(2z)^r}}{\sqrt{2\pi z}\sin(n+\frac{1}{2})\pi}.$$

Therefore,

$$K_{n+\frac{1}{2}}(z) \;\; = \;\; \sqrt{\frac{\pi}{2}} e^{-z} z^{-n-\frac{1}{2}} \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!2^r} z^{n-r}. \tag{4.53}$$

We know from (4.38) that

$$\theta_n(z) = \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!r!2^r} z^{n-r}.$$

So we have

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} e^{-z} z^{-n-\frac{1}{2}} \theta_n(z), \tag{4.54}$$

and

$$K_{n-\frac{1}{2}}(z) - K_{n+\frac{3}{2}}(z) = -\frac{(2n+1)}{z} K_{n+\frac{1}{2}}(z). \tag{4.55}$$

If we insert (4.55) in (4.54) we get

$$\begin{aligned} \sqrt{\pi/2} e^{-z} z^{-(n-1)-1/2} \theta_{n-1}(z) - \sqrt{\pi/2} e^{-z} z^{-(n+1)-1/2} \theta_{n+1}(z) \\ = -\frac{(2n+1)}{z} \sqrt{\pi/2} e^{-z} z^{-n-1/2} \theta_n(z), \end{aligned} \tag{4.56}$$

which leads to

$$z\theta_{n-1}(z) - \frac{1}{z}\theta_{n+1}(z) = -\frac{(2n+1)}{z}\theta_n(z) \tag{4.57}$$

or

$$\theta_{n+1}(z) = (2n+1)\theta_n(z) + z^2\theta_{n-1}(z). \tag{4.58}$$

This is a recurrence relation for $\theta_n(z)$. If we replace in (4.58) z by z^{-1} and multiply the result by z^{n+1},we have

$$z^{n+1}\theta_{n+1}(z^{-1}) = (2n+1)z^{n+1}\theta_n(z^{-1}) + z^{-2} \cdot z^{n+1}\theta_{n-1}(z^{-1}). \tag{4.59}$$

By using (4.22) we can obtain the following recurrence relation for the BP $y_n(z)$

as

$$y_{n+1}(z) = (2n+1)zy_n(z) + y_{n-1}(z). \tag{4.60}$$

By starting from $\theta_0(z) = 1$ and $\theta_1(z) = 1 + z$ it can be written, recursively, that

$$\begin{aligned} \theta_2(z) &= 3\theta_1(z) + z^2\theta_0(z) = z^2 + 3z + 3, \\ \theta_3(z) &= 5\theta_2(z) + z^2\theta_1(z) = z^3 + 6z^2 + 15z + 15. \end{aligned} \tag{4.61}$$

Similarly, starting from $y_0(z) = 1$ and $y_1(z) = 1 + z$ we have

$$\begin{aligned} y_2(z) &= 3zy_1(z) + y_0(z) = 3z^2 + 3z + 1, \\ y_3(z) &= 5zy_2(z) + y_1(z)15z^3 + 15z + 6z + 1. \end{aligned} \tag{4.62}$$

Let us differentiate both sides of (4.54) with respect to z. We obtain

$$\begin{aligned} K'_{n+\frac{1}{2}}(z) = -\sqrt{\frac{\pi}{2}}e^{-z}z^{-n-1/2}\theta_n(z) - (n+\frac{1}{2})\sqrt{\frac{\pi}{2}}e^{-z}z^{-n-\frac{3}{2}}\theta_n(z) \\ +\sqrt{\frac{\pi}{2}}e^{-z}z^{-n-1/2}\theta'_n(z). \end{aligned} \tag{4.63}$$

In recurrence relation (4.46) if we take $v = n + \frac{1}{2}$ we obtain

$$K'_{n+\frac{1}{2}}(z) = -K_{n-\frac{1}{2}}(z) - (n+\frac{1}{2})\frac{1}{2}K_{n+\frac{1}{2}}(z) \tag{4.64}$$

Using (4.54) and (4.64)

$$\theta'_n(z) = \theta_n(z) - z\theta_{n-1}(z) \tag{4.65}$$

can be found. If we replace z in (4.65) by z^{-1} and multiply the result in z^n we have

$$z^2y'_n(z) = (nz-1)y_n(z) + y_{n-1}(z). \tag{4.66}$$

So, we found the differentiation formula for $y_n(z)$ and $\theta_n(z)$. Similarly by using (4.46) we have the following recurrence relations:

$$2z\theta_n'(z) = (2z + 2n + 1)\theta_n(z) - (z^2\theta_{n-1}(z) + \theta_{n+1}(z)), \tag{4.67}$$

$$2z^2y_n'(z) = y_{n-1}(z) - (2 + z)y_n(z) + y_{n+1}(z), \tag{4.68}$$

and finally combining (4.65) and (4.67) we get

$$z\theta n'(z) = (z + 2n + 1)\theta_n(z) - \theta_{n+1}(z). \tag{4.69}$$

Also by using (4.66) and (4.68) we get the corresponding relation

$$z^2y_{n-1}'(z) = y_n(z) - (1 + nz)y_{n-1}(z). \tag{4.70}$$

By the explicit formulas for generalized Bessel polynomials $y_n(z, a, b)$ the first four of these polynomials are therefore given by

