AN IMPROVED FINITE GRID SOLUTION FOR PLATES ON GENERALIZED FOUNDATIONS

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ABSTRACT

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In many engineering structures transmission of vertical or horizontal forces to the foundation is a major challenge. As a first approach to model it may be assumed that the foundation behaves elastically. For generalized foundations the model assumes that at the point of contact between plate and foundation there is not only pressure but also moments caused by interaction between the springs. In this study, the exact stiffness, geometric stiffness and consistent mass matrices of the beam element on two-parameter elastic foundation are extended to solve plate problems. Some examples of circular and rectangular plates on two-parameter elastic foundation including bending, buckling and free vibration problems were solved by the finite grid solution. Comparison with known analytical solutions and other numerical solutions yields accurate results.

Keywords: Winkler Foundation, Plates on Generalized Foundation, Bending, Free Vibration, Buckling, Finite Grid Solution

ÖZ

GENELLEŞTİRİLMİŞ TEMELLER ÜZERİNE OTURAN PLAKLAR İÇİN GELİŞTİRİLEN BİR SONLU IZGARA ÇÖZÜMÜ

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Birçok mühendislik yapılarında yatay ve dikey yüklerin zemine aktarılması önemli bir problem olarak karşımıza çıkmaktadır. Genelleştirilmiş zemin modellerinde plak ve zemin arasındaki temas noktasında sadece basınç degil aynı zamanda yayılı momentlerin de oldugu göz önüne alınmaktadır. Bu çalışmada iki parametreli elastik zeminlerle taşınan kiriş elemanları için bulunan rijitlik matrisleri geliştirilerek plakların ızgara şeklinde modellenmesi sağlanmıştır. Bu modelleme ile iki parametreli zeminlere oturan plaklar için bir sonlu ızgara çözümü geliştirilmiştir. Bu sayısal metot ile zeminin süreksiz ve gelişigüzel değişimi gibi parametrik değişimlerin bulunması halinde de uygulanabilir olması önemli bir avantajdır. Bu metot kulanılarak çeşitli sınır ve yükleme tiplerine sahip, eğilme, burkulma ve serbest titreşim dahil dairesel ve dikdörtgen plak problemleri çözümlerinde makul sonuçlar elde edilmiştir.

Anahtar Kelimeler: Winkler Zemini, Genelleştirilmiş Zeminde Plak problemleri, Dairesel Plak, Dikdörtgen Plak, Eğilme, Burkulma, Serbest Titreşim, Sonlu Izgara Çözümü

Dedicated to my family

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b	Width of plate or beam element
D	Plate flexural rigidity
E	Elasticity modulus
G	Shear modulus
I	Moment of inertia of a section
k_1	Winkler parameter
k_2	Second foundation parameter
k_θ	Second foundation parameter for generalized foundations
$\underline{k}$	Element stiffness matrix
$\underline{k}_g$	Element geometric stiffness matrix
k_{ii}	Indices of a specified term of stiffness matrix
L	Length of a beam
$\underline{M}$	Element consistent mass matrix
$\overline{m}$	Distributed moment per unit length
M_1	Moment at the left of a beam
M_2	Moment at the right of a beam
$\underline{N}$	Matrix of shape functions for beam elements
N_x	In-plane load in x-direction
N_y	In-plane load in y-direction
p	Reaction force intensity
q_o	Uniformly distributed force
T_1	Torque at the left of a beam
T_2	Torque at the right of a beam
U	Strain energy functional
V_1	Shear at the left of a beam

V_2 Shear at the right of a beam

w Transverse deflection

λ Eigenvalues indicting the buckling loads

μ Mass of a plate per unit area

ν Poisson ratio

θ Rotation due to bending

ω Eigenvalues indicting the free vibration frequencies

ψ_{ij} Indices of a specified shape function

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

Treatment of soil and structure as a whole is a major concern of many engineering applications. In many engineering structures rational estimation of the manner for transmission of vertical or horizontal forces to the foundation is an important and frequently recurring problem. Foundations very often represent a complex medium. It is often difficult to find suitable analytical models for foundation problems. An acceptable analysis must include behavior of foundation properly. By using certain assumptions there exist some simplified models to represent the behavior of foundations. One of the most elementary models is based on the assumption that the foundation behaves elastically. This implies not only that the foundation elements return to their original position after removing loads, but it is also accepted that their resistance is proportional to the deformation they experience. This assumption can be acceptable if displacement and pressure underneath foundation are small and approximately linearly related to each other. For "generalized" foundations the model assumes that at the point of contact between plate and foundation there is not only pressure but also distributed moments caused by the interaction between linear springs. In a generalized sense, translational and rotational deformations of the beam invoke reactions from the supporting foundation. The moments are assumed to be proportional to the slope of the elastic curve and a second parameter for foundation is then necessary for defining its response. This point will be utilized in the derivation of the corresponding equations in Chapter 2.

1.2 REVIEW OF PAST WORK

1.2.1 Studies of Models for the Supporting Medium

Plates on elastic foundations have received considerable attention due to their wide applicability in civil engineering. Since the interaction between structural foundations and supporting soil has a great importance in many engineering applications, a considerable amount of research has been conducted on plates on elastic foundations. Much research has been conducted to deal with bending, buckling and vibration problems of beam and plates on elastic foundation. The aim of most of these is to solve some real engineering problem such as structural foundation analysis of buildings, pavements of highways, water tanks, airport runways and buried pipelines, etc. Because the intent of this subsection is to give a synoptic overview of research accomplishments to date, it is necessarily brief.

Many studies have been done to find a convenient representation of physical behaviour of a real structural component supported on a foundation. The usual approach in formulating problems of beams, plates, and shells continuously supported by elastic media is based on the inclusion of the foundation reaction in the corresponding differential equation of the beam, plate, or shell.

In order to include behaviour of foundation properly into the mathematically simple representation it is necessary to make some assumptions. One of the most useful simplified models known as the Winkler model assumes the foundation behaves elastically, and that the vertical displacement and pressure underneath it are linearly related to each other. That is, it is assumed that the supporting medium is isotropic, homogeneous and linearly elastic, provided that the displacements are "small". This simplest simulation of an elastic foundation is considered to provide vertical reaction by a composition of closely spaced independent vertical linearly elastic springs. Thus the relation between the pressure and deflection of the foundation can be written as:

$$p(x,y)=k_1\, w(x,y) \tag{1.1}$$

where:

$p(x,y)$	: distributed reaction from the foundation due to applied load at point x, y
k_1	: Winkler parameter
$w(x,y)$	: vertical deflection at point x, y

The governing differential equation or Lagrange's equation of a plate subjected to lateral loads may be derived as:

$$D * \left(\frac{\partial^4 w(x,y)}{\partial x^4} + 2 * \frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4}\right) = q(x,y) \tag{1.2}$$

where:

$D = E\, t^3 / (12\,(1-v^2))$	: flexural rigidity of the plate
t	: thickness of the plate
v	: Poisson's ratio
E	: modulus of elasticity of the plate
$q(x,y)$	: external loads on the plate

In most cases, as a concentrated load applied to the surface of a linearly elastic layer it must not deflect only under the load, but it also must deflect with displacements diminishing with distance in the areas adjacent to the load. In contrast, Winkler model assumes that only the loaded points can settle while the adjacent areas remains unchanged. That is, the one – parameter way of modelling the soil underneath plates (the Winkler model) leads to a discontinuity of the deformation along the plate boundary. Therefore, in order to provide a continuity of vertical displacements there must be a relationship between the closely spaced spring elements. For satisfying the continuity, Hetényi (1967) suggested to use an elastic plate at the top of the independent spring elements to improve an interaction between them. So, the response function for this model is to modify Equation (1.2) by re-defining the external load acting in lateral direction as the difference between the surface load of the plate and the reaction of the elastic foundation given in Equation (1.1) can be derived in a general form as:

$$\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4} = \frac{q(x,y) - p(x,y)}{D} \tag{1.3}$$

There are several more realistic foundation models as well as their proper mathematical formulations, e.g. Selvadurai (1979) and Scott (1981). Representing the soil response underneath plates by two independent elastic parameters is a more refined model of having an inter-connected continuum. The main advantage of the two-parameter elastic foundation model is to provide a mechanical interaction between the individual spring elements. To have a relationship between the springs eliminates the discontinuous behaviour of Winkler model. Such physical models of soil behaviour have been suggested by a number of authors. A second foundation parameter defined by Filonenko-Boroditch (1940), Pasternak (1954) and Kerr (1964) ensures in effect that the tops of Winkler springs are inter-linked by a thin elastic membrane, a layer of compressible vertical element and rotational springs, respectively. The two-parameter models can be summarized as follows:

First, Filonenko-Boroditch introduced a model that indicated to connect the individual springs as the representation of the soil in Winkler model by a thin stretched elastic membrane under a constant tension T that provides the continuity. Then the reaction of soil in Equation (1.1) can be modified as:

$$p(x,y) = k_1 w(x,y) - T\left(\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}\right) \tag{1.4}$$

The soil behaviour model suggested by Pasternak assumes that insertion of shear interaction between the spring elements to satisfy continuity. The model implies the end of spring elements are connected by a layer of incompressible vertical elements, G, that only deform in transverse shear.

$$p(x,y) = k_1 w(x,y) - G(\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}) \tag{1.5}$$

The next model introduced by Kerr, known also as generalized foundation model, assumes that at the end of each spring element resisting pressure as in the Winkler model, there must be also a rotational spring to produce a reaction moment (k_θ) proportional to the local angle of rotation at that point. This model implies that the soil reaction in Equation (1.1) can be written as:

$$p(x,y) = k_1 w(x,y) - k_\theta (\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}) \tag{1.6}$$

Equations (1.3), (1.4) and (1.5) are all similar to each other except in the interpretation of the second parameter. Since the second parameters are constant the properties of the equations are same. Therefore, the second parameters of Filonenko-Boroditch model (T), Pasternak model (G) and Kerr model (k_θ) can be replaced by a single second parameter as (k_2). For two-parameter foundation models the soil reaction in Equation (1.1) can be redefined in a general form as:

$$p(x,y) = k_1 w(x,y) - k_2 (\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}) \tag{1.7}$$

The two-parameter elastic foundation model that provides a mechanical interaction between the individual spring elements shows a more realistic behaviour of the soil reaction. By using the soil reaction, Equation (1.3) derived by Hetényi's suggestion can be modified in a more general form as:

$$D(\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4})$$
$$+ k_1 w(x,y) - k_2 (\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}) = q(x,y) \tag{1.8}$$

This equation is applicable to all types of plates resting on two-parameter elastic foundation problems.

1.2.2 Studies on the Solution Methods

The solution of plate problems with classical methods that provide mathematically exact solutions are available for a limited number of limited cases. There are a few load and boundary conditions that permit Equations (1.2) and (1.3) to be solved exactly. For arbitrary load and boundary conditions, there is no exact solution of Equation (1.8) for plates resting on two-parameter elastic foundation problems because it is too complex,. In another words the two-parameter elastic foundation soil model underneath plate boundary problems cannot be solved analytically in readily understood format for all load and boundary conditions (Sladek et al. 2002).

Currently, there exist approximate and numerical methods to solve the governing differential equations of plates resting on one-parameter and two-parameter elastic foundation for transverse displacement *w*. Many studies have been done related to such problems. Before embarking on a review of these results, it is useful to examine where we stand in relation to one-dimensional elements supported by generalized foundations.

Introducing the finite element method in 1960s and the developments in computers have had a great importance for the developments in applied mechanics. A broad range of the engineering problems has been solved by computer-based methods such as finite element, boundary elements methods, etc.

In the case of the beam analysis, the formulations based on interpolation (shape) functions have been used in solution by finite element method. In 1980's some authors have derived exact stiffness matrices such as Wang (1983) and Eisenberger (1985a) for beams on Winkler foundations and Cook and Zhaohua (1983) and Eisenberger (1987a) for two parameter foundations.

Razaqpur and Shah (1991) derived a new finite element to eliminate the limitations of the solution, such as the necessities of certain combinations of beam and foundation parameters, for beams on a two-parameter elastic foundation. They concluded that the derivation of explicit element stiffness matrix and nodal load vector makes the proposed element efficient and obviates the need for dividing the beam into many elements between the points of loading. They presented the complete solution of the governing equation corresponding to the most common types of load.

Gülkan and Alemdar (1999) reported an analytical solution for the shape functions of a beam segment supported on a generalized two-parameter elastic foundation. In that study it is pointed out that the exact shape functions can be utilized to derive exact analytic expressions for the coefficients of the element stiffness matrix, work equivalent nodal forces for arbitrary transverse loads and coefficients of the consistent mass and geometrical stiffness matrices.

When plates are considered, the solutions must become more sophisticated, and mathematically less familiar for most engineering applications.

Zafrany and Fadhill (1996) derived boundary integral equations with three degrees-of-freedom per boundary node, thus avoiding the generation of unknown corner terms for plates with non-smooth boundaries. For thin plates resting on a two-parameter elastic foundation, based on a modified Kirchhoff theory in which the transverse normal stress is considered. The explicit expressions of kernel functions are provided in terms of complex Bessel functions. Additional boundary element derivations for plates with free-edge conditions are presented, and reduction of loading domain integral terms for cases with concentrated loads and moments, and uniformly- or linearly-distributed loading is included. They concluded that the three degrees-of-freedom approach has led to very accurate results for plates with corners and the transverse normal stress has a minor effect on plate deflection, but it has some effect on stresses and moments, which increases with the thickness of the plate.

Wang, et al. (1997) presented relationships between the buckling loads determined using classical Kirchhoff plate theory and shear deformable plate theories on Pasternak foundation. The relationships of Kirchhoff, Mindlin and Reddy polygonal plates resting on a Pasternak foundation obtained are exact for isotropic, simply supported under an isotropic in-plane load. The relationships are also applicable for the Winkler foundation as this foundation model is a special case of the Pasternak foundation.

Tameroğlu (1996) studied a different solution technique for free vibrations of rectangular plates with clamped boundaries resting on elastic foundations and subjected to uniform and constant compressive, unidirectional forces in the mid-plane. The method is based on the use of a non-orthogonal series expansion consisting of some specially chosen trigonometric functions for the deflection surface w of the plate. The orthogonalization of the series and other calculations are performed using Fourier expansion of Bernoulli polynomials under some realistic approximations for the limiting values of the boundary conditions. It is concluded that by this method one need not use the solution of the differential equation of the problem. The results obtained for the problem are consistent with the well-known solutions.

Saha, et al. (1997) studied the dynamic stability of a rectangular plate on non-homogeneous foundation, subjected to uniform compressive in-plane bi-axial dynamic loads and supported on completely elastically restrained boundaries. In that study, non-homogeneous foundation consists of two regions having different stiffness but symmetric about the centre lines of the plate. They derived the equation governing the small amplitude motion of the system by a variational method. They also studied the effects of stiffness and geometry of the foundation, boundary conditions, static load factor, in-plane load ratio and aspect ratio on the stability boundaries of the plate for first- and second-order simple and combination resonance.

Ramesh, et al. (1997) analyzed the behavior of flexible rectangular plates resting on tensionless elastic foundations using finite-element method (FEM) techniques. They adopted a nine-noded Mindlin element for modeling the plate to account for transverse shear effects. The model can be effectively used to analyze plates on tensionless elastic foundations with any type of common boundary conditions and loading combinations. The model also accounts for realistic design conditions, namely, the tensionless nature of the foundations, transverse shear effects, and effects of attachment. They concluded that in case the plate dimension to thickness ratios are very small, the shear effect dominates, and deflections are highly underestimated if the problem is analyzed assuming thin plate behavior. The contacting region is only dependent on the relative stiffness and plate thickness.

Omurtag and Kadıoğlu (1997) studied a functional and a plate element capable of modelling the Kirchhoff type orthotropic plate resting on Winkler / Pasternak (isotropic/orthotropic) elastic foundation are given and numerical results of a free vibration analysis is performed by using the Gateaux Differential Method (GDM) that successfully applied to various structural problems such as space bars, plates and shells by Omurtag and Aköz (1997). Their PLTEOR4 element has four nodes with 4 x 4 DOF. Natural angular frequency results of the orthotropic plate are justified by the analytical expressions present in the literature and some new problems for orthotropic plates on elastic foundation (Winkler and Pasternak type foundation) are solved. The Pasternak foundation, as a special case, converges to Winkler type foundation if shear layer is neglected. Results that they report are quite satisfactory. They concluded that when the results of the foundation models Winkler and Pasternak are compared, it is observed that natural angular frequency results obtained by Pasternak type foundation modelling are higher than Winkler model for a constant k in each case.

Kocatürk (1997) presented an elastoplastic analysis of rotationally symmetric reinforced concrete plates resting on elastoplastic subgrade under column load. The analysis is simplified by the assumption that any plate element is either entirely elastic or entirely plastic. This assumption is practically fulfilled for a sandwich plate. Differential equations that describe the behaviour of plastic zones during the deformation process are derived and solved in closed form. Interaction between the plate and the foundation is investigated for dimensionless load-moment relations.

Trifunac (1997) investigated stiff structures with large plan dimensions, on soft soil and supported by columns on separate foundations. Differential motion of the column foundations may lead to large moments and shear forces in the first-story columns, during near field moderate and large earthquakes. These forces will augment the effects of the concurrently occurring dynamic response, causing larger than expected ductility, larger inter-story drift, and thus larger and more dangerous participation of vertical acceleration. When the design conditions call for the connecting beams and slabs between individual column foundations, some components of motion of the first-story columns may be reduced. He concluded that the foundation should be designed to withstand the forces created by deformation of soil. He presented approximate criteria for estimating the relative significance of these additional effects.

Chung at al. (2000) investigated finite strip method for the free vibration and buckling analysis of plates with abrupt changes in thickness and complex support conditions. The free vibration problem of a stepped plate is modelled by finite strip method supported on non-homogeneous Winkler elastic foundation with elastically mounted masses is formulated based on Hamilton's principle. The method is further extended to the buckling analysis of rectangular stepped plates. Numerical results also show that the method is versatile, efficient and accurate.

In the study of Huang and Thambiratnam (2001) a procedure incorporating the finite strip method together with spring systems is proposed for treating plates on elastic supports. The spring systems can simulate different elastic supports, such as elastic foundation, line and point elastic supports, and also mixed boundary conditions. As a numerical example a three-span simply supported plate is first considered and the effects of support stiffness on the static and free vibration responses and on the critical buckling stress are discussed. A plate resting on a Winkler foundation is studied next, and the effects of dimension ratio on the static and free vibration responses are discussed. Numerical results show that the spring system can successfully simulate different kinds of elastic supports.

1.3 OBJECTS AND SCOPE OF THIS STUDY

The aim of this proposed research is to investigate an improved finite grid solution of plates on a two-parameter elastic foundation. This is an extension of the so-called discrete parameter approach where the physical continuous domain is broken down into discrete sub-domains, each endowed with a response suitable for the purpose of mimicking problem at hand. Conceptually, it is similar to the finite element method, except that each discrete element utilized is equipped with an exact solution. Therefore, errors are attributable only to the effects of discretization.

The governing equation for plates resting on two-parameter elastic foundation problems is quite complicated. Hence, an analytical solution is not feasible or easily formulated. Therefore, it is necessary to get an accurate and efficient numerical method for general applications. In order to simplify the problem it is possible to use a grid of beam elements to model plates. After all, within limitations of simplified formulation as Wilson (2000) indicated, plate bending is an extension of beam theory. In this dissertation, plates on generalized foundations will be represented by grillages of beams that resemble the original plates such as rectangular, circular or annular plates when they are assembled. No limits are placed on the geometrical properties of the plate boundaries, or on their displacement boundary conditions. Because the plate is discretized, discontinuous foundation, abrupt changes in plate thickness and other types of irregularities are easily accommodated.

1.4 ORGANIZATION OF THE STUDY

There are six chapters in this dissertation. A general discussion and overview of the study, a review of past studies and objectives of this study are presented in Chapter 1.

In Chapter 2 analytical solutions of the discrete beam element resting on one- or two-parameter elastic foundation are obtained. These analytic solutions include derivation of the governing differential equations and exact shape functions. Then the exact shape functions are used to form element stiffness matrices and work equivalent load vectors for finite element applications. Some graphical comparisons have been done to observe the influences of foundation parameters on the work equivalent nodal loads, stiffness terms and the shape functions

In Chapter 3 the problem is extended to the solution of stability and vibration problems. The geometric stiffness matrices and consistent mass matrices of the discrete beam element on one- or two-parameter elastic foundation are derived. The influences of foundation parameters are portrayed graphically for geometric stiffness and consistent mass terms.

In Chapter 4 a general description of the representation of rectangular and circular plates by beam elements is given. A proper transformation matrix is used for assembling the discretized plate element. Then the system stiffness, consistent mass and the geometric stiffness matrices are generated to solve the plate bending, buckling and vibration problems.

Chapter 5 contains the solution of the bending, buckling and vibration problems of rectangular and circular plates resting on one- or two-parameter elastic foundation. The results are compared with the well known analytical and the other numerical solutions.

Chapter 6 presents the conclusions and the suggestions for further studies

In the Appendix explicit forms of the element based consistent mass and consistent geometric stiffness matrices are presented.

CHAPTER 2

FORMULATION OF THE PROBLEM

2.1 INTRODUCTION

A differential part of a plate supported by a generalized foundation, which terminates at the ends of the plate is shown in Figure 2.1. In this Figure the first of these parameters is representative of the foundation's resistance to transverse translations, and is called the Winkler parameter k_1 in force per unit length per unit area (e.g. kN/m/m^2= kN/m^3 units). In the Winkler formulation each translational spring can deflect independently of springs immediately adjacent to it. In this model it is assumed that there is both pressure and moment at the points of contact between plate and foundation. These moments are assumed to be proportional to the angle of rotation, so that the second foundation parameter is representative of the foundation's resistance to rotational deformations, and is denoted by k_θ.

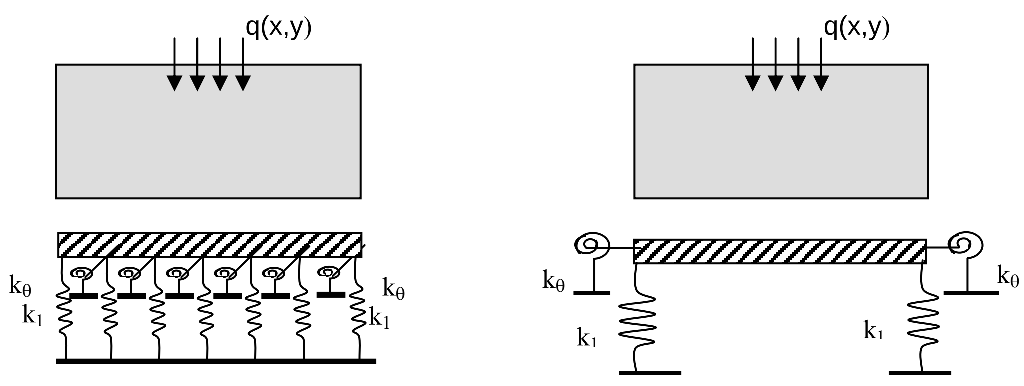


Figure 2.1: Plate and Model Foundation Representation by (a) Consistent Springs, (a) Lumped Springs

By a generalized foundation modelling, influence of moment reaction of foundation will be inserted into the formulation by a distributed rotational spring element (k_θ) in addition to the vertical spring element (k_1). For all types of plates resting on two-parameter elastic foundations related to Figure 2.1 the governing differential equation derived in Section 1.2 can be rewritten as:

$$\begin{aligned}
&D\left(\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4}\right) \\
&+ k_1 w(x,y) - k_\theta\left(\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}\right) = q(x,y)
\end{aligned} \tag{2.1}$$

With most elements developed to date, there exist no rigorous solution for this equation except in the form of infinite Fourier series for a Levy-type solution. The series solutions are valid for very limited cases such as when the second parameter has been eliminated, and simple loading and boundary conditions exist.

As an alternative for different types of loading and boundary conditions it is possible to extend the exact solution for a beam supported on a one- or two-parameter elastic foundation to plates on generalized foundations when the plate is represented by a discrete number of intersecting beams.

In the following sections finite element based matrix methods will be used to determine the exact shape, fixed end forces and stiffness matrices of beam elements resting on elastic foundations. These individual element matrices will be used to form the system exact load and stiffness matrices for plates.

2.2 PROPERTIES OF BEAM ELEMENTS RESTING ON ONE-PARAMETER ELASTIC FOUNDATION

The solution of plate problems cannot be solved analytically for all load and boundary condition combinations. Instead, grillages of beam elements that have no such limitations can represent the plates. The properties of beam elements resting on elastic foundations will be a very useful tool to solve such complicate problems. A representation of the foundation with independent closely linear springs underlying a beam element is shown in Figure 2.2

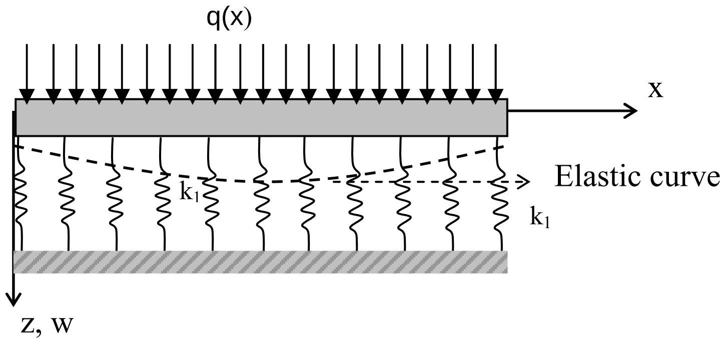


Figure 2.2: Representation of the Beam Element Resting on One-Parameter (Winkler) Foundation

The analysis bending of beam elements resting on an elastic foundation is developed, by the Winkler assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point.

2.2.1 Derivation of the Differential Equation

Consider the straight beam supported along its entire length by an elastic medium and subjected to uniform distributed load as shown in Figure 2.2. The reaction forces will be assumed to be acting opposing to the vertical deflection of the beam due to distributed load and this will cause compression in the supporting medium. Assuming the medium's material follows Hooke's law let us the fundamental assumption that the reaction force intensity (p) at any point is proportional to the deflection (w) of the beam at that point:

$$p(x) = k_1 w(x) \tag{2.2}$$

The constant for supporting medium, which is known as modulus of foundation (k_0), has a dimension of force per unit displacement per unit area. Since beam elements have no second dimension, its width must be taken into consideration to determine the Winkler parameter.

$k_1 = k_0$ for plate elements with F/L^3 dimensions.

$k_1 = b \times k_0$ for beam elements with F/L^2 dimensions.

