DUALISATION OF SUPERGRAVITY THEORIES

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ABSTRACT

DUALISATION OF SUPERGRAVITY THEORIES

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By using the Kaluza-Klein reduction, the derivation of the maximal supergravities from the $D = 11$ supergravity theory, as well as the Abelian Yang-Mills supergravities from the $D = 10$ type I supergravity theory are discussed. After a thorough review of the symmetric spaces the symmetric space sigma model is studied in detail. The first-order formulation of both the pure and the matter coupled symmetric space sigma model is presented in a general formalism. The dualisation of the non-gravitational Bosonic sectors of the $D = 11$, IIB and the maximal supergravities are also reviewed in a concise but a self-contained formulation. As an example of the dualisation of the matter coupled supergravities, the doubled formalism is constructed for the $D = 8$ Salam-Sezgin supergravity.

Keywords: Kaluza-Klein Reduction, Supergravity, Symmetric Spaces, Non-Linear Sigma Models, Matter Coupling, Dualisation, First-Order Formulations.

ÖZ

SÜPER KÜTLE ÇEKİM KURAMLARININ İKİLLEŞTİRİLMESİ.

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Kaluza ve Klein indirgemesi kullanılarak onbir boyutlu süper kütle çekim kuramından sınırda olan süper kütle çekim kuramları, aynı zamanda on boyutlu birinci tip süper kütle çekim kuramından da Abelsel Yang ve Mills süper kütle çekim kuramlarının türetilmesi tartışıldı. Bakışımlı uzayların geniş bir tekrarından sonra bakışımlı uzay sigma örnek kuramı ayrıntılı olarak çalışıldı. Yalın ve madde eklenmiş bakışımlı uzay sigma örnek kuramının birinci derece simgesel biçimlendirmesi genel bir yöntem içinde sunuldu. Onbir boyutlu, 2B ve sınırda olan süper kütle çekim kuramlarının kütle çekimsel olmayan Bozonsal kesimlerinin ikilleştirmesi de kısa ve öz ama yeterli bir simgesel biçimlendirme içerisinde tekrarlandı. Madde eklenmiş süper kütle çekim kuramlarının ikilleştirilmesine bir örnek olarak da sekiz boyutlu Salam ve Sezgin süper kütle çekim kuramının çifte biçimlendirmesi kuruldu.

Anahtar Sözcükler: Kaluza ve Klein İndirgemesi, Süper Kütle Çekim Kuramı, BakışımlıUzaylar, Doğrusal Olmayan Sigma Örnek Kuramları, Madde Eklenmesi, İkilleştirme, Birinci Derece Simgesel Biçimlendirmeler.

To My Family

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CHAPTER 1

INTRODUCTION

The supergravity theory which has the highest spacetime dimension is the $D = 11$, $N = 1$ supergravity [1]. There are three types of supergravity theories in ten dimensions namely the IIA, [2, 3, 4], the IIB, [5, 6, 7] and as the third supergravity, the ten dimensional type I supergravity theory which is coupled to the Yang-Mills theory [8, 9]. One can obtain the $D = 10$, IIA supergravity theory by the Kaluza-Klein dimensional reduction of the $D = 11$ supergravity on the circle, S^1. Owing to the fact that there exists a self-dual five-form field strength, there is not a straight forward method to construct a covariant Lagrangian for the IIB supergravity in ten dimensions although covariant equations of motion can be formulated. For the IIB supergravity, if one relaxes the self-duality condition on the five-form field strength, by using the Lagrange multiplier methods, a covariant Lagrangian can be constructed [10, 11]. When the self-duality is imposed which is a consistent condition with both the Bianchi identity of the five-form field strength and the equations of motion of the constructed Lagrangian, it becomes a truncation of the equations of motion which turns them into the field equations of the IIB supergravity. The supergravity theories for $D < 10$ dimensions can be obtained from the $D = 11$ and the $D = 10$ supergravities by the dimensional reduction and the truncation of fields. A general treatment of the supergravity theories can be found in [12, 13, 14, 15].

The ten dimensional IIA supergravity and the IIB supergravity theories are the massless sectors or the low energy effective limits of the type IIA and the type IIB superstring theories respectively. The type I supergravity theory in ten dimensions on the other hand is the low energy effective limit of the type I superstring theory and the heterotic string theory. The eleven dimensional supergravity is conjectured to be the low energy effective theory of the eleven dimensional M theory.

The symmetries of the supergravity theories have been studied in the recent years to gain insight in the symmetries and the duality transformations of the string theories. Especially the global symmetries of the supergravities contribute to the knowledge of the non-perturbative U-duality symmetries of the string theories and the M theory [16, 17]. An appropriate restriction of the global symmetry group G of the supergravity theory to the integers $\mathbb{Z}$, namely $G(\mathbb{Z})$, is conjectured to be the U-duality symmetry of the relative string theory which unifies the T-duality and the S-duality [16]. Therefore the dualisation and the coset formulation of the supergravities have not only enabled us to study the symmetries of the supergravities in detail but also provided a better understanding of the dualities and the symmetries of the relative string theories.

In supergravity theories which possess scalar fields, the global (rigid) symmetries of the scalar sector whose action do not depend on the spacetime coordinates are essential to have a deeper understanding of these theories. One can define the action of the global symmetry group of the scalars on the other fields as well, thus the global symmetry of the scalars can be extended to be the global symmetry of the entire Bosonic sector of the theory. For a majority of the supergravities, the scalar manifolds are homogeneous, symmetric spaces which are in the form of cosets G/K [18], and the scalar Lagrangians can be formulated as the non-linear coset sigma models, in particular the symmetric space sigma models. The dimension of the coset space G/K is equal to the number of the scalars of the theory. In the symmetric space sigma models the scalars have non-linear transformation properties. We will basically deal with two classes of scalar coset manifolds G/K. When we apply the Kaluza-Klein dimensional reduction for the Bosonic sector of the $D = 11$ supergravity [1] over the tori T^n, where $n = 11 - D$, we obtain the D-dimensional maximal supergravity theories [10, 19, 20]. The global (rigid) symmetry groups of the Bosonic sector of the reduced Lagrangians are in split real form (maximally non-compact whose Cartan subalgebras can be chosen along the non-compact directions). For the scalar coset manifolds, G/K of the maximal supergravities, G is the global symmetry group which is a semi-simple, split real form and K is its maximal compact subgroup. Thus the coset spaces G/K can be parameterized by using the Borel gauge and the scalar sectors of the maximal supergravities can be formulated as the symmetric space sigma models. However we should remark that in certain dimensions in order to formulate the scalar sectors as symmetric space sigma models one has to make use of the dualisation methods to replace the higher-order fields with the newly defined scalars. On the other hand when one considers the Kaluza-Klein compactification of the

Bosonic sector of the ten dimensional simple supergravity which is coupled to N Abelian gauge multiplets (type I supergravity) on the tori T^{10-D} [8, 9], one can show that after a single scalar is decoupled, the rest of the scalars of the lower, D-dimensional theories can be formulated as the G/K symmetric space sigma models [21]. One can even enlarge the coset formulations of the scalars by using partial dualisations. Unlike the maximal supergravities for this class of supergravities, the global symmetry group G is not necessarily a split real form but it is in general a semi-simple, non-compact real form and again K is a maximal compact subgroup of G. Therefore one makes use of the more general solvable Lie algebra gauge [18] which covers the Borel gauge as a special case to parameterize the scalar coset manifolds for these supergravity theories. In general in the non-linear G/K coset sigma models, the scalars transform non-linearly in the linear representations of G while the higher rank potentials always transform linearly. Although the global symmetry scheme of the supergravity theories are consequences of the supersymmetry, the study of the coset symmetries of the supergravity theories provides a broader understanding of the underlying structure of these theories [22, 23].

The method of non-linear realizations [24, 25, 26, 27, 28] is used in [29, 30] to formulate the gravity as a non-linear realization in which the gravity and the gauge fields appear on equal footing. Later, the dualisation of the Bosonic fields has provided the non-linear realization formulation of the Bosonic sectors of the maximal supergravity theories [10, 19].

By introducing auxiliary fields for a subset of the field content and by using the coset formulation, the global symmetries of the scalar sectors of the maximal supergravities are studied in [10]. As we have mentioned before these symmetries can also be realized on the Bosonic fields. A general, dimension-independent formalism is developed for the Bosonic sectors of the maximal supergravities in [31]. The coset realizations of the non-gravitational Bosonic sectors of the $D = 11$ supergravity, the maximal supergravities as well as the IIB supergravity, are introduced by the dualisation of the scalars and the higher-order gauge fields in [19]. In the same work, the twisted self-duality structure of the supergravities [32, 33] is generalized to regain the first-order equations of the corresponding theories from the Cartan forms of the dualized coset. Therefore in [19] it is shown that the non-linear coset formulation of the scalars can be improved to include the other non-gravitational Bosonic fields, resulting in the first-order formulation of the relative theories. The mainline of [19] is to introduce dual fields for the non-gravitational Bosonic fields and to construct the Lie superalgebra which will generate the coset representatives that realize the original field equations both in first and second-order by means of the Cartan form. The dualisation method is another manifestation of the Lagrange multiplier methods which are used for the scalar sectors of the maximal supergravities in [10, 20].

The discussion about the symmetry groups of the Cartan forms (the doubled field strengths) of the coset formulation as well as the symmetry groups of the twisted self-duality equations (i.e. first-order equations) which are larger than the symmetry groups of the Cartan forms is given in [10] and [19].

By following the outline of [19], in [34] the complete coset formulations of the

$D = 11$ and the IIA supergravity theories are performed for the entire Bosonic sector including the gravity which is missing in [19]. The symmetries of these theories are also discussed in detail in [34]. The Bosonic sector of the IIB supergravity is derived as a non-linear coset realization by dualizing the non-gravitational Bosonic fields in parallel with the formulation given in [19], also by including the gravity sector in [35]. The method of [34] and [35] which includes the gravity sector in the coset formulation is a standard one which can be extended to the coset realizations of the other supergravities.

The non-linear, coset realizations of the complete Bosonic sectors of the $D = 11$ supergravity and the IIA supergravity lead to finite dimensional Lie algebras whose corresponding groups are denoted as G_{11} and G_{IIA}, [34] respectively, which are not Kac-Moody algebras. In [35] the algebra corresponding to the symmetry group G_{IIB} of IIB is constructed as an element of the complete non-linear realization of the Bosonic sector of the IIB supergravity. The equations of motion of the relative theories in [34, 35] are obtained as a result of the simultaneous non-linear realizations of the symmetry groups G_{11}, G_{IIA} and G_{IIB} by taking the Lorentz group as the local subgroup and the non-linear realizations of the conformal groups. The IIA, IIB and the $D = 11$ supergravities are in fact the non-linear realizations of the groups which are the closures of the symmetry groups G_{IIA}, G_{IIB} and G_{11} in the conformal groups, respectively. However instead of dealing with the non-linear realizations of the infinite dimensional closure groups, one follows the easier method of simultaneous non-linear realizations as explained in detail in [34].

In the simultaneous non-linear realizations of the complete Bosonic sectors of the IIA, IIB and the $D = 11$ supergravities the local subgroups are chosen to be the Lorentz group so that the general coset representatives of the entire Bosonic sector can not be parameterized by a solvable Lie algebra or a Borel subalgebra of some larger group. These coset formulations are not like the scalar cosets which arise for the maximal supergravities obtained from the $D = 11$ supergravity by dimensional reduction, also which all give rise to Kac-Moody algebras and whose general coset representatives can be parameterized by a Borel subgroup of a larger group. In [35, 36] it is discussed that the non-linear realizations of the $D = 11$ and the maximal supergravities, in particular the IIA supergravity, can be enlarged to include a Kac-Moody algebra which contains the Borel subalgebra of E_8 and whose corresponding group is identified as E_{11}. This can be done either by introducing a larger local subgroup than the Lorentz group or by describing the gravity by two fields which are duals of each other. Furthermore in [35] for the IIB supergravity theory, it is also argued that the non-linear realization can be enlarged to include the Kac-Moody algebra of the group E_{11} of the IIA and the $D = 11$ supergravities.

In [37] the method of non-linear realizations is used to derive the dynamics of the M theory branes. Furthermore the M theory branes when they are in a background are also described as non-linear realizations in [34]. In the light of the non-linear realizations of the $D = 11$ and the IIA supergravities, it is mentioned in [34] that $Osp(1/64)$ may be a symmetry of the M theory since it is present in the IIA and the $D = 11$ supergravities, also it is the unique extension of the conformal group to include the supersymmetry. It automatically contains all the automorphisms of the Poincare supersymmetry algebras with all their central charges.

The different symmetry groups G_{IIA} and G_{11} appearing in the dualisations of the IIA and the $D = 11$ supergravity theories may be interpreted as the different contractions of the full automorphism group of the M theory. However as we have pointed out above, the common Kac-Moody group which appears in the non-linear realizations of the IIA, IIB and the $D = 11$ supergravities is E_{11}. Since the IIA and the $D = 11$ supergravities are related by the Kaluza-Klein reduction on the circle, it is not puzzling to have a common symmetry group, on the contrary IIB is not related to the $D = 11$ supergravity in a simple way, for this reason finding a common symmetry scheme is a source of motivation. Thus in [35] it is discussed that E_{11} can be a symmetry of the M theory and the IIA, IIB also the $D = 11$ supergravities can be different manifestations of the M theory.

We also encounter the non-linear realizations when there is spontaneous symmetry breaking and they formulate the theories which govern the low energy excitations of the original theory. The local subgroup of the non-linear coset formulation of the low energy regime is the part of the original symmetry which is preserved during the symmetry breaking. If the original theory has various vacua, the corresponding theories of the low energy regimes share the same rigid symmetry group but they have different local symmetry groups. Since the $D = 11$, the maximal, in particular the IIA and the IIB supergravity theories all share the same rigid symmetry E_{11} with various local symmetry subgroups, in [35] it is conjectured that these theories could correspond to the various vacua of the M theory.

The general perspective of the following thesis is to study the framework of the dualisation of the scalars and the higher-order fields excluding the gravity in the supergravity theories. As we have mentioned above, the complete coset construction of any supergravity theory can be obtained by following the standard method given in [34, 35] which enables the inclusion of the gravity sector. Therefore we will primarily be interested in the dualisation of the non-gravitational Bosonic fields.

We will start with the discussion of the supergravities in two categories in Chapter two. First we will introduce the Kaluza-Klein reduction of the pure gravity and the matter fields on the tori T^n and mention about how the spacetime symmetries propagate to the lower dimensions. Next we will study the Kaluza-Klein compactification of the $D = 11$ supergravity and present the construction of the D-dimensional maximal supergravities. The Kaluza-Klein reduction over the tori will again be applied to derive the Abelian Yang-Mills supergravities from the ten dimensional type I supergravity or more correctly from its subtheory which is obtained by coupling N Abelian gauge multiplets to the ten dimensional simple $N = 1$, supergravity. In these two classes of supergravities we will focus on the scalar sectors to study the coset constructions and the partial dualisations. The coset constructions of the scalar sectors will lead us to the symmetric space sigma models. The partial dualisations which increase the number of the scalars are needed to construct the coset Lagrangians or to enhance the symmetries of the scalars in certain dimensions. Such dualisations which are based on the Lagrange multiplier methods direct us to the idea of the complete dualisation and the non-linear realization of the Bosonic sectors of the supergravities.

The non-linear coset constructions of the scalar sectors of the supergravities have a distinctive place in our discussion since in general, dualisation is basically the method of extending the coset formulation already existing for the scalars to the other Bosonic fields which leads to the non-linear realization of the complete Bosonic theory. As will be clear in Chapter two, the scalar sectors of a wide class of supergravities can be formulated as symmetric space sigma models. For this reason in Chapter three we will discuss the formal construction of the symmetric spaces from both the geometrical and the algebraic perspectives. We will also introduce the Cartan and the Iwasawa decompositions of the semi-simple Lie algebras which provide the basic tool of solvable Lie algebra gauge in the general scalar coset constructions.

In Chapter four after a concise introduction of the general non-linear sigma models we will present a detailed covering of the symmetric space sigma model. We will perform the first-order formulation of the symmetric space sigma model by using the dualisation method of [10, 19]. The formulation will be based on two different coset parameterizations which both make use of the solvable Lie algebra gauge defined in Chapter three. We will also mention about the transformations between these two parameterizations.

Chapter five is reserved for the discussion of the dualisation of the non-gravitational Bosonic sectors of the $D = 11$, IIA, IIB and the D-dimensional maximal supergravities.

In Chapter six we will consider the coupling of m-form matter fields to the symmetric space sigma model which we study in detail in Chapter four and we will perform the dualisation and the first-order formulation of the matter coupled symmetric space sigma models in a general formalism.

The dualisation of the $D = 8$ Salam-Sezgin supergravity [38] which is constructed by coupling N vector multiplets to the $D = 8$, $N = 1$ supergravity is the subject of Chapter seven. In carrying out the dualisation and the first-order formulation we will refer to the results derived in chapters four and six which are applicable to any supergravity theory which has a symmetric space scalar coset manifold and which is coupled to matter fields.

CHAPTER 2

THE SUPERGRAVITY THEORIES AND THEIR SCALAR COSETS

The scalar fields have a distinctive place among the field content of the supergravity theories. One can explore the non-local (global) symmetry properties of the entire theory by just looking at the scalar sector. The scalars of the supergravity theories have a non-linear nature in the sense that the scalar Lagrangian can be basically formulated as a non-linear sigma model [14]. There is also a reflection of the non-linearity in the transformation properties of the scalars [19]. In particular the scalar sectors of a wide class of supergravities [13, 18] are governed by the symmetric space sigma models where the scalar fields parameterize a coset G/K which is a symmetric space. In general the space G/K is called the scalar coset manifold. In the following chapters we will give a detailed discussion of both the symmetric spaces and the symmetric space sigma model.

In this chapter we will introduce two main classes of supergravity theories whose scalar fields parameterize symmetric spaces G/K, where G is the global internal symmetry group of the Bosonic sector of the corresponding theory. We will leave the task of constructing the Lagrangian from the coset G/K in a general formulation to the following chapters. Instead, in this chapter we will discuss the origin of these theories and we will classify the cosets which are intimately related to the global symmetries of these theories. The first class of supergravity theories we will deal with is composed of the maximally extended supergravities [10, 13, 19, 20]. The D-dimensional maximal supergravity theory and its symmetries can be systematically obtained as a result of the Kaluza-Klein reduction [20] of the eleven dimensional supergravity [1] which is the low energy effective limit of the M-theory on the n-torus $T^n = S^1 \times \cdots \times S^1$ where $n = 11 - D$. The elements of the second class of supergravities we will discuss in this chapter are the Kaluza-Klein descendants of the ten dimensional simple supergravity which is coupled to N Abelian gauge multiplets on the tori T^n [21]. When $N = 16$, the ten dimensional simple supergravity which is coupled to 16 Abelian gauge multiplets corresponds to the low energy effective limit of the ten dimensional heterotic string theory [21]. We will keep our interest only in the Bosonic sectors which contain the scalar content of these supergravity theories. We will give an introduction to the Kaluza-Klein compactification on S^1 and we will shortly mention how it can be used step by step to handle the reduction on T^n without going into detail in both cases. However our main objective in this chapter is the presentation of the scalar cosets which arise in lower dimensions in both classes of supergravities. Apart from their higher dimensional origins, there is a major difference between the two different classes of supergravity theories we uttered above. The scalar coset manifolds of the maximal supergravities are always based on the maximally non-compact (split) real form global symmetry groups, on the other hand the scalar cosets coming from the Kaluza-Klein reduction of the ten dimensional simple supergravity which is coupled to N Abelian gauge multiplets have a more general structure, namely their global symmetry groups are non-compact real forms (being maximally non-compact for certain choice of D and N [20, 21]). We will give a formal construction of the real forms of semi-simple Lie groups and discuss their classification in the next chapter.

In some dimensions, in order to formulate the scalar Lagrangian of the Kaluza-Klein descendant theories as symmetric space sigma models, one needs to dualize certain higher order fields to introduce new scalar fields by using Lagrange multiplier methods. We will give the details of how this is done for both classes of supergravity theories in relevant dimensions. The scalar cosets obtained after the dualisation of certain higher order fields are called maximal, since one can not obtain more scalar fields upon dualisation in the corresponding dimensions. The partial dualisation method forms a motivation for the complete dualisation of the Bosonic sectors of these theories which includes the introduction of dual fields for the entire set of the original Bosonic fields of the theory in a systematical way. This will be the main topic of the following chapters.

2.1 The Kaluza-Klein Compactification

The Kaluza-Klein reduction or the compactification is simply assigning particular spacetime, metric and potential fields ansatz to a theory so that the equations of motion of the fields which are assumed to be constant in certain directions of the spacetime manifold can be obtained from a lower dimensional Lagrangian defined on a component of the spacetime manifold. We need to mention about the mechanism of the Kaluza-Klein reduction on the torus $T^n = S^1 \times \cdots \times S^1$ to understand the nature of the scalar fields in the dimensionally reduced theories. Since the torus T^n is composed of n-times Cartesian product of circles S^1, one can first consider the S^1-reduction of a D-dimensional theory into a $(D-1)$-dimensional one then step by step construct the $T^n = S^1 \times \cdots \times S^1$ reduction. We will give a summary of the reduction of the D-dimensional metric and the potential fields on the circle S^1, a more detailed analysis can be referred in [10, 20]. On a local coordinate chart of the total D-dimensional manifold which we assume, is composed of the Cartesian product of a circle S^1 of radius L and a $(D-1)$-dimensional sub-manifold, we will denote the coordinate on S^1 by z and the rest of the D-dimensional coordinates by x_i. One can make a Fourier series expansion of the components of the D-dimensional metric tensor field as

$$g_{km}^{D}(x_i, z) = \sum_n g_{km}^{(n)}(x_i) e^{inz/L}. \tag{2.1}$$

We assume that L is of the order of the Planck length, 10^{-33}cm, since for physical reasons one assumes that the circle constituent of the spacetime is ignorably small. For this reason the massive modes $g_{km}^{(n)}(x_i)$ for which $n \neq 0$ gain large mass values which are out of the ranges we can detect. Therefore we assume that in (2.1) the contribution from the massive modes are ignorable and we will only consider the massless modes which correspond to the terms, $n = 0$. This means that our ansatz for the D-dimensional metric tensor is such that its components do not depend on the coordinate z. In light of this discussion we will assume that the

D-dimensional metric has the form

$$g^D = e^{2\alpha\phi} g_{\mu\nu} dx^\mu \otimes dx^\nu + e^{2\beta\phi}(dz + \mathcal{A}) \otimes (dz + \mathcal{A}), \tag{2.2}$$

where $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. In (2.2) α and β are constants which will be determined later and the indices μ, ν correspond to the tangent space indices of the $(D-1)$-dimensional sub-manifold of the original D-dimensional one. All of the fields in (2.2) are independent of z due to the discussion we have given above, thus they are all $(D-1)$-dimensional fields. The scalar field ϕ is called the dilaton and the one-form field $\mathcal{A}$ is called the Kaluza-Klein potential. We also have the $(D-1)$-dimensional metric components $g_{\mu\nu}$ which determine the $(D-1)$-dimensional branch of the D-dimensional spacetime. In terms of the $(D-1)$-dimensional fields the components of the D-dimensional metric can be given as

$$g^D_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu, \qquad g^D_{\mu z} = e^{2\beta\phi}\mathcal{A}_\mu, \qquad g^D_{zz} = e^{2\beta\phi}. \tag{2.3}$$

We will give the basic definitions about the Riemannian geometry when we discuss the symmetric spaces in the next chapter but for our purposes of deriving the lower dimensional Lagrangian we need to mention shortly about the elements of the D-dimensional gravity. Firstly for an affine connection ∇ defined on the tangent bundle of the D-dimensional spacetime [39] one defines the connection coefficients Γ^c_{ab} as [40, 41]

$$\nabla_{X_a} X_b = \Gamma^c_{ab}\, X_c, \tag{2.4}$$

where $\{X_a\}$ is a D-dimensional frame field for the D-dimensional manifold. If we define the dual, co-frame field (vielbein) $e^a(X_b) = \delta^a_b$, then we can introduce the connection one-forms as

$$\omega^a_b = \Gamma^a_{cb}\, e^c. \tag{2.5}$$

By using the affine connection ∇, if we define the map $T(X, Y) : T^1M \times T^1M \longrightarrow T^1M$,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{2.6}$$

then we can introduce the torsion tensor $T \in T^1_2 M$ as

$$T : (\omega, X, Y) \longrightarrow \omega(T(X, Y)), \tag{2.7}$$

$\forall X, Y \in T^1M$ and $\omega \in \omega^1(M)$. The torsion two-forms $T^a \in \omega^2(M)$ are defined as [42]

$$T = 2T^a \otimes X_a. \tag{2.8}$$

One can also define the map $R(X, Y) : T^1M \times T^1M \longrightarrow Hom(T^1M, T^1M)$ as

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \tag{2.9}$$

from which the curvature tensor $R \in T^1_3 M$ can be defined as

$$R : (\omega, Z, X, Y) \longrightarrow \omega(R(X, Y) \cdot Z), \tag{2.10}$$

$\forall X, Y, Z \in T^1M$ and $\omega \in \omega^1(M)$. Similarly we can introduce the curvature two-forms $R^d_c \in \omega^2(M)$ as [42]

$$R = 2R^d_c \otimes e^c \otimes X_d. \tag{2.11}$$

One can calculate the connection one-forms and the curvature two-forms by using a frame field $\{X_a\}$ and the corresponding dual vielbein $\{e^a\}$ from the first and the second Cartan structure equations

$$T^a = de^a + \omega^a_b \wedge e^b,$$

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \tag{2.12}$$

respectively [40, 42]. Alternatively one can calculate the structure functions $\{C^c_{ab}\}$ from

$$[X_a, X_b] = C^c_{ab} X_c, \tag{2.13}$$

for an arbitrary frame field $\{X_a\}$. Then one can read the connection coefficients $\{\Gamma^p_{ab}\}$ by using the identity [42]

$$2\,\Gamma^p_{ab} = g^{cp}(X_a(g_{bc}) + X_b(g_{ca}) - X_c(g_{ab}) - C^d_{bc} g_{ad}$$

$$+ C^d_{ca} g_{bd} + C^d_{ab} g_{cd} - T^m_{bc} g_{ma} + T^m_{ca} g_{mb} + T^m_{ab} g_{mc}), \tag{2.14}$$

where the metric components, $g_{ab} = g(X_a, X_b)$ are used to raise and lower the indices of the structure functions, $\{C^c_{ab}\}$ and the torsion tensor components, $T^c_{ab} = T(X_a, X_b, e^c)$. There are two special classes of frame fields which simplify (2.14), if the frame field is a coordinate basis then the structure functions $\{C^c_{ab}\}$ are zero, on the other hand if it is an orthonormal basis then the metric components are constant so that $X_i(g_{jk}) = 0$. An orthonormal basis frame field can be read from the explicit form of the metric tensor. After the calculation of the connection coefficients $\{\Gamma^p_{ab}\}$, the connection one-forms and the curvature two-forms can be obtained from (2.5) and (2.12).

Since we are concerned with the Einstein-Hilbert action in the Kaluza-Klein reduction procedure, we will assume the unique metric compatible, torsion-free, Levi-Civita connection ∇, [39] so that $\{T^a\}$ and T will be zero in (2.12) and in (2.14) respectively. Choosing an orthonormal frame field will simplify the calculation of the connection one-forms and the curvature two-forms in both of the methods we have discussed. By following either of the methods above, when one derives the curvature two-forms $\{R^b_a\}$, one can calculate the curvature tensor R from (2.11) and then calculate the Ricci tensor $\mathcal{R} \in T^0_2 M$ which is obtained by a contraction on R as

$$\mathcal{R}(X, Y) = R(X_a, X, Y, e^a), \tag{2.15}$$

$\forall X, Y \in T^1 M$ and for an arbitrary frame field $\{X_a\}$ and its dual, co-frame field (vielbein) $\{e^a\}$. The D-dimensional Ricci scalar R^D can then be calculated as

$$R^D = \mathcal{R}(X_a, X^a), \tag{2.16}$$

where $X^a = g^{ab} X_b$, with the inverse matrix g^{ab} of the metric tensor components $g_{ab} = g(X_a, X_b)$. Then the D-dimensional Einstein-Hilbert Lagrangian can be expressed as

$$\mathcal{L}_D = R^D * 1, \tag{2.17}$$

where $*$ is the Hodge-star operator [39] and its action on a p-form basis is defined as

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{1}{q!} \epsilon_{\nu_1 \cdots \nu_q}{}^{\mu_1 \cdots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_q}, \tag{2.18}$$

here $q = D - p$ and $\epsilon_{\nu_1 \cdots \nu_D} = \sqrt{|g|}\, \varepsilon_{\nu_1 \cdots \nu_D}$ with g, being the determinant of the D-dimensional metric tensor components g_{ab} and $\varepsilon_{\nu_1 \cdots \nu_D}$ is the totally antisymmetric

Levi-Civita tensor density [41]. We again use the metric tensor components to raise and lower the indices. Thus in (2.17) we have

$$*1 = \sqrt{|g|}\, dx^D, \tag{2.19}$$

with dx^D, being the canonical volume element D-form whose orientation is fixed. If we go back to the Kaluza-Klein ansatz (2.2) now, we have defined the entire machinery to calculate the Einstein-Hilbert Lagrangian in terms of the $(D-1)$-dimensional fields ϕ, $g_{\mu\nu}$, $\mathcal{A}$. We will assume a D-dimensional veilbein of the form

$$(e^D)^a = e^{\alpha\phi} e^a, \qquad (e^D)^z = e^{\beta\phi}(dz + \mathcal{A}), \tag{2.20}$$

where e^a correspond to a $(D-1)$-dimensional veilbein so that the index a runs on the $(D-1)$-dimensional subspace directions and $(e^D)^z$ is the additional basis element along a mixed direction of the S^1 tangent space direction and the $(D-1)$-dimensional subspacetime tangent space directions. When we define the dual, moving frame $\{X^D_\alpha\}$ such that $(e^D)^\beta(X^D_\alpha) = \delta^\beta_\alpha$ for $\alpha, \beta = 1, ..., D$, we can define the metric components $g^D(X^D_\alpha, X^D_\gamma) = \eta_{\alpha\gamma}$. From the Kaluza-Klein ansatz (2.2) for the D-dimensional metric we immediately see that $\eta_{zz} = 1$ and $\eta_{za} = 0$ for $a = 1, ..., (D-1)$. By following the discussion above, one can use the vielbein (2.20) to calculate the D-dimensional connection one-forms ω^{ab}_D in terms of the $(D-1)$-dimensional fields as

$$\omega^{ab}_D = \omega^{ab} + \alpha e^{-\alpha\phi}(\partial^b\phi(e^D)^a - \partial^a\phi(e^D)^b) - \frac{1}{2}\mathcal{F}^{ab} e^{(\beta-2\alpha)\phi}(e^D)^z,$$

$$\omega_D^{az} = -\omega_D^{za} = -\beta e^{-\alpha\phi}\partial^a\phi(e^D)^z - \frac{1}{2}\mathcal{F}^a{}_b e^{(\beta-2\alpha)\phi}(e^D)^b, \tag{2.21}$$

where we define $\partial_a\phi = E_a^\mu\partial_\mu\phi$. If we consider the local components of the $(D-1)$-dimensional sub-vielbein as $e^a = e_\mu^a dx^\mu$ then E_a^μ is the inverse of the components e_μ^a. We also define

$$\mathcal{F} = d\mathcal{A} = \frac{1}{2}\mathcal{F}_{ab}\, e^a \wedge e^b. \tag{2.22}$$

In (2.21) the indices a, b correspond to the $(D-1)$-dimensional sub-directions and the index z is along a mixed direction as we have pointed out above, thus ω^{ab} are the $(D-1)$-dimensional connection one-forms corresponding to the Levi-Civita connection of the $(D-1)$-dimensional metric $g_{\mu\nu}$ defined in (2.2). We use the $(D-1)$-dimensional metric components $\eta_{\mu\nu}$ for $\mu, \nu = 1, ..., (D-1)$ to raise and lower the indices. From the second Cartan structure equation in (2.12) and from the D-dimensional vielbein choice (2.20) one can calculate the D-dimensional curvature two-forms $(R_D)_b^a$. We will not display them here since we only need the Ricci tensor which generates the Ricci scalar in the Einstein-Hilbert Lagrangian. The components of the D-dimensional Ricci tensor can be calculated from (2.11) and (2.15) by using the D-dimensional curvature two-forms and they can be given in terms of the $(D-1)$-dimensional fields as

$$\mathcal{R}_{ab}^D = e^{-2\alpha\phi}(\mathcal{R}_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi) - \frac{1}{2}e^{-2(D-1)\alpha\phi}\mathcal{F}_a{}^c\mathcal{F}_{bc},$$

$$\mathcal{R}_{az}^D = \mathcal{R}_{za}^D = \frac{1}{2}e^{(D-4)\alpha\phi}\nabla^b(e^{-2(D-2)\alpha\phi}\mathcal{F}_{ab}),$$

$$\mathcal{R}^D_{zz} = (D-3)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}_{bc}\mathcal{F}^{bc}, \tag{2.23}$$

where $\Box = \partial^a\partial_a$ with $a = 1, ..., (D-1)$ and $\mathcal{R}$ is the $(D-1)$-dimensional Ricci tensor corresponding to the Levi-Civita connection of the $(D-1)$-dimensional metric $g_{\mu\nu}$ in (2.2). In deriving (2.23) we have chosen the constants α and β in (2.2) as

$$\alpha^2 = \frac{1}{2(D-2)(D-3)}, \qquad \beta = -(D-3)\alpha. \tag{2.24}$$

This choice of the constants in the Kaluza-Klein ansatz (2.2) is necessary to obtain a lower dimensional Lagrangian with the canonical and the usual normalizations of the Einstein-Hilbert and the dilaton kinetic terms [20]. Now we can calculate the the Ricci scalar

$$R^D = \eta^{ab}\mathcal{R}^D_{ab}$$

$$= \eta^{\mu\nu}\mathcal{R}^D_{\mu\nu} + \mathcal{R}^D_{zz}, \tag{2.25}$$

where μ, ν correspond to the $(D-1)$-dimensional indices. In terms of the $(D-1)$-dimensional fields we have [20]

$$R^D = e^{-2\alpha\phi}(R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + (D-4)\alpha\Box\phi)$$

$$-\frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^{ab}\mathcal{F}_{ab}, \tag{2.26}$$

here R is the $(D-1)$-dimensional Ricci scalar. Moreover from (2.2) and (2.24) we have

$$\sqrt{|g|} = \sqrt{-g} = e^{2\alpha\phi}\sqrt{-g'}, \tag{2.27}$$

where g' is the determinant of the $(D-1)$-dimensional metric $g_{\mu\nu}$ over the $(D-1)$-dimensional subspacetime which is defined in (2.2). Finally, now that we have calculated the elements of the D-dimensional Einstein-Hilbert Lagrangian, by omitting the $\Box\phi$ term in (2.26) which contributes a total derivative, we can express the D-dimensional Einstein-Hilbert Lagrangian in terms of the $(D-1)$-dimensional fields R, ϕ, $g_{\mu\nu}$, $\mathcal{F} = d\mathcal{A}$ as

$$\mathcal{L}_D = (R *^{(D-1)} 1 - \frac{1}{2} *^{(D-1)} d\phi \wedge d\phi$$

$$- \frac{1}{2} e^{-2(D-2)\alpha\phi} *^{(D-1)} \mathcal{F} \wedge \mathcal{F}) \wedge dz$$

$$= \mathcal{L}_{(D-1)} \wedge dz, \tag{2.28}$$

where $*^{(D-1)}1 = \sqrt{-g'} dx^{(D-1)}$.

Our next task will be to built up an ansatz to reduce a general D-dimensional $(n-1)$-form potential field $A^D_{(n-1)}$, again with the assumption that the D- dimensional spacetime is composed of the Cartesian product of a $(D-1)$-dimensional subspacetime and S^1. The ansatz will be chosen as [20]

$$A^D_{(n-1)}(x,z) = A_{(n-1)}(x) + A_{(n-2)}(x) \wedge dz, \tag{2.29}$$

where the coordinates x are on the $(D-1)$-dimensional spacetime. When we take the exterior derivative of (2.29) we get

$$F^D_{(n)}(x,z) = dA^D_{(n-1)}(x,z)$$

$$= dA_{(n-1)}(x) + dA_{(n-2)}(x) \wedge dz. \tag{2.30}$$

While choosing the $(D-1)$-dimensional field strength of $A_{(n-2)}$ as $F_{(n-1)} = dA_{(n-2)}$, we will not simply choose the $(D-1)$-dimensional field strength of $A_{(n-1)}$ as $F_{(n)} = dA_{(n-1)}$ but for the purpose of obtaining a nice looking lower dimensional Lagrangian we define the lower dimensional field strengths through the transgression relations

$$F_{(n)}(x) = dA_{(n-1)}(x) - dA_{(n-2)}(x) \wedge \mathcal{A}(x),$$

$$F_{(n-1)}(x) = dA_{(n-2)}(x). \tag{2.31}$$

In terms of these $(D-1)$-dimensional field strengths the D-dimensional field strength can be given as

$$F^D_{(n)}(x,z) = F_{(n)}(x) + F_{(n-1)}(x) \wedge (dz + \mathcal{A}(x)). \tag{2.32}$$

We can now express the D-dimensional kinetic term of $F^D_{(n)}$ in terms of the $(D-1)$-dimensional field strengths we have defined

$$\mathcal{L}^D_F = -\frac{1}{2} * F^D_{(n)} \wedge F^D_{(n)}$$

$$= (-\frac{1}{2} e^{-2(n-1)\alpha\phi} *^{(D-1)} F_{(n)} \wedge F_{(n)}$$

$$-\frac{1}{2} e^{2(D-n-1)\alpha\phi} *^{(D-1)} F_{(n-1)} \wedge F_{(n-1)}) \wedge dz$$

$$= \mathcal{L}_F^{(D-1)} \wedge dz. \tag{2.33}$$

As a final remark we will also discuss how the symmetries of the lower dimensional theory originate from the coordinate transformation symmetries of the D-dimensional theory and the scaling symmetry of the field equations of the D-dimensional theory in the S^1-reduction. The D-dimensional pure gravitational theory has a general coordinate covariance whose infinitesimal action on the coordinates and the D-dimensional metric tensor can be defined as

$$\delta x_D^a = -\xi_D^a, \quad \delta g_{ab}^D = \xi_D^p \, \partial_p \, g_{ab}^D + g_{pb}^D \, \partial_a \, \xi_D^p + g_{ap}^D \, \partial_b \, \xi_D^p, \tag{2.34}$$

where ξ_D^a are the infinitesimal transformation parameters which are functions of the D-dimensional coordinates. Thus the transformations (2.34) are local. The form of the Kaluza-Klein ansatz (2.2) will not be preserved by the entire group of the D-dimensional coordinate transformations rather, the subgroup of the infinitesimal transformations (2.34) which preserves the Kaluza-Klein ansatz (2.2) is generated by the infinitesimal parameters of the form

$$\xi_D^\mu = \xi^\mu(x), \quad \xi_D^z = cz + \lambda(x), \tag{2.35}$$

where ξ_D^μ is along the $(D-1)$-dimensional subspacetime directions and ξ_D^z is along the S^1 direction [20]. As we have done before, we also split the coordinates x_D^a as (x^μ, z). We will show that when one chooses $c = 0$ in (2.35), the corresponding restriction of the general D-dimensional coordinate transformations will not only preserve the form of the Kaluza-Klein ansatz but they will also leave the form of the D-dimensional Lagrangian (2.28) invariant, thus they form the local part of the symmetry group of the $(D-1)$-dimensional Lagrangian defined in (2.28). The field equations of the D-dimensional Einstein-Hilbert Lagrangian (2.17) namely the Einstein equations, $\mathcal{R}_{km} - \frac{1}{2}R^D g^D_{km} = 0$, for the vacuum, have also the global scaling symmetry in which the metric, the Ricci tensor and the Ricci scalar transform as

$$g^D_{ab} \longrightarrow k^2 g^D_{ab}, \quad \mathcal{R} \longrightarrow \mathcal{R}, \quad R^D \longrightarrow k^{-2} R^D, \tag{2.36}$$

where k is a real parameter. The symmetry transformations in (2.36) however do not leave the D-dimensional Einstein-Hilbert Lagrangian (2.17) invariant. If we choose $c = 0$ in (2.35) then we may show that from (2.34) by using (2.3) the $(D-1)$-dimensional fields transform as

$$\delta\phi = \xi^p \, \partial_p \phi,$$

$$\delta\mathcal{A}_\mu = \xi^p \, \partial_p \, \mathcal{A}_\mu + \mathcal{A}_p \, \partial_\mu \, \xi^p + \partial_\mu \lambda,$$

$$\delta g_{\mu\nu} = \xi^p \, \partial_p \, g_{\mu\nu} + g_{p\nu} \, \partial_\mu \, \xi^p + g_{\mu p} \, \partial_\nu \, \xi^p, \tag{2.37}$$

under the restricted D-dimensional infinitesimal coordinate transformations which are generated by the infinitesimal parameters of the form (2.35). The transformation rules (2.37) denote that $c = 0$ branch of (2.35) correspond to the $(D-1)$-dimensional local coordinate transformations when λ is chosen to be zero. This is because when $\lambda(x) = 0$ in (2.37) the dilaton transforms like a scalar field, the

Kaluza-Klein potential $\mathcal{A}$ transforms as a one-form and the metric tensor components transform properly as a second rank tensor. On the other hand when $\xi^p(x) = 0$ in (2.37) then the dilaton ϕ and the $(D-1)$-dimensional metric g are inert while the Kaluza-Klein potential $\mathcal{A}$ transforms as a $U(1)$-gauge field under the transformation of the local infinitesimal parameter λ. This is the origin of the name Kaluza-Klein-Maxwell potential field for $\mathcal{A}$.

If one considers special combinations of the rest of the restricted D- dimensional transformations when $\xi^\mu(x) = \lambda(x) = 0$ but $c \neq 0$ in (2.35) with the scaling symmetry (2.36), one may obtain the global symmetry sector of the reduced $(D-1)$-dimensional theory. The infinitesimal form of the scaling transformation of the D-dimensional metric tensor field g^D in (2.36) can be given as $\delta g^D_{ab} = 2ag^D_{ab}$ where a is a real infinitesimal parameter. When one considers the infinitesimal transformations (2.34) by choosing $\xi^\mu(x) = \lambda(x) = 0$ in (2.35) together with $\delta g^D_{ab} = 2ag^D_{ab}$ then by using (2.3), the $(D-1)$-dimensional fields can be shown to transform as

$$\beta\delta\phi = a + c, \quad \delta\mathcal{A}_\mu = -c\mathcal{A}_\mu, \quad \delta g_{\mu\nu} = 2ag_{\mu\nu} - 2\alpha g_{\mu\nu}\delta\phi. \tag{2.38}$$

We observe that solely the transformation generated through (2.35) by choosing $\xi^\mu(x) = \lambda(x) = 0$ does not correspond to the global infinitesimal symmetry transformations of the lower dimensional theory which do not depend on the $(D-1)$-dimensional coordinates. However we can show that if we choose

$$a = -\frac{c}{D-2}, \tag{2.39}$$

then the metric becomes inert under the transformations (2.38) and they become

$$\delta\phi = -\frac{c}{\alpha(D-2)}, \quad \delta\mathcal{A}_\mu = -c\mathcal{A}_\mu, \quad \delta g_{\mu\nu} = 0. \tag{2.40}$$

Thus (2.40) correspond to the global symmetry of the lower dimensional Lagrangian in which the dilaton has a constant shift transformation $\phi \longrightarrow \phi + c'$. Since $\delta g_{\mu\nu} = 0$, (2.40) is also called the global internal symmetry. This global transformation group of the $(D-1)$-dimensional Lagrangian defined in (2.28) is the real line $\mathbb{R}$ which is considered as an additive group. On the other hand if we choose

$$a = -c, \tag{2.41}$$

then the transformations in (2.38) become

$$\delta\phi = 0, \quad \delta\mathcal{A}_\mu = -c\mathcal{A}_\mu, \quad \delta g_{\mu\nu} = -2cg_{\mu\nu}, \tag{2.42}$$

and they correspond to the scaling symmetry of the equations of motion of the $(D-1)$-dimensional Lagrangian which leave the dilaton ϕ invariant. This symmetry group is also equal to $\mathbb{R}$. Thus the lower dimensional theory has two non-coordinate dependent global symmetry groups. One of them leaves the $(D-1)$-dimensional field equations invariant and the other one leaves the $(D-1)$-dimensional Lagrangian invariant.

2.2 D=11 and the Maximal Supergravities

In the previous section we have given the details of the Kaluza-Klein reduction of the Einstein-Hilbert Lagrangian and the kinetic terms which correspond to the general m-form field strengths on the circle S^1. One can improve this procedure by applying it step by step to find the Kaluza-Klein reduction of a general $(D+n)$-dimensional theory on the n-torus, $T^n = S^1 \times \cdots \times S^1$ [20]. The spirit of the reduction is similar to the S^1-reduction so that basically one assumes a Cartesian product structure for the $(D+n)$-dimensional spacetime which is made up of a D-dimensional subspacetime and T^n. One also assumes a $(D+n)$-dimensional Kaluza-Klein metric ansatz, like we have done for the S^1-reduction. Then the procedure of the S^1-reduction is applied step by step to express the $(D+n)$-dimensional quantities in terms of the D-dimensional ones which are assumed to be constant along the torus T^n and zero in the torus directions. The $(D+n)$-dimensional Lagrangian can then be expressed in terms of the D-dimensional one and the equations of motion for the D-dimensional fields can entirely be obtained from the D-dimensional Lagrangian. The maximally extended supergravity theories in D-dimensions can be obtained by the Kaluza-Klein reduction of the eleven-dimensional supergravity theory [1] on the tori T^n where $n + D = 11$, [10, 20]. The $D = 11$ supergravity is the low energy limit, effective theory of the M-theory. We will only consider the reduction of the Bosonic sector of the eleven dimensional supergravity which contains the scalars. In [20] the reduction of the Fermionic fields as well as the supersymmetry transformation rules are also discussed shortly to imply that the entire reduction corresponds properly to the D-dimensional maximal supergravity. The Bosonic

Lagrangian of the $D = 11$ supergravity theory is

$$\mathcal{L}_{11} = R * 1 - \frac{1}{2} * F_{(4)} \wedge F_{(4)} + \frac{1}{6} dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}, \tag{2.43}$$

where R is the eleven dimensional Ricci scalar. We see that we have the eleven dimensional metric, g_{11} and the 3-form potential field, $A_{(3)}$ as the constituents of the Bosonic field content of the eleven dimensional supergravity [1]. If we consider the Kaluza-Klein reduction on the n-torus, T^n, starting from the eleven dimensional metric, g_{11} and the 3-form potential field, $A_{(3)}$, at each i'th step of the S^1-reduction of the metric we will obtain a lower dimensional metric, g_D, a dilaton ϕ_i and a Kaluza-Klein potential $\mathcal{A}^i_{(1)}$. At the next S^1-reduction step from the ansatz (2.29) each Kaluza-Klein potential $\mathcal{A}^i_{(1)}$ will generate a lower dimensional one-form field which is again denoted as $\mathcal{A}^i_{(1)}$ and a scalar field $\mathcal{A}^i_{(0)i+1}$ which is called an axion. The sequence will continue in the same manner such that each Kaluza-Klein potential $\mathcal{A}^i_{(1)}$ will be carried to the desired dimension by generating an axion $\mathcal{A}^i_{(0)j}$ at each step following the step which $\mathcal{A}^i_{(1)}$ first appears (the i'th step). The axions $\mathcal{A}^i_{(0)j}$ do not generate more fields upon further S^1-reductions from (2.29), they only generate lower dimensional scalars which we will denote again by $\mathcal{A}^i_{(0)j}$. In other words they carry themselves to the desired dimension. We should also state that obviously $i < j$ for the axions $\mathcal{A}^i_{(0)j}$.

On the other hand starting from the eleven dimensional 3-form field $A_{(3)}$, from the ansatz (2.29) at the first S^1-reduction step, a ten dimensional three-form $A_{(3)}$ and a ten dimensional two-form $A_{(2)}$ will be generated. Then if further reductions to the desired dimension D are concerned there will be a final three-form $A_{(3)}$, two-forms $A_{(2)i}$, coming from the i'th step of the reduction of $A_{(3)}$, one-forms $A_{(1)ij}$ coming from the j'th-reduction step of the two-forms $A_{(2)i}$ and finally further axionic scalars $A_{(0)ijk}$ coming from the k'th-reduction step of the one-forms $A_{(1)ij}$. All these fields are D-dimensional and again clearly $i < j < k$. One can show that after performing a series of S^1-reductions by using the Kaluza-Klein ansatz, the eleven dimensional Lagrangian (2.43) can be written as

$$\mathcal{L}_{11} = \mathcal{L}_D \wedge dz^1 \wedge \cdots \wedge dz^n, \tag{2.44}$$

where the coordinates $\{z^i\}$ are the coordinates on the n-torus T^n. The D-dimensional Lagrangian defined in (2.44) can, in terms of the D-dimensional fields we have mentioned above be given as

$$\begin{aligned}
\mathcal{L}_D = {} & R * 1 - \frac{1}{2} * d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} e^{\vec{a}\cdot\vec{\phi}} * F_{(4)} \wedge F_{(4)} \\
& - \frac{1}{2} \sum_i e^{\vec{a}_i\cdot\vec{\phi}} * F_{(3)i} \wedge F_{(3)i} - \frac{1}{2} \sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} * F_{(2)ij} \wedge F_{(2)ij} \\
& - \frac{1}{2} \sum_i e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}^i_{(2)} \wedge \mathcal{F}^i_{(2)} - \frac{1}{2} \sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}} * F_{(1)ijk} \wedge F_{(1)ijk} \\
& - \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} * \mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j} + \mathcal{L}^D_{FFA},
\end{aligned} \tag{2.45}$$

where the dilaton vectors $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$, $\vec{a}_{ijk}$, $\vec{b}_i$, $\vec{b}_{ij}$, which couple to the dilatons $\vec{\phi}$

in various field strength kinetic terms are

$$\vec{a} = -\vec{g} \quad , \quad \vec{a}_i = \vec{f}_i - \vec{g} \quad , \quad \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g},$$

$$\vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g} \quad , \quad \vec{b}_i = -\vec{f}_i \quad , \quad \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j. \tag{2.46}$$

We express the dilatons compactly as a vector $\vec{\phi}$ and make use of the vector notation. In (2.46) we have defined the $(11-D)$ component vectors $\vec{g}$ and $\vec{f}_i$ as [10]

$$\vec{g} = 3(s_1, s_2, ..., s_{(11-D)}),$$

$$\vec{f}_i = (0, ..., 0, (10-i)s_i, s_{i+1}, ..., s_{(11-D)}), \tag{2.47}$$

where in the second line there are $(i-1)$ zeros and $s_i = \sqrt{2/((10-i)(9-i))}$. The properties of the dilaton vectors in (2.46) and the vectors $\vec{g}$, $\vec{f}_i$ can be found in [10, 20]. The Kaluza-Klein ansatz for the eleven dimensional metric can also be given as

$$ds_{11}^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds_D^2 + \sum_i e^{2\vec{\gamma}_i\cdot\vec{\phi}} (h^i)^2, \tag{2.48}$$

where $\vec{\gamma}_i = \frac{1}{6}\vec{g} - \frac{1}{2}\vec{f}_i$ and

$$h^i = dz^i + \mathcal{A}^i_{(1)} + \mathcal{A}^i_{(0)j} dz^j. \tag{2.49}$$

The transgression relations which express the D-dimensional field strengths introduced in (2.45) in terms of the D-dimensional potential fields can be given as

[20],

$$F_{(3)i} = \gamma^j{}_i \widetilde{F}_{(3)j} + \gamma^j{}_i \gamma^k{}_l \widetilde{F}_{(2)jk} \wedge \mathcal{A}^l_{(1)} + \frac{1}{2}\gamma^j{}_i \gamma^k{}_m \gamma^l{}_n \widetilde{F}_{(1)jkl} \wedge \mathcal{A}^m_{(1)} \wedge \mathcal{A}^n_{(1)},$$

$$F_{(2)ij} = \gamma^k{}_i \gamma^l{}_j \widetilde{F}_{(2)kl} - \gamma^k{}_i \gamma^l{}_j \gamma^m{}_n \widetilde{F}_{(1)klm} \wedge \mathcal{A}^n_{(1)} \quad , \quad \mathcal{F}^i_{(1)j} = \gamma^k{}_j \widetilde{\mathcal{F}}^i_{(1)k},$$

$$F_{(1)ijk} = \gamma^l{}_i \gamma^m{}_j \gamma^n{}_k \widetilde{F}_{(1)lmn} \quad , \quad \mathcal{F}^i_{(2)} = \widetilde{\mathcal{F}}^i_{(2)} - \gamma^j{}_k \widetilde{\mathcal{F}}^i_{(1)j} \wedge \mathcal{A}^k_{(1)},$$

$$F_{(4)} = \widetilde{F}_{(4)} - \gamma^i{}_j \widetilde{F}_{(3)i} \wedge \mathcal{A}^j_{(1)} + \frac{1}{2}\gamma^i{}_k \gamma^j{}_l \widetilde{F}_{(2)ij} \wedge \mathcal{A}^k_{(1)} \wedge \mathcal{A}^l_{(1)}$$

$$- \frac{1}{6}\gamma^i{}_l \gamma^j{}_m \gamma^k{}_n \widetilde{F}_{(1)ijk} \wedge \mathcal{A}^l_{(1)} \wedge \mathcal{A}^m_{(1)} \wedge \mathcal{A}^n_{(1)}, \tag{2.50}$$

where $\widetilde{F}_{(n)} = dA_{(n-1)}$ and γ^i_j is defined as

$$\gamma^i_j = [(1 + \mathcal{A}_{(0)})^{-1}]^i_j. \tag{2.51}$$

Notice that we define the $n \times n$ matrix $(\mathcal{A}_{(0)})^i_j$ from the fields $\mathcal{A}^i_{(0)j}$ for $n = 11 - D$, however since the fields $\mathcal{A}^i_{(0)j}$ exist only when $j > i$ we define null entries for the matrix $(\mathcal{A}_{(0)})^i_j$ when $j \leq i$. The $\mathcal{L}_{FFA}$ terms in (2.45) are the Kaluza-Klein descendants of the $\frac{1}{6} dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}$ term of the eleven dimensional Lagrangian (2.43). A straightforward reduction results in

$$\mathcal{L}^{10}_{FFA} = \frac{1}{2}\widetilde{F}_{(4)} \wedge \widetilde{F}_{(4)} \wedge A_{(2)},$$
$$\mathcal{L}^{9}_{FFA} = (\frac{1}{4}\widetilde{F}_{(4)} \wedge \widetilde{F}_{(4)} \wedge A_{(1)ij} - \frac{1}{2}\widetilde{F}_{(3)i} \wedge \widetilde{F}_{(3)j} \wedge A_{(3)})\epsilon^{ij},$$
$$\mathcal{L}^{8}_{FFA} = (\frac{1}{12}\widetilde{F}_{(4)} \wedge \widetilde{F}_{(4)} A_{(0)ijk} - \frac{1}{6}\widetilde{F}_{(3)i} \wedge \widetilde{F}_{(3)j} \wedge A_{(2)k}$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\widetilde{F}_{(4)} \wedge \widetilde{F}_{(3)i} \wedge A_{(1)jk})\epsilon^{ijk}, \\
\mathcal{L}^7_{FFA} &= (\frac{1}{6}\widetilde{F}_{(4)} \wedge \widetilde{F}_{(3)i} A_{(0)jkl} - \frac{1}{4}\widetilde{F}_{(3)i} \wedge \widetilde{F}_{(3)j} \wedge A_{(1)kl} \\
&\quad + \frac{1}{8}\widetilde{F}_{(2)ij} \wedge \widetilde{F}_{(2)kl} \wedge A_{(3)})\epsilon^{ijkl}, \\
\mathcal{L}^6_{FFA} &= (\frac{1}{12}\widetilde{F}_{(4)} \wedge \widetilde{F}_{(2)ij} A_{(0)klm} - \frac{1}{12}\widetilde{F}_{(3)i} \wedge \widetilde{F}_{(3)j} A_{(0)klm} \\
&\quad + \frac{1}{8}\widetilde{F}_{(2)ij} \wedge \widetilde{F}_{(2)kl} \wedge A_{(2)m})\epsilon^{ijklm}, \\
\mathcal{L}^5_{FFA} &= (\frac{1}{12}\widetilde{F}_{(3)i} \wedge \widetilde{F}_{(2)jk} A_{(0)lmn} + \frac{1}{48}\widetilde{F}_{(2)ij} \wedge \widetilde{F}_{(2)kl} A_{(1)mn} \\
&\quad - \frac{1}{72}\widetilde{F}_{(1)ijk} \wedge \widetilde{F}_{(1)lmn} \wedge A_{(3)})\epsilon^{ijklmn}, \\
\mathcal{L}^4_{FFA} &= (\frac{1}{48}\widetilde{F}_{(2)ij} \wedge \widetilde{F}_{(2)kl} A_{(0)mnp} - \frac{1}{72}\widetilde{F}_{(1)ijk} \wedge \widetilde{F}_{(1)lmn} A_{(2)p})\epsilon^{ijklmnp}, \\
\mathcal{L}^3_{FFA} &= -\frac{1}{144}\widetilde{F}_{(1)ijk} \wedge \widetilde{F}_{(1)lmn} \wedge A_{(1)pq}\epsilon^{ijklmnpq}, \\
\mathcal{L}^2_{FFA} &= -\frac{1}{1296}\widetilde{F}_{(1)ijk} \wedge \widetilde{F}_{(1)lmn} A_{(0)pqr}\epsilon^{ijklmnpqr}.
\end{aligned} \tag{2.52}
$$

Similar to the S^1-reduction case we may inspect the symmetries of the lower dimensional Lagrangians and the field equations starting from the eleven dimensional symmetries. It can be shown that if one considers the general coordinate transformations of the eleven dimensional supergravity theory whose Bosonic Lagrangian is given in (2.43) then we should consider only the transformations which leave the form of the Kaluza-Klein metric ansatz (2.48) invariant to explore the lower dimensional symmetries. The subset

$$
\xi^\mu_D(x^\nu, z^i) = \xi^\mu(x^\nu), \quad \xi^i_D(x^\nu, z^j) = \Lambda^i_j z^j + \lambda^i(x^\nu), \tag{2.53}
$$

of the eleven dimensional infinitesimal transformation parameters preserves the ansatz (2.48) when used in (2.34) with $D = 11$. In (2.53) x^ν are the coordinates on the D-dimensional spacetime while z^i are the coordinates on the n-torus

T^n and Λ^i_j for $i, j = 1, ..., n$ are real constant parameters. The infinitesimal parameters $\xi^\mu(x^\nu)$ generate the D-dimensional coordinate transformations whereas the infinitesimal parameters $\lambda^i(x^\nu)$ generate the $U(1)$-gauge invariance of the n Kaluza-Klein potential fields $\mathcal{A}^i_{(1)}$. These two kinds of transformations are the local symmetry transformations of the D-dimensional Lagrangian defined in (2.45). The Λ^i_j transformations choosing $\xi^\mu(x^\nu) = \lambda^i(x^\nu) = 0$ in (2.53) will generate a global symmetry group of $SL(n, \mathbb{R})$ for the D-dimensional Lagrangian. In order to find the full set of the lower dimensional global symmetry transformations, one should take a combination of the Λ^i_j transformations choosing $\xi^\mu(x^\nu) = \lambda^i(x^\nu) = 0$ in (2.53) and the scaling symmetry transformations (2.36) of the eleven dimensional Einstein field equations in a way that the combination of the transformations will leave the D-dimensional metric invariant. These transformations form the full set of the global internal symmetry transformations of the D-dimensional Lagrangian and they are nothing but the group $GL(n, \mathbb{R})$. Thus the global symmetry group of the D-dimensional Lagrangian becomes $GL(n, \mathbb{R}) \sim SL(n, \mathbb{R}) \times \mathbb{R}$ where the $\mathbb{R}$ factor is a result of mixing the eleven dimensional scaling symmetry. One can find another combination of the eleven dimensional transformations Λ^i_j and the eleven dimensional scaling symmetry to generate the D-dimensional scaling symmetry of the D-dimensional field equations. These local and the global symmetries of the fields coming from the reduction of the eleven dimensional metric can also be extended to the fields coming from the reduction of the three-form potential field of the eleven dimensional supergravity [20]. Thus we may expect to extend the D-dimensional local and the global symmetries mentioned above to the entire Bosonic sector of the D-dimensional theory. This is a partially legitimate expectation since although these symmetries exist we will see that there are more mysterious symmetries of the dimensionally reduced theories.