$$\begin{aligned}
y_0(z) &= 1, \\
y_1(z) &= 1 + a(\frac{z}{b}), \\
y_2(z) &= 1 + 2(a+1)(\frac{z}{b}) + (a+1)(a+2)(\frac{z}{b})^2, \\
y_3(z) &= 1 + 3(a+2)(\frac{z}{b}) + 3(a+2)(a+3)(\frac{z}{b})^2 + \\
&\qquad (a+2)(a+3)(a+4)(\frac{z}{b})^2,
\end{aligned} \tag{4.71}$$

and by using (4.22) we get

$$\begin{aligned}
\theta_0(z) &= 1, \\
\theta_1(z) &= z + \frac{a}{b}, \\
\theta_2(z) &= z^2 + \frac{2(a+1)z}{b} + \frac{(a+1)(a+2)}{b^2}, \\
\theta_3(z) &= z^3 + 3\frac{(a+2)}{b}z^2 + 3\frac{(a+2)(a+3)}{b^2}z + \frac{(a+2)(a+3)(a+4)}{b^3}.
\end{aligned} \tag{4.72}$$

4.3 Orthogonality and the Weight Function of Bessel Polynomials

The classical sets of orthogonal polynomials of Jacobi, Laguerre and Hermite satisfy second order differential equations, and also have the property that their derivatives form orthogonal system. There is a fourth class of polynomials with these two properties and similar in other ways to the other three classes. This fourth class of polynomials are Bessel polynomials. They are orthogonal, but not in quite the same sense as the other three systems [18].

In the earlier part we show the orthogonality of Jacobi, Laguerre and Hermite polynomials by using Theorem 3.3. From this theorem it can be seen that Bessel polynomials are not orthogonal in the real-axis since we can not find an interval in the real-axis. As we know the BP satisfy the differential equation

$$z^2y'' + (2+2z)y' - n(n+1)y = 0. \tag{4.73}$$

The weighting function $\rho(x)$ satisfies

$$[z^2\rho(z)]' = (2+2z)\rho(z)$$

resulting in

$$\rho(z) = e^{\frac{-2}{z}}.$$

From Theorem 3.3 the condition of orthogonality in the real-axis is

$$\sigma(x)\rho(x)x^k\bigg|_a^b = 0 \quad k = 0, 1 \cdots .$$

If we substitute $\sigma(x)$ and $\rho(x)$ of BP into the condition we can not find an interval in the real-axis. We shall now show that the Bessel polynomials form an orthogonal system, the weight function being $e^{-2/z}$ and the path of integration the unit circle in the complex plane. The weight function and the path are not unique; an arbitrary analytic function may be added to the weight function, it may be multiplied by a non-zero constant, and the unit circle may be replaced by an arbitrary curve around the origin [18]. To prove orthogonality we first note that the differential equation (4.73) may be written in the form

$$e^{-2/z}z^2y'' + (2+2z)e^{-2/z}y' = n(n+1)e^{-2/z}y.$$

Furthermore,

$$(z^2e^{-2/z}y_n')' = n(n+1)e^{-2/z}y_n. \tag{4.74}$$

If we multiply both sides of (4.74) by y_m and integrate around the unit circle we find:

$$\int_C (z^2e^{-2/z}y_n')'y_m dz = \int_C n(n+1)e^{-2/z}y_ny_m dz$$

and by integrating by parts we have

$$\int_C (z^2e^{-2/z}y_n')'y_m dz = z^2e^{-2/z}y_n'y_m\bigg|_C - \int_C z^2e^{-2/z}y_n'y_m' dz$$

From fundemental theorem of calculus, the first term vanishes since it is analytic in the unit circle then we obtain

$$n(n+1)\int_C e^{-2/z}y_my_n dz = -\int_C z^2e^{-2/z}y_n'y_m' dz. \tag{4.75}$$

Interchanging m and n, that is

$$n(n+1)\int_C y_m y_n e^{-2/z}dz = -\int_C z^2 e^{-2/z} y_n' y_m' dz$$

$$m(m+1)\int_C y_n y_m e^{-2/z}dz = -\int_C z^2 e^{-2/z} y_m' y_n' dz$$

and substracting we get

$$[n(n+1)-m(m+1)]\int_C y_m y_n e^{-2/z}dz = 0. \tag{4.76}$$

So, for $m \neq n$ we have

$$\int_C y_m y_n e^{-2/x}dz = 0. \tag{4.77}$$

This is the reguired orthogonality relation. Let us discuss the orthogonality of generalized bessel polynomials. For arbitrary real or complex $a \neq 0, -1, -2, \cdots$ and $z \neq 0$, set $\rho(z) = \rho(z; a, b)$

$$\rho(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a+n-1)}(\frac{-b}{z})^n \tag{4.78}$$

which satisfies the related nonhomogeneous equation [18]:

$$(z^2\rho)' = (az+b)\rho - \frac{(a-1)(a-2)}{2\pi i}z. \tag{4.79}$$

We shall show that the generalized Bessel polynomials form an orthogonal system with path of integration an arbitrary curve surrounding the origin, and with the weight function $\rho(z)$ given by (4.78). Expanding (4.78) gives:

$$\rho(z) = \frac{1}{2\pi i}[a-1+(\frac{-b}{z})+\frac{1}{a}(\frac{-b}{z})^2+\frac{1}{a(a+1)}(\frac{-b}{z})^3+ \\ +\frac{1}{a(a+1)(a+2)}(\frac{-b}{z})^4+\cdots]. \tag{4.80}$$