For the derivation of the differential equation let us take an infinitesimal element as in Figure 2.2. The forces exerted on such an element are shown in Figure 2.3. Considering the equilibrium of the element by the summation of the forces in vertical direction gives:

$$(V + dV) - V + q(x)dx - k_1 w(x)dx = 0 \tag{2.3}$$

$$\frac{dV}{dx} + q(x) - k_1 w(x) = 0 \tag{2.4}$$

By ignoring infinitesimal quantities and taking moments about O:

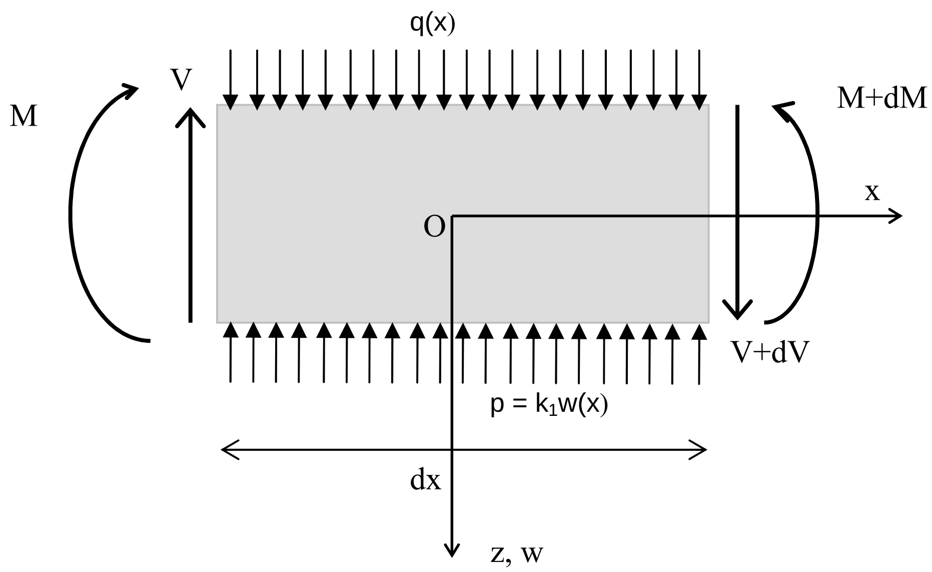


Figure 2.3: Forces Exerted on an Infinitesimal Element of the Beam Element Resting on One-Parameter (Winkler) Foundation.

$$V = \frac{dM}{dx} \tag{2.5}$$

is obtained. Using the differential equation of a beam in bending

$$M = -EI\frac{d^2 w(x)}{d^2 x} \tag{2.6}$$

where EI is the flexural rigidity and differentiating Equation (2.6) twice to obtain:

$$\frac{dV}{dx} = \frac{d^2 M}{dx^2} \tag{2.7}$$

By substituting Equations (2.6) and (2.7) into Equation (2.4) the differential equation of a beam element resting on one-parameter foundation is obtained as:

$$EI\frac{d^4 w(x)}{d^4 x} + k_1 w(x) = q(x) \tag{2.8}$$

2.2.2 Derivation of Exact Shape Functions of the Beam Elements

By equating $q(x)$=0; the homogeneous form of Equation (2.8) is:

$$EI\,\frac{d^4 w(x)}{dx^4} + k_1 w(x) = 0 \tag{2.9}$$

Let us rewrite Equation (2.9) as:

$$\frac{d^4 w(x)}{dx^4} + \frac{k_1}{EI} w(x) = 0 \tag{2.10}$$

$$\frac{d^4 w(x)}{dx^4} + 4\lambda^4 w(x) = 0 \tag{2.11}$$

where

$$\lambda = \sqrt[4]{\frac{k_1}{4EI}} \tag{2.12}$$

By the operator method using $\frac{d^n}{dx^n} = D^n$ then the characteristic Equation (2.11) can be written as:

$$(D^4 + 4\lambda^4)w(x) = 0 \tag{2.13}$$

The roots of the characteristic equation are

$$\begin{aligned} D_1 &= \lambda + i\lambda \\ D_2 &= -\lambda + i\lambda \\ D_3 &= -\lambda - i\lambda \\ D_4 &= \lambda - i\lambda \end{aligned} \tag{2.14}$$

where i is the imaginary number. Using Equation (2.14), the closed form solution of Equation (2.11) is

$$\begin{aligned} w(x) &= a_1 e^{\lambda x}(Cos[\lambda x] + Sin[\lambda x]) + a_2 e^{-\lambda x}(Cos[\lambda x] + Sin[\lambda x]) \\ &+ a_3 e^{-\lambda x}(Cos[\lambda x] - Sin[\lambda x]) + a_4 e^{\lambda x}(Cos[\lambda x] - Sin[\lambda x]) \end{aligned} \tag{2.15}$$

using hyperbolic functions

$$\begin{aligned} e^{\lambda x} &= Cosh[\lambda x] + Sinh[\lambda x] \\ e^{-\lambda x} &= Cosh[\lambda x] - Sinh[\lambda x] \end{aligned} \tag{2.16}$$

Substituting the above hyperbolic functions and rearranging Equation (2.15) with defining the new constants, the closed form solution of Equation (2.11) is obtained as:

$$w(x) = \begin{matrix} c_1 Sin[\lambda x] Sinh[\lambda x] + c_2 Sin[\lambda x] Cosh[\lambda x] + \\ c_3 Cos[\lambda x] Sinh[\lambda x] + c_4 Cos[\lambda x] Cosh[\lambda x] \end{matrix} \tag{2.18a}$$

By neglecting foundation effects, a linear description of the angular displacement at any point along the element can be expressed as $Ø(x) = a_1 + a_2 x$. Inserting the angular displacements due to torsional effects, Equation (2.18) that had been derived by Alemdar and Gülkan (1997) can be rearranged as follows:

$$w(x) = \begin{matrix} c_1 + c_2 Sin[\lambda x]\cosh[\lambda x] + c_3 Sin[\lambda x]\sinh[\lambda x] + \\ c_4 x + c_5 Cos[\lambda x]\cosh[\lambda x] + c_6 Cos[\lambda x]\sinh[\lambda x] \end{matrix} \tag{2.18b}$$

Then, the closed form equation can be expressed in matrix form as:

$$w = \underline{B}^T \underline{C} \tag{2.19}$$

where

$$\underline{B}^T = \{\ 1 \quad Sin[\lambda x]\cosh[\lambda x] \quad Sin[\lambda x]\sinh[\lambda x] \quad x \quad Cos[\lambda x]\cosh[\lambda x] \quad Cos[\lambda x]\sinh[\lambda x]\ \}$$

$$\underline{C} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}$$

The arbitrary constants c_2, c_3, c_5 and c_6 subscript of the vector $\underline{C}$ can be determined by relating them to the end displacements which forms boundary conditions shown in Figure 2.4. In this figure:

$$\{\underline{d}\}^T = \{\phi_1, \theta_1, w_1, \phi_2, \theta_2, w_2\} \tag{2.20}$$

$$\{\underline{F}\}^T = \{T_1, M_1, V_1, T_2, M_2, V_2\} \tag{2.21}$$

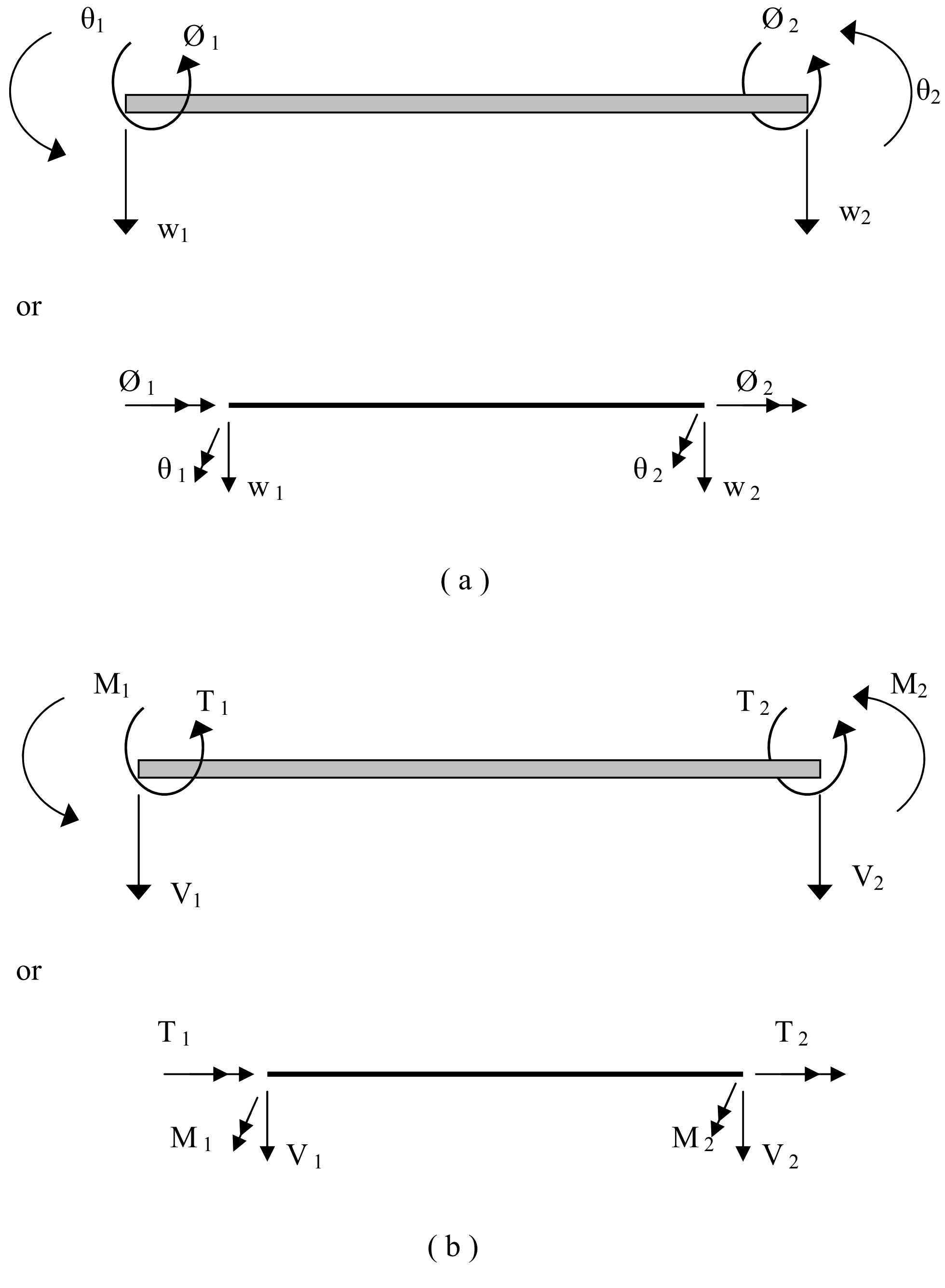


Figure 2.4: A Finite Element of a Beam (a) Generalized Displacements (b) Loads Applied to Nodes

Vectors { $\underline{d}$ } and { $\underline{F}$ } represent the generalized displacements and the loads applied to the nodes, respectively. In order to relate the rotational elements of the displacement vector to the constant vector, it is necessary to differentiate the bending part of Equation (2.18) with respect to x.

$$\frac{dw(x)}{dx}=\begin{array}{l} c_3\left(\lambda Sin[\lambda x]Cosh[\lambda x]+\lambda Sinh[\lambda x]Cos[\lambda x]\right)+ \\ c_2\left(\lambda Sin[\lambda x]Sinh[\lambda x]+\lambda Cos[\lambda x]Cosh[\lambda x]\right)+ \\ c_6\left(\lambda Cos[\lambda x]Cosh[\lambda x]-\lambda Sin[\lambda x]Sinh[\lambda x]\right)+ \\ c_5\left(\lambda Sinh[\lambda x]Cos[\lambda x]-\lambda Sin[\lambda x]Cosh[\lambda x]\right) \end{array} \tag{2.22}$$

The generalized displacement vector given in Equation (2.20) can be determined with x= 0 and x= L values of Equations (2.18) and (2.22) in terms of the constants as follows,

Bending case:

$$\frac{dw}{dx}(x=0)=\theta_1=c_2\lambda+c_6\lambda$$

$$w(x=0)=w_1=-c_5$$

$$\frac{dw}{dx}(x=L)=\begin{array}{l} c_3\left(\lambda Sin[\lambda L]Cosh[\lambda L]+\lambda Sinh[\lambda L]Cos[\lambda L]\right)+ \\ c_2\left(\lambda Sin[\lambda L]Sinh[\lambda L]+\lambda Cos[\lambda L]Cosh[\lambda L]\right)+ \\ c_6\left(\lambda Cos[\lambda L]Cosh[\lambda L]-\lambda Sin[\lambda L]Sinh[\lambda L]\right)+ \\ c_5\left(\lambda Sinh[\lambda L]Cos[\lambda L]-\lambda Sin[\lambda L]Cosh[\lambda L]\right) \end{array} \tag{2.23a}$$

$$w(x=L)=w_2=\begin{array}{l} -\left(c_3 Sin[\lambda L]Sinh[\lambda L]+c_2 Sin[\lambda L]Cosh[\lambda L]+\right. \\ \left. c_6 Cos[\lambda L]Sinh[\lambda L]+c_5 Cos[\lambda L]Cosh[\lambda L]\right) \end{array}$$

Torsional case:

$$\phi(x)=c_1+c_4x$$

$$\phi(x=0)=\phi_1=c_1 \tag{2.23b}$$

$$\phi(x=L)=\phi_2=c_1+c_4L$$

Equation (2.23) can be rewritten in matrix form,

$$\begin{Bmatrix} \phi_1 \\ \theta_1 \\ w_1 \\ \phi_2 \\ \theta_2 \\ w_2 \end{Bmatrix} = [\underline{H}] \cdot \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} \tag{2.24}$$

or

$$[\underline{d}] = [\underline{H}] \cdot [\underline{C}] \tag{2.25}$$

where [H] is a 6x6 matrix from Equation (2.23). The arbitrary constant vector C can be defined as:

$$[\underline{C}] = [\underline{H}]^{-1} \cdot [\underline{d}] \tag{2.26}$$

Substitute Equation (2.26) into Equation (2.18) then the closed form solution of the differential equation can be written in matrix form as:

$$[\underline{w}] = [\underline{B}]^T \cdot [\underline{H}]^{-1} \cdot [\underline{d}] \tag{2.27}$$

Equation (2.27) can be redefined by introducing matrix N that includes four shape functions and the generalized displacements defined in Figure 2.4 as follows,

$$[\underline{w}] = [\underline{N}] \cdot \begin{Bmatrix} \phi(x=0) \\ \dfrac{dw}{dx}(x=0) \\ w(x=0) \\ \phi(x=L) \\ \dfrac{dw}{dx}(x=L) \\ w(x=L) \end{Bmatrix} \tag{2.28}$$

After performing the necessary symbolic calculations, the shape functions are obtained. Each shape (interpolation) function defines the elastic curve equation of the beam elements for a unit displacement applied to the element in one of the generalized displacement direction as the others are set equal to zero. The elements of the shape functions matrix N are:

$$\psi_1 = \left(1 - \frac{x}{L}\right) \tag{2.29a}$$

$$\psi_2 = \left(\frac{\begin{array}{l}\sin[\lambda x]\cosh[\lambda(2L-x)] - \cosh[\lambda x]\sin[\lambda x] + \\ \cos[\lambda(2L-x)]\sinh[\lambda x] - \cos[\lambda x]\sinh[\lambda x]\end{array}}{\lambda(-2+\cos[2\lambda L]+\cosh[2\lambda L]} \right) \tag{2.29b}$$

$$\psi_3 = \left(\frac{\begin{array}{l}\cos[\lambda x]\cosh[\lambda(2L-x)] + \cosh[\lambda x]\cos[\lambda(2L-x)] - \\ 2\cos[\lambda x]\cosh[\lambda x] + \sin[\lambda x]\sinh[\lambda(2L-x)] - \\ \sinh[\lambda x]\sin[\lambda(2L-x)]\end{array}}{2-(\cos[2\lambda L]+\cosh[2\lambda L])} \right) \tag{2.29c}$$

$$\psi_4 = \left(\frac{x}{L}\right) \tag{2.29d}$$

$$\psi_5 = \left(\frac{\begin{array}{l}\sin[\lambda(L-x)]\cosh[\lambda(L-x)] - \cosh[\lambda(L+x]\sin[\lambda(L-x)] + \\ \cos[\lambda(L-x)]\sinh[\lambda(L-x)] - \cos[\lambda(L+x)]\sinh[\lambda(L-x)]\end{array}}{\lambda(-2+\cos[2\lambda L]+\cosh[2\lambda L]} \right) \tag{2.29e}$$

$$\psi_6 = \left(\frac{\begin{array}{l}-2\cos[\lambda(L-x]\cosh[\lambda(L-x)] + \cosh[\lambda(L-x]\cos[\lambda(L+x)] + \\ \cos[\lambda(L-x]\cosh[\lambda(L+x)] - \sinh[\lambda(L-x]\sin[\lambda(L+x)] + \\ \sin[\lambda(L-x]\sinh[\lambda(L+x)]\end{array}}{2-(\cos[2\lambda L]+\cosh[2\lambda L])} \right) \tag{2.29f}$$

The bending shape functions are directly affected by the foundation parameter. It is possible to redefine them in non-dimensional forms for comparing the functions with the corresponding Hermitian polynomials. To have non-dimensional forms, let us insert the following relations into Equation (2.29).

$$\xi = \frac{x}{L} \qquad \text{for} \qquad 0 \le x \le L \tag{2.30}$$

and

$$p = \lambda L = \sqrt[4]{\frac{k_1}{4EI}}\,L \tag{2.31}$$

where L is the length of the beam. Note that both p and ξ are non-dimensional quantities. Since the torsional shape functions are not affected, then only the non-dimensional forms of the bending shape functions will be considered as follows:

$$\frac{\psi_2}{L} = \left(\frac{\sin[p\xi]\cosh[p(2-\xi)] - \cosh[p\xi]\sin[p\xi] + \cos[p(2-\xi)]\sinh[p\xi] - \cos[p\xi]\sinh[p\xi]}{p(-2+\cos[2p]+\cosh[2p]} \right) \tag{2.32a}$$

$$\psi_3 = \left(\frac{\cos[p\xi]\cosh[p(2-\xi)] + \cosh[p\xi]\cos[p(2-\xi)] - 2\cos[p\xi]\cosh[p\xi] + \sin[p\xi]\sinh[p(2-\xi)] - \sinh[p\xi]\sin[p(2-\xi)]}{2-(\cos[2p]+\cosh[2p])} \right) \tag{2.32b}$$

$$\frac{\psi_5}{L} = \left(\frac{\sin[p(1-\xi)]\cosh[p(1-\xi)] - \cosh[p(1+\xi)]\sin[p(1-\xi)] + \cos[p(1-\xi)]\sinh[p(1-\xi)] - \cos[p(1+\xi)]\sinh[p(1-\xi)]}{p(-2+\cos[2p]+\cosh[2p]} \right) \tag{2.32c}$$

$$\psi_6 = \left(\frac{-2\cos[p(1-\xi)]\cosh[p(1-\xi)] + \cosh[p(1-\xi)]\cos[p(1+\xi)] + \cos[p(1-\xi)]\cosh[p(1+\xi)] - \sinh[p(1-\xi)]\sin[p(1+\xi)] + \sin[p(1-\xi)]\sinh[p(1+\xi)]}{2-(\cos[2p]+\cosh[2p])} \right) \tag{2.32d}$$

On the other hand, the shape functions for flexure of uniform beam element without any foundation, that is the limits of Equation (2.29) as k_1 tends to zero, are:

$$\psi_2 = x\left(1-\frac{x}{L}\right)^2 \tag{2.33a}$$

$$\psi_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 - 1 \tag{2.33b}$$

$$\psi_5 = x\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right] \tag{2.33c}$$

$$\psi_6 = 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 \tag{2.33d}$$

Substitute Equation (2.30) into Equation (2.33) to find out the non-dimensional forms of the shape functions as Hermitian polynomials.

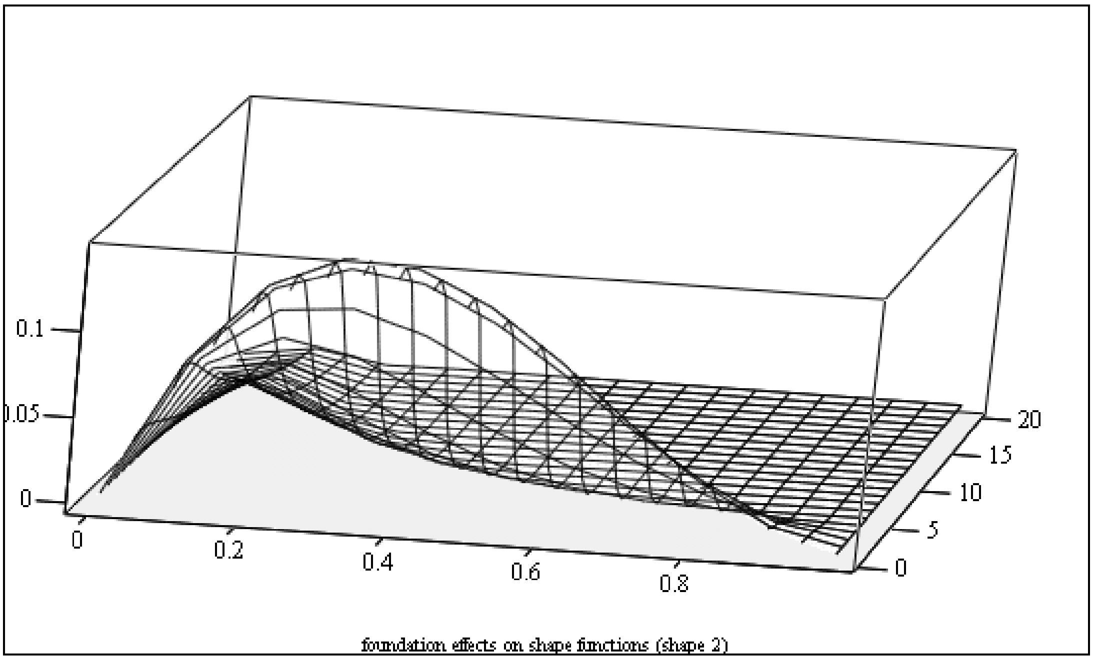
$$\frac{\psi_2}{L} = \xi - 2\xi^2 + \xi^3 \tag{2.34a}$$

$$\psi_3 = 3\xi^2 - 2\xi^3 - 1 \tag{2.34b}$$

$$\frac{\psi_5}{L} = \xi^3 + \xi^2 \tag{2.34c}$$

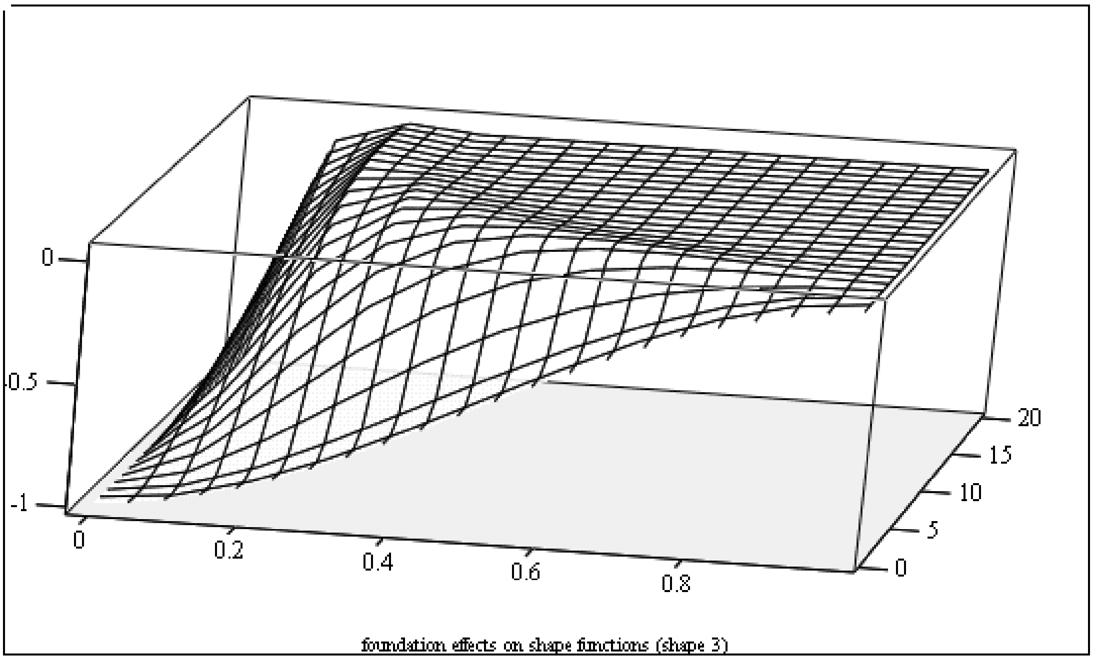
$$\psi_6 = 2\xi^3 - 3\xi^2 \tag{2.34d}$$

In order to observe the foundation parameter effects, the expressions in Equations (2.32) and (2.34) are portrayed graphically in Figures 2.5 to 2.8 for comparison.



(x,y,z)

Figure 2.5: Effects of One-Parameter Foundation on the Shape Function ψ_2



(x3,y3,z3)

Figure 2.6: Effects of One-Parameter Foundation on the Shape Function ψ_3

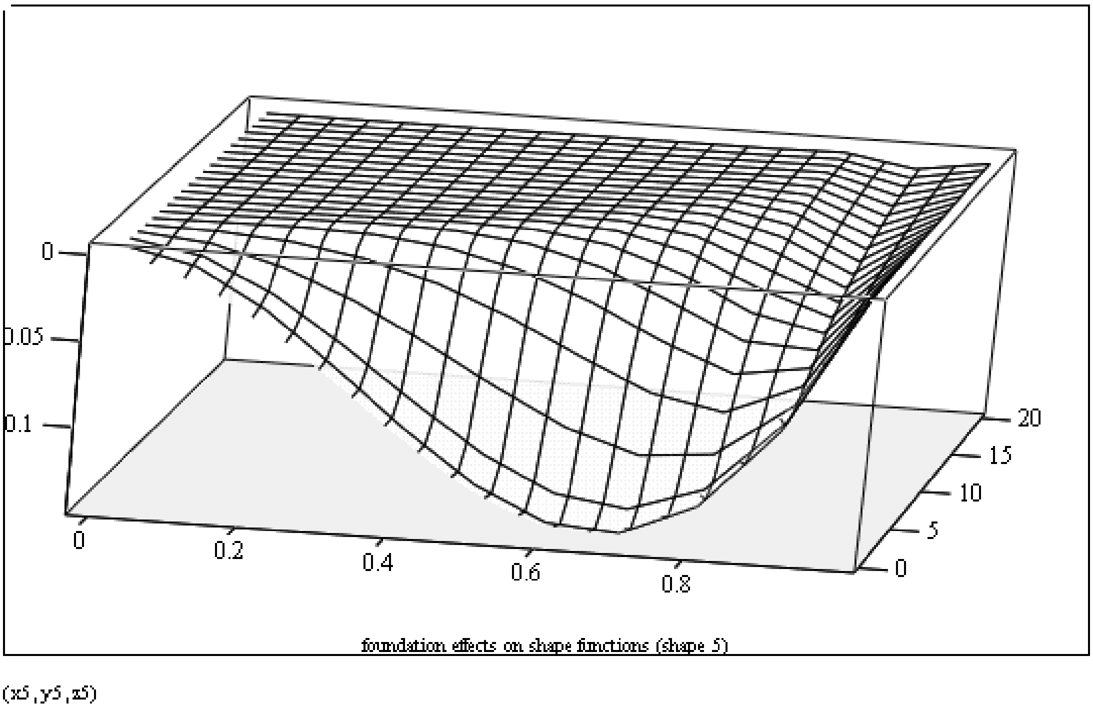


Figure 2.7: Effects of One-Parameter Foundation on the Shape Function ψ_5

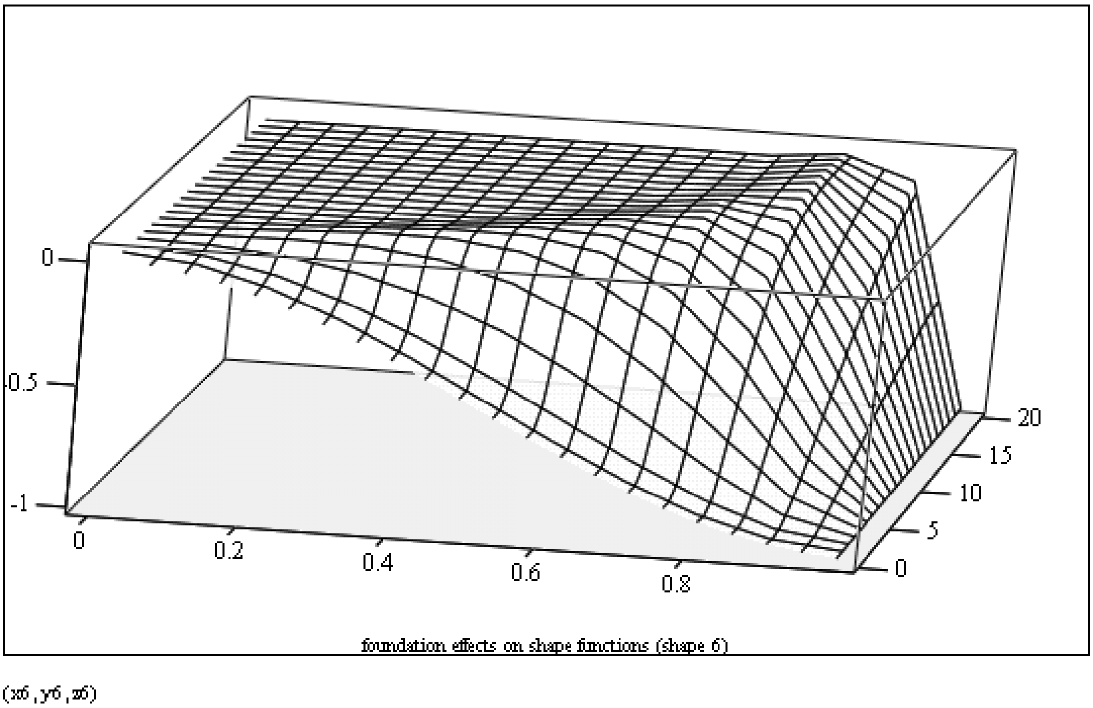


Figure 2.8: Effects of One-Parameter Foundation on the Shape Function ψ_6

2.2.3 Derivation of the Element Stiffness Matrix

The element stiffness matrix relates the nodal forces to the nodal displacements. Once the displacement function has been determined as in previous section for beam elements resting on one parameter elastic foundation, it is possible to formulate the stiffness matrix. The element stiffness matrix for the prismatic beam element shown in Figure 2.4 can be obtained from the minimization of strain energy functional U as follows:

The governing differential equation for beam elements on one-parameter elastic foundation Equation (2.8) can be rewritten as:

$$EI\frac{d^4w(x)}{d^4x} + k_1w(x) - q(x) = 0 \tag{2.35}$$

Let Equation (2.35) be multiplied by a test or weighting function, *v(x)* which is a continuous function over the domain of the problem. The test function *v(x)* viewed as a variation in *w* must be consistent with the boundary conditions. The variation in w as a virtual change vanishes at points where w is specified, and it is an arbitrary elsewhere.