The scalar sectors of the dimensionally reduced maximal supergravities are important since one can show that if the scalar sector has a global symmetry then this symmetry can be extended to the entire Lagrangian [20]. Therefore to detect the global symmetries of the D-dimensional Lagrangian (2.45) it is sufficient to study merely the scalar sector. However there is an ambiguity in the Kaluza-Klein compactification of the $D = 11$ supergravity theory. As we discussed above the dimensionally reduced theories inherit an internal global symmetry of $GL(n, \mathbb{R})$ from the eleven dimensional general coordinate transformations and the scaling symmetry. However when one inspects the global internal symmetries of the D-dimensional theories one finds bigger symmetry groups than $GL(n, \mathbb{R})$ in each dimension. This denotes that there is a hidden symmetry scheme of the $D = 11$ supergravity containing the metric and the three-form potential field. We will study the scalar Lagrangians and their symmetries for each dimension for revealing the global internal symmetries of the maximal supergravities as well as for presenting the non-linear nature of the scalar sectors which will be the main reference point when we consider the dualisation to extend these non-linear sigma model constructions to the entire Bosonic sector.

If one reduces the $D = 11$ supergravity on S^1 one obtains the IIA supergravity [2, 3, 4] which is the low energy effective limit of the type IIA string theory and whose Bosonic Lagrangian can be written from (2.45) as

$$\mathcal{L}_{IIA} = R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{-\frac{1}{2}\phi} * F_{(4)} \wedge F_{(4)}$$

$$- \frac{1}{2} e^{\phi} * F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} + \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)}. \quad (2.54)$$

The global internal symmetry group of the IIA Lagrangian is $O(1,1)$.

Before giving the general scalar Lagrangian structures we will take a look at the so-called dilaton-axion system of the ten dimensional IIB supergravity theory [5, 6, 7] which is the low energy effective limit of the type IIB string theory. The IIB supergravity theory is not a Kaluza-Klein descendant of the $D = 11$ supergravity. The covariant field equations of the IIB supergravity theory can not be derived from a simple covariant Lagrangian due to the presence of a self-dual 5-form field strength, the complete Bosonic field equations can be found in [19]. However one can construct a Lagrangian for the IIB supergravity by increasing the degrees of freedom and by removing the self duality condition on the five-form field, then one can construct a doubled Lagrangian which contains the extra degrees of freedom. Solving the field equations and truncating the extra degrees of freedom by imposing the self-duality as a consistent constraint will lead to the original Bosonic equations of motion of the IIB supergravity [10]. The scalar sector of the IIB supergravity contains two scalar fields, the dilaton ϕ and the axion χ and their field equations can be obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} * d\chi \wedge d\chi. \quad (2.55)$$

This scalar Lagrangian is called the dilaton-axion system which has a more general notion than being just the scalar sector of the IIB supergravity, we will also encounter with it in other theories, thus we will consider it as a separate system in D-dimensions to reveal its non-linear sigma model construction. Now if we define the complex scalar field $\tau = \chi + ie^{-\phi}$, then the scalar Lagrangian functional in (2.55) becomes

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2$$

$$= -\frac{\partial\tau \cdot \partial\bar{\tau}}{2(Im(\tau))^2}. \tag{2.56}$$

One can show that [20] the Lagrangian (2.56) is invariant under the transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \tag{2.57}$$

where a, b, c, d are constants which satisfy

$$ad - bc = 1. \tag{2.58}$$

If we define the matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.59}$$

then the condition (2.58) becomes $det\Lambda = 1$ so as a result the global symmetry group of the dilaton-axion system (2.55) becomes $SL(2, \mathbb{R})$. If we consider the action (2.57) directly on the dilaton ϕ and the axion χ then we have

$$e^{\phi} \longrightarrow e^{\phi'} = (c\chi + d)^2 e^{\phi} + c^2 e^{-\phi},$$

$$\tag{2.60}$$

$$\chi e^{\phi} \longrightarrow \chi' e^{\phi'} = (a\chi + b)(c\chi + d) e^{\phi} + ace^{-\phi},$$

we see that $SL(2,\mathbb{R})$ acts non-linearly on the scalars. The Lie algebra of $SL(2,\mathbb{R})$ namely $sl(2,\mathbb{R})$ is isomorphic to $su(2)$ and it has three generators $\{H, E_+, E_-\}$ so its Cartan subalgebras are one dimensional (we will give a detailed covering of the Lie algebras in the next chapter) and there is one positive root and one negative root. Therefore the Borel subalgebra is generated by two generators $\{H, E_+\}$ and they satisfy the commutation relation $[H, E_+] = 2E_+$. For $sl(2,\mathbb{R})$ one can choose the following representation

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.61}$$

We will consider the map $\nu(x)$

$$\nu = e^{\frac{1}{2}\phi H} e^{\chi E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}, \tag{2.62}$$

which is from the D-dimensional spacetime into the group $SL(2,\mathbb{R})$ and we will define the matrix

$$\mathcal{M} = \nu^T \nu = \begin{pmatrix} e^{\phi} & \chi e^{\phi} \\ \chi e^{\phi} & e^{-\phi} + \chi^2 e^{\phi} \end{pmatrix}. \tag{2.63}$$

Now we can express the Lagrangian (2.55) in terms of $\mathcal{M}$ as

$$\mathcal{L} = \frac{1}{4} tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}). \tag{2.64}$$

If we consider the map $\nu(x)$, then the action of $SL(2,\mathbb{R})$ on the scalar fields which is given in (2.60) can be realized on $\nu(x)$ as

$$\nu' = \Delta(x)\nu\Lambda, \tag{2.65}$$

where $\Lambda \in SL(2,\mathbb{R})$ as defined in (2.59) and $\Delta(x) \in O(2)$ which, for a specific choice of the matrix Λ in (2.59) explicitly can be given as

$$\Delta = (c^2 + e^{2\phi}(c\chi + a)^2)^{-1/2} \begin{pmatrix} e^{\phi}(c\chi + a) & c \\ -c & e^{\phi}(c\chi + a) \end{pmatrix}. \tag{2.66}$$

Notice that the group $O(2)$ is a subgroup of $SL(2,\mathbb{R})$. We see that $\Delta(x)$ depends on the D-dimensional spacetime coordinates and it is a local compensation factor to regularize the global (non-coordinate dependent) $SL(2,\mathbb{R})$ action from the right in (2.65). Basically this compensation is necessary so that the transformed map ν' can be still written in the form of the so-called Borel gauge (2.62) in terms of the transformed scalar fields in (2.60). The possibility of finding such a compensating factor so that one can define the map $\nu(x)$ and by defining the action of $SL(2,\mathbb{R})$ on it one can construct a Lagrangian in terms of it, is a direct consequence of the Iwasawa decomposition [10, 20] which we will study in detail in the next chapter where we also reveal the geometrical and the algebraical details of the Borel gauge. Furthermore we can show that under the transformation (2.65), $\mathcal{M}$ transforms as

$$\mathcal{M} \longrightarrow \mathcal{M}' = \Lambda^T \mathcal{M} \Lambda. \tag{2.67}$$

This way of constructing the scalar Lagrangian teaches us the non-linear sigma model (precisely speaking, the symmetric space sigma model) nature of the scalars

[14]. This construction also denotes that [20] the scalar fields parameterize a manifold (each scalar field can be thought of the parametrization of a coordinate in $\mathbb{R}^m$ for some m which implies that the scalar fields obey some constraint equations which are the defining equations of the scalar manifold in $\mathbb{R}^m$, [21]) which is the coset space $SL(2,\mathbb{R})/O(2)$ and which is a symmetric space. As it will be clear in the following chapter, this manifold can also be considered as $SL(2,\mathbb{R})/SO(2)$. The scalar sectors of all the maximal supergravities and the supergravities which are the Kaluza-Klein descendants of the ten dimensional simple supergravity coupled to N Abelian gauge multiplets [21] share the same symmetric space sigma model structure with various global G and the local K transformation groups and the G/K scalar coset manifolds. In all of these scalar cosets K is the maximal compact subgroup of G. As we will study in detail in the next chapter, although the maximal supergravity global symmetry groups G fall into a special class of Lie groups namely the split real forms, the descendants of the ten dimensional simple supergravity which is coupled to N Abelian gauge multiplets have a more general class of global symmetry groups G which are in general the non-compact real forms. Thus we can make use of the Borel gauge for the first set while the more general solvable Lie algebra gauge must be used for the second.

The dilaton-axion system which is the scalar sector of the IIB supergravity as we have discussed above appears in many supergravities in a way that after certain field redefinitions two scalars can be decoupled from the rest of the scalar Lagrangian and their contribution to the scalar Lagrangian can be formulated as

(2.55). One also encounters the $SL(2,\mathbb{R})/O(2)$ scalar coset manifold in general when one reduces the $(D+2)$-dimensional pure gravity on T^2. It can be shown that when the $(D+2)$-dimensional Einstein-Hilbert Lagrangian is written in the form

$$\mathcal{L}_{(D+2)} = \mathcal{L}_D \wedge dz^1 \wedge dz^2, \tag{2.68}$$

by using the Kaluza-Klein ansatz (2.2) after two S^1-reduction steps then the D-dimensional Lagrangian becomes

$$\begin{aligned} \mathcal{L}_D = R * 1 - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{(\phi + q\varphi)} * \mathcal{F}^1_{(2)} \wedge \mathcal{F}^1_{(2)} \\ - \frac{1}{2} e^{(-\phi + q\varphi)} * \mathcal{F}^2_{(2)} \wedge \mathcal{F}^2_{(2)} - \frac{1}{2} e^{2\phi} * d\chi \wedge d\chi, \end{aligned} \tag{2.69}$$

where $q = \sqrt{D/(D-2)}$ after defining new dilatons ϕ, φ from the original Kaluza-Klein ones ϕ_1, ϕ_2 by performing an orthogonal transformation on them [20]. The field strengths above are defined in terms of the potentials as

$$\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)} - d\chi \wedge \mathcal{A}^2_{(1)}, \quad \mathcal{F}^2_{(2)} = d\mathcal{A}^2_{(1)}. \tag{2.70}$$

It is apparent that the scalar sector of (2.69) contains one scalar φ decoupled from the other two scalar fields ϕ and χ which form a dilaton-axion system. The decoupled scalar φ has a global internal constant shift symmetry $\varphi \longrightarrow \varphi + c$ which can be extended to the other fields and the dilaton-axion system has the $SL(2,\mathbb{R})$ global internal symmetry whose action can also be defined on the other fields to leave the entire Bosonic Lagrangian invariant. Thus the scalar sector of (2.69) has the group $SL(2,\mathbb{R}) \times \mathbb{R}$ as its global internal symmetry group. It is also the global internal symmetry of the entire Bosonic Lagrangian, there is also the scaling symmetry of the D-dimensional equations of motion. We have mentioned that the global internal symmetry group of the scalar sector can be extended to the other fields (leaving the D-dimensional metric inert). For the $SL(2,\mathbb{R})$ case if we introduce a new potential $\mathcal{A}'^{1}_{(1)}$ by the definition $\mathcal{A}^{1}_{(1)} = \mathcal{A}'^{1}_{(1)} + \chi\mathcal{A}^{2}_{(1)}$ instead of $\mathcal{A}^{1}_{(1)}$ then from (2.70) we have

$$\mathcal{F}^{1}_{(2)} = d\mathcal{A}'^{1}_{(1)} + \chi d\mathcal{A}^{2}_{(1)}. \tag{2.71}$$

The $SL(2,\mathbb{R})$ action which is the global symmetry of the Lagrangian (2.69) acts on the potentials $\mathcal{A}'^{1}_{(1)}$ and $\mathcal{A}^{2}_{(1)}$ as

$$\begin{pmatrix} \mathcal{A}^{2}_{(1)} \\ \mathcal{A}'^{1}_{(1)} \end{pmatrix} \longrightarrow (\Lambda^{T})^{-1} \begin{pmatrix} \mathcal{A}^{2}_{(1)} \\ \mathcal{A}'^{1}_{(1)} \end{pmatrix}, \tag{2.72}$$

where $\Lambda \in SL(2,\mathbb{R})$. We observe that the group $SL(2,\mathbb{R})$ has a non-linear action (2.60) on the dilaton ϕ and the axion χ while it has a linear action (2.72) on the Kaluza-Klein potentials.

We have given the details of the dilaton-axion system. The scalar sectors of the maximal supergravities which are the Kaluza-Klein descendants of the $D = 11$ supergravity exhibit similar structures whose mainline have the common construction pattern we have given for the $SL(2,\mathbb{R})/O(2)$ scalar coset. We will present the resulting scalar cosets without going into the details of the field re-definitions [20] which are used to simplify the construction of the corresponding scalar Lagrangian in each dimension.

We will give a formal construction of the Lie algebra structures and the parametrization of the general scalar cosets in the next chapter. However we should mention about how the Lie algebra structure of the global symmetry group G can be identified from the dimensionally reduced Lagrangian (2.45) for the Kaluza-Klein reduction of the $D = 11$ supergravity for $n > 1$. First of all we should state that the dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ which are defined in (2.46) and which are associated with the kinetic terms of the field strengths of the axions $\mathcal{A}^i_{(0)j}$ and $A_{(0)ijk}$ in (2.45) can be identified as the representatives of the positive roots of the Lie algebra of the global symmetry group G under some representation for each dimension [10]. Their negatives are the representatives of the negative roots so that we have the complete root system of G from the Lagrangian (2.45). The number of the dilatons $\vec{\phi}$ is equal to the dimension of the Cartan subalgebra of the Lie algebra of G which is equal to n. If one considers the Cartan subalgebra generators $\{H_i\}$ which in vector notation we will also denote by $\vec{H}$ and which are associated with the dilatons $\vec{\phi}$ then the representation of the roots $\{\alpha_i\}$ as we have mentioned above namely $\alpha_i \equiv (\alpha_i(H_1), \alpha_i(H_2), ..., \alpha_i(H_n))$ fixes the choice of the Cartan subalgebra of the Lie algebra of G and the basis $\vec{H}$. We also have the positive and the negative root generators which are associated with the positive and the negative roots. In the next chapter we will see that the Cartan generators, the positive and the negative root generators form a basis to span the Lie algebra of G. Since the Borel gauge is used to parameterize the scalar coset G/K in each dimension we are only interested in the positive root generators which we denote by E^i_j and E^{ijk} where the indices are $1 \leq i < j < k \leq n$. The representative root vectors are more precisely defined from the commutation relations,

$$[H_i, E_{\alpha_j}] = \alpha_j(H_i)E_{\alpha_j}, \tag{2.73}$$

where E_{α_j} is the positive or the negative root generator associated with the positive or the negative root α_j. If the first non-zero component of the representative of the root is a positive number then the root is called positive and if it is negative then it is called negative. The simple roots which form a subset of the positive roots and which constitute a basis for the dual space of the Cartan subalgebra of the Lie algebra of G can be given as

$$\vec{b}_{i,i+1} \quad \text{for} \quad 1 \leq i \leq n-1 \quad \text{and} \quad \vec{a}_{123}. \tag{2.74}$$

It can be shown that all the other roots can be written as sums of these roots with integer coefficients. Since all the roots have the same length within the representation, the global symmetry groups we encounter in the Kaluza-Klein reduction of the $D = 11$ supergravity are simply-laced. We may give the Dynkin diagram representation of these groups by using the simple roots as

$$\begin{array}{ccccccc}
\vec{b}_{12} & & \vec{b}_{23} & & \vec{b}_{34} & & \vec{b}_{(n-1)n} \\
\circ & - & \circ & - & \circ & \cdots & \circ \\
 & & & & | & & \\
 & & & & \circ\ \vec{a}_{123} & &
\end{array} \tag{2.75}$$

We observe that for $n = 2$, the only simple root we have is $\vec{b}_{12}$ and we have two Cartan generators, one of them generates the constant shift symmetry $\mathbb{R}$ of the decoupled dilaton. Thus for the case $n = 2$, the Dynkin diagram (2.75) generates the Lie algebra of $GL(2,\mathbb{R}) \sim SL(2,\mathbb{R}) \times \mathbb{R}$ which becomes the global internal symmetry group of the nine dimensional reduced theory. We have seen before that in the Kaluza-Klein reduction on T^2, one dilaton is decoupled from the other two scalars which form up a dilaton-axion system. The $\mathbb{R}$ factor comes from the constant shift symmetry of the decoupled dilaton and $SL(2,\mathbb{R})$ is the global symmetry of the dilaton-axion system. Thus the scalar coset manifold of the $D = 9$ maximal supergravity becomes

$$\frac{GL(2,\mathbb{R})}{O(2)}. \tag{2.76}$$

For $n = 3$ the simple roots are $(\vec{b}_{12}, \vec{b}_{23}, \vec{a}_{123})$ and the global internal symmetry group of the scalars and the $D = 8$ Bosonic Lagrangian can be identified as $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ from the Dynkin diagram (2.75). One can start from the dimensionally reduced $D = 8$ Lagrangian (2.45) for $n = 3$ then after suitable scalar redefinitions, two scalar fields $A_{(0)123}, \phi_1$ which form a dilaton-axion system can be decoupled from the rest of the scalar Lagrangian and the remaining scalar sector can be constructed in the form of the Lagrangian (2.64) where in this case $\nu(x)$ is a parametrization of the scalar coset manifold $SL(3,\mathbb{R})/O(3)$. On the other hand there will also be a contribution to the total scalar coset and the global internal symmetry group from the dilaton-axion system. The particular representation for the Cartan and the positive root generators of $SL(3,\mathbb{R})$ and the explicit construction of the $SL(3,\mathbb{R})$-invariant part of the total scalar Lagrangian can be referred in [20]. Therefore for $n = 3$ the scalar coset manifold becomes

$$\frac{SL(3,\mathbb{R})}{O(3)} \times \frac{SL(2,\mathbb{R})}{O(2)}. \tag{2.77}$$

We might also choose the identity components of $O(2)$ and $O(3)$ instead. Since for the $n = 3$ case we make a transformation on the dilatons while leaving the axions unchanged to decouple two scalar fields from the rest, the coupling coefficients of the transformed dilatons in the eight dimensional scalar Lagrangian (2.45) are changed. Thus the eight dimensional dilaton vectors are modified as

$$\vec{b}_{12} = (0, 1, \sqrt{3}), \quad \vec{b}_{23} = (0, 1, -\sqrt{3}), \quad \vec{b}_{13} = (0, 2, 0), \quad \vec{a}_{123} = (2, 0, 0). \quad (2.78)$$

If we neglect the dilaton ϕ_1 which is coupled to $A_{(0)123}$ to form a dilaton-axion system then we are left with two dilatons (ϕ_2, ϕ_3) and from (2.78) we have their restricted dilaton vectors

$$\vec{b'}_{12} = (1, \sqrt{3}), \quad \vec{b'}_{23} = (1, -\sqrt{3}), \quad \vec{b'}_{13} = (2, 0). \quad (2.79)$$

The part of the eight dimensional scalar Lagrangian which governs the dilatons $\vec{\phi'} = (\phi_2, \phi_3)$ and the axions $\mathcal{A}^i_{(0)j}$ can now be given as

$$\mathcal{L} = -\frac{1}{2} * d\vec{\phi'} \wedge d\vec{\phi'} - \frac{1}{2} \sum_{i<j} e^{\vec{b'}_{ij} \cdot \vec{\phi'}} * \mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j}. \quad (2.80)$$

We observe that this Lagrangian is invariant under the action of the global symmetry group $SL(3, \mathbb{R})$. For $n = 4$ the simple roots are $(\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}, \vec{a}_{123})$ and from the Dynkin diagram the global internal symmetry group of the $D = 7$ maximal supergravity becomes $SL(5, \mathbb{R})$. The scalar coset manifold is

$$\frac{SL(5, \mathbb{R})}{SO(5)}, \quad (2.81)$$

where the denominator group can again be chosen as either $O(5)$ or $SO(5)$. The simple roots of the six dimensional maximal supergravity for $n = 5$ are

$(\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}, \vec{b}_{45}, \vec{a}_{123})$ and the Dynkin diagram for $n = 5$ corresponds to the Lie algebra of the global internal symmetry group $O(5, 5)$. The scalar coset manifold in this case becomes

$$\frac{O(5,5)}{O(5) \times O(5)}. \tag{2.82}$$

For $1 < n \leq 5$, the commutation relations of the n Cartan generators $\vec{H}$ and the positive root generators E^i_j for $1 \leq i < j \leq n$ and E^{ijk} for $1 \leq i < j < k \leq n$ are

$$[E^{ijk}, E^{lmn}] = 0 \quad , \quad [\vec{H}, E^j_i] = \vec{b}_{ij} E^j_i \quad , \quad [\vec{H}, E^{ijk}] = \vec{a}_{ijk} E^{ijk}, \quad \text{no sum},$$

$$[E^j_i, E^l_k] = \delta^j_k E^l_i - \delta^l_i E^j_k \quad , \quad [E^m_l, E^{ijk}] = -3\delta^{[i}_l E^{|m|jk]}. \tag{2.83}$$

In (2.83) if we omit the generators E^{ijk} then the commutation relations of $\vec{H}$ and E^j_i generate the $gl(n, \mathbb{R})$ subalgebra of the corresponding Lie algebra of the global internal symmetry group G in each dimension. This can also be seen from the Dynkin diagram (2.75), if one omits the simple root $\vec{a}_{123}$ which is associated with the positive root generator E^{123} then the remaining portion of the Dynkin diagram corresponds to $gl(n, \mathbb{R})$ in each dimension. This is an expected result since the generators $\vec{H}$ and E^j_i are associated with the fields $\vec{\phi}$ and $\mathcal{A}^i_{(0)j}$ which originate from the dimensional reduction of the pure gravity, thus as we have mentioned before there will be an $SL(n, \mathbb{R})$ global internal symmetry of the lower dimensional theory which is a descendant of the eleven dimensional coordinate covariance of the $D = 11$ supergravity and an additional $\mathbb{R}$ global symmetry which is a contribution from mixing the scaling symmetries of the eleven dimensional Einstein equations to the eleven dimensional coordinate transformations, thus our expectation of a full global symmetry of $GL(n,\mathbb{R}) \sim SL(n,\mathbb{R}) \times \mathbb{R}$ of the fields generated by the $D = 11$ pure gravity is justified. The extra $\mathbb{R}$ factor in $GL(n,\mathbb{R})$ can not be directly seen from the Dynkin diagram since although the simple root systems of $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$ are the same they have n and $(n-1)$ Cartan generators respectively thus as we have n Cartan generators for each dimension $D = 11 - n$, we have the sub- $GL(n,\mathbb{R})$ global internal symmetry group embedded in the commutation relations (2.83) and the Dynkin diagram (2.75) where we observe that there is a greater global internal symmetry group as a result of a mixture symmetry of the eleven dimensional metric and the extra degree of freedom, $A_{(3)}$ of the $D = 11$ supergravity. For the dimensions $D = 9, 8, 7, 6$, the parametrization of the corresponding total scalar coset can be given by the coset representative

$$\nu = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}\Big(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}E^j_i}\Big)exp\Big(\sum_{i<j<k} A_{(0)ijk}E^{ijk}\Big), \tag{2.84}$$

where in the product the ordering of the factors are in anti-lexigraphical order $(\cdots(24)(23)\cdots(14)(13)(12))$ [10, 20]. If one defines the map $\mathcal{M} = \nu^T\nu$ whose image values are also group elements in a fundamental representation of the Lie algebra of the global internal symmetry group G in the corresponding dimension then the scalar Lagrangian can be expressed as (2.64). One can also show that, when one transforms the scalar fields under the action of the global internal symmetry group in each dimension the action on the coset representative (2.84) must be in the form

$$\nu' = \Delta(x)\nu\Lambda, \tag{2.85}$$

where $\Lambda \in G$ and the coordinate-dependent $\Delta(x) \in K$, the denominator group of the scalar coset manifold which is also the maximal compact subgroup of G. The transformation (2.67) is also valid for all the dimensions.

For the reductions when $n > 5$ we have an additional point to mention. We have already assumed that the simple roots are given as in (2.74) for all of the reductions on T^n tori. When we reduce the $D = 11$ supergravity on n torus we get $\frac{1}{2}n(n-1)$ axions, $\mathcal{A}^i_{(0)j}$ and also $\frac{1}{6}n(n-1)(n-2)$ axions, $A_{(0)ijk}$. Thus for $n = (2, 3, 4, 5, 6, 7, 8)$ we have $(1, 4, 10, 20, 35, 56, 84)$ axions in total, however when one counts the positive roots of the global internal symmetry groups whose Dynkin diagrams are given in (2.75) they yield $(1, 4, 10, 20, 36, 63, 120)$ for the n values $(2, 3, 4, 5, 6, 7, 8)$ respectively. We see that there is a mismatch starting from $n = 6$. At fist glance we observe that the number of the axionic scalars is not enough to generate the positive roots of the global internal symmetry groups we propose in (2.75) (we assign the positive roots to the above mentioned axionic scalars in one to one correspondence). The ambiguity is resolved when we realize that for the reductions $n > 5$ one may handle field redefinitions and use the higher order fields to define new axionic scalar fields [10, 20] by using Lagrange multiplier methods. This general method of replacing the original field with a Lagrange multiplier field is called the dualisation method.

For the case $n = 6$ when the reduced theory is the $D = 5$ maximal supergravity, we have one missing scalar field from the counting above. In this dimension if we consider the Hodge-dual of the field strength of the three-form potential $A_{(3)}$ we get a one-form field and one can reformulate the total Lagrangian in terms of this one-form field strength instead of the field strength of $A_{(3)}$ so that its potential can be interpreted as an additionally defined scalar field which is the Lagrange multiplier indeed. For the case when $n = 7$ in $D = 4$, there are seven two-form potentials $A_{(2)i}$, when one rebuilds the entire Lagrangian in terms of the Hodge-duals of the field strengths of these two-form potentials by introducing Lagrange multipliers, one replaces the $A_{(2)i}$ terms with the ones consisting of one-form field strengths which again can be interpreted as the field strengths of the new scalar fields which are the Lagrange multipliers themselves. Finally for the case when $n = 8$ in $D = 3$ we have 28 $A_{(1)ij}$ potential fields as well as 8 $\mathcal{A}^i_{(1)}$ potentials. Similarly by making field redefinitions and by using Lagrange multipliers when we construct an equivalent $D = 3$ Bosonic Lagrangian in terms of the Hodge-duals of the field strengths of these potentials we get terms coming from 36 one-form fields which can be interpreted as the kinetic terms of the field strengths of newly defined scalars which are also Lagrange multipliers.

As a starting point to show how one dualize a higher dimensional potential field to formulate a Bosonic Lagrangian in terms of the field strength of a newly defined scalar which is a Lagrange multiplier, we will consider the situation in $D = 5$, [20] which is the simplest case. The $n = 7$ and $n = 8$ cases have similar structures with more degrees of freedom. In all of these formulations one basically considers the Bianchi identities of the field strengths of the corresponding potentials defined in (2.50) as constraint equations and introduces new scalar fields as Lagrange multipliers then one constructs an extended dualized Lagrangian by adding the Bianchi terms which couple the Lagrange multipliers to the Bianchi identities. If we return to the $D = 5$ case, from (2.50) we have

$$F_{(4)} = \widetilde{F}_{(4)} - \gamma^i{}_j \widetilde{F}_{(3)i} \wedge \mathcal{A}^j_{(1)} + \frac{1}{2}\gamma^i{}_k \gamma^j{}_l \widetilde{F}_{(2)ij} \wedge \mathcal{A}^k_{(1)} \wedge \mathcal{A}^l_{(1)}$$

$$- \frac{1}{6}\gamma^i{}_l \gamma^j{}_m \gamma^k{}_n \widetilde{F}_{(1)ijk} \wedge \mathcal{A}^l_{(1)} \wedge \mathcal{A}^m_{(1)} \wedge \mathcal{A}^n_{(1)}. \tag{2.86}$$

We will treat $F_{(4)}$ as a fundamental field and use its Bianchi identity to extend the original $D = 5$ Lagrangian (2.45) by introducing a scalar field as a Lagrange multiplier. The Bianchi identity for the field strength $F_{(4)}$ is obtained by taking the exterior derivative of (2.86) and it becomes $dF_{(4)} = 0 + f$ where f comes from taking the exterior derivative of the rest of the terms in (2.86) except the first term which gives zero [10]. The Lagrange multiplier χ will be a zero degree scalar field since the degree of the field strength of the Lagrange multiplier and the degree of the field strength $F_{(4)}$ must sum up to $D = 5$. Thus we will add a term of $-\chi(dF_{(4)} - f)$ to the $D = 5$ Lagrangian (2.45) and if we consider the extended Lagrangian

$$\mathcal{L}'_{(5)} = \mathcal{L}_{(5)} - \chi(dF_{(4)} - f), \tag{2.87}$$

in addition with the Bianchi identity $dF_{(4)} - f = 0$ which becomes the constraint equation, we get an equivalent system since although we have increased the degrees of freedom by introducing a new scalar field, its equation of motion is nothing but the original Bianchi identity which acts as a compensating and consistent constraint equation. We will not in detail give the explicit calculations but we will only present the resulting Lagrangian containing the field strength $F_{(4)}$ so that we may formulate it in terms of the field strength of χ.

We may show that [10, 20] the terms in (2.87) which contain the field strength $F_{(4)}$ can be given as

$$\mathcal{L}'_{(5)}(F_{(4)}) = -\frac{1}{2}e^{\vec{a}\cdot\vec{\phi}} * F_{(4)} \wedge F_{(4)} - \frac{1}{72}A_{(0)ijk}dA_{(0)lmn} \wedge F_{(4)}\epsilon^{ijklmn} - \chi dF_{(4)}, \quad (2.88)$$

where the origin of the second term is the $D = 5$ $\mathcal{L}_{FFA}$ in (2.52). If we vary (2.87) with respect to χ we regain the Bianchi identity of $F_{(4)}$. On the other hand if we vary (2.87) or (2.88) with respect to $F_{(4)}$ then we obtain the purely algebraic first-order equation of motion for $F_{(4)}$ as

$$e^{\vec{a}\cdot\vec{\phi}} * F_{(4)} = d\chi - \frac{1}{72}A_{(0)ijk}dA_{(0)lmn}\epsilon^{ijklmn}. \quad (2.89)$$

If we define the right hand side of this equation as the field strength

$$G_{(1)} = d\chi - \frac{1}{72}A_{(0)ijk}dA_{(0)lmn}\epsilon^{ijklmn}, \quad (2.90)$$

of χ which is our Lagrange multiplier scalar field then in the Lagrangian (2.87) and (2.88) we can eliminate the field strength $F_{(4)}$ in terms of the field strength $G_{(1)}$ resulting

$$\mathcal{L}'_{(5)}(F_4 \to G_{(1)}) = -\frac{1}{2}e^{-\vec{a}\cdot\vec{\phi}} * G_{(1)} \wedge G_{(1)}, \quad (2.91)$$

which corresponds to the kinetic term of χ. We may insert (2.91) back into the dualized Lagrangian (2.87) so that we have a proper kinetic term of the additional scalar χ replacing the terms containing $F_{(4)}$. Thus we have constructed an equivalent, dualized theory in which the role of the potential $A_{(3)}$ is exchanged with the Lagrange multiplier scalar potential χ while the rest of the Lagrangian (2.87) is unchanged. To summarize; we have introduced the Bianchi identity

$dF_{(4)} - f = 0$ and constructed the dualized Lagrangian (2.87) by considering $F_{(4)}$ as an independent field and then we have eliminated the terms containing $F_{(4)}$ by using the field strength of the Lagrange multiplier scalar field χ. The main reason of why we have applied the dualisation method by defining a new Lagrange multiplier scalar field and formulated an equivalent Lagrangian in terms of this field is that we have obtained an enlarged scalar sector which we can formulate as a symmetric space sigma model likewise we have done for the cases when $n < 6$. The scalar coset manifold we obtain after this dualisation process is called the maximal scalar coset since we may not introduce more scalar Lagrange multipliers by dualizing other fields to enlarge it. The definition (2.90) is the transgression relation of the scalar field χ. We will take the dilaton vector $-\vec{a}$ which couples to the dilatons in (2.91) as the representative of the additional positive root to formulate the scalar sector as a symmetric space sigma model. It can be shown that $-\vec{a}$ can also be expressed in terms of the simple roots (2.74) with integer coefficients. The next task is to introduce the extra positive root generator J associated with χ and $-\vec{a}$ in order to construct the maximal scalar coset representative and the enlarged scalar Lagrangian as the symmetric space sigma model Lagrangian. We have the extra commutation relations

$$[\vec{H}, J] = -\vec{a}J \quad , \quad [E_i^j, J] = [E^{ijk}, J] = 0 \quad , \quad [E^{ijk}, E^{lmn}] = -\epsilon^{ijklmn}J, \quad (2.92)$$

which are additional to the ones,

$$[\vec{H}, E_i^j] = \vec{b}_{ij}E_i^j \quad , \quad [\vec{H}, E^{ijk}] = \vec{a}_{ijk}E^{ijk}, \quad \text{no sum,}$$

$$[E_i^j, E_k^l] = \delta_k^j E_i^l - \delta_i^l E_k^j \quad , \quad [E_l^m, E^{ijk}] = -3\delta_l^{[i} E^{|m|jk]}, \tag{2.93}$$

where the generators E_i^j, E^{ijk} are associated with the other positive roots $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ respectively and $\vec{H}$ are the 6 Cartan generators. The above commutation relations correspond to the Borel subalgebra of the the global internal symmetry group $E_{6(+6)}$ which is the maximally non-compact real form of E_6. This result is consistent with (2.75) since the Dynkin diagram of $E_{6(+6)}$ corresponds to the one in (2.75) when $n = 6$. The $D = 5$ enlarged scalar coset manifold becomes

$$\frac{E_{6(+6)}}{USp(8)}, \tag{2.94}$$

where $USp(8)$ is the intersection of $SU(8)$ and $Sp(8)$ and it is the maximal compact subgroup of $E_{6(+6)}$. The coset representative for the $D = 5$ maximal scalar coset manifold above can be constructed by using the Borel gauge as

$$\nu = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}(\prod_{i<j} e^{\mathcal{A}^i_{(0)j} E_i^j})exp(\sum_{i<j<k} A_{(0)ijk} E^{ijk})e^{\chi J}, \tag{2.95}$$

where we have added an extra factor of the exponentiation of the new axionic scalar χ and the new positive root generator J to (2.84) which is the general coset representative for $n < 6$.

The scalar Lagrangian of the $D = 5$ maximal supergravity, after the dualisation can be constructed from the above coset representative of the scalar manifold. However we have to mention one generalization which is valid for all of the three cases $n = 6, 7, 8$. We have previously defined the map $\mathcal{M} = \nu^T \nu$ from the D-dimensional spacetime into the global internal symmetry group G by using a fundamental representation of the Lie algebra of G to construct the general scalar Lagrangian (2.64). The transpose of ν can be used in $\mathcal{M} = \nu^T \nu$ because as it will be clear in the following chapters, a more general notion, namely the generalized transpose map $\#$ on G coincides with the ordinary matrix transpose under the fundamental representations of the Lie algebras of the global symmetry groups for $n < 6$. It will be possible to give a complete definition of this Lie group operation after we discuss the symmetric spaces in the next chapter. The groups we encounter in the Kaluza-Klein reduction of the $D = 11$ supergravity are maximally non-compact (split) real forms of semi-simple Lie groups. For such special class of Lie groups we may define the generalized transpose $\#$ on a representation of the global internal symmetry group, as a map which has the usual properties of the matrix transpose (i.e. $(AB)^\# = B^\# A^\#$, $(e^A)^\# = e^{A^\#}$) and which has the action

$$\# : (E_\alpha, E_{-\alpha}, H_i) \longrightarrow (E_{-\alpha}, E_{+\alpha}, H_i), \tag{2.96}$$

on the Cartan $\{H_i\}$ and the positive and the negative root generators $\{E_\alpha, E_{-\alpha}\}$, [10, 20, 43]. Therefore in general the scalar Lagrangian is based on the map $\mathcal{M} = \nu^\# \nu$ and similar to the one for $n < 6$, it can be constructed as

$$\mathcal{L} = \frac{1}{4} tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}). \tag{2.97}$$

For $n = 7$, when we reduce the $D = 11$ supergravity on T^7 we get the maximal supergravity in $D = 4$. We may similarly dualize the seven $A_{(2)i}$ potentials to write the dualized extended Lagrangian in terms of seven Lagrange multiplier scalar fields χ^i. The field strengths of $A_{(2)i}$ are defined in (2.50) as

$$F_{(3)i} = \gamma^j_{\ i} \widetilde{F}_{(3)j} + \gamma^j_{\ i} \gamma^k_{\ l} \widetilde{F}_{(2)jk} \wedge \mathcal{A}^l_{(1)} + \frac{1}{2} \gamma^j_{\ i} \gamma^k_{\ m} \gamma^l_{\ n} \widetilde{F}_{(1)jkl} \wedge \mathcal{A}^m_{(1)} \wedge \mathcal{A}^n_{(1)}. \tag{2.98}$$

Similar to the $n = 6$ case, we will again treat the field strengths $F_{(3)i}$ as fundamental fields and consider their Bianchi identities to construct the dualized Lagrangian. If we define $\widetilde{\gamma}^j_i = \delta_i^j + \mathcal{A}^j_{(0)i}$, the inverse matrix of γ^j_i which is defined in (2.51), multiply (2.98) by $\widetilde{\gamma}^j_i$ and then take the exterior derivative we find the Bianchi identities as

$$d(\widetilde{\gamma}^i_k F_{(3)i}) = 0. \tag{2.99}$$

We will follow the standard procedure of the Lagrange multiplier method which we have already used in $D = 5$. We may introduce seven Lagrange multipliers χ^i and extend the $D = 4$, Bosonic Lagrangian (2.45) by adding the Bianchi terms which are the couplings of the Lagrange multipliers χ^i and the Bianchi identities (2.99). In the resulting dualised Lagrangian, the field strengths; $F_{(3)i}$ appear to be fundamental fields since we cast their Bianchi identities into the modified Lagrangian by introducing Lagrange multiplier scalar fields whose equations of motion will be the corresponding Bianchi identities. The part of the $D = 4$ Lagrangian in (2.45) which contains the potentials $A_{(2)i}$ is

$$\mathcal{L}_4(F_{(3)i}) = -\frac{1}{2}\sum_{i=1}^{7} e^{\vec{a}_i\cdot\vec{\phi}} * F_{(3)i} \wedge F_{(3)i} - \frac{1}{72} A_{(0)ijk} dA_{(0)lmn} \wedge dA_{(2)p} \epsilon^{ijklmnp}, \tag{2.100}$$

where we have modified the $\mathcal{L}_{FFA}$ term coming from (2.52) by adding a total derivative to the Lagrangian. After a couple of steps basically, by applying the integration by parts one may show that the part of the Lagrangian that is modified by the Bianchi terms, which contains the field strengths $F_{(3)i}$ is [10]

$$\mathcal{L}'_4(F_{(3)i}) = -\frac{1}{2}\sum_{i=1}^{7} e^{\vec{a}_i\cdot\vec{\phi}} * F_{(3)i} \wedge F_{(3)i}$$

$$-\frac{1}{72}A_{(0)ijk}dA_{(0)lmn}\wedge\widetilde{\gamma}^{q}_{p}F_{(3)q}\epsilon^{ijklmnp}-d\chi^{i}\wedge\widetilde{\gamma}^{j}_{i}F_{(3)j}. \qquad (2.101)$$

If we vary (2.101) with respect to $F_{(3)i}$ we find the algebraic equations of motion for $F_{(3)i}$

$$F_{(3)i}=e^{-\vec{a}_{i}\cdot\vec{\phi}}\ast G^{i}_{(1)}, \qquad (2.102)$$

where we have defined the field strengths $G^{i}_{(1)}$ of the Lagrange multiplier scalar fields χ^{i} as

$$G^{i}_{(1)}=\widetilde{\gamma}^{i}_{j}(d\chi^{j}+\frac{1}{72}A_{(0)klm}dA_{(0)npq}\epsilon^{jklmnpq}). \qquad (2.103)$$

Thus we may express the $F_{(3)i}$ terms in terms of the field strengths $G^{i}_{(1)}$ in (2.101) to formulate it as

$$\mathcal{L}'_{4}(F_{(3)i}\rightarrow G^{i}_{(1)})=-\frac{1}{2}\sum_{i=1}^{7}e^{-\vec{a}_{i}\cdot\vec{\phi}}\ast G^{i}_{(1)}\wedge G^{i}_{(1)}. \qquad (2.104)$$

This Lagrangian may be inserted back into the total $D=4$ Lagrangian which is modified by the Bianchi terms to yield an equivalent Lagrangian in which the role of $F_{(3)i}$ are exchanged with the field strengths, $G^{i}_{(1)}$ of the seven Lagrange multiplier scalars χ^{i}. Thus we have obtained an equivalent $F_{(3)i}$ free formulation where the contribution from the fields $F_{(3)i}$ is governed implicitly by the Lagrange multiplier scalar fields χ^{i}. We have an enlarged scalar field content and the scalar sector Lagrangian can now be constructed as the symmetric space sigma model Lagrangian (2.97). We take the dilaton vectors $-\vec{a}_{i}$ which couple to the dilatons in the kinetic terms of the field strengths $G^{i}_{(1)}$ in (2.104) as the extra positive root vectors we need to formulate the enlarged scalar sector as a symmetric space sigma model whose global internal symmetry group we have foreseen by defining the simple roots and by constructing the Dynkin diagram in (2.75). Next we introduce the extra positive root generators J_i associated with the positive roots $-\vec{a}_i$. It can be shown that $-\vec{a}_i$ can also be written in terms of the simple roots (2.74) with integer coefficients. In addition to the dimension-independent commutation relations (2.93) we have the extra commutation relations

$$[\vec{H}, J_i] = -\vec{a}_i J_i \quad , \quad [E_j^k, J_i] = \delta_i^k J_j \quad , \quad [E^{ijk}, J_m] = 0,$$

$$[E^{ijk}, E^{lmn}] = \epsilon^{ijklmnp} J_p. \tag{2.105}$$

We should remind the reader that we keep the generators E_i^j and E^{ijk} which are associated with the other positive roots $\vec{b}_{ij}$ and $\vec{a}_{ijk}$, respectively, and $\vec{H}$ are the 7 Cartan generators.

We observe that the commutation relations (2.93) and (2.105) define the Borel subalgebra of $E_{7(+7)}$ which is the maximally non-compact real form of E_7. Thus the global internal symmetry group is $E_{7(+7)}$ and the maximal scalar coset manifold in four dimensions becomes

$$\frac{E_{7(+7)}}{SU(8)}, \tag{2.106}$$

where the denominator group $SU(8)$ is the maximal compact subgroup of $E_{7(+7)}$. The coset representative of the enlarged scalar coset manifold in four dimensions can now be given as

$$\nu = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}(\prod_{i<j} e^{\mathcal{A}^i_{(0)j} E^j_i}) exp(\sum_{i<j<k} A_{(0)ijk} E^{ijk}) e^{\chi^i J_i}, \tag{2.107}$$

where we have used the Borel gauge. Once more we observe that in the four dimensional scalar coset representative there is an extra factor from the right to the original coset representative (2.84) for $n < 6$. Finally the scalar Lagrangian for the $D = 4$ enlarged set of scalars can be formulated as (2.97), based on the coset representative map (2.107) by using the generalized transpose # and by constructing $\mathcal{M} = \nu^{\#}\nu$ as we have mentioned before.

When we consider the last Kaluza-Klein reduction of the $D = 11$ supergravity which is over T^8, we have to dualize 8 two-forms $\mathcal{F}^i_{(2)}$ and 28 two-forms $F_{(2)ij}$ to obtain the necessary scalar fields in order to formulate the enlarged scalar Lagrangian as a symmetric space sigma model. The part of the $D = 3$ Lagrangian (2.45) which contains these field strengths is

$$\mathcal{L}_3(F_{(2)ij}, \mathcal{F}^i_{(2)}) = -\frac{1}{2}\sum_i e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}^i_{(2)} \wedge \mathcal{F}^i_{(2)} - \frac{1}{2}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} * F_{(2)ij} \wedge F_{(2)ij}$$

$$- \frac{1}{144}\widetilde{F}_{(1)ijk} \wedge \widetilde{F}_{(1)lmn} \wedge A_{(1)pq}\epsilon^{ijklmnpq}. \tag{2.108}$$

Following the same path we performed for the $n = 6, 7$ cases, first we have to construct the Bianchi identities. From (2.50) we have

$$F_{(2)ij} = \gamma^k_{\ i}\gamma^l_{\ j}\widetilde{F}_{(2)kl} - \gamma^k_{\ i}\gamma^l_{\ j}\gamma^m_{\ n}\widetilde{F}_{(1)klm} \wedge \mathcal{A}^n_{(1)},$$

$$\mathcal{F}^i_{(2)} = \widetilde{\mathcal{F}}^i_{(2)} - \gamma^j_{\ k}\widetilde{\mathcal{F}}^i_{(1)j} \wedge \mathcal{A}^k_{(1)}. \tag{2.109}$$

One can show that, [10] by multiplying the first field strengths $F_{(2)ij}$ with $\widetilde{\gamma}^i_p\widetilde{\gamma}^j_q$ and the second ones $\mathcal{F}^i_{(2)}$ with γ^j_i and then by taking the exterior derivative of the results, the Bianchi identities become

$$d(\widetilde{\gamma}^i_p \widetilde{\gamma}^j_q F_{(2)ij} - A_{(0)pqm} \gamma^m_n \mathcal{F}^n_{(2)}) = 0,$$

$$d(\gamma^j_i \mathcal{F}^i_{(2)}) = 0. \tag{2.110}$$

By introducing 8 scalars λ_i and another 28 scalar fields λ^{ij} as Lagrange multipliers and by coupling them to (2.110), we may construct the Bianchi terms which will be added to the $D = 3$ Lagrangian in (2.45). As usual if we vary the dualized Lagrangian with respect to λ_i and λ^{ij} we regain the Bianchi identities (2.110). When we vary the part of the dualized Lagrangian which contains $F_{(2)ij}$ and $\mathcal{F}^i_{(2)}$ (namely which is the Lagrangian (2.108) with the addition of the Bianchi terms coming from (2.110)) with respect to $F_{(2)ij}$ and $\mathcal{F}^i_{(2)}$ we get the purely algebraic equations of motion for $F_{(2)ij}$ and $\mathcal{F}^i_{(2)}$ in terms of the Lagrange multiplier scalars λ_i and λ^{ij}. As we have done before we may define the field strengths $G_{(1)i}$, $G^{ij}_{(1)}$ of the scalars λ_i and λ^{ij} from these algebraic equations of motion. Then we can express the dualized Lagrangian in terms of these field strengths. The part of the dualized Lagrangian which contains $F_{(2)ij}$ and $\mathcal{F}^i_{(2)}$ then becomes

$$\mathcal{L}'_3(F_{(2)ij} \to G^{ij}_{(1)}, \mathcal{F}^i_{(2)} \to G_{(1)i}) = -\frac{1}{2}\sum_i e^{-\vec{b}_i \cdot \vec{\phi}} \ast G_{(1)i} \wedge G_{(1)i}$$

$$-\frac{1}{2}\sum_{i<j} e^{-\vec{a}_{ij} \cdot \vec{\phi}} \ast G^{ij}_{(1)} \wedge G^{ij}_{(1)}, \tag{2.111}$$

where the field strengths $G_{(1)i}$ and $G^{ij}_{(1)}$ are defined as

$$G_{(1)i} = \gamma^j_i (d\lambda_j - \frac{1}{2} A_{(0)jkl} d\lambda^{kl} - \frac{1}{432} dA_{(0)klm} A_{(0)npq} A_{(0)rsj} \epsilon^{klmnpqrs}),$$

$$G^{ij}_{(1)} = \widetilde{\gamma}^i_k \widetilde{\gamma}^j_l (d\lambda^{kl} + \frac{1}{72} dA_{(0)mnp} A_{(0)qrs} \epsilon^{klmnpqrs}). \tag{2.112}$$

The dilaton vectors $-\vec{b}_i$ and $-\vec{a}_{ij}$ which are coupled to the dilatons in the kinetic terms of $G_{(1)i}$ and $G^{ij}_{(1)}$ in (2.111) will be taken as the additional positive root vectors beside the original ones, $\vec{b}_{ij}$ and $\vec{a}_{ijk}$. We may show that $-\vec{b}_i$ and $-\vec{a}_{ij}$ can also be expressed in terms of the simple roots in (2.74) with integer coefficients. The corresponding positive root generators will be denoted as J^i and J_{ij}. There are $8 + 28 = 36$ of them. Apart from the dimension-independent commutation relations in (2.93) we have the additional commutation relations

$$[\vec{H}, J_{ij}] = -\vec{a}_{ij} J_{ij}, \quad [\vec{H}, J^i] = -\vec{b}_i J^i, \quad [E^j_i, J_{kl}] = -2\delta^j_{[k} J_{l]i},$$

$$[E^j_i, J_k] = -\delta^k_i J^j, \quad [E^{ijk}, J_{lm}] = -6\delta^{[i}_{[l} \delta^j_{m]} J^{k]}, \quad [E^{ijk}, J_l] = 0,$$

$$[E^{ijk}, E^{lmn}] = -\frac{1}{2} \epsilon^{ijklmnpq} J_{pq}, \tag{2.113}$$

where again E^j_i, E^{ijk} are associated with the other positive roots $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ respectively and $\vec{H}$ are the 8 Cartan generators. The commutation relations (2.93) and (2.113) define the Borel subalgebra of $E_{8(+8)}$ which is the maximally non-compact real form of E_8. The global internal symmetry group becomes $E_{8(+8)}$ and after dualisation the $D = 3$ enlarged scalar coset manifold becomes

$$\frac{E_{8(+8)}}{SO(16)}, \tag{2.114}$$

where the denominator group $SO(16)$ is the maximal compact subgroup of $E_{8(+8)}$. The coset representative of the enlarged scalar coset manifold in three dimensions can be given as

$$\nu = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}(\prod_{i<j} e^{\mathcal{A}^i_{(0)j}E^j_i})exp(\sum_{i<j<k} A_{(0)ijk}E^{ijk})e^{\lambda_i J^i}e^{\frac{1}{2}\lambda^{ij}J_{ij}}, \tag{2.115}$$

where we have used the Borel gauge. There are two extra factors from the right to the original coset representative (2.84) for $n < 6$. The scalar Lagrangian for the $D = 3$ enlarged scalar sector can be given as (2.97), by using the generalized transpose # and constructing $\mathcal{M} = \nu^{\#}\nu$ where ν is defined as (2.115).

In all of the dualized cases for $n = 6, 7, 8$ with the maximal scalar coset manifolds G/K, one can show that the coset representatives again are transformed as; $\nu \longrightarrow \Delta(x)\nu\Lambda$ upon the direct global action of the global internal symmetry groups on the scalars in each dimension. Here $\Delta(x) \in K$ corresponds to the local action of the maximal compact denominator subgroup K which is a compensator to keep the coset representative in the Borel gauge when it is transformed globally (rigidly) by Λ from the right so that the transformed coset representative ν' can legitimately be generated by the transformed set of scalars. We can not omit $\Delta(x)$ since then the transformation of $\nu(x)$ by Λ from the right can not correspond to a transformation on the scalars which is the basic transformation. For each Λ the existence of $\Delta(x)$ is ensured as a result of the Iwasawa decomposition which we have already referred to and will discuss in detail in the following chapters. Similarly the symmetry scheme for $n < 6$, the global internal symmetry group G can be extended to the entire Bosonic sector by defining the action on the other fields for the cases $n = 6, 7, 8$. As a final remark we are denoting the global symmetry groups for $n \geq 6$ as $E_{n(+n)}$ [20] in order to emphasize that they are the maximally non-compact (split) real forms of E_n having n non-compact generators which are the Cartan generators. We will come back to this point and analyze the real forms and the compactness in detail in the next chapter.

2.3 The Abelian Yang-Mills Supergravities

In this section we will focus on the Kaluza-Klein reduction on the Euclidean torus T^{10-D} of the Bosonic sector of the ten dimensional simple $N = 1$, supergravity which is coupled to N Abelian gauge multiplets [21]. We have given the details of the S^1-reductions of the metric and the other fields in section 2.1, similar to the Kaluza-Klein reduction of the $D = 11$ supergravity we may reduce the ten dimensional simple supergravity which is coupled to N Abelian gauge multiplets step by step to the lower dimensions. We will not go into the details of the reduction steps but again our main concern will be the structure of the scalar cosets which appear in each dimension upon the dimensional reduction of the basic fields in the ten dimensional theory. The scalar cosets of the lower dimensional theories can again be formulated as the non-linear sigma models, more specifically as the symmetric space sigma models. The study of the scalar cosets is useful in two ways; first they exhibit a nature of non-linear sigma model which we will extend to the entire Bosonic sector later, second as we have done previously, one needs to apply dualisations in certain dimensions to construct the scalar cosets as symmetric space sigma models and such a partial dualisation will be the main perspective of the complete dualisations of the supergravity theories. As we have already discussed, the global symmetry of the scalar Lagrangian can be extended to the entire Bosonic sector of the theory. The scalar cosets G/K which we will introduce in this section are based on the global internal symmetry groups G which are in general non-compact real forms of semi-simple Lie groups. Under certain conditions the global internal symmetry groups may be maximally non-compact (split) real forms but in general they are elements of a bigger class of Lie groups which contains the global internal symmetry groups of the maximal supergravities namely the split real forms as a special subset. The main difference between the scalar cosets based on the non-compact and the maximally non-compact numerator groups is the parametrization one can choose for the coset representative. For the general non-compact real forms one can make use of the solvable Lie algebra gauge [18] to parameterize the scalar coset. We will give a complete analysis of the symmetric spaces and the solvable Lie algebra parameterizations in the next chapter. However as it will be an element of the language we will use in this section, we can simply state that the solvable Lie algebra is a subalgebra of the Borel algebra of a Lie algebra which is composed of certain Cartan and the positive root generators. The Kaluza-Klein compactification of the Bosonic sector of the ten dimensional simple $N = 1$ supergravity which is coupled to N Abelian gauge multiplets on the Euclidean torus T^{10-D} is given in [21]. When as a special case, the number of the $U(1)$ gauge fields is chosen to be 16, the ten dimensional supergravity which is coupled to 16 Abelian $U(1)$ gauge multiplets becomes the low energy effective limit of the ten dimensional heterotic string theory. Thus the formulation corresponds to the dimensional reduction of the low-energy effective Bosonic Lagrangian of the ten dimensional heterotic string theory. We will however keep the number of $U(1)$ gauge fields unspecified in order to obtain a general matter coupled supergravity formulation.