The series in (4.80) converges for all z except zero. And this formula would clearly have to be modified for the excluded cases $a = 0, -1, -2, \cdots$. It can be seen that $\rho(z)$ reduces to the previous weight function $\frac{e^{-2/z}}{2\pi i}$ for the case $a = b = 2$ of Bessel polynomials proper.

As we know generalized Bessel polynomials satisfy the differential equation:

$$z^2 y'' + (az + b)y' = n(n + a - 1)y. \tag{4.81}$$

The function $\rho(z)$ differs, except when $a = 1$ or $a = 2$, from $\sigma(z)$ given by

$$\sigma(z) = \frac{x^{a-2}e^{-b/x}}{2\pi i}.$$

It is the factor needed to make equation (4.81) self-adjoint, and it is therefore a natural candidate for a weight function. When a is not an integer, it is inconvenient if we wish to integrate around the point $z = 0$. The function $\sigma(z)$ satisfies the differential equation

$$(z^2\sigma)' = (az + b)\sigma. \tag{4.82}$$

The equations (4.82) and (4.79) are the same if a is 1 or 2 and we treated the case $a = 2$ in the earlier part. If equation (4.81) is multiplied by $\rho(z)$, we have

$$(z^2\rho y')' - (z^2\rho)'y' + (az + b)\rho y' = n(n + a - 1)y\rho.$$

and using (4.79)

$$(z^2\rho y')' - (az + b)\rho y' + (az + b)\rho y' + \frac{(a-1)(a-2)}{2\pi i} zy' = n(n + a - 1)y\rho$$

can be obtained. Futhermore

$$(z^2\rho y_n')' + \frac{z(a-1)(a-2)}{2\pi i} y_n' = n(n + a - 1)\rho y_n. \tag{4.83}$$

If we multiply equation (4.83) by y_k and integrate around the unit circle we get

$$\int_C (z^2\rho y_n')' y_k dz + \int_C \frac{z(a-1)(a-2)}{2\pi i} y_n' y_k dz = n(n+a-1)\int_C \rho y_n y_k dz. \quad (4.84)$$

By integrating by parts we have

$$z^2\rho y_n' y_k\Big|_C - \int_C z^2\rho y_n' y_k' dz + \int_C \frac{z(a-1)(a-2)}{2\pi i} y_n' y_k dz = n(n+a-1)\int_C \rho y_n y_k dz.$$

From Cauchy's Theorem, one obtains

$$\int_C \frac{z(a-1)(a-2)}{2\pi i} y_n' y_k dz = 0$$

and $z^2\rho y_n' y_k$ is analytic so by the fundemental theorem it vanishes. Then (4.84) becomes

$$n(n+a-1)\int_C \rho y_n y_k dz = -\int_C z^2\rho y_n' y_k' dz. \quad (4.85)$$

Interchanging n and k , that is

$$n(n+a-1)\int_C y_n y_k \rho dz = -\int_C z^2\rho y_n' y_k' dz$$

$$k(k+a-1)\int_C y_k y_n \rho dz = -\int_C z^2\rho y_k' y_n' dz$$

and substracting gives:

$$[n(n+a-1) - k(k+a-1)]\int_C y_k y_n \rho dz = 0. \quad (4.86)$$

Finally, for $n \neq k$ we have

$$\int_C y_k y_n \rho dz = 0,$$

which shows that the polynomials are orthogonal with weight function $\rho(z)$. As we show before, in real-axis Bessel polynomials are not orthogonal and also generalized Bessel polynomials are not orthogonal in the real-axis with the function

$\rho(z) = e^{-b/z}z^{a-2}$ because we can not find an interval that satisfies condition which is given in Theorem 3.3. Let us calculate for $n = k$,the value of

$$\int_C y_n(z;a,b)^2\rho(z;a,b)dz$$

and

$$\int_C z^k y_n(z;a,b)\rho(z;a,b)dz.$$

Suppose $p_n(z) = \sum_{m=0}^{n} c_m z^m$ be an arbitrary polynomial of degree n. The weight function ρ is defined by

$$\rho(z;a,b) = \frac{1}{2\pi i}\sum_{r=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a+r-1)}(\frac{-b}{z})^r.$$

So,

$$\int_{|z|=1} z^k p_n(z)\rho(z)dz = \frac{1}{2\pi i}\int_{|z|=1} z^k p_n(z)(\sum_{r=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a+r-1)}(\frac{-b}{z})^r)dz.$$