First step is to integrate the product over the domain,

$$\int_0^L v(x)\left[EI\frac{d^4w(x)}{dx^4} + k_1w(x) - q(x)\right]dx = 0 \tag{2.36a}$$

$$\int_0^L v(x)e(x)dx = 0 \tag{2.36b}$$

The purpose of the *v(x)* is to minimize the function *e(x)*, the residual of the differential equation, in weighted integral sense. Equation (2.36) is the weighted residual statement equivalent to the original differential equation. Then it can be rewritten as:

$$\int_0^L EI\nu(x)\frac{d^4w(x)}{dx^4}dx + \int_0^L \nu(x)k_1w(x)dx - \int_0^L \nu(x)q(x)dx = 0 \tag{2.37}$$

The first part of the Equation (2.37) can be transferred from dependent variable *w(x)* to the weight function *v(x)* by integration by parts as follows:

$$\frac{d}{dx}\left(\nu(x)\frac{d^3w(x)}{dx^3}\right) = \frac{d\nu(x)}{dx}\frac{d^3w(x)}{dx^3} + \nu(x)\frac{d^4w(x)}{dx^4}$$

$$\nu(x)\frac{d^4w(x)}{dx^4} = \frac{d}{dx}\left(\nu(x)\frac{d^3w(x)}{dx^3}\right) - \frac{d\nu(x)}{dx}\frac{d^3w(x)}{dx^3}$$

$$\frac{d}{dx}\left(\frac{d\nu(x)}{dx}\frac{d^2w(x)}{dx^2}\right) = \frac{d^2\nu(x)}{dx^2}\frac{d^2w(x)}{dx^2} + \frac{d\nu(x)}{dx}\frac{d^3w(x)}{dx^3}$$

$$\frac{d\nu(x)}{dx}\frac{d^3w(x)}{dx^3} = \frac{d}{dx}\left(\frac{d\nu(x)}{dx}\frac{d^2w(x)}{dx^2}\right) - \frac{d^2\nu(x)}{dx^2}\frac{d^2w(x)}{dx^2}$$

$$\int_0^L EI\nu(x)\frac{d^4w(x)}{dx^4}dx = EI\int_0^L\left[\frac{d}{dx}\left(\nu(x)\frac{d^3w(x)}{dx^3}\right) - \frac{d}{dx}\left(\frac{d\nu(x)}{dx}\frac{d^2w(x)}{dx^2}\right) + \frac{d^2\nu(x)}{dx^2}\frac{d^2w(x)}{dx^2}\right]dx$$

$$\int_0^L EI\nu(x)\frac{d^4w(x)}{dx^4}dx = EI\nu(x)\frac{d^3w(x)}{dx^3}\Bigg|_0^L - EI\frac{d}{dx}\left(\frac{d\nu(x)}{dx}\frac{d^2w(x)}{dx^2}\right)\Bigg|_0^L + EI\int_0^L\frac{d^2\nu(x)}{dx^2}\frac{d^2w(x)}{dx^2}dx$$

since $v(x)$ is the variation in $w(x)$, it has to satisfy homogeneous form of the essential boundary condition:

$$w(x)\Big|_{x=0} = w_1 \qquad \rightarrow \qquad v(x)\Big|_{x=0} = 0 \tag{2.38a}$$

$$\frac{dw(x)}{dx}\Big|_{x=0} = \theta_1 \qquad \rightarrow \qquad \frac{dv(x)}{dx}\Big|_{x=0} = 0 \tag{2.38b}$$

$$w(x)\Big|_{x=L} = w_2 \qquad \rightarrow \qquad v(x)\Big|_{x=L} = 0 \tag{2.38c}$$

$$\frac{dw(x)}{dx}\Big|_{x=L} = \theta_2 \qquad \rightarrow \qquad \frac{dv(x)}{dx}\Big|_{x=L} = 0 \tag{2.38d}$$

$$v(x)\frac{d^3w(x)}{dx^3}\Big|_0^L = 0 \qquad \rightarrow \qquad \frac{dv(x)}{dx}\frac{d^2w(x)}{dx^2}\Big|_0^L = 0 \tag{2.38e}$$

Then, Equation (2.36) takes the form of only twice differentiable in contrast to Equation (2.35), which is in fourth order differential equation, as follows:

$$\begin{aligned} &\int_0^L v(x)\left[EI\frac{d^4w(x)}{dx^4} + k_1 w(x) - q(x)\right]dx = \\ &EI\int_0^L \frac{d^2v(x)}{dx^2}\frac{d^2w(x)}{dx^2}dx + k_1\int_0^L v(x)w(x)dx - \int_0^L v(x)q(x)dx = 0 \end{aligned} \tag{2.39}$$

Equation (2.39) is called the weak, generalized or variational equation associated with Equation (2.35). The variational solution is not differentiable enough to satisfy the original differential equation. However it is differentiable enough to satisfy the variational equation equivalent to Equation (2.35). In order to obtain the stiffness matrix, the displacement fields can be defined as follows:

$$w(x) = \sum_{j=1}^{6} \psi_j w_j \tag{2.40}$$

$$v(x) = \psi_i$$

Substituting them into Equation (2.39)

$$EI \int_0^L \frac{d^2\psi_i}{dx^2} \frac{d^2\psi_j}{dx^2} w_j dx + k_1 \int_0^L \psi_i \psi_j w_j dx - \int_0^L \psi_i q(x) dx = 0$$

$$\left\{ EI \int_0^L \frac{d^2\psi_i}{dx^2} \frac{d^2\psi_j}{dx^2} dx + k_1 \int_0^L \psi_i \psi_j dx \right\} w_j = \int_0^L \psi_i q(x) dx \tag{2.41}$$

$$\{\underline{K_e}\}\{\underline{w_j}\} = \{\underline{F_e}\}$$

The shape functions, ψ_1, ψ_2, ψ_3, ψ_4, ψ_5 and ψ_6, are already known from Equation (2.29). The nodal displacements are $\{\underline{w_j}\}^T = \{\phi_1, \theta_1, w_1, \phi_2,, \theta_2, w_2\}$ referring to sign convention in Figure 2.4. After performing the necessary symbolic calculations, the stiffness terms are obtained as:

$$k_{11} = \left(\frac{GJ}{L} \right) \tag{2.42a}$$

$$k_{14} = \left(-\frac{GJ}{L} \right) \tag{2.42b}$$

$$k_{22} = \left(\frac{2EI\lambda(\sinh[2L\lambda] - \sin[2L\lambda])}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42c}$$

$$k_{23} = \left(\frac{2EI\lambda^2(\cos[2L\lambda] - \cosh[2L\lambda])}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42d}$$

$$k_{25} = \left(\frac{4EI\lambda(\cosh[L\lambda]\sin[L\lambda] - \cos[L\lambda]\sinh[L\lambda])}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42e}$$

$$k_{26} = \left(\frac{8EI\lambda^2 \sinh[L\lambda]\sin[L\lambda]}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42f}$$

$$k_{32} = \left(\frac{2EI\lambda^2 (\cos[2L\lambda] - \cosh[2L\lambda])}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42g}$$

$$k_{33} = \left(\frac{4EI\lambda^3 (\sin[2L\lambda] + \sinh[2L\lambda])}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42h}$$

$$k_{35} = \left(\frac{-8EI\lambda^2 \sin[L\lambda]\sinh[L\lambda]}{-2 + \cosh[2L\lambda] + \cos[2L\lambda]} \right) \tag{2.42i}$$

$$k_{12} = k_{13} = k_{15} = k_{16} = k_{42} = k_{43} = k_{45} = k_{46} = 0 \tag{2.42j}$$

$$k_{41} = k_{14} \tag{2.42k}$$

$$k_{44} = k_{11} \tag{2.42l}$$

$$k_{52} = k_{25} \tag{2.42m}$$

$$k_{53} = k_{35} \tag{2.42n}$$

$$k_{55} = k_{22} \tag{2.42o}$$

$$k_{56} = k_{26} \tag{2.42p}$$

$$k_{62} = k_{26} \tag{2.42r}$$

$$k_{63} = k_{36} \tag{2.42s}$$

$$k_{65} = k_{56} \tag{2.42t}$$

$$k_{66} = k_{33} \tag{2.42u}$$

It is obvious that when foundation parameter k_1 tends to zero (or $\lambda \rightarrow 0$), the terms in Equation (2.42) must reduce to the conventional beam stiffness terms obtained by Hermitian functions. As a measure of the correctness of the terms in Equation (2.42), it is verified that the terms reduces to the following conventional terms in matrix form.

$$\lim_{k_1 \to 0}\left[K_e\right] = \begin{bmatrix} \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} \\ -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{12EI}{L^3} \end{bmatrix} \tag{2.43}$$

the unequal terms of the matrix are

$$\lim_{k1 \to 0} k_{22} = \frac{4EI}{L} \tag{2.44a}$$

$$\lim_{k1 \to 0} k_{23} = -\frac{6EI}{L^2} \tag{2.44b}$$

$$\lim_{k1 \to 0} k_{25} = \frac{2EI}{L} \tag{2.44c}$$

$$\lim_{k1 \to 0} k_{26} = \frac{6EI}{L^2} \tag{2.44d}$$

$$\lim_{k1 \to 0} k_{33} = \frac{12EI}{L^3} \tag{2.44e}$$

$$\lim_{k1 \to 0} k_{36} = -\frac{12EI}{L^3} \tag{2.44f}$$

The effect of the foundation parameter k_1 on the stiffness terms given in Equation (2.44) and corresponding terms of Equation (2.44) is portrayed in Figures 2.9 to 2.14.

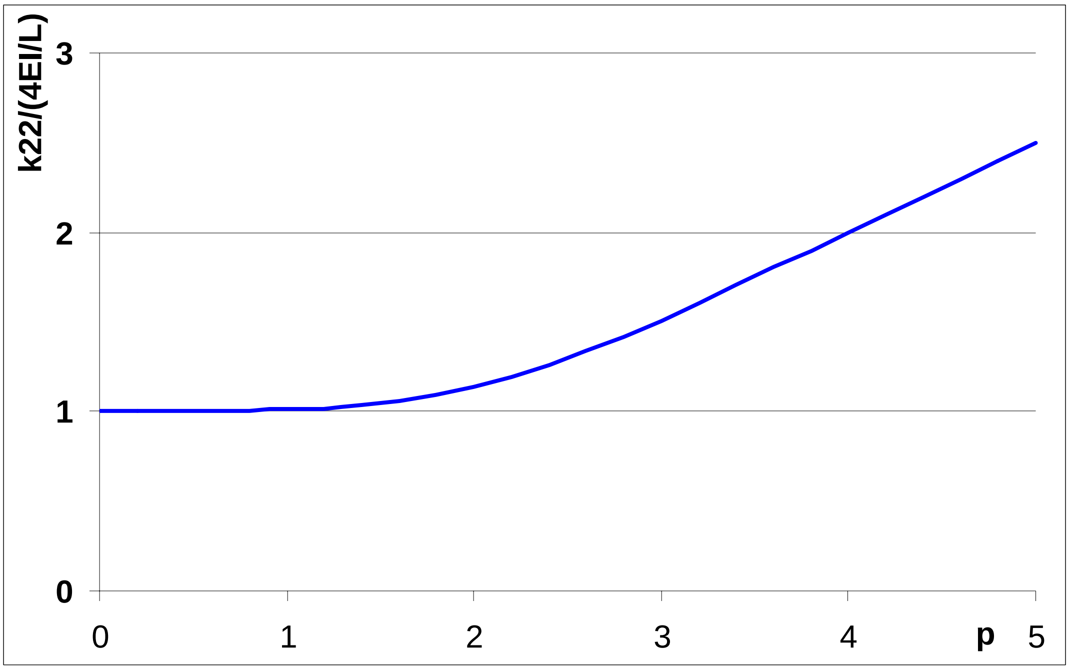


Figure 2.9: Influence of One-Parameter Foundation on the Normalized Stiffness term k_{22}

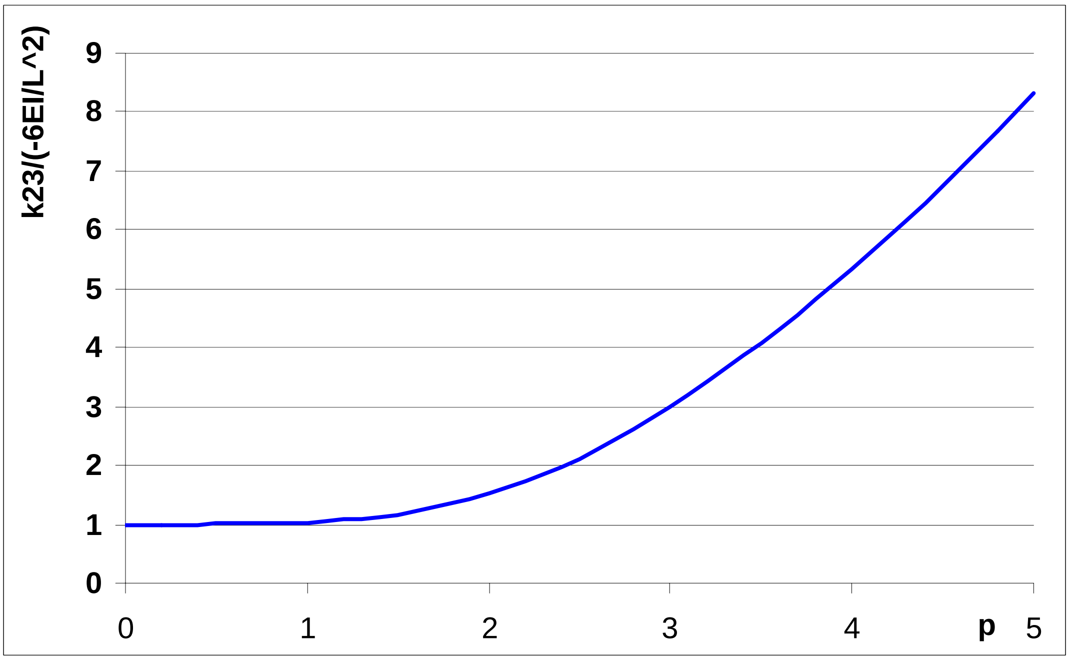


Figure 2.10: Influence of One-Parameter Foundation on the Normalized Stiffness term k_{23}

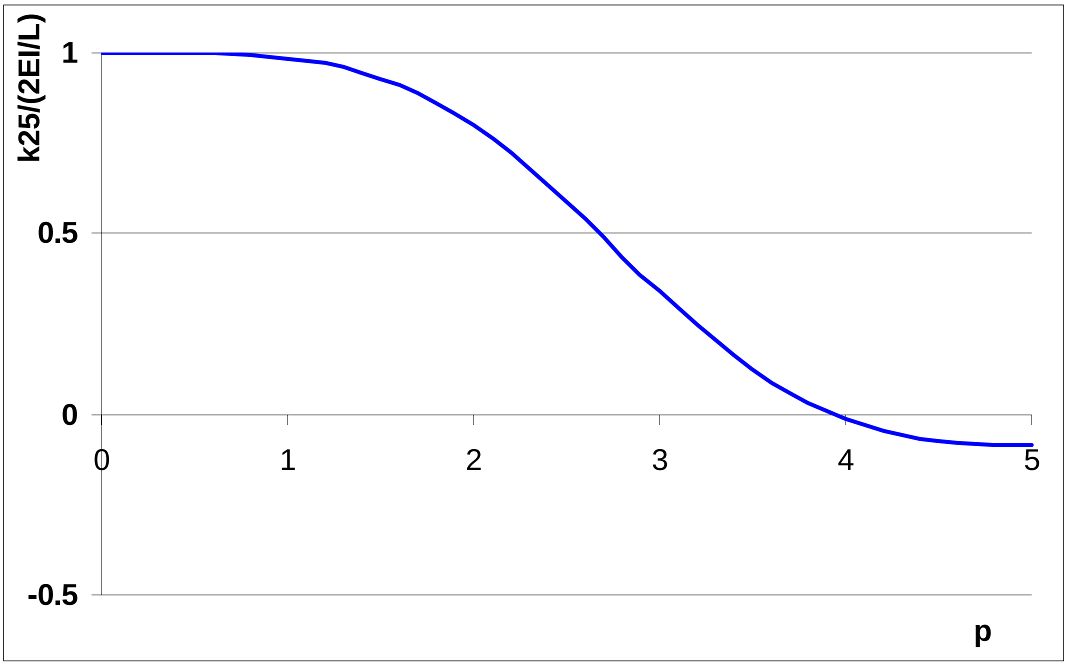


Figure 2.11: Influence of One-Parameter Foundation on the Normalized Stiffness term k_{25}

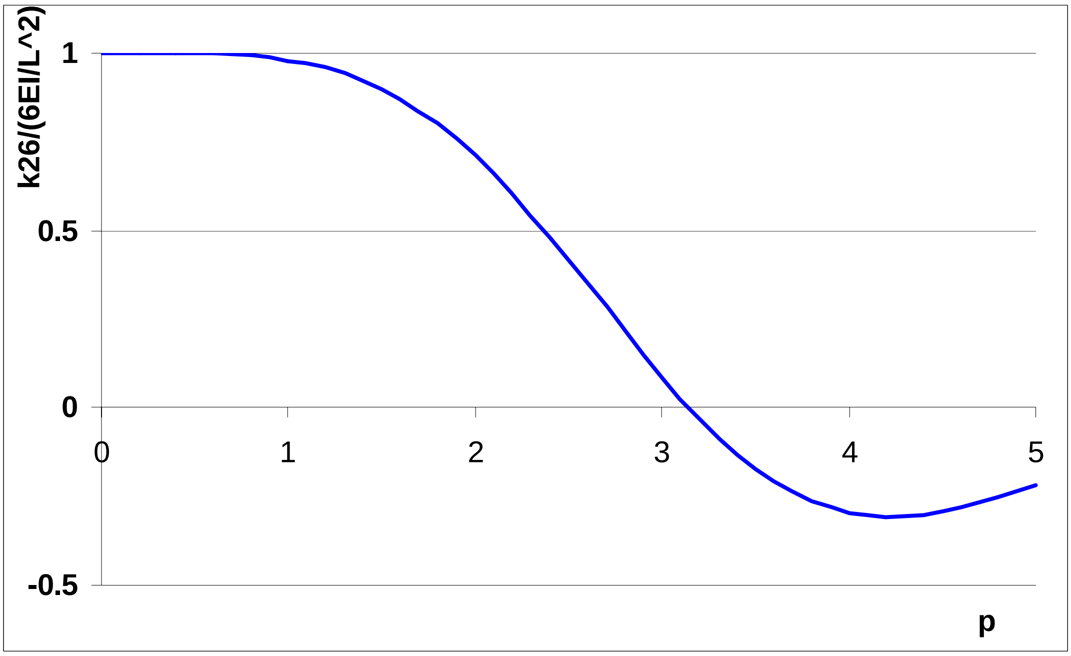


Figure 2.12: Influence of One-Parameter Foundation on the Normalized Stiffness term k_{26}

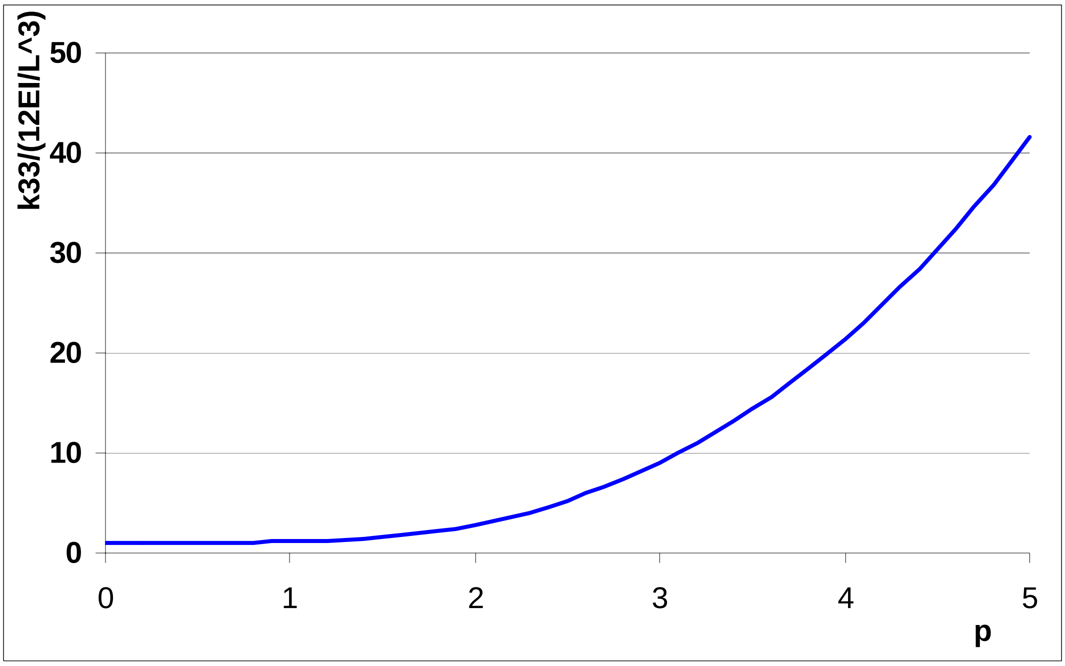


Figure 2.13: Influence of One-Parameter Foundation on the Normalized Stiffness term k_{33}

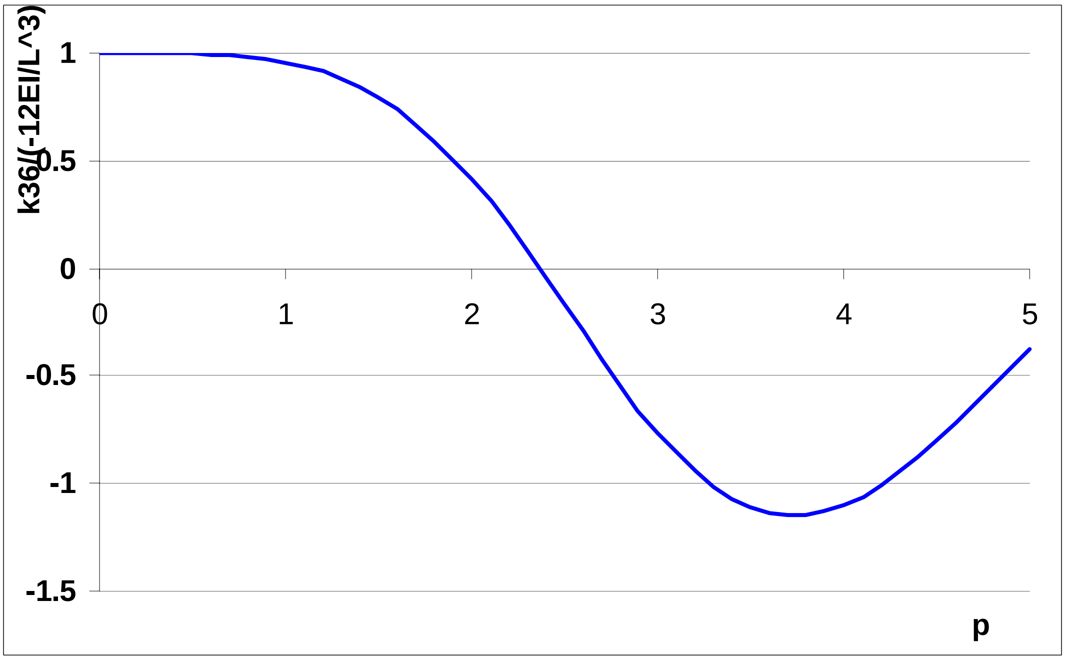


Figure 2.14: Influence of One-Parameter Foundation on the Normalized Stiffness term k_{36}

2.2.4 Derivation of Work Equivalent Nodal Loads

Fixed end moments and forces obtained with conventional cases are not valid for beam elements resting on elastic foundations. As seen in Figure 2.15, it is obvious that the foundation reaction will affect the fixed end bending moments and forces. In some cases influence of foundation has a great importance.

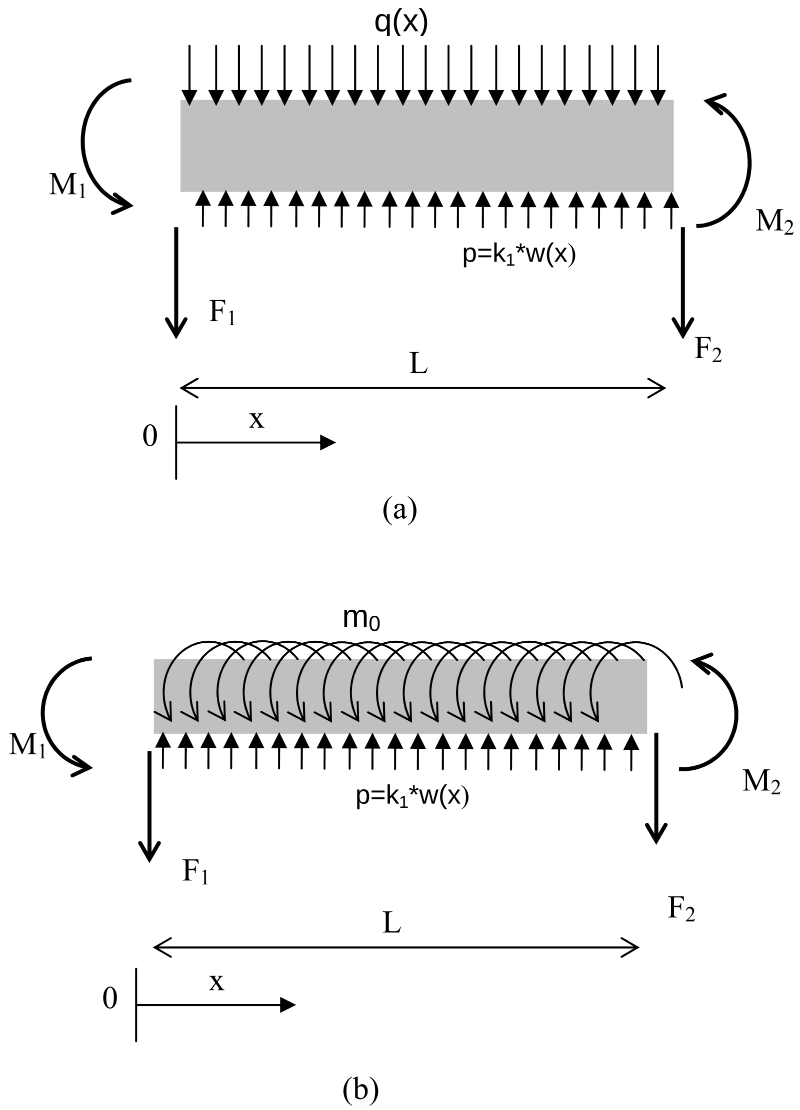


Figure 2.15: Nodal Forces due to Uniform Loading of a Beam Element Resting on One-Parameter (Winkler) Foundation.

The nodal load vector corresponding to the loading function, q(x), acting on the span L shown in Figure 2.15a is given by

$$\{\underline{P}\} = \int_0^L [\underline{N}] q(x) dx \tag{2.45}$$

where [$\underline{N}$] is the shape functions for beam elements resting on one-parameter elastic foundation.

For a distributed moment m(x) acting along the element as shown in Figure 2.15b, the load vector can be rewritten as:

$$\{\underline{P}\} = \int_0^L \frac{d[\underline{N}]}{dx} m(x) dx \tag{2.46}$$

The above equations can be used to determine the load vectors for many common loading types. As stated earlier the plate will be represented in this study by a discrete number of intersecting beams. Since beam elements can be accepted as infinitesimal elements of plates, many types of loading can be represented with uniformly distributed loads or point loads applied at the nodes. Therefore, the nodal load vector will be derived only for ($q(x) = q_0$) uniformly distributed loading of the beam elements.

Referring to Figure 2.15a for uniform distributed loading, q_0, the equivalent nodal loads can be obtained by rewriting Equation (2.45) as:

$$\{\underline{P}\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \int_0^L q_0 \begin{Bmatrix} N_2 \\ N_3 \\ N_5 \\ N_6 \end{Bmatrix} dx \tag{2.47}$$

Inserting the corresponding shape functions from Equation (2.29) into Equation (2.47), the nodal loads obtained as:

$$F_1 = F_2 = \left(\frac{q_0(\cosh[\lambda L] - \cos[\lambda L])}{\lambda(\sin[\lambda L] + \sinh[\lambda L])} \right) \tag{2.48a}$$

$$M_1 = -M_2 = \left(\frac{q_0(\sinh[\lambda L] - \sin[\lambda L])}{2\lambda^2(\sin[\lambda L] + \sinh[\lambda L])} \right) \tag{2.48b}$$

It is obvious that when foundation parameter k_1 tends to zero, the terms in Equations (2.48a) and (2.48b) must reduce to the conventional beam fixed end forces obtained by Hermitian functions. The well known terms are obtained as:

$$F_1 = F_2 = \lim_{\lambda \to 0} \left(\frac{q_0(\cosh[\lambda L] - \cos[\lambda L])}{\lambda(\sin[\lambda L] + \sinh[\lambda L])} \right) = \frac{q_0 L}{2} \tag{2.49a}$$

$$M_1 = -M_2 = \lim_{\lambda \to 0} \left(\frac{q_0(\sinh[\lambda L] - \sin[\lambda L])}{2\lambda^2(\sin[\lambda L] + \sinh[\lambda L])} \right) = \frac{q_0 L^2}{12} \tag{2.49b}$$

In order to compare the influence of the foundation parameter k_1 on fixed end forces, the normalized terms of Equation (2.48) with those of Equation (2.49) are portrayed in Figures 2.16 and Figure 2.17.

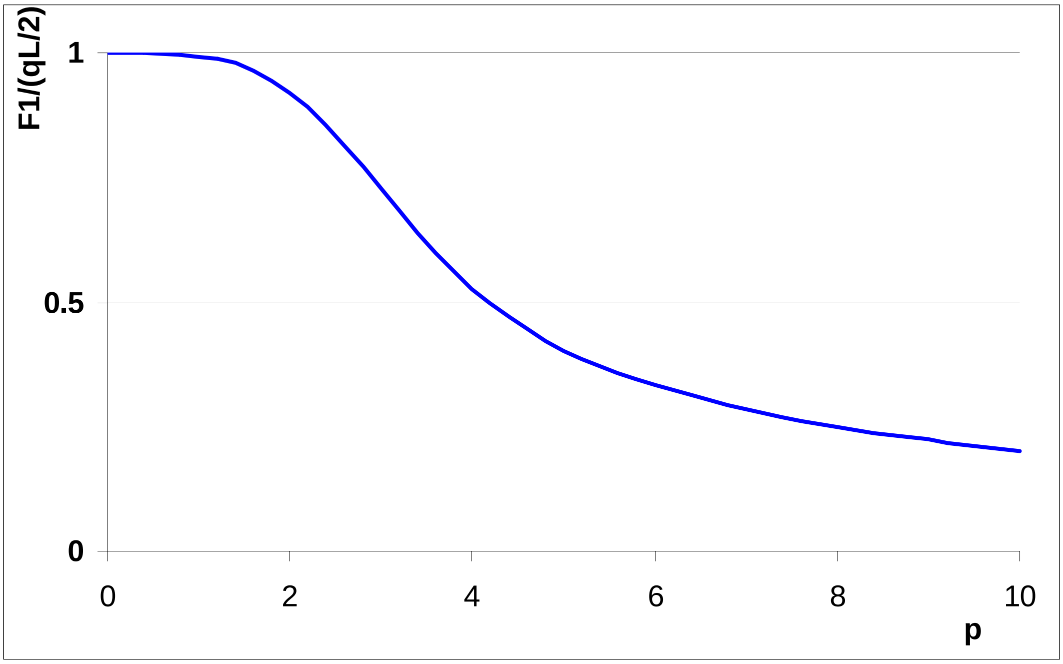


Figure 2.16: Normalized Nodal Force F_1 due to Continuous Loading of a Beam Element Resting on One-Parameter (Winkler) Foundation.