When $N = 16$, the $D = 10$ Yang-Mills supergravity [8] has the $E_8 \times E_8$ Yang-Mills gauge symmetry, however the general Higgs vacuum structure causes a spontaneous symmetry breakdown so that the full symmetry $E_8 \times E_8$ is broken down to its maximal torus subgroup $U(1)^{16}$, whose Lie algebra is the Cartan subalgebra of $E_8 \times E_8$. Thus the ten dimensional Yang-Mills supergravity reduces to its maximal torus subtheory which is an Abelian gauge supergravity theory. The Bosonic sector of this Abelian Yang-Mills supergravity corresponds to the low energy effective Lagrangian of the Bosonic sector of the fully Higgsed ten dimensional heterotic string theory [8]. Therefore we will consider the Abelian Yang-Mills supergravity in ten dimensions and we will in general take N $U(1)$ gauge field multiplets. The Bosonic Lagrangian of the $D = 10$, $N = 1$ Abelian Yang-Mills supergravity which is coupled to N $U(1)$ gauge multiplets is [8, 21]

$$\mathcal{L}_{10} = R * 1 - \frac{1}{2} * d\phi_1 \wedge d\phi_1 - \frac{1}{2} e^{\phi_1} * F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{\frac{1}{2}\phi_1} \sum_{I=1}^{N} *G^I_{(2)} \wedge G^I_{(2)}, \tag{2.116}$$

where $G^I_{(2)} = dB^I_{(1)}$ are the N $U(1)$ gauge field strengths. We also define the field strength

$$F_{(3)} = dA_{(2)} + \frac{1}{2} B^I_{(1)} \wedge dB^I_{(1)}. \tag{2.117}$$

The pure supergravity sector of (2.116) can be truncated from IIA supergravity by choosing the extra R-R fields zero in the IIA Lagrangian (2.54). The Kaluza-Klein ansatz for the ten dimensional spacetime metric upon T^n-reduction where $n = 10 - D$ is [31]

$$ds_{10}^2 = e^{\vec{s}\cdot\vec{\phi}'} ds_D^2 + \sum_{i=2}^{11-D} e^{2\vec{\gamma}_i\cdot\vec{\phi}'}(h^i)^2. \tag{2.118}$$

Similar to the reduction of the eleven dimensional metric in section 2.2 we have

$$\vec{s} = (s_2, s_3, ..., s_{(11-D)}),$$

$$\vec{f}_i = (0, ..., 0, (10-i)s_i, s_{i+1}, ..., s_{(11-D)}), \tag{2.119}$$

in the second line there are $i-2$ zeros instead of $i-1$ for the reduction of the eleven dimensional metric and $s_i = \sqrt{2/((10-i)(9-i))}$. Also we define $\vec{\gamma}_i = \frac{1}{2}\vec{s} - \frac{1}{2}\vec{f}_i$ and h^i is already given in (2.49). As we have done for the reduction of the $D = 11$ supergravity, we define the vector $\vec{\phi}' = (\phi_2, \phi_3, ..., \phi_{(11-D)})$ whose components are the dilatons of the Kaluza-Klein reduction. If we perform the S^1-reduction step by step on the ten dimensional metric with the ansatz (2.118) and on the other fields in (2.116) then the ten dimensional Lagrangian (2.116) can be written as

$$\mathcal{L}_{10} = \mathcal{L}_D \wedge dz^1 \wedge \cdots \wedge dz^n, \tag{2.120}$$

where the coordinates $\{z^i\}$ are the coordinates on the n-torus T^n as we have defined before. Similarly the D-dimensional Lagrangian in (2.120) can be calculated in terms of the D-dimensional fields as

$$\mathcal{L}_D = R * 1 - \frac{1}{2} * d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} e^{\vec{a}_1\cdot\vec{\phi}} * F_{(3)} \wedge F_{(3)}$$

$$-\frac{1}{2}\sum_i e^{\vec{a}_{1i}\cdot\vec{\phi}} * F_{(2)i} \wedge F_{(2)i} - \frac{1}{2}\sum_{i<j} e^{\vec{a}_{1ij}\cdot\vec{\phi}} * F_{(1)ij} \wedge F_{(1)ij}$$

$$-\frac{1}{2}\sum_I e^{\vec{c}\cdot\vec{\phi}} * G^I_{(2)} \wedge G^I_{(2)} - \frac{1}{2}\sum_{i,I} e^{\vec{c}_i\cdot\vec{\phi}} * G^I_{(1)i} \wedge G^I_{(1)i}$$

$$-\frac{1}{2}\sum_i e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}^i_{(2)} \wedge \mathcal{F}^i_{(2)} - \frac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} * \mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j}, \qquad (2.121)$$

where $i, j = 2, ..., 11 - D$ and $\vec{\phi} = (\phi_1, \phi_2, ..., \phi_{(11-D)})$ in which except ϕ_1, the rest of the scalars are the Kaluza-Klein scalars which originate from the ten dimensional spacetime metric. The transgression relations which define the field strengths in (2.121) in terms of the potentials are

$$F_{(3)} = dA_{(2)} + \frac{1}{2}B^I_{(1)}dB^I_{(1)} - (dA_{(1)i} + \frac{1}{2}B^I_{(0)i}dB^I_{(1)} + \frac{1}{2}B^I_{(1)}dB^I_{(0)i})\gamma^i{}_j\mathcal{A}^j_{(1)}$$

$$+\frac{1}{2}(dA_{(0)ij} - B^I_{(0)[i}dB^I_{(0)j]})\gamma^i{}_k\mathcal{A}^k_{(1)}\gamma^j{}_m\mathcal{A}^m_{(1)},$$

$$F_{(2)i} = \gamma^k_i(dA_{(1)k} + \frac{1}{2}B^I_{(0)k}dB^I_{(1)} + \frac{1}{2}B^I_{(1)}dB^I_{(0)k}$$

$$+ (dA_{(0)kj} - B^I_{(0)[k}dB^I_{(0)j]})\gamma^j{}_l\mathcal{A}^l_{(1)}),$$

$$F_{(1)ij} = \gamma^l{}_i\gamma^m{}_j(dA_{(0)lm} - B^I_{(0)[l}dB^I_{(0)m]}), \quad \mathcal{F}^i_{(2)} = \widetilde{\gamma}^i{}_j d(\gamma^j_k\mathcal{A}^k_{(1)}),$$

$$G^I_{(2)} = dB^I_{(1)} - dB^I_{(0)i}\gamma^i{}_j\mathcal{A}^j_{(1)}, \quad G^I_{(1)i} = \gamma^j{}_i dB^I_{(0)j}, \quad \mathcal{F}^i_{(1)j} = \gamma^k_j d\mathcal{A}^i_{(0)k}, \quad (2.122)$$

where we have omitted the wedge product for saving space. The dilatons $\vec{\phi}'$, the Kaluza-Klein-Maxwell potentials $\mathcal{A}^j_{(1)}$ as well as the axions $\mathcal{A}^i_{(0)k}$ in each dimension are the descendants of the ten dimensional spacetime metric as we have discussed in the last section. The potentials $A_{(0)lm}$, $A_{(1)k}$ and $A_{(2)}$ are the Kaluza-Klein descendants of the two-form potential in $D = 10$. The potentials $B^I_{(0)j}$ and $B^I_{(1)}$ are the D-dimensional remainders of the ten dimensional Yang-Mills potentials $B^I_{(1)}$ as a result of the ansatz (2.29) applied in each S^1-reduction step. We have already defined the matrix $\gamma^i{}_j$ in (2.51) and $\widetilde{\gamma}^i{}_j$ is the inverse of it, notice that in this case $n = 10 - D$. For our further analysis we will make the field redefinitions [21]

$$F_{(3)} = dA'_{(2)} + \frac{1}{2}B'^I_{(1)}dB'^I_{(1)} - \frac{1}{2}\gamma^i{}_j\mathcal{A}^j_{(1)}dA'_{(1)i} - \frac{1}{2}A'_{(1)i}d(\gamma^i{}_j\mathcal{A}^j_{(1)}),$$

$$F_{(2)i} = \gamma^k_i(dA'_{(1)k} + B^I_{(0)k}dB'^I_{(1)} - A_{(0)kl}d(\gamma^l{}_j\mathcal{A}^j_{(1)}) + \frac{1}{2}B^I_{(0)k}B^I_{(0)l}d(\gamma^l{}_j\mathcal{A}^j_{(1)})),$$

$$G^I_{(2)} = dB'^I_{(1)} + B^I_{(0)i}d(\gamma^i{}_j\mathcal{A}^j_{(1)}), \quad \mathcal{F}^i_{(2)} = \widetilde{\gamma}^i_j d(\gamma^j{}_m\mathcal{A}^m_{(1)}), \qquad (2.123)$$

where the primed potentials are defined as

$$B^I_{(1)} = B'^I_{(1)} + B^I_{(0)i}\gamma^i{}_j\mathcal{A}^j_{(1)},$$

$$A_{(1)i} = A'_{(1)i} - A_{(0)ij}\gamma^j{}_m \mathcal{A}^m_{(1)} + \frac{1}{2} B^I_{(0)i} B'^I_{(1)},$$

$$A_{(2)} = A'_{(2)} + \frac{1}{2} A_{(0)ij}\gamma^i{}_m \mathcal{A}^m_{(1)} \gamma^j{}_n \mathcal{A}^n_{(1)} + \frac{1}{2} B^I_{(0)i} B'^I_{(1)} \gamma^i{}_m \mathcal{A}^m_{(1)} + \frac{1}{2} A'_{(1)j} \gamma^j{}_m \mathcal{A}^m_{(1)}. \tag{2.124}$$

The dilaton vectors which couple to the scalars $\vec{\phi}$ in various field strength terms in (2.121) are defined as

$$\vec{a}_1 = (1, -2\vec{s}) \quad , \quad \vec{a}_{1i} = (1, \vec{f}_i - 2\vec{s}) \quad , \quad \vec{a}_{1ij} = (1, \vec{f}_i + \vec{f}_j - 2\vec{s}),$$

$$\vec{b}_i = (0, -\vec{f}_i) \quad , \quad \vec{b}_{ij} = (0, -\vec{f}_i + \vec{f}_j),$$

$$\vec{c} = (\frac{1}{2}, -\vec{s}) \quad , \quad \vec{c}_i = (\frac{1}{2}, \vec{f}_i - \vec{s}). \tag{2.125}$$

The properties of the dilaton vectors as well as the vectors $\vec{s}$, $\vec{f}_i$ can be found in [21].

We will first discuss how the global symmetry group is identified in the S^1-reduction of (2.116) resulting in the $D = 9$ Abelian Yang-Mills supergravity theory. If we consider the nine dimensional Bosonic Lagrangian in (2.121), as a first step we introduce the field redefinitions

$$\phi_1 = \frac{1}{2\sqrt{2}}\varphi - \frac{1}{2}\sqrt{\frac{7}{2}}\phi \quad , \quad \phi_2 = \frac{1}{2}\sqrt{\frac{7}{2}}\varphi + \frac{1}{2\sqrt{2}}\phi, \tag{2.126}$$

and then we define the $N+2$ scalar fields X, Y, Z^I as

$$X + Y = e^{\frac{1}{\sqrt{2}}\varphi}, \quad X - Y = e^{-\frac{1}{\sqrt{2}}\varphi} + \frac{1}{2} B^I_{(0)} B^I_{(0)} e^{\frac{1}{\sqrt{2}}\varphi}, \quad Z^I = \frac{1}{\sqrt{2}} B^I_{(0)} e^{\frac{1}{\sqrt{2}}\varphi}, \tag{2.127}$$

from φ and $B^I_{(0)}$. After the field redefinitions we end up with four scalars namely, ϕ, X, Y, Z^I but since we originally had three scalar fields, we also have a resulting constraint equation which suppresses the degrees of freedom of X, Y, Z^I,

$$X^2 - Y^2 - Z^I Z^I = 1. \tag{2.128}$$

The nine dimensional Lagrangian in terms of the newly defined fields become

$$\begin{aligned}\mathcal{L}_9 = R * 1 + \mathcal{L}_{scalar} - \frac{1}{2}e^{-\sqrt{\frac{8}{7}}\phi} * F_{(3)} \wedge F_{(3)} + \frac{1}{4}e^{-\sqrt{\frac{2}{7}}\phi}(*dA_X \wedge dA_X \\ - *dA_Y \wedge dA_Y - \sum_{I,J=1}^{N}(*dB^I_{(1)} \wedge dB^I_{(1)} + 2*(XdA_X + YdA_Y \\ + Z^J dB^J_{(1)}) \wedge (XdA_X + YdA_Y + Z^J dB^J_{(1)}))),\end{aligned} \tag{2.129}$$

where the scalar Lagrangian $\mathcal{L}_{scalar}$ is

$$\mathcal{L}_{scalar} = *dX \wedge dX - *dY \wedge dY - *dZ^I \wedge dZ^I - \frac{1}{2} * d\phi \wedge d\phi. \tag{2.130}$$

We use the transformed potentials defined in (2.124), but we omit the prime on the potentials. We have also defined the one-form fields A_X, A_Y, as

$$A_X = \frac{1}{\sqrt{2}}(A_{(1)} + \mathcal{A}_{(1)}), \quad A_Y = \frac{1}{\sqrt{2}}(A_{(1)} - \mathcal{A}_{(1)}), \tag{2.131}$$

where we have used the single Kaluza-Klein-Maxwell potential $\mathcal{A}_{(1)}$ and the one-form descendant of $A_{(2)}$ namely $A_{(1)}$ which come with the single S^1-reduction. The scalar field ϕ can be considered as decoupled from the rest of the scalars in (2.130). Thus it has a constant shift symmetry $\phi \longrightarrow \phi + c$ which is a global internal symmetry of the scalar Lagrangian (2.130) and which can be extended to the entire Bosonic Lagrangian (2.129). If we define a further action on the other $N+2$ scalar fields X, Y, Z^I as

$$\begin{pmatrix} X \\ Y \\ Z^I \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} X \\ Y \\ Z^I \end{pmatrix}, \tag{2.132}$$

where $\Lambda \in O(1,N+1)$ then the scalar Lagrangian (2.130) remains invariant. The group $O(1,N+1)$ is also the global internal symmetry group of (2.130) and if we define the associated action

$$\begin{pmatrix} A_X \\ A_Y \\ B^I_{(1)} \end{pmatrix} \longrightarrow (\Lambda^T)^{-1} \begin{pmatrix} A_X \\ A_Y \\ B^I_{(1)} \end{pmatrix}, \tag{2.133}$$

on the fields A_X, A_Y, $B^I_{(1)}$ then $O(1,N+1)$ becomes the global internal symmetry group of the nine dimensional Bosonic Lagrangian (2.129). We should note that the fields ϕ and $F_{(3)}$ are singlets under this action. Thus the total global symmetry group of both the nine dimensional scalar sector (2.130) and the Bosonic sector Lagrangian (2.129) becomes $O(1,N+1)\times\mathbb{R}$ where $\mathbb{R}$ corresponds to the constant shift symmetry group of ϕ. We observe that if we disregard the decoupled scalar ϕ then the other two scalars namely φ and $B^I_{(0)}$ parameterize a hypersurface in the flat space $\mathbb{R}^{1,N+1}$ which is defined by (2.128) where X, Y, Z^I are the $N+2$ coordinates of the points on the hypersurface. Besides, the constraint equation (2.128) defining the hypersurface is covariant under the global $O(1,N+1)$ action defined in (2.132) and the points of the hypersurface are mapped onto each other under the action of $O(1,N+1)$. This hypersurface is nothing but the scalar manifold and it is diffeomorphic to the coset space $O(1,N+1)/O(N+1)$. Therefore together with the decoupled scalar ϕ, the total scalar manifold in nine dimensions become

$$\frac{O(1,N+1)}{O(N+1)} \times \mathbb{R}. \tag{2.134}$$

The coset representatives for $O(1,N+1)/O(N+1)$ can be parameterized as

$$\nu = e^{\frac{1}{2}\varphi H} e^{B^I_{(0)} E_I}, \tag{2.135}$$

where the generators H, E_I satisfy the commutation relations

$$[H, E_I] = \sqrt{2} E_I, \quad [H,H] = [E_I, E_J] = 0, \tag{2.136}$$

and they are the generators of the solvable Lie algebra of $O(1,N+1)$. If we define the map $\mathcal{M} = \nu^T \nu$ then the scalar Lagrangian excluding ϕ can be constructed as (2.64).

When one considers further reductions, the scalar Lagrangian of the D- dimensional compactified theories for $8 \geq D \geq 3$ can be described in the form of (2.64) with an additional decoupled dilatonic kinetic term after certain field redefinitions. In [21] it is shown that when a single dilaton is decoupled from the rest of the scalars then the G/K coset representatives ν and the internal metric $\mathcal{M} = \nu^T \nu$ generated by the remaining scalars are elements of $O(10-D, 10-D+N)$. The scalar manifold for the D-dimensional compactified theory with N gauge multiplet couplings becomes

$$\frac{O(10-D, 10-D+N)}{O(10-D) \times O(10-D+N)} \times \mathbb{R}. \tag{2.137}$$

The extra $\mathbb{R}$ factor arises since there is an additional dilaton which is decoupled from the rest of the scalars in the scalar Lagrangian. The group $O(10-D,10-D+N)\times\mathbb{R}$ is the global internal symmetry of not only the scalar Lagrangian but the entire D-dimensional Bosonic Lagrangian (2.121) as well. Here again $\mathbb{R}$ corresponds to the constant shift symmetry of the decoupled dilaton. Furthermore the $D=4$ and the $D=3$ cases can be studied separately [21], since they have global symmetry enhancements in addition to the general scheme of $O(10-D,10-D+N)\times\mathbb{R}$. We have already seen in the previous section that one may define additional scalars in appropriate dimensions by dualizing certain fields by applying the Lagrange multiplier methods. When the two-form potential $A_{(2)}$ is dualized with an additional axion in $D=4$, an axion-dilaton, $SL(2,\mathbb{R})$ system which we have studied in the previous section is decoupled from the rest of the scalars in the scalar Lagrangian and the enlarged $D=4$ scalar manifold becomes

$$\frac{O(6,6+N)}{O(6)\times O(6+N)}\times\frac{SL(2,\mathbb{R})}{O(2)}. \tag{2.138}$$

On the other hand in $D=3$, the Bosonic fields $(\mathcal{A}^i_{(1)},A_{(1)i},B^I_{(1)})$ are dualised to give $7+7+N$ additional axions. The $D=3$ enlarged scalar manifold then becomes

$$\frac{O(8,8+N)}{O(8)\times O(8+N)}. \tag{2.139}$$

We see that all of the global internal symmetry groups in (2.137), (2.138) and (2.139) apart from the contributions of the decoupled scalars namely the groups $O(10-D,10-D+N)$, $O(6,6+N)$, $O(8,8+N)$ are non-compact real forms of a semisimple Lie group, and as we will study in detail in the next chapter, they enable solvable Lie algebra parameterizations of the cosets generated by the denominator groups $O(10-D)\times O(10-D+N)$, $O(6)\times O(6+N)$, $O(8)\times O(8+N)$, respectively.

We observe that the complete or the part of the scalar cosets we encounter in the T^n Kaluza-Klein compactification of the ten dimensional Lagrangian (2.116) of the Abelian Yang-Mills supergravity with N $U(1)$ gauge multiplets are in the general form of $O(p,q)/(O(p)\times O(q))$. We will concisely discuss the construction of these cosets now. The elements W of the matrix group $O(p,q)$ satisfy the relation

$$W^T \eta W = \eta, \tag{2.140}$$

where η =diag$(-1,-1,...,-1,1,1,...,1)$ is the indefinite signature metric with p -1 and q $+1$ entries. We consider the self right action of $O(p,q)$ onto the subset M of $O(p,q)$ which is composed of the symmetric matrices $W^T = W$ for $W \in O(p,q)$. We define the action on M as

$$W \longrightarrow \Lambda^T W \Lambda, \tag{2.141}$$

$\forall \Lambda, W \in O(p,q)$ (We will study the action of Lie groups in detail when we study the symmetric spaces). The action (2.141) on the symmetric matrices in $O(p,q)$ induces an equivalence relation on M and the corresponding equivalence classes form the orbit space $M/O(p,q)$ of the action under the quotient topology transferred from $O(p,q)$. One can show that [21] the orbits, the equivalence classes which are points in the orbit space $M/O(p,q)$, can be classified by the corresponding equivalence class representatives

$$W_0 = \mathrm{diag}(-1, ..., -1, 1, ..., 1, -1, ..., -1, 1, ...1), \tag{2.142}$$

where there are $m - 1$ entries, $p - m + 1$ entries and $n - 1$ entries, $q - n + 1$ entries from the left to the right respectively for $0 \leq m \leq p$ and $0 \leq n \leq q$. The stability or the isotropy group G_{W_0} of the fiducial point W_0 can be given as

$$G_{W_0} = O(m, q - n) \times O(p - m, n). \tag{2.143}$$

Due to the disjoint orbit structure it is obvious that $O(p, q)$ does not act transitively on M however as we will study in detail in the next chapter, $O(p, q)$ has an induced transitive action on each orbit whose representative W_0 is given by (2.142) and whose specific stability group is as in (2.143). Therefore each orbit is diffeomorphic to the coset space $O(p, q)/(O(m, q-n) \times O(p-m, n))$. If we choose the particular orbit whose representative fiducial point is W_0 =diag$(1, ..., 1)$ then we get the coset space

$$M_{p,q} = \frac{O(p, q)}{O(p) \times O(q)}, \tag{2.144}$$

which is diffeomorphic to the corresponding orbit of the $O(p, q)$ action on M thus a point in $M/O(p, q)$ as we have described above.

The denominator groups $O(p) \times O(q)$ are the maximal compact subgroups of $O(p, q)$ in each case. As a manifold the dimension of $M_{p,q}$ is

$$\begin{aligned} \dim(M_{p,q}) &= \dim(O(p, q)) - \dim(O(p) \times O(q)) \\ &= \frac{1}{2}(p + q)(p + q - 1) - \frac{1}{2}p(p - 1) - \frac{1}{2}q(q - 1) = pq. \end{aligned} \tag{2.145}$$

We have seen before that the dimensions of the scalar cosets are equal to the number of the scalar fields. When constructing the coset representatives in a particular gauge (the solvable Lie algebra gauge for the non-compact real forms and the Borel gauge for the maximally non-compact (split) real forms) the generators corresponding to the gauge which generate a subalgebra of the Lie algebra of the global internal symmetry group are assigned to the scalars in one to one correspondence. Thus the number of the generators of the coset representative gauge is equal to the dimension of the coset. The Lie group $O(p,q)$ has $\frac{1}{2}p(p-1)+\frac{1}{2}q(q-1)$ compact generators while the rest of the generators are non-compact. The dimension of the Cartan subalgebra of $o(p,q)$ is $\frac{1}{2}n$ when $p+q=n$ is even and the number of the positive roots (thus the positive root generators) is $\frac{1}{2}(\frac{1}{2}n(n-1)-\frac{n}{2})$. Thus the dimension of the Borel subalgebra of $o(p,q)$ when $p+q=n$ is even becomes $\frac{1}{2}n+\frac{1}{2}(\frac{1}{2}n(n-1)-\frac{n}{2})=\frac{1}{4}n^2$. Therefore when n is even the dimension of the coset can be equal to the dimension of the Borel subalgebra of $o(p,q)$ only when $p=q$, this means that the coset representatives of $O(p,p)/(O(p)\times O(p))$ must be generated by the Borel gauge as we have explained above and $O(p,p)$ is a maximally non-compact real form in this case. Similarly when n is odd the global internal symmetry groups $O(p+1,p)$ and $O(p,p+1)$ are maximally non-compact real forms and the coset representatives of $O(p+1,p)/(O(p+1)\times O(p))$ and $O(p,p+1)/(O(p)\times O(p+1))$ can be generated by using the Borel gauge. The rest of the cases are all non-compact real forms and one has to use the solvable Lie algebra gauge to construct the coset representatives.

The orthogonal algebras $o(p,q)$ are elements (real forms) of the D_n-series when $p+q=2n$ and the B_n-series when $p+q=2n+1$. As we have done before a representation based on the chosen Cartan subalgebra can be used to express the roots of $o(p,q)$. If we choose the orthonormal basis $e_i \cdot e_j = \delta_{ij}$ such that $e_i = (0,0,...,1,0,...,0)$, where the $+1$ entry is at the i'th position for $i,j=1,...,n$, then the representatives of the positive roots can be given as

$$D_n : \qquad e_i \pm e_j, \quad i < j \leq n,$$

$$B_n : \qquad e_i \pm e_j, \quad i < j \leq n \quad \text{and} \quad e_i. \tag{2.146}$$

There are the associated positive root generators $E_{(e_i \pm e_j)}$ and $E_{(e_i)}$ and there are n Cartan generators h_i for both of the series. By considering the meaning of the representation of the roots namely, (2.73) we can write $[h_i, E_{e_j \pm e_k}] = (\delta_{ij} \pm \delta_{ik}) E_{e_j \pm e_k}$. Min$(p,q)$ of the Cartan generators h_i are non-compact and the remaining are compact. Using the conventions of [21] we will enumerate the Cartan generators h_i for $1 \leq i \leq$min(p,q) as the non-compact ones.

We have already mentioned about the nine dimensional theory, we will give the general construction of the scalar cosets for the descendant theories $D \geq 5$ now. We may decouple the single scalar field

$$\phi = -\sqrt{\frac{D-2}{8}}\, \vec{a}_{123} \cdot \vec{\phi}, \tag{2.147}$$

which is a linear combination of the original dilatons, from the rest of the scalar Lagrangian in (2.121), by making a transformation on the original dilatons. In other words by rotating the dilatons one may define new fields among which one of them will be (2.147) and its kinetic term is decoupled from the rest [21]. ϕ is the only scalar which the three form field strength $F_{(3)}$ couples to. The scalar Lagrangian for the other scalars become

$$\mathcal{L}^D_{scalar} = -\frac{1}{2} * d\vec{\varphi} \wedge d\vec{\varphi} - \frac{1}{2} \sum_{i<j} e^{\vec{a}_{ij} \cdot \vec{\varphi}} * F_{(1)ij} \wedge F_{(1)ij}$$

$$- \frac{1}{2} \sum_{i,I} e^{\vec{c}_i \cdot \vec{\varphi}} * G^I_{(1)i} \wedge G^I_{(1)i} - \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\varphi}} * \mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j}. \qquad (2.148)$$

As we omit the decoupled dilaton (2.147) we have used the dilaton scalar vector $\vec{\varphi}$ which has $10 - D$ components that are the remaining transformed dilatons. We have also modified the dilaton vectors. This is because we have performed a rotation transformation on the original dilatons to decouple ϕ in (2.147) from the rest and we have defined new dilatonic scalars thus their coupling coefficients in the original scalar Lagrangian (2.121) are transformed to the ones in (2.148). Therefore in (2.148) we have

$$\vec{b}_{ij} = \sqrt{2}\,(-\vec{e}_i + \vec{e}_j), \quad \vec{a}_{ij} = \sqrt{2}\,(\vec{e}_i + \vec{e}_j), \quad \vec{c}_i = \sqrt{2}\,\vec{e}_i, \qquad (2.149)$$

where $i, j = 1, ..., (10 - D)$. We have mentioned before that for $D \geq 5$ the scalars whose Lagrangian is (2.148) parameterize the scalar coset manifold

$$\frac{O(10 - D, 10 - D + N)}{O(10 - D) \times O(10 - D + N)}. \qquad (2.150)$$

We should inspect the realizations of these scalar cosets from a closer point of view. We will follow a slightly different method than the one we have used before. We will identify the global internal symmetry group by defining its solvable Lie algebra. We will introduce generators which are assumed to generate the coset representative and calculate their commutation relations from the requirement that they should lead to the scalar Lagrangian (2.148) in the form of (2.64). A similar method will be our basic argument when we discuss the complete dualisation of the Bosonic sectors of the supergravity theories where we will also dualize all the Bosonic fields but there we will use the equations of motion as a tester instead of the Lagrangian. The details of how one identifies the commutation relations from the coset construction of the Lagrangian can be found in [10, 19, 20, 21]. We remind the reader of the fact that for the orthogonal groups $O(p, q)$, the generalized transpose # coincides with the ordinary matrix transpose in the fundamental representation of the Lie algebra $o(p, q)$. Thus in constructing the scalar Lagrangian (2.64) we will take $\mathcal{M} = \nu^T \nu$.

If one assigns the set of generators $\{H_i,\, E_i^j,\, V^{ij},\, U_I^i\}$ to the scalar fields $\{\varphi^i,\, A^i_{(0)j},\, A_{(0)ij},\, B^I_{(0)i}\}$ resulting from the dimensional reduction respectively and by intuition if we assume the coset parametrization

$$\nu = e^{\frac{1}{2}\varphi^i H_i} e^{A^i_{(0)j} E_i^j} e^{\frac{1}{2}A_{(0)ij} V^{ij}} e^{B^I_{(0)i} U_I^i}, \tag{2.151}$$

then the scalar Lagrangian, (2.148) of the compactified D-dimensional theory apart from the decoupled scalar can be constructed in the form of (2.64) only if the Cartan form $\mathcal{G} = d\nu\nu^{-1}$ which can be calculated from (2.151) is in the form

$$\mathcal{G} = d\nu\nu^{-1}$$

$$= \frac{1}{2} d\vec{\varphi} \cdot \vec{H} + \sum_{i<j} e^{\frac{1}{2} \vec{a}_{ij} \cdot \vec{\varphi}} F_{(1)ij} V^{ij}$$

$$+ \sum_{i,I} e^{\frac{1}{2} \vec{c}_i \cdot \vec{\varphi}} G^I_{(1)i} U^i_I + \sum_{i<j} e^{\frac{1}{2} \vec{b}_{ij} \cdot \vec{\varphi}} \mathcal{F}^i_{(1)j} E^j_i . \qquad (2.152)$$

Now if one explicitly calculates the Cartan form $\mathcal{G}$ from (2.151) in terms of the unknown structure constants of the commutation relations and then compare it with (2.152) by inserting the field strengths in terms of the potentials from their transgression relations (2.122) the non-vanishing commutators of the generators can be calculated as

$$[\vec{H}, E^j_i] = \vec{b}_{ij} E^j_i \quad , \quad [\vec{H}, V^{ij}] = \vec{a}_{ij} V^{ij} \quad , \quad [\vec{H}, U^i_I] = \vec{c}_i U^i_I ,$$

$$[E^j_i, E^l_k] = \delta^j_k E^l_i - \delta^l_i E^j_k \quad , \quad [E^j_i, V^{kl}] = -\delta^k_i V^{jl} - \delta^l_i V^{kj} ,$$

$$[E^j_i, U^k_I] = -\delta^k_i U^j_I \quad , \quad [U^i_I, U^j_J] = \delta_{IJ} V^{ij} . \qquad (2.153)$$

The embedding of the algebra we have calculated in (2.153) into the Lie algebra $o(10-D, 10-D+N)$ can be given as

$$H_i = \sqrt{2}\, h_i \quad , \quad E^j_i = E_{-\widetilde{e}_i + \widetilde{e}_j} \quad , \quad V^{ij} = E_{\widetilde{e}_i + \widetilde{e}_j} ,$$

$$U^i_{2k-1} = \frac{1}{\sqrt{2}} \, (E_{\widetilde{e}_i + e_{k+m}} + E_{\widetilde{e}_i - e_{k+m}}) ,$$

$$U^i_{2k} = \frac{i}{\sqrt{2}} \, (E_{\widetilde{e}_i + e_{k+m}} - E_{\widetilde{e}_i - e_{k+m}}) , \qquad (2.154)$$

where $1 \leq i < j \leq 10 - D$. We have also defined $m = 10 - D$ and $1 \leq k \leq [N/2]$. When N is odd in addition to (2.154) we also have

$$U_N^i = E_{\widetilde{e}_i}. \tag{2.155}$$

In (2.154) and (2.155) $\{e_i\}$ is the orthonormal basis which we have defined to characterize the positive roots of $o(10 - D, 10 - D + N)$ before and $\{\widetilde{e}_i\}$ are defined as

$$\widetilde{e}_i = e_{11-D-i}, \qquad 1 \leq i \leq 10 - D. \tag{2.156}$$

As it is mentioned in [21] owing to their definitions the generators $\{h_i\}$, $\{E_{\widetilde{e}_i \pm e_j}$, $E_{\widetilde{e}_i \pm \widetilde{e}_j}$, $E_{\widetilde{e}_i}\}$ (the last set $\{E_{\widetilde{e}_i}\}$ is included when N is odd and excluded when N is even) generate the solvable Lie algebra of $o(10-D, 10-D+N)$ in each dimension thus the parametrization of the coset in (2.151) is nothing but an example of the solvable Lie algebra gauge. This result justifies the prediction that the global internal symmetry group of the scalar Lagrangian (2.148) is $O(10-D, 10-D+N)$ and the scalar coset manifold is $O(10 - D, 10 - D + N)/(O(10 - D) \times O(10 - D + N))$ for $D \geq 5$. As a final remark we should state that when the global symmetry group $O(10 - D, 10 - D + N)$ acts on the scalars in a non-linear fashion which is the general rule as we have seen in the maximal supergravities, the coset representative (2.151) transforms as $\nu \longrightarrow \Delta(x)\nu\Lambda$ where $\Lambda \in O(10 - D, 10 - D + N)$ is a global and $\Delta(x) \in O(10 - D) \times O(10 - D + N)$ is a local transformation. As it will be clear in the next chapter this is a result of the Iwasawa decomposition of the non-compact real forms which is also responsible for the solvable Lie algebra parametrization.

We note that since in the D-dimensional T^{10-D}- compactified Lagrangian (2.121) the scalars are coupled to the one-form potentials which form the $(20-2D+N)$-dimensional linear representation of $O(10-D,10-D+N)$, both the coset representative (2.151), the internal metric $\mathcal{M}$ and the generators in (2.153) must be represented by $(20-2D+N)$-dimensional matrices in order to construct the Bosonic Lagrangian (2.121). We will choose a fundamental representation of $o(10-D,10-D+N)$ in which the generators $\{H_i, E_i^j, V^{ij}, U_I^i\}$ are represented by the $(20-2D+N)$-dimensional matrices

$$\vec{H}_i = \begin{pmatrix} \sum_i \vec{c}_i e_{ii} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sum_i \vec{c}_i e_{ii} \end{pmatrix}, \quad E_i^j = \begin{pmatrix} -e_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_{ij} \end{pmatrix},$$

$$V^{ij} = \begin{pmatrix} 0 & 0 & e_{ij}-e_{ji} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_I^i = \begin{pmatrix} 0 & e_{iI} & 0 \\ 0 & 0 & e_{Ii} \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.157}$$

where the matrices are partitioned with appropriate dimensions which can be read from the non-zero entries and e_{ab} is a matrix with appropriate dimensions and which has zero entries, except a $+1$ entry at the a'th row and the b'th column. Notice that the indices $i,j=1,...,10-D$ and $I=1,...,N$ determine the dimensions of the matrices e_{ab}. By using this representation, the coset representative (2.151) and by defining $\mathcal{M}=\nu^T\nu$, the Bosonic Lagrangian (2.121) for $D\geq 5$ can now be reconstructed as

$$\mathcal{L}_D = R * 1 - \frac{1}{2} * d\phi \wedge d\phi + \frac{1}{4} tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M})$$

$$- \frac{1}{2} e^{-\sqrt{8/(D-2)}\phi} * F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{-\sqrt{2/(D-2)}\phi} * H^T_{(2)} \wedge \mathcal{M} H_{(2)}, \qquad (2.158)$$

where $H_{(2)} = dC_{(1)}$ and

$$C_{(1)} = \begin{pmatrix} A_{(1)i} \\ B^I_{(1)} \\ \mathcal{A}^i_{(1)} \end{pmatrix}, \qquad (2.159)$$

is a column vector of dimension $(20-2D+N)$. We use the transformed potentials which are the primed potentials in (2.124) by omitting the primes. We should note that in (2.158) taking the exterior derivative of a matrix or a vector means taking the exterior derivative of all the elements and we use the matrix notation in the sense that the matrix-matrix or the matrix-vector product must be taken by using the wedge product between the components. When one only considers the scalar Lagrangian in any dimension then one does not have to fix the dimension of the representation in fact one does not have to use any matrix representation at all, the formulation can be based on merely the abstract generators and the generalized transpose # by specifying an abstract basis through its commutation relations namely the structure constants. For the abstract construction the choice of the basis and the coset representative defines the fields which are associated to the generators indeed. On the other hand when any fundamental representation is chosen then the matrix transpose can be used instead of the generalized transpose. The choice of the basis defines how the terms of the scalar Lagrangian will look in terms of the potentials associated to the generators in the basis. The transformations can always be found between the original fields of the Kaluza-Klein descendant scalar Lagrangian and the ones in the Lagrangian which results from the general form (2.97) by choosing an arbitrary abstract basis. We also observe that there is a particular basis which generates, from (2.97), the original scalar Lagrangian including the Kaluza-Klein descendant fields. As we see from (2.158) the other fields can also be coupled to the scalars in a compact form however in this case as the number of the higher rank fields in (2.159) is fixed the dimension of the representation we have to use is fixed too. One can use different representations or basis with the same dimension, as we have discussed above various representations or basis mean different field definitions and they will lead to the transformations between the fields. There is again a particular basis and a representation which gives the original Kaluza-Klein descendant form of the Bosonic Lagrangian when used in (2.158). The analysis we have performed and the formulas we have obtained for $D \geq 5$ are also in general terms valid for the cases when $D = 3, 4$, however as we have mentioned before we can do more in the three and four dimensions. Basically by applying the dualisation method we can enlarge the scalar coset and show that the global internal symmetry group is larger than the expected $O(6, 6+N)$ for $D = 4$ and the expected $O(7, 7+N)$ for $D = 3$. First we consider the $D = 4$ case, as we have discussed before one can dualize the three-form field to obtain an extra axionic scalar. To see this let us consider the Bianchi identity for $F_{(3)}$

$$dF_{(3)} = \frac{1}{2}dB^I_{(1)} \wedge dB^I_{(1)} - dA_{(1)i} \wedge d\mathcal{A}^i_{(1)}, \tag{2.160}$$

which may be obtained by taking the exterior derivative of the transgression expression for $F_{(3)}$ in (2.123). The expression for the field strength $F_{(3)}$ in (2.123) can also be given in a more compact form for any dimension as

$$F_{(3)} = dA_{(2)} + \frac{1}{2}C^T_{(1)}\,\Omega\,dC_{(1)}, \tag{2.161}$$

we have defined $C_{(1)}$ in (2.159) and the $(20-2D+N)\times(20-2D+N)$ matrix Ω is

$$\Omega = \begin{pmatrix} 0 & 0 & -\mathbf{1}_{(10-D)} \\ 0 & \mathbf{1}_{(N)} & 0 \\ -\mathbf{1}_{(10-D)} & 0 & 0 \end{pmatrix}, \tag{2.162}$$

where $\mathbf{1}_n$ is the $n\times n$ unit matrix. Notice that we are using the primed potentials we have introduced in (2.124) but we omit the primes for the sake of clearness. Thus the Bianchi identity can also be expressed as

$$dF_{(3)} = \frac{1}{2}dC^T_{(1)}\,\Omega\,dC_{(1)}. \tag{2.163}$$

By adding the Bianchi term $-\chi(dF_{(3)} - \frac{1}{2}dC^T_{(1)}\,\Omega\,dC_{(1)})$ to the original four dimensional Lagrangian (2.158), where we introduce the Lagrange multiplier χ, we construct the dualized Lagrangian $\mathcal{L}^4_{dual}$ as

$$\mathcal{L}^4_{dual} = R*1 - \frac{1}{2}*d\phi\wedge d\phi + \frac{1}{4}tr(*d\mathcal{M}^{-1}\wedge d\mathcal{M})$$

$$-\frac{1}{2}e^{-2\phi}*F_{(3)}\wedge F_{(3)}-\frac{1}{2}e^{-\phi}*H^T_{(2)}\mathcal{M}H_{(2)}$$

$$-\chi(dF_{(3)}-\frac{1}{2}dC^T_{(1)}\,\Omega\,dC_{(1)}). \tag{2.164}$$

We can make use of the Lagrangian (2.158) bearing in mind that its scalar coset without the decoupled dilaton is defined as $O(6,6+N)/(O(6)\times O(6+N))$ before the dualisation. If (2.164) is varied with respect to χ then the Bianchi identity is obtained if one varies it with respect to $F_{(3)}$ which is considered as a fundamental field now then an algebraic equation of motion is obtained for $F_{(3)}$ as

$$F_{(3)}=e^{2\phi}*d\chi. \tag{2.165}$$

We should remind the reader of the fact that we use the convention for the Hodge map which is introduced in (2.18). The three-form $F_{(3)}$ can be eliminated from (2.164) in terms of χ by using this algebraic equation. Then apart from the rest of the scalars which are described by the internal metric $\mathcal{M}$ in (2.164) and whose global internal symmetry group is $O(6,N+6)$, the remaining scalars (ϕ,χ) form up a dilaton-axion system

$$\mathcal{L}^4_{dual}(\phi,F_{(3)}\rightarrow d\chi)=-\frac{1}{2}*d\phi\wedge d\phi-\frac{1}{2}e^{2\phi}*d\chi\wedge d\chi, \tag{2.166}$$

where ϕ has been defined in (2.147) and χ is the Lagrange multiplier we have introduced for the Bianchi identity (2.163). We have studied this Lagrangian in detail in the previous section and we have seen that its global internal symmetry group is $SL(2,\mathbb{R})$. Therefore the global symmetry group of the enlarged scalar Lagrangian is $O(6,N+6)\times SL(2,\mathbb{R})$ and this verifies the scalar coset manifold already given in (2.138). It can be shown that $O(6, N+6)$ can be extended to be the global symmetry of the entire Bosonic Lagrangian while $SL(2,\mathbb{R})$ can be extended to be a global symmetry of the Bosonic field equations [21]. Finally the $D = 4$ dualized Lagrangian becomes

$$\mathcal{L}^4_{dual} = R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2}e^{2\phi} * d\chi \wedge d\chi$$

$$+ \frac{1}{4}tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}) - \frac{1}{2}e^{-\phi} * H^T_{(2)}\mathcal{M}H_{(2)}$$

$$+ \frac{1}{2}\chi(H^T_{(2)}\Omega H_{(2)}), \tag{2.167}$$

where we have eliminated an exact form from the Lagrangian. In $D = 3$ we will dualize the Bosonic fields $(\mathcal{A}^i_{(1)}, A_{(1)i}, B^I_{(1)})$ by introducing 7+7+N axionic scalars $(\widetilde{\chi}_i, \chi^i, \lambda_I)$ which are the associated Lagrange multipliers for the corresponding Bianchi identities. From (2.123) the Bianchi identities of the field strengths of $(\mathcal{A}^i_{(1)},\ A_{(1)i},\ B^I_{(1)})$ can be calculated as

$$d\mathcal{F}^i_{(2)} = \mathcal{F}^i_{(1)j} \wedge \mathcal{F}^j_{(2)},$$

$$dF_{(2)i} = -\mathcal{F}^j_{(1)i} \wedge F_{(2)j} - F_{(1)ij} \wedge \mathcal{F}^j_{(2)} + G^I_{(1)i} \wedge G^I_{(2)},$$

$$dG^I_{(2)} = G^I_{(1)i} \wedge \mathcal{F}^i_{(2)}. \tag{2.168}$$

As usual we use the primed potentials of (2.124) without the primes. If we introduce the Lagrange multipliers $(\widetilde{\chi}_i, \chi^i, \lambda_I)$ we can construct the Bianchi Lagrangian from the identities above as

$$\mathcal{L}_{Bianchi} = \widetilde{\chi}_i(d\mathcal{F}^i_{(2)} - \mathcal{F}^i_{(1)j} \wedge \mathcal{F}^j_{(2)}) + \lambda_I(dG^I_{(2)} - G^I_{(1)i} \wedge \mathcal{F}^i_{(2)})$$

$$- \chi^i(dF_{(2)i} + \mathcal{F}^j_{(1)i} \wedge F_{(2)j} + F_{(1)ij} \wedge \mathcal{F}^j_{(2)} - G^I_{(1)i} \wedge G^I_{(2)}). \qquad (2.169)$$

One can add this Lagrangian to the $D = 3$ Lagrangian in (2.121) (one has to transform it in terms of the primed potentials first) to construct the dualized Lagrangian as usual. When the dualized Lagrangian is varied with respect to the Lagrange multipliers $(\widetilde{\chi}_i, \chi^i, \lambda_I)$ it gives the Bianchi identities (2.168) and when it is varied with respect to the field strengths $\mathcal{F}^i_{(2)}, F_{(2)i}, G^I_{(2)}$ we get the purely algebraic equations of motion for $\mathcal{F}^i_{(2)}, F_{(2)i}, G^I_{(2)}$ which are

$$-e^{\vec{b}_i \cdot \vec{\phi}} * \mathcal{F}^i_{(2)} = d\widetilde{\chi}_i + \widetilde{\chi}_j \mathcal{F}^j_{(1)i} - \chi^k F_{(1)ik} + \lambda_I G^I_{(1)i},$$

$$e^{\vec{a}_{1i} \cdot \vec{\phi}} * F_{(2)i} = d\chi^i - \chi^j \mathcal{F}^i_{(1)j},$$

$$-e^{\vec{c} \cdot \vec{\phi}} * G^I_{(2)} = d\lambda_I - \chi^i G^I_{(1)i}. \qquad (2.170)$$

We define the right hand sides of these algebraic equations as the field strengths ($\mathcal{F}_{(1)i}$, $F^i_{(1)}$, $G_{(1)I}$) which are associated with the Lagrange multiplier scalars ($\widetilde{\chi}_i$, χ^i, λ_I) respectively. Furthermore by using the Lagrange multiplier axionic scalars we can define the extended set of potentials $\widetilde{A}_{(0)AB}$, $\widetilde{\mathcal{A}}^A_{(0)B}$, $\widetilde{B}^I_{(0)A}$, [21]

$$\widetilde{A}_{(0)ij} = A_{(0)ij}, \quad \widetilde{A}_{(0)i\,9} = \widetilde{\gamma}^j_i \widetilde{\chi}_j + \frac{1}{2} B^I_{(0)i} \lambda_I, \quad \widetilde{\mathcal{A}}^i_{(0)j} = \mathcal{A}^i_{(0)j},$$

$$\widetilde{\mathcal{A}}^i_{(0)9} = \chi^i, \quad \widetilde{B}^I_{(0)i} = B^I_{(0)i}, \quad \widetilde{B}^I_{(0)9} = \lambda_I. \tag{2.171}$$

If we define the matrix $(\gamma'^{-1})^A_B = \delta^A_B + \widetilde{\mathcal{A}}^A_{(0)B}$ then the corresponding extended set of field strengths can be given as

$$\widetilde{F}_{(1)AB} = \gamma'^C_A \gamma'^D_B (d\widetilde{A}_{(0)CD} - \widetilde{B}^I_{(0)[C} d\widetilde{B}^I_{(0)D]}),$$

$$\widetilde{\mathcal{F}}^A_{(1)B} = \gamma'^C_B d\widetilde{\mathcal{A}}^A_{(0)C},$$

$$\widetilde{G}^I_{(1)A} = \gamma'^B_A d\widetilde{B}^I_{(0)B}. \tag{2.172}$$

Since we have dualized certain higher rank Bosonic fields and exchanged them with the scalar fields the enlarged scalar Lagrangian of the $D = 3$ Bosonic Lagrangian in terms of the extended set of field strengths (2.172) can be given as

$$\mathcal{L}'^3_{scalar} = -\frac{1}{2} * d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_{A<B} e^{\vec{a}_{AB} \cdot \vec{\phi}} * \widetilde{F}_{(1)AB} \wedge \widetilde{F}_{(1)AB}$$

$$- \frac{1}{2} \sum_{A,I} e^{\vec{c}_A \cdot \vec{\phi}} * \widetilde{G}^I_{(1)A} \wedge \widetilde{G}^I_{(1)A} - \frac{1}{2} \sum_{A<B} e^{\vec{b}_{AB} \cdot \vec{\phi}} * \widetilde{\mathcal{F}}^A_{(0)B} \wedge \widetilde{\mathcal{F}}^A_{(0)B}, \tag{2.173}$$

where $2 \leq A, B \leq 9$ and the extended dilaton vectors are

$$\vec{c}_A = (\vec{c}_i, \vec{c}_9) = (\vec{c}_i, -\vec{c}),$$

$$\vec{b}_{AB} = -\vec{c}_A + \vec{c}_B,$$

$$\vec{a}_{AB} = \vec{c}_A + \vec{c}_B, \tag{2.174}$$

where $2 \leq A, B \leq 9$ and $2 \leq i \leq 8$. We have defined $\vec{c}_i$ and $\vec{c}$ in (2.125). Notice that we are not using the modified dilaton vectors given in (2.149) as we need not do a dilaton rotation to decouple one dilaton from the rest in $D = 3$. We make the observation that (2.173) is in the form of (2.148) with the index range $2 \leq A, B \leq 9$ rather than $1 \leq i \leq (10 - D)$ in (2.148). Therefore by following the results we have found for the scalar Lagrangian (2.148) we conclude that the global internal symmetry group for the $D = 3$ Bosonic Lagrangian which is also the symmetry of the enlarged scalar Lagrangian becomes $O(8, 8 + N)$. Then the scalar coset manifold is $O(8, 8 + N)/(O(8) \times O(8 + N))$.

CHAPTER 3

SYMMETRIC SPACES

In Chapter two we have introduced the scalar coset manifolds G/K of the maximal and the gauge multiplet coupled supergravity theories in various dimensions. In this chapter we will give a rigorous algebraic, group theoretical and geometrical construction of these coset manifolds. We will first mention the formal construction of the Riemannian symmetric spaces and the symmetric pairs. We will also introduce the fundamentals of the Lie groups and their Lie algebras. Then after presenting a revision of the structure of the semi-simple Lie algebras we will discuss two important decompositions namely the Cartan and the Iwasawa decompositions [44]. These decompositions are important in the analysis of the sigma models which are based on the symmetric spaces. Apart from the construction of the symmetric spaces the mainline of this chapter can be considered as the introduction of the Cartan and the Iwasawa decompositions because as it will be clear by the end of the chapter, the numerator non-compact semi-simple Lie group G of the scalar coset manifold G/K is parameterized by using these decompositions. Such a parametrization will lead us to the so called solvable Lie algebra parametrization [18] of the scalar manifolds of a wide class of supergravity theories. When we give the classification of the compact, non-compact and the split semi-simple real forms we will finally discuss how the scalar coset manifolds of the supergravity theories are oriented in this mathematical picture. The details of a wide class of geometrical and algebraic concepts which take part in the formal definitions of the symmetric spaces presented here may be found in the references [39, 41, 42] and [44]-[56].

3.1 Riemannian Symmetric Spaces

We need to start with a series of definitions of the point-set topology and the differentiable manifolds [54]. The power set PX of any set X is the collection of all the subsets of X. A topology T on X is a subset of PX, $T \subset PX$ such that if $x_1, x_2 \in T$ then $x_1 \cap x_2 \in T$, if $\{x_\alpha \mid \alpha \in J\} \subset T$ then $\bigcup_\alpha x_\alpha \in T$ also $\emptyset \in T$ and $X \in T$. The space (X, T) is called a topological space if it satisfies the requirements given above. The elements of T are called the open sets [54]. The complements of the open sets in X are called the closed sets. The topology T on X may equivalently be defined in terms of the closed sets. If $A \subset X$ then the interior A^0 of A is defined as the union of all the open sets contained in A. A neighborhood of a point $p \in X$ is any $A \subset X$ such that $p \in A^0$. The topological space (X, T) is called a Hausdorff topological space if for each pair of points $x, y \in X$ there exists corresponding neighborhoods $U \subset X$ of x and $V \subset X$ of y such that $U \cap V = \emptyset$. Now a map $\varphi : X \longrightarrow Y$ between two topological spaces is called continuous if it maps the open sets of X onto the open sets of Y. Moreover if φ is one to one it is called a homeomorphism. A chart at a point $p \in X$ is a homeomorphism, $\mu : U \longrightarrow \mathbb{R}^d$ where U is an open set ($p \in U$) and μ is onto an open subset of $\mathbb{R}^d$. Here d is called the dimension of the chart. The maps $x^i(\mu(p)) \in \mathbb{R}$ for $i = 1, ...d$ are called the coordinates of $p \in X$ with respect to the chart (U, μ). The basis of neighborhoods of a point $p \in X$ is a collection of neighborhoods at p such that every neighborhood of p contains at least one of the basis neighborhoods. A specification of basis neighborhoods at each point $p \in X$ provides a basis neighborhood for X. A separable topological space is a topological space which has a countable set of basis neighborhoods at each point $p \in X$.

A topological C^0-manifold M is a separable Hausdorff topological space such that there is a d-dimensional chart at every point $p \in M$ [54]. Here d is called the dimension of the manifold. For a topological C^0-manifold M there exists an atlas which is a collection of charts $\{\mu_\alpha : U_\alpha \longrightarrow \mathbb{R}^d \mid \alpha \in I\}$ such that $\{U_\alpha \mid \alpha \in I\}$ is a covering of M meaning that $\bigcup_\alpha U_\alpha = M$. We define the C^∞-compatibility of two charts (U, φ) and (V, ψ) as either $U \cap V = \emptyset$ or the coordinate transformation map $\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$ which is a homeomorphism from a subset of $\mathbb{R}^d$ onto a subset of $\mathbb{R}^d$ is an infinitely differentiable map for all the points in $\psi(U \cap V)$ in the ordinary differentiability sense [51]. If any pair of charts are compatible in an atlas then the atlas is said to be C^∞-differentiable and two atlases are compatible if all the members of them are compatible with each other. Compatibility defines an equivalence relation on the set of the atlases of M. An equivalence class of compatible atlases defines a differentiable structure and if C^0-manifold M has such a structure then it is called a C^∞-manifold (a smooth manifold) with the associated differentiable structure [42]. Furthermore M is called an analytical manifold if we replace the differentiability condition in the definition of the compatibility of two charts with the analyticity which means that there should exist a convergent power series expansion in a neighborhood of each point $p \in \psi(U \cap V)$ for the coordinate change map $\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$.

A map $\alpha : M \longrightarrow N$ from a smooth manifold M into another smooth manifold N is called a smooth map at a point $p \in M$ if there exists charts (U, φ) in M and (V, ψ) in N such that $p \in U$ and $\alpha(p) \in V$ and $\psi \circ \alpha \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$ is infinitely differentiable at $\varphi(p)$. If α is differentiable at every point $p \in M$ then it is called a differentiable or a smooth map. In addition if α is one to one then it is called a diffeomorphism. Let M be a C^∞-manifold, a curve in M is a smooth map $\phi : (-\epsilon, \epsilon) \subset \mathbb{R} \longrightarrow M$. The tangent space T_pM of a point $p \in M$ is an $\mathbb{R}$-vector space composed of the equivalence classes of curves whose images in M contain $p \in M$ where two curves ϕ_1 and ϕ_2 are equivalent if $\phi_1(0) = \phi_2(0) = p$ and $\dot{x}^i(\phi_1)|_{t=0} = \dot{x}^i(\phi_2)|_{t=0}$ where $\{x^i\}$ are the coordinate maps with respect to any chart (U, φ) which contains p. The $\mathbb{R}$-vector space structure can be defined on T_pM by using any local chart of M containing p [53]. The dimension of the $\mathbb{R}$-vector space T_pM is equal to the dimension of the manifold M. The elements of the tangent space T_pM may also be considered as the derivations from the ring of smooth functions $C^\infty(M)$ $(M \longrightarrow \mathbb{R})$ into $\mathbb{R}$ [53]. A derivation is a map, $\nu : C^\infty(M) \longrightarrow \mathbb{R}$ such that $\nu(f+g) = \nu(f) + \nu(g)$ and $\nu(rf) = r\nu(f)$ also ν satisfies the Leibniz rule $\nu(fg) = f\nu(g) + \nu(f)g$, $\forall r \in \mathbb{R}$ and $\forall f, g \in C^\infty(M)$. Clearly it is more straightforward to put an $\mathbb{R}$-vector space structure on the tangent space T_pM when one considers the tangent vectors as derivations. A vector field on M is a derivation of the ring $C^\infty(M)$ onto itself and for each point $p \in M$ it corresponds to an element of T_pM. The set of vector fields of M will be denoted as T^1M which has a module structure globally [42]. One can define the dual space T_p^*M of T_pM and similarly the differential one-forms $\Omega^1(M)$ are the maps from T^1M into $C^\infty(M)$ such that at each point $p \in M$ they correspond to an element of T_p^*M. The set of tensor fields T_s^rM of certain types on M are the multi-linear maps from the Cartesian product set of s-times T^1M and r-times $\Omega^1(M)$ into $C^\infty(M)$ such that at each point $p \in M$ they correspond to the elements of the mixed tensor algebra of $T_{(p)s}^rM$. For each pair of (r, s) the set T_s^rM becomes a module. In particular higher order differential forms $\Omega^m(M)$ are the maps from the set of m Cartesian products of T^1M into $C^\infty(M)$ in a way that at each $p \in M$ they correspond to an element of the m-th exterior power space of T_p^*M [42].

An affine connection on a C^∞-manifold M is a rule which assigns to each $X \in T^1M$ a linear map $\nabla_X : T^1M \longrightarrow T^1M$ such that

$$\nabla_{(fX+gY)} = f\nabla_X + g\nabla_Y,$$

$$\nabla_X(fY) = f(\nabla_X Y) + X(f)Y, \tag{3.1}$$

$\forall f, g \in C^\infty(M)$ and $\forall X, Y \in T^1M$. For a smooth map $\phi : M \longrightarrow N$ the push-forward (differential) map $\phi_* : T_pM \longrightarrow T_{\phi(p)}N$ at a point $p \in M$ is defined as $\phi_*x(f) = x(f \circ \phi)$, $\forall f \in C^\infty(N)$ and $\forall x \in T_pM$. If ϕ is a diffeomorphism then ϕ_* maps T^1M into T^1N. Now if ∇ is an affine connection of a C^∞-manifold

M and ϕ is a diffeomorphism of M onto itself then ∇' which can be defined as $\nabla'_X(Y) = \phi_*^{-1}(\nabla_{\phi_* X}(\phi_* Y))$, $\forall X, Y \in T^1M$ is also an affine connection and if $\nabla' = \nabla$ then ϕ is called an affine transformation [44]. An affine connection ∇ always induces an affine connection ∇_U on an open submanifold U of M as its restriction on U.