We may interchange summation and integration since $\sum_{r=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a+r-1)}(\frac{-b}{z})^r$ converges.

$$\int_{|z|=1} z^k p_n(z)\rho(z)dz = \frac{1}{2\pi i}\sum_{r=0}^{\infty}\frac{(-1)^r b^r\Gamma(a)}{\Gamma(a+r-1)}\sum_{m=0}^{n}\int_{|z|=1} c_m z^{k+m-r}dz$$

Using Cauchy's Integral Formula and Cauchy's Teorem we obtain,

$$\int_{|z|=1} z^{k+m-r}dz = 0, \quad \text{for} \quad k+m-r \neq -1$$

and

$$\int_{|z|=1} z^{k+m-r}dz = 2\pi i \quad \text{for} \quad k+m-r = -1.$$

Therefore,

$$\begin{aligned}\int_{|z|=1} z^k p_n(z)\rho(z)dz &= \sum_{k+m-r=-1} c_m \frac{\Gamma(a)}{\Gamma(a+r-1}(-1)^r b^r \\ &= \sum_{r=k+1}^{n+k+1} (-1)^r b^r c_{r-k-1} \frac{\Gamma(a)}{\Gamma(a+r-1).}\end{aligned}$$

We are interested in the case when $p_n(z) = y_n(z;a,b) = \sum_{m=0}^{n} f_m^{(n)} z^m$. From (4.39) we have

$$f_m^{(n)} = \binom{n}{m} b^{-m}(n+m+a-2)(n+m+a-3)\cdots(n+a-1),$$

so that the sum may be written as [17], [24]

$$\int_{|z|=1} z^k y_n(z;a,b)\rho(z;a,b)dz = \sum_{r=k+1}^{n+k+1}(-1)^r b^r \binom{n}{r-k-1}(n+r-k-1+a-2)$$

$$(n+r-k-1+a-3)\cdots(n+a-1)b^{-(r-k-1)}\frac{\Gamma(a)}{\Gamma(a+r-1)}$$

$$\begin{aligned}&= \sum_{r=k+1}^{n+k+1}(-1)^r b^r \frac{n!(n+r-k+a-3)\cdots(n+a-1)\cdot(a-1)!}{(r-k-1)!(n-r+k+1)!(a+r-2)\cdots(a+1)a(a-1)!} \\ &= \sum_{v=r-k-1=0}^{n} (-1)^{k+1} b^{k+1}(-1)^v \frac{n!(n+v+a-2)\cdots(n+a-1)}{v!(n-v)!(k+v+a-1)\cdots(a+1)a}.\end{aligned}$$

This leads to

$$\int_{|z|=1} z^k y_n(z;a,b)\rho(z;a,b)dz = \frac{(-b)^{k+1}}{(k+n+a-1)\cdots(a+1)(a)}$$

$$\sum_{v=0}^{n}(-1)^v \binom{n}{v}(n+v+a-2)\cdots(n+a-1)(k+v+a)\cdots(k+n+a-1) \quad (4.87)$$

Lemma 4.1.
$$\sum_{v=0}^{n} \binom{n}{v} \binom{k-n}{t-v} = \binom{k}{t}$$

Proof.

$$\begin{aligned}
\sum_{t=0}^{k} \binom{k}{t} x^t = (x+1)^k &= (x+1)^n (x+1)^{k-n} \\
&= \sum_{v=0}^{n} \binom{n}{v} x^v \sum_{w=0}^{k-n} \binom{k-n}{w} x^w \\
&= \sum_{v=0}^{n} \binom{n}{v} \sum_{w=0}^{k-n} \binom{k-n}{w} x^{w+v}
\end{aligned}$$

By changing $v + w = t$,

$$\sum_{t=0}^{k} \binom{k}{t} x^t = \sum_{t=0}^{k} x^t \sum_{v=0}^{n} \binom{n}{v} \binom{k-n}{t-v} \text{[17]} \tag{4.88}$$

□

Lemma 4.2. *The function*

$$f(x) = \sum_{v=0}^{n} (-1)^v \binom{n}{v} \frac{\Gamma(k+n+x+1)}{\Gamma(n+x)(k+v+x)(k-1+v+x)\cdots(n+v+x)} \tag{4.89}$$

is independent of x, and has the value $k(k-1)\cdots(k-n+1)$; in particular, for $k < n$ one has $f(x) = 0$.

Proof. For $k \geq n$ (4.89) is equivalent to

$$f(x) = \sum_{v=0}^{n} (-1)^v \binom{n}{v} \frac{(k+n+x)\cdots(k+v+x)\cdots(n+v+x)\cdots(n+x)}{(k+v+x)\cdots(n+v+x)} \tag{4.90}$$

Clearly $f(x)$ is a rational function and could have poles at most for $x = -m$, $n \leq m \leq k+n$. In fact, as seen from (4.90), all these poles cancel and $f(x)$ is a polynomial of degree at most n. We shall show that $\lim\limits_{x\to -m} f(x)$ exists for all these $k+1$ values of m in the given range and that all these limits equal $\frac{k!}{(k-n)!}$, independently of m. It then follows that the polynomial $f(x)$ of degree $n<k+1$ equals $\frac{k!}{(k-n)!}$ identically, as claimed.