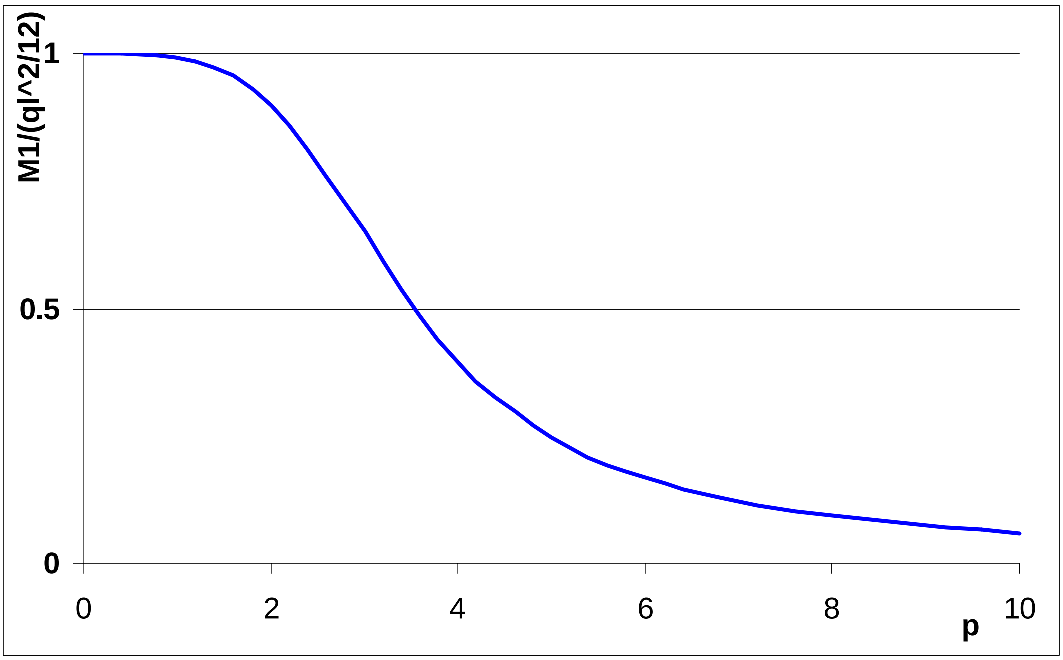


Figure 2.17: Normalized Nodal Force M_1 due to Continuous Loading for a Beam Element Resting on One-Parameter (Winkler) Foundation.

2.3 PROPERTIES OF BEAM ELEMENTS RESTING ON TWO-PARAMETER ELASTIC FOUNDATION

The main advantage of the two-parameter elastic foundation model is to provide a mechanical interaction between the individual spring elements. This relationship between the springs that shows a more realistic behaviour of the soil reaction eliminates the discontinuous behaviour of Winkler model.

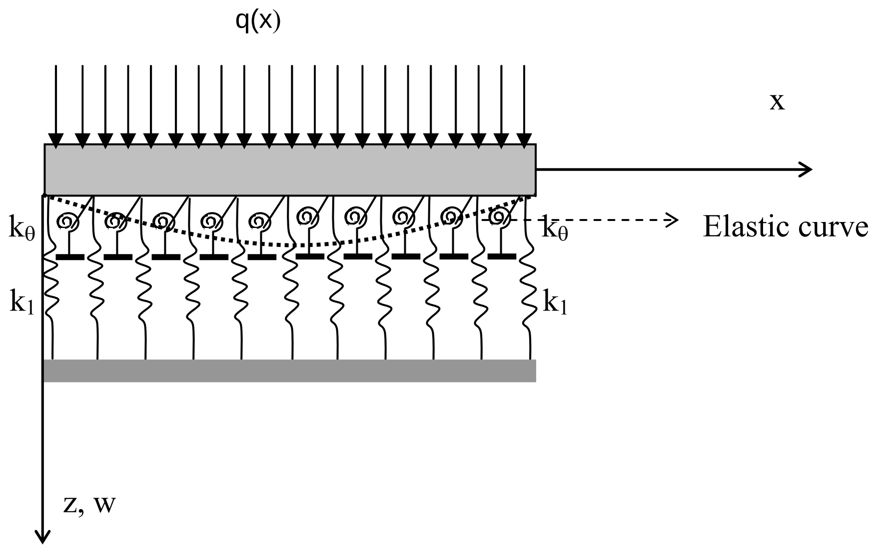


Figure 2.18: Representation of the Beam Element Resting on Two-Parameter (Generalized) Foundation

The generalized foundation as a representation of two-parameter model implies that at the end of each translational spring element there must be also a rotational spring to produce a reaction moment (k_θ) proportional to the local slope at that point. A representation of the foundation with closely linear translational and rotational springs underlying a beam element is shown in Figure 2.18.

2.3.1 Derivation of the Differential Equation of the Elastic Line

For generalized foundations the model assumes that at the point of contact between plate and foundation there is not only pressure but also distributed moments caused by the interaction between linear springs. These moments are assumed to be proportional to the slope of the elastic curve by a second parameter for foundation. That is, the reaction force intensity (p) at any point for generalized foundation can be rewritten for beam elements as:

$$p(x) = k_1 w(x) - k_\theta \frac{d^2 w(x)}{dx^2} \tag{2.50}$$

To determine the basic differential equation of the beam elements, the same procedures used for plate elements in Section 1.2 will be re-examined. However, the equation of the elastic curve derived for a beam element resting on a two-parameter elastic foundation from the equilibrium equations of an infinitesimal segment of the structural member in as:

$$EI \frac{d^4 w(x)}{dx^4} + k_1 w(x) - k_\theta \frac{d^2 w(x)}{dx^2} = q(x) \tag{2.51}$$

As defined in previous section for beam elements k_1 is the Winkler parameter with the unit of force per unit length/per length and k_θ is the second parameter that is defined as the reaction moment proportional to the local angle of rotation in generalized foundation model with unit of moment per unit length.

2.3.2 Derivation of the Exact Shape Functions

For a beam element resting on two-parameter elastic foundation, the homogeneous form of Equation (2.51) is obtained by equating $q(x)$ =0.

$$EI\frac{d^4w(x)}{dx^4} - k_\theta\frac{d^2w(x)}{dx^2} + k_1w(x) = 0 \qquad (2.52)$$

to solve the above equation firstly let us introduce

$$A = \frac{k_\theta}{EI} \quad \text{and} \quad B = \frac{k_1}{EI} \qquad (2.53)$$

Equation (2.52) can then be rearranged as:

$$\frac{d^4w(x)}{dx^4} - A\frac{d^2w(x)}{dx^2} + Bw(x) = 0 \qquad (2.54)$$

Let $\frac{d^n}{dx^n} = D^n$ then the characteristic Equation (2.54) can be written as:

$$(D^4 - AD^2 + B)w(x) = 0 \qquad (2.55)$$

The roots of the characteristic equation are

$$\begin{aligned} D_1 &= \frac{\sqrt{A+\sqrt{(A^2-4B)}}}{\sqrt{2}} \\ D_2 &= -\frac{\sqrt{A+\sqrt{(A^2-4B)}}}{\sqrt{2}} \\ D_3 &= \frac{\sqrt{A-\sqrt{(A^2-4B)}}}{\sqrt{2}} \\ D_4 &= -\frac{\sqrt{A-\sqrt{(A^2-4B)}}}{\sqrt{2}} \end{aligned} \qquad (2.56)$$

There are three possible combinations of parameters A and B that must be considered to define Equation (2.56). The cases are

$$\begin{aligned} A &< 2\sqrt{B} \\ A &= 2\sqrt{B} \\ A &> 2\sqrt{B} \end{aligned} \qquad (2.57)$$

Since the case $A = 2\sqrt{B}$ (or $k_\theta = \sqrt{4k_1 EI}$) is a very special one it is not necessary to obtain solution of the equation for this case. It is possible to obtain an accurate solution by increasing k_θ a very small amount that let to use the solution for $A > 2\sqrt{B}$ case. Therefore, solution of the differential equation would be obtained for the other possible cases.

2.3.2.1 The Shape Functions for the Case $A < 2\sqrt{B}$

For this case Equation (2.56) yields

$$
\begin{aligned}
D_1 &= \frac{\sqrt{A + i\sqrt{(4B - A^2)}}}{\sqrt{2}} \\
D_2 &= \frac{\sqrt{-A - i\sqrt{(4B - A^2)}}}{\sqrt{2}} \\
D_3 &= \frac{\sqrt{A - i\sqrt{(4B - A^2)}}}{\sqrt{2}} \\
D_4 &= \frac{\sqrt{-A + i\sqrt{(4B - A^2)}}}{\sqrt{2}}
\end{aligned}
\tag{2.58}
$$

Utilizing Equation (2.12) for the first parameter and a new auxiliary quantity for the second parameter as

$$
\begin{aligned}
\lambda &= \sqrt[4]{\frac{B}{4}} = \sqrt[4]{\frac{k_1}{4EI}} \\
\delta &= \frac{A}{4} = \frac{k_\theta}{4EI}
\end{aligned}
\tag{2.59}
$$

then the first root can be expressed in the following way:

$$
\begin{aligned}
D_1 &= \frac{\sqrt{4\delta + i\sqrt{(16\lambda^4 - 16\delta^4)}}}{\sqrt{2}} \\
&= \frac{2\sqrt{\delta + i\sqrt{(\lambda^4 - \delta^4)}}}{\sqrt{2}} \\
&= \sqrt{2}\sqrt{\delta + i(\sqrt{\lambda^2 - \delta^2})(\sqrt{\lambda^2 + \delta^2})}
\end{aligned}
\tag{2.60}
$$

by defining new quantities to simplify the term

$$
\begin{aligned}
\alpha &= \sqrt{\lambda^2 + \delta} \\
\beta &= \sqrt{\lambda^2 - \delta}
\end{aligned}
\qquad \rightarrow \qquad \delta = \frac{\alpha^2 - \beta^2}{2}
\tag{2.61}
$$

Both α and β have dimension of 1/L. Then substitute the new quantities into Equation (2.60), the first root can be written in simplified form as:

$$
\begin{aligned}
D_1 &= \sqrt{(\alpha^2 - \beta^2) + 2i\alpha\beta} \\
&= \sqrt{(\alpha + i\beta)^2} \\
&= \alpha + i\beta
\end{aligned}
$$

The other roots also can be found by the same procedures. Then the roots are:

$$
\begin{aligned}
D_1 &= \alpha + i\beta \\
D_2 &= -\alpha - i\beta \\
D_3 &= \alpha - i\beta \\
D_4 &= -\alpha + i\beta
\end{aligned}
\tag{2.62}
$$

Considering the above roots solution of Equation (2.52) is:

$$
w(x) = \begin{array}{l}
a_1 e^{\alpha x}\left(\cos[\beta x] + \sin[\beta x]\right) + a_2 e^{-\alpha x}\left(\cos[\beta x] - \sin[\beta x]\right) + \\
a_3 e^{\alpha x}\left(\cos[\beta x] - \sin[\beta x]\right) + a_4 e^{-\alpha x}\left(\cos[\beta x] + \sin[\beta x]\right)
\end{array}
\tag{2.63}
$$

Using hyperbolic functions

$$
\begin{aligned}
e^{\alpha x} &= Cosh[\alpha x] + Sinh[\alpha x] \\
e^{-\alpha x} &= Cosh[\alpha x] - Sinh[\alpha x]
\end{aligned}
\tag{2.64}
$$

Substituting the above hyperbolic functions and rearrange Equation (2.63) with defining the new constants, the closed form of the solution in terms of hyperbolic and trigonometric functions is obtained as:

$$
w(x) = \begin{array}{l} c_1 \cos[\beta x]\cosh[\alpha x] + c_2 \cos[\beta x]\sinh[\alpha x] + \\ c_3 \sin[\beta x]\cosh[\alpha x] + c_4 \sin[\beta x]\sinh[\alpha x] \end{array}
\tag{2.65}
$$

By neglecting foundation effects for torsional degree of freedoms, a linear description of the angular displacement at any point along the element can be expressed as Ø(x)= a_1+a_2x. Inserting the angular displacements due to torsional effects, Equation (2.65) can be rearranged as follows:

$$
w(x) = \begin{array}{l} c_1 + c_2 \cos[\beta x]\cosh[\alpha x] + c_3 \cos[\beta x]\sinh[\alpha x] + \\ c_4 x + c_5 \sin[\beta x]\cosh[\alpha x] + c_6 \sin[\beta x]\sinh[\alpha x] \end{array}
\tag{2.66}
$$

then, the closed form equation can be expressed in matrix form as:

$$
w = \underline{B}^T \underline{C}
\tag{2.67}
$$

where

$$
\underline{B}^T = \left\{ 1 \quad \cos[\beta x]\cosh[\alpha x] \quad \cos[\beta x]\sinh[\alpha x] \quad x \quad \sin[\beta x]\cosh[\alpha x] \quad \sin[\beta x]\sinh[\alpha x] \right\}
$$

and

$$
\underline{C} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}
$$

The generalized displacement vector which forms boundary conditions shown in Figure 2.4 is obtained with x= 0 and x= L as in Section 2.2.2. Then the arbitrary constant elements of the vector C can be related to the end displacements in matrix form as follows:

$$\begin{Bmatrix} \phi_1 \\ \theta_1 \\ w_1 \\ \phi_2 \\ \theta_2 \\ w_2 \end{Bmatrix} = [\underline{H}] \cdot \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} \tag{2.68}$$

or

$$[\underline{d}] = [\underline{H}] \cdot [\underline{C}] \tag{2.69}$$

The arbitrary constant vector C can be defined by

$$[\underline{C}] = [\underline{H}]^{-1} \cdot [\underline{d}] \tag{2.70}$$

Here [H] is a 6x6 matrix. Substituting Equation (2.70) into Equation (2.67) leads to the closed form solution of the differential equation that can be written in matrix form as:

$$[\underline{w}] = [\underline{B}]^{T} \cdot [\underline{H}]^{-1} \cdot [\underline{d}] \tag{2.71}$$

Equation (2.71) can be redefined by introducing vector N that includes six shape functions. Then the closed form of the solution in terms of shape functions and the generalized displacements defined in Figure 2.4 is

$$
\underline{[w]} = \underline{[N]} \cdot \begin{Bmatrix} \phi(x=0) \\ \dfrac{dw}{dx}(x=0) \\ w(x=0) \\ \phi(x=L) \\ \dfrac{dw}{dx}(x=L) \\ w(x=L) \end{Bmatrix} \tag{2.72a}
$$

or

$$
\underline{[w]} = \underline{[N]} \cdot \underline{[d]} \tag{2.72b}
$$

where

$$
\underline{[N]} = \underline{[B]}^T \cdot \underline{[H]}^{-1} \tag{2.73}
$$

For $A < 2\sqrt{B}$ the shape functions can be obtained by the same procedures followed for one-parameter case. After the necessary evaluations the shape functions determined as follows:

$$
\psi_1 = \left(1 - \frac{x}{L}\right) \tag{2.74a}
$$

$$
\psi_2 = \left(\frac{\begin{array}{l} \beta \cosh[\alpha x]\sin[\beta x] - \beta \sin[\beta x]\cosh[\alpha(2L-x)] - \\ \alpha \sinh[\alpha x]\cos[\beta(2L-x)] + \alpha \sinh[\alpha x]\cos[\beta x] \end{array}}{\left(\alpha^2 + \beta^2 - \alpha^2 \cos[2\beta L] - \beta^2 \cosh[2\alpha L]\right)} \right) \tag{2.74b}
$$

$$
\psi_3 = \left(\frac{\begin{array}{l} \alpha^2 \cos[\beta x]\cosh[\alpha x] + \beta^2 \cos[\beta x]\cosh[\alpha x] - \\ \beta^2 \cos[\beta x]\cosh[\alpha(2L-x)] - \alpha^2 \cosh[\alpha x]\cos[\beta(2L-x)] - \\ \alpha\beta \sin[\beta x]\sinh[\alpha(2L-x)] + \alpha\beta \sinh[\alpha x]\sin[\beta(2L-x)] \end{array}}{\left(\alpha^2 + \beta^2 - \alpha^2 \cos[2\beta L] - \beta^2 \cosh[2\alpha L]\right)} \right) \tag{2.74c}
$$

$$\psi_4 = \left(\frac{x}{L}\right) \tag{2.74d}$$

$$\psi_5 = \left(\frac{\begin{array}{l}\beta\cosh[\alpha(L+x)]\sin[\beta(L-x)] - \beta\cosh[\alpha(L-x)]\sin[\beta(L-x)] - \\ \alpha\sinh[\alpha(L-x)]\cos[\beta(L-x)] + \alpha\sinh[\alpha(L-x)]\cos[\beta(L+x)]\end{array}}{\left(\alpha^2 + \beta^2 - \alpha^2\cos[2\beta L] - \beta^2\cosh[2\alpha L]\right)}\right) \tag{2.74e}$$

$$\psi_6 = \left(\frac{\begin{array}{l}\alpha^2\cosh[\alpha(L-x)]\cos[\beta(L-x)] + \beta^2\cos[\beta(L-x)]\cosh[\alpha(L-x)] - \\ \alpha^2\cosh[\alpha(L-x)]\cos[\beta(L+x)] - \beta^2\cos[\beta(L-x)]\cosh[\alpha(L+x)] + \\ \alpha\beta\sin[\beta(L+x)]\sinh[\alpha(L-x)] - \alpha\beta\sinh[\alpha(L+x)]\sin[\beta(L-x)]\end{array}}{\left(\alpha^2 + \beta^2 - \alpha^2\cos[2\beta L] - \beta^2\cosh[2\alpha L]\right)}\right) \tag{2.74f}$$

2.3.2.2 The Shape Functions for the Case $A > 2\sqrt{B}$

For $A > 2\sqrt{B}$ the roots of Equation (2.56) are definite. Therefore, by substituting the auxiliary parameters defined in Equation (2.59) into Equation (2.56) the first root can be expressed in the following way:

$$\begin{aligned} D_1 &= \frac{\sqrt{4\delta + \sqrt{(16\delta^2 - 16\lambda^4)}}}{\sqrt{2}} \\ &= \frac{2\sqrt{\delta + \sqrt{(\delta^2 - \lambda^4)}}}{\sqrt{2}} \\ &= \sqrt{2}\sqrt{\delta + (\sqrt{\delta - \lambda^2})(\sqrt{\delta + \lambda^2})} \end{aligned} \tag{2.75}$$

Redefining the β term of Equation (2.61) to simplify the above equation:

$$\begin{aligned} \alpha &= \sqrt{\lambda^2 + \delta} \\ \beta &= \sqrt{\delta - \lambda^2} \end{aligned} \qquad \rightarrow \qquad \delta = \frac{\alpha^2 + \beta^2}{2} \tag{2.76}$$

Both α and β as previously mentioned have dimension of 1/L. substituting the quantities into Equation (2.75), the first root can be written as:

$$\begin{aligned} D_1 &= \sqrt{2}\sqrt{\frac{\alpha^2+\beta^2}{2}+\alpha\beta} \\ &= \sqrt{\alpha^2+\beta^2+2\alpha\beta} \\ &= \alpha+\beta \end{aligned} \tag{2.77}$$

The other roots also can be found by the same procedures. Then the full set is

$$\begin{aligned} D_1 &= \alpha+\beta \\ D_2 &= -\alpha-\beta \\ D_3 &= \alpha-\beta \\ D_4 &= -\alpha+\beta \end{aligned} \tag{2.78}$$

and the solution of Equation (2.52) for $A > 2\sqrt{B}$ is

$$w(x) = a_1 e^{(\alpha+\beta)x} + a_2 e^{-(\alpha+\beta)x} + a_3 e^{(\alpha-\beta)x} + a_4 e^{(-\alpha+\beta)x} \tag{2.79}$$

Substituting the hyperbolic functions and inserting the angular displacements due to torsional effects, the closed form solution can be rearranged as follows:

$$w(x) = \begin{matrix} c_1 + c_2\cos[\beta x]\cosh[\alpha x] + c_3\cos[\beta x]\sinh[\alpha x] + \\ c_4 x + c_5\sin[\beta x]\cosh[\alpha x] + c_6\sin[\beta x]\sinh[\alpha x] \end{matrix} \tag{2.80}$$

After the necessary evaluations as previously done the shape functions for the case $A > 2\sqrt{B}$ determined as follows:

$$\psi_1 = \left(1-\frac{x}{L}\right) \tag{2.81a}$$

$$\psi_2 = \left(\frac{\begin{array}{l} \alpha \sinh[(\alpha - \beta)x] + \beta \sinh[(\alpha - \beta)x] + \alpha \sinh[(\alpha + \beta)x] - \\ \beta \sinh[(\alpha + \beta)x] - \beta \sinh[2\alpha L - \alpha x - \beta x] + \alpha \sinh[2\beta L - \alpha x - \beta x] - \\ \alpha \sinh[2\beta L + \alpha x - \beta x] + \beta \sinh[2\alpha L - \alpha x + \beta x] \end{array}}{2\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right) \quad \text{2. 81b)}$$

$$\psi_3 = \left(\frac{\begin{array}{l} \cosh[\alpha x]\cosh[\beta x] - \beta^2 \cosh[\beta x]\sinh[2\alpha L] - \alpha \sinh[\alpha x]\sinh[2\beta L] \\ (\alpha\beta \cosh[\alpha x]\sinh[2\alpha L] + \alpha \sinh[2\beta L]\sinh[\beta x]) + \\ \alpha\beta(\cosh[2\beta L]\sinh[\alpha L]\sinh[\alpha x]\sinh[\beta x] - \cosh[2\alpha L]) \end{array}}{\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right) \quad (2.\ 81c)$$

$$\psi_4 = \left(\frac{x}{L} \right) \quad (2.\ 81d)$$

$$\psi_5 = \left(\frac{\begin{array}{l} -\alpha \sinh[(\alpha + \beta)(L - x)] + \beta \sinh[(\alpha + \beta)(L - x)] + \\ \alpha \sinh[\alpha L - \beta L - \alpha x - \beta x] - \beta \sinh[\alpha L + \beta L + \alpha x - \beta x] - \\ \alpha \sinh[\alpha L - \beta L - \alpha x + \beta x] - \beta \sinh[\alpha L - \beta L - \alpha x + \beta x] + \\ \alpha \sinh[\alpha L + \beta L - \alpha x + \beta x] + \beta \sinh[\alpha L - \beta L + \alpha x + \beta x] \end{array}}{2\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right) \quad (2.\ 81e)$$

$$\psi_6 = \left(\frac{\begin{array}{l} 2\cosh[\beta x](\beta^2 \cosh[\beta L]\sinh[\alpha L] + \alpha\beta \cosh[\alpha L]\sinh[\beta L]\sinh[\alpha x]) - \\ 2\cosh[\alpha x](\alpha\beta \cosh[\beta L]\sinh[\alpha L] + \alpha^2 \cosh[\alpha L]\sinh[\beta L]\sinh[\beta x]) + \\ 2(\alpha^2 - \beta^2)(\cosh[\alpha L]\sinh[\beta L]\sinh[\alpha x]\sinh[\beta x]) \end{array}}{\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right) \quad (2.\ 81f)$$

For both $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ cases, when foundation parameter k_1 and k_θ tends to zero (dependently $\lambda\to 0$, $\delta\to 0$, $\alpha\to 0$, $\beta\to 0$), the terms in Equation (2.74) and Equation (2.81) must reduce to Hermitian functions.

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}\psi_2 \to x\left(1-\frac{x}{L}\right)^2 \tag{2.82a}$$

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}(\psi_3) \to 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 - 1 \tag{2.82b}$$

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}(\psi_5) \to x\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right] \tag{2.82c}$$

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}(\psi_6) \to 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 \tag{2.82d}$$

To observe the influence of the foundation parameters, it is necessary to compare the expressions in Equations (2.67) and (2.81) with the Hermitian polynomials in Equation (2.82). For clarifying the comparison let Equations (2.61) and (2.76) be rearranged as follows:

For $A < 2\sqrt{B}$
$$\begin{aligned}\alpha &= \sqrt{\lambda^2+\delta} = \lambda\sqrt{1+t}\\ \beta &= \sqrt{\lambda^2-\delta} = \lambda\sqrt{1-t}\end{aligned} \tag{2.83a}$$

and

For $A > 2\sqrt{B}$
$$\begin{aligned}\alpha &= \sqrt{\lambda^2+\delta} = \lambda\sqrt{1+t}\\ \beta &= \sqrt{\delta-\lambda^2} = \lambda\sqrt{t-1}\end{aligned} \tag{2.83b}$$

where t is dimensionless

$$t = \frac{\delta}{\lambda^2} = \frac{\dfrac{k_\theta}{4EI}}{\sqrt{\dfrac{k_1}{4EI}}} \tag{2.84}$$

The effect of the foundation parameters k_1 and k_θ on the shape function terms given in Equation (2.74) for $A < 2\sqrt{B}$ and Equation (2.81) for $A > 2\sqrt{B}$ with corresponding terms of Equation (2.82) is portrayed in Figure 2.19 to Figure 2.30.

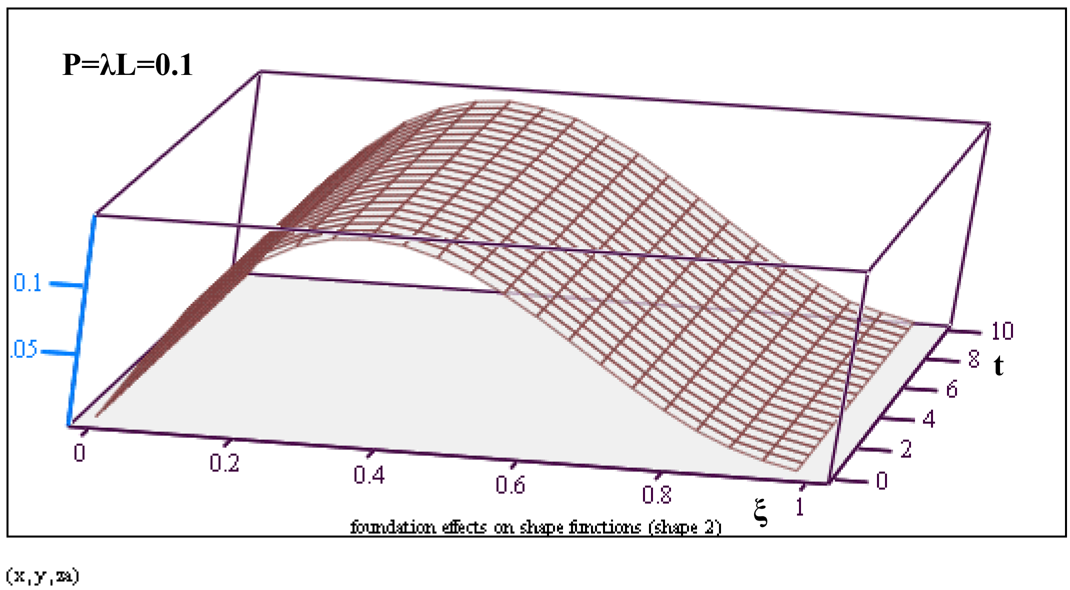


Figure 2.19: Variation of the Shape Function $\psi_{2,}$ for P=0.1

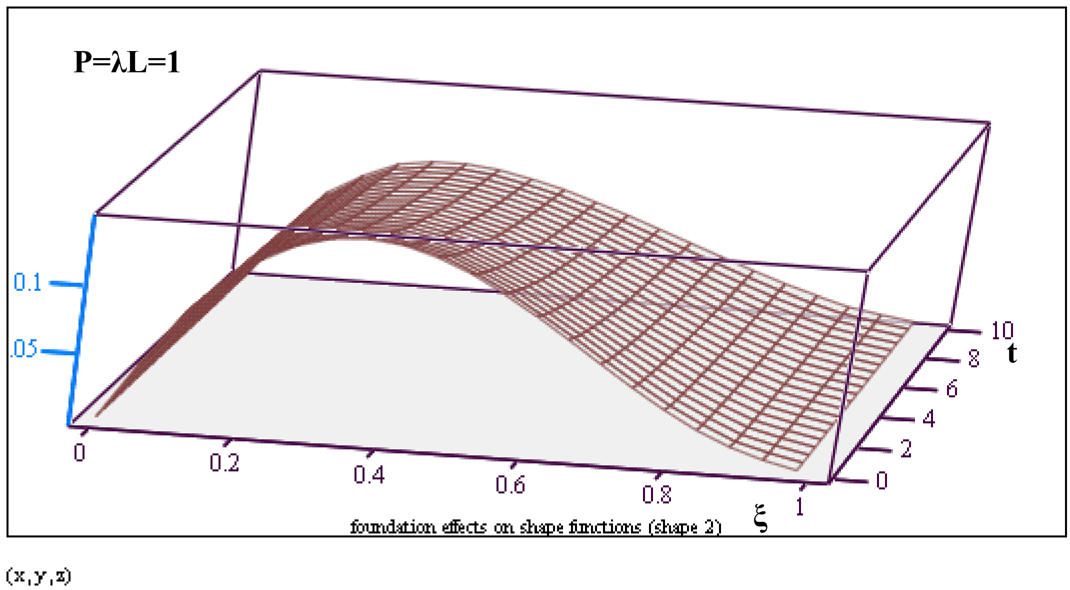
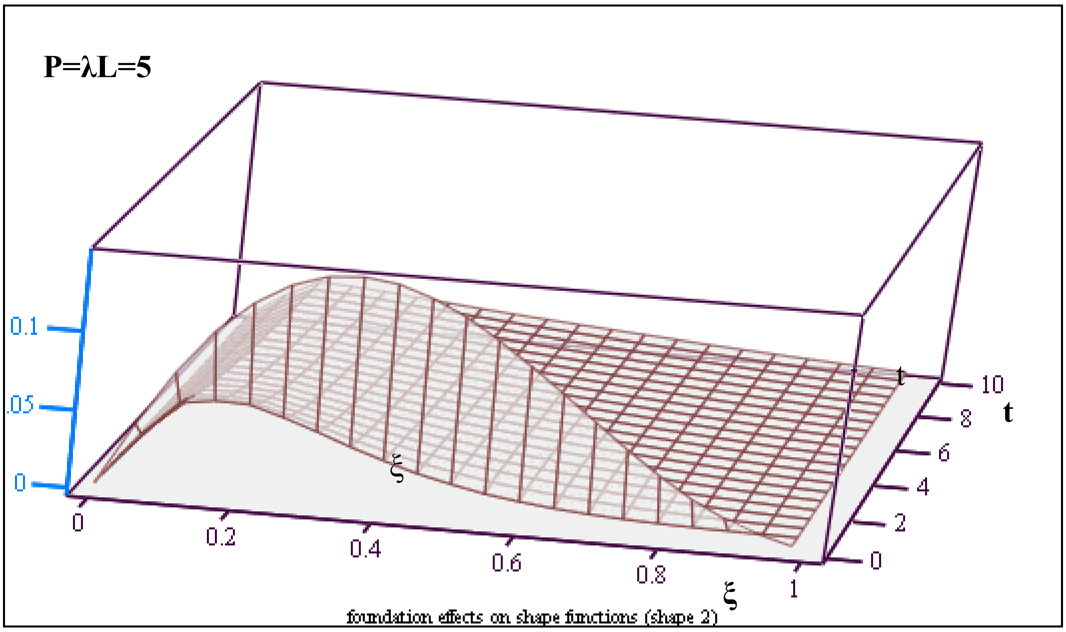
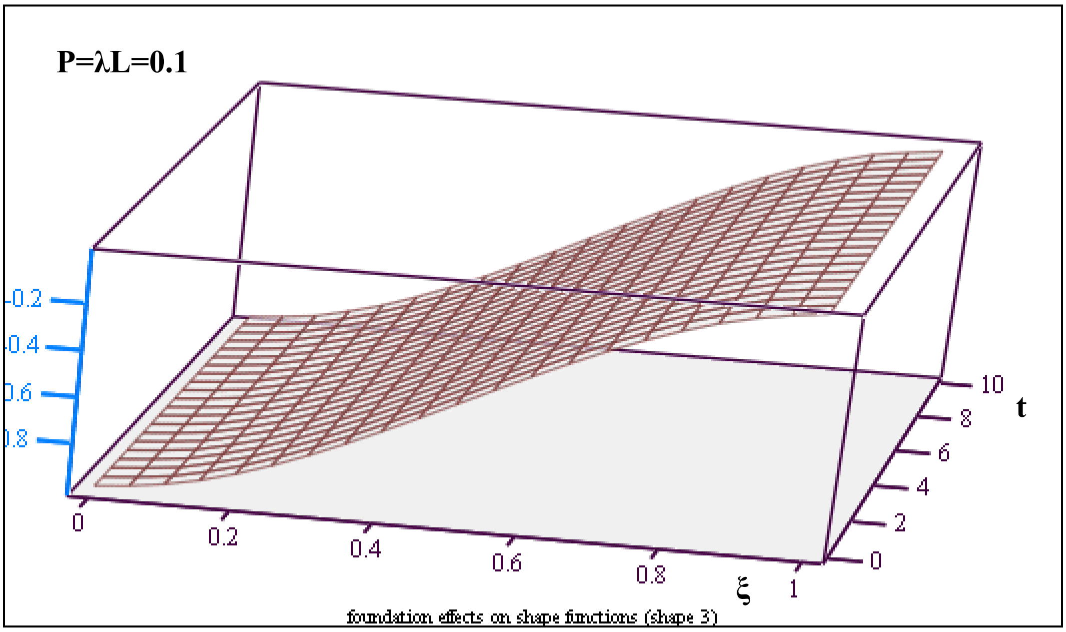