Now consider a C^∞-manifold M with an affine connection ∇. Let $\gamma : (-\epsilon, \epsilon) \subset \mathbb{R} \longrightarrow \gamma(t) \subset M$ be a curve, if we consider any family of tangent vectors $Z(t) \in T^1\gamma(t)$ on $\gamma(t)$ which is a restriction $Y|_{\gamma(t)} = Z(t)$ for some $Y \in T^1M$ also the family $X(t) = \gamma_*(\frac{d}{dt})|_t \in T^1\gamma(t)$ (where $\frac{d}{dt} \in T_{t'}\mathbb{R}$ is the tangent vector at $t' \in (-\epsilon, \epsilon)$ which corresponds to the ordinary derivation on the smooth functions on $(-\epsilon, \epsilon)$), there always exist [44] $X \in T^1M$ such that $X|_{\gamma(t)} = X(t)$. If $\nabla_X Y|_{\gamma(t)} = 0$ then the family $Z(t)$ is said to be parallel with respect to the curve γ. If in particular $\nabla_X X|_{\gamma(t)} = 0$ then the curve γ is called a geodesic. If a geodesic is not a proper restriction of any other geodesic then it is called maximal. For every point $p \in M$ if $X \in T_pM$ and $X \neq 0$ then there exists a maximal geodesic γ_X such that $\gamma_X(0) = p$ and $\gamma_{X*}(\frac{d}{dt})|_{t=0} = X$ [44]. We also assume that $\gamma_X(t) = p$, $\forall t \in (-\epsilon, \epsilon)$ when $X = 0$. The map $Exp : T_pM \longrightarrow M$, such that $X \longrightarrow \gamma_X(1)$ is called the exponential map at $p \in M$. Since T_pM is isomorphic to $\mathbb{R}^{dimM}$ it can always be equipped with the differentiable structure of $\mathbb{R}^{dimM}$ and there always exist the so called normal neighborhoods N_0 and N_p of $0 \in T_pM$ and $p \in M$ respectively which are open and on which the exponential map Exp is a diffeomorphism ($N_p = ExpN_0$). If a basis $\{X_i \in T_pM\}$ for $i = 1, ..., dimM$ is specified for T_pM then the coordinate maps $x^i(p') : (p' = Exp(x^i X_i) \in M) \longrightarrow$

$((x^i) \in \mathbb{R}^{dimM})$ of $p' \in N_p$ are called the normal coordinates of the point p'. In the previous chapter we have defined the maps $T' : T^1M \times T^1M \longrightarrow T^1M$ and $R' : T^1M \times T^1M \longrightarrow Hom(T^1M, T^1M)$ as

$$T'(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R'(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \tag{3.2}$$

where M is a C^∞-manifold, we have also introduced the torsion tensor $T \in T^1_2M$ and the curvature tensor $R \in T^1_3M$ as

$$T : (\omega, X, Y) \longrightarrow \omega(T'(X,Y)),$$

$$R : (\omega, Z, X, Y) \longrightarrow \omega(R'(X,Y) \cdot Z), \tag{3.3}$$

$\forall X, Y, Z \in T^1M$ and $\omega \in \omega^1(M)$. The torsion and the curvature tensors will completely characterize the affine locally symmetric spaces which we will introduce after we make a further definition of the covariant derivative. Now if $\gamma : (-\epsilon, \epsilon) \in \mathbb{R} \longrightarrow M$ is a curve in a C^∞-manifold M and if (U, ψ) is a local chart with the coordinate functions x^i such that $\gamma(t) \in U$ then there exists m= $dimM$ differentiable functions $\varphi_i : ([a,b] \subset (-\epsilon, \epsilon)) \times \mathbb{R}^{dimM} \longrightarrow M$ such that for each m-tuple $(y_1, ..., y_m)$, $Y^i(t) = \varphi_i(t, y_1, ..., y_m)$ satisfy

$$\frac{dY^k}{dt} + \Gamma^k_{ij} \frac{dx^i(t)}{dt} Y^j = 0,$$

$$Y^i(a) = y_i, \tag{3.4}$$

where we have defined $x^i(t) = x^i(\gamma(t))$ and $(\nabla_U)_{\partial/\partial x^i}(\frac{\partial}{\partial x^j}) = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$ [44]. Here $\{\frac{\partial}{\partial x^i}\}$ for $i = 1, ..., dimM$ is the set of basis vector fields on U induced by the chart (U, ψ) as a result of applying the differential map ψ_*^{-1} on a standard basis of $T_{\psi(U)}\mathbb{R}^{dimM}$ which corresponds to the partial derivative operators in $T_{\psi(U)}\mathbb{R}^{dimM}$ [42]. If we define the vector field $Y(t) = Y^i(t)\frac{\partial}{\partial x^i}|_{\gamma(t)}$ which is parallel with respect to γ then the map $\tau : Y(a) \longrightarrow Y(b)$ where $Y(a) = y^i\frac{\partial}{\partial x^i}|_{\gamma(a)}$ for the previously chosen m-tuple $(y_1, ..., y_m)$ is an isomorphism from $T_{\gamma(a)}M$ onto $T_{\gamma(b)}M$ for $a, b \in (-\epsilon, \epsilon)$. The isomorphism τ is called the parallel translation with respect to the curve γ.

We will define the covariant differentiation for the general tensor fields. An integral curve of a vector field $X \in T^1M$ at a point $p \in M$ is a curve $\varphi : t \longrightarrow \varphi(t)$ such that $\varphi_*\frac{d}{dt}|_t = X|_{\varphi(t)}$ and $\varphi(0) = p$. If $X_p \neq 0$ there always exists an integral curve through p. The covariant differentiation ∇_X of a general tensor field $T \in T^r_sM$ with respect to $X \in T^1M$ at a point $p \in M$ is defined as

$$(\nabla_X T)_p = \lim_{s \longrightarrow 0}\frac{1}{s}(\tau^{-1}\cdot T_{\varphi(s)} - T_p), \tag{3.5}$$

where φ is an integral curve of X at the point p. We define the parallel translation of T at $p \in M$ as $\tau \cdot T_p(F_1, .., F_r, A_1, .., A_s) = T_p(\tau^{-1}F_1,.., \tau^{-1}F_r,\tau^{-1}A_1, .., \tau^{-1}A_s)$, $\forall F_i \in \Omega^1_q(M)$, $\forall A_j \in T_qM$ where $\tau \cdot T_p \in T^r_{s\,q}M$ and we have defined τF_i for $F_i \in \Omega^1_q(M)$ as $\tau F_i(A) = F_i(\tau^{-1}(A))$, $\forall A \in T_qM$ also $\tau F_i \in T^*_qM$. We also define $(\nabla_X T)_p = 0$ when $X_p = 0$. The covariant differentiation ∇_X is a linear map of the module T^r_sM also it is a derivation of the associative tensor algebra $T(M)$ [44].

Let M be C^∞-manifold, a pseudo-Riemannian structure on M is a tensor field $g \in T_2^0 M$ such that $g(X, Y) = g(Y, X)$, $\forall X, Y \in T^1 M$ and $\forall p \in M$, g_p is a non-degenerate bilinear form on $T_p M \times T_p M$. Moreover if g_p is positive definite for all $p \in M$ then g is called a Riemannian structure. A topological space M is called connected if the only sets which are both open and closed in M are $\emptyset$ and M. A pseudo-Riemannian (Riemannian) manifold is a connected C^∞-manifold with a pseudo-Riemannian (Riemannian) structure. The pseudo-Riemannian (Riemannian) connection is the unique affine connection on a pseudo-Riemannian (Riemannian) manifold which satisfies

$$T = 0 \qquad , \qquad \nabla_Z(g) = 0, \tag{3.6}$$

$\forall Z \in T^1 M$. If M is an analytical manifold the smooth map $f \in C^\infty(M)$ is called analytical at $p \in M$ whenever there is a local chart (U, φ) with $p \in U$ such that the map $f \circ \varphi^{-1} : \mathbb{R}^{dim M} \longrightarrow \mathbb{R}$ is analytical at $\varphi(p)$. If f is analytical at every point of M then f is called an analytical function on M. A vector field $X \in T^1 M$ is called analytical if it maps analytical functions into analytical functions. Similarly a differential one-form $\omega \in \Omega^1(M)$ is said to be analytical if it maps analytical vector fields into the set of analytical functions. The definition may be generalized to the modules of general mixed tensor fields. An affine connection on M is called an analytical connection if for each $p \in M$, $\nabla_X Y$ is analytical at p whenever X, Y are analytical at $p \in M$ [44]. Finally if the pair (M, g) is analytical then M is called an analytical pseudo-Riemannian (Riemannian) manifold and in this case the pseudo-Riemannian (Riemannian) connection is analytical.

Let (M,g) and (N,h) be two Riemannian manifolds, a map $\varphi : M \longrightarrow N$ is called an isometry if it is a diffeomorphism of M onto N and $\varphi^* h = g$. We have used the pull-back map φ^* which is associated to a C^∞-map $\varphi : M \longrightarrow N$ and which is defined on the modules $\Omega^1(N)$ and $T^0_n N$. For the one-form $\omega \in \Omega^1(N)$, $\varphi^*\omega(X) = \omega(\varphi_* X)$ for each $X \in T^1 M$ where $\varphi^*\omega \in \Omega^1(M)$. Similarly if $\beta \in T^0_n N$ then it can be written as $\beta = (\sum_i A_{i_1,...,i_n}(p)\, \omega_{i_1} \otimes, ..., \otimes \omega_{i_n})$ where $\omega_{i_j} \in \Omega^1(N)$ and we define $\varphi^*\beta = \sum_i A_{i_1,...,i_n}(p)\, \varphi^*\omega_{i_1} \otimes, ..., \otimes \varphi^*\omega_{i_n})$. The isometry definition can be extended for the subsets of the manifolds as well.

Now that we have covered the outline of the basic notions about the Riemannian geometry we are at a stage to introduce the definitions about the symmetric spaces. Let (M, ∇) be a C^∞-manifold with an affine connection and consider the normal neighborhood N_p of a point $p \in M$, there exists a geodesic $\gamma : t \longrightarrow \gamma(t)$ for each $q \in N_p$ such that $\gamma(0) = p$ and $\gamma(1) = q$ [44]. If we let $q' = \gamma(-1)$ then the map $s_p : q \longrightarrow q'$ which is from N_p onto itself is a diffeomorphism and it is called the geodesic symmetry with respect to $p \in M$. In normal coordinates defined before $s_p : (x_1, ..., x_{dimM}) \longrightarrow (-x_1, ..., -x_{dimM})$. Also the induced linear map is $(s_{p*})_p = -Id|_p$. Now if for each point $p \in M$ there is an open neighborhood N'_p where s_p is an affine transformation then M is called an affine locally symmetric space. M is an affine locally symmetric space if and only if $T = 0$ and $\nabla_Z R = 0$, $\forall Z \in T^1 M$. On the other hand if M is a Riemannian manifold with an affine connection ∇ and if there is a normal neighborhood N'_p for each point $p \in M$ such that s_p is an isometry on it then M is called a Riemannian locally symmetric space. A Riemannian locally symmetric space is always an analytical

Riemannian manifold.

Finally we will give the definition of the Riemannian globally symmetric spaces which we will refer to as the symmetric spaces. Let M be an analytical Riemannian manifold, M is called a Riemannian globally symmetric space if for each point $p \in M$, p is an isolated fixed point of an involutive isometry S_p of M ($S_p^2 = I$ and $S_p \neq I$, also an isolated fixed point means that there exists a neighborhood of p on which p is the only point such that $S_p(p) = p$). If M is a Riemannian globally symmetric space then S_p is unique for each $p \in M$ and there exists a normal neighborhood N'_p such that $S_p|_{N'_p} = s_p|_{N'_p}$ for some affine connection ∇ that is assigned on M. If M is a Riemannian globally symmetric space then M is a Riemannian locally symmetric space for the assigned affine connection ∇. A more detailed and formal discussion of the symmetric spaces can be found in [44].

3.2 Analytical Lie Groups and Symmetric Pairs

An analytical Lie group G is a group and an analytical manifold for which the maps

$$\begin{aligned} \mu : G \times G &\longrightarrow G, & \gamma : G &\longrightarrow G, \\ (g_1, g_2) &\longrightarrow g_1 g_2, & g &\longrightarrow g^{-1}, \end{aligned} \tag{3.7}$$

are smooth maps where $G \times G$ is the Cartesian product manifold. The right and the left translations

$$\begin{aligned} r_g : G &\longrightarrow G, & l_g : G &\longrightarrow G, \\ g' &\longrightarrow g'g, & g' &\longrightarrow gg', \end{aligned} \tag{3.8}$$

respectively which are defined $\forall g \in G$ are diffeomorphisms of G. If a group G is a real smooth manifold and if it satisfies the conditions in (3.7) it is called a real Lie group and it has an analytical structure [53], therefore every real Lie group is analytical. Despite of this fact we will keep on referring the real Lie group G as an analytical Lie group to emphasize on its analytical structure. If a vector field $X \in T^1G$ satisfies $l_{g*}X = X,\ \forall g \in G$ in other words $l_{g*}(X_{g'}) = X_{gg'}$ then X is called a left-invariant vector field. On the other hand if it satisfies $r_{g*}X = X,\ \forall g \in G$ such that $r_{g*}(X_{g'}) = X_{g'g}$ then it is called a right-invariant vector field. The left-invariant vector fields $L(G)$ and the right-invariant vector fields $R(G)$ form $dimG$-dimensional $\mathbb{R}$ vector spaces and under the map $X, Y \longrightarrow [X, Y] = (XY - YX),\ \forall X, Y \in T^1G$ they become a Lie algebra. Here we define $XY(F) = X(Y(F)),\ \forall F \in C^\infty(G)$. A Lie algebra g_0 is a real vector space equipped with a binary operation $[\ ,\ ]$ which is linear for both of its entries and which satisfies the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y[Z, X]] = 0, \tag{3.9}$$

also $[X, X] = 0,\ \forall X, Y, Z \in g_0$. The tangent space T_eG at the identity element of G is isomorphic to $L(G)$ and $R(G)$ as a $\mathbb{R}$ vector space. We can construct the isomorphism if for each $A \in T_eG$ we assign an $L^A \in L(G)$ such that $L^A_g = l_{g*}A,\ \forall g \in G$. Similarly we can assign an $R^A \in R(G)$ for each $A \in T_eG$ such that $R^A_g = r_{g*}A,\ \forall g \in G$ to establish the isomorphism from T_eG to $R(G)$ [53]. These isomorphisms can be used to induce a Lie algebra structure over T_eG. We will refer to T_eG with a structure induced by $L(G)$ as the Lie algebra of the analytical Lie group G and we will denote it as g. An affine connection ∇ on G is said to be left-invariant or right-invariant correspondingly if the left translations l_g or the right translations r_g are affine transformations. For the left-invariant and the right-invariant affine connections on G the maximal geodesic γ_X at each point $g' \in G$ is an analytical homomorphism from $\mathbb{R}$ into G. There exists a unique analytical homomorphism θ_X from the additive group $\mathbb{R}$ into $G,\ \forall X \in T_eG$ such that $\theta(0) = e$ (which is obvious from the homomorphism map) and $\theta_*(d/dt)|_{t=0} = X$. The set of such analytical homomorphisms θ_X is called the set of one parameter subgroups of G [48] and there are one to one correspondences between the elements of the Lie algebra g of G, the one parameter subgroups of G and the maximal geodesics γ_X at the identity element e of G corresponding to the left-invariant affine connections. If we consider the maximal integral curves σ^{L^A} of the left-invariant vector fields L^A through the identity element e of G then $\sigma^{L^A} \equiv \theta_A \equiv \gamma_A$ indeed. The exponential map $Exp : g \longrightarrow G$ for a Lie group G is defined as usual $Exp : A \longrightarrow \gamma_A(1)$. The one parameter subgroups of G can be expressed by the use of the exponential map [53] namely $\sigma^{L^A}(t) = \theta_A(t) = \gamma_A(t) = Exp(tA)$. There are normal neighborhoods N_o and N_e of $0 \in g$ and $e \in G$ on which Exp becomes a diffeomorphism and the definition of the normal coordinates has already been given in the last section. The exponential map Exp coincides with the matrix exponential function for the matrix Lie groups or the matrix representations of the analytical Lie groups [49].

Let H be submanifold of an analytical Lie group G, if H is a subgroup of G in group theory sense or if it is homeomorphic to a topological group with a different group structure then H is itself a Lie group and it is called a Lie subgroup of G. A connected Lie subgroup of G is called an analytical subgroup of G. If H is a Lie subgroup of G then h is isomorphic (an algebra isomorphism ϕ is an $\mathbb{R}$-linear one-to-one map such that $\phi([X,Y]) = [\phi(X), \phi(Y)]$) to a Lie subalgebra of g. Conversely for each subalgebra k of g there is only one analytical subgroup of G such that its Lie algebra is isomorphic to k. If H is a subgroup of an analytical Lie group G and if it is a closed subset in G then H has a unique analytical structure and it is analytically a submanifold of G therefore it is a Lie subgroup of G. Now we will mention about the Cartan-Maurer forms which will be one of the basic tools of the analysis of the sigma models on the symmetric spaces. Similar to the invariant vector fields on an analytical Lie group G, if a differential n-form $\omega \in \Omega^n(G)$ is left-invariant, $l_g^*\omega = \omega$ then it is called a left-invariant n-form. Also if ω is invariant under the right translation pull-back map, $r_g^*\omega = \omega$ then it is called a right-invariant n-form. We will denote the set of left-invariant and the right-invariant n-forms as $L^n(G)$ and $R^n(G)$ respectively. If ω is both left and right-invariant then it is called a bi-invariant n-form. A bi-invariant n-form ω is always closed, $d\omega = 0$. The left and the right-invariant one-forms form vector spaces over $\mathbb{R}$ which we will denote as $L^*(G)$ and $R^*(G)$ and which are the dual spaces of $L(G)$ and $R(G)$ respectively. We should imply that $\omega(X) \in \mathbb{R}$, $\forall X \in L(G)$ and $\forall \omega \in L^*(G)$ or $\forall X \in R(G)$ and $\forall \omega \in R^*(G)$. If we choose a basis $\{X_i \mid X_i \in g\}$ for g then $\{L^{X_i}\}$ is a basis for $L(G)$ and the dual basis $\{\omega^i \in L^*(G) \mid \omega^i(L^{X_j}) = \delta^i_j\}$ satisfy the Cartan-Maurer equation [47]

$$d\omega^i = -\frac{1}{2}\sum_{j,k} C^i_{jk}\omega^j \wedge \omega^k, \tag{3.10}$$

where the structure constants $\{C^i_{jk}\}$ are defined as $[X_j, X_k] = C^i_{jk}X_i$. The exterior derivative d is an $\mathbb{R}$-linear map from $\Omega^n(G)$ into $\Omega^{n+1}(G)$ and the wedge product $\wedge$ is the alternating tensor product of two differential forms on G. The details of the exterior algebra of differential forms can be referred in [41].

The Cartan-Maurer form Ω is an $L(G)$-valued or a Lie algebra g-valued one-form, formally $\Omega \in L(G) \otimes L^*(G) \simeq g \otimes L^*(G)$. Alternatively the Cartan-Maurer form can also be considered as a map $\Omega(X) : T^1G \longrightarrow L(G),\ \forall X \in T^1G$. More specifically we can define the Cartan-Maurer form Ω as the $L(G)$-valued one-form such that at each point $g' \in G$ it is the map $\Omega_{g'}(v) : T_{g'}G \longrightarrow L(G),\ \forall v \in T_{g'}G$ such that $(\Omega_{g'}(v))_{g''} = l_{g''*}(l_{g'^{-1}*}v)$. The Cartan-Maurer form Ω is left-invariant when the maps l_{g*} and l^*_g are applied on different tensor product components of $\Omega = \sum_{i,j} A^i_j Y_i \otimes \beta^j$ where $\{Y_i\}$ is a basis for $L(G)$ and $\{\beta^j\}$ is a basis for $L^*(G)$. If we consider the action of Ω on the left-invariant vector field L^A then we get $\Omega(L^A) = L^A$. We can equivalently define $\widetilde{\Omega} \in g \otimes L^*(G)$, in this case $\widetilde{\Omega}_x = (l_{x^{-1}*})_x$ where $\widetilde{\Omega}_x : T_xG \longrightarrow T_eG$ is such that $\widetilde{\Omega}_x(v) = l_{x^{-1}*}v \in T_eG,\ \forall v \in T_xG$. Therefore $\widetilde{\Omega} = \omega^i \otimes X_i$ where $\{X_i\}$ is a basis for g and $\{\omega^i\}$ is the dual basis such that $\omega^i(L^{X_j}) = \delta^i_j$. Locally for each point $p \in G,\ \widetilde{\Omega}_p = \omega^i_p \otimes X_i$. Equivalently we have the exact definition of Ω as $\Omega = \omega^i \otimes L^{X_i}$. A Lie algebra valued left-invariant n-form ψ can be defined in a similar way ($\Psi \in g \otimes L^n G \simeq L(G) \otimes L^n G$). By using the Cartan-Maurer form Ω we define the $L(G)$-valued left-invariant two-form $[\Omega, \Omega] \in L(G) \otimes L^2 G$ as $[\Omega, \Omega]_x(v_1, v_2) = [\Omega_x(v_1), \Omega_x(v_2)], \ \forall v_1, v_2 \in T_x G$. Then one can show that [44] the Cartan-Maurer form satisfies the equation

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0. \tag{3.11}$$

The exterior derivative operator d acts on the $L^*(G)$ part of Ω. If we consider a differentiable map (a gauge map) $\varphi : M \longrightarrow G$ where M is a C^∞-manifold and G is an analytical Lie group then we can define the images of the Cartan-Maurer forms under the pull-back of φ as $\widetilde{\mathcal{G}} = \varphi^* \widetilde{\Omega} = \varphi^*(\omega^i) \otimes X_i$. Similarly $\mathcal{G} = \varphi^* \Omega = \varphi^*(\omega^i) \otimes L^{X_i}$. If G is a matrix Lie group or an $(n \times n)$ $\mathcal{F}$-matrix representation is used for G then $\mathcal{G} \in GL(n, \mathcal{F}) \otimes \Omega^1(M)$. In particular one may consider a representation in which $\mathcal{G}$ is an $n \times n$ matrix of one-forms of M. For a local chart (U, ψ) on M if we consider a matrix representation of G then for $\psi^{-1}(x) \in U$ and $x \in \mathbb{R}^{dimM}$ there exists representations on M such that the Cartan-Maurer forms $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ can be represented as

$$\mathcal{G} \simeq \widetilde{\mathcal{G}} \simeq L^{-1} dL, \tag{3.12}$$

where $L(p) = \varphi(p)$ is the matrix function on M, $\forall p \in U$ taking values in the matrix representation of G and if we consider the map $L(x) = \varphi \circ \psi^{-1}(x)$ on $\psi(U) \subset \mathbb{R}^{dimM}$ then

$$(dL)^j_i = \frac{\partial L^j_i}{\partial x^k} dx^k. \tag{3.13}$$

Owing to its local matrix definition $\mathcal{G}$ satisfies the Cartan-Maurer equation on $U \subset M$ [44]

$$d\mathcal{G} + \mathcal{G} \wedge \mathcal{G} = 0. \tag{3.14}$$

If we consider the identity map I on G itself we can obtain the local expression of Ω and $\widetilde{\Omega}$ on G as well

$$\Omega \simeq \widetilde{\Omega} \simeq L^{-1} dL, \tag{3.15}$$

where $L(x) = \psi^{-1}(x)$ in this case.

We will define the left action of a Lie group G on a smooth manifold M now. A group G is said to act on a set M from the left (right) if there is a homomorphism (anti-homomorphism) $\mu : G \longrightarrow Perm(M)$ where $Perm(M)$ is the group of bijections of M. Let M be a Hausdorff space and G a topological group, a homomorphism (anti-homomorphism) μ from G into the group of homeomorphisms of M is called a left (right) action of G on M if the map

$$G \times M \longrightarrow M,$$

$$(g, p) \longrightarrow \mu(g) \cdot p, \tag{3.16}$$

is a continuous map onto M. We assume that $G \times M$ has the Cartesian product topology. The group G in this case is called the topological transformation group of M. If G is an analytical Lie group and M is a smooth manifold and the map (3.16) is a differentiable map, also if μ is into the diffeomorphism group of M (which is the group of diffeomorphisms of M under the composition map) then G is called the Lie transformation group of M. We will use the notation $\mu(g) \cdot p \equiv gp$

for the left actions and $\mu(g) \cdot p \equiv pg$ for the right actions. If in general μ is one-to-one then it is an isomorphism and the left (right) action is called effective, also if for all $p \in M$, $\{g \in G \mid \mu(g) \cdot p = p\} = \{e\}$ then the left (right) action is said to be free. A free action is effective but the converse is not necessarily true. The left (right) action is called transitive if $\forall p, q \in M$ there exists a $g \in G$ such that $q = \mu(g) \cdot p$. There is a one-to-one correspondence between the left actions and the right actions of any set (a Hausdorff space or a smooth manifold) therefore we will use simply the action when it is not necessary to differ between the left and the right. At the same time G/H will correspond to both of the left and the right cosets unless they are not specified as left or right. If a Hausdorff space M has a homeomorphism group which allows any topological group G to act transitively on M then M is called a homogenous space. The set $O_p = \{q \in M \mid p = \mu(g) \cdot q$ for some $g \in G\}$ is called the orbit of M at the point p under the action of G. If we define the relation $p \sim q$ when $p = \mu(g) \cdot q$ for some $g \in G$ then $\sim$ is an equivalence relation on M [53] and $\{O_p\}$ are the equivalence classes in M. We will denote the set of all the orbits $\{O_p\}$ of M as M/G. The coset space M/G carries a quotient topology induced by the projection map of the equivalence classes in M [56]. If the action is transitive then M has a single orbit and M/G has a single point so it is trivial. The group G has an induced transitive action on each of the orbits M/G. The left cosets $G/H \equiv \{gH \mid \forall g \in G\}$, generated by a subgroup H of G are the orbits of the right action of H on G as $\mu(h) = gh$, $\forall g \in G$ and $\forall h \in H$ (if G is a topological group then H is considered with the relative topology). Similarly the right cosets $G/H \equiv \{Hg \mid \forall g \in G\}$ with respect to a subgroup H are the orbits of the left action $\mu(h) = hg$, $\forall g \in G$ and $\forall h \in H$ of H on G. The adjoint left action of G on itself namely $Ad_g : g' \longrightarrow gg'g^{-1}$, $\forall g,\, g' \in G$ has the orbits as the conjugacy classes. If H is a closed subgroup of a topological group G then G/H has a Hausdorff quotient topology. Moreover if H is a closed subgroup of an analytical Lie group G then G/H has a unique analytical structure. If we define the left action $\tau : G \longrightarrow G/H$ of a group G on the left cosets with respect to a subgroup H such that $\tau(g)(g'H) = gg'H$ then τ is a transitive action. When G is a topological group and H is its closed subgroup then G becomes a transitively acting topological transformation group of G/H under τ and G/H becomes a homogeneous space. If G is a Lie group and H is again one of its closed subgroups then G is a transitively acting Lie transformation group of G/H under τ and G/H is a homogeneous space. The subgroups $G_p \equiv \{g \in G \mid \mu(g) \cdot p = p\}$, $\forall p \in M$ of G are called the isotropy groups at the points p. For a transitive action of a Lie transformation group G on a smooth manifold M, G_p is always a closed subgroup of G for every point $p \in M$. Besides if G is a locally compact (there is a compact neighborhood of every point of G) topological group with a countable basis such that $G = \bigcup_n H_n$ where $\{H_n\}$ are closed subsets of G and if G is a transitively acting topological transformation group of a locally compact Hausdorff space M then G_p are closed subgroups of G for every point $p \in M$ therefore G/G_p is a Hausdorff space. As explained above G is acting transitively on this Hausdorff space G/G_p as a result G/G_p becomes a homogeneous space. Similarly if G is a transitively acting Lie group of a smooth manifold M then G/G_p has a unique analytical structure for any G_p and G again acts transitively on G/G_p thus G/G_p becomes a homogeneous space. Now let G act transitively from the left on a set M then for each point $p \in M$ the map

$$J_p : G/G_p \longrightarrow M,$$

$$gG_p \longrightarrow gp, \tag{3.17}$$

is a bijection. If G is a transitively acting topological transformation group of a locally compact Hausdorff space M and if G is locally compact with a countable basis then (3.17) becomes a homeomorphism. For the transitive action of a Lie group G on a smooth manifold M if (3.17) is a homeomorphism then it becomes a diffeomorphism. Thus when G is a locally compact (with a countable basis) transitive Lie transformation group of a locally compact smooth manifold M then (3.17) is a diffeomorphism. Also if (3.17) is a homeomorphism and M is connected then the identity component (a topological space splits into a countable number of disjoint maximally connected subsets; the connected component including the identity element of a topological group is called the identity component [54]) G_0 of G acts transitively on M.

Let M be a Riemannian manifold the set of isometries of M namely $I(M)$ forms a group under the composition map. We may equip $I(M)$ with a compact open topology such that for each pair of open and compact subsets C and U of M, if we define the open sets in $I(M)$ as $W(C, U) = \{g \in I(M) \mid g \cdot C \subset U\}$ then the collection of sets $\{W(C, U)\}$ defines a Hausdorff topology on $I(M)$. $I(M)$ has a countable basis of open sets such that each open set in $I(M)$ contains one of the basis elements. $I(M)$ becomes a topological group and it is the locally compact (with a countable basis) transitive topological transformation group of M. Every isotropy group of this action is a compact subgroup of $I(M)$ thus they are closed (every compact subset of a Hausdorff space is closed). By following the discussion above since $I(M)$ is a locally compact transitive topological transformation group of M with a countable basis of open sets and since M is a Riemannian manifold so that it is a locally compact Hausdorff space we conclude that for any $p \in M$, $I(M)/G_p$ with its Hausdorff quotient topology, is homeomorphic to M through (3.17). Moreover if M is a Riemannian globally symmetric space then $I(M)$ has a unique analytical structure compatible with its compact open topology (a topological group may have at most one analytical structure which turns it into a Lie group) and $I(M)$ becomes the transitive Lie transformation group in this case. Therefore when M is a Riemannian globally symmetric space since (3.17) is a homeomorphism for the Lie transformation group as mentioned above it is a diffeomorphism for the unique analytical structure of $I(M)/G_p$. Also since the Riemannian manifold M is connected the identity component $I_0(M)$ of $I(M)$ acts transitively on M and $I_0(M)/G_p$ is also diffeomorphic to M.

We will give the definition of a symmetric pair now. Beforehand we will shortly mention about the adjoint representation of a Lie group. The set of endomorphisms namely the linear maps on a vector space a over the field $\mathbb{R}$ form a vector space with the addition and the scalar product induced from a. They also form a Lie algebra denoted as $gl(a)$ under the product $[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha$.

The non-singular (invertible) endomorphisms of a form an analytical Lie group which we will refer as $GL(a)$. Naturally $gl(a)$ is isomorphic to the Lie algebra of $GL(a)$. If a has a Lie algebra structure and if we assign the map $ad_X =[X,\ ]$, $\forall\, X \in a$ such that $ad_X \cdot Y = [X, Y],\ \forall\, Y \in a$ then ad_X is an endomorphism. The map $ad_g(a) \equiv ad_X : X \longrightarrow ad_X$ is an algebra homomorphism and it is called the adjoint representation of the algebra a. The image of ad_X in $gl(a)$ is a subalgebra and we will denote it as $ad(a)$. The kernel (the subalgebra of a which is mapped to the identity element of $gl(a)$) of $ad(a)$ is the center of a. The center z of an algebra a is the subset $z \subset a$ such that $[z, a] = 0$ and z is maximal in a. If one chooses a basis $\{x_i\}$ in a then each endomorphism in $gl(a)$ corresponds to a $\dim a \times \dim a$ real matrix which acts on the components of the elements of a with respect to the basis chosen. The analytical subgroup $Int(a)$ of $GL(a)$ whose Lie algebra is isomorphic to $ad(a)$ is called the adjoint group. When a is an algebra the set of automorphisms (algebra isomorphisms of a onto itself) $Aut(a)$ of a is a closed subgroup of $GL(a)$. The set of derivations $\partial(a)$ of a is a subalgebra of $gl(a)$ and it is isomorphic to the Lie algebra of $Aut(a)$. We should remind that a derivation D of an algebra a is an endomorphism of a satisfying $D(\alpha\beta) = \alpha D(\beta) + D(\alpha)\beta$. Moreover $ad(a)$ is a subalgebra of $\partial(a)$ and its elements are called the inner derivations of a also $Int(a)$ is a normal analytical subgroup of $Aut(a)$ which is an analytical subgroup of $gl(a)$ and the elements of $Int(a)$ are called the inner automorphisms of a.

When G is an analytical Lie group then the map $I(\sigma) : G \longrightarrow G \mid g \longrightarrow \sigma g \sigma^{-1},\ \forall g \in G$ and $\forall \sigma \in G$ is an analytical isomorphism of G onto itself.

Therefore $\forall \sigma \in G$ the map $Ad(\sigma) \equiv Ad_G(\sigma) = I_*(\sigma)|_e$ is an automorphism of g and the map $\sigma \longrightarrow Ad(\sigma)$ is an analytical homomorphism of G into $GL(g)$. This analytical homomorphism is called the adjoint representation of the group G. If g is a Lie algebra over $\mathbb{R}$ and t is its Lie subalgebra then $ad(t)$ is a subalgebra of $ad(g)$. The subalgebra t is said to be a compactly imbedded subalgebra of g if the analytical subgroup K^* of $Int(g)$ whose Lie algebra is isomorphic to $ad(t)$ is compact. A topological space X is called compact if every open covering $X = \bigcup_i V_i$ for the open sets $\{V_i\}$ has an open subcovering. A Lie algebra g is called compact if it can be compactly imbedded into itself in other words if $Int(g)$ is compact.

The pair (G, H) which is composed of a connected analytical Lie group G and one of its closed subgroups H is called a symmetric pair if there exists an involutive analytical automorphism σ of G (a group isomorphism onto itself which is an analytical map and whose square is the identity map) such that $(H_\sigma)_0 \subset H \subset H_\sigma$ where H_σ is the set of fixed points of σ which is a topological subgroup of G and $(H_\sigma)_0$ is the identity component of H_σ. If H is a closed subgroup of an analytical Lie group G then there is a unique analytical structure on H and it becomes an analytical Lie group itself. In addition to the above definition of the symmetric pairs if $Ad_G(\sigma)|_H$ is compact then (G, H) is called a Riemannian symmetric pair. If (G, H) is a Riemannian symmetric pair then the left cosets G/H with the quotient topology has a unique analytical structure [44] and for each G-invariant Riemannian structure on G/H (a Riemannian structure t on G/H is G-invariant if $\tau^*(g) \cdot t = t, \ \forall g \in G$) the left coset space G/H becomes a

Riemannian globally symmetric space. The Riemannian connection ∇ on G/H and the geodesic symmetry s_p are independent of the choice of the G-invariant structure t on G/H. Furthermore if G is semi-simple (we will give a rigorous definition of a semi-simple Lie algebra and a Lie group in the next section) and if it acts effectively on G/H then there is an analytical isomorphism between $I_0(G/H)$ and G for all the G-invariant Riemannian structures on G/H.

Similar to the pair of Lie groups, (g, s) is called an orthogonal symmetric Lie algebra if g is a Lie algebra over $\mathbb{R}$ and if s is an involutive automorphism of g, $(s \in Aut(g))$ whose set of fixed points $t = \{u \in g \mid su = u\}$ is a compactly imbedded subalgebra of g. If in addition $t \cap z = \{0\}$ where z is the center of g then (g, s) is called an effective orthogonal symmetric Lie algebra. A pair (G, H) is said to be associated to (g, s) if G is a connected analytical Lie group with a Lie algebra isomorphic to g and if H is a Lie subgroup of G whose Lie algebra is isomorphic to t. We should stress on the point that an associated pair does not have to be a symmetric pair. On the other hand if (G, H) is an associated pair and if H is closed and connected then G/H is a Riemannian locally symmetric space for all the G-invariant Riemannian structures, also it has a unique analytical structure, in addition if G is simply connected (from the image of every loop $f : [0, 1] \in \mathbb{R} \longrightarrow G$ in G there is a continuous map to a point p of G) then (G, H) is a Riemannian symmetric pair. An involutive automorphism of an algebra has two eigenspaces corresponding to the two possible eigenvalues ± 1 respectively. Namely t which is introduced above is the eigenspace with eigenvalue 1. Thus in an orthogonal symmetric Lie algebra (g, s) the Lie algebra g can be written as a vector space direct sum of these two eigenspaces. In an effective orthogonal symmetric Lie algebra (g, s) if g is semi-simple and compact (non-compact) then (g, s) is called a compact (non-compact) type. Accordingly the associated pair is also called a compact (non-compact) type. A left or a right-ideal b of an algebra a is a vector subspace such that $ba \subset b$ or $ab \subset b$ respectively. If b is both a left and a right-ideal then it is simply called an ideal. For a Lie algebra every left-ideal is a right-ideal and the ideals are subalgebras. For an effective orthogonal symmetric Lie algebra (g, s) if the eigenspace of s which corresponds to the eigenvalue -1 is an Abelian ideal then (g, s) is called an Euclidean type. For a Riemannian globally symmetric space M the pair $(I_0(M), K_p)$ is a Riemannian symmetric pair where K_p is the isotropy group of any point p in M. If we define the map $\varepsilon : \gamma \longrightarrow S_p\gamma S_p$ which is an involutive automorphism of $I_0(M)$ the differential map θ of ε at the identity element e of $I_0(M)$ so as to say $\theta : g_0 \longrightarrow g_0$ is an involutive automorphism of g_0 the Lie algebra of $I_0(M)$. The pair (g_0, θ) is associated to $(I_0(M), K_p)$ and it is an orthogonal symmetric Lie algebra. Let M and M' be two Riemannian globally symmetric spaces such that $I(M)$ and $I(M')$ have the same Lie algebra g. Since $I_0(M)$ and $I_0(M')$ are the Lie subgroups of $I(M)$ and $I(M')$ respectively their Lie algebras g_0 are subalgebras of g and they are isomorphic to each other. If g_0 is non-compact then $(I_0(M), K_p)$ is of non-compact type. In this case M and M' are isometric to each other (where the isometry corresponds to an analytical diffeomorphism whose push-forward map preserves the Riemannian structures on M and M').

3.3 The Semi-Simple Lie Algebras and the Iwasawa Decomposition

In this section we will discuss the general framework of the semi-simple Lie algebras and we will construct the Cartan and the Iwasawa decompositions of the semi-simple real Lie algebras.

A field $\mathcal{F}$ is said to be characteristic p if there is a prime number p such that $a + \ldots + a = 0,\ \forall a \in \mathcal{F}$ where there are p-terms in the sum. If there is no such degree p then $\mathcal{F}$ is called a characteristic zero field. If g is a Lie algebra we will denote the Lie algebra product space as $g \times g$ which is a Cartesian product set and a direct sum vector space, the algebra product is $[(X, Y), (V, Z)] = ([X, V], [Y, Z])$, $\forall\,(X, Y)$ and $(V, Z) \in g \times g$. The subsets $(0, X)$ and $(X, 0)$, $\forall\, X \in g$ are ideals in $g \times g$. If g is a Lie algebra over a characteristic zero field then the symmetric bilinear form

$$B(X, Y) = tr(adX\ adY), \tag{3.18}$$

$\forall X, Y \in g$ is called the Killing form. The trace in (3.18) is over the matrix representation which is based on a basis which we have mentioned before. For each $\sigma \in Aut(g)$, $B(\sigma X, \sigma Y) = B(X, Y)$ and if a is an ideal in g then the restriction of B on a is the Killing form on $a \times a$. A Lie algebra over a characteristic zero field is called semi-simple if its Killing form B is non-degenerate (such that $B(X, X) = 0$ if and only if $X = 0$) equivalently if it contains no Abelian ideals except $\{0\}$. If a Lie algebra $g \neq \{0\}$ is semi-simple and if it has no ideals except $g = \{0\}$ or g then it is called simple. A Lie group is called semi-simple (simple) if its Lie algebra is semi-simple (simple). The center of a semi-simple Lie algebra is $\{0\}$. An algebra A is said to be a direct sum of two algebras B and C if it is a direct sum vector space, $A = B \oplus C$ and $BC = CB = 0$. If A is a direct sum of its subalgebras then it is called reducible and the subalgebras are called the components. The components of a reducible algebra are ideals. A semi-simple Lie algebra is reducible and its components are simple ideals, this is an essential result predicting that the classification of the semi-simple Lie algebras can be based on the classification of the simple Lie algebras. For a semi-simple Lie algebra g every derivation is an inner derivation that is to say $ad(g) = \partial(g)$ also the adjoint group $Int(g)$ is the identity component of $Aut(g)$, it is a closed topological subgroup of $Aut(g)$ thus $Int(g)$ has a unique analytical structure and it is an analytical Lie group. A semi-simple Lie algebra g is compact if and only if the Killing form B is strictly negative definite. On the other hand a Lie algebra g over $\mathbb{R}$ is compact if and only if there is a compact Lie group whose Lie algebra is isomorphic to g.

The nilpotent and the solvable Lie algebras will be the building blocks of our analysis of the semi-simple real Lie algebras. An endomorphism $N \in Hom(V, V)$ of a vector space V over a field $\mathcal{F}$ is called nilpotent if $N^k = 0$ for some $k \in \mathbb{Z}$. There always exists a basis of V such that the matrix representative of a nilpotent endomorphism N which is defined by the action of N on the basis of V has null entries on and below the diagonal. Conversely if an endomorphism has such a matrix representative then it is nilpotent. Let P be a subset of $Hom(V, V)$, then a vector subspace W of V is called invariant if $SW \subset W,\ \forall S \in P$. The set P is called semi-simple if each invariant subspace W of it has an invariant complement in V. Each $A \in Hom(V,V)$ can be uniquely written as $A = N + S$ where the addition belongs to the Lie algebra structure $gl(V)$ of $Hom(V,V)$,N is nilpotent and the complement S is semi-simple. A Lie algebra a is called nilpotent if $\forall Z \in a$, $ad_a Z$ is nilpotent. Relatively a Lie group is called nilpotent if its Lie algebra is nilpotent. If $\mathcal{K}$ is a characteristic zero field and a is a vector space and a Lie algebra on $\mathcal{K}$ then the set $\mathcal{D}a = \{[X,Y]\}$ which is generated by all the elements $X, Y \in a$ is an ideal of a. Since an ideal is a subalgebra $\mathcal{D}a = [a,a]$ is called the derived algebra of a. The higher order derived algebras are defined inductively in the same way over one less rank derived algebra and they are denoted as $\mathcal{D}^n a$ where $\mathcal{D}^n a = \mathcal{D}(\mathcal{D}^{n-1}a)$ and $\mathcal{D}^0 a = a$. A Lie algebra is called solvable if there exists an integer $n \geq 0$ such that $\mathcal{D}^n\, a = \{0\}$. A Lie group is called solvable if its Lie algebra is solvable. A nilpotent Lie algebra is always solvable. If a Lie algebra is solvable then it is not semi-simple [44]. A Lie algebra is solvable if and only if it satisfies the chain condition $g_0 \supset g_1 \supset ...g_{n-1} \supset g_n = \{0\}$ where g_i is an ideal of g_{i-1} with $dimg_{i-1} - dimg_i = 1$. If g is a solvable Lie algebra over a field $\mathcal{K}$ and if $V \neq \{0\}$ is an m-dimensional $\mathcal{K}$-vector space then there exists a representation $\pi : g \longrightarrow gl(V)$ such that each element of g is represented by upper triangular $m \times m$ matrices of $\mathcal{K}$ when a basis is specified in V. A similar case occurs for a nilpotent Lie algebra n so that each element of n is represented by $m \times m$ $\mathcal{K}$ matrices with null entries on and below the diagonal. Similar to the derived algebras of a Lie algebra g if we define the ideals $\varphi^{p+1}g = [g, \varphi^p g]$ where $\varphi^0 g = g$ then the series $\varphi^0 g \supset \varphi^1 g \supset \varphi^2 \supset ...$ is called the central descending series. A Lie algebra g over $\mathcal{K}$ is nilpotent if and only if the central descending series terminates with $\varphi^m g = \{0\}$ for some $m \geq dim g$. A nilpotent Lie algebra $g \neq \{0\}$ has a non-zero center.

The next step will be the construction of the root subspace decomposition of a complex semi-simple Lie algebra. The Cartan subalgebra h of a semi-simple Lie algebra g over $\mathbb{C}$ is a maximal Abelian subalgebra of g (which is not a subalgebra of another Abelian subalgebra) such that $\forall H \in h,\ adH \in gl(g)$ is semi-simple. Each semi-simple Lie algebra g over $\mathbb{C}$ has a set of Cartan subalgebras and for any pair of Cartan subalgebras h_1, h_2 there is an automorphism $\sigma \in Aut(g)$ such that $\sigma(h_1) = h_2$.

We will construct the root space decomposition of g which is associated with a chosen Cartan subalgebra now. Let g be a semi-simple Lie algebra over $\mathbb{C}$ and let h be one of its Cartan subalgebras also let $\alpha \in h^*$, if we define the vector subspace

$$g^\alpha = \{X \in g \mid [H, X] = \alpha(H)X, \quad \forall H \in h\}, \tag{3.19}$$

of g then α is called a root with respect to h if $g^\alpha \neq \{0\}$. When α is a root then g^α is called the root subspace. If $\alpha = 0$ then $g^0 = h$ since h is a maximal Abelian subalgebra of g. We will denote the set of non-zero roots as Δ, formally

$$\Delta = \{\alpha \in h^* \mid g^\alpha \neq 0,\ \alpha \neq 0\}. \tag{3.20}$$

For a pair of roots α and β we have $[g^\alpha, g^\beta] \subset g^{\alpha+\beta}$. Also for each $\alpha \in \Delta$, $dim g^\alpha = 1$ and there is a unique $H_\alpha \in h$ such that $B(H, H_\alpha) = \alpha(H),\ \forall H \in h$. If $\alpha \in \Delta$ then $-\alpha \in \Delta$ too. Thus we have the identity $[g^\alpha, g^{-\alpha}] = \mathbb{C}H_\alpha$ for all $\alpha \in \Delta$. If $\alpha + \beta \neq 0$ for $\alpha, \beta \in \Delta$, $B(g^\alpha, g^\beta) = 0$ which we interpret as the orthogonality of the roots α and β, also $[g^\alpha, g^\beta] = g^{\alpha+\beta}$. The restriction of B on h is non-degenerate and $\forall H, H' \in h,\ B(H, H') = \sum_{\beta\in\Delta} \beta(H)\beta(H')$. The only roots which are proportional to α are $-\alpha$ and 0. We also define the product of two roots λ and μ as $< \lambda, \mu >= B(H_\lambda, H_\mu)$. After giving a summary of the properties of the roots we can write down the root space decomposition of the complex Lie algebra g as

$$g = h \oplus \sum_{\alpha\in\Delta} g^\alpha, \tag{3.21}$$

where the sum is a direct vector space sum. We may also define a subset of h as

$$h_{\mathbb{R}} = \sum_{\alpha\in\Delta} \mathbb{R} H_\alpha, \tag{3.22}$$

where $\mathbb{R}$ is the real subfield of $\mathbb{C}$. The restriction of B on $h_{\mathbb{R}}$ is in the $\mathbb{R}$ subfield of $\mathbb{C}$ and it is strictly positive definite. Another subset $ih_{\mathbb{R}}$ of h can be defined and the elements of h can uniquely be written as the decomposition

$$h = h_{\mathbb{R}} + ih_{\mathbb{R}}. \tag{3.23}$$

Before introducing the positive and the negative roots we should mention about the ordering of a vector space. In general a set M is called totally ordered if we define a relation $\{>\} \subset M \times M$ such that either $a > b$ or $b > a$ so as to say either $(a, b) \in \{>\}$ or $(b, a) \in \{>\}$ also if $a > b$ and $b > c$ then $a > c,\ \forall a, b, c \in M$. If $\{>\} \subset S \times S$ for some $S \subset M$ then M is called partially ordered. If V is a finite dimensional vector space over $\mathbb{R}$ then V is called an ordered vector space if V is an ordered set and $\{>\}$ satisfies

$$\begin{aligned} &\text{(i)} \qquad \text{if } X > 0 \text{ and } Y > 0 \text{ then } X + Y > 0, \quad \forall X, Y \in V, \\ &\text{(ii)} \qquad \text{if } X > 0 \text{ then } \ aX > 0, \quad \forall\, a > 0 \text{ and } \forall X \in V. \end{aligned} \tag{3.24}$$

The first condition (i) in (3.24) implies that $X > 0$ if and only if $0 > -X$. Let $\{X_i\}$ be a basis for an n-dimensional real vector space V, we will define a special relation $\{>\}$, $\forall X, Y \in V$ such that $X > Y$ when the first non-zero component A^m in the expansion $X - Y = \sum_i A^i X_i$ is a positive real number. With this relation V becomes an ordered vector space and $\{>\}$ is called the lexicographic ordering with respect to the basis $\{X_i\}$. One can also put a lexicographic ordering on the dual space V^* of V as follows; if $\mu, \lambda \in V^*$ then $\lambda > \mu$ when the first non-zero number in the sequence $(\lambda(X_1) - \mu(X_1)), ..., (\lambda(X_n) - \mu(X_n))$ is positive. The dual space V^* turns into an ordered vector space with this ordering. An element $\lambda \in V^*$ is called positive if $\lambda > 0$ and negative if $0 > \lambda$, this is a legitimate definition since according to the definition of an ordering they can not be valid at the same time.

We will define the realization of a complex vector space in order to be able to induce an ordering on the nonzero roots of a semi-simple complex Lie algebra. If V is an even dimensional vector space over $\mathbb{R}$ then a complex structure J over V is an endomorphism, $J \in Hom(V, V)$ such that $J \cdot J = -I$ where I is the identity map on V. An even dimensional real vector space V can be turned into a complex vector space $\widetilde{V}$ by keeping the addition rule and by defining the complex scalar product as $(a+ib)X = aX + bJX,\ \forall(a+ib) \in \mathbb{C}$ and $\forall\, X \in V$. We have the dimensional relation $dim_{\mathbb{C}}\widetilde{V} = \frac{1}{2}dim_{\mathbb{R}}V$. Set theoretically two spaces are the same $V = \widetilde{V}$. Conversely if E is a vector space over $\mathbb{C}$ we define the real vector space E^R which is called the realization of E by keeping the addition rule and by identifying the complex structure $J \cdot J = -I$ from the complex scalar product $(a+ib)X = aX + bJX,\ \forall(a+ib) \in \mathbb{C}$ and $\forall\, X \in E$ so that we can define the real product $aX,\ \forall\, a \in \mathbb{R}$ and $\forall\, X \in E^R$. Clearly we have $\widetilde{E^R} = E$. A Lie algebra g over $\mathbb{R}$ is said to have a complex structure J if J is a complex structure on the vector space g and if it satisfies

$$[X, JY] = J[X, Y], \tag{3.25}$$

$\forall X, Y \in g$. The complex vector space $\widetilde{g}$ becomes a complex Lie algebra by inheriting the real Lie algebra product $[\ ,\ ]$ from g. On the other hand if E is a complex Lie algebra the real vector space E^R, constructed by the complex structure J of E which obeys (3.25) becomes a real Lie algebra by inheriting the algebra product of E.

If we return to our discussion about the roots now we can make use of the realization $(h_{\mathbb{R}})^R$ for ordering the roots in (3.20). We consider the image of $h_{\mathbb{R}}$ through the realization g^R of g. We should emphasize on the point that Δ and $(h_{\mathbb{R}})^R$ completely specify the complex Lie algebras since a real vector space isomorphism of $(h_{\mathbb{R}})^R$s' of two complex Lie algebras can always be extended to a $\mathbb{C}$ Lie algebra isomorphism [44]. Basically $h_{\mathbb{R}}$ is a not a complex vector subspace but a subset of h, under the realization h^R of h the image $(h_{\mathbb{R}})^R$ becomes a real vector subspace of h^R. The action of the roots $\alpha \in \Delta$ on $h_{\mathbb{R}}$ have real values so as to say; when $\alpha(X) = a$ the complex number a is an element of the real subfield of $\mathbb{C}$ which is isomorphic to $\mathbb{R}$ for all the roots $\alpha \in \Delta$ and for all $X \in h_{\mathbb{R}}$. We have to make two observations; the first one is since $h_{\mathbb{R}}$ is a subset of h the restriction of the roots $\alpha \in \Delta$ on $h_{\mathbb{R}}$ are elements of the set of $\mathbb{R}$-valued and $\mathbb{R}$-linear maps on $h_{\mathbb{R}}$ where $\mathbb{R}$ must be again taken as the real subfield of $\mathbb{C}$, secondly set theoretically the set of $\mathbb{R}$-valued and $\mathbb{R}$-linear maps on $h_{\mathbb{R}}$ is equal to the dual space $(h^*_{\mathbb{R}})^R$. Therefore there is a one to one correspondence between the roots $\alpha \in \Delta$ and the realizations of the restricted roots which we will refer as Δ_{res}. Obviously if we put a lexicographic ordering on the real vector space $(h_{\mathbb{R}})^R$ we have the induced ordering on $(h^*_{\mathbb{R}})^R$ then we obtain the ordering of the realization correspondents of the restrictions of the roots into the set of $\mathbb{R}$-valued and $\mathbb{R}$-linear maps on $h_{\mathbb{R}}$. Thus indirectly we order the non-zero roots $\alpha \in \Delta$ of g which are in one to one correspondence with their restrictions on $h_{\mathbb{R}}$. We point out that the analysis is with respect to a chosen Cartan subalgebra h of g. According to this ordering if $\alpha > 0$ then α will be called a positive root and if $0 > \alpha$ then it is called a negative root. The set of positive roots will be denoted as Δ^+ and the set of negative roots will be referred as Δ^-. Thus $\Delta = \Delta^+ \cup \Delta^-$. Since if $\alpha \in \Delta$ then $-\alpha \in \Delta$, from the definition of the vector space ordering we can predict that if α is positive then $-\alpha$ is negative. Therefore when $\alpha \in \Delta^+$ we also have $-\alpha \in \Delta^-$ so the roots form positive-negative pairs as $(\beta, -\beta)$ and if we represent the positive roots as $\{\beta\}$ the negative roots may be represented as

$\{-\beta\}$. For each $\alpha \in \Delta$ a set of vectors $\{E_\alpha \in g^\alpha\}$ can be chosen such that [44]

$$[E_\alpha, E_{-\alpha}] = H_\alpha \quad , \quad [H, E_\alpha] = \alpha(H)E_\alpha, \quad \forall H \in h,$$

$$[E_\alpha, E_\beta] = 0, \quad \text{if } \alpha+\beta \neq 0 \text{ and if } \alpha+\beta \notin \Delta,$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}, \quad \text{if } \alpha+\beta \in \Delta, \tag{3.26}$$

$\forall \alpha, \beta \in \Delta$. In (3.26), $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Since $dimg^\alpha = 1$ each of the vectors $\{E_\alpha\}$ form a basis for the corresponding root subspaces $\{g^\alpha\}$.

A positive root $\alpha \in \Delta^+$ is called a simple root if it can not be written as a sum $\alpha = \beta + \gamma$ of any other two positive roots $\beta, \gamma \in \Delta^+$. If $\alpha \neq \beta$ are two simple roots then $\beta - \alpha \notin \Delta$ and $B(H_\alpha, H_\beta) \leq 0$. The roots $\alpha \in \Delta$ are not all linearly independent [49] but if $\{\alpha_1, \alpha_2, ..., \alpha_l\} \subset \Delta^+$ are the simple roots then $\{H_{\alpha_1}, H_{\alpha_2}, ..., H_{\alpha_l}\}$ form a basis for $(h_\mathbb{R})^R$ in the real sense and a basis for h in the complex sense. Therefore from (3.21) we conclude that $\{H_{\alpha_1}, H_{\alpha_2}, ..., H_{\alpha_l}, E_\beta\}$ where $\beta \in \Delta$ form a basis for the complex semi-simple Lie algebra g. Here we assume that $\{E_\beta\}$ is the specific set of basis vectors introduced in (3.26). Evidently the set of simple roots $\{\alpha_1, \alpha_2, ..., \alpha_l\} \subset \Delta^+$ form a basis for h^*. If $\beta \in \Delta$ then in terms of the simple roots $\{\alpha_i\}$, β can be expanded as $\beta = \sum_i^l n^i \alpha_i$ where $n^i \in \mathbb{Z}$. The integer coefficients n^i are either all positive in which case β is a positive root or they are all negative then β is a negative root. Finally we should remark that the root space analysis of a semi-simple complex Lie algebra is not unique and it effectively depends on the particular Cartan subalgebra h, the basis chosen for the lexicographic ordering of the non-zero roots, the basis of (3.26) and the simple roots. However the basic notions and the functioning of the root space decomposition do not alter.