To find the limit, set $m=n+t$, $0\le t\le n$, $t\in\mathbb{Z}$, $x=-n-s$. Then by (4.90)

$$\lim_{s\to t} f(-n-s) = \lim_{s\to t}\frac{\Gamma(k-s+1)}{(t-s)(t-1-s)(-s)\Gamma(-s)}\sum_{v=0}^{n}(-1)^v\binom{n}{v}\frac{(t-s)\cdots(-s)}{(k+v-n-s)\cdots(v-s)}$$

$$= \lim_{s\to t}\frac{\Gamma(k-s+1)}{\Gamma(t-s)}\sum_{v=0}^{n}(-1)^v\binom{n}{v}\frac{(t-s)\cdots(-s)\Gamma(-s)}{(k+v-n-s)\cdots(v-s)\Gamma(-s)}$$

$$= \Gamma(k-1+1)\lim_{s\to t}\sum_{v=0}^{n}(-1)^v\binom{n}{v}\frac{(t-s)\cdots(-s)\Gamma(-s)}{(k+v-n-s)\cdots(v-s)r(-s)}$$

because for $t<v\le n$, $-s<t-s<v-s$ and the vanishing factor of the numerator not cancelled by a corresponding factor of the denominator. It follows that therefore

$$\begin{aligned}
\lim_{s\to t} f(-n-s) &= \Gamma(k-t+1)\lim_{s\to t}\sum_{v=0}^{t}(-1)^v\binom{n}{v}\frac{\Gamma(t-s)(v-s-1)\cdots(-s)}{\Gamma(k+v-n-s)}\\
&= \Gamma(k-t+1)\sum_{v=0}^{t}(-1)^v\binom{n}{v}\frac{(v-t-1)\cdots(1-t)(-t)}{\Gamma(k+v-n-t)}\\
&= (k-t)!\sum_{v=0}^{t}\binom{n}{v}\frac{t!}{(t-v)!(k+v-n-t)!}\\
&= \frac{(k-t)!t!}{(k-n)!}\sum_{v=0}^{t}\binom{n}{v}\binom{k-n}{t-v}.
\end{aligned}$$

By Lemma 4.1 the last sum equals to $\binom{k}{t}$, so that

$$\lim_{x\to -m} f(x) = \frac{(k-t)!t!}{(k-n)!}\,\frac{k!}{t!(k-t)!} = \frac{k!}{(k-n)!}.$$

as claimed. The lemma is proved [17]. $\square$

Using Lemma 4.2, let us calculate the sum in (4.87): The sum becomes

$$\sum_{v=0}^{n}(-1)^v\binom{n}{v}(n+v+x-1)\cdots(n+x-1)(n+x)(k+v+x-1)\cdots(k+n+x)$$

$$=\sum_{v=0}^{n}(-1)^v\binom{n}{v}\frac{(k+n+x)(k+n+x-1)\cdots(k+v+x)\cdots(n+v+x)\cdots(n+x)}{(k+v+x)\cdots(n+v+x)}$$

$$=\sum_{v=0} n(-1)^v\binom{n}{v}\frac{\Gamma(k+n+x+1)}{\Gamma(n+x)}\frac{1}{(k+v+x)(k+v+x-1)\cdots(n+v+x)}$$

where $x=a-1$. This leads to

$$k(k-1)\cdots(k-n+1)=\frac{k!}{(k-n)!}$$

by Lemma 4.2. If we inset this value in (4.87) we obtain

$$\begin{aligned}\int_{|z|=1} z^k y_n(z;a,b)\rho(z;a,b)dz &= \frac{(-b)^{k+1}}{(k+n+a-1)\cdots(a+1)a}\frac{k!}{(k-n)!}\\ &= \frac{(-b)^{k+1}\Gamma(k+1)\Gamma(a)}{\Gamma(k+n+a)\Gamma(k+1-n)}. \qquad (4.91)\end{aligned}$$

As a consequences of from Lemma 4.2 we have for $0\le k<n$,

$$\int_{|z|=1} z^k y_n(z;a,b)\rho_n(z;a,b)dz=0. \qquad (4.92)$$

If $a>0$

$$\int_{|z|=1} z^k y_n(z;a,b)\rho(z;a,b)dz=(-b)^{k+1}\frac{(a-1)!k!}{(k+n+a-1)!(k-n)!}$$

For $a=2$,

$$\rho(z;2,b)=\sum_{n=0}^{\infty}\frac{(-1)^n(b/z)n}{n!}=e^{-b/z}.$$

Thus when $a = 2$ and $k \geq n$,

$$\int_{|z|=1} z^k y_n(z; 2, b)\rho(z; 2, b)dz = (-b)^{k+1}\frac{k!}{(k+n+1)!(k-n)!}. \tag{4.93}$$

For $a = b = 2$ and $k \geq n$, we obtain

$$\int_{|z|=1} z^k y_n(z; 2, 2)\rho(z; 2, 2)dz = \frac{(-2)^{k+1}k!}{(k+n+1)!(k-n)!} \tag{4.94}$$