Figure 2.20: Variation of the Shape Function $\psi_{2,}$ for P=1



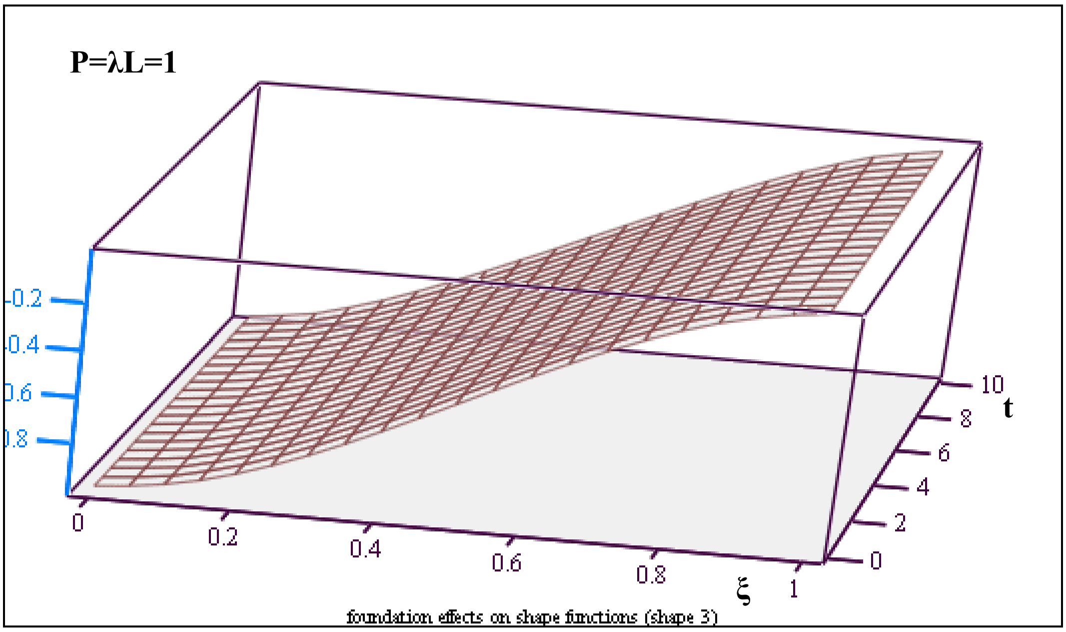
(x2,y2,z2)

Figure 2.21: Variation of the Shape Function $\psi_{2,}$ for P=5



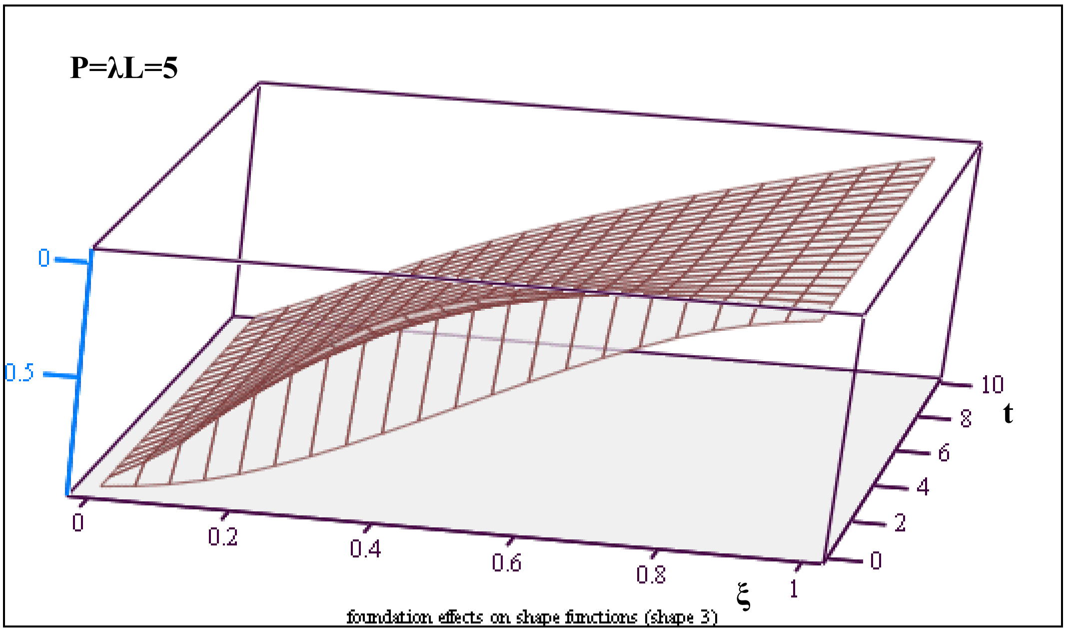
(x3,y3,z3a)

Figure 2.22: Variation of the Shape Function $\psi_{3,}$ for P=0.1



(x3 , y3 , z3)

Figure 2.23: Variation of the Shape Function $\psi_{3,}$ for P=1



(x33 , y33 , z33)

Figure 2.24: Variation of the Shape Function $\psi_{3,}$ for P=5

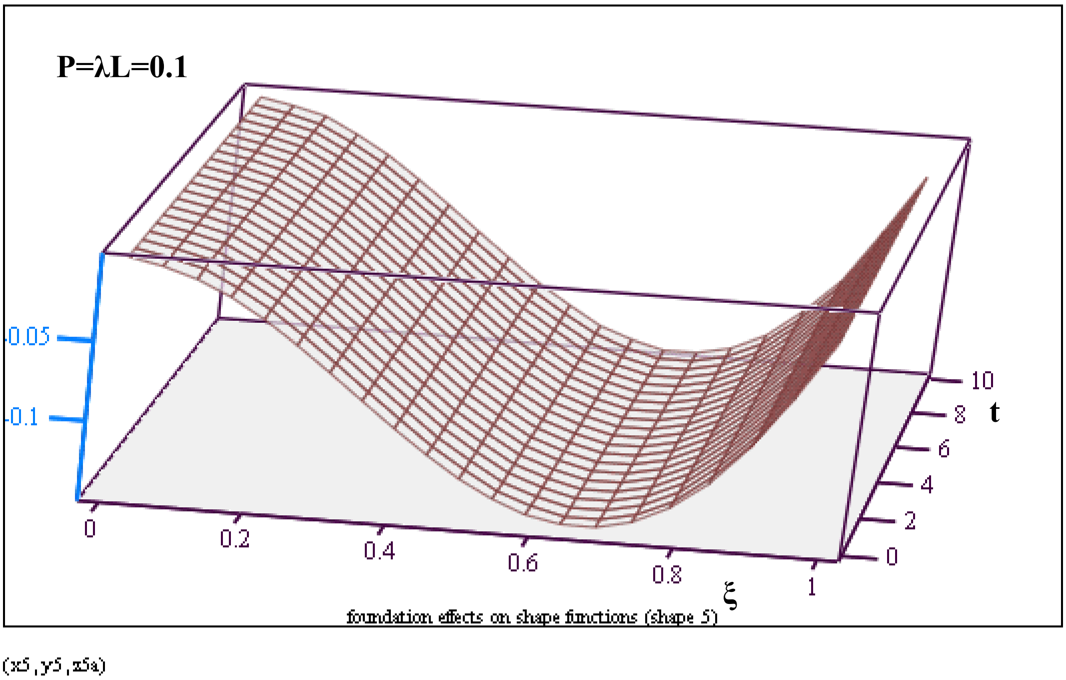


Figure 2.25: Variation of the Shape Function $\psi_{5,}$ for P=0.1

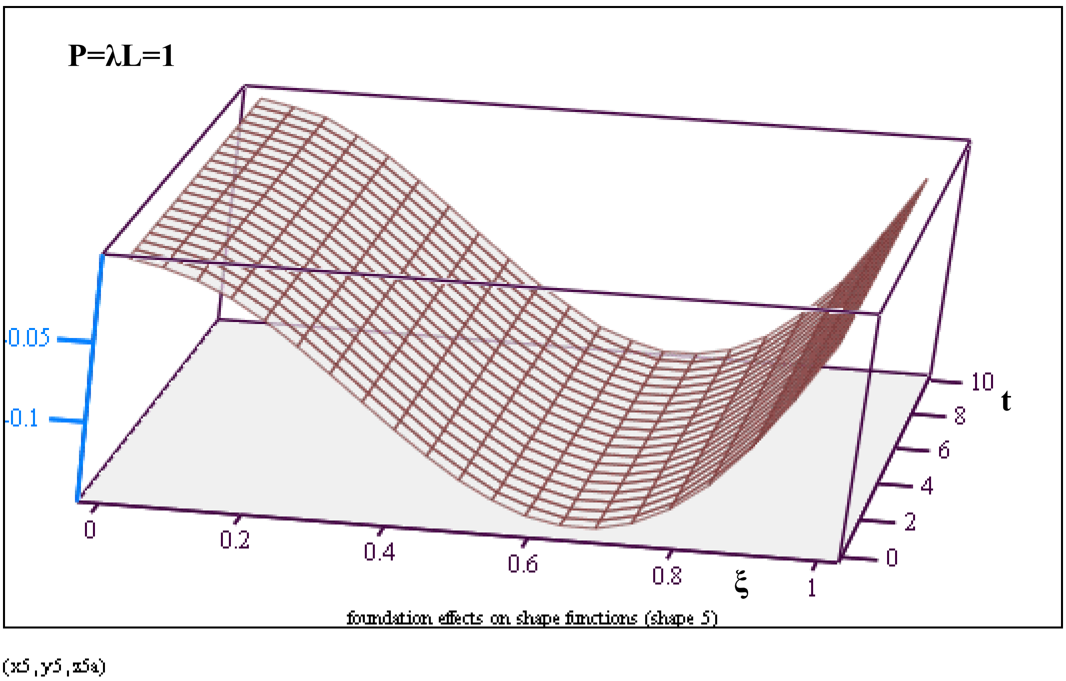
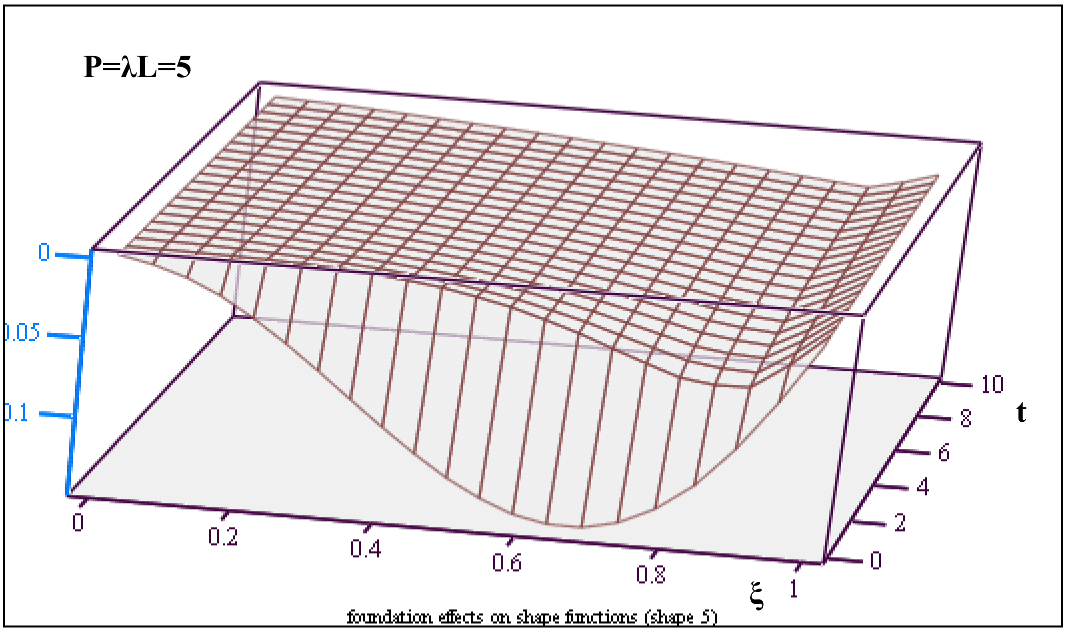
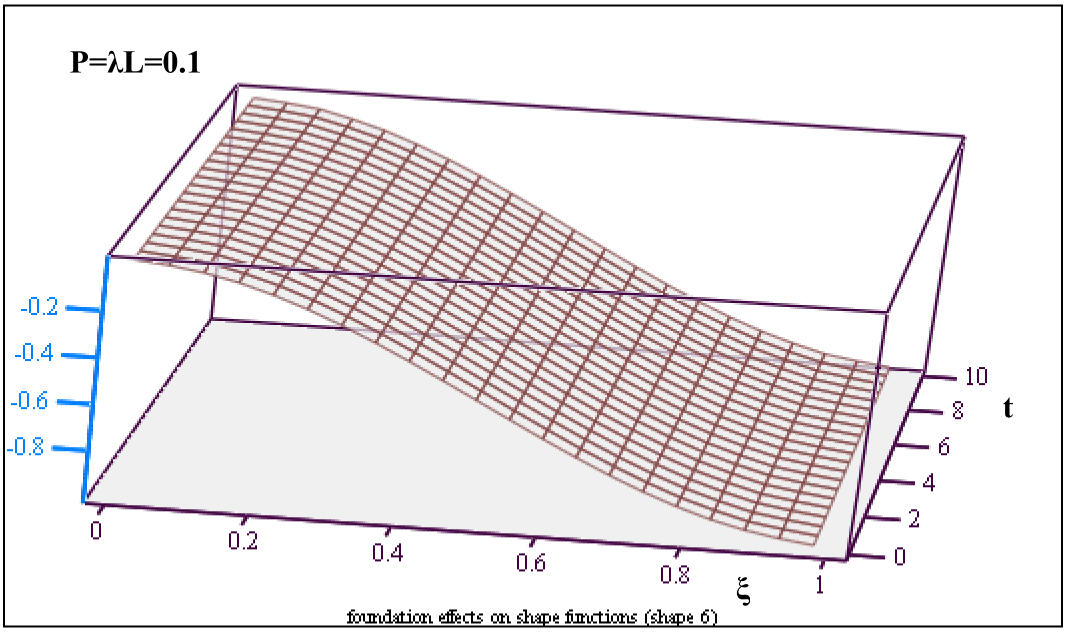


Figure 2.26: Variation of the Shape Function $\psi_{5,}$ for P=1



(x55,y55,z55)

Figure 2.27: Variation of the Shape Function $\psi_{5,}$ for P=5



(x6,y6,z6a)

Figure 2.28: Variation of the Shape Function $\psi_{6,}$ for P=0.1

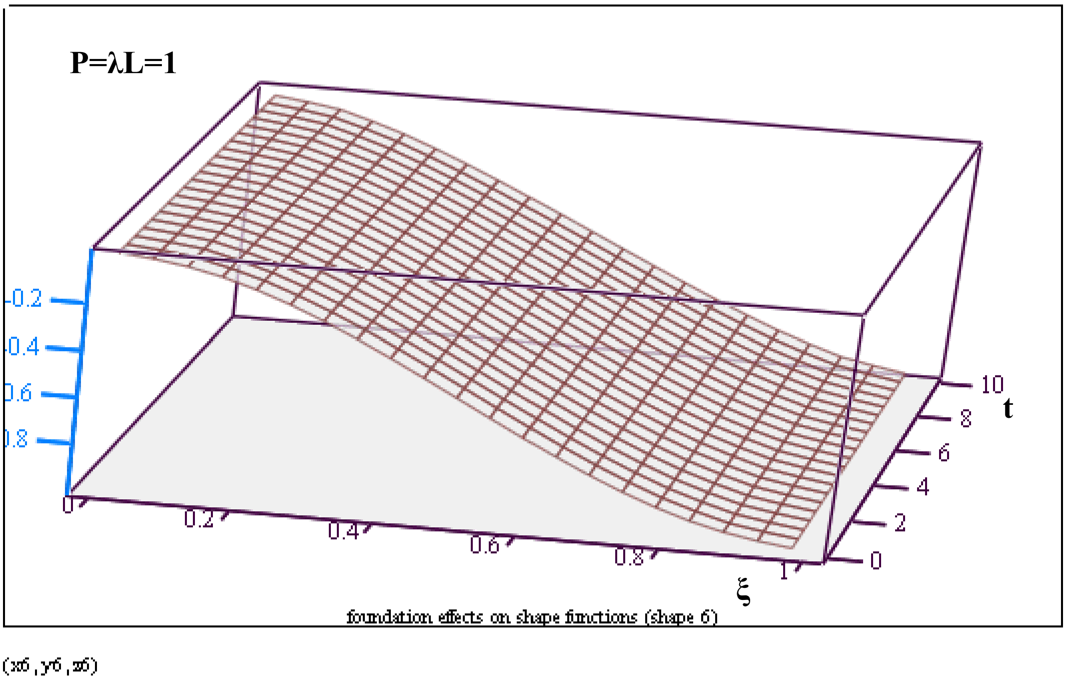


Figure 2.29: Variation of the Shape Function $\psi_{6,}$ for P=1

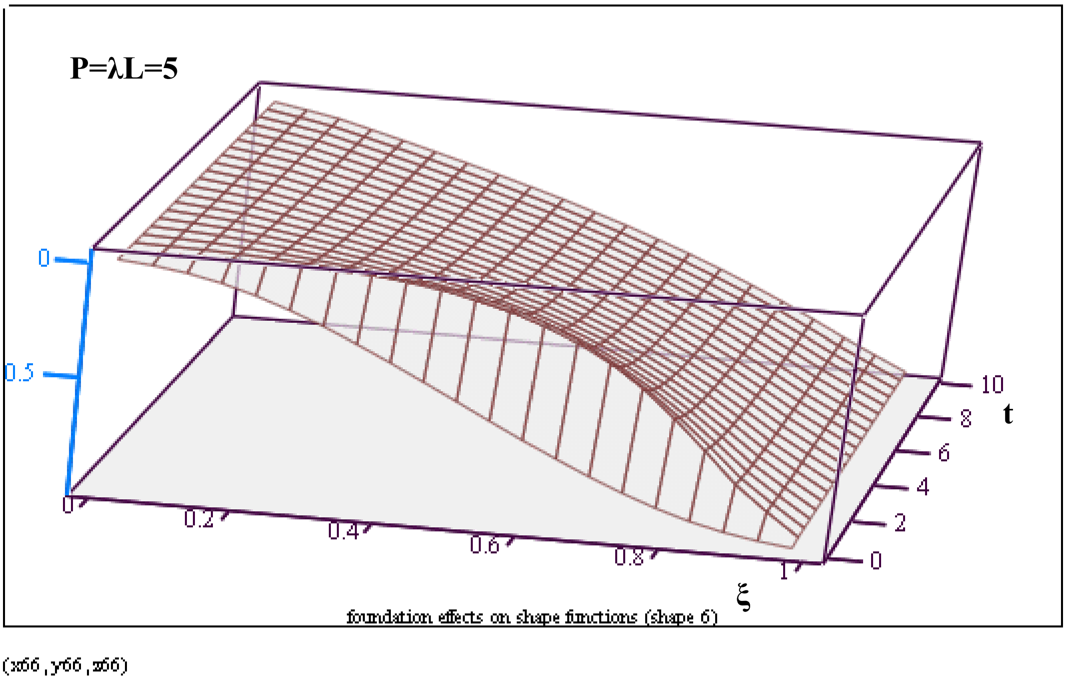


Figure 2.30: Variation of the Shape Function $\psi_{6,}$ for P=5

2.3.3 Derivation of the Element Stiffness Matrix

The element stiffness matrix of a beam element, which relates the nodal forces to the nodal displacements resting on two-parameter elastic foundation can be obtained by the same procedures as in Section 2.2.3. As a summary, the stiffness matrix, [K_e], for the prismatic beam element shown in Figure 2.18 can be obtained from the minimization of strain energy functional U as follows:

$$\left[\underline{K_e}\right] = \frac{\partial U}{\partial\{\underline{d}\}} \tag{2.85}$$

where

$$U = \frac{EI}{2}\int_0^L \frac{d^2w(x)}{dx^2}\frac{d^2w(x)}{dx^2}dx + \frac{k_1}{2}\int_0^L w(x)w(x)dx - \frac{k_\theta}{2}\int_0^L \frac{dw(x)}{dx}\frac{dw(x)}{dx}dx \tag{2.86}$$

Substituting w(x) and its derivatives from Equation (2.72) into Equation (2.85), the stiffness matrix can be written in the following form

$$\begin{aligned}\left[\underline{K_e}\right] = EI\int_0^L \left\{\frac{d^2\{N\}}{dx^2}\right\}^T \left\{\frac{d^2\{N\}}{dx^2}\right\}dx + k_1\int_0^L \{N\}^T\{N\}dx - \\ k_\theta\int_0^L \left\{\frac{d\{N\}}{dx}\right\}^T \left\{\frac{d\{N\})}{dx}\right\}dx\end{aligned} \tag{2.87}$$

where N is, a 6x1 matrix of the exact shape functions, given in Equation (2.74) for $A < 2\sqrt{B}$ and Equation (2.81) for $A > 2\sqrt{B}$. Their first and second derivatives in matrix forms are

$$\left\{\frac{d\{N\}}{dx}\right\} = \left[\underline{H}^{-1}\right]^T \left\{\frac{d\{B\}}{dx}\right\} \tag{2.88a}$$

$$\left\{\frac{d^2\{N\}}{dx^2}\right\} = \left[\underline{H}^{-1}\right]^T \left\{\frac{d^2\{B\}}{dx^2}\right\} \tag{2.88b}$$

Substituting N from Equations (2.74) and (2.81) and their derivatives, using Equations (2.88a) and (2.88b), into Equation (2.87) and carrying out the necessary integrals and procedures the stiffness terms are obtained for the cases:

$A < 2\sqrt{B}$

$$k_{22} = \left(\frac{2EI\alpha\beta\left(\beta \sinh[2\alpha L] - \alpha \sin[2\beta L]\right)}{\alpha^2 \cos[2\beta L] + \beta^2 \cosh[2\alpha L] - \alpha^2 - \beta^2} \right) \tag{2.89.a}$$

$$k_{23} = \left(\frac{EI(\alpha^2 + \beta^2)\left(\beta^2 - \alpha^2 + \alpha^2 \cos[2\beta L] - \beta^2 \cosh[2\alpha L]\right)}{\alpha^2 \cos[2\beta L] + \beta^2 \cosh[2\alpha L] - \alpha^2 - \beta^2} \right) \tag{2.89.b}$$

$$k_{25} = \left(\frac{4EI\alpha\beta\left(\beta \cos[\beta L]\sinh[\alpha L] - \alpha \cosh[\alpha L]\sin[\beta L]\right)}{\alpha^2 \cos[2\beta L] + \beta^2 \cosh[2\alpha L] - \alpha^2 - \beta^2} \right) \tag{2.89.c}$$

$$k_{26} = \left(\frac{4EI\alpha\beta\left(\alpha^2 + \beta^2\right)\left(\sin[\beta L]\sinh[\alpha L]\right)}{\alpha^2 \cos[2\beta L] + \beta^2 \cosh[2\alpha L] - \alpha^2 - \beta^2} \right) \tag{2.89.d}$$

$$k_{33} = \left(\frac{2EI\alpha\beta\left(\alpha^2 + \beta^2\right)\left(\alpha \sin[2\beta L] + \beta \sinh[2\alpha L]\right)}{\alpha^2 \cos[2\beta L] + \beta^2 \cosh[2\alpha L] - \alpha^2 - \beta^2} \right) \tag{2.89.e}$$

$$k_{36} = \left(-\frac{4EI\alpha\beta\left(\alpha^2 + \beta^2\right)\left(\alpha \cosh[\alpha L]\sin[\beta L] + \beta \cos[\beta L]\sinh[\alpha L]\right)}{\alpha^2 \cos[2\beta L] + \beta^2 \cosh[2\alpha L] - \alpha^2 - \beta^2} \right) \tag{2.89.f}$$

$A > 2\sqrt{B}$

$$k_{22} = \left(\frac{2EI\alpha\beta\left(\beta \sinh[\alpha L]\cosh[\alpha L] - \alpha \sinh[\beta L]\cosh[\beta L]\right)}{\beta^2 \sinh\left[(\alpha L)^2\right] - \alpha^2 \sinh\left[(\beta L)^2\right]} \right) \tag{2.90.a}$$

$$k_{23} = EI\left(\left(\alpha^2 + \beta^2\right) + \frac{\alpha^2\beta^2\left(\cosh(\alpha L)^2 \sinh(\beta L)^2 - \cosh(\beta L)^2 \sinh(\alpha L)^2\right)}{\beta^2 \sinh\left[(\alpha L)^2\right] - \alpha^2 \sinh\left[(\beta L)^2\right]} \right) \tag{2.90.b}$$

$$k_{25} = \left(\frac{2EI\alpha\beta\left(\alpha \sinh[\beta L]\cosh[\alpha L] - \beta \sinh[\alpha L]\cosh[\beta L]\right)}{\beta^2 \sinh\left[(\alpha L)^2\right] - \alpha^2 \sinh\left[(\beta L)^2\right]} \right) \tag{2.90.c}$$

$$k_{26} = \left(\frac{2EI\alpha\beta\left(\alpha^2 - \beta^2\right)\sinh[\alpha L]\sinh[\beta L]}{\beta^2 \sinh\left[(\alpha L)^2\right] - \alpha^2 \sinh\left[(\beta L)^2\right]} \right) \tag{2.90.d}$$

$$k_{33} = \left(\frac{EI\alpha\beta\left(\alpha^2 - \beta^2\right)\left(\alpha \sinh[2\beta L] + \beta \sinh[2\alpha L]\right)}{\beta^2 \sinh\left[(\alpha L)^2\right] - \alpha^2 \sinh\left[(\beta L)^2\right]} \right) \tag{2.90.e}$$

$$k_{36} = \left(\frac{-2EI\alpha\beta\left(\alpha^2 - \beta^2\right)\left(\alpha \cosh[\alpha L]\sinh[\beta L] + \beta \cosh[\beta L]\sinh[\alpha L]\right)}{\beta^2 \sinh\left[(\alpha L)^2\right] - \alpha^2 \sinh\left[(\beta L)^2\right]} \right) \tag{2.90.f}$$

the other terms of the stiffness matrix for both cases are

$$k_{11} = k_{44} = -k_{14} = -k_{41} = \frac{GJ}{L} \tag{2.91.a}$$

$$k_{12} = k_{13} = k_{15} = k_{16} = k_{42} = k_{43} = k_{45} = k_{46} = 0 \tag{2.91.b}$$

$$k_{32} = k_{35} = k_{53} = k_{23} \tag{2.91.c}$$

$$k_{56} = k_{65} = k_{62} = k_{26} \tag{2.91.d}$$

$$k_{52} = k_{25} \tag{2.91.e}$$

$$k_{55} = k_{22} \tag{2.91.f}$$

$$k_{63} = k_{36} \tag{2.91.g}$$

$$k_{66} = k_{33} \tag{2.91.h}$$

For cases $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$, the terms in Equations (2.89) and (2.90) must reduce to the conventional stiffness terms when foundation parameter k_1 and k_θ tends to zero ($\lambda \to 0$ and $\delta \to 0$ or $\alpha \to 0$ and $\beta \to 0$). They are verified for the both cases:

$$\underset{\alpha\to 0}{Lim}\,\underset{\beta\to 0}{Lim}\, k_{22} \to \frac{4EI}{L} \tag{2.92.a}$$

$$\underset{\alpha\to 0}{Lim}\,\underset{\beta\to 0}{Lim}\, k_{23} \to -\frac{6EI}{L^2} \tag{2.92.b}$$