Another concept which will prove to be a basic element of our analysis of the symmetric spaces through the Iwasawa decomposition is the complexification of a real vector space. Let W be a finite dimensional vector space over $\mathbb{R}$, we can equip the Cartesian product set $W \times W$ with a real vector space structure by defining the scalar product as $a(X,Y) = (aX, aY)$ and the addition rule as $(X,Y) + (Z,T) = (X+Z, Y+T)$, $\forall\, a \in \mathbb{R}$ and $\forall\, X,Y,T,Z \in W$. Then $J : (X,Y) \longrightarrow (-Y,X)$ is a complex structure on $W \times W$. The complex vector space $\widetilde{W \times W}$ generated by using J is called the complexification of W and it is denoted as W^C. We have $dim_{\mathbb{C}} W^C = dim_{\mathbb{R}} W$. Since W^C is the set equivalent of $W \times W$ there are subsets in W^C whose equivalent subsets in $W \times W$ are isomorphic to W in the real sense. Each element (X,Y) of W^C can be uniquely written as $(X,Y) = (X,0) + J(Y,0) = (X,0) + i(Y,0)$, therefore we can use the representation $X+iY$ instead of (X,Y). In this representation $(a+ib)(X+iY) = aX - bY + i(aY + bX)$. Every finite dimensional complex vector space E is isomorphic to a complexification W^C of some real vector space W. In fact if $\{Z_i\}$ is a basis for E then we can take W to be the realization $(E_{\mathbb{R}})^R$ of $E_{\mathbb{R}} = \sum_i A^i Z_i$ where $\{A^i\}$ run over the elements of the subfield of $\mathbb{C}$ which is isomorphic to $\mathbb{R}$, more explicitly we can write $E \simeq ((E_{\mathbb{R}})^R)^C$. We have already encountered such a decomposition of a complex vector space in the example of h and $h_{\mathbb{R}}$. In general if $\{X_i\}$ is a basis for W in the real sense then $\{(X_i, 0)\}$ is a basis for W^C in the complex sense according to the complex structure J we have defined for $W \times W$ above. The subset $W^C_{\mathbb{R}} = \sum_i \mathbb{R}(X_i, 0)$ of W^C has an image $(W^C_{\mathbb{R}})^R$ which is a real vector subspace of $W \times W$ under the realization of W^C. Evidently $(W^C_{\mathbb{R}})^R$ is isomorphic to W. We will not generally differ between $(W^C_{\mathbb{R}})^R$ and the set equivalent $(W, 0)$ of W in $W \times W$ or in W^C. One can consider other isomorphic images of W in $W \times W$.

On the other hand if W is a Lie algebra over $\mathbb{R}$ with the Lie algebra product $[\,,]$ then we can define a Lie algebra structure over W^C as

$$[X + iY, Z + iT] = [X, Z] - [Y, T] + i([Y, Z] + [X, T]), \tag{3.27}$$

$\forall X, Y, T, Z \in W$. With the Lie algebra product in (3.27), W^C is called the complexification of the Lie algebra W. If W is a semi-simple Lie algebra then W^C is semi-simple too.

Let g be a Lie algebra over $\mathbb{C}$, a real form g_0 of g is a subalgebra of the real Lie algebra g^R such that

$$g^R = g_0 \oplus J g_0, \tag{3.28}$$

where the sum is a direct vector space sum and J is the complex structure on g^R identified through the definition of g^R which can also be thought as an $\mathbb{R}$-subfield linear endomorphism of g. Set theoretically $g = g^R \supset g_0$. As a result of (3.28) each element $Z \in g^R$ or each element of g can uniquely be written as $Z = X + iY = X + JY,\ \forall X, Y \in g_0$. In this case $g \simeq g_0^C$ that is to say g is the complexification of its real form g_0. The map $\sigma : g \longrightarrow g$, such that $\sigma(X + iY) = X - iY,\ \forall (X + iY) \in g$ where $X, Y \in g_o$ is called the conjugation of g with respect to g_0. The map σ is not an automorphism of g but of g^R when one considers an equivalent action of it on g^R which is the set equivalent of g. It satisfies the rules; $\sigma(\sigma(X)) = X,\ \sigma(X+Y) = \sigma(X) + \sigma(Y),\ \sigma(\alpha X) = \bar{\alpha}\sigma(X)$, $\sigma([X,Y]) = [\sigma(X), \sigma(Y)],\ \forall X, Y \in g$ and $\alpha \in \mathbb{C}$. If any onto map σ' on g satisfies these conditions then the set of its fixed points is a real form of g and σ' is the conjugation with respect to its fixed points. We should also observe that the subset ig_0 of g which is the set equivalent of the real vector subspace Jg_0 of g^R is an eigenspace of σ with the eigenvalue -1 and the subset g_0 of g is the eigenspace of σ with the eigenvalue $+1$. Since g_0 is a real subalgebra of g^R, $ad(g_0)$ is a subalgebra of $ad(g^R)$. If W is a real Lie algebra and W^C is its complexification then W is a real form of W^C. In general every image $W^C_{\mathbb{R}}$ of W in W^C whose definition is given above is a real form of W^C. The analytical subgroup G_0 of $Int(g^R)$ whose Lie algebra is isomorphic to $ad(g_0)$ is a closed subgroup and $G_0 \simeq Int(g_0)$. Every semi-simple Lie algebra over $\mathbb{C}$ has a set of compact real forms. We have come to a level at which we can give a formal definition of the Cartan decomposition of a semi-simple real Lie algebra. Let g_0 be a semi-simple real Lie algebra and let g be its complexification $(g = (g_0)^C)$, since g_0 is a real form of g we also assume that σ is the conjugation of g with respect to g_0. Basically the Cartan decomposition is to express g_0 as a direct sum of two components. These components are defined by using another real form of g. The vector space direct sum decomposition of the semi-simple real Lie algebra g_0

$$g_0 = t_0 \oplus p_0, \tag{3.29}$$

which is composed of a subalgebra t_0 and a vector subspace p_0 is called a Cartan decomposition if there exits a real form g_k of g such that

$$\sigma(g_k) \subset g_k \quad , \quad t_0 = g_0 \cap g_k \quad , \quad p_0 = g_0 \cap (ig_k). \tag{3.30}$$

We should imply that set theoretically $t_0, p_0 \subset g_0 \subset g$ and $g_k \subset g^R = g$. Each semi-simple Lie algebra over $\mathbb{R}$ has a set of inequivalent Cartan decompositions. For any two Cartan decompositions; $g_0 = t_1 \oplus p_1$ and $g_0 = t_2 \oplus p_2$ there exists an inner automorphism $\psi \in Int(g_0)$ such that $\psi(t_1) = t_2$ and $\psi(p_1) = p_2$ thus Cartan decompositions of a semi-simple real Lie algebra g_0 are adjoint under the elements of $Int(g_0)$. The subalgebra t_0 in (3.29) is a maximal compactly imbedded subalgebra of g_0. However this means neither that the analytical subgroup of $Int(g_0)$ whose Lie algebra is isomorphic to $ad(t_0)$ is maximally compact nor that the analytical subgroup of a Lie group G (we assume that the Lie algebra of G is g_0) whose Lie algebra is isomorphic to t_0 is maximally compact. It even does not mean that t_0 is a maximally compact subalgebra of g_0. The above maximal condition for t_0 simply states that there is no larger subalgebra of g_0 which contains t_0 and which is compactly imbedded in g_0. Furthermore every compactly imbedded subalgebra of a semi-simple real Lie algebra is a part of a Cartan decomposition.

An involutive automorphism $\theta \in Aut(g_0)$ of a semi-simple real Lie algebra g_0 is called a Cartan involution if the induced bilinear form $B_\theta(X,Y) = -B(X, \theta(Y))$ is strictly positive definite $\forall X, Y \in g_0$. Since according to the Cartan decomposition (3.29) for each element $Z \in g_0$ there is a unique decomposition

$Z = (T, X)$ where $T \in t_0$ and $X \in p_0$, we can define an involutive map $s_0 : (T, X) \longrightarrow (T, -X)$ of g_0. The involution s_0 is a Cartan involution. On the other hand every Cartan involution leads to a Cartan decomposition in which the involution can be expressed by means of the decomposition it induces in a similar way given above. If g is a complex Lie algebra and J is the complex structure which corresponds to multiplying by i and which is identified in defining g^R from g then $g^R = u \oplus Ju$ is a Cartan decomposition for any compact real form u of g.

For the Cartan decomposition (3.29), $B(t_0, p_0) = 0$ and essentially we have the following closures which contribute a major simplicity in the structure of the symmetric space sigma models we will introduce in the next chapter

$$[t_0, t_0] \subset t_0 \quad , \quad [t_0, p_0] \subset p_0 \quad , \quad [p_0, p_0] \subset t_0. \tag{3.31}$$

We should make the observation that t_0 is the set of fixed points of s_0. Furthermore $s_0(t_0) = t_0$ and $s_0(p_0) = -p_0$ so that (3.29) is nothing but the eigenspace decomposition of g_0 with respect to s_0 and the corresponding eigenvalues are ± 1. We see that (g_0, s_0) is an effective orthogonal symmetric Lie algebra [44]. In general for a non-compact type effective orthogonal Lie algebra (l, s) the eigenspace decomposition $l = u \oplus e$ is a Cartan decomposition where u and e are eigenspaces of s with the eigenvalues $+1$ and -1 respectively. We should also remark that in an eigenspace decomposition $l = u \oplus e$ of any involutive automorphism θ of any algebra l, the mixed elements do not belong to any eigenspace of θ.

When g_0 is a non-compact, semi-simple Lie algebra over $\mathbb{R}$, for any Cartan decomposition $g_0 = t_0 \oplus p_0$ the pair (g_0, s_0) is an orthogonal symmetric Lie algebra of non-compact type. If (G, K) is associated with (g_0, s_0) then it is of non-compact type too. Basically (G, K) is a Riemannian symmetric pair [44]. The involutive automorphism σ of the Riemannian symmetric pair (G, K) which is defined in section 3.2 is related to s_0 as $s_0 = \sigma_*|_e$. We have $B(s_0(X), s_0(Y)) = B(X, Y)$, $\forall X, Y \in g_0$ which is valid for any automorphism $\phi \in Aut(g_0)$. In the Cartan decomposition above the Killing form B is strictly negative definite on t_0 and strictly positive definite on p_0. The subgroup K is connected and closed therefore it is an analytical subgroup of G. The center Z of G is a subset of K. The analytical subgroup K is compact if and only if the center Z is finite if so then K is a maximal compact subgroup of G. By following the previous discussion about the Riemannian symmetric pairs we conclude that G/K is a Riemannian globally symmetric space for all the G-invariant Riemannian structures on G/K. An essential result which will provide a parametrization of the symmetric space G/K is that the restriction of the exponential map on p_0 namely $Exp : p_0 \longrightarrow G$ produces an induced difeomorphism from p_0 onto G/K since it maps the elements of p_0 on the representatives of the left cosets G/K in G.

Each Cartan decomposition of a non-compact, semi-simple real Lie algebra g_0 generates a Riemannian globally symmetric space G/K as described above. All of these spaces are isometric to each other. Since all the Cartan decompositions are conjugate to each other under the elements of $Int(g_0)$, the isometries can be constructed from these inner automorphisms.

In general if (G, K) is a Riemannian symmetric pair of non-compact type then K has a unique maximal compact subgroup K' and K' is maximally compact in G

as well. All maximal compact subgroups in a connected, semi-simple, analytical Lie group G are connected and they are conjugate under the inner automorphisms of G. If K' is any maximal compact subgroup of a connected, semi-simple Lie group G then there is a submanifold E of G which is difeomorphic to $\mathbb{R}^m$ for some m and the map $(a,k) \longrightarrow ak$ is a difeomorphism of $E \times K'$ onto G, $\forall\, a \in E$ and $\forall\, k \in K'$.

The Iwasawa decomposition combines the Cartan decomposition (3.29) and the root space decomposition (3.21) of a semi-simple real Lie algebra. Let $g_0 = t_0 \oplus p_0$ be a Cartan decomposition of a semi-simple real Lie algebra g_0, if the semi-simple complex Lie algebra $g = g_0^C$ is the complexification of g_0 then the set of elements u of g which is generated as

$$u = t_0 + ip_0, \tag{3.32}$$

through the complexification of g_0, is a compact real form of g whose conjugation will be denoted by τ. We should bear in mind that g_0 has the set equivalent images in $g = g_0^C$ whose realizations in $g_0 \times g_0$ are isomorphic to g_0. In this way t_0 and p_0 can be considered as subsets of g then u is a subset of g and it is also a subset of one of the images of g_0 in g. Thus under the realization of g, u^R corresponds to a subalgebra of g_0. The real semi-simple Lie algebra g_0 is also a real form of its complexification g so that we may define σ as the conjugation of g with respect to g_0. The map $\theta = \sigma \cdot \tau = \tau \cdot \sigma$ is an involutive automorphism of g. In fact θ is a Cartan involution of g. Now let h_{p_0} be any maximal Abelian vector subspace of p_0 (it is also an Abelian subalgebra of g_0). We will also take

h_0 as the maximal Abelian subalgebra of g_0 containing h_{p_0}. Basically we have $\theta(h_0) \subset h_0$ and $h_{p_0} = h_0 \cap p_0$. We also define $h_{t_0} = h_0 \cap t_0$. When we consider the complexifications h, h_p, h_t, t, p of $h_0, h_{p_0}, h_{t_0}, t_0, p_0$ respectively we conclude that [44], h is a Cartan subalgebra of g also h_0 is isomorphic to the image of $h_{\mathbb{R}}$ under the realization of g.

$$h_{\mathbb{R}} = h_{p_0} + ih_{t_0}, \tag{3.33}$$

where as the subsets of g_0, the sets h_{p_0} and h_{t_0} can also be considered as the subsets of g. This is similar to the case of u we have discussed above.

Let us consider a finite dimensional real vector space V and one of its vector subspaces W with their corresponding dual spaces V^* and W^* which is a vector subspace of V^*. Let V^* and W^* are turned into ordered vector spaces, the orderings are said to be compatible if the restriction $\lambda_{res} \in W^*$ of every $\lambda \in V^*$ is positive (negative) whenever λ is positive (negative). If $\{X_1, X_2, ..., X_n\}$ is a basis for V^* and the restricted basis $\{X_1, X_2, ..., X_m\}$ is a basis for the m-dimensional vector subspace W^* then the lexicographic orderings of V^* and W^* are compatible. The Cartan subalgebra h is a complex vector subspace of g and $h_{\mathbb{R}}$ is a subset of h and g. As mentioned before $h_{p_0} \subset p_0 \subset g_0$ also $h_{p_0} \subset h_0 \subset g_0$. The real vector spaces $h_0, h_{p_0}, h_{t_0}, p_0, t_0, g_0$ can be visualized as the isomorphic images of the subsets in $g_0 \times g_0$ which are set equivalent to the same subsets of the complex vector space g. Since h_{p_0} is a real vector subspace of $(h_{\mathbb{R}})^R = h_0$ we can define compatible orderings on $h^*_{p_0}$ and $(h^*_{\mathbb{R}})^R$. Besides the set of $\mathbb{R}$-valued and $\mathbb{R}$-linear maps on $h_{\mathbb{R}}$ and the dual space $(h^*_{\mathbb{R}})^R$ are set equivalent thus we indirectly have defined an ordering on the elements of the former. Eventually the restriction Δ_{res} of the non-zero roots Δ on $h_{\mathbb{R}}$ are elements of the set of $\mathbb{R}$-valued and $\mathbb{R}$-linear maps on $h_{\mathbb{R}}$ and they are in one-to-one correspondence with the unrestricted non-zero roots Δ as discussed before. As a result we basically achieve an ordering of the elements of Δ by using this one to one correspondence.

For each element $\beta \in h^*$ we define the element $\beta^\theta \in h^*$ such that $\beta^\theta(H) = \beta(\theta(H))$, $\forall\, H \in h$ where $\theta = \sigma \cdot \tau$ as defined before. If $\alpha \in \Delta$ then $\alpha^\theta \in \Delta$ as well. A non-zero root α has the null action $\alpha(\widetilde{h_{p_0}}) = 0$ on $\widetilde{h_{p_0}}$, the image of h_{p_0} in g if and only if $\alpha = \alpha^\theta$. The set of non-zero roots which do not vanish on $\widetilde{h_{p_0}}$ $(\alpha \neq \alpha^\theta)$ are called the non-compact roots and they will be denoted as Δ_{nc}. Conversely if $\alpha \in \Delta$ such that $\alpha = \alpha^\theta$ then α is called a compact root and their set will be shown as Δ_c. As a result we have

$$\Delta = \Delta_c \cup \Delta_{nc}. \tag{3.34}$$

We also classify the positive roots in Δ as

$$\begin{aligned} \Delta_c^+ &= \{\alpha \mid \alpha \in \Delta^+,\, \alpha = \alpha^\theta\}, \\ \Delta_{nc}^+ &= \{\alpha \mid \alpha \in \Delta^+,\, \alpha \neq \alpha^\theta\}, \end{aligned} \tag{3.35}$$

which implies that

$$\Delta^+ = \Delta_c^+ \cup \Delta_{nc}^+. \tag{3.36}$$

If $\alpha \in \Delta_{nc}^+$ then $-\alpha^\theta \in \Delta_{nc}^+$. The subalgebra

$$n = \sum_{\alpha \in \Delta_{nc}^+} g^\alpha, \tag{3.37}$$

of g is a nilpotent Lie algebra. As mentioned before if we let the image of g_0 in $g_0 \times g_0$ or equivalently in g be g_0' then

$$n_0 = g_0' \cap n, \tag{3.38}$$

is a nilpotent Lie algebra in g_0. The direct sum vector space

$$s_0 = h_{p_0} \oplus n_0, \tag{3.39}$$

becomes a solvable Lie subalgebra of g_0 under the Lie algebra product of g_0. As a real vector space the solvable Lie algebra s_0 is isomorphic to p_0 which is the element of the Cartan decomposition (3.29). As a result we have the Iwasawa decomposition

$$g_0 = t_0 \oplus s_0$$

$$= t_0 \oplus h_{p_0} \oplus n_0, \tag{3.40}$$

of a semi-simple real Lie algebra g_0 which is based on an arbitrary Cartan decomposition and the root space decomposition of g_0 and g. The sum in (3.40) is a vector space direct sum.

We have defined the particular Cartan involution $\theta = \sigma \cdot \tau = \tau \cdot \sigma$ on $g = g_0^C$ before. As an involutive automorphism it has two eigenspaces g^+ and g^- with the eigenvalues ± 1 resulting in the decomposition

$$g = g^+ \oplus g^-. \tag{3.41}$$

The Cartan involution θ is also closed on the Cartan subalgebra h of g meaning that $\theta(H) \subset h,\ \forall\, H \in h$. The Cartan subalgebra h of g likewise g itself can be written as a vector space direct sum of two eigenspaces h^+ and h^- with the eigenvalues $+1$ and -1 under θ. Therefore

$$h = h^+ \oplus h^-. \tag{3.42}$$

As we have defined earlier we also have the induced map $\beta \longrightarrow \beta^\theta$ on the dual space h^* of h. Thus the dual space also has a decomposition

$$h^* = h^{*+} \oplus h^{*-}, \tag{3.43}$$

under θ. We remind the reader of the fact that $\Delta_c = h^{*+} \cap \Delta$ and $\Delta_{nc} = (h^* - h^{*+}) \cap \Delta$. Furthermore $\Delta_c^+ = h^{*+} \cap \Delta^+$ and $\Delta_{nc}^+ = (h^* - h^{*+}) \cap \Delta^+$. We will also denote the set of roots which satisfy $\beta = -\beta^\theta$ as Δ_s which can be written as $\Delta_s = h^{*-} \cap \Delta$. We observe that $\Delta_s \subset \Delta_{nc}$. The remaining roots $\Delta_{nc} - \Delta_s$ mix under the Cartan involution θ. In general the elements of g which are invariant (with $+1$ eigenvalue) under a Cartan involution are called compact and the elements of g which only change sign under the Cartan involution (with -1 eigenvalue) are called non-compact. The Cartan subalgebra h is already decomposed into its compact, h^+ and the non-compact, h^- components in (3.42) according to θ. In general there are algebra elements which neither stay invariant nor change sign but which mix under the Cartan involution. The Cartan subalgebra generators $\{H_{\alpha_i}\}$ where α_i are simple roots are also the generators of $h_{\mathbb{R}}$ in the real sense thus they form a basis for the realization h_0 of $h_{\mathbb{R}}$. The real maximal Abelian subalgebra h_0 of g_0 which contains p_0 has the decomposition

$$h_0 = h_{t_0} \oplus h_{p_0}. \tag{3.44}$$

We should remark that for the Cartan decomposition $g_0 = t_0 \oplus p_0$, the associated Cartan involution s_0 which we have defined before also has the compact-non-compact decomposition on g_0 which coincides with the Cartan decomposition itself. The non-compact part of h_0 in (3.44) is h_{p_0} (with respect to the associated Cartan involution s_0) and similarly the compact part of h_0 is h_{t_0}. The compactness (non-compactness) on g_0 with respect to θ can be obtained by considering the compact (non-compact) part in g generated by the compact (non-compact) generators and then the set equivalent of g_0 in g can be decomposed into its compact (non-compact) components. Equivalently one can consider the $\mathbb{R}$-linear restriction of θ on the image of g_0 in g which induces an involution on g_0 since g_0 is a real form of g, as a result the compact-non-compact decomposition with respect to this restricted and the induced Cartan involution is formed. One should be careful when speaking about the compact or the non-compact elements of g_0, the compactness or the non-compactness depends on the particular Cartan involution we specify. One more point about the compactness is that in general the root space decomposition of an algebra may not be compatible with the eigenspace decomposition with respect to a Cartan involution, in this case it is possible that the elements of the basis in (3.26) may be neither compact nor non-compact.

When we consider a semi-simple Lie group G with a semi-simple Lie algebra g_0 we may define special classes of real forms which have distinct properties. Let g' be the complex semi-simple Lie algebra obtained from g_0 as $g' = \widetilde{g_0}$. We will also assume a certain decomposition of the roots of g' with respect to any Cartan involution ψ of g' like we have performed for the specific Cartan involution

$\theta = \tau \cdot \sigma$ on $g = g_0^C$ before. The lattice Δ_c' in $(h')^{*+}$ coincides with the root lattice of a compact analytical subgroup G_c of G whose Lie algebra is isomorphic to a compact real form g_c^R of g' which is a subalgebra of g_0. The real form g_c^R is the image of the subset g_c of g' under the realization of g' as it is clear from our former remarks about the set equivalences. One can construct g_c in g' as

$$g_c = \sum_{\alpha \in \Delta'} \mathbb{R}(iH_\alpha + (E_\alpha - E_{-\alpha}) + i(E_\alpha + E_{-\alpha})). \tag{3.45}$$

We use the basis introduced in (3.26) for g'. The analytical subgroup G_c is called the compact real form of G however it is not a maximal compact subgroup [57].

Similarly the root lattice Δ_s' in $(h')^{*-}$ corresponds to the roots of the so called split real form (maximally non-compact real form) G_s which is an analytical subgroup of G. The elements of G_s commute with G_c. The Lie algebra of G_s is a Lie subalgebra g_s^R of g_0 and it is a real form of g'. The real form g_s^R is the image of the subset g_s of g' under the realization of g'. We can construct g_s as follows

$$g_s = \sum_{\alpha \in \Delta'} \mathbb{R}(H_\alpha + (E_\alpha - E_{-\alpha}) + (E_\alpha + E_{-\alpha})). \tag{3.46}$$

The analytical subgroup G_s is not maximally split in G. The Lie algebra of the maximal compact subgroup of the split real form G_s is generated by taking the real combinations of the generators $(E_\alpha - E_{-\alpha})$ and then considering their image under the realization of g' as usual.

The non-compact real forms of $g' = \widetilde{g}_0$ where g_0 is a semi-simple real Lie algebra can be constructed from the compact real forms g_c^R or G_c of G by using the Cartan involution ψ of g' [57]. One may define the image of the root subspace generators of g' in (3.26) under any specified Cartan involution ψ of g' by considering the action of ψ on the root lattice of g'. The action on the root subspace generators of g' can simply be written as $\psi(E_\alpha) = C_\alpha E_{\psi(\alpha)}$ where $C_\alpha = \pm 1$. When $\alpha \in \Delta'_c$ then $C_\alpha = 1$ and when $\alpha \in \Delta'_s$ then $C_\alpha = -1$. If $\alpha \notin \Delta'_c \cup \Delta'_s$ then the sign is arbitrary but the coefficients satisfy the rule [57]

$$C_{\alpha+\beta} N_{\alpha,\beta} = C_\alpha C_\beta N_{\theta(\alpha),\theta(\beta)}. \tag{3.47}$$

To construct a non-compact real form of g' by using the action of ψ on the generators of g', one first identifies the compact and the non-invariant generators of the compact real form G_c then after multiplying the non-invariant generators by i, they are used together with the compact generators as linear combinations with the real subfield coefficients to generate a subset of $g' = \widetilde{g_0}$ [57]. Then the realization image of this subset corresponds to the Lie algebra of the non-compact real form which is an analytical subgroup of G. The compact generators solely generate a compact subgroup of this non-compact real form. We assume that the analytical Lie groups we deal with are linear so that an analytical Lie group is compact (non-compact) if and only if its Lie algebra is compact (non-compact). Each time by taking a different Cartan involution ψ of g' one can generate a series of compact, split and non-compact real forms of g'.

When the analytical Lie group G is a compact, split or a non-compact real form of any other analytical Lie group D, it has certain algebraic and topological properties which help us to identify its type. We should remark that although it is more common to speak about the real forms of semi-simple complex Lie algebras we use the same classification for the analytical Lie groups G in a parallel way that one should consider algebraically the corresponding Lie algebra g_0 of G as a real form of the complex Lie algebra $\widetilde{d}$ where d is the Lie algebra of D. We will consider the complexification $g = g_0^C$ of the semi-simple real Lie algebra g_0 which is isomorphic to the Lie algebra of the Lie group G. We also consider an arbitrary Cartan, (3.29) and the corresponding Iwasawa, (3.40) decompositions of g_0. Therefore we assume the compact-non-compact decompositions on g with respect to s_0 and θ as usual. The realization of $h_{\mathbb{R}}$ namely h_0 which is a maximal Abelian subalgebra that contains p_0 in g_0 generates an Abelian subgroup in G which is called the maximal torus. It is an analytical subgroup of G whose Lie algebra is isomorphic to h_0. Although we call it torus it is not the ordinary torus topologically, in fact it has the topology $(S^1)^m \times \mathbb{R}^n$ for some m and n and if it is diagonalizable in $\mathbb{R}$ (such that $m = 0$) then it is called an R-split torus. These definitions can be generalized for the subalgebras of h_0 as well. If g_0 is one of the three special classes of semi-simple real forms then the analytical subgroup of G which is generated by h_{p_0} is the maximal R-split torus in G in the sense defined above and its dimension is called the R-rank of G which we will denote by r. The R-rank is equal to the dimension of h_{p_0} which is the non-compact component of the eigenspace decomposition of h_0 with respect to s_0. The R-rank is also the multiplicity of the roots in Δ_s ($\alpha = -\alpha^\theta$). For the split real forms, G_s (maximally non-compact) the R-rank, r is maximal such that $r = l$ where l is the rank of G_s ($l = dim_{\mathbb{R}}(h_{0s}) = dim_{\mathbb{C}}(h_s)$); which also means that $h_{p_{0s}} = h_{0s}$. On the other hand for the compact real forms G_c, r is minimal such that $r = 0$

and $h_{0c} = h_{t_{0c}}$. For the non-compact semi-simple real forms r is in between 0 and l so that $h_{0nc} = h_{t_{0nc}} \oplus h_{p_{0nc}}$ with $dimh_{p_{0nc}} = r$. Therefore the number of the non-compact roots in Δ_s and the real dimension of h_{p_0} is l for a split real form, it is r for a non-compact real form and there are no non-compact roots for a compact real form so that $h_{p_{0c}}$ is an empty set. We see that all the roots for a split real form become non-compact because a basis of l roots are non-compact.

For a compact real form Δ_s is empty. Also for a split real form $\Delta = \Delta_s$, $\Delta^+ = \Delta^+_{nc} = \Delta^+_s$ and Δ_c is empty. The non-compact real form has r split non-compact roots. In all of the three cases h_{p_0}, in the real sense or h_p, in the complex sense are generated by the generators $\{H_i\}$ where we include only the Cartan generators corresponding to the roots in Δ_s. One can also define the compactness on h_0 with respect to θ by restricting θ into the $\mathbb{R}$-valued, $\mathbb{R}$-linear maps on $h_{\mathbb{R}}$ then by considering the action of it on the set equivalent h_0 of $h_{\mathbb{R}}$; this restriction would reveal the compact-non-compact parts of h_0 with respect to θ. On the other hand h_{p_0} is non-compact with respect to s_0 due to the Iwasawa decomposition (3.40).

When the Lie group G is in split real form then for a Cartan decomposition and a resulting Iwasawa decomposition the solvable algebra s_0 in (3.40) coincides with the Borel subalgebra which is generated by all the simple root Cartan generators $\{H_\alpha\}$ (or any other basis) of h_0 and the positive root generators $\{E_\alpha \mid \alpha \in \Delta^+\}$ [10]. This result is obvious since for a split real form $dimh_{p_{0s}} = l$ and $\Delta^+_{nc} = \Delta^+$. For a split real form since $r = l$ with respect to θ, all the Cartan generators are non-compact and the combinations $\{(E_\alpha - E_{-\alpha})\}$ are compact while $\{(E_\alpha + E_{-\alpha})\}$

are non-compact. Also $h_{p_{0s}} = h_{0s}$ is non-compact with respect to both of s_0 and θ. Since all the roots and the corresponding Cartan generators are non-compact we can conclude that h is non-compact too (with respect to θ). The root space generators $\{E_\alpha\}$ for $\alpha \in \Delta_{nc}^+$ are non-compact with respect to s_0 but since $\theta(E_\alpha) = -E_{-\alpha}$ for all the roots in the split case they are mixed with respect to θ. Therefore the compactness with respect to s_0 and θ are not compatible. The compactness of the other root generators with respect to s_0 must be studied case by case.

When the Lie group G is a compact real form there are no non-compact roots and non-compact Cartan generators with respect to θ then $h_{0c} = h_{t_{0c}}$ and h_{0c} is compact with respect to s_0. For a non-compact real form G, from the Iwasawa decomposition (3.40), $h_{p_{0nc}}$ is non-compact with respect to s_0 and θ since it is generated by the non-compact Cartan generators $\{H_\alpha\}$ whose number is r. From the Cartan decomposition (3.29), $h_{t_{0nc}}$ is compact as usual. The root subspace generators $\{E_\alpha \mid \alpha \in \Delta_{nc}^+\}$ are non-compact with respect to s_0 but the set $\{E_\alpha \mid \alpha \in \Delta_s \subset \Delta_{nc}^+\}$ is mixed with respect to θ therefore the compact-non-compact decompositions of s_0 and θ are not compatible. The compactness (non-compactness) of the rest of the roots, the Cartan and the root subspace generators with respect to s_0 and θ must be analyzed case by case. We should also remark that the analysis we have performed for the real forms by using the decompositions with respect to Cartan involutions is an important tool in the study of general compact and non-compact Lie algebras [44].

The classification of the Riemannian globally symmetric spaces reduces to the classification of the real simple Lie algebras. There are four infinite sequences, the classical simple complex Lie algebras (A_n, B_n, C_n, D_n) and five exceptional simple complex Lie algebras $(G_2, F_4, E_6, E_7, E_8)$. These complex Lie algebras correspond to the classification of all the simple complex Lie algebras and their constructions can be found in any of the references about the Lie algebras in [44]-[49]. Since a Lie algebra is semi-simple if and only if it can be written as a direct sum of its simple ideals [49], the classification of the complex simple Lie algebras enables us to work out a method of construction of all the semi-simple complex Lie algebras thus it leads to the classification of the complex semi-simple Lie algebras indirectly. The complete set of real forms of a given complex semi-simple Lie algebra g in addition to the discussion we have performed above can be obtained from the identification of all the Cartan involutions of g. The classification of the real simple Lie algebras is entirely based on the classification of the complex simple Lie algebras [48]. Eventually the different classes of the real simple Lie algebras appear as the compact and the non-compact (including the split case which is also called the maximally non-compact; meaning that it is a limit case of the non-compact real forms as we have explained before) real forms of the complex simple Lie algebras or in a few cases as the real forms of their complex semi-simple Lie algebra direct sums [48]. The entire classification can be found in [44] for example. Likewise the complex case, the real simple Lie algebra classification gives us the material to construct all the real semi-simple Lie algebras in a systematical way. We also emphasize that although all the simple real Lie algebras can be obtained from the complex simple Lie algebras or their semi-simple direct sums, the entire set of semi-simple real Lie algebras is not necessarily composed of the real forms of the semi-simple complex Lie algebras. Therefore the semi-simple real forms whose structures we have discussed form a special class of real semi-simple Lie algebras. The general set of real semi-simple Lie algebras may have richer topological and algebraical structures.

Before getting to the point how the scalar coset manifolds fit into this picture of the symmetric spaces and the real forms we will clarify one point about the basis we use. As mentioned earlier in this section the basis in (3.26) which is generated by the root subspace decomposition of g can be used to generate g. If one takes l Cartan generators corresponding to a basis of h, for example the Cartan generators corresponding to the simple roots then according to (3.21) the basis $\{H_{\alpha_1}, H_{\alpha_2}, ..., H_{\alpha_l}, E_\beta\}$ for $\beta \in \Delta$ is a legitimate basis for the complex semi-simple Lie algebra g. We will consider the complexification $g = g_0^C$ of a semi-simple real Lie algebra now. If $\{X_i\}$ is a basis for g_0 then since

$$\begin{aligned}(a+ib)(X_i,0) &= a(X_i,0) + Jb(X_i,0)\\ &= (aX_i,0) + (0,bX_i)\\ &= (aX_i,bX_i),\end{aligned} \tag{3.48}$$

for all $a, b \in \mathbb{R}$, $\{(X_i, 0)\}$ which generates the entire set of $g = g_0 \times g_0$ is a basis for the complex Lie algebra g in the complex sense. Now if we let

$$[X_i, X_j] = C_{ij}^k X_k, \tag{3.49}$$

then we have

$$([(X_i,0),(X_j,0)]) = ([X_i,X_j],0)$$

$$= (C_{ij}^k X_k, 0), \tag{3.50}$$

where we have used (3.27). Thus we see that for the basis specified above g has the same real structure constants with g_0. However notice that the basis $\{(X_i,0)\}$ for g is in the complex sense and the structure constants $\{C_{ij}^k\}$ for g are in some chosen real subfield of $\mathbb{C}$. This is not surprising as g_0 is a real form of g. In fact if we let

$$g_{\mathbb{R}} = \sum_i \mathbb{R}(X_i,0), \tag{3.51}$$

then $g_0 = g_{\mathbb{R}}^R$. One can construct a real form of g by choosing a basis whose structure constants are in the real subfield of $\mathbb{C}$ and then by taking the linear combinations of this basis with the elements of the real subfield. The resulting real structure constants in the real form (more correctly in the realization of the set described above) are inherited from the complex Lie algebra g. This way one constructs a subset of g whose image under the realization of g is isomorphic to g_0. There is a set of such images of g_0 or images of any real form of g in g. Thus if we chose the basis $\{H_{\alpha_1}, H_{\alpha_2}, ..., H_{\alpha_l}, E_\beta\}$ within one of such images $g_{\mathbb{R}}$ of g_0 (for example in a representation corresponding to the representatives $\{(X_i,0)\}$) then the structure constants in (3.26) are real and the same basis generates g_0 with real coefficients (keeping in mind that we borrow the addition rule of g and we determine the scalar product with $\mathbb{R}$ from the complex structure J on g which are the requirements of the realization of $g_{\mathbb{R}}$). We will always assume this basis as the standard basis for g_0 and its subalgebras.

3.4 Scalar Cosets as Symmetric Spaces

The scalar manifolds of a wide class of supergravities, in particular of all the pure and the matter coupled, $N > 2$, extended supergravities in $D = 4, 5, 6, 7, 8, 9$ dimensions [13] as well as the maximally extended supergravities in $D \leq 11$ [10, 19] are homogeneous spaces [46], in the sense that there exists a transitive action of a Lie group G on these manifolds. These homogeneous spaces are in the form of a coset manifold G/K where G is in general a non-compact real form of any other semi-simple Lie group and K is a maximal compact subgroup of G. When G is a compact real form its maximal compact subgroup K is G itself thus the coset space G/G is a single point and the corresponding sigma model is an empty set. The numerator group may as well be a split real form (maximally non-compact) of a semi-simple Lie group, this is the case for the maximal supergravities in [10, 19].

The Lie algebra k_0 of the analytical subgroup K is a subalgebra of the semi-simple Lie algebra g_0 which is the Lie algebra of G. The Lie algebra k_0 is a maximal compactly imbedded Lie subalgebra of g_0 therefore it is an element of a Cartan decomposition of g_0

$$g_0 = k_0 \oplus p_0. \tag{3.52}$$

Since G is a linear analytical Lie group the corresponding Lie algebra g_0 is non-compact for both of the cases when G is a non-compact or a split real form (although we consider it separately we should not forget that it is a limiting non-compact case). The map $s_0 : k + p \longrightarrow k - p$, for all $k \in k_0$ and $p \in p_0$ is the Cartan involution which generates (3.52). The pair (g_0, s_0) is an orthogonal symmetric Lie algebra of the non-compact type. By following the outline of the previous sections we conclude that (G, K) is associated with (g_0, s_0) and it is of non-compact type too. The Cartan decomposition (3.52) is the eigenspace decomposition of s_0 where the elements of k_0 have $+1$ eigenvalues and the elements of p_0 have -1 eigenvalues under the involution s_0. We also know that (G, K) is a Riemannian symmetric pair therefore the scalar manifold which is the coset space G/K has a unique analytical structure induced by the quotient topology of G as stated before. The scalar manifold G/K is a Riemannian globally symmetric space for all the G-invariant Riemannian structures on G/K. This is the origin of the name; symmetric space sigma model which governs the scalar sectors of the supergravities mentioned above. The crucial result of this identification of the scalar manifolds as symmetric spaces is that the exponential map $Exp : g_0 \longrightarrow G$ induces a difeomorphism

$$Exp : p_0 \longrightarrow G/K, \tag{3.53}$$

from the $\mathbb{R}^{dim p_0}$ manifold p_0 onto G/K since it maps the elements of p_0 onto the representatives of the left cosets G/K. This result will enable us to define a parametrization of the scalar manifold G/K on which the sigma model will be constructed in the next chapter.

Furthermore one may use the Iwasawa decomposition of g_0 to express p_0 in terms of the root space decomposition basis introduced in (3.26). The Iwasawa decomposition reads

$$g_0 = k_0 \oplus h_{p_0} \oplus n_0, \tag{3.54}$$

where h_{p_0} is generated by r non-compact Cartan generators $\{H_i\}$. Here we use the notation we have established before thus r is the R-rank of G. The nilpotent algebra n_0 is generated by the root subspace generators $\{E_\beta\}$ corresponding to the roots in Δ^+_{nc} such that any element $\eta \in n_0$ can be written as $\eta = \sum_{\beta \in \Delta^+_{nc}} X^\beta E_\beta$ where $X^\beta \in \mathbb{R}$. The direct sum $s_0 = h_{p_0} \oplus n_0$ is the solvable subalgebra of g_0 as we have discussed in the last section. When G is a split real form then as we have mentioned before, $r = l$ where l is the rank of g_0 thus $h_{p_0} = h_0$ is generated by all the Cartan generators $\{H_i\}$ and $\Delta^+_{nc} = \Delta^+$ so that the root subspace generators $\{E_\beta\}$ correspond to the entire set of positive roots. Therefore the solvable Lie subalgebra s_0 coincides with the Borel subalgebra which is generated by the entire Cartan and the positive root generators of g_0 for the split real form case.

For the non-compact real form G if we use the basis $\{H_i, E_\beta\}$ in (3.53) to generate p_0 which is isomorphic to s_0 we have the parametrization

$$Exp : \sum \mathbb{R}\{H_i, E_\beta\} \longrightarrow G/K, \tag{3.55}$$

then (3.55) is called the solvable Lie algebra parametrization or the solvable Lie algebra gauge of the scalar manifold G/K [18]. On the other hand when g is a split real form (maximally non-compact) and if use the Borel subalgebra basis which is made up of the entire set of the Cartan generators and the positive root generators; also which is a maximal case of (3.55) in terms of the generators then (3.55) is called the Borel parametrization or the Borel gauge of the scalar manifold G/K.

In conclusion we have obtained a legitimate parametrization of the scalar coset manifold G/K by using the solvable Lie subalgebra s_0 of g_0. If we use the notation $\{T_i\}$ for the basis vectors $\{H_j, E_\beta \mid j = 1, ..., r \;;\; \beta \in \Delta_{nc}^+\}$ of s_0 and if $\{\varphi^i(x)\}$ are C^∞-maps over the D-dimensional spacetime then the map

$$\nu(x) = e^{\varphi^i(x)T_i}, \tag{3.56}$$

is an onto C^∞-map from the D-dimensional spacetime to the Riemannian globally symmetric space G/K. The gauge map, (3.56) which depends on the scalar functions $\{\varphi^i(x)\}$ will be the building block of our construction of the symmetric space sigma model in the next chapter.

CHAPTER 4

THE SYMMETRIC SPACE SIGMA MODEL

In the previous chapter we have given the formal construction of the Riemannian globally symmetric spaces which we will simply refer to as the symmetric spaces. We have studied the algebraic properties of the symmetric spaces in detail, especially the Cartan and the Iwasawa decompositions. In this chapter the symmetric space sigma model is constructed and studied in two equivalent formulations and for different coset parametrizations. Our formalism will be in parallel with the ones given in [58, 59]. We have already covered the symmetric space sigma model constructions of the scalar sectors of certain supergravities in Chapter two. Although we have given the Lagrangian, we have not mentioned the details of how it is constructed nor how one derives the equations of motion. First we will introduce the general, non-linear sigma models. We will consider the formulation of the symmetric space sigma model in the veilbein formalism. Then by making use of the algebraic outline, i.e. by using the solvable Lie algebra parametrization introduced in the last chapter, we will basically study the symmetric space sigma model Lagrangian which is constructed in terms of the internal metric $\mathcal{M}$, whose rigid symmetry group G is a real form of a non-compact semi-simple Lie group and G is not necessarily in split real form. The semi-simple split real forms have already appeared as the global symmetry groups of the Bosonic sectors of the maximal supergravities and we have encountered the more general semi-simple non-compact real forms when we discussed the scalar sectors of the Kaluza-Klein descendant supergravity theories of the ten dimensional Abelian Yang-Mills supergravity in Chapter two. The formalism presented here covers the split symmetry group case as a particular limit in the possible choice of the non-compact rigid symmetry groups which is clear from the definitions we have made in the last chapter. We will introduce two coset maps, one being the parametrization used in [10, 19]. A transformation law between the two sets of scalar fields will also be established. The second-order equations of the vielbein formulation and the internal metric formulation will be derived by using these coset maps. The dualisation and the local first-order formulation will also be performed for the symmetric space sigma model whose scalar fields parameterize the coset G/K, where the rigid symmetry group G is a real form of a non-compact semi-simple Lie group (not necessarily split) and the local symmetry group K is G's maximal compact subgroup as we have discussed in the last chapter. The transformation between the two different coset parametrizations we have mentioned above will be used to relate the corresponding first-order equations of motion.

4.1 The Sigma Model

In this section we will present two equivalent formulations of the symmetric space sigma model which governs the scalar sectors of a wide class of supergravities including the supergravity theories we have discussed in Chapter two. The homogeneous coset scalar manifolds are the common features of these supergravities. The first of these formulations does not specify a coset parametrization while the second one makes use of the results of the Iwasawa decomposition. The first formulation is a more general one which is valid for any Lie group G and its subgroup K, namely it is the G/K non-linear sigma model. In particular it is applicable to the symmetric space sigma models.

In order to construct the non-linear sigma model we first consider the set of G-valued maps $\nu(x)$ which are from the D-dimensional spacetime to a generic Lie group G and which transform onto each other as $\nu \rightarrow k(x)\nu g$, $\forall g \in G$, $k(x) \in K$ for some Lie subgroup K of G. We can calculate $\mathcal{G} = \nu^{-1}d\nu$ which is the pull back of the Lie algebra g_0-valued Cartan form over G through the map $\nu(x)$. For the construction of the non-linear coset sigma model we assume that the map $\nu(x)$ corresponds to a parametrization of the coset G/K (for convenience we will consider the left cosets). Thus we will assume that the map $\nu(x)$ is from the D-dimensional spacetime into the group G and its range is composed of the the representatives of the left cosets of G/K. Moreover if G is a non-compact semi-simple Lie group and K is a maximal compact subgroup of G then G/K becomes a symmetric space in the light of the last chapter and $\nu(x)$ can be taken as the map (3.56) such that $\nu(x) = e^{\varphi^i(x)T_i}$ by using the Cartan and the Iwasawa decompositions. In this case the transformation rule $\nu \rightarrow k(x)\nu g$, $\forall g \in G$, $k(x) \in K$ we assign on $\nu(x)$ which is in the form of (3.56) becomes a manifestation of the Iwasawa decomposition. As it is clear from the previous chapters, for the scalar cosets G/K of the supergravity theories we have considered, the global symmetry groups are semi-simple non-compact real forms so that we can make use of the map (3.56) which functions the Iwasawa decomposition resulting in the solvable Lie algebra gauge. For the most general case of the Lie group G and its subgroup K, by using the most general map $\nu(x)$ we have

$$\mathcal{G}_\mu dx^\mu = (f^a_\mu(x) T_a + \omega^i_\mu(x) K_i) dx^\mu, \tag{4.1}$$

where $T_a \in p_0$ and $K_i \in k_0$. Here k_0 is the Lie algebra of K and p_0 is the orthogonal complement of k_0 in g_0. In particular if G is a non-compact real form of a semi-simple Lie group and K is a maximal compact subgroup of G then k_0 and p_0 are elements of the Cartan decomposition (3.52). When G/K is a Riemannian globally symmetric space then the fields $\{f^a_\mu\}$ form a vielbein of the G-invariant Riemann structures on G/K and $\{\omega^i_\mu\}$ can be considered as the components of the connection one forms of the gauge theory over the K-bundle. We should also bear in mind that $[k_0, k_0] \subset k_0$ and if furthermore we have $[k_0, p_0] \subset p_0$ then we will have a simpler theory. Let $P_\mu \equiv f^a_\mu T_a$ and $Q_\mu \equiv \omega^i_\mu K_i$ then we can construct a Lagrangian [14, 15]

$$\mathcal{L} = \frac{1}{2} tr(P_\mu P^\mu), \tag{4.2}$$

where the trace is over the particular representation we choose. The Lagrangian, $\mathcal{L}$ is invariant when $\nu(x)$ is transformed under the rigid (global) action of G from the right and the local action of K from the left which we have assumed as the transformation law for the maps $\nu(x)$. The elements of (4.1), P_μ and Q_μ are invariant under the rigid action of G but under the local action of K they transform as

$$Q_\mu \to k(x) Q_\mu k^{-1}(x) + k(x)\partial_\mu k^{-1}(x),$$
$$\tag{4.3}$$
$$P_\mu \to k(x) P_\mu k^{-1}(x).$$

The field equations corresponding to (4.2) are

$$D_\mu P^\mu \equiv \partial_\mu P^\mu + [Q_\mu, P^\mu] = 0, \tag{4.4}$$

where we have introduced the covariant derivative $D_\mu \equiv \partial_\mu + [Q_\mu, \quad]$.

Beside having more general applications the above formalism covers the symmetric space sigma model in it. From now on we will assume that the global symmetry group G is a semi-simple non-compact real form and the local transformation group K is a maximal compact subgroup of G. We will also introduce another parametrization of the coset G/K, differing from (3.56) which is a result of the solvable Lie algebra gauge of the Iwasawa decomposition that is discussed in the last chapter. Thus when G is a real form of a non-compact semi-simple Lie group and K is its maximal compact subgroup the coset G/K can locally be parameterized as

$$\nu(x) = \mathbf{g}_H(x)\mathbf{g}_N(x)$$

$$= e^{\frac{1}{2}\phi^i(x)H_i} e^{\chi^m(x)E_m}, \tag{4.5}$$

where $\{H_j, E_\beta \mid j = 1, ..., r\ ;\ \beta \in \Delta^+_{nc}\}$ is the basis for the solvable Lie algebra s_0 of the semi-simple Lie algebra g_0 which we have introduced in the last chapter.

Here $\{H_i\}$ is a basis for h_{p_0} for $i = 1, ..., r$ and $\{E_m\}$ is a basis for n_0. The map (4.5) is obtained by considering a map from the spacetime onto a neighborhood of the identity of s_0. Thus it defines locally a diffeomorphism from $h_{p_0} \times n_0$ into the space G/K, so it is a local parametrization of the coset G/K. We assume that the locality is both over the spacetime and over the space $h_{p_0} \times n_0$ in order to write two products of exponentials instead of the map (3.56). Thus the local definition of the map (4.5) is a legitimate result of the existence of the parametrization (3.56). If G is in split real form then $h_{p_0} = h_0$ and $\Delta^+_{nc} = \Delta^+$ thus the solvable Lie algebra s_0 becomes the Borel subalgebra of g_0. The fields $\{\phi^i\}$ are called the dilatons and $\{\chi^m\}$ are called the axions. The scalar fields namely the dilatons and the axions as it is clear from the construction of (4.5) parameterize the coset space G/K which is also called the scalar coset manifold. At this stage we can calculate the field equations (4.4) in terms of these newly defined scalar fields under the parametrization (4.5). By using the explicit definition of the map $\nu(x)$ in (4.5) the Cartan form $\mathcal{G} = \nu^{-1} d\nu$ can be decomposed into two components which can be calculated separately. Thus by applying simply the matrix algebra we have

$$\begin{aligned}\mathcal{G} &= \nu^{-1} d\nu \\ &= (\mathbf{g}_N^{-1}\mathbf{g}_H^{-1})(d\mathbf{g}_H\mathbf{g}_N + \mathbf{g}_H d\mathbf{g}_N) \\ &= \mathbf{g}_N^{-1} d\mathbf{g}_N + \mathbf{g}_N^{-1}\mathbf{g}_H^{-1} d\mathbf{g}_H \mathbf{g}_N.\end{aligned} \tag{4.6}$$

If we make use of the identity

$$e^{-C}de^{C} = dC - \frac{1}{2!}[C, dC] + \frac{1}{3!}[C, [C, dC]] - ...., \qquad (4.7)$$

in a matrix representation, the first term in (4.6) can be calculated as

$$\begin{aligned}
\mathbf{g}_N^{-1}d\mathbf{g}_N &= e^{-\chi^m E_m} de^{\chi^m E_m} \\
&= d\chi^m E_m - \frac{1}{2!}[\chi^m E_m, d\chi^n E_n] + \frac{1}{3!}[\chi^m E_m, [\chi^l E_l, d\chi^n E_n]] - .... \\
&= d\chi^m E_m - \frac{1}{2!}\chi^m d\chi^n K^v_{mn} E_v + \frac{1}{3!}\chi^m \chi^l d\chi^n K^v_{ln} K^u_{mv} E_u - .... \\
&= \overrightarrow{\mathbf{E}}\,\mathbf{\Sigma}\,\overrightarrow{d\chi}, \qquad (4.8)
\end{aligned}$$

where we have defined the row vector, $(\overrightarrow{\mathbf{E}})_\alpha = E_\alpha$ and the column vector $(\overrightarrow{d\chi})^\alpha = d\chi^\alpha$. We have also introduced $\mathbf{\Sigma}$ as the $\dim n_0 \times \dim n_0$ matrix

$$\mathbf{\Sigma} = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^n}{(n+1)!}, \qquad (4.9)$$

where ω is composed of the axions coupled to the structure constants, $\omega^\gamma_\beta = \chi^\alpha K^\gamma_{\alpha\beta}$ where the structure constants $K^\gamma_{\alpha\beta}$ are defined as $[E_\alpha, E_\beta] = K^\gamma_{\alpha\beta} E_\gamma$. If we consider the commutator $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ then $K^\alpha_{\beta\beta} = 0$ also $K^\alpha_{\beta\gamma} = N_{\beta,\gamma}$ if in the root sense $\beta + \gamma = \alpha$ and $K^\alpha_{\beta\gamma} = 0$ if $\beta + \gamma \neq \alpha$. Similarly since the Cartan generators commute with each other we have

$$\mathbf{g}_H^{-1}d\mathbf{g}_H = e^{-\frac{1}{2}\phi^i H_i} de^{\frac{1}{2}\phi^i H_i}$$

$$= \frac{1}{2} d\phi^i H_i. \tag{4.10}$$

The second term in (4.6) can now be calculated as

$$\mathbf{g}_N^{-1} (\frac{1}{2} d\phi^i H_i) \mathbf{g}_N = e^{-\chi^m E_m} (\frac{1}{2} d\phi^i H_i) e^{\chi^m E_m}$$

$$= \frac{1}{2} d\phi^i H_i - [\chi^m E_m, \frac{1}{2} d\phi^i H_i]$$

$$+ \frac{1}{2!} [\chi^m E_m, [\chi^l E_l, \frac{1}{2} d\phi^i H_i]] - ....$$

$$= \frac{1}{2} d\phi^i H_i + \chi^m \frac{1}{2} d\phi^i m_i E_m - \frac{1}{2!} \chi^m \chi^l \frac{1}{2} d\phi^i l_i K^u_{ml} E_u + ...$$

$$= \frac{1}{2} d\phi^i H_i + \overset{\rightharpoonup}{\mathbf{E}}\, \mathbf{\Sigma}\, \overset{\rightharpoonup}{U}, \tag{4.11}$$

where we have used the Campbell-Hausdorff formula

$$e^{-X} Y e^X = Y - [X, Y] + \frac{1}{2!} [X, [X, Y]] - ...., \tag{4.12}$$

and we have defined the column vector $(\overset{\rightharpoonup}{U})^m = \frac{1}{2} \chi^m m_i d\phi^i$. We have $[H_i, E_m] = m_i E_m$. Therefore the Cartan form $\mathcal{G} = \nu^{-1} d\nu$ in (4.6) becomes

$$\mathcal{G} = \frac{1}{2} d\phi^i H_i + \overset{\rightharpoonup}{\mathbf{E}}\, \mathbf{\Sigma}\, (\overset{\rightharpoonup}{U} + \overset{\rightharpoonup}{d\chi}). \tag{4.13}$$

Since the expansion of $\mathcal{G}$ consists of only the generators of s_0 but not the generators of k_0 which is a direct result of (4.5) where the parametrization is derived locally from the solvable Lie algebra parametrization, we have $Q_\mu = 0$ and from

(4.13) P_μ is

$$P_\mu = \frac{1}{2}\partial_\mu \phi^i H_i + \Sigma_m^\alpha (\frac{1}{2}\chi^m m_i \partial_\mu \phi^i + \partial_\mu \chi^m) E_\alpha. \tag{4.14}$$

Since $Q_\mu = 0$ from (4.4) the equations of motion become

$$\partial^\mu P_\mu = 0. \tag{4.15}$$

Thus we have

$$\begin{aligned} \partial^\mu \partial_\mu \phi^i &= 0, \\ \partial^\mu \left(\Sigma_m^\alpha \left(\frac{1}{2}\chi^m m_i \partial_\mu \phi^i + \partial_\mu \chi^m\right)\right) &= 0, \end{aligned} \tag{4.16}$$

for $i = 1, ...r$ and $m, \alpha \in \Delta_{nc}^+$. Notice that we enumerate the roots in Δ_{nc}^+.