For $n = k$ (4.94) can be written as

$$\begin{aligned} \int_{|z|=1} y_n^2(z; a, b)\rho(z; a, b)dz &= \int_{|z|=1} f_n^{(n)} z^n y_n(z; a, b)\rho(z; a, b)dz \\ &= f_n^{(n)} \int_{|z|=1} z^n y_n(z; a, b)\rho(z; a, b)dz \end{aligned}$$

From (4.37) we have

$$f_n^{(n)} = b^{-n}(2n+a-2)(2n+a-3)\cdots(n+a-1) = b^{-n}\frac{\Gamma(2n+a-1)}{\Gamma(n+a-1)}.$$

So by substituting $f_n^{(n)}$ it follows that

$$\begin{aligned} f_n^{(n)} \int_{|z|=1} z^n y_n(z; a, b)\rho(z; a, b)dz &= f_n^{(n)}(-b)^{n+1}\frac{\Gamma(n+1)\Gamma(a)}{\Gamma(2n+a)} \\ &= \frac{b^{-n}\Gamma(2n+a-1)\Gamma(a)\Gamma(n+1)(-b)^{n+1}}{\Gamma(2n+a)\Gamma(n+a-1)} \\ &= \frac{(-1)^{n+1}bn!\Gamma(a)}{(2n+a-1)\Gamma(n+a-1)}. \end{aligned} \tag{4.96}$$

In particular, if $a = b = 2$, then

$$\int_{|z|=1} y_n^2(z; 2, 2)e^{-2/z}dz = (-1)^{n+1}\frac{2}{2n+1}. \tag{4.97}$$

By using $y_n(z) = z^n\theta_n(z^{-1}; a, b)$ and setting

$$\rho_1(z; a, b) = \frac{z^2}{2\pi i}\sum_{r=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a+r-1)}(-bz)^r \tag{4.98}$$

we obtain, for $k \geq n$, that

$$\int_{|z|=1} z^{-k}(z^{-n}\theta_n(z; a, b))\rho_1(z; a, b)dz = \frac{(-1)^{k+1}b^{k+1}\Gamma(a)\Gamma(k+1)}{\Gamma(k+a+n)\Gamma(k+1-n)}. \tag{4.99}$$

It follows that for $k = n$,

$$\int_{|z|=1}(z^{-n}\theta_n(z; a, b))^2\rho_1(z; a, b)dz = (-1)^{n+1}\frac{bn!}{2n+a-1}\frac{\Gamma(a)}{\Gamma(n+a-1)} \tag{4.100}$$

4.4 Relations of the Bessel Polynomials to the Classical Orthogonal Polynomials and to Other functions

4.4.1 Relations to Hypergeometric Functions

In Chapter 2 we give the definition of generalized hypergeometric series as

$${}_AF_B(\alpha_1, \alpha_2, \cdots \alpha_A; \gamma_1, \gamma_2, \cdots, \gamma_B; z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_A)_n}{(\gamma_1)_n(\gamma_2)_n\cdots(\gamma_B)_n}\frac{z^n}{n!}. \tag{4.101}$$

For $A = 2$ and $B = 0$, (4.101) gives

$${}_2F_0(\alpha_1, \alpha_2; z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n z^n}{n!}.$$

Let us take $\alpha_1 = -n$, $\alpha_2 = n + a - 1$ and $z \to \frac{-z}{b}$ in ${}_2F_0(\alpha_1, \alpha_2, z)$, where n is a natural number so ${}_2F_0(-n, n+a-1; \frac{-z}{b})$ is

$$\sum_{k=0}^{n} \frac{(-n)(1-n)(2-n)\cdots(k-n-1)(a+n-1)\cdots(a+n+k-2)}{k!}$$
$$(\frac{-z}{b})^k$$
$$= \sum_{k=0}^{n} \frac{(-1)^k n(n-1)\cdots(n-(k-1))(a+n-1)\cdots(a+n+k-2)\cdot z^k}{(-1)^k b^k k!}$$
$$= \sum_{k=0}^{n} \frac{n!(n+k+a-2)^{(k)} z^k}{k!(n-k)!b^k}.$$

This gives the generalized Bessel polynomials

$$y_n(z;a,b) = {}_2F_0(-n, n+a-1; \frac{-z}{b}). \tag{4.102}$$

In particular, taking $a = b = 2$, we obtain

$$y_n(z;2,2) = y_n(z) = {}_2F_0(-n, n+1; \frac{-z}{2}). \tag{4.103}$$

4.4.2 Relations to Laguerre Polynomials

As we identified before Bessel polynomials can be defined by using Laguerre polynomials. They may be defined by

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-1)^m (1+\alpha)_n x^m}{m!(n-m)!(1+\alpha)_m}. \tag{4.104}$$