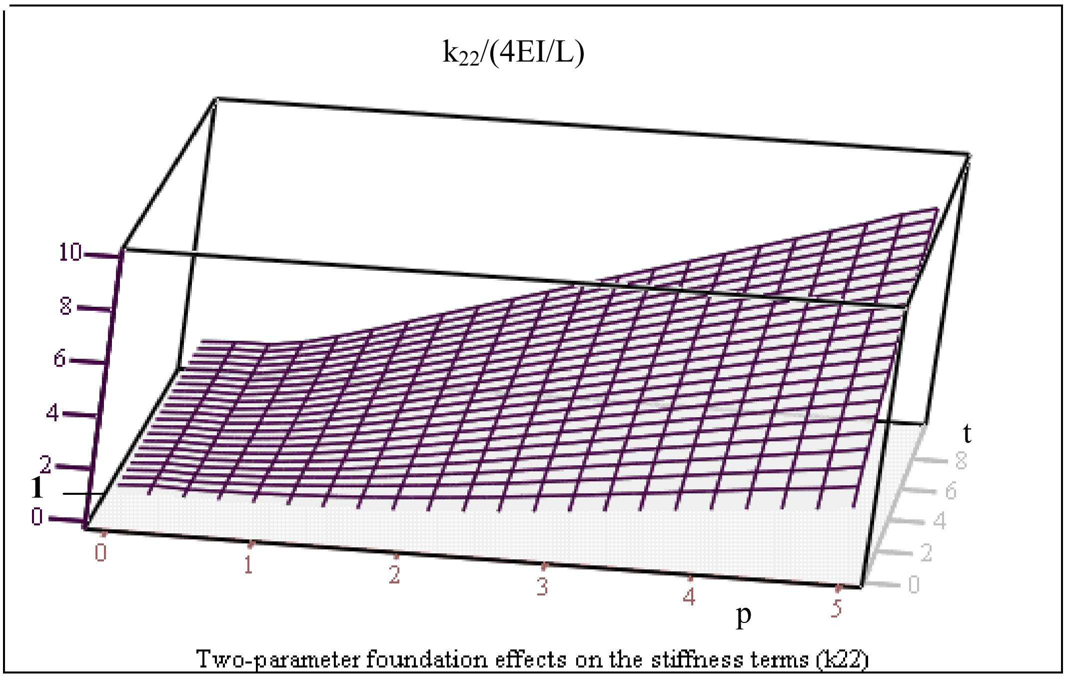
$$\underset{\alpha\to 0}{Lim}\,\underset{\beta\to 0}{Lim}\, k_{25} \to \frac{2EI}{L} \tag{2.92.c}$$

$$\lim_{\alpha \to 0} \lim_{\beta \to 0} k_{26} \to \frac{6EI}{L^2} \tag{2.92.d}$$

$$\lim_{\alpha \to 0} \lim_{\beta \to 0} k_{33} \to \frac{12EI}{L^3} \tag{2.92.e}$$

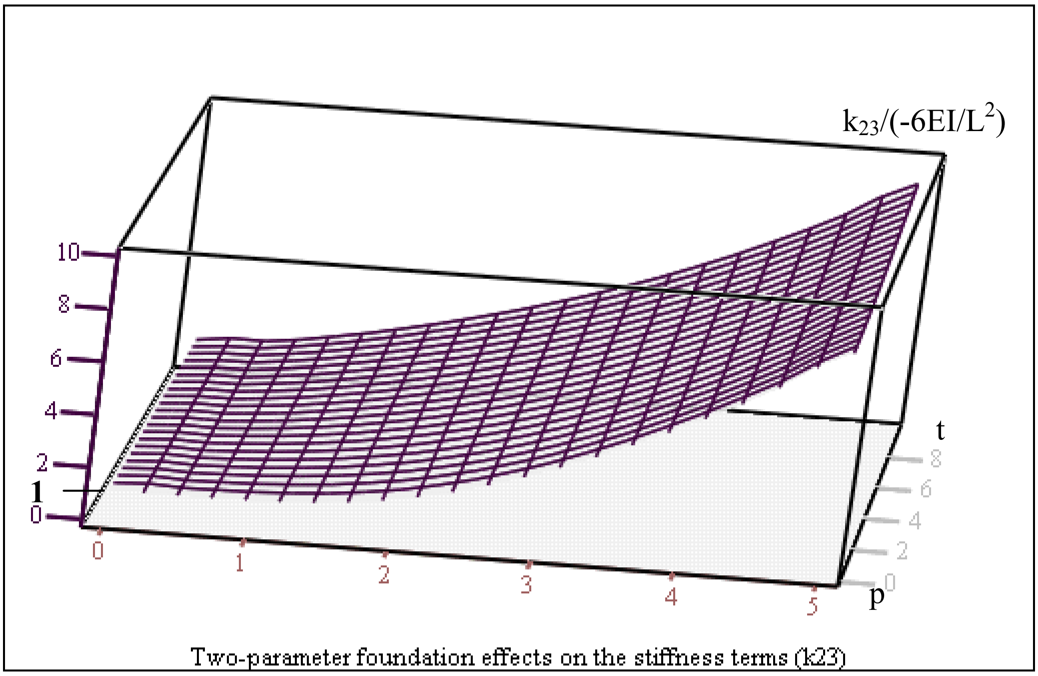
$$\lim_{\alpha \to 0} \lim_{\beta \to 0} k_{36} \to -\frac{12EI}{L^3} \tag{2.92.f}$$

The normalized terms as the ratio of the Equations (2.89) and (2.90) to Equation (2.92) are plotted in three-dimensional view to observe the influence of the foundation parameters. The p and t terms given in Figures 2.31 to 2.36 represents dominant effects of the first and the second foundation parameters respectively. Note that as t sets to zero the same curves of the Figures 2.9 - 2.14 obtained.



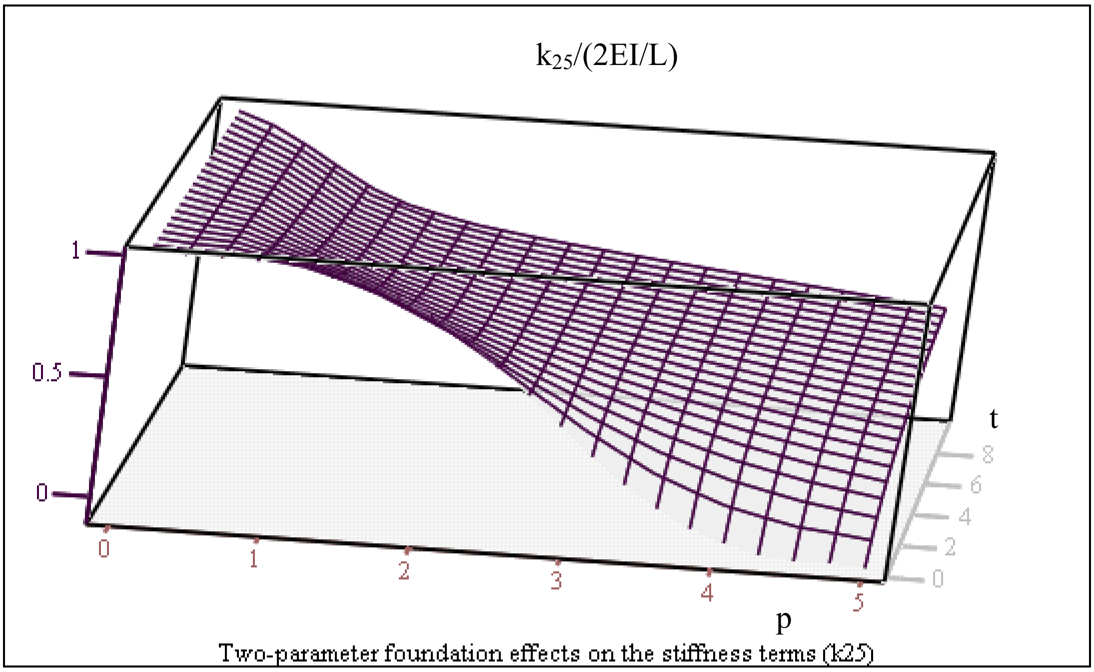
(x,y,z)

Figure 2.31: Normalized k_{22} Term for Two-Parameter Elastic Foundation



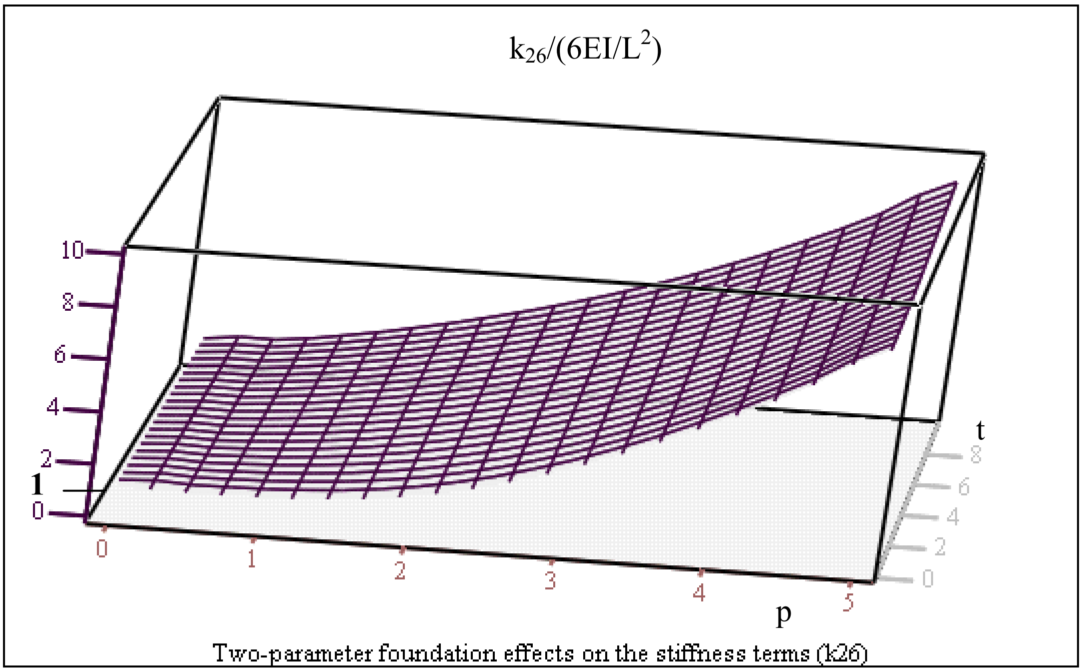
(x1 ,y1 ,z1)

Figure 2.32: Normalized k_{23} Term for Two-Parameter Elastic Foundation



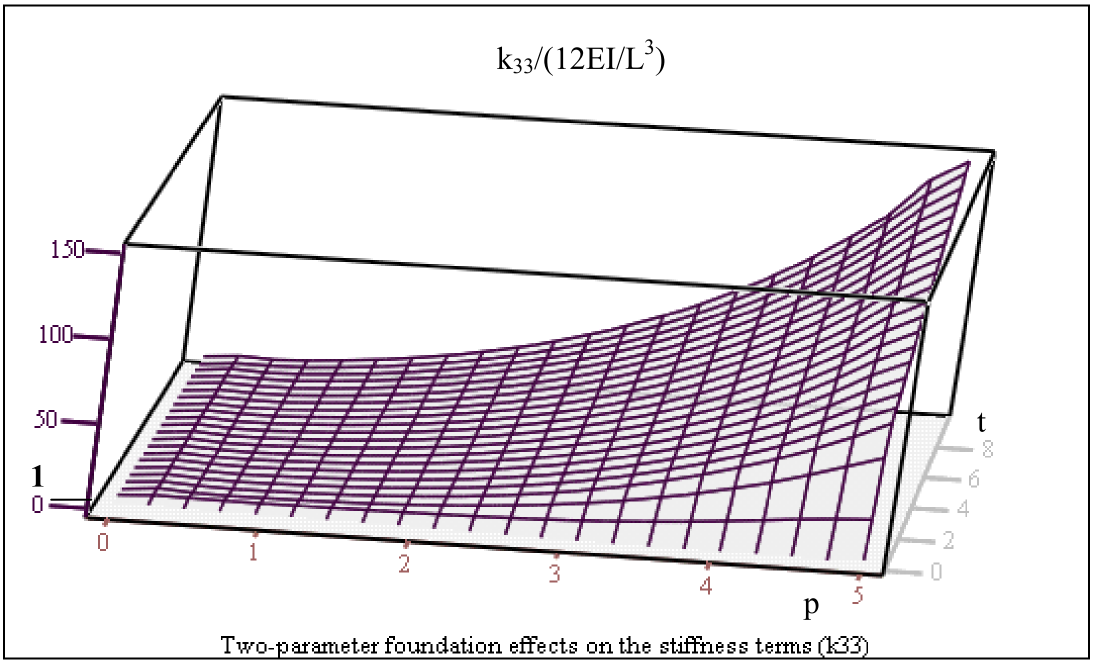
(x2 ,y2 ,z3)

Figure 2.33: Normalized k_{25} Term for Two-Parameter Elastic Foundation



(x4,y4,z4)

Figure 2.34: Normalized k_{26} Term for Two-Parameter Elastic Foundation



(x5,y5,z5)

Figure 2.35: Normalized k_{33} Term for Two-Parameter Elastic Foundation

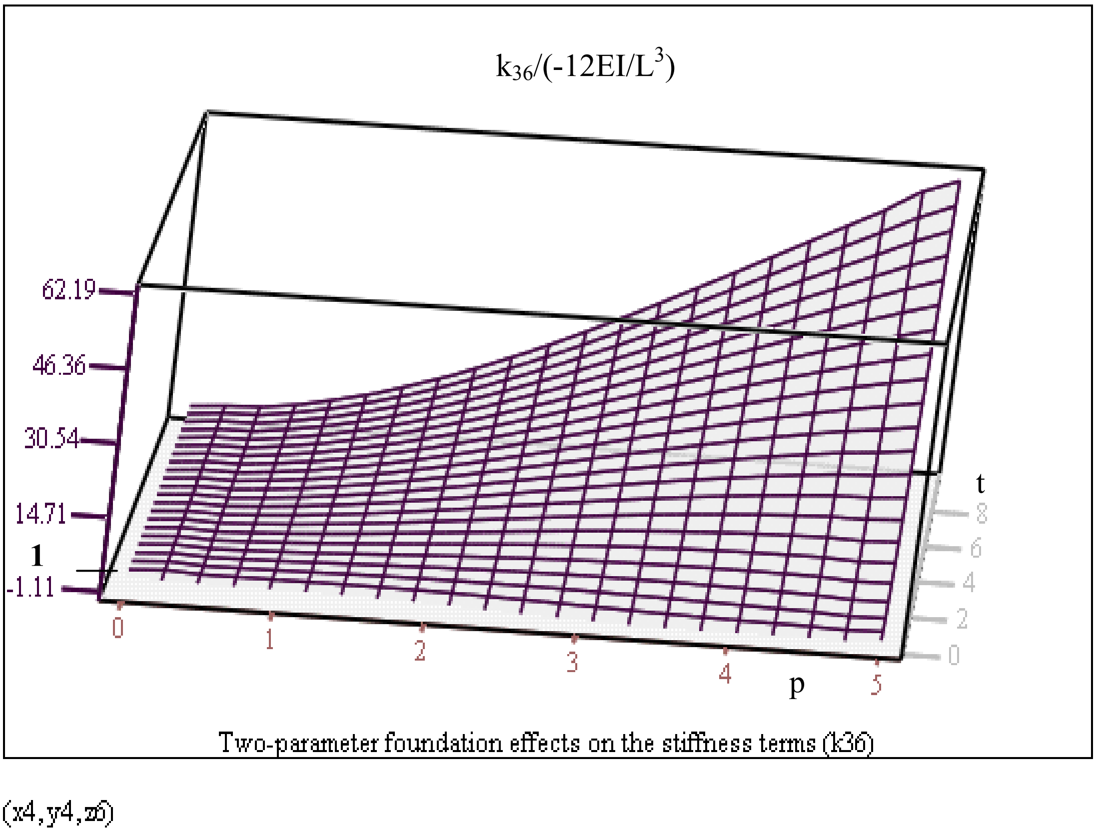


Figure 2.36: Normalized k_{36} Term for Two-Parameter Elastic Foundation

2.3.4 Derivation of the Work Equivalent Nodal Load Vector

The work equivalent nodal loads of a beam element resting on generalized foundation as shown in Figure 2.87a can be represented by Figure 2.87b. The fixed end forces are formed by shape functions. Since the shape functions are vary by foundation parameters the conventional cases are not valid for beam elements resting on elastic foundations. That is, foundation reactions will affect the equivalent nodal loads.

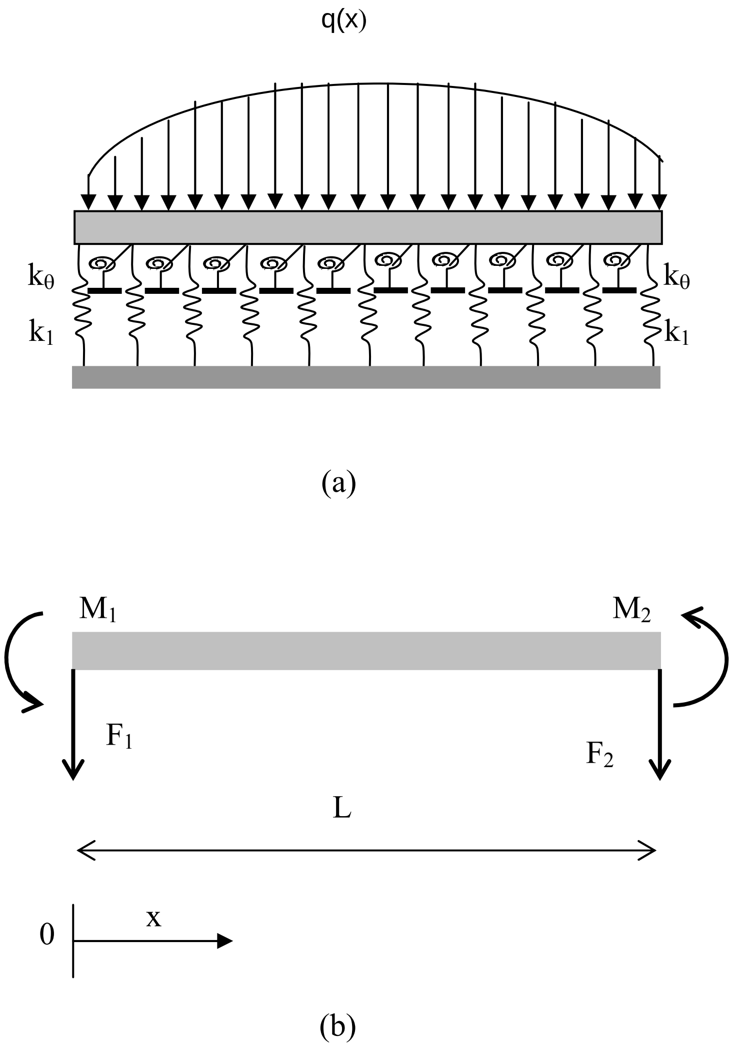


Figure 2.37: (a) Continuous Loading of a Beam Element Resting on Two-Parameter (Generalized) Foundation **(b)** Nodal Forces due to the Loading.

In this study, as mentioned in Section 2.3.4, the plate will be represented by a discrete number of intersecting beams. Since beam elements can be accepted as infinitesimal elements of plates, many types of loading can be represented with uniformly distributed loads or point loads applied at the nodes. Therefore, the nodal load vector will be derived only for ($q(x) = q_0$) uniformly distributed loading of the beam elements.

The equivalent nodal loads referring to Figure 2.37 for uniform distributed loading, q_0, can be obtained by rewriting Equation (2.45) as:

$$\{\underline{P}\} = \int_0^L q_0 [\underline{N}] dx \tag{2.93a}$$

$$\{\underline{P}\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \int_0^L q_0 \begin{Bmatrix} N_2 \\ N_3 \\ N_5 \\ N_6 \end{Bmatrix} dx \tag{2.93b}$$

where [$\underline{N}$] is the shape functions for either region $A < 2\sqrt{B}$ or $A > 2\sqrt{B}$ for beam elements resting on two-parameter elastic foundation.

Inserting the corresponding shape functions from Equation (2.74) into Equation (2.93), the nodal loads yields, for $A < 2\sqrt{B}$

$$F_1 = F_2 = \frac{2q_0\alpha\beta(\cosh[\alpha L] - \cos[\beta L])}{(\alpha^2 + \beta^2)(\beta \sinh[\alpha L] + \alpha \sin[\beta L])} \tag{2.94a}$$

$$M_1 = -M_2 = \frac{q_0(\beta \sinh[\alpha L] - \alpha \sin[\beta L])}{(\alpha^2 + \beta^2)(\beta \sinh[\alpha L] + \alpha \sin[\beta L])} \tag{2.94b}$$

In the second region, inserting the corresponding shape functions from Equation (2.81) into Equation (2.93), the nodal loads yields, for $A > 2\sqrt{B}$

$$F_1 = F_2 = \left(\frac{\begin{array}{c} 2q_0\alpha\beta(\beta \sinh[(\alpha - \beta)L] - \beta \sinh[2L\alpha] - \alpha \sinh[(\alpha - \beta)L] - \\ \alpha \sinh[2\beta L] + \alpha \sinh[(\alpha + \beta)L] + \beta \sinh[(\alpha + \beta)L]) \end{array}}{(\alpha^2 - \beta^2)(\beta^2 - \alpha^2 - \beta^2 \cosh[2\alpha L] + \alpha^2 \cosh[2\beta L])} \right) \tag{2.95a}$$

$$M_1 = -M_2 = \left(\frac{\begin{matrix} q_0(\alpha^2 + \beta^2 - \beta^2 \cosh[2\alpha L] - 2\alpha\beta \cosh[(\alpha-\beta)L] - \\ -\alpha^2 \cosh[2\beta L] + 2\alpha\beta \cosh[(\alpha+\beta)L]) \end{matrix}}{(\alpha^2-\beta^2)(\beta^2 - \alpha^2 - \beta^2 \cosh[2\alpha L] + \alpha^2 \cosh[2\beta L])} \right) \tag{2.95b}$$

For both $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ cases, when both of the foundation parameters k_1 and k_θ tend to zero, the terms in Equations (2.94) and (2.95) will reduce to the conventional beam fixed end forces obtained by Hermitian functions. That is

$$F_1 = F_2 = \underset{\alpha\to 0}{Lim}\,\underset{\beta\to 0}{Lim}\left(Equation(2.94a \quad or \quad 2.95a) \right) \to \frac{q_0 L}{2} \tag{2.96a}$$

$$M_1 = -M_2 = \underset{\alpha\to 0}{Lim}\,\underset{\beta\to 0}{Lim}\left(Equation(2.94b \quad or \quad 2.95b) \right) \to \frac{q_0 L^2}{12} \tag{2.96b}$$

Normalizing Equation (2.94) for $A < 2\sqrt{B}$ and Equation (2.95) for $A > 2\sqrt{B}$ with conventional terms of Equation (2.96) can be used to observe the effect of the foundation parameters, k_1 and k_θ, on the nodal forces. For clarifying the comparison let foundation parameters be rewritten as:

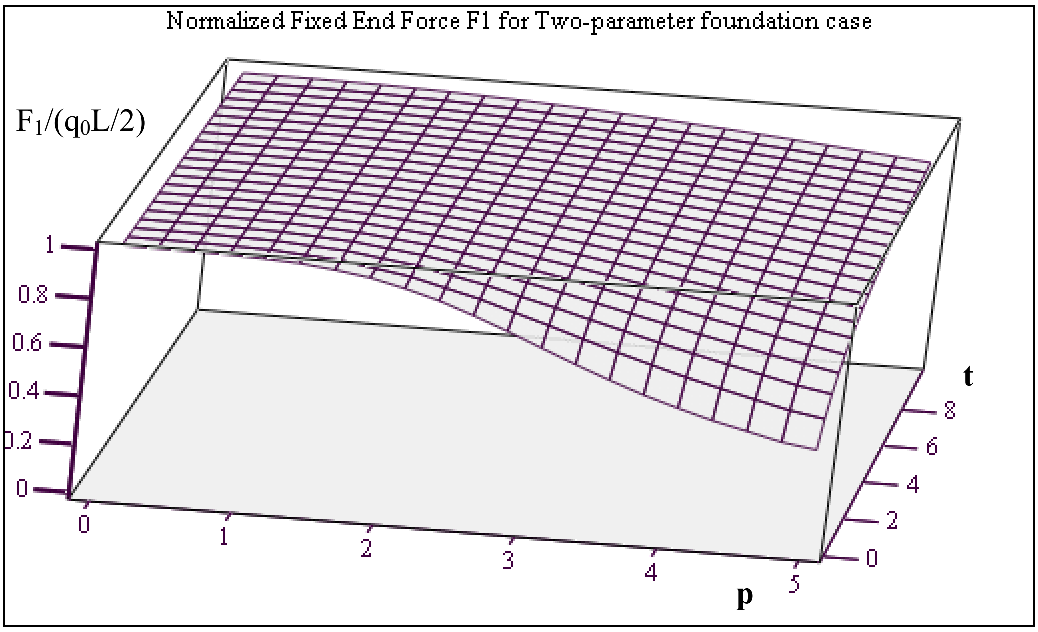
For $A < 2\sqrt{B}$
$$\begin{aligned} \alpha &= \sqrt{\lambda^2 + \delta} = \lambda\sqrt{1+t} \\ \beta &= \sqrt{\lambda^2 - \delta} = \lambda\sqrt{1-t} \end{aligned} \tag{2.97a}$$

and

For $A > 2\sqrt{B}$
$$\begin{aligned} \alpha &= \sqrt{\lambda^2 + \delta} = \lambda\sqrt{1+t} \\ \beta &= \sqrt{\delta - \lambda^2} = \lambda\sqrt{t-1} \end{aligned} \tag{2.97b}$$

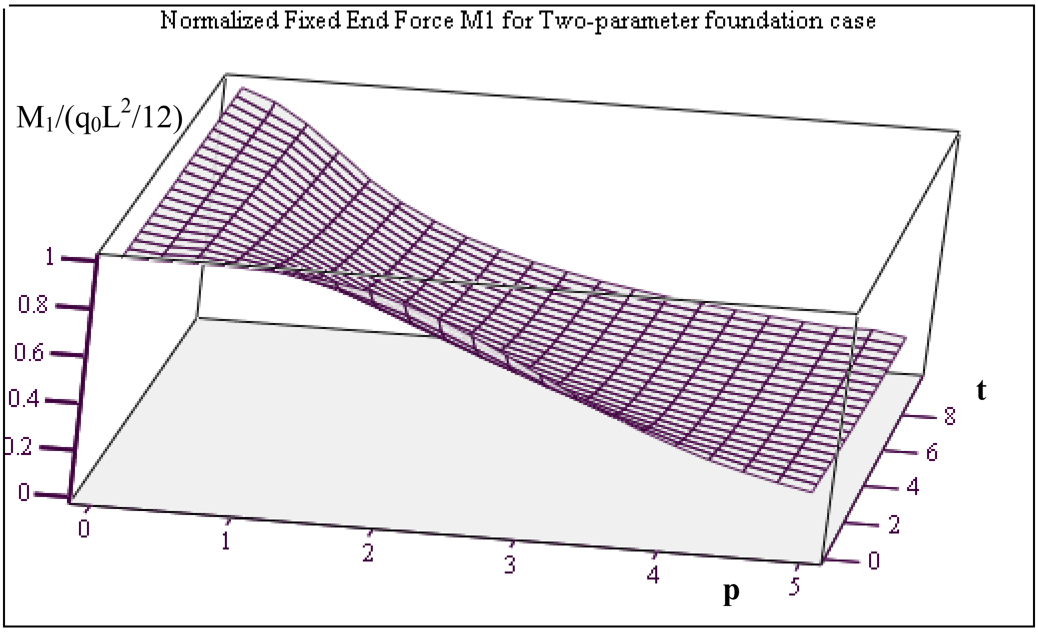
where
$$t = \frac{\delta}{\lambda^2} = \frac{\dfrac{k_\theta}{4EI}}{\sqrt{\dfrac{k_1}{4EI}}} \quad \text{and} \quad p=\lambda L \tag{2.98}$$

The normalized terms with respect to indirectly foundation parameters are shown in Figures 2.37 and 2.38.



(x7,y7,z7)

Figure 2.38: Normalized F_1 Term for Uniform Distributed Loaded Beam Elements Resting on Two-Parameter Elastic Foundation



(x8,y8,z8)

Figure 2.39: Normalized M_1 Term for Uniform Distributed Loaded Beam Elements Resting on Two-Parameter Elastic Foundation

CHAPTER 3

EXTENSION TO VIBRATION AND STABILITY PROBLEMS

3.1 INTRODUCTION

Since response of structures frequently involves a dynamic process or stability, in some cases it will be more realistic to extend the formulation in Chapter 2 to enable buckling and stability solutions. Therefore, in engineering practice, beside static case often one or both of the stability and dynamic effects must be taken into consideration to the plate analysis and design problems. It will be necessary to describe the governing equation of motion of plates in a general mathematical form for such cases. This can be achieved by inserting both of the inertia force due to the lateral translation and in-plane loading simultaneously, in an appropriate way, into the governing differential equation for static case. Referring to Figure 3.1, the governing equation for plates resting on generalized foundation under the combined action of transverse load and biaxial in-plane loading can be obtained by rearranging Equation (2.1) as:

$$\begin{aligned} & D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+k_1 w- \\ & k_\theta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+N_x\frac{\partial^2 w}{\partial y^2}+N_y\frac{\partial^2 w}{\partial x^2}-\overline{m}\frac{\partial^2 w}{\partial t^2}=q(x,y) \end{aligned} \tag{3.1}$$

where $w = w(x, y, t)$ is the transverse deflection of the plate, $D = Eh^3/12(1-\nu^2)$ is the plate flexural rigidity, $\overline{m}$ is the mass of the plate per unit area, N_x and N_y are in-plane loads in x and y directions respectively, and k_1, k_θ are the foundation parameters defined in the previous chapter.