Another formulation of the G/K symmetric space sigma model (with G not necessarily a split real form) can be done by introducing the internal metric $\mathcal{M}$. The Lagrangian is again invariant when ν is transformed under the global action of G from the right and the local action of K from the left as $\nu \rightarrow k(x)\nu g$. In this formulation the symmetric space sigma model Lagrangian is given as

$$\mathcal{L} = \frac{1}{4} tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}), \tag{4.17}$$

where the internal metric $\mathcal{M}$ is defined as $\mathcal{M} = \nu^\# \nu$ and # is the generalized transpose over the Lie group G such that $(exp(g))^\# = exp(g^\#)$. It is induced by the Cartan involution θ over g_0 ($g^\# = -\theta(g)$ for $g \in g_0$). As mentioned in [43] it is possible to find a high dimensional representation of the Lie algebra g_0 in which # coincides with the matrix transpose operator. For this reason one can define an induced $\#$ map over the group G as $(exp(g))^{\#} = exp(g^{\#})$. If the subgroup of G generated by the compact generators is an orthogonal group then in the fundamental representation of g_0 the generators can be chosen such that $g^{\#} = g^T$ for $g \in g_0$, notice that the great majority of the global symmetry groups we have encountered in Chapter two are in this category. If the subgroup of G generated by the compact generators is a unitary group then in the fundamental representation $g^{\#} = g^{\dagger}$ for $g \in g_0$. By using $\nu^{-1}d\nu = -d\nu^{-1}\nu$ and the properties of the generalized transpose $\#$ also the fact that the cyclic permutations are permissable under the trace, the scalar coset Lagrangian (4.17) can be expressed as

$$\mathcal{L} = -\frac{1}{2}tr(*d\nu\nu^{-1} \wedge (d\nu\nu^{-1})^{\#} + *d\nu\nu^{-1} \wedge d\nu\nu^{-1}). \qquad (4.18)$$

We will again assume the parametrization given in (4.5). The Lie algebra valued 1-form $\mathcal{G}_0 = d\nu\nu^{-1}$ is the pullback of a solvable Lie algebra, s_0-valued 1-form on G through the map ν. For this reason it can be expressed in the solvable Lie subalgebra, s_0-basis

$$\begin{aligned}\mathcal{G}_0 &= d\nu\nu^{-1} \\ &= (d\mathbf{g}_H\mathbf{g}_N + \mathbf{g}_H d\mathbf{g}_N)(\mathbf{g}_N^{-1}\mathbf{g}_H^{-1}) \\ &= d\mathbf{g}_H\mathbf{g}_H^{-1} + \mathbf{g}_H d\mathbf{g}_N\mathbf{g}_N^{-1}\mathbf{g}_H^{-1}. \qquad (4.19)\end{aligned}$$

Now in order to simplify the first term, like we have done in the calculation of $\mathcal{G}$

we will use the formula

$$de^{X}e^{-X} = dX + \frac{1}{2!}[X, dX] + \frac{1}{3!}[X, [X, dX]] + ..., \tag{4.20}$$

Therefore we have

$$\begin{aligned} d\mathbf{g}_H\mathbf{g}_H^{-1} &= de^{\frac{1}{2}\phi^i H_i} e^{-\frac{1}{2}\phi^i H_i} \\ &= \frac{1}{2} d\phi^i H_i, \end{aligned} \tag{4.21}$$

where we have used the fact that $[H_i, H_j] = 0$. The other commutation relations of the solvable Lie algebra, s_0 are $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ for $i = 1, ..., r$ and $\alpha, \beta \in \Delta_{nc}^+$. By using the above expansion for $de^X e^{-X}$ we can also calculate $d\mathbf{g}_N\mathbf{g}_N^{-1}$ as

$$\begin{aligned} d\mathbf{g}_N\mathbf{g}_N^{-1} &= de^{\chi^m E_m} e^{-\chi^m E_m} \\ &= d\chi^m E_m + \frac{1}{2!}[\chi^m E_m, d\chi^n E_n] + \frac{1}{3!}[\chi^m E_m, [\chi^l E_l, d\chi^n E_n]] + ... \\ &= d\chi^m E_m + \frac{1}{2!}\chi^m d\chi^n K^v_{mn} E_v + \frac{1}{3!}\chi^m \chi^l d\chi^n K^v_{ln} K^u_{mv} E_u + ... \\ &= \overset{\rightharpoonup}{\mathbf{E}}\, \mathbf{\Omega}\, \overset{\rightharpoonup}{d\chi}. \end{aligned} \tag{4.22}$$

In the compact expression on the right hand side of the last line $\overset{\rightharpoonup}{\mathbf{E}}$ is the row vector of the positive root generators; $\mathbf{E}_\alpha = E_\alpha$ for $\alpha \in \Delta_{nc}^+$ and $\overset{\rightharpoonup}{d\chi}$ is the column vector of the 1-form field strengths $\{d\chi^\alpha\}$ of the axions. The matrix $\mathbf{\Omega}$ is a series which arises from the infinite sum given above

$$\mathbf{\Omega} = \sum_{n=0}^{\infty} \frac{\omega^n}{(n+1)!}$$

$$= (e^{\omega} - I)\, \omega^{-1}. \tag{4.23}$$

$\mathbf{\Omega}$ is a $\dim n_0 \times \dim n_0$ matrix and we have already defined ω before. Now if we use the Campbell-Hausdorff formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + ..., \tag{4.24}$$

we can explicitly write the second term in (4.19) as

$$\mathbf{g}_H d\mathbf{g}_N \mathbf{g}_N^{-1} \mathbf{g}_H^{-1} = e^{\frac{1}{2}\phi^i H_i} (\overrightarrow{\mathbf{E}}\, \mathbf{\Omega}\, \overrightarrow{d\chi}) e^{-\frac{1}{2}\phi^i H_i}$$

$$= \overrightarrow{\mathbf{E}'}\, \mathbf{\Omega}\, \overrightarrow{d\chi}. \tag{4.25}$$

The primed row vector $\overrightarrow{\mathbf{E}'}$ is defined as $\mathbf{E}'_\beta = e^{\frac{1}{2}\beta_i \phi^i} E_\beta$. Finally we can now write $\mathcal{G}_0$ expanded in the solvable Lie subalgebra generators

$$\mathcal{G}_0 = d\nu \nu^{-1}$$

$$= \frac{1}{2} d\phi^i H_i + e^{\frac{1}{2}\alpha_i \phi^i} F^\alpha E_\alpha$$

$$= \frac{1}{2} d\phi^i H_i + \overrightarrow{\mathbf{E}'}\, \mathbf{\Omega}\, \overrightarrow{d\chi}, \tag{4.26}$$

where $\{H_i\}$ for $i = 1, ..., r$ are the generators of h_{p_0} and $\{E_\alpha\}$ for $\alpha \in \Delta_{nc}^+$ are the generators of n_0. We have also introduced the vector $F^\alpha = \mathbf{\Omega}_\beta^\alpha\, d\chi^\beta$. We will express the equations of motion in terms of $\overrightarrow{\mathbf{F}}$. If we use (4.26) in (4.18) we find that

$$\mathcal{L}=-\frac{1}{2}\sum_{i=1}^{r}*d\phi^{i}\wedge d\phi^{i}-\frac{1}{2}\sum_{\alpha\in\Delta_{nc}^{+}}e^{\alpha_{i}\phi^{i}}*F^{\alpha}\wedge F^{\alpha}. \tag{4.27}$$

We have already defined that $\dim h_{p_0}\equiv r$ before and $\alpha\in\Delta_{nc}^{+}$. By following the outline of [43, 57] one can derive the equations of motion of the symmetric space Lagrangian (4.27). We should first observe that

$$d(d\nu\nu^{-1})=-d\nu\wedge d\nu^{-1}$$

$$=d\nu\nu^{-1}\wedge d\nu\nu^{-1}. \tag{4.28}$$

If (4.26) is substituted into this identity one gets the Bianchi identities for $\overrightarrow{\mathbf{F}}$,

$$dF^{\gamma}=\frac{1}{2}\sum_{\alpha+\beta=\gamma}N_{\alpha,\beta}F^{\alpha}\wedge F^{\beta}, \tag{4.29}$$

where $\alpha,\beta,\gamma\in\Delta_{nc}^{+}$. We have already used the Lagrange multiplier method in Chapter two, we will also make use of it here to find the equations of motion. We consider the fields $\{F^{\gamma}\}$ as independent fields. Thus we propose that the Bianchi identities are constraint equations. Then the Lagrange multipliers ($(D-2)$-forms) can be introduced and the additional Lagrangian corresponding to the Bianchi identities can be constructed as

$$\mathcal{L}_{Bianchi}=(dF^{\alpha}-\frac{1}{2}\sum_{\alpha=\beta+\gamma}N_{\beta,\gamma}F^{\beta}\wedge F^{\gamma})\wedge A_{(D-2),\alpha}. \tag{4.30}$$

The new Lagrangian becomes

$$\mathcal{L}'=\mathcal{L}+\mathcal{L}_{Bianchi}. \tag{4.31}$$

The variation with respect to $A_{(D-2),\alpha}$ for $\alpha \in \Delta_{nc}^{+}$ will give back the Bianchi identities. If we vary $\mathcal{L}'$ with respect to F^{γ} and then take the exterior derivative of the resulting field equation we achieve the second-order equations of motion for F^{γ}

$$d(e^{\gamma_i\phi^i} * F^{\gamma}) = \sum_{\alpha-\beta=-\gamma} N_{\alpha,-\beta} F^{\alpha} \wedge e^{\beta_i\phi^i} * F^{\beta}. \tag{4.32}$$

By varying (4.27) with respect to the dilatons $\{\phi^i\}$ (since $\mathcal{L}_{Bianchi}$ does not depend on $\{\phi^i\}$) we can also find the equations of motion for ϕ^i as

$$d(*d\phi^i) = \frac{1}{2} \sum_{\alpha\in\Delta_{nc}^{+}} \alpha_i e^{\frac{1}{2}\alpha_i\phi^i} F^{\alpha} \wedge e^{\frac{1}{2}\alpha_i\phi^i} * F^{\alpha}. \tag{4.33}$$

The details of the formulation given above can be found in [43, 57]. We slightly change (4.32) into a form which is more appropriate for the dualisation analysis

$$\begin{aligned}
d(e^{\frac{1}{2}\gamma_i\phi^i} * F^{\gamma}) &= d(e^{-\frac{1}{2}\gamma_i\phi^i} e^{\gamma_i\phi^i} * F^{\gamma}) \\
&= -\frac{1}{2}\gamma_j e^{-\frac{1}{2}\gamma_i\phi^i} d\phi^j \wedge e^{\gamma_i\phi^i} * F^{\gamma} \\
&\quad + \sum_{\alpha-\beta=-\gamma} e^{-\frac{1}{2}\gamma_i\phi^i} N_{\alpha,-\beta} F^{\alpha} \wedge e^{\beta_i\phi^i} * F^{\beta} \\
&= -\frac{1}{2}\gamma_j e^{\frac{1}{2}\gamma_i\phi^i} d\phi^j \wedge *F^{\gamma} + \sum_{\alpha-\beta=-\gamma} e^{\frac{1}{2}\alpha_i\phi^i} e^{\frac{1}{2}\beta_i\phi^i} N_{\alpha,-\beta} F^{\alpha} \wedge *F^{\beta}.
\end{aligned} \tag{4.34}$$

The second-order equations (4.33) and (4.34) are the ones which may be referred to derive the commutation relations of the dualized generators when we construct the doubled field strength which will give the correct first-order equations by satisfying a twisted self-duality condition.

We will now construct the transformation between the two parametrizations given in (3.56) and (4.5) which are based on two different sets of scalar function definitions. We can derive a method to calculate the transformations between these two sets. We may assume that the coset valued maps in (3.56) and (4.5) are chosen to be equal. This is possible if we restrict the scalars to the ones which generate ranges in the sufficiently small neighborhoods around the identity element of g_0 when they are coupled to the algebra generators in (3.56) and (4.5), [44]-[49]. This local equality is sufficient since our aim is to obtain the local first-order formulation of the parametrization of (3.56) from the first-order formulation which is based on (4.5) in the next section. We will firstly discuss a method through which one can calculate the exact transformations from $\{\varphi^i\}$ to $\{\phi^j, \chi^m\}$. We will not attempt to solve the explicit transformation functions which are dependent on the structure constants in a complicated way. One can solve these set of differential equations when the structure constants are specified. Let us first define the function

$$f(\lambda) = e^{\lambda(\frac{1}{2}\phi^i H_i)} e^{\lambda(\chi^\alpha E_\alpha)}. \tag{4.35}$$

Taking the derivative of $f(\lambda)$ with respect to λ gives

$$\frac{\partial f(\lambda)}{\partial \lambda} f^{-1}(\lambda) = \frac{1}{2}\phi^i H_i + e^{\frac{\lambda}{2}\phi^i \alpha_i} \chi^\alpha E_\alpha. \tag{4.36}$$

We have used the fact that

$$e^{\frac{\lambda}{2}\phi^i H_i} \chi^\alpha E_\alpha e^{-\frac{\lambda}{2}\phi^i H_i} = e^{\frac{\lambda}{2}\phi^i \alpha_i} \chi^\alpha E_\alpha. \tag{4.37}$$

Now if we let $f(\lambda) = e^{C(\lambda)}$ where we define $C(\lambda) = \varphi^i(\lambda)T_i$ and use the formula $de^C e^{-C} = dC + \frac{1}{2!}[C, dC] + \frac{1}{3!}[C, [C, dC]] + ...$ we find that

$$\frac{1}{2}\phi^i H_i + e^{\frac{\lambda}{2}\phi^i \alpha_i}\chi^\alpha E_\alpha = \vec{\mathbf{T}}\, \mathbf{S}(\lambda)\, \vec{\partial\varphi}, \tag{4.38}$$

where the components of the row vector $\vec{\mathbf{T}}$ are $T_i = H_i$ for $i = 1, ..., r$ and $T_{\alpha+r} = E_\alpha$ for $\alpha = 1$,...,dimn_0. Notice that in general we assume that we enumerate the roots in Δ^+_{nc}. Besides, the column vector $\vec{\partial\varphi}$ is defined as $\{\frac{\partial \varphi^i(\lambda)}{\partial \lambda}\}$. We have also introduced the dim$s_0\times$dims_0 matrix $\mathbf{S}(\lambda)$ as

$$\mathbf{S}(\lambda) = \sum_{n=0}^{\infty} \frac{V^n(\lambda)}{(n+1)!}$$

$$= (e^{V(\lambda)} - I)V^{-1}(\lambda). \tag{4.39}$$

The matrix $V(\lambda)$ is $V^\beta_\alpha(\lambda) = \varphi^i(\lambda)C^\beta_{i\alpha}$ for $[T_i, T_j] = C^k_{ij}T_k$. The calculation of the right hand side of (4.38) is similar to (4.26). If the structure constants are given for a particular s_0-basis one can obtain the functions $\{\varphi^i(\lambda)\}$ from the set of differential equations (4.38). Then setting $\lambda = 1$ will yield the desired set of transformations $\{\varphi^i(\phi^j, \chi^\alpha)\}$. We might also make use of a direct calculation namely the Lie's theorem [44]-[49]. For a matrix representation and in a neighborhood of the identity if we let $e^C = e^A e^B$ then

$$C = B + \int_0^1 g(e^{tadA}e^{adB})Adt$$

$$= A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] - [B, [B, A]]) + ..., \tag{4.40}$$

where $g \equiv lnz/z - 1$. In the above equation if we choose $A = \frac{1}{2}\phi^i H_i$, $B = \chi^\alpha E_\alpha$ and $C = \varphi^i T_i$ we can calculate the transformations we need.

We may also derive the differential form of the transformations between the two parametrizations which are more essential for our purpose of obtaining the local first-order formulation of the parametrization in (3.56) in the next section. From (3.56) keeping in mind that $\{T_i\}$ is a basis of s_0, similar to the previous calculations we can calculate the s_0-valued Cartan form $d\nu\nu^{-1}$ as

$$d\nu\nu^{-1} = de^{\varphi^i T_i} e^{-\varphi^i T_i}$$

$$= \overrightarrow{\mathbf{T}}\, \mathbf{\Delta}\, \overrightarrow{\mathbf{d}\varphi}. \tag{4.41}$$

We have defined $\overrightarrow{\mathbf{T}}$ before, $\overrightarrow{\mathbf{d}\varphi}$ is a column vector of the field strengths $\{d\varphi^i\}$ and the $\dim s_0 \times \dim s_0$ matrix $\mathbf{\Delta}$ can be given as

$$\mathbf{\Delta} = \sum_{n=0}^{\infty} \frac{M^n}{(n+1)!}$$

$$= (e^M - I)M^{-1}, \tag{4.42}$$

where $M_\alpha^\beta = \varphi^i C_{i\alpha}^\beta$. We should imply that $\mathbf{\Delta} = \mathbf{S}(\lambda = 1)$ and $M = V(\lambda = 1)$. If we refer to the equation (4.26), we have already calculated the s_0-valued Cartan form $d\nu\nu^{-1}$ for the parametrization of (4.5). Therefore if we compare (4.26) and (4.41), since locally they must be equal we find that

$$\mathbf{\Delta}_i^\gamma d\varphi^i = \frac{1}{2} d\phi^\gamma,$$

$$\mathbf{\Delta}_i^\beta d\varphi^i = e^{\frac{1}{2}\beta_j\phi^j}\mathbf{\Omega}_k^\beta d\chi^k. \tag{4.43}$$

The indices above are $\gamma = 1, ..., r$ and $\beta, k = r+1, ...,$dimn_0 also $i = 1$,...,dims_0 $= r+$dimn_0. As a result we have obtained the differential form of the transformation between $\{\varphi^i\}$ and $\{\phi^j, \chi^\alpha\}$. It can be seen that the relation between the two scalar parametrizations is dependent on the structure constants in a very complicated way. One can also integrate (4.43) to obtain the explicit form of the transformations as an alternative to the equation (4.38).

Finally we will calculate the field equations (4.4) for the parametrization (3.56) by assuming the Iwasawa decomposition. Similar to (4.41) from (3.56) the s_0-valued Cartan form $\nu^{-1}d\nu$ is

$$\nu^{-1}d\nu = e^{-\varphi^i T_i} de^{\varphi^i T_i}$$

$$= \vec{\mathbf{T}}\,\mathbf{W}\,\vec{\mathbf{d}}\varphi, \tag{4.44}$$

where we have

$$\mathbf{W} = \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n+1)!}$$

$$= (I - e^{-M})M^{-1}. \tag{4.45}$$

From (4.44) like in (4.13) we see that $Q_\mu = 0$ due to the solvable Lie algebra parametrization. On the other hand P_μ is

$$P_\mu = P^i_\mu T_i$$

$$= (\mathbf{W})^{\ l}_{k}\, \partial_\mu \varphi^k\, T_l. \tag{4.46}$$

Thus in terms of the fields $\{\varphi^i\}$ the second-order equations (4.4) become

$$\partial^\mu P_\mu = \partial^\mu \left((\mathbf{W})^{\ l}_{k}\, \partial_\mu \varphi^k T_l \right) = 0. \tag{4.47}$$

4.2 Dualisation and the First-Order Formulation

In this section we will introduce the local, first-order formulation of the G/K symmetric space sigma model when G is a semi-simple, non-compact real form and K is its maximal compact subgroup. We will apply the standard dualisation method of [10, 19] by introducing dual generators for the solvable Lie subgroup generators (i.e. the basis we use for the solvable Lie algebra s_0) and also new auxiliary fields, $(D-2)$-forms for the scalar fields. Then the enlarged Lie superalgebra which contains the original solvable Lie algebra will be inspected so that it would realize the original second-order equations in an enlarged coset model. After calculating the extra commutation relations coming from the new generators, locally the first-order equations will be given as a twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ where $\mathcal{G}'$ is the doubled field strength (the Cartan form generated by the new coset representative) and $\mathcal{S}$ is a pseudo-involution of the enlarged Lie superalgebra which maps the original generators onto the dual ones and the dual generators onto the original scalar generators with a sign factor depending on the dimension D and the signature s of the spacetime. We will see more general dualisation examples in the following chapters. The split, rigid symmetry group symmetric space sigma model is a limiting case as it is clear from the constructions of Chapter three. The solvable Lie algebra is a subalgebra of the Borel algebra in general. Therefore the results we present here cover the split symmetric space sigma model as a special case.

Firstly we will introduce dual $(D-2)$-form fields for the dilatons and the axions which are defined in (4.5). The dual fields will be denoted as $\{\widetilde{\phi}^i\}$ and $\{\widetilde{\chi}^m\}$. For each scalar generator we will also define the dual generators which will extend the solvable Lie algebra s_0 to a Lie superalgebra which generates a differential algebra together with the local differential form algebra [19]. These generators are $\{\widetilde{E}_m\}$ as duals of $\{E_m\}$ and $\{\widetilde{H}_i\}$ for $\{H_i\}$. If we define a new parametrization into the enlarged group as

$$\nu'(x) = e^{\frac{1}{2}\phi^i H_i} e^{\chi^m E_m} e^{\widetilde{\chi}^m \widetilde{E}_m} e^{\frac{1}{2}\widetilde{\phi}^i \widetilde{H}_i}, \tag{4.48}$$

we can calculate the doubled field strength $\mathcal{G}' = d\nu'\nu'^{-1}$ in terms of the unknown structure constants and then use the twisted self-duality condition $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ primarily to calculate the structure constants of the dual generators from the comparison of the Cartan-Maurer equation with the second-order equations and finally we can derive the first-order equations of motion. The general form of the commutation relations in addition to the ones of s_0 can be given as [10, 19]

$$[E_\alpha, \widetilde{T}_m] = \widetilde{f}^n_{\alpha m} \widetilde{T}_n \quad , \quad [H_i, \widetilde{T}_m] = \widetilde{g}^n_{im} \widetilde{T}_n,$$

$$[\widetilde{T}_m, \widetilde{T}_n\} = 0, \tag{4.49}$$

where $\widetilde{T}_i = \widetilde{H}_i$ for $i = 1, ..., r$ and $\widetilde{T}_{\alpha+r} = \widetilde{E}_\alpha$ for $\alpha = 1, ...,\dim n_0$. In general the pseudo-involution $\mathcal{S}$ maps the original generators onto the dual ones and when $\mathcal{S}^2$ acts on the dual generators it has the same eigenvalues ± 1 with the action of the operator $* \cdot *$ on the corresponding dual field strength of the coupling dual potential. Therefore $\mathcal{S}T_i = \widetilde{T}_i$ and $\mathcal{S}\widetilde{T}_i = (-1)^{(p(D-p)+s)}T_i$ where p is the degree of the dual field strength and s is the signature of the spacetime. The degree of the dual field strengths corresponding to the dual generators is $(D-1)$ in our case and the signature is $s = 1$. Thus the sign factor above is dependent on the spacetime dimension D and we have $\mathcal{S}\widetilde{T}_i = (-1)^D T_i$. Now by following the same steps in [10, 19] and using the twisted self-duality condition we can express the doubled field strength as

$$\mathcal{G}' = d\nu\nu^{-1} + \frac{1}{2}(-1)^D * d\phi^i \widetilde{H}_i + (-1)^D e^{\frac{1}{2}\alpha_i\phi^i} * F^\alpha \widetilde{E}_\alpha. \tag{4.50}$$

The Cartan form $\mathcal{G}_0 = d\nu\nu^{-1}$ is already calculated in (4.26). By following the discussion in [19] we conclude that the generators $\{T_i\}$ are even and $\{\widetilde{T}_i\}$ are even or odd depending on the spacetime dimension D within the context of the differential algebra generated by the solvable Lie algebra generators, their duals and the differential forms. A generator K_i or $\widetilde{K}_i$ is even or odd whether the corresponding potentials, m-forms or $(D-m-2)$-forms are even or odd rank in the dualisation analysis in general. Thus since scalars are zero-forms $\{T_i\}$ are even. The even-odd separation of the generators is due to the $\mathbb{Z}_2$ grading of the Lie superalgebra we construct. A pair of even generators or an even and an odd generator obey the commutation relations while a pair of odd generators obey the anti-commutation relations. For our analysis we keep track of the evenness or the oddness of the dual generators within the differential algebra structure however since the dual generators commute with themselves we only have commutation relations in our enlarged algebra structure. By using the properties of this differential algebra and the fact that from its definition $\mathcal{G}'$ obeys the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0, \tag{4.51}$$

we can show that if we choose

$$[T_i, \widetilde{H}_j] = 0,$$

$$[H_j, \widetilde{E}_\alpha] = -\alpha_j \widetilde{E}_\alpha \quad , \quad [E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r} \alpha_j \widetilde{H}_j,$$

$$[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta} \widetilde{E}_\gamma, \qquad \alpha - \beta = -\gamma, \quad \alpha \neq \beta, \tag{4.52}$$

for $i = 1, ..., \dim s_0$, $j = 1, ..., r$ and $\alpha, \beta, \gamma \in \Delta_{nc}^+$, then (4.51) by inserting (4.50) gives the correct second-order equations (4.33) and (4.34). As a matter of fact if we choose in general $[E_\alpha, \widetilde{H}_j] = a_{\alpha j} \alpha_j \widetilde{E}_\alpha$ and $[H_j, \widetilde{E}_\alpha] = b_{j\alpha} \alpha_j \widetilde{E}_\alpha$ with $a_{j\alpha}$, $b_{\alpha j}$ arbitrary but obeying the constraint $a_{j\alpha} + b_{\alpha j} = -1$ in addition to the rest of the commutators in (4.52) we can satisfy the second-order equations (4.33) and (4.34). However for simplicity we will choose $a_{j\alpha} = 0$ as seen in (4.52). Therefore the structure constants in (4.49) become

$$\widetilde{f}^{n}_{\alpha m} = 0, \qquad m \leq r \quad , \quad \widetilde{f}^{i}_{\alpha,\alpha+r} = \frac{1}{4}\alpha_i, \qquad i \leq r,$$

$$\widetilde{f}^{i}_{\alpha,\alpha+r} = 0, \qquad i > r \quad , \quad \widetilde{f}^{i}_{\alpha,\beta+r} = 0, \qquad i \leq r, \quad \alpha \neq \beta,$$

$$\widetilde{f}^{\gamma+r}_{\alpha,\beta+r} = N_{\alpha,-\beta}, \qquad \alpha - \beta = -\gamma, \quad \alpha \neq \beta,$$

$$\widetilde{f}^{\gamma+r}_{\alpha,\beta+r} = 0, \qquad \alpha - \beta \neq -\gamma, \quad \alpha \neq \beta. \tag{4.53}$$

Also

$$\widetilde{g}^{n}_{im} = 0, \qquad m \leq r \quad , \quad \widetilde{g}^{n}_{im} = 0, \qquad m > r, \quad m \neq n,$$

$$\widetilde{g}^{\alpha}_{i\alpha} = -\alpha_i, \qquad \alpha > r. \tag{4.54}$$

These relations with the commutation relations of the solvable Lie algebra s_0 form the complete algebraic structure of the enlarged Lie superalgebra. In (4.50) $\mathcal{G}'$ has been given only in terms of the original scalar fields as the twisted self-duality condition has been used primarily. From the definition of $\nu'(x)$ in (4.48), since we have obtained the full set of commutation relations without using the twisted self-duality condition we can explicitly calculate the Cartan form $\mathcal{G}'$ in terms of both the scalar fields and their duals.

$$\mathcal{G}' = d\nu'\nu'^{-1}$$

$$= \frac{1}{2} d\phi^i H_i + \overrightarrow{\mathbf{E}'}\, \mathbf{\Omega}\, \overrightarrow{d\chi} + \overrightarrow{\widetilde{\mathbf{T}}}\, e^{\mathbf{\Gamma}}\, e^{\mathbf{\Lambda}}\, \overrightarrow{\mathbf{A}}. \tag{4.55}$$

We have introduced the matrices

$$\begin{aligned} (\Gamma)_n^k &= \frac{1}{2} \phi^i\, \widetilde{g}_{in}^k, \\ \\ (\Lambda)_n^k &= \chi^m\, \widetilde{f}_{mn}^k. \end{aligned} \tag{4.56}$$

The components of the row vector $\overrightarrow{\widetilde{\mathbf{T}}}$ are defined as $\{\widetilde{\mathbf{T}}^i\}$ and the components of the column vector $\overrightarrow{\mathbf{A}}$ are $\mathbf{A}^i = \frac{1}{2} d\widetilde{\phi}^i$ for $i = 1, ..., r$ and $\mathbf{A}^{\alpha+r} = d\widetilde{\chi}^\alpha$, $\alpha \in \Delta_{nc}^+$ in other words if we enumerate the roots in Δ_{nc}^+ then $\alpha = 1, ..., \mathrm{dim} n_0$. When we apply the twisted self-duality condition $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ on (4.55) we may achieve the first-order equations locally whose exterior derivative will give the second-order equations (4.33) and (4.34). Therefore the locally integrated first-order field equations of the Lagrangian (4.17) are

$$*\overrightarrow{\mathbf{\Psi}} = (-1)^D e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}} \overrightarrow{\mathbf{A}}. \tag{4.57}$$

The components of the column vector $\overrightarrow{\mathbf{\Psi}}$ are defined as $\mathbf{\Psi}^i = \frac{1}{2} d\phi^i$ for $i = 1, ...r$ and $\mathbf{\Psi}^{\alpha+r} = e^{\frac{1}{2}\alpha_i \phi^i} \mathbf{\Omega}_l^\alpha d\chi^l$ where $\alpha = 1, ..., \mathrm{dim} n_0$. Due to the assumed signature of the spacetime these equations have a sign factor depending on the spacetime dimension.

Following the discussion in the previous section we can also find the first-order equations for the set $\{\varphi^i\}$. Firstly we can define the transformed matrices

$\mathbf{\Gamma}'(\varphi^j) \equiv \mathbf{\Gamma}(\phi^i(\varphi^j))$ and $\mathbf{\Lambda}'(\varphi^j) \equiv \mathbf{\Lambda}(\chi^m(\varphi^j))$ which can be obtained by calculating the local transformation rules from (4.38) or (4.43). If we make the observation that the right hand side of the differential form of the transformations between $\{\varphi^i\}$ and $\{\phi^i, \chi^\alpha\}$ namely (4.43) are the components of $\overrightarrow{\mathbf{\Psi}}$, from (4.57) we can write down the first-order equations for $\{\varphi^i\}$ as

$$\mathbf{\Delta} * \overrightarrow{\mathbf{d}\varphi} = (-1)^D e^{\mathbf{\Gamma}'} e^{\mathbf{\Lambda}'} \overrightarrow{\mathbf{A}}. \tag{4.58}$$

We may also transform (4.58) so that we do not have to calculate the explicit transformations between the fields. Firstly we should observe that the structure constants $\{\widetilde{g}^k_{in}\}$ and $\{\widetilde{f}^k_{mn}\}$ of (4.49) form a representation for s_0 as the representatives of the basis $\{H_i\}$ and $\{E_m\}$ respectively. Thus under the representation

$$\begin{aligned} e^{\frac{1}{2}\phi^i H_i} e^{\chi^m E_m} &\equiv e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}} \\ &= e^{\frac{1}{2}\phi^i \widetilde{g}^k_{in}} e^{\chi^m \widetilde{f}^k_{mn}}. \end{aligned} \tag{4.59}$$

In the last section we have assumed that locally

$$e^{\varphi^i T_i} = e^{\frac{1}{2}\phi^i H_i} e^{\chi^m E_m}. \tag{4.60}$$

Therefore the first-order equations (4.58) for $\{\varphi^i\}$ can be written as

$$\mathbf{\Delta} * \overrightarrow{\mathbf{d}\varphi} = (-1)^D e^{\mathbf{\Pi}} \overrightarrow{\mathbf{A}}, \tag{4.61}$$

where we have defined

$$(\mathbf{\Pi})^k_n = \sum_{i=1}^{r} \varphi^i \widetilde{g}^k_{in} + \sum_{m=r+1}^{\dim n_0} \varphi^m \widetilde{f}^k_{mn}, \tag{4.62}$$

by using the representation established by (4.49).

One may also obtain the first-order equations presented above independently by applying the dualisation method on the parametrization (3.56). We again assume the solvable Lie algebra gauge. Let us first define the doubled coset map

$$\nu'' = e^{\varphi^i T_i} e^{\widetilde{\varphi}^i \widetilde{T}_i}, \tag{4.63}$$

in which we have introduced the dual fields and the dual generators as usual. If we calculate the Cartan form $\mathcal{G}'' = d\nu''\nu''^{-1}$ by carrying out similar calculations like we have done before we find that

$$\begin{aligned}\mathcal{G}'' &= de^{\varphi^i T_i} e^{-\varphi^i T_i} + e^{\varphi^i T_i} de^{\widetilde{\varphi}^i \widetilde{T}_i} e^{-\widetilde{\varphi}^i \widetilde{T}_i} e^{-\varphi^i T_i} \\ &= \overrightharpoon{\mathbf{T}}\, \mathbf{\Delta}\, \overrightharpoon{\mathbf{d}\varphi} + \overrightharpoon{\widetilde{\mathbf{T}}}\, e^{\mathbf{\Pi}}\, \overrightharpoon{\mathbf{d}\widetilde{\varphi}}.\end{aligned} \tag{4.64}$$

The first term has already been calculated in (4.41). One can show that the standard method of the derivation of the structure constants in (4.64) which are left unknown yields the same results given in (4.53) and (4.54). This is due to the local transformation laws given in (4.43), if one uses (4.43) in (4.64), similar to the transformation we have done to obtain (4.58) from (4.57) one can show that (4.64) is equal to (4.50) indeed by also using the twisted self-duality condition. Therefore one would reach the same structure constants related to the dual generators which we have already calculated in (4.53) and (4.54). If we apply the twisted self-duality equation $*\mathcal{G}'' = \mathcal{S}\mathcal{G}''$ above we find the first-order equations as

$$\mathbf{\Delta} * \overrightharpoon{\mathbf{d}\varphi} = (-1)^D e^{\mathbf{\Pi}} \overrightharpoon{\mathbf{d}\widetilde{\varphi}}. \tag{4.65}$$

Since the dual fields are auxiliary fields we can always choose $(\mathbf{d}\widetilde{\varphi})^i = \mathbf{A}^i$, this corresponds to a dual field redefinition in the doubled coset element (4.63). Thus the equations (4.61) and (4.65) are the same equations. This result verifies the validity of (4.61) which is obtained by using the transformation law (4.43) in (4.57).

We should point out the fact that the case when the global symmetry group G is in split real form can be obtained by choosing $r = l$ (the rank of G) and $\Delta^+_{nc} = \Delta^+$ in the expressions given in this section.

In conclusion starting from the general non-linear sigma model concepts we have presented the symmetric space sigma model Lagrangian and formulated the second and the first-order equations of motion for two different coset maps which are based on the usual solvable Lie algebra gauge.

CHAPTER 5

DUALISATION OF THE MAXIMAL SUPERGRAVITIES

In Chapter two we have introduced and studied the Kaluza-Klein descendant supergravity theories which are produced by the dimensional reduction of the $D = 11$ supergravity as well as the Abelian Yang-Mills supergravities which are the lower dimensional descendants of the ten dimensional simple supergravity which is coupled to N Abelian $U(1)$ gauge fields. We have also presented the symmetric space sigma model construction of the scalar sectors of these theories and later we have discussed the symmetric spaces and the symmetric space sigma model in detail in general terms. A remarkable feature of these theories is that in certain dimensions one has to introduce Lagrange multiplier scalar fields and eliminate some of the higher order fields in terms of the field strengths of these scalars in order to increase the number of scalars in the theory to formulate the scalar sector as a symmetric space sigma model or enhance the already existing coset structure. Therefore we have the motivation that by the method of dualisation and by introducing auxiliary fields one can establish a non-linear coset formulation at least partially, merely for the scalars in the above mentioned supergravities. The non-linear nature of the scalar sectors of the maximal supergravities has been improved to include the matter fields in order to formulate the non-gravitational Bosonic field equations as generations of enlarged coset formulations in [10, 19]. The method of [10, 19] includes the dualisation of the entire non-gravitational Bosonic field content in a systematical way. The route followed in the dualisation can be in various equivalent ways. In all of these routes basically one starts by introducing dual fields for the non-gravitational original Bosonic fields of the relative theory excluding the graviton and then one defines generic algebra generators for the original fields and their duals. A coset element $\nu(x)$ which takes values in a generic group and which assumes an intuitively chosen form of the exponentiation of the generators coupled to the fields may be constructed first. We assume that the Cartan form $\mathcal{G}$ calculated from this doubled coset element obeys the twisted self-duality condition $*\mathcal{G} = \mathcal{S}\mathcal{G}$. When one uses this fact to express the Cartan form in terms of the original fields only, the result is intended to yield the correct original second-order field equations when inserted in the Cartan-Maurer equation. In order to calculate the Cartan form one needs to know the algebra structure of the generators which parameterize the proposed coset representative when coupled to the fields. The algebra which is used to construct the coset representative is a differential graded algebra generated by the differential forms and the field generators. It covers the Lie superalgebra of the field generators which is the Lie algebra of the symmetry group of the Cartan form. The correct choice of the commutation and the anti-commutation relations of the Lie superalgebra is a result of the direct comparison of the second-order field equations and the Cartan-Maurer equation expressed in terms of the unknown structure constants of the commutators and the anti-commutators of the generators. When the commutators and the anti-commutators are calculated, one may derive the Cartan form $\mathcal{G}$ explicitly and show that the twisted self-duality equation $*\mathcal{G} = \mathcal{S}\mathcal{G}$, [10, 19], when applied on the Cartan form leads to the first-order formulation of the equations of motion. The first-order formulation can be obtained from the original second-order field equations by integration which results in introducing auxiliary fields which are nothing but the dual fields used in the construction of the doubled coset representative.

As we have done when we applied the Lagrange multiplier method to dualize the higher rank fields with the Lagrange multiplier scalars in the second chapter, we will assign a $(D-m-2)$-form dual potential $\widetilde{A}^i$ for each original m-form field A^i in the relative D-dimensional theory. The generators K_i and $\widetilde{K}_i$ are assigned to the fields A^i, $\widetilde{A}^i$ respectively. We assume that the generators K_i and $\widetilde{K}_i$ together with the differential forms generate a differential graded algebra and further assume that the generators K_i and $\widetilde{K}_i$ merely, generate a Lie superalgebra which has a $\mathbb{Z}_2$ grading. We should bear in mind that the generators K_i and $\widetilde{K}_i$ are of even or odd degree depending on whether their corresponding potentials, m-forms and $(D-m-2)$-forms, are of even or odd rank. A pair of even generators and a pair of even and an odd generator obey commutation relations which we denote as $[\ ,\ ]$ while a pair of odd generators obey an anti-commutation relation which we denote by $\{\ ,\ \}$. In the differential graded algebra structure even or odd generators behave like even or odd degree forms when commuting with the differential forms. Also if T is an odd generator $d(TA) = -TdA$ and if it is even $d(TA) = TdA$, where d is the differential operator of the differential graded algebra. The next task is to construct the map $\nu(A^i, \widetilde{A}^i, K_i, \widetilde{K}_i) : M \longrightarrow G$ from the D-dimensional spacetime M to a generic group G. We will see that we may construct ν in a particular, dimension-independent way. The Cartan form $\mathcal{G}$ is defined as

$$\mathcal{G} = d\nu\nu^{-1}. \tag{5.1}$$

From its definition the Cartan form $\mathcal{G}$ satisfies the Cartan-Maurer equation

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0. \tag{5.2}$$

As we have mentioned before the Cartan form $\mathcal{G}$ can be expanded in terms of the generators $K_i, \widetilde{K}_i$ if one knows the commutation and the anti-commutation relations of the generators. Following the outline of [19] we assume that the Cartan form $\mathcal{G}$ satisfies a twisted self-duality equation

$$*\mathcal{G} = \mathcal{S}\mathcal{G}. \tag{5.3}$$

In general the pseudo-involution $\mathcal{S}$ maps the original generators onto the dual ones and $\mathcal{S}^2$, when applied on the dual generators, has the same eigenvalues ± 1 with the action of the operator $* \cdot *$ on the corresponding field strength of the coupling dual potential. Therefore $\mathcal{S}K_i = \widetilde{K}_i$ and $\mathcal{S}\widetilde{K}_i = (-1)^{(p(D-p)+s)}K_i$, where $p = D - m - 1$ is the degree of the dual field strength of the dual potential which the generator $\widetilde{K}_i$ is assigned to, and s is the signature of the spacetime which is equal to 1.

When one expands the Cartan form in terms of the generators $K_i, \widetilde{K}_i$, uses the twisted self-duality condition (5.3) to eliminate the dual fields and inserts the result in (5.2) one obtains the original second-order field equations for the potentials A^i [19]. We may use this framework to calculate the unknown commutation and anti-commutation relations. As we have pointed out before, the sequence we have described above may be followed by expressing everything in terms of the unknown structure constants and then the results may be compared with the second-order equations of motion to read the structure constants of the commutation and the anti-commutation relations. Equivalently we may first integrate the second-order equations locally by writing them in the form of a closed form. Since locally a closed form is an exact one, we may introduce auxiliary fields to pull out the exterior derivative and to obtain the first-order equations of motion. The twisted self-duality equation (5.3) gives the first-order equations and the dual potentials are nothing but the auxiliary fields which are defined in the integration process. One may also compare the first-order equations with the twisted self-duality equation to read the structure constants which justify this coset formulation.

There is another route of dualisation which differs from the one mentioned above in the calculation of the structure constants. Similarly one first integrates the second-order equations to find the first-order field equations. In this second method, one does not start with the coset representative but directly with the Cartan form which is constructed in a way that will give the first-order equations via the twisted self-duality equation. By using the twisted self-duality condition one may write the Cartan form in terms of the original fields only and again insert the result in the Cartan-Maurer equation. Then one calculates the structure constants from the comparison of the Cartan-Maurer equation with the second-order equations of motion. Finally one constructs the coset representative which will lead to the Cartan form by using the already calculated structure constants. This second method which will be used in this chapter together with the first one is similar to the method we have used for the construction of the $O(10-D, 10-D+N)/(O(10-D)\times O(10-D+N))$ scalar cosets of the D-dimensional theories which are the descendants of the ten dimensional Abelian Yang-Mills supergravity. An abstract set of generators is conjectured to yield the original theory through the Cartan form of an unknown coset map in each case. However, in the calculation of the algebraic structure of the generators, one makes use of the Lagrangian in the scalar coset case given in Chapter two while in the complete dualisation presented in this chapter the field equations are used to obtain the commutation and the anti-commutation relations of the generators.

As a result the twisted self-duality structure of the supergravities [32, 33] can be generalized to regain the first-order equations of motion of the $D=11$, IIA, IIB and the D-dimensional maximal supergravity theories from the Cartan forms of the dualized coset. In [34, 35, 36] more general coset formulations of the IIA, $D=11$ and the IIB supergravity theories are introduced to include the gravity as well. We will ignore the gravity sector in this chapter and we will focus on the complete dualisation of the non-gravitational Bosonic fields in parallel with the construction given in [10, 19].

We will start with the $D=11$ supergravity whose Bosonic Lagrangian is given in [19] as

$$\mathcal{L}_{11} = R * 1 - \frac{1}{2} * F_{(4)} \wedge F_{(4)} - \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)}, \tag{5.4}$$

where $F_{(4)} = dA_{(3)}$. If we vary (5.4) with respect to $A_{(3)}$ then we obtain the corresponding equation of motion as

$$d * F_{(4)} + \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0. \tag{5.5}$$

We may also write this equation as

$$d(*F_{(4)} + \frac{1}{2} A_{(3)} \wedge F_{(4)}) = 0. \tag{5.6}$$

Thus if we introduce the dual six-form potential $\widetilde{A}_{(6)}$ then we locally have the first-order equation of motion

$$\widetilde{F}_{(7)} \equiv *F_{(4)} = d\widetilde{A}_{(6)} - \frac{1}{2} A_{(3)} \wedge F_{(4)}, \tag{5.7}$$

where we have also defined the dual field strength $\widetilde{F}_{(7)}$ of the dual field $\widetilde{A}_{(6)}$. Now we will introduce the generators V and $\widetilde{V}$ to the fields $A_{(3)}$ and $\widetilde{A}_{(6)}$, respectively. Following the discussion above we conclude that V is an odd generator while $\widetilde{V}$ is an even one. We will assume a coset representative of the form

$$\nu(x) = e^{A_{(3)} V} e^{\widetilde{A}_{(6)} \widetilde{V}}. \tag{5.8}$$

The pseudo-involution has the action on the generators as

$$\mathcal{S}V = \widetilde{V} \quad , \quad \mathcal{S}\widetilde{V} = -V. \tag{5.9}$$

If we calculate the Cartan form $\mathcal{G} = d\nu\nu^{-1}$ and compare the twisted self-duality equation (5.3) with the first-order equation (5.7), then we derive the commutation and the anti-commutation relations as

$$\{V,V\} = -\widetilde{V} \quad , \quad [V,\widetilde{V}] = [\widetilde{V},\widetilde{V}] = 0. \tag{5.10}$$

By using these relations we can calculate the Cartan form explicitly as

$$\mathcal{G} = dA_{(3)}V + (d\widetilde{A}_{(6)} - \frac{1}{2}A_{(3)} \wedge F_{(4)})\widetilde{V}. \tag{5.11}$$

If we use the twisted self-duality condition or the first-order equation (5.7) with (5.11), we have

$$\mathcal{G} = F_{(4)}V + *F_{(4)}\widetilde{V}. \tag{5.12}$$

We also have

$$\mathcal{G} \wedge \mathcal{G} = -\frac{1}{2}F_{(4)} \wedge F_{(4)}\widetilde{V}. \tag{5.13}$$

Therefore the Cartan-Maurer equation $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$ gives the second-order equation of motion (5.5) and the Bianchi identity of $F_{(4)}$ which is $dF_{(4)} = 0$. These results justify the choice of the coset element (5.8) and the structure constants in (5.10) of the Lie superalgebra generated by V and $\widetilde{V}$. This concludes the coset construction of the Bosonic sector of the eleven dimensional supergravity.

When we consider the Kaluza-Klein reduction of (5.4) on S^1, we get the ten dimensional IIA supergravity theory whose Bosonic Lagrangian can be given as

$$\begin{aligned}\mathcal{L}_{IIA} = R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2}e^{-\frac{1}{2}\phi} * F_{(4)} \wedge F_{(4)} \\ - \frac{1}{2}e^{\phi} * F_{(3)} \wedge F_{(3)} - \frac{1}{2}e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \frac{1}{2}dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)},\end{aligned} \tag{5.14}$$

where $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}$, $F_{(3)} = dA_{(2)}$, $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$. The corresponding Bosonic field equations are

$$d(e^{-\frac{1}{2}\phi} * F_{(4)}) = -F_{(4)} \wedge F_{(3)},$$

$$d(e^{\phi} * F_{(3)}) = -\mathcal{F}_{(2)} \wedge (e^{-\frac{1}{2}\phi} * F_{(4)}) - \frac{1}{2}F_{(4)} \wedge F_{(4)},$$

$$d(e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)}) = -F_{(3)} \wedge (e^{-\frac{1}{2}\phi} * F_{(4)}),$$

$$d * d\phi = \frac{1}{4}F_{(4)} \wedge (e^{-\frac{1}{2}\phi} * F_{(4)}) + \frac{1}{2}F_{(3)} \wedge (e^{\phi} * F_{(3)})$$

$$+ \frac{3}{4}\mathcal{F}_{(2)} \wedge (e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)}). \tag{5.15}$$

We may integrate these second-order equations locally to obtain the first-order equations by introducing the dual fields $\{\psi, \widetilde{\mathcal{A}}_{(7)}, \widetilde{A}_{(6)}, \widetilde{A}_{(5)}\}$ which are nothing but the corresponding Lagrange multipliers for the Bianchi identities of the field strengths $\{d\phi, \mathcal{F}_{(2)}, F_{(3)}, F_{(4)}\}$ of the potentials $\{\phi, \mathcal{A}_{(1)}, A_{(2)}, A_{(3)}\}$, respectively. The result becomes

$$e^{-\frac{1}{2}\phi} * F_{(4)} = d\widetilde{A}_{(5)} - A_{(2)} \wedge dA_{(3)},$$

$$e^{\phi} * F_{(3)} = d\widetilde{A}_{(6)} - \frac{1}{2}A_{(3)} \wedge dA_{(3)} - \mathcal{A}_{(1)} \wedge (d\widetilde{A}_{(5)} - A_{(2)} \wedge dA_{(3)}),$$

$$e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)} = d\widetilde{\mathcal{A}}_{(7)} - A_{(2)} \wedge (d\widetilde{A}_{(5)} - \frac{1}{2}A_{(2)} \wedge dA_{(3)}),$$

$$
\begin{aligned}
*d\phi = d\psi &+ \frac{1}{2}A_{(2)} \wedge d\widetilde{A}_{(6)} + \frac{1}{4}A_{(3)} \wedge (d\widetilde{A}_{(5)} - A_{(2)} \wedge dA_{(3)}) \\
&+ \frac{3}{4}\mathcal{A}_{(1)} \wedge (d\widetilde{\mathcal{A}}_{(7)} - A_{(2)} \wedge (d\widetilde{A}_{(5)} - \frac{1}{2}A_{(2)} \wedge dA_{(3)})).
\end{aligned} \tag{5.16}
$$

We define the righthand side of these equations as the corresponding field strengths $\{\widetilde{F}_{(6)}, \widetilde{F}_{(7)}, \widetilde{\mathcal{F}}_{(8)}, \widetilde{P}\}$ of the dual fields $\{\widetilde{A}_{(5)}, \widetilde{A}_{(6)}, \widetilde{\mathcal{A}}_{(7)}, \psi\}$ respectively. Therefore the first-order equations (5.16) can also be considered as the transgression relations of the dual field strengths. Following the general discussion we have made before we will introduce the generators $\{H, V, V^1, W_1\}$ for the original potentials $\{\phi, A_{(3)}, A_{(2)}, \mathcal{A}_{(1)}\}$ also the dual generators $\{\widetilde{H}, \widetilde{V}, \widetilde{V}_1, \widetilde{W}^1\}$ for the dual potentials $\{\psi, \widetilde{A}_{(5)}, \widetilde{A}_{(6)}, \widetilde{\mathcal{A}}_{(7)}\}$ respectively. The generators $H, V^1, \widetilde{V}_1, \widetilde{H}$ are even while the ones $V, W_1, \widetilde{V}, \widetilde{W}^1$ are odd. We may inquire the correct form of the Cartan form $\mathcal{G}$ which would lead to the second-order field equations when inserted in the Cartan-Maurer equation (5.2). One may find out how the Cartan form will look by the trial and error method, also by considering the Bianchi identities of the original and the dual field strengths we have defined above [19]. However since we have the first-order equations at hand, the starting point for the construction of the Cartan form is the fact that it should give the first-order equations when the twisted self-duality is used. Thus our task becomes easier by using the first-order equations (5.16). In fact we will see that a very simple ansatz which will give (5.16) upon the application of the twisted self-duality equation will work.

We have made a similar analysis in Chapter two when we have calculated the algebra structure for the coset formulation of the D-dimensional scalar sector in the Kaluza-Klein reduction of the ten dimensional Abelian Yang-Mills supergravity. The general forms of the Cartan forms have common features. For the IIA supergravity we propose the Cartan form $\mathcal{G}$ as

$$\begin{aligned}\mathcal{G} &= d\nu\nu^{-1} \\ &= \frac{1}{2}d\phi H + e^{\frac{1}{2}\phi}F_{(3)}V^1 + e^{-\frac{3}{4}\phi}\mathcal{F}_{(2)}W_1 + e^{-\frac{1}{4}\phi}F_{(4)}V \\ &\quad + e^{\frac{1}{4}\phi}\widetilde{F}_{(6)}\widetilde{V} + e^{-\frac{1}{2}\phi}\widetilde{F}_{(7)}\widetilde{V}_1 + e^{\frac{3}{4}\phi}\widetilde{\mathcal{F}}_{(8)}\widetilde{W}^1 + \frac{1}{2}\widetilde{P}\widetilde{H}.\end{aligned} \tag{5.17}$$

If we use the twisted self-duality once more to express (5.17) merely in terms of the original fields and then insert the result in the Cartan-Maurer equation (5.2), we can read the commutation and the anti-commutation relations of the generators $\{H, V, V^1, W_1, \widetilde{H}, \widetilde{V}, \widetilde{V}_1, \widetilde{W}^1\}$ as

$$[H, W_1] = -\frac{3}{2}W_1 \quad , \quad [H, V^1] = V^1 \quad , \quad [H, V] = -\frac{1}{2}V,$$

$$[H, \widetilde{W}_1] = \frac{3}{2}\widetilde{W}_1 \quad , \quad [H, \widetilde{V}^1] = -\widetilde{V}^1 \quad , \quad [H, \widetilde{V}] = \frac{1}{2}\widetilde{V},$$

$$[W_1, V^1] = -V \quad , \quad \{W_1, \widetilde{V}\} = -\widetilde{V}_1 \quad , \quad [V^1, V] = -\widetilde{V},$$

$$[V^1, \widetilde{V}] = -\widetilde{W}^1 \quad , \quad \{V, V\} = -\widetilde{V}_1 \quad , \quad \{W_1, \widetilde{W}^1\} = \frac{3}{8}\widetilde{H},$$

$$[V^1, \widetilde{V}_1] = \frac{2}{8}\widetilde{H} \quad , \quad \{V, \widetilde{V}\} = \frac{1}{8}\widetilde{H}. \tag{5.18}$$

The rest of the commutators and the anti-commutators which are not listed above vanish. After the calculation of the algebra structure one can also construct the coset element which would lead to the Cartan form in (5.17) as

$$\nu = e^{\frac{1}{2}\phi H} e^{\mathcal{A}_{(1)} W_1} e^{A_{(2)} V^1} e^{A_{(3)} V} e^{\widetilde{A}_{(5)} \widetilde{V}} e^{\widetilde{A}_{(6)} \widetilde{V}_1} e^{\widetilde{\mathcal{A}}_{(7)} \widetilde{W}^1} e^{\frac{1}{2}\psi \widetilde{H}}. \tag{5.19}$$

The action of the pseudo-involution $\mathcal{S}$ on the original and the dual generators can be determined by following the discussion we have made before. We remind the reader of the point that applying the twisted self-duality equation (5.3) on (5.17) gives the first-order equations in (5.16), since that is how we have constructed (5.17). As a result we have a consistent reformulation of the first and the second-order equations of motion by using the Cartan form which is based on the appropriate choice of the coset element (5.19) and the algebra structure given in (5.18).

The Kaluza-Klein reduction of the eleven dimensional supergravity whose Bosonic Lagrangian is given in (5.4) on the n-torus T^n results in the D-dimensional maximal supergravities. The D-dimensional Lagrangian (2.45) has been given in Chapter two, but since we will follow the convention used in [19] in this chapter, one has to consider the modification of the Chern-Simons $\mathcal{L}_{FFA}$ terms given in Chapter two with a sign factor. We will use a slightly different notation than the one which is introduced for the field strengths in (2.50). We define the various field strengths corresponding to the potentials introduced in Chapter two as

$$\mathcal{F}^i_{(2)} = \widetilde{\gamma}^i{}_j \hat{\mathcal{F}}^j_{(2)} \quad , \quad \mathcal{F}^i_{(1)j} = \gamma^k{}_j d\mathcal{A}^i_{(0)k} \quad , \quad F_{(4)} = \hat{F}_{(4)},$$

$$F_{(2)ij} = \gamma^k{}_i \gamma^l{}_j \hat{F}_{(2)kl} \quad , \quad F_{(3)i} = \gamma^j{}_i \hat{F}_{(3)j}, \tag{5.20}$$

where γ^i_j is defined in (2.51) and $\widetilde{\gamma}^i_j$ is the inverse of it. The hatted field strengths are defined as

$$\hat{\mathcal{F}}^i_{(2)} = d\hat{\mathcal{A}}^i_{(1)} \quad , \quad \hat{F}_{(2)ij} = dA_{(1)ij} - dA_{(0)ijk} \wedge \hat{\mathcal{A}}^k_{(1)},$$

$$\hat{F}_{(3)i} = dA_{(2)i} + dA_{(1)ij} \wedge \hat{\mathcal{A}}^j_{(1)} + \frac{1}{2} dA_{(0)ijk} \wedge \hat{\mathcal{A}}^j_{(1)} \wedge \hat{\mathcal{A}}^k_{(1)},$$

$$\hat{F}_{(4)} = dA_{(3)} - dA_{(2)i} \wedge \hat{\mathcal{A}}^i_{(1)} + \frac{1}{2} dA_{(1)ij} \wedge \hat{\mathcal{A}}^i_{(1)} \wedge \hat{\mathcal{A}}^j_{(1)}$$

$$- \frac{1}{6} dA_{(0)ijk} \wedge \hat{\mathcal{A}}^i_{(1)} \wedge \hat{\mathcal{A}}^j_{(1)} \wedge \hat{\mathcal{A}}^k_{(1)}. \tag{5.21}$$

We also define $\hat{\mathcal{A}}^i_{(1)} = \gamma^i_j \mathcal{A}^j_{(1)}$. The D-dimensional first-order equations of motion of (2.45) from the second-order field equations have been derived in a semi-dimension independent way in [19]. They have partially a dimension independent structure but there are also varying contributions in each dimension. The D-dimensional first-order equations can locally be given as

$$e^{\vec{a}\cdot\vec{\phi}} \ast F_{(4)} = \widetilde{F}_{(D-4)} \quad , \quad e^{\vec{a}_i\cdot\vec{\phi}} \ast F_{(3)i} = \widetilde{\gamma}^i_j \widetilde{F}^j_{(D-3)},$$

$$e^{\vec{a}_{ij}\cdot\vec{\phi}} * F_{(2)ij} = \widetilde{\gamma}^i_k \widetilde{\gamma}^j_l \widetilde{F}^{kl}_{(D-2)} \quad , \quad e^{\vec{a}_{ijk}\cdot\vec{\phi}} * F_{(1)ijk} = \widetilde{\gamma}^i_l \widetilde{\gamma}^j_m \widetilde{\gamma}^k_n \widetilde{F}^{lmn}_{(D-1)},$$

$$e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}^i_{(2)} = \gamma^j_i \widetilde{\mathcal{F}}_{(D-2)j} \quad , \quad e^{\vec{b}_{ij}\cdot\vec{\phi}} * \mathcal{F}^i_{(1)j} = \gamma^l_i \widetilde{\gamma}^j_k \widetilde{\mathcal{F}}^k_{(D-1)l}, \tag{5.22}$$

we also have the first-order equations for the dilatons as

$$\begin{aligned} *d\vec{\phi} = & -\frac{1}{2}(-1)^D \vec{a} A_{(3)} d\widetilde{A}_{(D-5)} + \frac{1}{2}\sum_i \vec{a}_i \gamma^j_i \widetilde{\gamma}^i_k A_{(2)j} d\widetilde{A}^k_{(D-4)} \\ & -\frac{1}{2}(-1)^D \sum_{i<j} \vec{a}_{ij} \gamma^l_i \gamma^m_j \widetilde{\gamma}^i_p \widetilde{\gamma}^j_q A_{(1)lm} d\widetilde{A}^{pq}_{(D-3)} \\ & +\frac{1}{2}\sum_{i<j<k} \vec{a}_{ijk} \gamma^l_i \gamma^m_j \gamma^n_k \widetilde{\gamma}^i_p \widetilde{\gamma}^j_q \widetilde{\gamma}^k_r A_{(0)lmn} d\widetilde{A}^{pqr}_{(D-2)} \\ & -\frac{1}{2}(-1)^D \sum_i \vec{b}_i \widetilde{\gamma}^i_j \gamma^k_i \hat{\mathcal{A}}^j_{(1)} (d\widetilde{\mathcal{A}}_{(D-3)k} - A_{(2)k} d\widetilde{A}_{(D-5)} \\ & + (-1)^D A_{(1)lk} d\widetilde{A}^l_{(D-4)} - \frac{1}{2} A_{(0)klm} d\widetilde{A}^{lm}_{(D-3)} - Y_k) \\ & +\frac{1}{2}\sum_{i<j} \vec{b}_{ij} \gamma^k_i \mathcal{A}^i_{(0)j} d\widetilde{\mathcal{A}}^j_{(D-2)k} - \frac{1}{2}\vec{Q}. \end{aligned} \tag{5.23}$$

In the first-order equations above, we have introduced the dual auxiliary fields $\{\widetilde{A}_{(D-5)}, \widetilde{A}^i_{(D-4)}, \widetilde{A}^{ij}_{(D-3)}, \widetilde{A}^{ijk}_{(D-2)}, \widetilde{\mathcal{A}}_{(D-3)i}, \widetilde{\mathcal{A}}^j_{(D-2)i}, \vec{\psi}\}$ which are the Lagrange multipliers. We have expressed the first-order equations (5.22) in terms of the dual field strengths of these dual potentials and the dual field strengths are defined as

$$\widetilde{F}_{(D-4)} = d\widetilde{A}_{(D-5)} + (-1)^D X, \quad \widetilde{\mathcal{F}}^k_{(D-1)l} = d\widetilde{\mathcal{A}}^k_{(D-2)l} + (-1)^D W^k_l,$$

$$\widetilde{F}^j_{(D-3)} = d\widetilde{A}^j_{(D-4)} - d\widetilde{A}_{(D-5)}\hat{\mathcal{A}}^j_{(1)} - (-1)^D(X^j + X\hat{\mathcal{A}}^j_{(1)}),$$

$$\widetilde{F}^{kl}_{(D-2)} = d\widetilde{A}^{kl}_{(D-3)} - d\widetilde{A}^k_{(D-4)}\hat{\mathcal{A}}^l_{(1)} + d\widetilde{A}^l_{(D-4)}\hat{\mathcal{A}}^k_{(1)} + d\widetilde{A}_{(D-5)}\hat{\mathcal{A}}^k_{(1)}\hat{\mathcal{A}}^l_{(1)}$$

$$+ (-1)^D(X^{kl} + X^k\hat{\mathcal{A}}^l_{(1)} - X^l\hat{\mathcal{A}}^k_{(1)} + X\hat{\mathcal{A}}^k_{(1)}\hat{\mathcal{A}}^l_{(1)}),$$

$$\widetilde{F}^{lmn}_{(D-1)} = d\widetilde{A}^{lmn}_{(D-2)} - 3d\widetilde{A}^{[lm}_{(D-3)}\hat{\mathcal{A}}^{n]}_{(1)} + 3d\widetilde{A}^{[l}_{(D-4)}\hat{\mathcal{A}}^m_{(1)}\hat{\mathcal{A}}^{n]}_{(1)}$$

$$- d\widetilde{A}_{(D-5)}\hat{\mathcal{A}}^l_{(1)}\hat{\mathcal{A}}^m_{(1)}\hat{\mathcal{A}}^n_{(1)} - (-1)^D(X^{lmn} + 3X^{[lm}\hat{\mathcal{A}}^{n]}_{(1)}$$

$$+ 3X^{[l}\hat{\mathcal{A}}^m_{(1)}\hat{\mathcal{A}}^{n]}_{(1)} + X\hat{\mathcal{A}}^l_{(1)}\hat{\mathcal{A}}^m_{(1)}\hat{\mathcal{A}}^n_{(1)}),$$

$$\widetilde{\mathcal{F}}_{(D-2)j} = d\widetilde{\mathcal{A}}_{(D-3)j} - A_{(2)j}d\widetilde{A}_{(D-5)} + (-1)^D A_{(1)kj}d\widetilde{A}^k_{(D-4)}$$

$$- \frac{1}{2}A_{(0)jkl}d\widetilde{A}^{kl}_{(D-3)} - Y_j, \tag{5.24}$$

where we have also defined

$$W^k_j = Q^k_j + A_{(2)j}\hat{\mathcal{A}}^k_{(1)}d\widetilde{A}_{(D-5)} - (-1)^D A_{(2)j}d\widetilde{A}^k_{(D-4)}$$

$$
\begin{aligned}
&-(-1)^D A_{(1)jl}\hat{\mathcal{A}}^k_{(1)} d\widetilde{A}^l_{(D-4)} + A_{(1)jl} d\widetilde{A}^{kl}_{(D-3)} \\
&+\frac{1}{2} A_{(0)jlm}\hat{\mathcal{A}}^k_{(1)} d\widetilde{A}^{lm}_{(D-3)} - \frac{1}{2}(-1)^D A_{(0)jlm} d\widetilde{A}^{klm}_{(D-2)} \\
&-\hat{\mathcal{A}}^k_{(1)} d\widetilde{\mathcal{A}}_{(D-3)j} + Y_j \hat{\mathcal{A}}^k_{(1)}.
\end{aligned}
\tag{5.25}
$$

It should be clear that we have omitted the wedge product for the sake of simplicity in the expressions above. The dimension dependent quantities X, X^i, X^{ij}, X^{ijk}, Y_k, Q^k_j, $\overrightarrow{Q}$ are given in Appendix A for each dimension.