It follows that, if we take $\alpha = -2n - a + 1$ and $x = \frac{b}{z}$,

$$L_n^{(-2a-a+1)}(b/z) = \sum_{m=0}^{n} (-1)^m \frac{(-2n-a+2)\cdot b^m}{m!(n-m)!(-2n+2-a)_m} z^{-m} \tag{4.105}$$

and multiply both sides by $n!(-z/z)^n$ then $n!(\frac{-z}{b})^n L_n^{(-2n-a+1)}(\frac{b}{z})$ becomes

$$\sum_{m=0}^{n} \frac{(-1)^{m+n} n!(-2n-a+2)(-2n-a+3)\cdots(-n-a+1)}{(n-m)!(2n+a-2)\cdots(2n+a-m-1)}(\frac{z}{b})^{n-m}$$
$$= \sum_{m=0}^{n} \frac{(-1)^{m+n}(-1)^n(2n+a-2)\cdots(2n+a-m-1)\cdots(n+a-1)n!}{(-1)^m m!(n-m)!(2n+a-2)\cdots(2n+a-m-1)}(\frac{z}{b})^{n-m}$$
$$= \sum_{m=0}^{n} \frac{(2n+a-m-2)\cdots(n+a-1)n!}{m!(n-m)!}(\frac{z}{b})^{n-m}.$$

Here by changing the indices $n-m \rightarrow k$ we have

$$\sum_{k=0}^{m} \frac{(n+k+a-2)\cdots(n+a-1)n!}{k!(n-k)!b^k} z^k = \sum_{k=0}^{m} \frac{(n+k+a-2)_k}{k!(n-k)!b^k} z^k. \quad (4.106)$$

This leads to

$$y_n(z;a,b) = n!(\frac{-z}{b})^n L_n^{(-2n-a+1)}(\frac{b}{z}).$$

In particular,

$$y_n(z;a,2) = n!(\frac{-z}{2})^n L_n^{(-2n-a+1)}(\frac{2}{z}).$$

For reversed generalized Bessel polynomials, we have

$$\theta_n(z;a,b) = z^n y_n(z^{-1};a,b) = (-1)^n n! b^n L_n^{(-2n-a+1)}(bz) \quad (4.107)$$

4.4.3 Relations to Jacobi Polynomials

Jacobi polynomials may be defined as

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{(1-z)}{2})\ [23].$$

Setting $\alpha = t-1$, $\beta = a-1-t$ and $(1-z)/2 = -t(y/b)$ it follows

$${}_2F_1(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-z}{2}) = {}_2F_1(-n, a+n-1; t; -t(y/b))$$

$$= \frac{n!}{(1+\alpha)_n} P_n^{(t-1,a-t-1)}(1+2t(y/b)) = \frac{\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+1)} P_n^{(t-1,a-t-1)}(1+2t(y/b)$$

which is equal to

$$n\frac{\Gamma(t)\Gamma(n)}{\Gamma(n+t)}P_n^{(t-1,a-t-1)}(1+2t(y/b)). \tag{4.108}$$

Here, it is convenient to recall that,

$$\lim_{t\to\infty} {}_2F_1(a_1,a_2;t;tz) = {}_2F_0(a_1,a_2;-;z).$$

If we take the limit as $t \to \infty$, in (4.108)

$$\lim_{t\to\infty} n\frac{\Gamma(x)\Gamma(n)}{\Gamma(n+x)}P_n^{(x-1,a-x-1)}(1+2x(y/b)) = {}_2F_0(-n,n+a-1;-;\frac{-y}{b}).$$

By using the identity

$$B(t,n) = \frac{\Gamma(t)\Gamma(n)}{\Gamma(n+t)}$$

we get

$$y_n(z;a,b) = \lim_{t\to\infty} nB(n,t)P_n^{(t-1,a-t-1)}(1+2t(z/b)) \tag{4.109}$$

with z instead of y [17].

4.5 Generating Function

Generalized Bessel polynomials are the solutions of differential equation,

$$z^2y'' + (b+az)y' - n(a+n-1)y = 0 \tag{4.110}$$

where the weight function is,

$$\rho(z) = z^{a-2}e^{-b/2}.$$

From the definition of generating function in Chapter 3,

$$\Phi(z,t) = \sum_{n=0}^{\infty} \tilde{y}_n(z)\frac{t^n}{n!}$$

where $\tilde{y}_n(z) = \frac{1}{B_n} y_n(z)$ with B_n as normalization constant and

$$\Phi(z,t) = \frac{\rho(s)}{\rho(z)} \frac{1}{1-\sigma'(s)t}\bigg|_{s=\xi_0}.$$

From equation (4.110), $\sigma(s) = 2s$, $\sigma'(s) = 2$, we have

$$\Phi(z,t) = \frac{\rho(s)}{\rho(z)} \frac{1}{1-\sigma'(s)t}\bigg|_{s=\xi_0} = \frac{s^{a-2}e^{b/s}}{z^{a-2}e^{-b/z}} \frac{1}{1-2st}\bigg|_{s=\xi_0}.$$