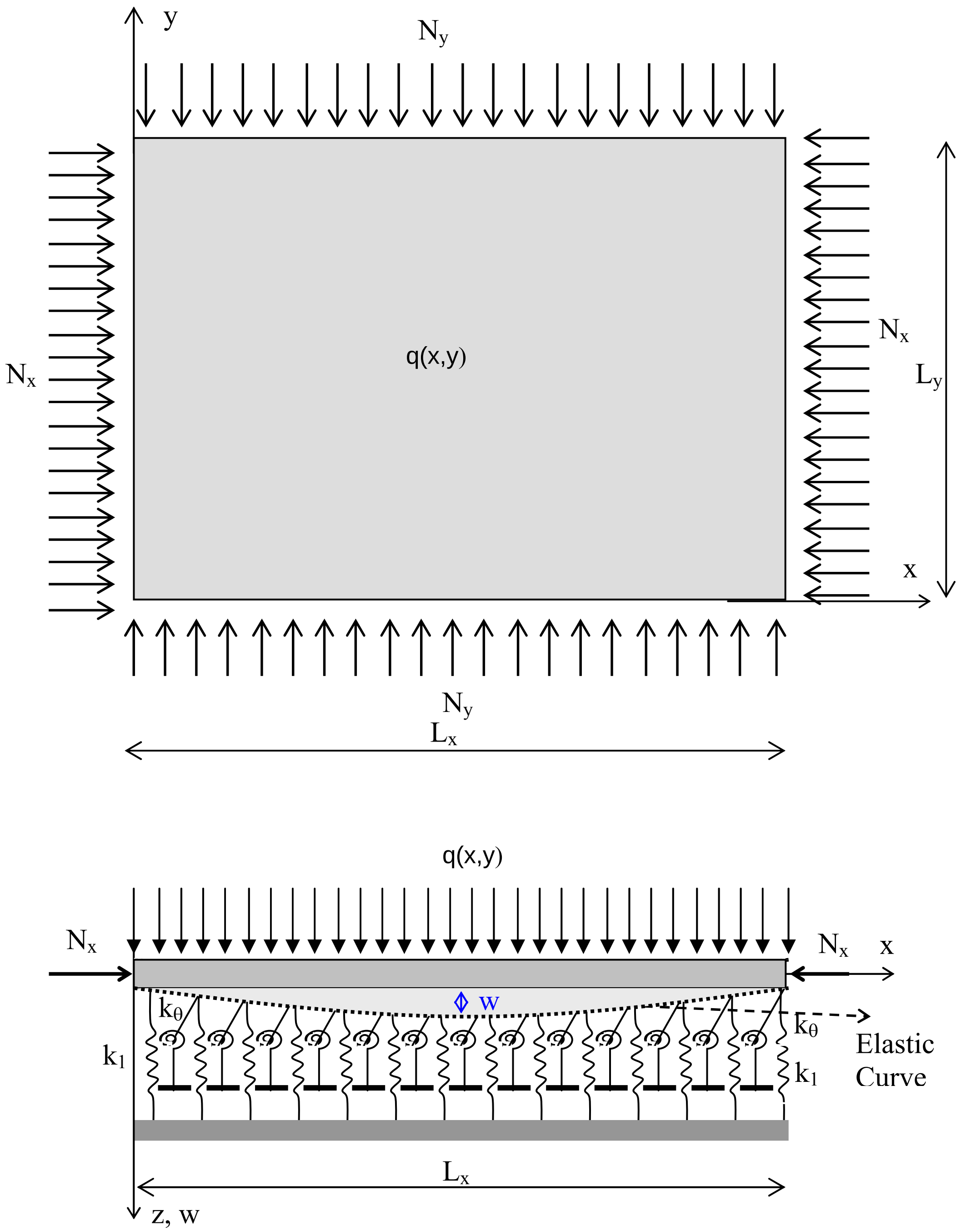


Figure 3.1: The Representation of a Modal Plates Resting on Generalized Foundation Under the Combined Action of Transverse Load and Biaxial In-Plane Loads.

Equation (3.1) is a linear fourth-order partial differential equation for the unknown displacement function $w = w(x, y, t)$. If there are no any loads other than the in-plane loads the equation will define an eigenvalue problem which allows us to find out the critical buckling loads. However if the in-plane and the transverse loads are set to zero then it will be again an eigenvalue problem that describes the case of a freely vibrating plate. Accordingly the governing differential equation of the plates under static buckling of plates is:

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) + k_1 w - k_\theta (\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) + N_x \frac{\partial^2 w}{\partial y^2} + N_y \frac{\partial^2 w}{\partial x^2} = 0 \tag{3.2}$$

On the other hand, the governing differential equation of the freely vibrating plate is:

$$D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) + k_1 w - k_\theta (\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) - \overline{m}\frac{\partial^2 w}{\partial t^2} = 0 \tag{3.3}$$

Bazant (1989) has stated that buckling of plates is analogous to buckling of columns and frames. The similarities are bifurcation type of buckling with similar critical load and the possibility of solving the critical loads from linear eigenvalue problem. Dynamic problems of the plates with arbitrary contours and arbitrary boundary condition are very difficult or often impossible to solve in closed form by the classical methods based on Equations (3.1) to (3.3). In some respects dynamic behaviour of plates resembles that of beams. Therefore plates can be modeled as an assemblage of individual beam elements interconnected at their neighboring joints as represented in Figures 3.2 and 3.3. By representing the plate with assemblage of individual beam elements interconnected at their neighboring joints, the system cannot truly be equal to the continuous structure, however sufficient accuracy can be obtained similar to the static case. By representing the plate shown in Figure 3.1 with individual beam elements the problem will be reduced to one-dimensional one. Then Equations (3.2) and (3.3) can be rewritten for one-dimensional beam elements as:

$$D\frac{d^4w}{dx^4}+k_1w-k_\theta\frac{d^2w}{dx^2}+N\frac{d^2w}{dy^2}=0 \tag{3.4}$$

$$D\frac{d^4w}{dx^4}+k_1w-k_\theta\frac{d^2w}{dx^2}+\overline{m}\frac{d^2w}{dt^2}=0 \tag{3.5}$$

The main advantage of the reduction is that both of the exact geometric stiffness matrix and consisting mass matrix can be determined for the beam elements. These matrices will be used as a basis in Chapter 4 for assembling the plate problems in a proper way. Then dynamic problems of the plates resting on two-parameter foundation with arbitrary loading and boundary conditions could be solved approximately.

3.2 CONSISTENT MASS MATRIX

It is possible to evaluate mass influence coefficients of a structural element with the procedures similar to that obtaining the element stiffness matrix by making the use of finite element concept (Clough and Penzien, 1993). The consistent mass matrix of beam elements resting on elastic foundations can also be evaluated by the same procedures.

The degrees of freedom of the element are the torsion, rotation and translation at each end. Since the angular displacements are obtained from the pure torsion member, the torsional DOF's are independent. Then it can be assumed that the displacements within the span are defined again by the same interpolation functions those already derived in Chapter 2 for obtaining the element stiffness matrices.

Consider the beam element shown in Figure 3.2 having a mass distribution m(x). If it were subjected to a unit angular acceleration at point a, the acceleration would be developed along its length as follow:

$$\ddot{v}(x) = \psi_2(x)\ddot{v}_2 \tag{3.6}$$

By d'Alembert's principle, the inertial force due to this acceleration is:

$$f_I(x) = m(x)\ddot{v}(x) = m(x)\psi_2(x)\ddot{v}_2 \tag{3.7}$$

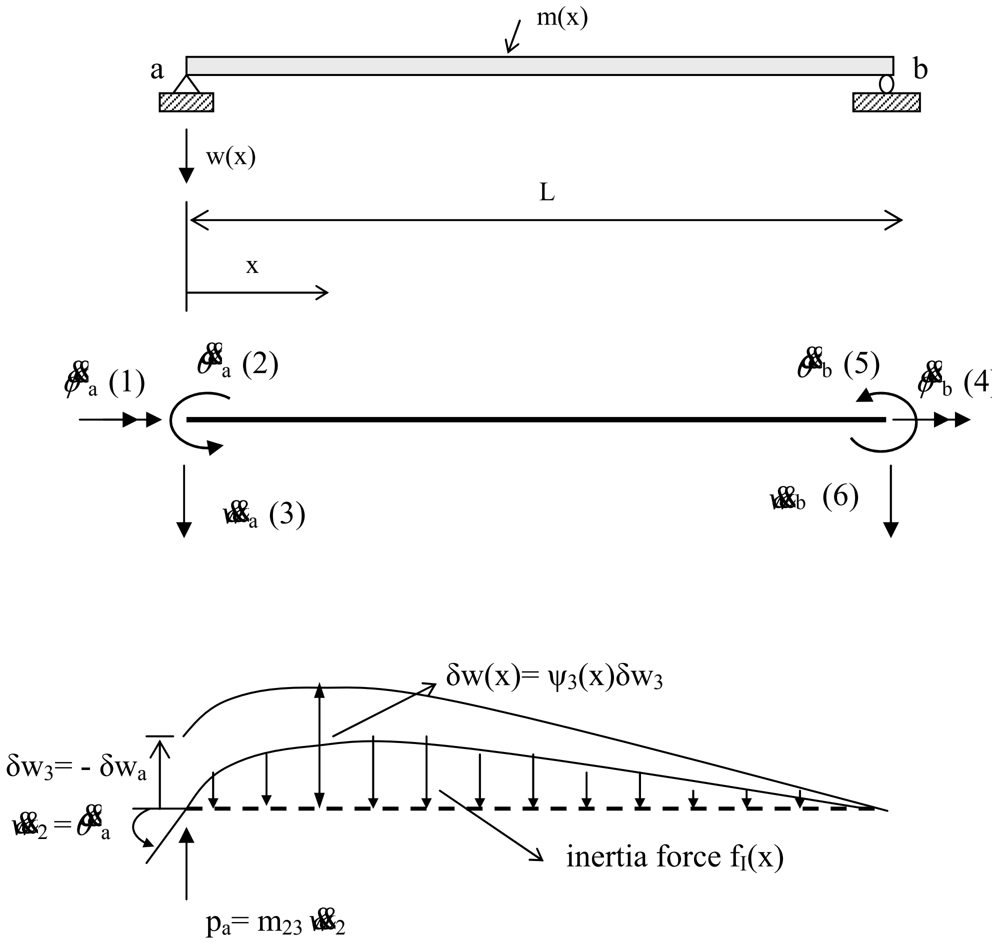


Figure 3.2: The Representation of a Beam Element Subjected to a Unit Real Acceleration and Virtual Translation at the Left Side.

By the principle of virtual displacements the mass influence coefficients associated with this acceleration as the nodal inertial forces can be evaluated from Equation (3.7). As an example, it is possible to evaluate the vertical force p_a, equating work done by the external force due to virtual displacement, to the work done on the distributed inertial forces $f_I(x)$. That is

$$p_a \delta w_3 = \int_0^L f_I(x) \delta w(x) dx \tag{3.8}$$

Substituting the vertical virtual displacement in terms of the shape functions into Equation (3.7) then,

$$m_{23} = \int_0^L m(x) \psi_2(x) \psi_3(x) dx \tag{3.9}$$

By this analogy, Equation (3.9) can be extended to evaluate for the other degrees of freedoms such as:

$$m_{ij} = \int_0^L m(x) \psi_i(x) \psi_j(x) dx \tag{3.10}$$

By using the proper shape functions for conventional beam or beam element resting on one or two parameter elastic foundations, Equation (3.10) enables us to evaluate all of the mass matrix terms. Computing the mass coefficients by the same shape functions with same procedures as done for determining the stiffness matrices is called consistent-mass matrices.

3.2.1 Consistent Mass Matrix for One-Parameter Foundation

Recalling the corresponding shape functions given in Equation (2.29) and substitute them into Equation (3.10) leads us to evaluate the consistent mass matrix for the beam elements resting on one-parameter elastic foundations. After evaluating the necessary integrations and introducing the constant mass distribution m(x)= μ as uniform mass per unit length, the mass matrix terms will be

$$m_{11} = \frac{\mu L}{3} \quad \textit{and} \quad m_{14} = \frac{\mu L}{6} \tag{3.11a}$$

$$m_{22} = \frac{\mu L^3 \begin{pmatrix} 8p(\cos[2p] - \cosh[2p]) + 4\sin[2p] - \sin[4p] \\ -2\cosh[2p]\sin[2p] - 4\sinh[2p] + 2\cos[2p]\sinh[2p] \\ +8p\sinh[2p]\sin[2p] + \sinh[4p] \end{pmatrix}}{16p^3(\cosh[2p] + \cos[2p] - 2)^2} \tag{3.11b}$$

$$m_{23} = \frac{\mu L^2 \begin{pmatrix} 4 + \cos[4p] - 4(1 - \cosh[2p])(1 - p\sin[2p]) - 4\cos[2p] \\ -\cosh[4p] + 4p\sinh[2p](1 - \cos[2p]) \end{pmatrix}}{8p^2(\cosh[2p] + \cos[2p] - 2)^2} \tag{3.11c}$$

$$m_{25} = \frac{\mu L^3 \begin{pmatrix} 2p(\cos[p]\cosh[3p] - \cosh[p]\cos[3p]) - 4\cosh[p]\sin[p] \\ +\cosh[3p]\sin[p] + \cosh[p]\sin[3p] + 4\sinh[p]\cos[p] \\ -\cos[3p]\sinh[p] - 16p\sinh[p]\sin[p] - \cos[p]\sinh[3p] \end{pmatrix}}{8p^3(\cosh[2p] + \cos[2p] - 2)^2} \tag{3.11d}$$

$$m_{26} = \frac{\mu L^2 \begin{pmatrix} p\cosh[p](\sin[3p] - 2\sin[p]) - p\cosh[3p]\sin[p] \\ +p\sinh[p](2\cos[p] - \cos[3p]) - 12\sinh[p]\sin[p] \\ +2\sinh[3p](\sin[3p] + \sin[p]) + p\cos[p]\sinh[3p] \end{pmatrix}}{4p^2(\cosh[2p] + \cos[2p] - 2)^2} \tag{3.11e}$$

$$m_{33} = \frac{\mu L \begin{pmatrix} 8p(1 - \cosh[2p])(1 - \cos[2p]) + 3\sin[4p] \\ +6\sin[2p]\cosh[2p] + 6\sinh[2p]\cos[2p] \\ -12(\sin[2p] + \sinh[2p]) + 3\sinh[4p] \end{pmatrix}}{8p(\cosh[2p] + \cos[2p] - 2)^2} \tag{3.11f}$$

$$m_{36} = \frac{\mu L \begin{pmatrix} 12\sin[p]\cosh[p] - 3\sin[p]\cosh[3p] - 3\sin[3p]\cosh[p] \\ +12\sinh[p]\cos[p] - 3\sinh[p]\cos[3p] - 2p\sin[3p]\sinh[p] \\ +2p\sinh[3p]\sin[p] - 3\sinh[3p]\cos[p] \end{pmatrix}}{4p(\cosh[2p] + \cos[2p] - 2)^2} \tag{3.11g}$$

where μ is mass per unit length and $p = \lambda L = \sqrt[4]{\frac{k_1}{4EI}}L$

When foundation parameter k_1 tends to zero (or $p \to 0$), the terms in Equation (3.11) must reduce to the conventional beam consistent mass terms obtained by

Hermitian functions. The correctness of the terms is verified that the terms reduce to the following conventional terms in matrix form.

$$\lim_{p\to 0}[\underline{M}] = \frac{\mu L}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 4L^2 & -22L & 0 & -3L^2 & -13L \\ 0 & -22L & 156 & 0 & 13L & 54 \\ 70 & 0 & 0 & 70 & 0 & 0 \\ 0 & -3L^2 & 13L & 0 & 4L^2 & 22L \\ 0 & -13L & 54 & 0 & 22L & 156 \end{bmatrix} \tag{3.12}$$

The normalized terms represent the influence of the foundation parameter k_1 on the mass matrix terms given in Equation (3.11) and corresponding terms of the matrix given in Equation (3.12) is portrayed in Figures 3.3 to 3.8.

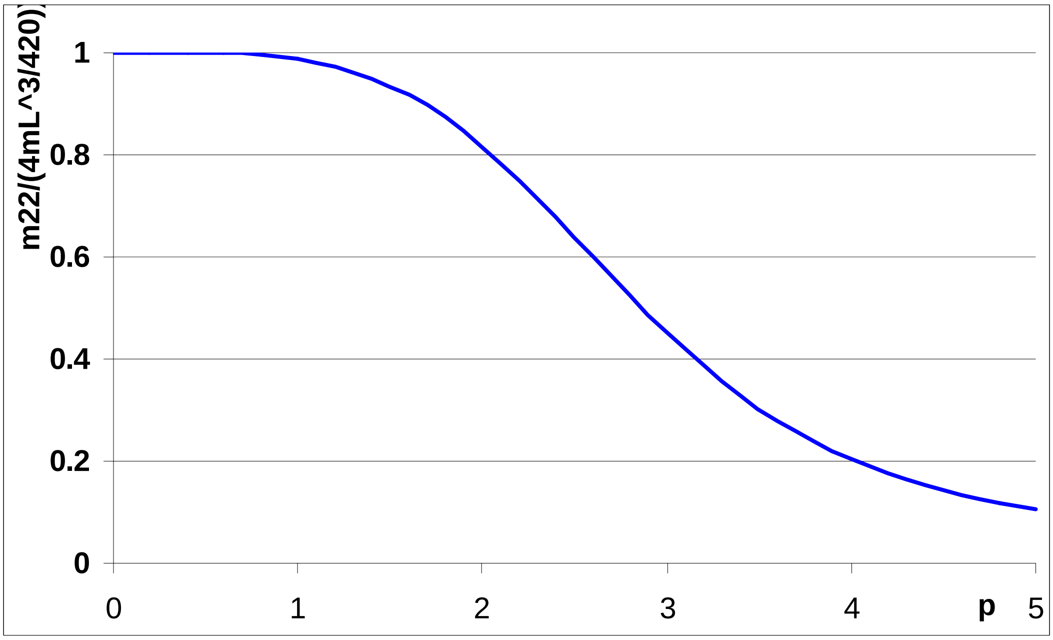


Figure 3.3: Influence of One-Parameter Foundation on the Normalized Consistent Mass Term m_{22}

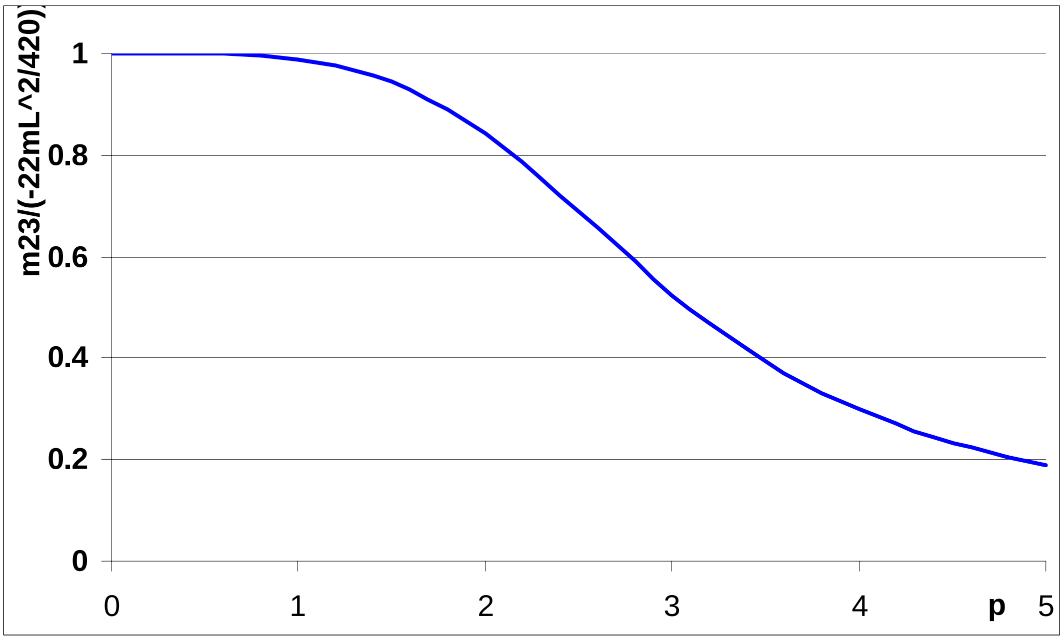


Figure 3.4: Influence of One-Parameter Foundation on the Normalized Consistent Mass Term m_{23}

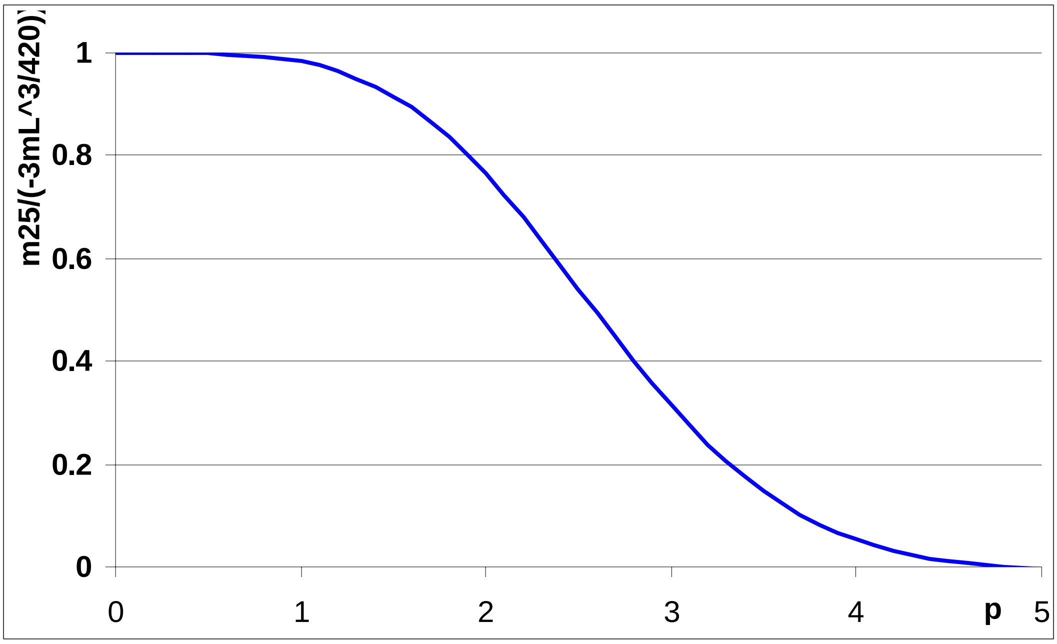


Figure 3.5: Influence of One-Parameter Foundation on the Normalized Consistent Mass Term m_{25}

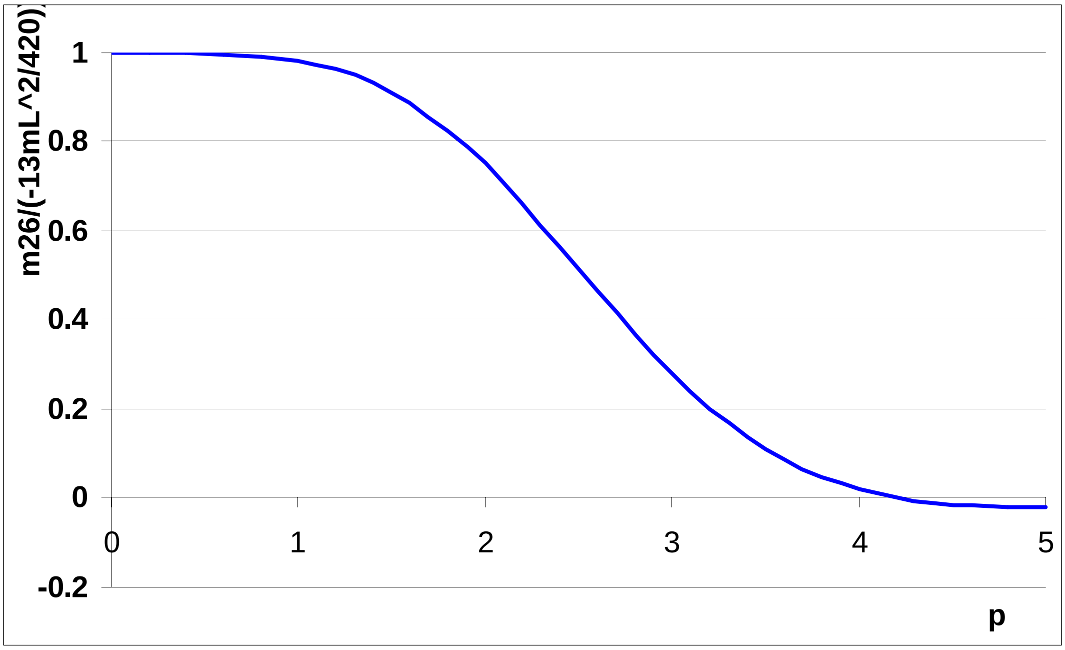


Figure 3.6: Influence of One-Parameter Foundation on the Normalized Consistent Mass Term m_{26}

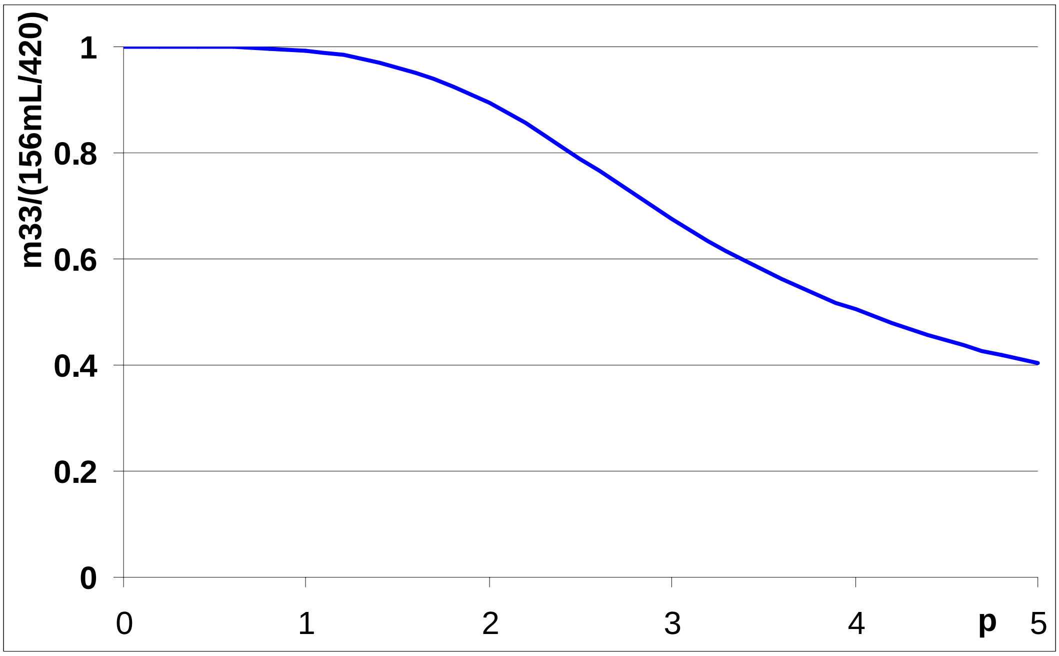


Figure 3.7: Influence of One-Parameter Foundation on the Normalized Consistent Mass Term m_{33}

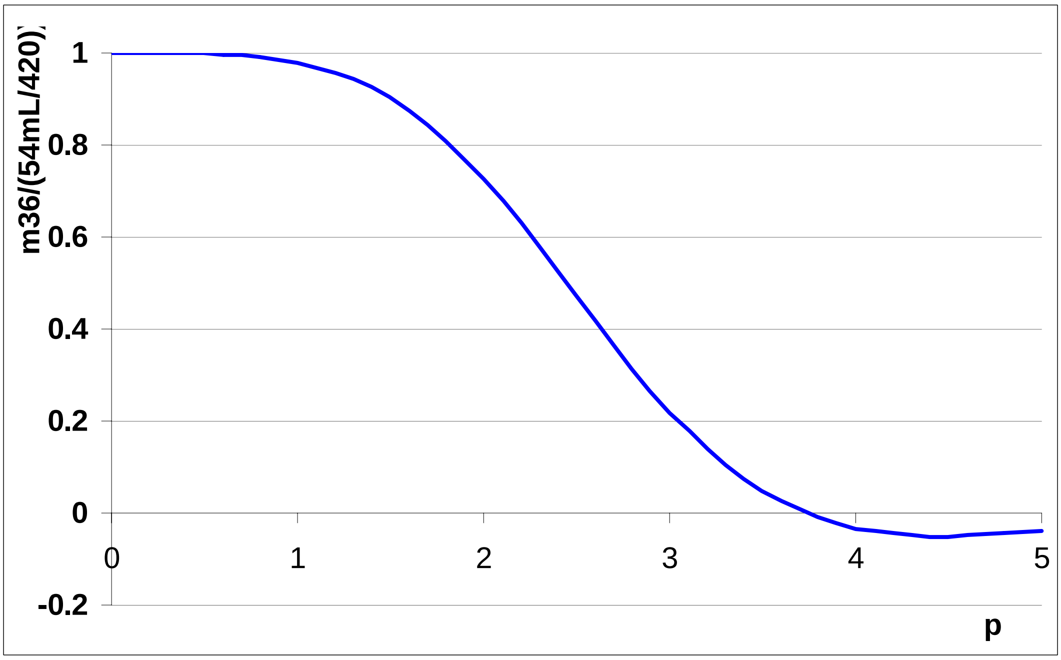


Figure 3.8: Influence of One-Parameter Foundation on the Normalized Consistent Mass Term m_{36}

3.2.2 Consistent Mass Matrix for Two-Parameter Foundation

When we substitute the proper shape functions for the beam elements resting on two-parameter for both $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ cases given in Equations (2.74) and (2.81) respectively, into Equation (3.10) it leads us to evaluate the terms of consistent mass matrices. Since the terms of the mass matrix for the two-parameter cases are too complex and extremely long functions, they are presented in Appendix A. However, for purposes of confidence in the result, by letting both of the foundation parameters tend to zero, the correctness of the terms is checked. The same conventional beam consistent mass terms are again obtained as given in Equation 3.12

The influence of the foundation parameters k_1 and k_θ on the consistent mass terms for $A < 2\sqrt{B}$ with corresponding terms of Equation (3.13) can be normalized as shown in Figures 3.9 to 3.14. Note that, as the second parameter tends to zero (i.e. $t \rightarrow 0$) the same two-dimensional curves of one-parameter case given in Figures 3.3 – 3.8 are obtained. Note that the p and t values given in the following figures are defined in Equations (2.97) and (2.98).