As we have mentioned before, there are various methods to formulate the dualisation. We may assign generators to the original and the dual fields and construct the coset element ν. Then we can calculate the commutators and the anti-commutators of the generators in several ways. We can either calculate the Cartan form $\mathcal{G} = d\nu\nu^{-1}$ in terms of the unknown structure constants and insert it in the Cartan-Maurer equation (5.2) and compare the result with the second-order equations, or we can directly use the Cartan form in the twisted self-duality equation (5.3) and compare the result with the first-order equations (5.22) and (5.23). On the other hand without constructing the coset element, one may construct the Cartan form directly from the first-order equations (5.22) and (5.23) in a way that it will lead to the first-order equations when inserted in the twisted self-duality equation. Then after using the Cartan form in the Cartan-Maurer equation one may calculate the commutation and the anti-commutation relations by comparing the result with the second-order equations. In both cases we may make use of the twisted self-duality of the Cartan form to write it in dual fields-free form. The coset element for the D-dimensional supergravities can be given as [19]

$$\nu = e^{\frac{1}{2}\phi^j H_j}(\prod_{i<j} e^{\mathcal{A}^i_{(0)j} E^j_i})e^{\hat{\mathcal{A}}^i_{(1)} W_i} e^{\frac{1}{6}A_{(0)ijk}E^{ijk}} e^{\frac{1}{2}A_{(1)ij}V^{ij}} e^{A_{(2)i}V^i} e^{A_{(3)}V}$$

$$e^{\widetilde{A}_{(D-5)}\widetilde{V}} e^{\widetilde{A}^i_{(D-4)}\widetilde{V}_i} e^{\frac{1}{2}\widetilde{A}^{ij}_{(D-3)}\widetilde{V}_{ij}} e^{\frac{1}{6}\widetilde{A}^{ijk}_{(D-2)}\widetilde{E}_{ijk}} e^{\widetilde{\mathcal{A}}_{(D-3)i}\widetilde{W}^i} e^{\widetilde{\mathcal{A}}^j_{(D-2)i}\widetilde{E}^i_j} e^{\frac{1}{2}\vec{\widetilde{\psi}}\cdot\vec{\widetilde{H}}}, \qquad (5.26)$$

where the factors in the product Π are in anti-lexical order i.e. $(i,j) = \cdots$ $(3,4)$, $(2,4)$, $(1,4)$, $(2,4)$, $(1,3)$, $(1,2)$. As we have discussed before the generators are even or odd, depending on whether the corresponding potentials are even or odd degree in each dimension. The pseudo-involution $\mathcal{S}$ acts on the generators in the way we have described before. By following either of the methods described above, one can calculate the commutators and the anti-commutators. We will first present the dimension independent commutators and the anti-commutators. The first group of dimension independent commutators are the ones related to E^j_i and they can be given as

$$[E^j_i, E^l_k] = \delta^j_k E^l_i - \delta^l_i E^j_k, \quad [E^j_i, E^{klm}] = -3\delta^{[k}_i E^{lm]j},$$

$$[E^j_i, V^k] = -\delta^k_i V^j, \quad [E^j_i, V^{kl}] = 2\delta^{[k}_i V^{l]j}, \quad [E^j_i, W_k] = \delta^j_k W_i,$$

$$[E_i^j, \widetilde{E}_{klm}] = 3\delta_{[k}^j \widetilde{E}_{lm]i}, \quad [E_i^j, \widetilde{W}^k] = -\delta_i^k \widetilde{W}^j,$$

$$[E_i^j, \widetilde{V}_k] = \delta_k^j \widetilde{V}_i, \quad [E_i^j, \widetilde{V}_{kl}] = -2\delta_{[k}^j \widetilde{V}_{l]i},$$

$$[E_i^j, \widetilde{E}_l^i] = -\widetilde{E}_l^j, \quad \text{no sum on } i,\ j \neq l,$$

$$[E_i^j, \widetilde{E}_j^k] = \widetilde{E}_i^k, \quad \text{no sum on } j,\ i \neq k. \tag{5.27}$$

The commutation relations of the Cartan generators $\vec{H}$, which are also dimension independent, are

$$[\vec{H}, E_i^j] = \vec{b}_{ij} E_i^j, \quad [\vec{H}, E^{ijk}] = \vec{a}_{ijk} E^{ijk}, \quad [\vec{H}, V^{ij}] = \vec{a}_{ij} V^{ij},$$

$$[\vec{H}, V^i] = \vec{a}_i V^i, \quad [\vec{H}, V] = \vec{a} V, \quad [\vec{H}, W_i] = \vec{b}_i W_i,$$

$$[\vec{H}, \widetilde{E}_j^i] = -\vec{b}_{ij} \widetilde{E}_j^i, \quad [\vec{H}, \widetilde{E}_{ijk}] = -\vec{a}_{ijk} \widetilde{E}_{ijk}, \quad [\vec{H}, \widetilde{V}_{ij}] = -\vec{a}_{ij} \widetilde{V}_{ij},$$

$$[\vec{H}, \widetilde{V}_i] = -\vec{a}_i \widetilde{V}_i, \quad [\vec{H}, \widetilde{V}] = -\vec{a} \widetilde{V}, \quad [\vec{H}, \widetilde{W}^i] = -\vec{b}_i \widetilde{W}^i. \tag{5.28}$$

The remaining dimension independent commutation and anti-commutation relations can be given as

$$[W_i, E^{jkl}] = -3\delta_i^{[j} V^{kl]}, \quad \{W_i, V^{jk}\} = -2\delta_i^{[j} V^{k]}, \quad [W_i, V^j] = -\delta_i^j V,$$

$$[W_i, \widetilde{V}\} = -\widetilde{V}_i, \quad [W_i, \widetilde{V}_j\} = \widetilde{V}_{ij}, \quad [W_i, \widetilde{V}_{jk}\} = -\widetilde{E}_{ijk},$$

$$[V^i, \widetilde{V}] = -\widetilde{W}^i, \quad [V^{ij}, \widetilde{V}_k\} = -2\delta_k^{[i}\widetilde{W}^{j]}, \quad [E^{ijk}, \widetilde{V}_{lm}] = -6\delta_l^{[i}\delta_m^j\widetilde{W}^{k]},$$

$$[V^i, \widetilde{V}_j] = -\widetilde{E}_j^i, \quad [V^{ij}, \widetilde{V}_{kl}\} = 4\delta_{[k}^{[i}\widetilde{E}_{l]}^{j]},$$

$$[E^{ijk}, \widetilde{E}_{lmn}] = -18\delta_{[l}^{[i}\delta_m^j\widetilde{E}_{n]}^{k]}, \quad [W_i, \widetilde{W}^j\} = -\widetilde{E}_i^j,$$

$$[V, \widetilde{V}\} = -\frac{1}{4}\vec{a} \cdot \vec{\widetilde{H}}, \quad [V^i, \widetilde{V}_i] = \frac{1}{4}\vec{a}_i \cdot \vec{\widetilde{H}}, \quad [V^{ij}, \widetilde{V}_{ij}\} = -\frac{1}{4}\vec{a}_{ij} \cdot \vec{\widetilde{H}},$$

$$[E^{ijk}, \widetilde{E}_{ijk}] = \frac{1}{4}\vec{a}_{ijk} \cdot \vec{\widetilde{H}}, \quad [W_i, \widetilde{W}^i\} = -\frac{1}{4}\vec{b}_i \cdot \vec{\widetilde{H}}, \quad [E_i^j, \widetilde{E}_j^i] = \frac{1}{4}\vec{b}_{ij} \cdot \vec{\widetilde{H}}. \tag{5.29}$$

There are also the dimension dependent commutators, we list them in Appendix B. The commutators and the anti-commutators which are not given above or in Appendix B vanish indeed. We have presented the resulting algebra structures in their basic notions. A more detailed treatment of the construction and the algebraic structure of the D-dimensional Lie superalgebras which arise in the dualisation of the maximal supergravities can be found in [19]. By using the coset element (5.26), the commutation and the anti-commutation relations one can construct the Cartan form $\mathcal{G} = d\nu\nu^{-1}$ explicitly as

$$\mathcal{G} = \frac{1}{2}d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\mathcal{F}^{i}_{(1)j}E_i^{\,j} + \sum_i e^{\frac{1}{2}\vec{b}_i\cdot\vec{\phi}}\mathcal{F}^{i}_{(2)}W_i$$

$$+ \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}F_{(1)ijk}E^{ijk} + \sum_{i<j} e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}F_{(2)ij}V^{ij} + \sum_i e^{\frac{1}{2}\vec{a}_i\cdot\vec{\phi}}F_{(3)i}V^i$$

$$+ e^{\frac{1}{2}\vec{a}\cdot\vec{\phi}}F_{(4)}V + e^{-\frac{1}{2}\vec{a}\cdot\vec{\phi}}\widetilde{F}_{(D-4)}\widetilde{V} + \sum_i e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}}\widetilde{\gamma}^i_j\widetilde{F}^j_{(D-3)}\widetilde{V}_i$$

$$+ \sum_{i<j} e^{-\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}\widetilde{\gamma}^i_k\widetilde{\gamma}^j_l\widetilde{F}^{kl}_{(D-2)}\widetilde{V}_{ij} + \sum_{i<j<k} e^{-\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}\widetilde{\gamma}^i_l\widetilde{\gamma}^j_m\widetilde{\gamma}^k_n\widetilde{F}^{lmn}_{(D-1)}\widetilde{E}_{ijk}$$

$$+ \sum_i e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\phi}}\gamma^j_i\widetilde{\mathcal{F}}_{(D-2)j}\widetilde{W}^i + \sum_{i<j} e^{-\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\gamma^k_i\widetilde{\gamma}^j_l\widetilde{\mathcal{F}}^l_{(D-1)k}\widetilde{E}^i_j + \frac{1}{2}F_\psi\widetilde{\vec{H}}. \quad (5.30)$$

We have already defined the field strengths in (5.20), (5.21) and (5.24). We can show that the above Cartan form leads to the D-dimensional second-order equations when used in the Cartan-Maurer equation. The Cartan form gives the first-order equations when the twisted self-duality condition is applied on it as well. We have constructed an algebra which generates a coset representative whose Cartan form gives the Bosonic field equations of the D-dimensional maximal supergravities. Thus we have reconstructed the D-dimensional maximal supergravities as non-linear sigma models.

As a final case of dualisation we will consider the dualisation of the ten dimensional IIB supergravity. We have already mentioned the IIB supergravity in

Chapter two, where we have studied the scalar sector of it (which is a dilaton-axion system). As we have discussed before, the construction of an $SL(2,\mathbb{R})$ globally invariant Lagrangian is not straightforward due to the presence of a self-dual five-form field $H_{(5)}$ for the IIB supergravity. One has to remove the self-duality condition on the five-form field $H_{(5)}$ and introduce more degrees of freedom to construct an $SL(2,\mathbb{R})$ globally invariant Lagrangian [10, 11]. This method is similar to the Lagrange multiplier methods we have used in Chapter two to formulate the scalar sectors as symmetric space sigma models. When one calculates the field equations of the Lagrangian mentioned above, one may impose the extra self-duality condition of the five-form field as $H_{(5)} = *H_{(5)}$, which can be shown to be consistent with both the field equations and the Bianchi identity of $H_{(5)}$ [10, 11]. The field equations of the globally $SL(2,\mathbb{R})$ invariant Lagrangian together with the self-duality condition which is a consistent truncation result in the desired field equations of the IIB supergravity. We will not present the Lagrangian formulation here, however the reader may find a construction in terms of the fields which are related to the fields we will use to express the field equations with simple redefinitions in [10]. The Bosonic, non-gravitational, matter field equations of the IIB supergravity can be given as [19, 60]

$$d * H_{(5)} = -\frac{1}{2}\epsilon_{ij} F^{i}_{(3)} \wedge F^{j}_{(3)},$$

$$d(\mathcal{M} * H_{(3)}) = H_{(5)} \wedge \Omega H_{(3)},$$

$$d(e^{2\phi} * d\chi) = -e^{\phi} F^2_{(3)} \wedge *F^1_{(3)},$$

$$d * d\phi = e^{2\phi} d\chi \wedge *d\chi + \frac{1}{2} e^{\phi} F^1_{(3)} \wedge *F^1_{(3)} - \frac{1}{2} e^{-\phi} F^2_{(3)} \wedge *F^2_{(3)}. \qquad (5.31)$$

We have already defined $\mathcal{M}$ in (2.63) when we constructed the scalar Lagrangian of the IIB supergravity as a symmetric space sigma model. The various field strengths in (5.31) are defined as

$$F^2_{(3)} = dA^2_{(2)} \quad , \quad F^1_{(3)} = dA^1_{(2)} - \chi dA^2_{(2)},$$

$$H_{(5)} = dB_{(4)} - \frac{1}{2} \epsilon_{ij} A^i_{(2)} \wedge dA^j_{(2)}. \qquad (5.32)$$

We have also defined

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad A_{(2)} = \begin{pmatrix} A^1_{(2)} \\ A^2_{(2)} \end{pmatrix}. \qquad (5.33)$$

In the field equations (5.31), we further define $H_{(3)} = dA_{(2)}$. If we introduce the auxiliary fields $\widetilde{A}^1_{(6)}, \widetilde{A}^2_{(6)}, \psi, \widetilde{\chi}$, we can locally express the first-order equations as

$$*H_{(5)} = dB_{(4)} - \frac{1}{2} \epsilon_{ij} A^i_{(2)} \wedge dA^j_{(2)},$$

$$\mathcal{M} * dA_{(2)} = d\widetilde{A}_{(6)} - \frac{1}{2} \Omega A_{(2)} \wedge (dB_{(4)} - \frac{1}{6} \epsilon_{ij} A^i_{(2)} \wedge dA^j_{(2)}),$$

$$ie^{\phi} * d\tau = P + \tau Q, \qquad (5.34)$$

where $\tau = \chi + ie^{-\phi}$. The fields P and Q are

$$P = d\psi + \frac{1}{2}A^1_{(2)}d\widetilde{A}^1_{(6)} - \frac{1}{2}A^2_{(2)}d\widetilde{A}^2_{(6)} - \frac{1}{4}A^1_{(2)}A^2_{(2)}dB_{(4)}$$

$$- \frac{1}{24}A^2_{(2)}A^2_{(2)}A^1_{(2)}dA^1_{(2)},$$

$$Q = d\widetilde{\chi} + A^2_{(2)}d\widetilde{A}^1_{(6)} - \frac{1}{4}A^2_{(2)}A^2_{(2)}dB_{(4)} - \frac{1}{36}A^2_{(2)}A^2_{(2)}A^2_{(2)}dA^1_{(2)}. \tag{5.35}$$

In order to define the dual field strengths, we write the first-order equations (5.34) in the form

$$H_{(5)} = *H_{(5)} = dB_{(4)} - \frac{1}{2}\epsilon_{ij}A^i_{(2)}dA^j_{(2)},$$

$$\widetilde{F}^1_{(7)} \equiv e^{\phi} * F^1_{(3)} = d\widetilde{A}^1_{(6)} - \frac{1}{2}A^2_{(2)}(dB_{(4)} - \frac{1}{6}\epsilon_{ij}A^i_{(2)}dA^j_{(2)}),$$

$$\widetilde{F}^2_{(7)} \equiv e^{-\phi} * F^2_{(3)} = d\widetilde{A}^2_{(6)} + \frac{1}{2}A^1_{(2)}(dB_{(4)} - \frac{1}{6}\epsilon_{ij}A^i_{(2)}dA^j_{(2)})$$

$$- \chi(d\widetilde{A}^1_{(6)} - \frac{1}{2}A^2_{(2)}(dB_{(4)} - \frac{1}{6}\epsilon_{ij}A^i_{(2)}dA^j_{(2)})),$$

$$*d\phi \equiv \widetilde{P} = P + \chi Q \quad , \quad e^{2\phi} * d\chi \equiv Q, \tag{5.36}$$

where we have given the dual field strengths corresponding to the auxiliary fields $\widetilde{A}^1_{(6)}, \widetilde{A}^2_{(6)}, \widetilde{\chi}, \psi$. By following our previous discussions, we can introduce generators for the original fields and for the auxiliary fields which arise during the integration of the second-order equations (5.31). Without constructing the coset element ν one can directly construct the appropriate Cartan form $\mathcal{G} = d\nu\nu^{-1}$ which will lead to the first-order field equations when inserted in the twisted self-duality equation $*\mathcal{G} = \mathcal{S}\mathcal{G}$. We have already discussed the general properties of the generators and how the pseudo-involution acts on them before. Thus we introduce the Cartan form

$$\mathcal{G} = \frac{1}{2}d\phi H + e^{\phi}d\chi E_+ + e^{\frac{1}{2}\phi}F^1_{(3)}V_+ + e^{-\frac{1}{2}\phi}F^2_{(3)}V_- + H_{(5)}U$$

$$+ e^{-\frac{1}{2}\phi}\widetilde{F}^1_{(7)}\widetilde{V}_+ + e^{\frac{1}{2}\phi}\widetilde{F}^2_{(7)}\widetilde{V}_- + e^{-\phi}Q\widetilde{E}_+ + \frac{1}{2}\widetilde{P}\widetilde{H}, \qquad (5.37)$$

which leads to the first-order equations as mentioned above. When we eliminate the dual fields by using the twisted self-duality property of the Cartan form, use the result in the Cartan-Maurer equation $d\mathcal{G} - \mathcal{G}\wedge\mathcal{G} = 0$ and compare with the second-order equations (5.31), we can read the commutation and the anti-commutation relations as

$$[H, E_+] = 2E_+ \quad , \quad [H, V_+] = V_+ \quad , \quad [H, V_-] = -V_-,$$

$$[H, \widetilde{E}_+] = -2\widetilde{E}_+ \quad , \quad [H, \widetilde{V}_+] = -\widetilde{V}_+ \quad , \quad [H, \widetilde{V}_-] = \widetilde{V}_-,$$

$$[E_+, V_-] = V_+ \quad , \quad [E_+, \widetilde{V}_+] = -\widetilde{V}_- \quad , \quad [V_+, V_-] = -U,$$

$$[V_+, U] = \frac{1}{2}\widetilde{V}_- \quad , \quad [V_-, U] = -\frac{1}{2}\widetilde{V}_+ \quad , \quad [V_-, \widetilde{V}_+] = \widetilde{E}_+,$$

$$[E_+,\widetilde{E}_+]=\frac{1}{2}\widetilde{H}\quad,\quad[V_+,\widetilde{V}_+]=\frac{1}{4}\widetilde{H}\quad,\quad[V_-,\widetilde{V}_-]=-\frac{1}{4}\widetilde{H}. \tag{5.38}$$

Finally we can show that the coset element which generates the Cartan form (5.37) via the algebra structure given in (5.38) is

$$\nu=e^{\frac{1}{2}\phi H}e^{\chi E_+}e^{(A^1_{(2)}V_++A^2_{(2)}V_-)}e^{B_{(4)}U}e^{(\widetilde{A}^1_{(6)}\widetilde{V}_++\widetilde{A}^2_{(6)}\widetilde{V}_-)}e^{\widetilde{\chi}\widetilde{E}_+}e^{\frac{1}{2}\psi\widetilde{H}}. \tag{5.39}$$

We have formulated a coset representative which is based on a particular gauge. The coset is generated by the generators whose algebraic structure is constructed in a way that when the Cartan form is calculated, it leads to the original first and second-order field equations of the IIB supergravity through the twisted self-duality and the Cartan-Maurer equations.

As we have seen, the dualisation of the Bosonic sectors of the supergravity theories leads to the non-linear coset sigma model constructions. Although we have constructed the coset representatives and the algebra structures which generate these cosets, we have not mentioned much about the symmetry properties. The discussion about the symmetry groups of the Cartan forms (the doubled field strengths) of the coset formulation as well as the symmetry groups of the twisted self-duality equations (i.e. first-order equations) is given in [10, 19]. The symmetry group of the twisted self-duality equation is larger than the symmetry group of the Cartan form since there may be transformations which change the Cartan form $\mathcal{G}$ but leave the twisted self-duality equation $*\mathcal{G}=\mathcal{S}\mathcal{G}$ invariant. In

[19] it is shown that the Lie algebra of the full symmetry group of the Cartan form corresponding to the dualized D-dimensional maximal supergravity can be obtained from the Lie superalgebra which generates the doubled coset. In the same work it is mentioned that the global parts of the full symmetry groups of the Cartan forms of the dualized maximal supergravities are equal to the Borel subgroups E_n^+ of the split real forms E_n for $n = 11 - D$ in each dimension. It is also conjectured that E_n are the global symmetry groups of the twisted self-duality equations so that the complement of the Borel subgroup E_n^+ in E_n is composed of the transformations which change $\mathcal{G}$ while leaving the twisted self-duality equation invariant. At least for the scalar sectors E_n is proven to be the global symmetry of the twisted self-duality (first-order) equations. In [19], it is also discussed that the original global symmetry group $SL(2, \mathbb{R})$ of the IIB supergravity leaves the first-order equations invariant in the coset formulation (dualisation) of the IIB supergravity. Thus it is a manifest symmetry of the twisted self-duality equation.

CHAPTER 6

DUALISATION OF THE MATTER COUPLED SYMMETRIC SPACE SIGMA MODEL

We have studied the symmetric space sigma model and its dualisation as well as the first-order formulation in detail in Chapter four. In this chapter we consider the coupling of other matter fields to the scalar coset Lagrangian of the symmetric space sigma model. Our discussion will be in parallel with the content of [61]. Basically we will construct the dualized coset element which will realize the field equations of the scalar coset coupled to the $(m-1)$-form fields. We will assume the most general non-split scalar coset case (by non-split we mean that G is a non-compact real form of a semi-simple Lie group as usual but it is not necessarily maximally non-compact [43, 57]). Apart from the scalar fields there will be an arbitrary number of m-form field strengths which contribute to the Lagrangian through an additional kinetic term which is casted by using an appropriate representation of the global symmetry group G. We will follow the standard dualisation method of [10, 19] by introducing auxiliary dual fields and by assigning generators to the original and the dual fields. The main objective of this chapter will be to derive the Lie superalgebra structure which generates the doubled coset element when there is matter field coupling to the scalar coset

Lagrangian. The first-order formulation will then be presented as the twisted self-duality equation (which we have effectively used before) by using the constructed algebra structure and by explicitly calculating the doubled field strength.

We start by discussing the Lagrangian and deriving the field equations. Next we work out the dualisation, calculate the algebraic structure and finally obtain the first-order field equations.

In Chapter four we have constructed the G/K symmetric space sigma model which governs the scalar fields. The group G is the global symmetry group of the corresponding scalar Lagrangian and it is a non-compact real form of a semi-simple Lie group. The local symmetry group K is a maximal compact subgroup of G. The scalars parameterize the coset space G/K which is called the scalar coset manifold. The coset space G/K is a Riemannian globally symmetric space for all the possible G-invariant Riemann structures on G/K as discussed in detail in Chapter three. We have also constructed a legitimate parametrization, $\nu(x)$ of the coset representatives of G/K in (4.5), by using the solvable Lie algebra s_0 of G [18] as

$$\nu(x) = e^{\frac{1}{2}\phi^i(x)H_i} e^{\chi^m(x)E_m}. \tag{6.1}$$

The scalar Lagrangian of the symmetric space sigma model is defined in terms of the internal metric $\mathcal{M} = \nu^{\#}\nu$. The globally G and the locally K-invariant scalar Lagrangian is

$$\mathcal{L}_{scalar} = \frac{1}{4} tr(d\mathcal{M}^{-1} \wedge *d\mathcal{M}). \tag{6.2}$$

We assume that $D > 2$ in order that the dualisation analysis we perform is meaningful and also for algebraic reasons which arise during the dualisation. We will keep on taking the signature of the spacetime as $(-,+,+,...)$. In Chapter four we have also calculated the Cartan form $\mathcal{G}_0 = d\nu\nu^{-1}$ generated by the map (6.1) as

$$\mathcal{G}_0 = \frac{1}{2}d\phi^i H_i + \overrightarrow{\mathbf{E}'}\,\mathbf{\Omega}\,\overrightarrow{d\chi}. \tag{6.3}$$

The row vector $\overrightarrow{\mathbf{E}'}$ and the column vector $\overrightarrow{d\chi}$ as well as the matrix $\mathbf{\Omega}$ are already defined in Chapter four.

We will consider the coupling of an arbitrary number of $(m-1)$-form potential fields $\{A^i\}$ to the G/K scalar coset. We further assume that $1 < m < \mathrm{D}/2$ and $3m - D \neq 1$ for algebraic reasons which we encounter when we apply the dualisation method. Basically by assuming these conditions on m and D, we restrict ourselves to a special class of scalar-matter couplings whose dualisations give rise to algebras in which the original generators generate a subalgebra. The quadratic terms due to the coupling of $(m-1)$-form potential fields $\{A^i\}$, which must be added to the scalar Lagrangian (6.2), are the combinations of the internal metric $\mathcal{M}$ and the field strengths $F^i = dA^i$. Thus we have the matter Lagrangian

$$\begin{aligned}\mathcal{L}_m &= -\frac{1}{2}\mathcal{M}_{ij}F^i \wedge *F^j \\ &= -\frac{1}{2}F \wedge \mathcal{M} * F.\end{aligned} \tag{6.4}$$

We assume that $\mathcal{M}$ and ν are in an appropriate representation which is obtained by choosing a representation for the Lie algebra g_0, thus for the generators

$\{H_j, E_\beta \mid j = 1, ..., r \ ; \ \beta \in \Delta^+_{nc}\}$ of s_0. The representation must be compatible with the number of the coupling fields. The total Lagrangian becomes

$$\mathcal{L} = \frac{1}{4} tr(d\mathcal{M}^{-1} \wedge *d\mathcal{M}) - \frac{1}{2} F \wedge \mathcal{M} * F. \tag{6.5}$$

If the subgroup of G generated by the compact generators is an orthogonal group, then the generators can be chosen such that $g^{\#} = g^T$ for $g \in g_0$ in the fundamental representation of g_0. Therefore # coincides with the ordinary matrix transpose and $\mathcal{M}$ becomes a symmetric matrix in the fundamental representation of g_0 we choose. We assume this is the case in our analysis bearing in mind that, for the general case, higher dimensional representations are possible in which we can still take $g^{\#} = g^T$ for $g \in g_0$ [43].

When we vary the Lagrangian (6.5) with respect to the dilatons $\{\phi^i\}$, the axions $\{\chi^m\}$ and the coupling potentials $\{A^i\}$, by following the analysis of [43, 57] and using (6.3), we can express the field equations for the Lagrangian (6.5) as

$$d(\mathcal{M}_{kl} * F^l) = 0,$$

$$d(e^{\frac{1}{2}\gamma_i\phi^i} * U^\gamma) = -\frac{1}{2}\gamma_j e^{\frac{1}{2}\gamma_i\phi^i} d\phi^j \wedge *U^\gamma$$

$$+ \sum_{\alpha-\beta=-\gamma} e^{\frac{1}{2}\alpha_i\phi^i} e^{\frac{1}{2}\beta_i\phi^i} N_{\alpha,-\beta} U^\alpha \wedge *U^\beta,$$

$$d(*d\phi^i) = \frac{1}{2} \sum_{\alpha \in \Delta^+_{nc}} \alpha_i e^{\frac{1}{2}\alpha_i\phi^i} U^\alpha \wedge e^{\frac{1}{2}\alpha_i\phi^i} * U^\alpha$$

$$-\frac{1}{2}((H_i)_{nl}\nu^n_m\nu^l_j)F^j\wedge *F^m, \tag{6.6}$$

where $i,j=1,...r$ and $\alpha,\beta,\gamma\in\Delta^+_{nc}$. The roots in Δ^+_{nc} and their corresponding generators $\{E_m\}$ are assumed to be enumerated as we have done in Chapter four. We have also defined the vector $U^\alpha=\mathbf{\Omega}^\alpha_\beta\, d\chi^\beta$. Furthermore the matrices $\{(H_i)_{nl}\}$ are the representatives of the Cartan generators $\{H_i\}$ under the representation chosen. We use the notation $[E_\alpha,E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}$. We should remark that in the dilaton equation (6.6), the contribution from the coupling fields $\{A^i\}$ is expressed in terms of the original fields rather than their weight expansions unlike the expressions in the formulation given in [43, 57]. In [43, 57] the Bianchi identities and the equations of motion for the field strengths F^i are expressed in terms of the components of νF and $\nu * F$ which are labelled by the weights of the representation that we use. It is also shown that when there is matter coupling the dilaton equations in (4.33) are modified whereas the equations for the axions (4.32) are unaltered as it can be seen in (6.6). For notational convenience we raise or lower the indices of the matrices by using an Euclidean metric.

We will adopt the method of [10, 19] as we have already used in Chapter four to establish a coset formulation and to derive the first-order field equations for (6.5), where the scalar fields are coupled to the $(m-1)$-form matter fields. We will first construct a Lie superalgebra which enables the dualized coset formulation by generating the doubled coset element. We assign the generators $\{H_i,E_m,V_j\}$ to the fields $\{\phi^i,\chi^m,A^j\}$, respectively. We assume that $\{H_i,E_m\}$ are even generators within the superalgebra structure, since the coupling fields are scalars that have even rank. The generators $\{V_j\}$ are even or odd depending on whether the rank of the coupling fields $\{A^j\}$, i.e. $(m-1)$, is even or odd. The next step is to introduce the dual fields $\{\widetilde{\phi}^i, \widetilde{\chi}^m, \widetilde{A}^j\}$ which are the Lagrange multipliers and which would arise as auxiliary fields as a result of the local integration of the field equations (6.6). The first two are $(D-2)$-forms and the last ones are $(D-m-1)$-forms. We also assign the dual generators $\{\widetilde{H}_i, \widetilde{E}_m, \widetilde{V}_j\}$ to these dual fields. The dual generators are even or odd depending on the spacetime dimension D and m; in other words according to the rank of the dual fields they are assigned to. We will derive the structure of the Lie superalgebra generated by the original and the dual generators we have introduced so that it will enable a coset formulation for (6.5). Similar to the non-linear coset structure of the scalars presented in Chapter four we can define the map

$$\nu' = e^{\frac{1}{2}\phi^i H_i} e^{\chi^m E_m} e^{A^j V_j} e^{\widetilde{A}^j \widetilde{V}_j} e^{\widetilde{\chi}^m \widetilde{E}_m} e^{\frac{1}{2}\widetilde{\phi}^i \widetilde{H}_i}, \tag{6.7}$$

which can be considered as the parametrization of a coset via the differential graded algebra [19] generated by the differential forms on the D-dimensional spacetime and the Lie superalgebra we propose. We are not intending to detect the group theoretical structure of this coset, instead we only aim to construct the Lie superalgebra of the original and the dual generators which function in the parametrization (6.7). If one knows the structure constants of this algebra, one can calculate the Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$. From the definition of ν' in (6.7) the

Cartan form $\mathcal{G}'$ obeys the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0. \tag{6.8}$$

As we have done before, in order to calculate the structure constants of the commutation and the anti-commutation relations of the original and the dual generators, following [10, 19] we assume that the Cartan form obeys the twisted self-duality equation

$$\ast\mathcal{G}' = \mathcal{S}\mathcal{G}', \tag{6.9}$$

where the action of the pseudo-involution $\mathcal{S}$ on the generators is taken as

$$\mathcal{S}E_m = \widetilde{E}_m \quad , \quad \mathcal{S}\widetilde{E}_m = (-1)^D E_m,$$

$$\mathcal{S}H_i = \widetilde{H}_i \quad , \quad \mathcal{S}\widetilde{H}_i = (-1)^D H_i,$$

$$\mathcal{S}V_j = \widetilde{V}_j \quad , \quad \mathcal{S}\widetilde{V}_j = (-1)^{m(D-m)+1} V_j. \tag{6.10}$$

If we calculate the Cartan form $\mathcal{G}'$ in terms of the unknown structure constants, use the twisted self-duality condition proposed above and insert the result in the Cartan-Maurer equation (6.8), we should obtain the second-order field equations (6.6) [19]. The Cartan form $\mathcal{G}'$ which is expressed only in terms of the original fields by using the twisted self-duality condition becomes

$$\mathcal{G}' = \frac{1}{2} d\phi^i H_i + \overrightarrow{\mathbf{E}'}\, \mathbf{\Omega}\, \overrightarrow{d\chi} + \overrightarrow{\mathbf{V}} e^{\mathbf{U}} e^{\mathbf{B}} \overrightarrow{\mathbf{dA}}$$

$$+\frac{1}{2}(-1)^D * d\phi^i \widetilde{H}_i + (-1)^D e^{\frac{1}{2}\alpha_i\phi^i}\mathbf{\Omega}^\alpha_\beta * d\chi^\beta \widetilde{E}_\alpha$$

$$+(-1)^{(m(D-m)+1)}\vec{\widetilde{\mathbf{V}}} e^{\mathbf{U}} e^{\mathbf{B}} * \vec{\mathbf{dA}}, \tag{6.11}$$

where $(\mathbf{U})^n_v = \frac{1}{2}\phi^i\theta^n_{iv}$ and $(\mathbf{B})^j_n = \chi^m\beta^j_{mn}$ in which we define the yet unknown structure constants as

$$[H_i, V_n] = \theta^t_{in} V_t \quad , \quad [E_m, V_j] = \beta^l_{mj} V_l. \tag{6.12}$$

We introduce the row vectors $\vec{\mathbf{V}}$ and $\vec{\widetilde{\mathbf{V}}}$ as (V_i) and $(\widetilde{V}_j)$, respectively, the column vector $\vec{\mathbf{dA}}$ is (dA^i). We have introduced the matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ such that $\mathbf{\Gamma}^k_n = \frac{1}{2}\phi^i\,\widetilde{g}^k_{in}$ and $\mathbf{\Lambda}^k_n = \chi^m\widetilde{f}^k_{mn}$ in Chapter four. We are not presenting the details of the calculation of the Cartan form, we have already discussed examples of how one calculates the Cartan forms in Chapter four. The correspondence of the Cartan-Maurer equation (6.8), in which we use (6.11) with the second-order field equations (6.6), enables us to determine the structure constants of the Lie superalgebra which generates the doubled coset element (6.7). After a direct calculation the resulting commutation and the anti-commutation relations are

$$[H_j, E_\alpha] = \alpha_j E_\alpha \quad , \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad , \quad [H_j, \widetilde{E}_\alpha] = -\alpha_j \widetilde{E}_\alpha,$$

$$[E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r}\alpha_j \widetilde{H}_j \quad , \quad [H_l, V_i] = (H_l)^k_i V_k \quad , \quad [E_\alpha, V_i] = (E_\alpha)^j_i V_j,$$

$$[H_i, \widetilde{V}_k] = -(H_i^T)^l_k \widetilde{V}_l \quad , \quad [E_\alpha, \widetilde{V}_k] = -(E_\alpha^T)^l_k \widetilde{V}_l,$$

$$[V_l, \widetilde{V}_k\} = (-1)^m \frac{1}{4} \sum_i (H_i)_{lk} \widetilde{H}_i,$$

$$[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta} \widetilde{E}_\gamma, \qquad \alpha - \beta = -\gamma, \quad \alpha \neq \beta, \tag{6.13}$$

where the indices of the Cartan generators and their duals, $i, j, l = 1, ..., r$ and $\alpha, \beta, \gamma \in \Delta^+_{nc}$. The matrices $((E_\alpha)^j_i,\ (H_l)^j_i)$ above are the representatives of the corresponding generators $((E_\alpha),\ (H_l))$. Also $((E^T_\alpha)^j_i,\ (H^T_l)^j_i)$ are the matrix transpose of $((E_\alpha)^j_i,\ (H_l)^j_i)$. We observe that the dimension of the matrices above, namely the dimension of the representation chosen, is equal to the number of the coupling fields and their corresponding generators. The remaining commutators or the anti-commutators which are not listed in (6.13) vanish. Now that we have determined the structure constants of the algebra generated by the original and the dual generators, we can explicitly calculate the Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$ in terms of both the original and the dual fields without using the twisted self-duality condition primarily. By using the identities

$$de^X e^{-X} = dX + \frac{1}{2!}[X, dX] + \frac{1}{3!}[X, [X, dX]] + ...,$$

$$\tag{6.14}$$

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + ...,$$

and the commutation and the anti-commutation relations given in (6.13) effectively, we have

$$\mathcal{G}' = \frac{1}{2} d\phi^i H_i + \overrightarrow{\mathbf{E}'}\, \mathbf{\Omega}\, \overrightarrow{d\chi} + \overrightarrow{\widetilde{\mathbf{T}}} e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}} \overrightarrow{\widetilde{\mathbf{S}}}$$

$$+\,\vec{\mathbf{V}}\,\nu\,\vec{\mathbf{dA}}+\vec{\widetilde{\mathbf{V}}}\,(\nu^T)^{-1}\,\vec{\mathbf{d}\widetilde{\mathbf{A}}}$$

$$+\,(-1)^{m(D-m)-D}\sum_{l=1}^{r}\frac{1}{4}\,(H_l)_{ij}A^i\wedge d\widetilde{A}^j\widetilde{H}_l. \tag{6.15}$$

In addition to the definitions we have given before, we have $\vec{\mathbf{d}\widetilde{\mathbf{A}}}$ as $(d\widetilde{A}^j)$. Besides, we have the row vector of the duals of the solvable Lie algebra generators of G as $\widetilde{\mathbf{T}}_i=\widetilde{H}_i$ for $i=1,...r$ and $\widetilde{\mathbf{T}}_{r+\alpha}=\widetilde{E}_\alpha$ for $\alpha\in\Delta_{nc}^+$. We have already defined the column vector $\vec{\widetilde{\mathbf{S}}}$ as $\vec{A}$ in Chapter four; however we change the notation in this chapter not to confuse it with the coupling potentials A_i, thus the vector $\vec{\widetilde{\mathbf{S}}}$ is defined as $\widetilde{\mathbf{S}}^i=\frac{1}{2}d\widetilde{\phi}^i$ for $i=1,...,r$ and $\widetilde{\mathbf{S}}^{r+\alpha}=d\widetilde{\chi}^\alpha$ for $\alpha\in\Delta_{nc}^+$.

Since we have obtained the explicit form of the Cartan form $\mathcal{G}'$ in (6.15) we can use the twisted self-duality equation $*\mathcal{G}'=\mathcal{S}\mathcal{G}'$ to find the first-order field equations of (6.5). The validity of the twisted self-duality equation is justified in the way that we have primarily used it when we derived the structure constants. The structure constants are chosen such that they give the correct Cartan form $\mathcal{G}'$ which leads to the second-order field equations (6.6) when used in (6.8). Therefore, directly from (6.15), the twisted self-duality equation $*\mathcal{G}'=\mathcal{S}\mathcal{G}'$ yields

$$\nu^i_j*dA^j=(-1)^{m(D-m)+1}((\nu^T)^{-1})^i_jd\widetilde{A}^j,$$

$$e^{\frac{1}{2}\alpha_i\phi^i}(\mathbf{\Omega})^{\alpha+r}_l*d\chi^l=(-1)^D(e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^{\alpha+r}_j\,\widetilde{\mathbf{S}}^j,$$

$$\frac{1}{2} * d\phi^k = (-1)^D (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})^k_j \widetilde{\mathbf{S}}^j$$

$$+ (-1)^{m(D-m)} \, \frac{1}{4} \, (H_k)_{ji} A^j \wedge d\widetilde{A}^i. \tag{6.16}$$

We should remark once more that the roots in Δ^+_{nc} and the corresponding generators $\{E_\alpha\}$ are enumerated. We can also express the equations (6.16) in a more compact form as

$$\mathcal{M} * \overrightarrow{\mathbf{dA}} = (-1)^{m(D-m)+1} \, \overrightarrow{\mathbf{d}\widetilde{\mathbf{A}}},$$

$$(6.17)$$

$$* \overrightarrow{\mathbf{\Psi}} = \overrightarrow{\mathbf{P}} + (-1)^D \, e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}} \overrightarrow{\widetilde{\mathbf{S}}},$$

where we use the column vector $\overrightarrow{\mathbf{\Psi}}$ which we have already defined in Chapter four as $\mathbf{\Psi}^i = \frac{1}{2} d\phi^i$ for $i = 1, ..., r$ and $\mathbf{\Psi}^{\alpha+r} = e^{\frac{1}{2}\alpha_i \phi^i} \mathbf{\Omega}^\alpha_l d\chi^l$ for $\alpha \in \Delta^+_{nc}$. We also define

$$\mathbf{P}^k = (-1)^{m(D-m)} \, \frac{1}{4} \, (H_k)_{ji} \, A^j \wedge d\widetilde{A}^i \quad \text{for} \quad k = 1, ..., r \; ,$$

$$(6.18)$$

$$\mathbf{P}^{\alpha+r} = 0 \quad \text{for} \quad \alpha \in \Delta^+_{nc}.$$

The indices α above correspond to the labelled roots in Δ^+_{nc}.

In summary, after defining the coupling of m-form field strengths to the scalar Lagrangian we have obtained the field equations following the outline of [43, 57]. We then have performed the dualisation method of [10, 19] to establish a coset formulation of the theory and to explore the Lie superalgebra which leads to the first-order equations of motion as a twisted self-duality condition whose validity is implicitly justified by the construction of the algebra, the Cartan-Maurer equation and the second-order field equations. Accordingly we have constructed a coset element by defining an algebraic structure and we have shown that the field equations can be directly obtained from the Cartan form of the coset element both in the second and the first-order form by using the Cartan-Maurer and the twisted self-duality equations. Therefore we have constructed a total coset formulation for the matter coupled symmetric space sigma model.

The formulation given in this chapter assumes a general non-split scalar coset G/K in D $>$ 2 spacetime dimensions. The coupling potentials are assumed to be $(m-1)$-forms for $1 < m <$ D/2 and $3m - D \neq 1$. We have also assumed a representation whose dimension is compatible with the number of the coupling fields.

CHAPTER 7

DUALISATION OF THE D=8, MATTER COUPLED SALAM-SEZGIN SUPERGRAVITY

In this chapter we formulate the Bosonic sector of the $D = 8$ Salam-Sezgin supergravity [38] in which there is the coupling of N vector multiplets as a non-linear coset model. The formulation we present in this chapter is mainly given in [62]. Here we will also discuss the details of the calculation of the algebra structure. If we focus on the scalar sector, the $2N$ vector multiplet scalars of the theory parameterize the coset $SO(N, 2)/(SO(N) \times SO(2))$ that is a Riemannian globally symmetric space and we have analyzed the Riemannian globally symmetric spaces in detail in Chapter three. Here $SO(N, 2)$ is a semi-simple, non-compact real form and $SO(N) \times SO(2)$ is a maximal compact subgroup of $SO(N, 2)$. We will follow a construction which is parallel with the ones we have presented in Chapter five for the maximal supergravities. We will also make use of the results of Chapter four and the last chapter. As usual we will define the Lie superalgebra which will be used to formulate the Bosonic fields as a non-linear coset model. The algebra structure will be constructed in such a way that it will yield the locally integrated first-order Bosonic field equations of the theory as a twisted self-duality equation of the Cartan form of the coset representative parametrization.

The first section is the introduction of the $D = 8$ matter coupled Salam-Sezgin supergravity. We will derive the equations of motion and then locally integrate them to find the first-order field equations. Whereas in the second section by following the outline of [10, 19], we will introduce dual fields for the Bosonic field content of the theory excluding the graviton and we will also define new algebra generators for the original fields and their duals. A coset element will then be constructed whose Cartan form is intended to yield the correct second-order equations when inserted in the Cartan-Maurer equation. In order to calculate the Cartan form one needs to know the algebra structure of the generators which parameterize the coset representatives when coupled to the fields. The algebra which generates the coset representatives is a differential graded algebra, similar to the examples we have studied before, and is composed of the differential forms and the field generators. It covers the Lie superalgebra of the field generators which is the Lie algebra of the symmetry group of the Cartan form. The correct choice of the commutators and the anti-commutators of the Lie superalgebra is again a result of the direct comparison of the field equations of the first section and the Cartan-Maurer equation. Finally we will show that the twisted self-duality equation $*\mathcal{G} = \mathcal{S}\mathcal{G}$ when applied on the Cartan form, which can explicitly be calculated after determining the algebra structure, leads to the first-order formulation of the equations of motion found in the first section.

7.1 The D=8 Matter Coupled Salam-Sezgin Supergravity

The eight dimensional Salam-Sezgin supergravity with matter coupling was constructed in [38]. N vector multiplets $\{\lambda, A, \varphi^i\}$ (λ is a Fermion and for $i = 1, 2$ $\{\varphi^i\}$ are the scalars whereas A is a one-form field) are coupled to the original field content $\{e_\mu^m, \psi^\mu, \chi, B_{\mu\nu}, A_i^\mu, \sigma\}$ where $\{e_\mu^m\}$ is the graviton, $\{\psi^\mu, \chi\}$ are the Fermionic fields, $\{\sigma\}$ is the dilaton, $B_{\mu\nu}$ is a two-form field and for $i = 1, 2$ $\{A_i^\mu\}$ are the one-form fields. We will combine the original one-forms $\{A_i^\mu; i = 1, 2\}$ and the N vector multiplet one-forms $\{A\}$ into a single set and we will denote them as $\{A^j; j = 1, ..., N + 2\}$. Later we will also classify the $2N$ vector multiplet scalars $\{\varphi^i\}$ as dilatons and axions as a result of the solvable Lie algebra parametrization we have defined in Chapter three and Chapter four. The single, decoupled dilaton $\{\sigma\}$ of the original theory and the $2N$ scalars $\{\varphi^i\}$ of the vector multiplet which parameterize the $SO(N, 2)/(SO(N) \times SO(2))$ Kähler coset manifold constitute the scalar sector of the matter coupled supergravity theory. An explicit representation parametrization for the scalar coset $SO(N, 2)/(SO(N) \times SO(2))$ is given in [38] where the scalar fields are not classified as dilatons and axions.

For odd N, $so(N, 2)$ is the non-compact real form of $B_{\frac{N+1}{2}}$ whereas for even N, it is the non-compact real form of $D_{\frac{N+2}{2}}$ and in general depending on N, $SO(N, 2)$ is not necessarily a split real form (maximally non-compact) as it is clear from Chapter two. We have already studied the cosets $M_{p,q} = O(p, q)/O(p) \times O(q)$ based on the orthogonal groups $O(p, q)$ in Chapter two and those conclusions can also be extended to the special orthogonal groups $SO(p, q)$ in light of Chapter three. Since $SO(N,2)$ is a non-compact real form of some semi-simple Lie group we will use the solvable Lie algebra parametrization established in Chapter three and Chapter four for the coset $SO(N,2)/(SO(N)\times SO(2))$. In reference with Chapter three and Chapter four, this parametrization introduces the dilatons, which are coupled to the generators of h_{p_0} the Lie algebra of the maximal R-split torus of $SO(N,2)$, and the axions, which are coupled to the positive root generators of $SO(N,2)$ that do not commute with the elements of h_{p_0}. Therefore, due to the nature of the semi-simple, non-compact real form $SO(N,2)$ and the fact that $SO(N)\times SO(2)$ is the maximal compact subgroup of $SO(N,2)$, by using the Iwasawa decomposition, the coset $SO(N,2)/(SO(N)\times SO(2))$ can be parameterized as

$$L = e^{\frac{1}{2}\phi^i(x)H_i}e^{\chi^m(x)E_m}, \tag{7.1}$$

where the Cartan generators $\{H_i\}$ for $i=1,...,\dim h_{p_0}=r$ are the generators of h_{p_0}, and $\{E_m\}$ (for $m\in\Delta_{nc}^+$) are the positive root generators which generate the orthogonal complement of h_{p_0} within the solvable Lie algebra s_0 of $so(N,2)$, namely n_0. Now the $2N$ scalars are classified as $\{\phi^i\}$, the dilatons for $i=1,...,r$, and $\{\chi^m\}$, the axions for $m\in\Delta_{nc}^+$. As usual we label the roots which are elements of Δ_{nc}^+ from 1 to n; then we have $r+n=2N$. By using the generalized transpose $\#$, we introduce the internal metric $\mathcal{M}=L^{\#}L$. For $SO(N,2)$, since the subgroup generated by the compact generators of the Lie algebra $so(N,2)$ is an orthogonal group, $\#$ coincides with the matrix transpose in the fundamental representation of $so(N,2)$.

Thus depending on the discussion we have made above, the Lagrangian of the vector multiplet scalar sector of the eight dimensional matter coupled Salam-Sezgin supergravity, that can be formulated as a symmetric space sigma model, can be written as

$$\mathcal{L}_{scalar} = \frac{1}{4}tr(d\mathcal{M}^{-1} \wedge *d\mathcal{M}), \tag{7.2}$$

where $\mathcal{M} = L^T L$. The Bosonic Lagrangian of the $D = 8$ matter coupled Salam-Sezgin supergravity can now be given as [38]

$$\mathcal{L} = \frac{1}{4}\hat{R} * 1 + \frac{3}{8}d\sigma \wedge *d\sigma - \frac{1}{2}e^{2\sigma}G \wedge *G$$

$$+ \frac{1}{4}tr(d\mathcal{M}^{-1} \wedge *d\mathcal{M}) - \frac{1}{2}e^{\sigma}F \wedge \mathcal{M} * F, \tag{7.3}$$

where we have assumed the $(N+2)$-dimensional matrix representation of $SO(N,2)$. We should also state that the last term in (7.3) can be explicitly written as

$$-\frac{1}{2}e^{\sigma}F \wedge \mathcal{M} * F = -\frac{1}{2}e^{\sigma}\mathcal{M}_{ij}F^i \wedge *F^j. \tag{7.4}$$

Under the $(N+2)$-dimensional representation we assume, due to its construction, $\mathcal{M} = L^T L$ is a symmetric matrix having the components

$$\mathcal{M}_{ij} = L_i^a L_j^a, \tag{7.5}$$

for $i, j, a = 1, ..., N+2$. The $(N+2)$ two-forms $\{F^i\}$ are the field strengths of $\{A^i\}$, $F^i = dA^i$, and the Chern-Simons term G is defined as

$$G = dB + \eta_{ij}F^i \wedge A^j. \tag{7.6}$$

Here B is the two-form field, the indices $\{i,j\}$ are running from 1 to $N+2$ and η is the metric corresponding to $SO(N,2)$, i.e.

$$\eta = (-,-,+,+,+,...). \tag{7.7}$$

We also have the standard formulas

$$L^T \eta L = \eta \qquad , \qquad L^{-1} = \eta L \eta, \tag{7.8}$$

of the orthogonal matrix groups for $SO(N,2)$. The second identity is due to the fact that the coset representatives L can be locally chosen as symmetric matrices, $L^T = L$, which is evident from the explicit parametrization given in [38] and the transformation we have established in Chapter four relating locally the parametrization of [38] and (7.1).

We derive certain identities of the Cartan generators $\{H_i\}$ under the $(N+2)$-dimensional representation which are used in deriving the first-order equations from the second-order ones. First we observe that

$$\partial_i L \equiv \frac{\partial L}{\partial \phi^i} = \frac{1}{2} H_i L,$$

$$\partial_i L^T \equiv \frac{\partial L^T}{\partial \phi^i} = \frac{1}{2} L^T H_i^T,$$

$$\partial_i L^{-1} \equiv \frac{\partial L^{-1}}{\partial \phi^i} = -\frac{1}{2} L^{-1} H_i. \tag{7.9}$$

We also have the explicit form of the differentiation of the internal metric $\mathcal{M}$ as

$$\partial_i \mathcal{M} \equiv \frac{\partial \mathcal{M}}{\partial \phi^i} = \frac{1}{2} L^T (H_i + H_i^T) L. \tag{7.10}$$

By differentiating the defining equations in (7.8) with respect to $\{\phi^i\}$ and by using the identities (7.8) and (7.9) effectively, we can show that

$$H_i^T \eta = -\eta H_i. \tag{7.11}$$

Moreover we have the symmetry relation

$$(H_i L)^T = H_i L, \tag{7.12}$$

for $i = 1, ..., r$. In obtaining (7.12), we have implicitly used the fact that the coset representatives can be chosen as symmetric matrices. The identities (7.9)-(7.12) are matrix equations and apparently they are valid in the $(N+2)$-dimensional representation we have assumed. We make use of these identities in the derivation of both the second-order and the first-order field equations. To find the field equations for the non-gravitational fields, we first vary the Bosonic Lagrangian (7.3) with respect to the fields σ, B, $\{A^i\}$ and find the corresponding field equations as

$$\frac{3}{4} d(*d\sigma) = -e^{2\sigma} G \wedge *G - \frac{1}{2} e^{\sigma} \mathcal{M}_{ij} F^i \wedge *F^j,$$

$$d(e^{2\sigma} * G) = 0,$$

$$d(e^{\sigma} \mathcal{M}_{ij} * F^j) = -2e^{2\sigma} (\eta_{ij} F^j \wedge *G). \tag{7.13}$$

The last equation above can also be expanded as

$$d(e^{\frac{1}{2}\sigma}L * F) = -\frac{1}{2}d\sigma \wedge (e^{\frac{1}{2}\sigma}L * F) - 2(L^T)^{-1}\eta e^{\frac{1}{2}\sigma}F \wedge e^{\sigma} * G$$

$$- \frac{1}{2}d\phi^i H_i^T \wedge e^{\frac{1}{2}\sigma}L * F - e^{\frac{1}{2}\alpha_i\phi^i}U^{\alpha}E_{\alpha}^T \wedge e^{\frac{1}{2}\sigma}L * F. \qquad (7.14)$$

(7.14) is a vector equation. The quantities $\{H_i^T, E_\alpha^T\}$ are the matrix transpose of the matrix representatives of the generators $\{H_i, E_\alpha\}$ similar to the matrix representatives $\{H_i, H_i^T\}$ in the identities (7.9)-(7.12). In deriving (7.14) we have made use of the explicitly calculated Cartan form (4.26). However as in Chapter six in order not to confuse F^α with the coupling field strengths, we have slightly changed the notation and defined $U^\alpha = \mathbf{\Omega}_\beta^\alpha d\chi^\beta$. (7.14) is in the form which one obtains as a result of the direct substitution of the dualized Cartan form into the Cartan-Maurer equation in the dualisation (This will be performed in the next section).