Here s is a root of $= s - z - s^2 t = 0$. So

$$s_1 = \frac{1-\sqrt{1-4zt}}{2t}, \quad s_2 = \frac{1+\sqrt{1-4zt}}{2t}.$$

We choose s_1 because it satisfies the condition $s_1 \to z$ as $t \to 0$,

$$\Phi(z,t) \quad = \quad (\frac{s}{z})^{a-2} e^{\frac{-b}{s}+\frac{b}{z}} \frac{1}{1-2t(\frac{1-\sqrt{1-4zt}}{2t})} \tag{4.111}$$

We now substitute s_1 in (4.111) to obtain

$$\Phi(z,t) = \left[\frac{1-\sqrt{1-4zt}}{2zt}\right]^{a-2} \exp\left\{b\frac{1-\sqrt{1-4zt}-2zt}{(1-\sqrt{1-4zt})z}\right\}(1-4zt)^{-1/2}$$

or

$$\Phi(z,t) = \left[\frac{1-\sqrt{1-4zt}}{2zt}\right]^{a-2} \exp\left\{b\frac{1-\sqrt{1-4zt}}{2z}\right\}(1-4zt)^{-1/2}.$$

For $y_n(z;a;b)$ $B_n = (\frac{1}{b})^n$. So, dividing and multiplying by 2^n,

$$\Phi(z,t) = \left[\frac{1-\sqrt{1-4zt}}{2zt}\right]^{a-2} e^{b(\frac{1-\sqrt{1-4zt}}{2z})}(1-4zt)^{-1/2} = \sum_{n=0}^{\infty}(\frac{b}{2})^n y_n(z)(2t)^n \frac{1}{n!}.$$

and replacing $2t$ by s

$$\left[\frac{1-\sqrt{1-2zs}}{zs}\right]^{a-2} e^{b(\frac{1-\sqrt{1-2zs}}{2z})}(1-2zs)^{-1/2} = \sum_{n=0}^{\infty}(\frac{b}{2})^n y_n(z)s^n\frac{1}{n!}.$$

Futhermore,

$$\left[\frac{(1-\sqrt{1-2zs})(1+\sqrt{1-2zs})}{zs(1+\sqrt{1-2zs})}\right]^{a-2} e^{\frac{b}{2z}(1-\sqrt{1-2zs})}(1-2zs)^{-1/2}$$
$$= \sum_{n=0}^{\infty}(\frac{b}{2})^n y_n(z)s^n\frac{1}{n!}.$$

and replacing s by t, we obtain the generating function for $y_n(z;a,b)$ as

$$\Phi(z,t) = (1-2zt)^{-1/2}\left[\frac{1+\sqrt{1-2zt}}{2}\right]^{2-a} \exp\left\{(\frac{b}{2z})(1-\sqrt{1-2zt})\right\}$$
$$= \sum_{n=0}^{\infty}(\frac{b}{2})^n t^n y_n(z;a,b)\frac{1}{n!}. \quad (4.112)$$

For $b=2$ generating function for $y_n(z;a,2)$ is

$$(1-2tz)^{-1/2}\left[\frac{1+\sqrt{1-2zt}}{2}\right]^{2-a} \exp\left\{\frac{1}{z}(1-\sqrt{1-2zt})\right\} = \sum_{n=0}^{\infty}\frac{y_n(z;a)t^n}{n!}.$$

In particular, if $a=b=2$, we have the generating function for Bessel polynomials as

$$(1-2zt)^{-1/2}\exp\left\{\frac{1}{z}(1-\sqrt{1-2tz})^{1/2}\right\} = \sum_{n=0}^{\infty}\frac{y_n(z)}{n!}t^n.$$

If we replace $y_n(z;a,b)$ by $z^n\theta_n(z^{-1};a,b)$ in (4.111) we get

$$(1-2zt)^{-1/2}\left[\frac{1+\sqrt{1-2zt}}{2}\right]^{2-a} \exp\left\{\frac{b}{2z}1-\sqrt{1-2zt}\right\}$$
$$= \sum_{n=0}^{\infty}(\frac{b}{2})^n z^n\theta_n(z^{-1};a,b)\frac{t^n}{n!}.$$

We now replace z by z^{-1} to get

$$(\frac{z}{z-2t})^{1/2}\left[\frac{1+(\frac{z}{z-2t})^{-1/2}}{2}\right]^{2-a}\exp\left\{\frac{bz}{2}(1-(\frac{z}{z-2t})^{-1/2}\right\}$$
$$=\sum_{n=0}^{\infty}(\frac{b}{2})^n\theta_n(z;a,b.) \tag{4.113}$$

If we insert $b=2$ in (4.113) we have

$$(\frac{z}{z-2t})^{1/2}\left[\frac{1+(\frac{z}{z-2t})^{-1/2}}{2}\right]^{2-a}\exp\left\{z(1-(\frac{z}{z-2t})^{-1/2}\right\}=\sum_{n=0}^{\infty}\theta_n(z;a)(\frac{t}{z})^n.$$

Particularly, for $a=b=2$ one obtains

$$(\frac{z}{z-2t})^{1/2}\exp\left\{z(1-(\frac{z}{z-2t})^{-1/2}\right\}=\sum_{n=0}^{\infty}\theta_n(z)(\frac{t}{z})^n.$$

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