We have already found the corresponding field equations for the scalars when there is matter field coupling in the previous chapter, where the formulation is done for a generic coset and for a general m-form coupling. For the present case, the coupling potentials have two-form field strengths F^i. Thus the field equations for $\{\phi^i\}$ and $\{\chi^m\}$ are

$$d(e^{\frac{1}{2}\gamma_i\phi^i} * U^\gamma) = -\frac{1}{2}\gamma_j e^{\frac{1}{2}\gamma_i\phi^i} d\phi^j \wedge *U^\gamma$$

$$+ \sum_{\alpha-\beta=-\gamma} e^{\frac{1}{2}\alpha_i\phi^i} e^{\frac{1}{2}\beta_i\phi^i} N_{\alpha,-\beta} U^\alpha \wedge *U^\beta,$$

$$
d(*d\phi^i) = \frac{1}{2} \sum_{\alpha \in \Delta_{nc}^+} \alpha_i e^{\frac{1}{2}\alpha_i \phi^i} U^\alpha \wedge e^{\frac{1}{2}\alpha_i \phi^i} * U^\alpha
$$

$$
- \frac{1}{2} e^\sigma ((H_i)^a_n L^n_m L^a_j) * F^m \wedge F^j, \tag{7.15}
$$

where the indices associated with the dilatons and the Cartan generators are from 1 to r and $\alpha, \beta, \gamma \in \Delta_{nc}^+$. We remind that the matrices $\{(H_i)^a_n\}$ are the ones corresponding to the generators $\{H_i\}$ in the $(N+2)$-dimensional representation. Notice that we have an extra e^σ factor which does not exist in (6.6) in the dilaton equation and which is the contribution of the decoupled dilaton in (7.3). To remind the reader, in the definition of $U^\alpha = \mathbf{\Omega}^\alpha_\beta d\chi^\beta$, $\mathbf{\Omega}$ is the matrix

$$
\mathbf{\Omega} = \sum_{n=0}^{\infty} \frac{\omega^n}{(n+1)!}
$$

$$
= (e^\omega - I)\, \omega^{-1}, \tag{7.16}
$$

with $\omega^\gamma_\beta = \chi^\alpha K^\gamma_{\alpha\beta}$. The structure constants $K^\gamma_{\alpha\beta}$ are defined as $[E_\alpha, E_\beta] = K^\gamma_{\alpha\beta} E_\gamma$ or, since $\{E_\alpha\}$ are the generators corresponding to a subset Δ_{nc}^+ of the roots of $so(N,2)$, we have $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$. In other words, $K^\alpha_{\beta\beta} = 0$, $K^\alpha_{\beta\gamma} = N_{\beta,\gamma}$ if $\beta + \gamma = \alpha$ and $K^\alpha_{\beta\gamma} = 0$ if $\beta + \gamma \neq \alpha$ in the root sense.

One can locally integrate the Bosonic field equations (7.13) and (7.15) by introducing auxiliary fields which can be considered as the Lagrange multipliers and by using the fact that locally a closed form is an exact one. By integration we mean to extract an exterior derivative on both sides of the equations. It is straightforward to write the first-order equations for (7.13). On the other hand for (7.15), the first-order formulation of the scalars must be treated separately due to their non-linear nature. Thus we will refer to Chapter four and Chapter six and use the results which are obtained for generic scalar cosets and couplings to obtain the first-order equations for the scalars. If we introduce the dual four-form $\widetilde{B}$, the set of five-forms $\{\widetilde{A}^j\}$ and the six-forms $\{d\widetilde{\sigma},\, d\widetilde{\phi}^i,\, d\widetilde{\chi}^m\}$, we can locally derive the first-order equations as

$$e^{2\sigma} * G = d\widetilde{B},$$

$$e^{\sigma}\mathcal{M}^i_j * F^j = -d\widetilde{A}^i + 2d\widetilde{B} \wedge \eta^i_j A^j,$$

$$*d\sigma = d\widetilde{\sigma} - \frac{4}{3} B \wedge d\widetilde{B} + \frac{2}{3}\delta_{ij} A^i \wedge d\widetilde{A}^j,$$

$$e^{\frac{1}{2}\alpha_i\phi^i}(\mathbf{\Omega})^{\alpha+r}_l * d\chi^l = (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})^{\alpha+r}_j \widetilde{S}^j,$$

$$\frac{1}{2} * d\phi^m = (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})^m_j \widetilde{S}^j + \frac{1}{4}(H_m)_{ji} A^j \wedge d\widetilde{A}^i$$

$$+ \frac{1}{4}\eta^k_i (H_m)_{jk} A^j \wedge A^i \wedge d\widetilde{B}. \tag{7.17}$$

The vector $\vec{\widetilde{S}}$ is defined in the last chapter as $\widetilde{S}^j = \frac{1}{2} d\widetilde{\phi}^j$ for $j = 1, ..., r$ and $\widetilde{S}^{\alpha+r} = d\widetilde{\chi}^\alpha$ for $\alpha \in \Delta^+_{nc}$. The matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ have been introduced before as

$\mathbf{\Gamma}_n^k = \frac{1}{2}\phi^i\,\widetilde{g}_{in}^k$ and $\mathbf{\Lambda}_n^k = \chi^m\widetilde{f}_{mn}^k$. The coefficients $\{\widetilde{g}_{in}^k\}$ and $\{\widetilde{f}_{mn}^k\}$ are already given in Chapter four. Notice that when we choose $D = 8$ and $m = 2$ in (6.16), we exactly get the first-order equations for the axions above however for the dilaton equation in (7.17) as well as the equation for the coupling potentials A^i, namely the second equation in (7.17), there are extra contributions containing $\widetilde{B}$ which result from the existence of the two-form field B. We should remind the reader of the fact that $\alpha, \beta \in \Delta_{nc}^+$ and we assume that these roots are enumerated from 1 to $n = 2N - r$.

The last term in the last equation of (7.17) needs attention: if its exterior derivative is taken in order to obtain the corresponding term in the second-order equations, one needs to make use of the fact that, as $H_i^T\eta = -\eta H_i$ and η is a diagonal matrix,

$$(H_i\eta)^T = -H_i\eta, \tag{7.18}$$

under the $(N+2)$-dimensional representation we have chosen. One also needs to make use of the identities (7.12) to reach the corresponding second-order equation in (7.15) when the last equation of (7.17) is differentiated. We are using the Euclidean metric to raise and lower the indices whenever necessary for notational purposes.

We should also remark that if we take the exterior derivative of the last equation of (7.17), we do not obtain exactly the second-order equations for $\{\phi^i\}$ given in (7.15) but instead those equations multiplied by $\frac{1}{2}$. These first-order equations will be re-formulated as a twisted self-duality equation after we introduce the full dualisation of the theory resulting in a coset model in the next section.

7.2 The Coset Formulation

In this section, by following the outline of [10, 19] we will generalize the coset formulation of the scalar manifold $SO(N,2)/(SO(N)\times SO(2))$ to the entire Bosonic sector of the $D=8$ matter coupled Salam-Sezgin supergravity. As in the other dualisation cases we have discussed before, our aim is to construct a coset representative which will be used to realize the original field equations. The group which is generated by the generators we introduce to construct the coset representative within the dualisation, in general, becomes the symmetry group of the Cartan form $\mathcal{G}$ which is generated by the coset representative map. We need to construct a new algebra which will lead to the non-linear realization of the Bosonic field equations. The first-order equations (7.17), will be formulated as a twisted self-duality equation $*\mathcal{G}=\mathcal{S}\mathcal{G}$, where $\mathcal{S}$ is a pseudo-involution of the Lie superalgebra which is generated by the original and the dual generators. The full symmetry group of the first-order equations is in general bigger than the symmetry group of the Cartan form.

We will assign a generator for each Bosonic field of the eight dimensional matter coupled Salam-Sezgin supergravity. The untilded, original generators are $\{K,\, V_i,\, Y,\, H_j,\, E_m\}$ for the fields $\{\sigma,\, A^i,\, B,\, \phi^j,\, \chi^m\}$, respectively. Their duals are introduced as $\{\widetilde{K},\, \widetilde{V}_i,\, \widetilde{Y},\, \widetilde{H}_j,\, \widetilde{E}_m\}$ for the dual fields $\{\widetilde{\sigma},\, \widetilde{A}^i,\, \widetilde{B},\, \widetilde{\phi}^j,\, \widetilde{\chi}^m\}$. We require that the Lie superalgebra to be constructed from the generators has the $\mathbb{Z}_2$ grading as usual. For this reason the generators will be chosen as odd if the corresponding potential is an odd degree differential form and even otherwise. In particular $\{V_i,\ \widetilde{V}_i\}$ are odd generators and the rest of the generators are even. In the construction of the general coset we will consider the differential graded algebra generated by the differential forms and the generators which are coupled to the fields. This algebra covers the Lie superalgebra of the field generators as we have studied in the other dualisation examples. Therefore the odd (even) generators behave like odd (even) degree differential forms under this graded differential algebra structure when they commute with the exterior product. The odd generators obey the anti-commutation relations while the even ones and the mixed couples obey the commutation relations.

The structure constants of this new algebra will be chosen so that they will lead to the correct second-order equations (7.13) and (7.15). Let us first consider the map

$$\nu = e^{\frac{1}{2}\phi^j H_j} e^{\chi^m E_m} e^{\sigma K} e^{A^i V_i} e^{\frac{1}{2} BY}. \tag{7.19}$$

The corresponding Cartan form $\mathcal{G} = d\nu\nu^{-1}$ can be expanded in terms of the original generators and it satisfies the Cartan-Maurer equation

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0, \tag{7.20}$$

by definition. The coset representative (including the dual generators as well) will be chosen as

$$\nu' = e^{\frac{1}{2}\phi^j H_j} e^{\chi^m E_m} e^{\sigma K} e^{A^i V_i} e^{\frac{1}{2} BY} e^{\frac{1}{2}\widetilde{B}\widetilde{Y}} e^{\widetilde{A}^i \widetilde{V}_i} e^{\widetilde{\sigma}\widetilde{K}} e^{\widetilde{\chi}^m \widetilde{E}_m} e^{\frac{1}{2}\widetilde{\phi}^j \widetilde{H}_j}. \tag{7.21}$$

The Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$ can also be calculated in the expansion of the original and the dual generators. For the explicit calculation of both of the

Cartan forms $\mathcal{G}$ and $\mathcal{G}'$, we need to know the algebra structure that the generators obey. This structure will be constructed in a way that the twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ will lead to the correct first-order equations. The action of the pseudo-involution $\mathcal{S}$ on the generators is as follows

$$\mathcal{S}Y = \widetilde{Y} \quad , \quad \mathcal{S}K = \widetilde{K} \quad , \quad \mathcal{S}E_m = \widetilde{E}_m \quad , \quad \mathcal{S}H_i = \widetilde{H}_i,$$

$$\mathcal{S}\widetilde{Y} = Y \quad , \quad \mathcal{S}\widetilde{K} = K \quad , \quad \mathcal{S}\widetilde{E}_m = E_m \quad , \quad \mathcal{S}\widetilde{H}_i = H_i,$$

$$\mathcal{S}V_i = \widetilde{V}_i \quad , \quad \mathcal{S}\widetilde{V}_i = -V_i. \tag{7.22}$$

The Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$ also obeys the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0. \tag{7.23}$$

By following our previous constructions, we know that one can primarily use the twisted self-duality condition $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ to construct the generator expansion of $\mathcal{G}'$, obtain an expression only in terms of the original fields and calculate (7.23). This equation must lead to the second-order field equations (7.13) and (7.15). In the calculation of the generator expansion of the Cartan form $\mathcal{G}'$, we first calculate $\mathcal{G}$ in terms of the unknown structure constants of the original field generators. $\mathcal{G}$ constitutes the part of $\mathcal{G}'$ which is composed of the original generators. We can use the twisted self-duality condition we propose on $\mathcal{G}'$ to generate the other part of $\mathcal{G}'$ which is composed of the dual generators. This is possible since when one inspects the Lie superalgebra structure, in order to obtain the correct field equations, one finds that the commutation or the anti-commutation relations of the original generators must lead to the original generators; a pair of an original and a dual generator lead to a dual generator, while the algebra product of two dual generators vanishes. When we calculate $\mathcal{G}'$, we find that

$$\mathcal{G}' = \mathcal{G} + \frac{1}{2} * d\phi^i \widetilde{H}_i + e^{\frac{1}{2}\alpha_i\phi^i} \mathbf{\Omega}^{\alpha}_{\beta} * d\chi^{\beta} \widetilde{E}_{\alpha}$$

$$+ *d\sigma\widetilde{K} - e^{c\sigma}\overset{\rightharpoonup}{\widetilde{\mathbf{V}}} e^{\mathbf{U}} e^{\mathbf{B}} * \overset{\rightharpoonup}{\mathbf{dA}} + *O\widetilde{Y}, \tag{7.24}$$

where the Cartan form $\mathcal{G} = d\nu\nu^{-1}$ is

$$\mathcal{G} = \frac{1}{2} d\phi^i H_i + \overset{\rightharpoonup}{\mathbf{E}'}\, \mathbf{\Omega}\, \overset{\rightharpoonup}{d\chi} + d\sigma K$$

$$+ e^{c\sigma}\overset{\rightharpoonup}{\mathbf{V}} e^{\mathbf{U}} e^{\mathbf{B}} \overset{\rightharpoonup}{\mathbf{dA}} + OY. \tag{7.25}$$

In (7.24) and (7.25) notice the contribution of the scalars of the theory which in general terms we have discussed in the previous chapters. Thus we keep the same notation which has already been established in Chapter four and Chapter six. Moreover we have already defined $(\mathbf{U})^n_v = \frac{1}{2}\phi^i\theta^n_{iv}$ and $(\mathbf{B})^j_n = \chi^m\beta^j_{mn}$ in Chapter six. The structure constants θ^n_{iv} and β^j_{mn} are given in (6.12). The three-form O is defined as

$$O = \frac{1}{2} e^{a\sigma} e^{\chi^m z_m} e^{\frac{1}{2}\phi^n v_n} (dB + A^i \wedge dA^j b_{ij}), \tag{7.26}$$

where we define the yet unknown structure constants as

$$[K, Y] = aY \quad , \quad \{V_i, V_j\} = b_{ij}Y,$$

$$[H_i, Y] = v_i Y \quad , \quad [E_m, Y] = z_m Y. \tag{7.27}$$

In (7.24) we have also defined the unknown structure constant c as

$$[K, V_i] = cV_i. \tag{7.28}$$

In the construction of (7.24) we have assumed that the commutators or the anti-commutators of the original generators which are not listed in (6.12), (7.27) and (7.28) vanish.

Now that we have obtained the Cartan form $\mathcal{G}'$ by using the twisted self-duality condition, solely in terms of the original fields, we can insert (7.24) into the Cartan-Maurer equation (7.23); the result can be compared with the second-order field equations (7.13) and (7.15) to read the desired commutation and the anti-commutation relations of the Lie superalgebra of the field generators. We will not give the details of this long calculation but only present the results. The resulting commutation and the anti-commutation relations of the original and the dual generators, apart from the purely scalar commutators are

$$[K, V_i] = \frac{1}{2} V_i \quad , \quad [K, Y] = Y \quad , \quad [K, \widetilde{Y}] = -\widetilde{Y},$$

$$[\widetilde{V}_k, K] = \frac{1}{2} \widetilde{V}_k \quad , \quad \{V_i, V_j\} = \eta_{ij} Y \quad , \quad [H_l, V_i] = (H_l)_i^k V_k,$$

$$[E_m, V_i] = (E_m)_i^j V_j \quad , \quad \{V_l, \widetilde{V}_k\} = \frac{2}{3} \delta_{lk} \widetilde{K} + \frac{1}{4} \sum_i (H_i)_{lk} \widetilde{H}_i,$$

$$[V_k, \widetilde{Y}] = -4\,\eta_k^l\,\widetilde{V}_l \quad , \quad [Y, \widetilde{Y}] = -\frac{16}{3}\widetilde{K} \quad , \quad [H_i, \widetilde{V}_k] = -(H_i^T)_k^m \widetilde{V}_m,$$

$$[E_\alpha, \widetilde{V}_k] = -(E_\alpha^T)_k^m \widetilde{V}_m. \tag{7.29}$$

The matrices $((H_m)_i^j,\ (E_\alpha)_i^j)$ are the images of the corresponding generators (H_m, E_α), respectively, under the representation chosen. Also the matrices $((H_m^T)_i^j$, $(E_\alpha^T)_i^j)$ are the matrix transpose of $((H_m)_i^j$, $(E_\alpha)_i^j)$. The scalar generators and the generators which are coupled to the six-form dual fields of the scalars, namely $\{H_i, E_m, \widetilde{E}_m, \widetilde{H}_i\}$, constitute a subalgebra with the following commutators

$$[H_j, E_\alpha] = \alpha_j E_\alpha \quad , \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta},$$

$$[H_j, \widetilde{E}_\alpha] = -\alpha_j \widetilde{E}_\alpha \quad , \quad [E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r} \alpha_j \widetilde{H}_j,$$

$$[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta} \widetilde{E}_\gamma, \qquad \alpha - \beta = -\gamma, \quad \alpha \neq \beta, \tag{7.30}$$

where $i, j = 1, ..., r$ and $\alpha, \beta, \gamma \in \Delta_{nc}^+$. As a reminder, Δ_{nc}^+ is the subset of the positive roots of $SO(N,2)$ whose corresponding generators do not commute with the elements of h_{p_0} in $so(N,2)$. The remaining commutators and the anti-commutators which are not listed in (7.29) and (7.30) vanish indeed.

We can now calculate the doubled field strength $\mathcal{G}' = d\nu'\nu'^{-1}$ explicitly, without using the twisted self-duality condition on it, in terms of both the original and the dual fields since we have determined the commutation and the anti-commutation relations of the original and the dual generators. From the definition of the coset element in (7.21), by using (7.29) and (7.30), the calculation of $\mathcal{G}' = d\nu'\nu'^{-1}$ yields

$$
\begin{aligned}
\mathcal{G}' = {} & \frac{1}{2}d\phi^i H_i + e^{\frac{1}{2}\alpha_i\phi^i}U^\alpha E_\alpha + d\sigma K + e^{\frac{1}{2}\sigma}L_i^k dA^i V_k \\
& + \frac{1}{2}e^\sigma GY + \frac{1}{2}e^{-\sigma}d\widetilde{B}\widetilde{Y} + (-\frac{4}{3}B\wedge d\widetilde{B} + \frac{2}{3}A^j\wedge d\widetilde{A}^i\delta_{ij} + d\widetilde{\sigma})\widetilde{K} \\
& + (e^{-\frac{1}{2}\sigma}((L^T)^{-1})_k^l d\widetilde{A}^k + 2e^{-\frac{1}{2}\sigma}((L^T)^{-1})_k^l\,\eta_i^k\,A^i\wedge d\widetilde{B})\widetilde{V}_l \\
& + \vec{\widetilde{\mathbf{T}}}(\vec{\mathbf{J}} + e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}}\vec{\widetilde{\mathbf{S}}}).
\end{aligned}
\tag{7.31}
$$

In the derivation of (7.31) we have used the matrix identities $de^Xe^{-X} = dX + \frac{1}{2!}[X,dX] + \frac{1}{3!}[X,[X,dX]] + ...$ and $e^XYe^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + ...$ which are the common tools in all Cartan form calculations as it is clear from our previous derivations. Although L is a symmetric matrix, we keep L^T in (7.31) for an easier comparison of the first-order equations we will obtain from (7.31) with the ones previously calculated in (7.17). As we have defined before, the row vector $\vec{\widetilde{\mathbf{T}}}$ is $\widetilde{\mathbf{T}}_i = \widetilde{H}_i$ for $i = 1,...,r$ and $\widetilde{\mathbf{T}}_{\alpha+r} = \widetilde{E}_\alpha$ for $\alpha \in \Delta_{nc}^+$, whereas the column vector $\vec{\widetilde{\mathbf{S}}}$ is $\widetilde{\mathbf{S}}^i = \frac{1}{2}d\widetilde{\phi}^i$ for $i = 1,...,r$ and $\widetilde{\mathbf{S}}^{\alpha+r} = d\widetilde{\chi}^\alpha$ for $\alpha \in \Delta_{nc}^+$ (We have already assumed that the roots in Δ_{nc}^+ are labelled by integer indices from 1 to n, therefore $r + n = 2\mathrm{N}$.).

The $\vec{\mathbf{J}}$ term in (7.31) is a consequence of the coupling between the scalars and the one-form potentials in (7.3) and we define it as

$$\mathbf{J}^m = \frac{1}{4}(H_m)_{ji}A^j \wedge d\widetilde{A}^i + \frac{1}{4}\,\eta_i^k (H_m)_{jk}A^j \wedge A^i \wedge d\widetilde{B} \quad , \quad m = 1, ..., r, \tag{7.32}$$

$$\mathbf{J}^{\alpha+r} = 0 \quad , \quad \alpha \in \Delta_{nc}^+.$$

In (7.31) more explicitly we have

$$\begin{aligned}\vec{\widetilde{\mathbf{T}}}(\vec{\mathbf{J}} + e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}}\vec{\widetilde{\mathbf{S}}}) = \sum_{m=1}^{r}((e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})_j^m \widetilde{\mathbf{S}}^j + \frac{1}{4}(H_m)_{ji}A^j \wedge d\widetilde{A}^i \\ + \frac{1}{4}\,\eta_i^k (H_m)_{jk}A^j \wedge A^i \wedge d\widetilde{B})\widetilde{H}_m \\ + \sum_{\alpha\in\Delta_{nc}^+}(e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})_j^{\alpha+r}\widetilde{\mathbf{S}}^j \widetilde{E}_\alpha.\end{aligned} \tag{7.33}$$

The next step is to show that if we apply the twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ on (7.31) by using (7.22), we get the correct first-order equations (7.17). Thus the twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ whose validity is implicitly justified in the construction we have presented gives

$$\begin{aligned}\frac{1}{2}e^{\sigma} * G = \frac{1}{2}e^{-\sigma}d\widetilde{B}, \\ e^{\frac{1}{2}\sigma}\text{Ł}_j^i * dA^j = -e^{-\frac{1}{2}\sigma}((L^T)^{-1})_j^i d\widetilde{A}^j \\ + 2e^{-\frac{1}{2}\sigma}((L^T)^{-1})_j^i\,\eta_k^j\,d\widetilde{B} \wedge A^k,\end{aligned}$$

$$*d\sigma = d\widetilde{\sigma} - \frac{4}{3} B \wedge d\widetilde{B} + \frac{2}{3} \delta_{ij} A^j \wedge d\widetilde{A}^i,$$

$$e^{\frac{1}{2}\alpha_i \phi^i} (\mathbf{\Omega})_l^{\alpha+r} * d\chi^l = (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})_j^{\alpha+r} \widetilde{\mathbf{S}}^j,$$

$$\frac{1}{2} * d\phi^m = (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})_j^m \widetilde{\mathbf{S}}^j + \frac{1}{4} (H_m)_{ji} A^j \wedge d\widetilde{A}^i$$

$$+ \frac{1}{4} \eta_i^k (H_m)_{jk} A^j \wedge A^i \wedge d\widetilde{B}. \tag{7.34}$$

These equations are the same with the first-order equations (7.17) which are obtained by directly integrating the second-order equations (7.13) and (7.15). We should also state that in (7.17), we have made use of the results of Chapter four and Chapter six to perform the first-order formulation of the scalar sector (7.2) excluding the matter coupling implicitly and separately by using the dualisation method only on the scalars. The second equation in (7.34) may seem to be different, however it is the second equation of (7.17) multiplied by $(L^T)^{-1}$ from the left. This result also separately justifies the proper choice of the commutation and the anti-commutation relations in (7.29) and (7.30).

In summary, we have presented the first-order formulation of the Bosonic sector of the $D = 8$ Salam-Sezgin supergravity coupled to N vector multiplets. The formulation is based on the dualisation of the fields and the construction of the Lie superalgebra of the symmetry group of the doubled field strength. After obtaining the second and the first-order equations of motion in the first section, we have doubled the non-gravitational Bosonic field content by defining dual fields which are Lagrange multipliers and we have constructed a Lie superalgebra which leads to a coset formulation of the Bosonic sector of the $D = 8$ matter coupled Salam-Sezgin supergravity in the second section. Finally we have regained the first-order equations of motion as a twisted self-duality equation of the doubled field strength namely the Cartan form $\mathcal{G}'$.

CHAPTER 8

CONCLUSION

We have introduced the Kaluza-Klein mechanism to discuss the construction of the D-dimensional maximal supergravities from the $D = 11$ supergravity theory besides, to derive the Abelian Yang-Mills supergravities from a subtheory of the ten dimensional type I supergravity. We have focussed on the scalar sectors and studied the scalar coset formulations of these supergravities from a closer point of view. The idea that in certain dimensions one can only formulate the scalar sectors as coset Lagrangians or sustain the already existing coset structures if one introduces dual scalars, replacing a subset of the higher rank fields by means of the Lagrange multiplier methods, suggests a complete dualisation of the Bosonic sectors leading to the extended coset formulations. In this direction, the dualisation of the non-gravitational Bosonic sectors of the $D = 11$, the maximal and the IIB supergravities are presented in a pattern which reveals the basics of the method of dualisation and the first-order formulation. We have devoted a whole chapter for a self-contained discussion of the symmetric spaces since, except for the $D = 4$ $N \leq 2$ and the $D = 5$ $N = 2$ supergravities, the scalar manifolds of all the other supergravities are homogeneous symmetric spaces. In accordance, we have given a rigorous formulation, including the dualisation, of the symmetric space sigma models which is based on two different approaches and parameterizations. Furthermore we have extended the dualisation of the symmetric space sigma models by coupling matter fields. Finally by mounting the general results of the dualisation of the matter coupled symmetric space sigma models in our analysis, we have developed the Bosonic doubled formalism for the $D = 8$ Salam-Sezgin supergravity which extends the results of the dualized maximal supergravities in [19] to the broader field of the matter coupled supergravities.

As we have mentioned before, the scalar coset manifolds of the maximal supergravities are symmetric spaces which are based on the semi-simple split real form global symmetry groups. We have given a thorough review of the symmetric spaces to explore how they emerge from the split (maximally non-compact) and, in general, the non-split (non-compact) real forms and to define the corresponding Borel and the solvable Lie algebra gauges for parameterizing the coset representatives. This detailed coverage has equipped us with the necessary tools to study the most general form of the symmetric space sigma models. The dualisation and the first-order formulation of the symmetric space sigma models are performed in an abstract derivation, for generic symmetric coset spaces G/K. The dualisation, when the symmetric space scalar coset manifolds are coupled to the matter fields is also carried out in D > 2 spacetime dimensions. The coupling potentials are assumed to be $(m-1)$-forms for $1 < m <$D$/2$ and $3m - D \neq 1$. As it is clear from our construction, the results are general and they are applicable to the entire set of supergravity theories which contain similar matter couplings. Thus we have constructed a general formalism which can be applied to the general scheme of the dualisation of the pure and matter coupled supergravity theories.

As a specific example of the dualisation of the matter coupled supergravities, the $D = 8$ Salam-Sezgin supergravity which is coupled to N vector multiplets [38] is reformulated as a coset model by constructing the Lie superalgebra which generates the Bosonic coset element. The formulation given in Chapter seven also explicitly denotes how the general conclusions of the chapters four and six can be adopted in the first-order formulations of the matter coupled supergravities.

Apart from reviewing the duality nature of the supergravities and the formal study of the symmetric space, scalar coset manifolds which are at the core of the supergravities, the main contribution of the thesis is two-fold. First we construct a general method to apply the dualisation techniques to the pure or matter coupled non-split scalar cosets. Secondly, we extend the range of application of the dualisation method from the maximal and the IIB supergravities [19] to the matter coupled supergravity theories. Therefore we have formed an assembly, in which we can study the symmetry schemes of the matter coupled supergravities. As a matter of fact, the matter coupling occurs as a course of the dimensional reduction [21], thus detecting the symmetries which appear in the Bosonic coset formulations also reveals the hidden symmetries of the original higher-dimensional supergravities and the relative string theories.

We have pointed out before that the coset constructions presented in chapters four, six and seven can be enlarged to include gravity by using the outline of [34] and [35]. Thus one may revise the coset formulations of the mentioned chapters to cover the entire Bosonic sector. Although we have derived the Lie superalgebras which generate the doubled coset elements of the pure and matter coupled symmetric space sigma models, as well as the Lie superalgebra of the $D = 8$ Salam-Sezgin supergravity, we have not mentioned about the related symmetry groups of the dualized Cartan forms and the twisted self-duality equations. Moreover, these symmetry groups can be studied in the general picture of the group theoretical construction of the cosets. In particular one may explore the group theoretical aspects of the doubled coset element of the $D = 8$ Salam-Sezgin supergravity by including gravity, which would provide the improved symmetry structure of the $D = 8$ Salam-Sezgin supergravity. For instance the group which will be obtained after the inclusion of the gravity sector would cover the Lorentz group and the Lie supergroup constructed in Chapter seven. The dualisation analysis we have considered here is based on the non-gravitational Bosonic field equations. Equivalently, one may work on the Lagrangian formulations which would give the twisted self-duality equations of the doubled field strengths, in other words the Cartan forms of the dualized cosets as equations of motion. The Lagrangian formalism would link the typical dualisation method presented in various examples here to its Lagrange multiplier method origins. As an example, one may find the dualized Lagrangian construction for the $SL(2, \mathbb{R})$ scalar coset in [19].

The doubled formalism or the dualisation may further be improved to include the Fermions. This would enable us to reformulate the supergravities as non-linear sigma model constructions completely, and this would reflect their full symmetries as explicit building blocks of their structures. The discussion about the dualized Lagrangian construction and the dualisation of the Fermionic sector of the $D = 11$ and the IIA supergravities can be found in [63] and [64], respectively.

Our formulation about the matter coupled supergravities does not cover the gauged supergravities on which an extensive research is going on recently. A general formalism can also be established for the dualisation of the gauged supergravities. In particular the dualized symmetries of the gauged form [38] of the $D = 8$ Salam-Sezgin supergravity can be studied on its own right.

In summary, we have plotted the outline of the dualisation and the doubled formalism of the supergravities starting from the scalar coset constructions. We have devised a general and an abstract formalism to dualize the matter coupled symmetric space sigma models, then we have used this formalism within the dualisation of the Bosonic sector of the $D = 8$ Salam-Sezgin supergravity. As a result we have formed the mainline of the dualisation of the matter coupled supergravities which can be improved in many directions as we have mentioned above.

REFERENCES

[1] E. Cremmer, B. Julia and J. Scherk, "*Supergravity theory in eleven dimensions*", Phys. Lett. **B76** (1978) 409.

[2] C. Campbell and P. West, "*N=2,D=10 non-chiral supergravity and its spontaneous compactification*", Nucl. Phys. **B243** (1984) 112.

[3] M. Huq and M. Namazie, "*Kaluza-Klein supergravity in ten dimensions*", Class. Quant. Grav. **2** (1985) 293.

[4] F. Giani and M. Pernici, "*N=2 supergravity in ten dimensions*", Phys. Rev. **D30** (1984) 325.

[5] J. Schwarz and P. C. West, "*Symmetries and transformation of chiral N=2 D=10 supergravity*", Phys. Lett. **126B** (1983) 301.

[6] P. Howe and P. West, "*The complete N=2 D=10 supergravity*", Nucl. Phys. **B238** (1984) 181.

[7] J. Schwarz, "*Covariant field equations of chiral N=2 D=10 supergravity*", Nucl. Phys. **B226** (1983) 269.

[8] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, "*Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields* ", Nucl. Phys. **B195** (1982) 97.

[9] G. Chapline and N. S. Manton, "*Unification of Yang-Mills theory and supergravity in ten dimensions* ", Phys. Lett. **B120** (1983) 105.

[10] E. Cremmer, B. Julia, H. Lü and C. N. Pope, "*Dualisation of Dualities. I.*", Nucl. Phys. **B523** (1998) 73, hep-th/9710119.

[11] E. Bergshoeff, H. J. Boonstra and T. Ortin, "*S duality and dyonic p-brane solutions in type II string theory*", Phys. Rev. **D53** (1996) 7206, hep-th/9508091.

[12] P. West, "**Introduction to Supersymmetry and Supergravity**", (World Scientific Pub. Co.; 2nd Edition 1990).

[13] A. Salam (Ed.), E. Sezgin (Ed.), "**Supergravities in Diverse Dimensions**", (World Scientific Pub. Co. 1989).

[14] P. C. West, "*Supergravity, brane dynamics and string duality*", hep-th/9811101.

[15] Y. Tanii, "*Introduction to supergravities in diverse dimensions*", hep-th/9802138.

[16] C. M. Hull and P. K. Townsend, "*Unity of superstring dualities*", Nucl. Phys. **B438** (1995) 109, hep-th/9410167.

[17] E. Witten, "*String theory dynamics in various dimensions*", Nucl. Phys. **B443** (1995) 85, hep-th/9503124.

[18] L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fre, M. Trigiante, "*R-R scalars, U-duality and solvable Lie algebras*", Nucl. Phys. **B496** (1997) 617, hep-th/9611014.

[19] E. Cremmer, B. Julia, H. Lü and C. N. Pope, "*Dualisation of dualities II : Twisted self-duality of doubled fields and superdualities*", Nucl. Phys. **B535** (1998) 242, hep-th/9806106.

[20] C. N. Pope, "**Lecture Notes on Kaluza-Klein Theory**", (unpublished).

[21] H. Lü, C. N. Pope and K. S. Stelle, "*M-theory/heterotic duality: A Kaluza-Klein perspective*", Nucl. Phys. **B548** (1999) 87, hep-th/9810159.

[22] E. Cremmer and B. Julia, "*The N=8 supergravity theory. The Lagrangian*", Phys. Lett. **B80** (1978) 48.

[23] E. Cremmer and B. Julia, "*The SO(8) supergravity*", Nucl. Phys. **B156** (1979) 141.

[24] S. Coleman, J. Wess and B. Zumino, "*Structure of phenomenological Lagrangians. 1*", Phys. Rev. **177** (1969) 2239.

[25] K. Callan, S. Coleman, J. Wess and B. Zumino, " *Structure of phenomenological Lagrangians. 2*", Phys. Rev. **177** (1969) 2247.

[26] D. V. Volkov, "*Phenomenological Lagrangians*", Sov. J. Part. Nucl. **4** (1973) 3.

[27] D. V. Volkov and V. P. Akulov, "*Possible universal neutrino interaction*", JETP Letters **16** (1972) 438.

[28] D. V. Volkov and V. P. Akulov, "*Is the neutrino a Goldstone particle?*", Phys. Lett. **B46** (1973) 109.

[29] V. Ogievetsky, "*Infinite-dimensional algebra of general covariance group as the closure of the finite dimensional algebras of conformal and linear groups*", Nuovo. Cimento **8** (1973) 988.

[30] A. Borisov and V. Ogievetsky, "*Theory of dynamical affine and conformal symmetries as the theory of gravitational field* ", Theor. Math. Phys. **21** (1974) 329.

[31] H. Lü and C. N. Pope, "*p-brane solitons in maximal supergravities*", Nucl. Phys. **B465** (1996) 127, hep-th/9512012.

[32] B. Julia and H. Nicolai, "*Conformal internal symmetry of 2d sigma-models coupled to gravity and a dilaton*", Nucl. Phys. **B482** (1996) 431, hep-th/9608082.

[33] D. Bernard and B. Julia, "*Twisted self-duality of dimensionally reduced gravity and vertex operators*", Nucl. Phys. **B547** (1999) 427, hep-th/9712254.

[34] P. West, "*Hidden superconformal symmetry in M theory*", JHEP **08** (2000) 007, hep-th/0005270.

[35] I. Schnakenburg and P. West, "*Kac-Moody symmetries of IIB supergravity*", Phys. Lett. **B517** (2001) 421, hep-th/0107181.

[36] P. West, "*E(11) and M theory*", Class. Quant. Grav. **18** (2001) 4443, hep-th/0104081.

[37] P. West, "*Automorphisms, non-linear realizations and branes*", JHEP **02** (2000) 024, hep-th/0001216.

[38] A. Salam, E. Sezgin, "*D=8 supergravity : Matter couplings, gauging and Minkowski compactification*", Phys. Lett. **B154** (1985) 37.

[39] M. Nakahara, "**Geometry, Topology and Physics**", (Adam Hilger, Philadelphia and New York, 1991).

[40] T. Dereli, "**Lecture Notes on Gravitation and Cosmology**", (unpublished).

[41] W. Thirring, "**A Course in Mathematical Physics I and II: Classical Dynamical Systems and Classical Field Theory**", (Springer-Verlag Berlin and Heidelberg GmbH and Co. KG 1991).

[42] R. W. Tucker, I. M. Benn, "**An Introduction to Spinors and Geometry with Applications in Physics**",(IOP Publishing, Ltd, 1987).

[43] A. Keurentjes, "*The group theory of oxidation*", Nucl. Phys. **B658** (2003) 303, hep-th/0210178.

[44] S. Helgason, "**Differential Geometry, Lie Groups and Symmetric Spaces**", (Graduate Studies in Mathematics 34, American Mathematical Society Providence R.I. 2001).

[45] A. L. Onishchik (Ed.), V. V. Gorbatsevich, E. B. Vinberg, "**Lie Groups and Lie Algebras I**", (Springer-Verlag New York Inc.1993).

[46] R. Carter, G. Segal, I. MacDonald, "**Lectures on Lie Groups and Lie Algebras**", (London Mathematical Society Student Texts 32, Cambridge University Press 1995).

[47] D. H. Sattinger, O. L. Weaver, "**Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics**", (Springer-Verlag, New York Inc.1986).

[48] J. F. Cornwell, "**Group Theory in Physics**", (Academic Press 1997).

[49] B. G. Wybourne, "**Classical Groups for Physicists**", (John Wiley and Sons New York 1974).

[50] J. G. Hocking and G. S. Young, "**Topology**", (Dover Publications New York 1988).

[51] R. W. R. Darling, "**Differential Forms and Connections**", (Cambridge University Press, 1995).

[52] N. Kobayashi and K. Nomizu, "**Foundations of Differential Geometry**", (Wiley - Interscience, New York, 1963).

[53] C. J. Isham, "**Modern Differential Geometry for Physicists**", (World Scientific Pub. Co. , 1989).

[54] R. L. Bishop, and S. I. Goldberg, "**Tensor Analysis on Manifolds**", (Dover Publications, Inc. New York 1980).

[55] B. Schutz, "**Geometrical Methods of Mathematical Physics**", (Cambridge University Press 1980).

[56] G. L. Naber, "**Topology, Geometry and Gauge Fields**", (Springer-Verlag, New York Inc. 1997).

[57] A. Keurentjes, "*The group theory of oxidation II : Cosets of non-split groups*", Nucl. Phys. **B658** (2003) 348, hep-th/0212024.

[58] N. T. Yılmaz, "*Dualisation of the general scalar coset in supergravity theories*", Nucl. Phys. **B664** (2003) 357, hep-th/0301236.

[59] N. T. Yılmaz, "*The non-split scalar coset in supergravity theories*", Nucl. Phys. **B675** (2003) 122.

[60] E. Cremmer, H. Lü, C. N. Pope, K. S. Stelle, "*Spectrum-generating symmetries for BPS solitons*", Nucl. Phys. **B520** (1998) 132, hep-th/9707207.

[61] T. Dereli and N. T. Yılmaz, "*Dualisation of the symmetric space sigma model with couplings*", (in preparation).

[62] T. Dereli and N. T. Yılmaz, "*Dualisation of the Salam-Sezgin d=8 supergravity*", (in preparation).

[63] A. J. Nurmagambetov, "*The sigma model representation for the duality-symmetric d=11 supergravity*", hep-th/0312157.

[64] I. A. Bandos, A. J. Nurmagambetov, D. P. Sorokin, "*Various faces of type IIA supergravity*", Nucl. Phys. **B676** (2004) 189, hep-th/0307153.

APPENDIX A

THE DIMENSION-DEPENDENT QUANTITIES OF THE MAXIMAL SUPERGRAVITIES

In Chapter five, when we give the first-order equations for the Bosonic sectors of the D-dimensional maximal supergravities which are Kaluza-Klein descendants of the eleven dimensional supergravity, we have written the first-order equations and the field strengths compactly in terms of the fields X, X^i, X^{ij}, X^{ijk}, Y_k, Q_j^k, $\vec{Q}$ for simplicity. We will give the definitions of these dimension dependent quantities for each dimension now. First we define the fields X, X^i, X^{ij}, X^{ijk} as

$$D = 11 : \qquad X = \frac{1}{2} A_{(3)} dA_{(3)}, \tag{A.1}$$

$$D = 10 : \qquad X = -A_{(2)1} dA_{(3)}, \quad X^1 = \frac{1}{2} A_{(3)} dA_{(3)}, \tag{A.2}$$

$$D = 9 : \qquad X = \frac{1}{2} \epsilon^{ij} (A_{(1)ij} dA_{(3)} - A_{(2)i} dA_{(2)j}),$$

$$X^i = -\epsilon^{ij} A_{(2)j} dA_{(3)},$$

$$X^{ij} = \frac{1}{2} \epsilon^{ij} A_{(3)} dA_{(3)}, \tag{A.3}$$

$$D=8: \qquad X = \epsilon^{ijk}(-\frac{1}{6}A_{(0)ijk}dA_{(3)} - \frac{1}{2}A_{(1)ij}dA_{(2)k}),$$

$$X^i = \frac{1}{2}\epsilon^{ijk}(A_{(1)jk}dA_{(3)} - A_{(2)j}dA_{(2)k}),$$

$$X^{ij} = -\epsilon^{ijk}A_{(2)k}dA_{(3)},$$

$$X^{ijk} = \frac{1}{2}\epsilon^{ijk}A_{(3)}dA_{(3)}, \tag{A.4}$$

$$D=7: \qquad X = \epsilon^{ijkl}(\frac{1}{8}A_{(1)ij}dA_{(1)kl} - \frac{1}{6}A_{(0)ijk}dA_{(2)l}),$$

$$X^i = -\epsilon^{ijkl}(\frac{1}{2}A_{(1)jk}dA_{(2)l} + \frac{1}{6}A_{(0)jkl}dA_{(3)}),$$

$$X^{ij} = \frac{1}{2}\epsilon^{ijkl}(A_{(1)kl}dA_{(3)} - A_{(2)k}dA_{(2)l}),$$

$$X^{ijk} = -\epsilon^{ijkl}A_{(2)l}dA_{(3)}, \tag{A.5}$$

$$D=6: \qquad X = -\frac{1}{12}\epsilon^{ijklm}A_{(0)ijk}dA_{(1)lm},$$

$$X^i = \epsilon^{ijklm}(\frac{1}{8}A_{(1)jk}dA_{(1)lm} - \frac{1}{6}A_{(0)jkl}dA_{(2)m}),$$

$$X^{ij} = -\epsilon^{ijklm}(\frac{1}{2}A_{(1)kl}dA_{(2)m} + \frac{1}{6}A_{(0)klm}dA_{(3)}),$$

$$X^{ijk} = \frac{1}{2}\epsilon^{ijklm}(A_{(1)lm}dA_{(3)} - A_{(2)l}dA_{(2)m}), \tag{A.6}$$

$$D = 5: \qquad X = -\frac{1}{72}\epsilon^{ijklmn}A_{(0)ijk}dA_{(0)lmn},$$

$$X^{i} = -\frac{1}{12}\epsilon^{ijklmn}A_{(0)jkl}dA_{(1)mn},$$

$$X^{ij} = \epsilon^{ijklmn}(\frac{1}{8}A_{(1)kl}dA_{(1)mn} - \frac{1}{6}A_{(0)klm}dA_{(2)n}),$$

$$X^{ijk} = -\epsilon^{ijklmn}(\frac{1}{2}A_{(1)lm}dA_{(2)n} + \frac{1}{6}A_{(0)lmn}dA_{(3)}), \tag{A.7}$$

$$D = 4: \qquad X^{i} = -\frac{1}{72}\epsilon^{ijklmnp}A_{(0)jkl}dA_{(0)mnp},$$

$$X^{ij} = -\frac{1}{12}\epsilon^{ijklmnp}A_{(0)klm}dA_{(1)np},$$

$$X^{ijk} = \epsilon^{ijklmnp}(\frac{1}{8}A_{(1)lm}dA_{(1)np} - \frac{1}{6}A_{(0)lmn}dA_{(2)p}), \tag{A.8}$$

$$D = 3: \qquad X^{ij} = -\frac{1}{72}\epsilon^{ijklmnpq}A_{(0)klm}dA_{(0)npq},$$

$$X^{ijk} = -\frac{1}{12}\epsilon^{ijklmnpq}A_{(0)lmn}dA_{(1)pq}, \tag{A.9}$$

$$D = 2: \qquad X^{ijk} = -\frac{1}{72}\epsilon^{ijklmnpqr}A_{(0)lmn}dA_{(0)pqr}. \tag{A.10}$$

As it is clear from the expressions we omit the wedge product. The next group of dimension dependent quantities are Y_k and they can be given for each dimension as

$$D = 10: \qquad Y_1 = -\frac{1}{2}A_{(2)1}A_{(2)1}dA_{(3)},$$

$$D = 9: \qquad Y_k = \epsilon^{ij}(\frac{1}{6}A_{(2)k}A_{(2)i}dA_{(2)j} - A_{(1)ik}A_{(2)j}dA_{(3)}),$$

$$D = 8: \qquad Y_k = \epsilon^{ijl}(-\frac{1}{2}A_{(0)jlk}A_{(2)i}dA_{(3)} - \frac{1}{4}A_{(1)jl}A_{(1)ik}dA_{(3)}$$

$$- \frac{1}{2}A_{(1)ik}A_{(2)j}dA_{(2)l}),$$

$$D = 7: \qquad Y_k = \frac{1}{4}\epsilon^{ijlm}(A_{(0)ijk}A_{(2)l}dA_{(2)m} - A_{(0)ijk}A_{(1)lm}dA_{(3)}$$

$$- A_{(1)ik}A_{(1)jl}dA_{(2)m}),$$

$$D = 6: \qquad Y_k = \epsilon^{ijlmn}(-\frac{1}{4}A_{(0)ijk}A_{(1)lm}dA_{(2)n} + \frac{1}{24}A_{(1)ik}A_{(1)jl}dA_{(1)mn}$$

$$- \frac{1}{12} A_{(0)ijk} A_{(0)lmn} dA_{(3)}),$$

$$D = 5: \qquad Y_k = \epsilon^{ijlmnp} (\frac{1}{24} A_{(0)ijk} A_{(0)lmn} dA_{(2)p} - \frac{1}{16} A_{(0)ijk} A_{(1)lm} dA_{(1)np}),$$

$$D = 4: \qquad Y_k = -\frac{1}{48} \epsilon^{ijlmnpq} A_{(0)ijk} A_{(0)lmn} dA_{(1)pq},$$

$$D = 3: \qquad Y_k = \frac{1}{432} \epsilon^{ijlmnpqr} A_{(0)ijk} A_{(0)lmn} dA_{(0)pqr},$$

$$D = 2: \qquad Y_k = 0. \qquad \text{(A.11)}$$

We also have the quantities Q_j^k

$$D = 9: \qquad Q_1^2 = -\frac{1}{2} A_{(2)1} A_{(2)1} dA_{(3)},$$

$$D = 8: \qquad Q_j^k = \epsilon^{klm} (-A_{(1)jl} A_{(2)m} dA_{(3)} - \frac{1}{6} A_{(2)j} A_{(2)l} dA_{(2)m}),$$

$$D = 7: \qquad Q_j^k = \epsilon^{klmn} (\frac{1}{2} A_{(0)jlm} A_{(2)n} dA_{(3)} - \frac{1}{4} A_{(1)mn} A_{(1)jl} dA_{(3)}$$

$$+ \frac{1}{2} A_{(1)nj} A_{(2)l} dA_{(2)m}),$$

$$D = 6: \qquad Q_j^k = \frac{1}{4} \epsilon^{klmnp} (A_{(0)jlm} A_{(1)np} dA_{(3)} - A_{(0)jlm} A_{(2)n} dA_{(2)p}$$

$$- A_{(1)jl}A_{(1)np}dA_{(2)m}),$$

$$D=5: \qquad Q^k_j = \epsilon^{klmnpq}(\frac{1}{4}A_{(0)jlm}A_{(1)np}dA_{(2)q} + \frac{1}{24}A_{(0)jlm}A_{(0)npq}dA_{(3)}$$

$$+ \frac{1}{24}A_{(1)jl}A_{(1)mn}dA_{(1)pq}),$$

$$D=4: \qquad Q^k_j = \epsilon^{klmnpqr}(\frac{1}{16}A_{(0)jlm}A_{(1)np}dA_{(1)qr} - \frac{1}{24}A_{(0)jlm}A_{(0)npq}dA_{(2)r}),$$

$$D=3: \qquad Q^k_j = \frac{1}{48}\epsilon^{klmnpqrs}A_{(0)jlm}A_{(0)npq}dA_{(1)rs},$$

$$D=2: \qquad Q^k_j = -\frac{1}{432}\epsilon^{klmnpqrst}A_{(0)jlm}A_{(0)npq}dA_{(0)rst}. \qquad \text{(A.12)}$$

Finally we have the vectors $\vec{Q}$,

$$D=10: \qquad \vec{Q} = \frac{1}{2}\vec{a}A_{(2)1}A_{(3)}dA_{(3)},$$

$$D=9: \qquad \vec{Q} = -\frac{1}{2}\vec{a}\epsilon^{ij}\hat{A}_{(1)ij}A_{(3)}dA_{(3)} + \frac{1}{2}\sum_i \vec{a}_i\hat{A}_{(2)i}\hat{A}_{(2)j}dA_{(3)}\epsilon^{ij},$$

$$D=8: \qquad \vec{Q} = -\frac{1}{6}\vec{a}\hat{A}_{(0)ijk}A_{(3)}dA_{(3)}\epsilon^{ijk} - \frac{1}{2}\sum_{i,j} \vec{a}_{ij}\hat{A}_{(1)ij}\hat{A}_{(2)k}dA_{(3)}\epsilon^{ijk}$$

$$-\frac{1}{3}\sum_{i}\vec{a}_i\hat{A}_{(2)i}\hat{A}_{(2)j}d\hat{A}_{(2)k}\epsilon^{ijk},$$

$$D=7: \qquad \vec{Q}=\sum_{i,j}\vec{a}_{ij}(\frac{1}{8}\hat{A}_{(1)ij}\hat{A}_{(1)kl}dA_{(3)}-\frac{1}{4}\hat{A}_{(1)ij}\hat{A}_{(2)k}\gamma_l^m dA_{(2)m})\epsilon^{ijkl}$$

$$+\frac{1}{6}\sum_{i,j,k}\vec{a}_{ijk}\hat{A}_{(0)ijk}\hat{A}_{(2)l}dA_{(3)}\epsilon^{ijkl},$$

$$D=6: \qquad \vec{Q}=\frac{1}{12}\sum_{i,j,k}\vec{a}_{ijk}(\hat{A}_{(0)ijk}\hat{A}_{(1)lm}dA_{(3)}-\hat{A}_{(0)ijk}\hat{A}_{(2)l}\gamma_m^n dA_{(2)n})\epsilon^{ijklm}$$

$$-\frac{1}{8}\sum_{i,j}\vec{a}_{ij}\hat{A}_{(1)ij}\hat{A}_{(1)kl}\gamma_m^n dA_{(2)n}\epsilon^{ijklm},$$

$$D=5: \qquad \vec{Q}=\sum_{i,j,k}\vec{a}_{ijk}(\frac{1}{72}\hat{A}_{(0)ijk}\hat{A}_{(0)lmn}dA_{(3)}$$

$$+\frac{1}{12}\hat{A}_{(0)ijk}\hat{A}_{(1)lm}\gamma_n^p dA_{(2)p})\epsilon^{ijklmn}$$

$$+\frac{1}{48}\sum_{i,j}\vec{a}_{ij}\hat{A}_{(1)ij}\hat{A}_{(1)kl}\gamma_m^p\gamma_n^q dA_{(1)pq}\epsilon^{ijklmn},$$

$$D=4: \qquad \vec{Q}=\sum_{i,j,k}\vec{a}_{ijk}(\frac{1}{24}\hat{A}_{(0)ijk}\hat{A}_{(1)lm}\gamma_n^q\gamma_p^r dA_{(1)qr}$$

$$- \frac{1}{72} \hat{A}_{(0)ijk} \hat{A}_{(0)lmn} \gamma_p^q dA_{(2)q}) \epsilon^{ijklmnp},$$

$$D = 3: \qquad \vec{Q} = \frac{1}{144} \sum_{i,j,k} \vec{a}_{ijk} \hat{A}_{(0)ijk} \hat{A}_{(0)lmn} \gamma_p^r \gamma_q^s dA_{(1)rs} \epsilon^{ijklmnpq}. \qquad \text{(A.13)}$$

APPENDIX B

THE DIMENSION-DEPENDENT COMMUTATORS OF THE MAXIMAL SUPERGRAVITIES

We have already given the dimension independent commutation and the anti-commutation relations in the coset formulation of the maximal supergravities. There are also the dimension dependent commutation and the anti-commutation relations in each dimension. We present them here:

$$D = 11: \quad \{V, V\} = -\widetilde{V}, \tag{B.1}$$

$$D = 10: \quad \{V, V\} = -\widetilde{V}_1 \quad , \quad [V, V^1] = \widetilde{V}, \tag{B.2}$$

$$D = 9: \quad \{V^{ij}, V\} = -\epsilon^{ij}\widetilde{V} \quad , \quad [V^i, V^j] = \epsilon^{ij}\widetilde{V},$$

$$[V^i, V] = \epsilon^{ij}\widetilde{V}_j \quad , \quad \{V, V\} = -\widetilde{V}_{12}, \tag{B.3}$$

$$D = 8: \quad [E^{ijk}, V] = -\epsilon^{ijk}\widetilde{V} \quad , \quad [V^{ij}, V^k] = \epsilon^{ijk}\widetilde{V},$$

$$\{V^{ij}, V\} = -\epsilon^{ijk}\widetilde{V}_k \quad , \quad [V^i, V^j] = \epsilon^{ijk}\widetilde{V}_k,$$

$$[V^i, V] = -\frac{1}{2}\epsilon^{ijk}\widetilde{V}_{jk} \quad , \quad \{V, V\} = -\widetilde{E}_{123}, \tag{B.4}$$

$$D = 7: \quad [E^{ijk}, V] = \epsilon^{ijkl}\widetilde{V}_l \quad , \quad [E^{ijk}, V^l] = \epsilon^{ijkl}\widetilde{V},$$

$$\{V^{ij}, V^{kl}\} = -\epsilon^{ijkl}\widetilde{V} \quad , \quad [V^{ij}, V^k] = -\epsilon^{ijkl}\widetilde{V}_l,$$

$$\{V^{ij}, V\} = -\frac{1}{2}\epsilon^{ijkl}\widetilde{V}_{kl} \quad , \quad [V^i, V^j] = \frac{1}{2}\epsilon^{ijkl}\widetilde{V}_{kl},$$

$$[V^i, V] = \frac{1}{6}\epsilon^{ijkl}\widetilde{E}_{jkl}, \tag{B.5}$$

$$D = 6: \quad [E^{ijk}, V^{lm}] = -\epsilon^{ijklm}\widetilde{V} \quad , \quad [E^{ijk}, V^l] = \epsilon^{ijklm}\widetilde{V}_m,$$

$$\{V^{ij}, V^{kl}\} = -\epsilon^{ijklm}\widetilde{V}_m \quad , \quad [V^{ij}, V^k] = \frac{1}{2}\epsilon^{ijklm}\widetilde{V}_{lm},$$

$$\{V^{ij}, V\} = -\frac{1}{6}\epsilon^{ijklm}\widetilde{E}_{klm} \quad , \quad [V^i, V^j] = \frac{1}{6}\epsilon^{ijklm}\widetilde{E}_{klm},$$

$$[E^{ijk}, V] = -\frac{1}{2}\epsilon^{ijklm}\widetilde{V}_{lm}, \tag{B.6}$$

$$D = 5: \quad [E^{ijk}, E^{lmn}] = \epsilon^{ijklmn}\widetilde{V} \quad , \quad [E^{ijk}, V^{lm}] = \epsilon^{ijklmn}\widetilde{V}_n,$$

$$\{V^{ij}, V^{kl}\} = -\frac{1}{2}\epsilon^{ijklmn}\widetilde{V}_{mn} \quad , \quad [V^{ij}, V^k] = -\frac{1}{6}\epsilon^{ijklmn}\widetilde{E}_{lmn},$$

$$[E^{ijk}, V^l] = \frac{1}{2}\epsilon^{ijklmn}\widetilde{V}_{mn} \quad , \quad [E^{ijk}, V] = \frac{1}{6}\epsilon^{ijklmn}\widetilde{E}_{lmn}, \tag{B.7}$$

$$D = 4: \quad [E^{ijk}, E^{lmn}] = \epsilon^{ijklmnp}\widetilde{V}_p \quad , \quad [E^{ijk}, V^{lm}] = -\frac{1}{2}\epsilon^{ijklmnp}\widetilde{V}_{np},$$

$$\{V^{ij}, V^{kl}\} = -\frac{1}{6}\epsilon^{ijklmnp}\widetilde{E}_{mnp} \quad , \quad [E^{ijk}, V^l] = \frac{1}{6}\epsilon^{ijklmnp}\widetilde{E}_{mnp}, \tag{B.8}$$

$$D = 3: \quad [E^{ijk}, E^{lmn}] = \frac{1}{2}\epsilon^{ijklmnpq}\widetilde{V}_{pq} \quad , \quad [E^{ijk}, V^{lm}] = \frac{1}{6}\epsilon^{ijklmnpq}\widetilde{E}_{npq}. \tag{B.9}$$

VITA

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