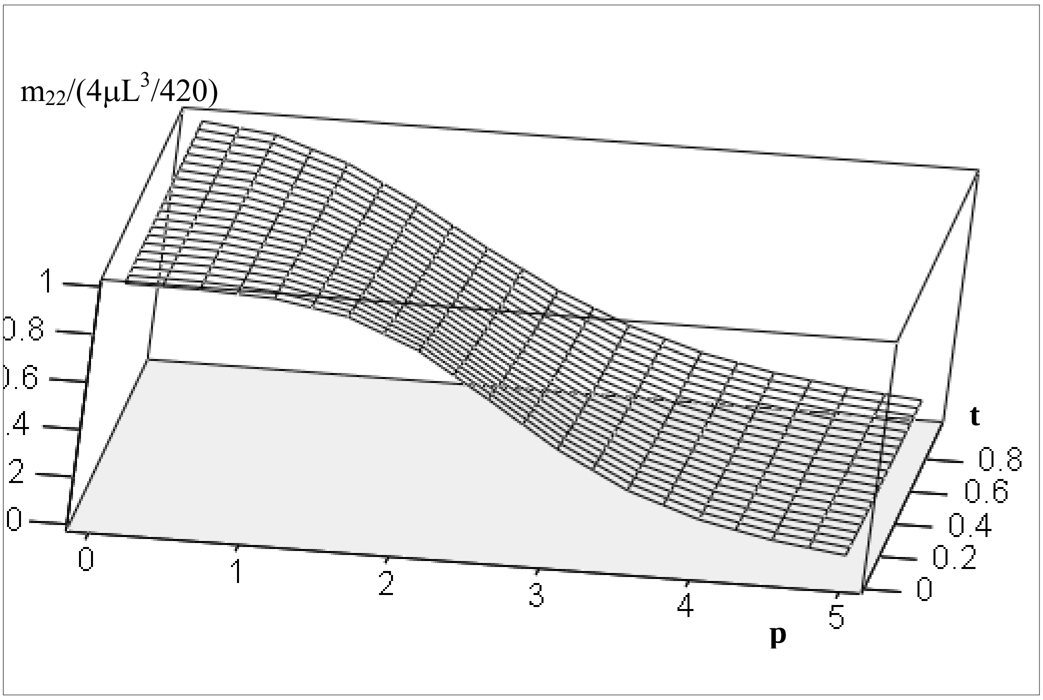


Figure 3.9: The Normalized Consistent Mass Term m_{22} for Beam Elements Resting on Two-Parameter Foundation.

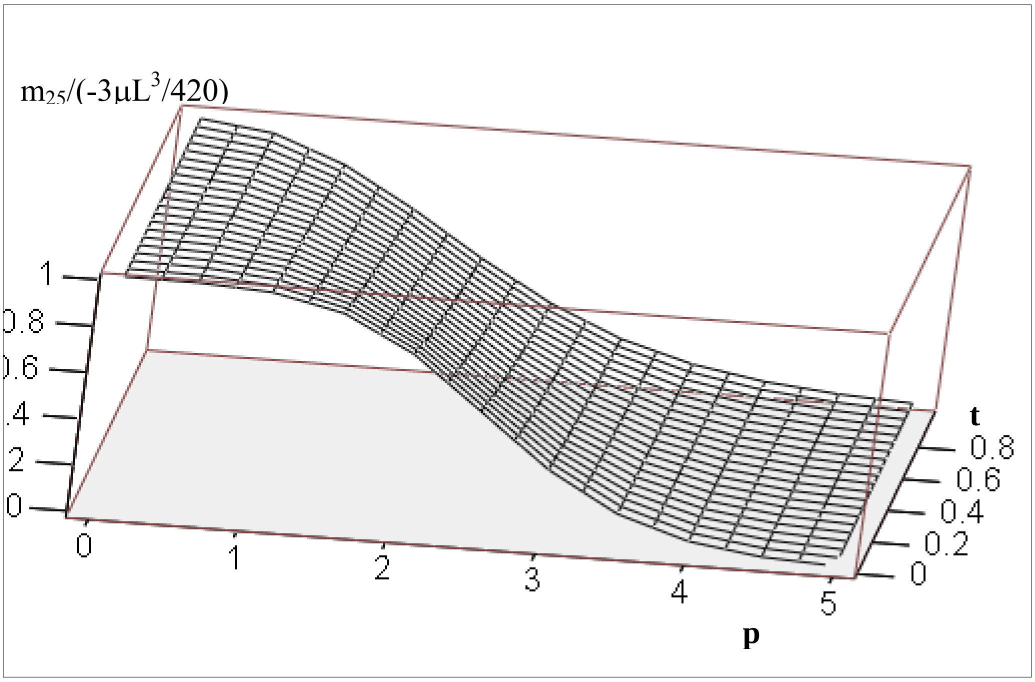


Figure 3.10: The Normalized Consistent Mass Term m_{25} for Beam Elements Resting on Two-Parameter Foundation.

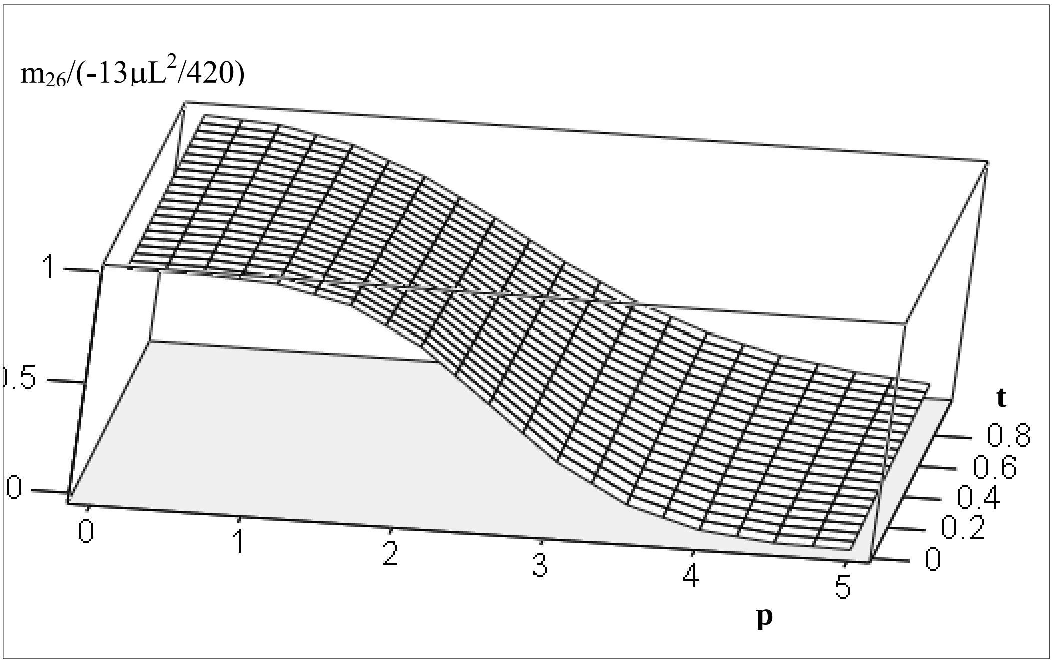


Figure 3.11: The Normalized Consistent Mass Term m_{26} for Beam Elements Resting on Two-Parameter Foundation.

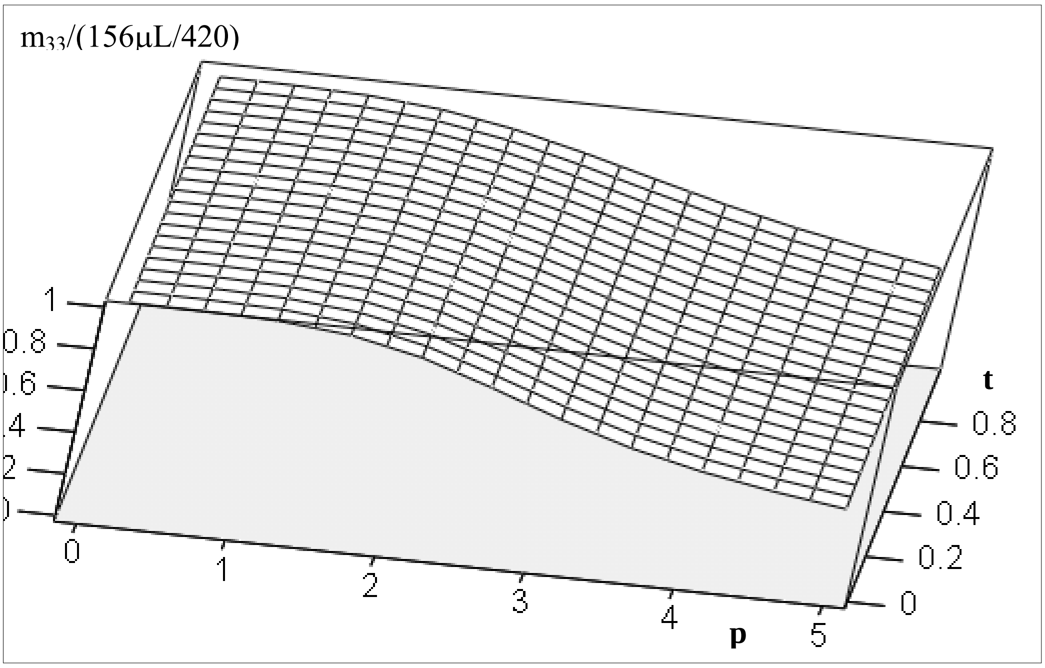


Figure 3.12: The Normalized Consistent Mass Term m_{33} for Beam Elements Resting on Two-Parameter Foundation.

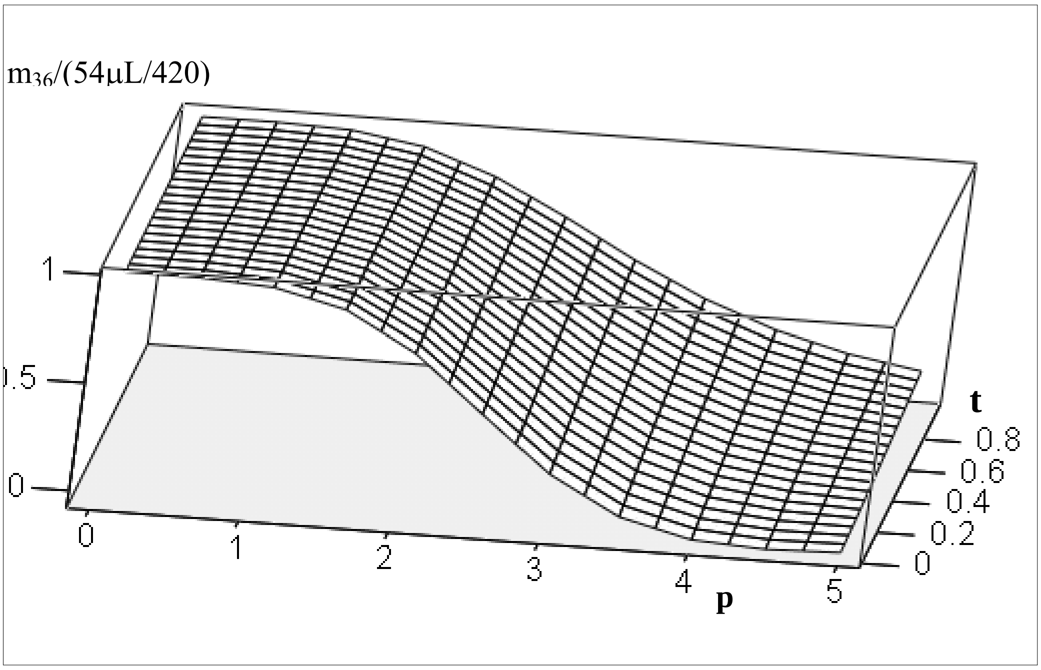


Figure 3.13: The Normalized Consistent Mass Term m_{36} for Beam Elements Resting on Two-Parameter Foundation.

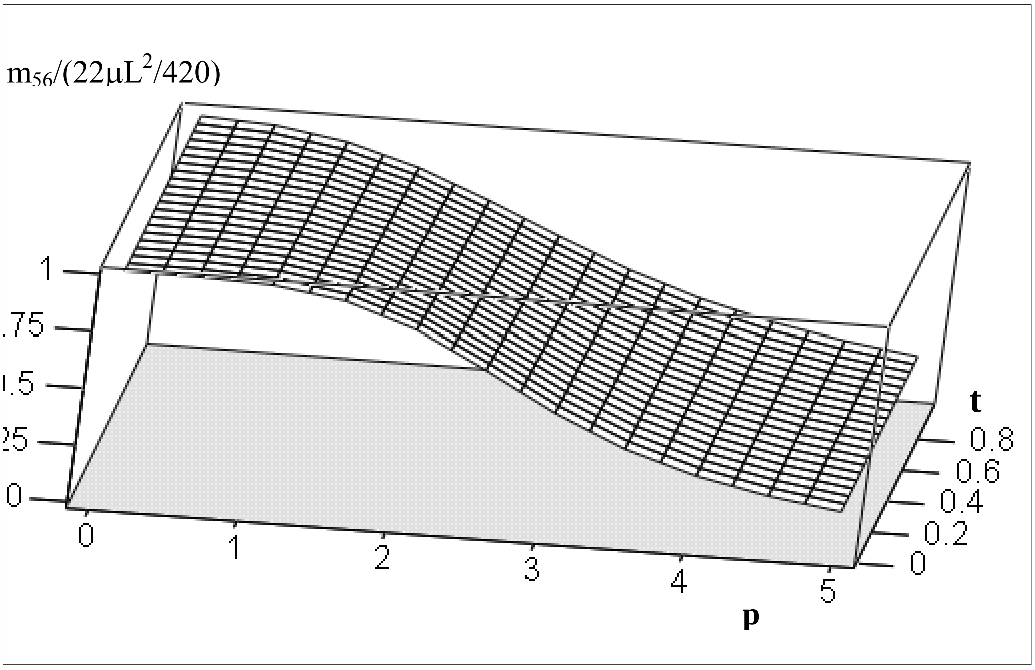


Figure 3.14: The Normalized Consistent Mass Term m_{56} for Beam Elements Resting on Two-Parameter Foundation.

3.3 CONSISTENT GEOMETRIC STIFFNESS MATRIX

As an compressive axial force applied to a beam element, it is obvious that its stiffness will reduce. The axial force influences can be included to the problem by the consistent geometric stiffness terms. It is possible to evaluate the terms, similar to the case of obtaining the consistent mass matrices, without introducing any terms due to axial force into the governing differential equation .

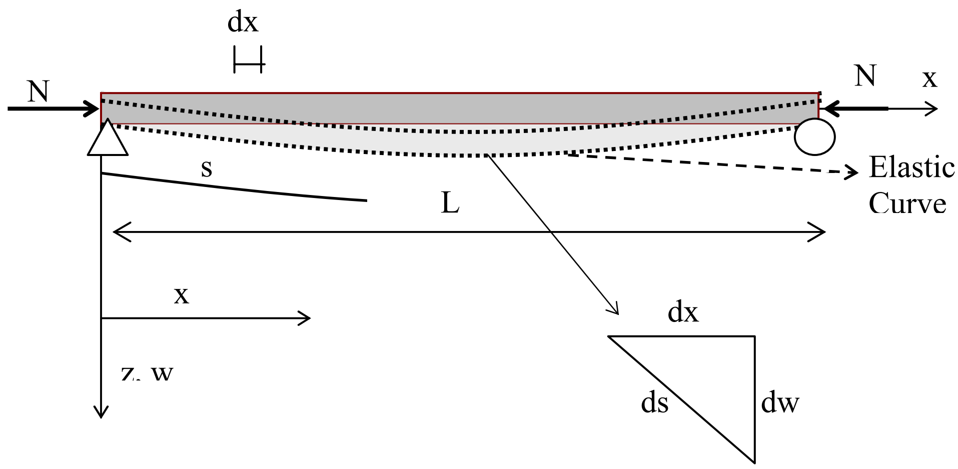


Figure 3.15: The Deformed Shape of a Simply Supported Axially Loaded Beam Element.

Consider a simply supported beam subjected to compressive axial load as shown in Figure 3.15. Due to the load the element will deformed, the change in length of the element can be obtained by the difference of the arc length and the horizontal length. From the Figure the arc length is

$$ds = \sqrt{dx^2 + dw^2} = \sqrt{1 + \left(\frac{dw}{dx}\right)^2} \tag{3.13}$$

The series solution is

$$ds = dx\left(1+\frac{1}{2}\left(\frac{dw}{dx}\right)^2 - \frac{1}{4}\left(\frac{dw}{dx}\right)^4 + ...\right) \tag{3.14}$$

Neglecting smaller terms

$$ds = dx\left(1+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) \tag{3.15}$$

Then

$$\Delta = ds - dx = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \tag{3.16}$$

Work done by the axial force N, the strain energy stored in the system, is

$$N \quad \Delta = \int_0^L N\frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \tag{3.17}$$

In this equation w can be defined as

$$w(x) = \{N\}^T \{w\} \tag{3.18}$$

where $\{w\}$ is the joint displacement vector and $\{N\}$ is the shape functions matrix of the beam element resting on one or two-parameter elastic foundation. For constant axial load, Equation (3.17) can be rewritten as:

$$\begin{aligned}\int_0^L N\frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx &= \frac{N}{2}\int_0^L \frac{dw(x)}{dx}\frac{dw(x)}{dx}dx \\ &= \frac{1}{2}\{w\}\int_0^L \{N'\}^T N \quad \{N'\}dx \quad \{w\} \\ &= \frac{1}{2}\{w\}[k_G] \quad \{w\}\end{aligned} \tag{3.19}$$

In Equation (3.19) $[k_G]$ is $\int_0^L \{N'\}^T N \quad \{N'\}dx$ represents consistent geometric stiffness matrix of the beam element. Using this equation, each terms of the matrix in general form can be evaluated by:

$$k_{G_{ij}} = N\int_0^L \psi_i' \quad \psi_j' dx = N\int_0^L \frac{d\psi_i}{dx} \quad \frac{d\psi_j}{dx} dx \tag{3.20}$$

By Equation (3.20) the consistent geometric stiffness using the proper shape functions terms for conventional beam or beam element resting on one or two parameter elastic foundations can be evaluated.

3.3.1 Consistent Geometric Stiffness Matrix for One-Parameter Foundation

The same procedures can be followed as done for determining the consistent-mass matrices to obtain the geometric stiffness terms. That is, the corresponding shape functions given in the Equations (2.29), for the beam elements resting on one-parameter elastic foundation can be substituted into Equation (3.20) to evaluate the geometric stiffness terms. After evaluating the necessary integrations, the terms will be obtained as:

$$k_{G22} = N\left(\frac{\begin{array}{l}8p(1-\cos(2p)(1-\cosh(2p)+4\sin(2p)-\\ 6\cosh(2p)\sin(2p)+\sin(4p)+4\sinh(2p)-\\ 6\cos(2p)\sinh(2p)+\sinh(4p)\end{array}}{8p(\cos(2p)+\cosh(2p)-2)^{2}}\right) \tag{3.21a}$$

$$k_{G23} = N\left(\frac{\begin{array}{l}4\sin(p)\sinh(p)(p\cosh(p)\sin(p)+\\ p\cos(p)\sinh(p)-2\sin(p)\sinh(p)\end{array}}{(\cos(2p)+\cosh(2p)-2)^{2}}\right) \tag{3.21b}$$

$$k_{G25} = N\left(\frac{\begin{array}{l}3\sin(p)\cosh(3p)-\cosh(p)\sin(3p)+3\cos(3p)\sinh(p)\\ +2p\sin(3p)\sinh(p)-\cos(p)\sinh(3p)-2p\sin(p)\sinh(3p)\end{array}}{4p(\cos(2p)+\cosh(2p)-2)^{2}}\right) \tag{3.21c}$$

$$k_{G26} = N\left(\frac{\begin{array}{l}-4p\sin(p)\sinh(p)\cosh(p)\sin(p)+\\ p\cos(p)\sinh(p)-2\sin(p)\sinh(p)\end{array}}{(\cos(2p)+\cosh(2p)-2)^{2}}\right) \tag{3.21d}$$

$$k_{G33} = N\left(\frac{\begin{array}{l} p(8p(\cos(2p)-\cosh(2p))+6\cosh(2p)\sin(2p) \\ +4\sinh(2p)-6\cos(2p)\sinh(2p)+\sinh(4p) \\ -8p\sin(2p)\sinh(2p)-4\sin(2p)-\sin(4p)) \end{array}}{4(\cos(2p)+\cosh(2p)-2)^2}\right) \tag{3.21e}$$

$$k_{G36} = N\left(\frac{\begin{array}{l} \mathrm{p(2p(cos(p)cosh(3p) - cosh(p)cos(3p)) -} \\ \mathrm{3cosh(3p)sin(p)} + s\mathrm{in(3p)cosh(p) - cos(p)sinh(3p)} \\ \mathrm{+ 3cos(3p)sinh(p) + 16psin(p)sinh(p))} \end{array}}{2(\cos(2p)+\cosh(2p)-2)^2}\right) \tag{3.21f}$$

where N is constant axial compressive force and $p = \lambda L = \sqrt[4]{\frac{k_1}{4EI}} L$

When foundation parameter k_1 tends to zero (or $p \rightarrow 0$), the terms in Equation (3.21) must reduce to the conventional beam consistent mass terms obtained by Hermitian functions. The correctness of the terms is verified that the terms reduce to the following conventional terms in matrix form.

$$\underset{p\rightarrow 0}{Lim}\left[k_G\right] = \frac{N}{30L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4L^2 & -3L & 0 & -L^2 & 3L \\ 0 & -3L & 36 & 0 & -3L & -36 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -L^2 & -3L & 0 & 4L^2 & 3L \\ 0 & 3L & -36 & 0 & 3L & 36 \end{bmatrix} \tag{3.22}$$

The normalized terms represent the influence of the foundation parameter k_1 on the geometric stiffness terms given in Equation (3.21) and corresponding terms of the matrix given in Equation (3.22) are portrayed in Figures 3.16 to 3.21.

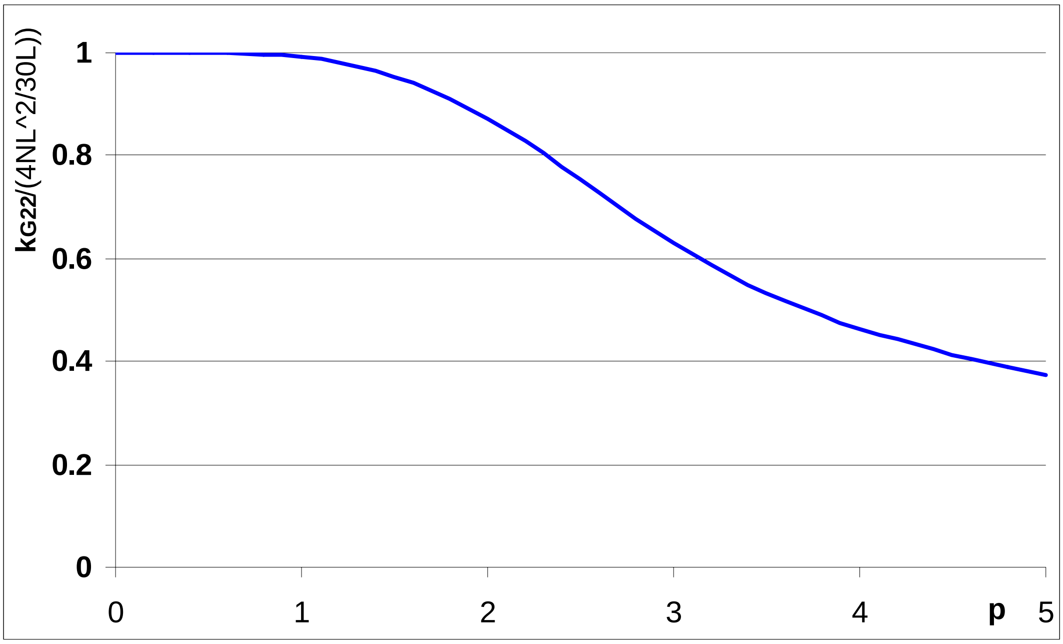


Figure 3.16: The Normalized Consistent Geometric Stiffness Term k_{G22} for Beam Elements Resting on One-Parameter Foundation.

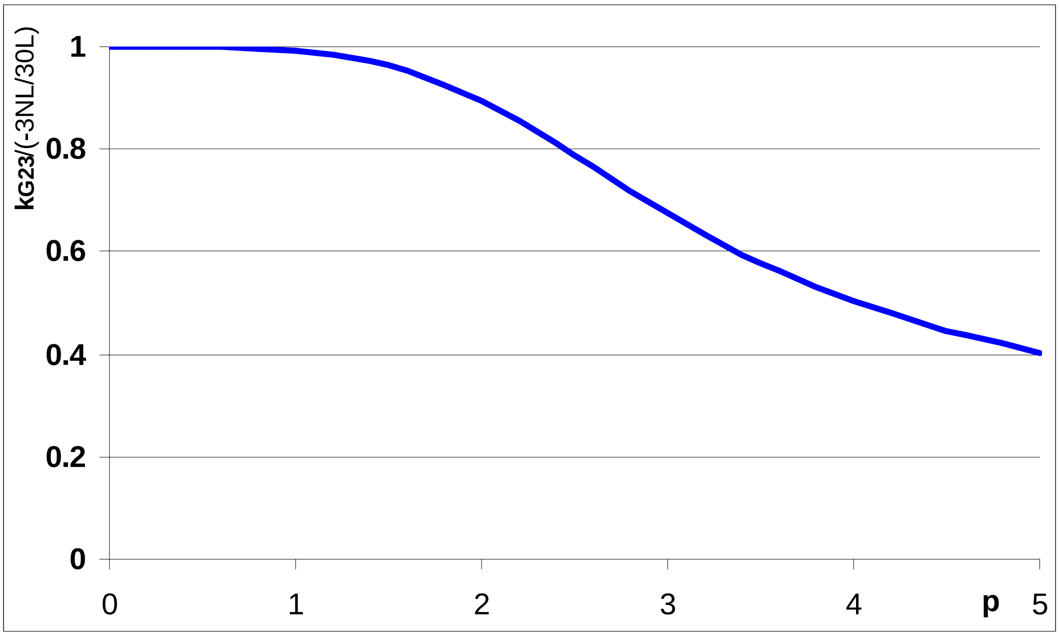


Figure 3.17: The Normalized Consistent Geometric Stiffness Term k_{G23} for Beam Elements Resting on One-Parameter Foundation.

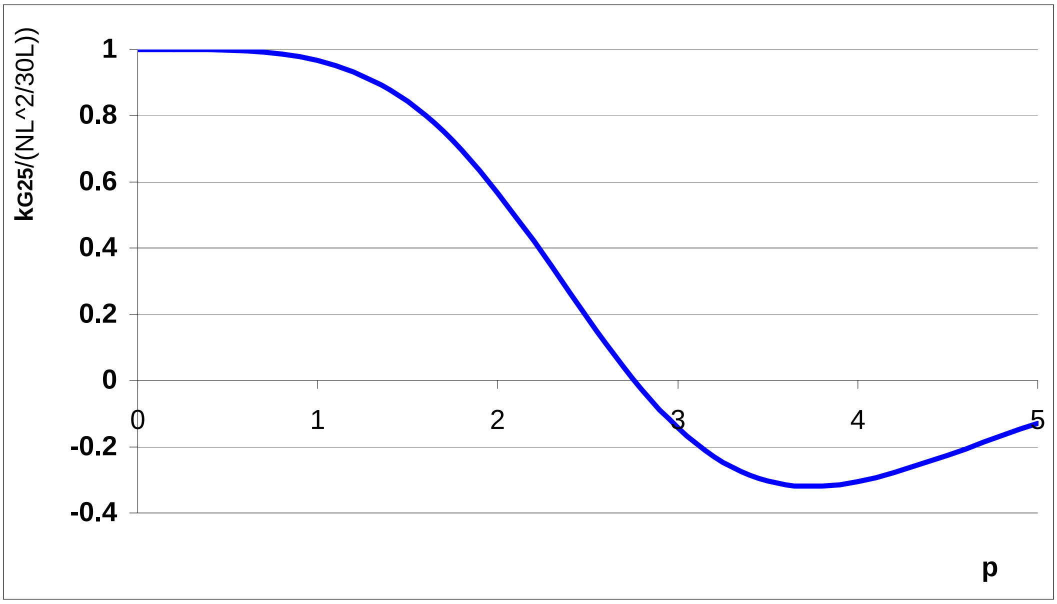


Figure 3.18: The Normalized Consistent Geometric Stiffness Term k_{G25} for Beam Elements Resting on One-Parameter Foundation

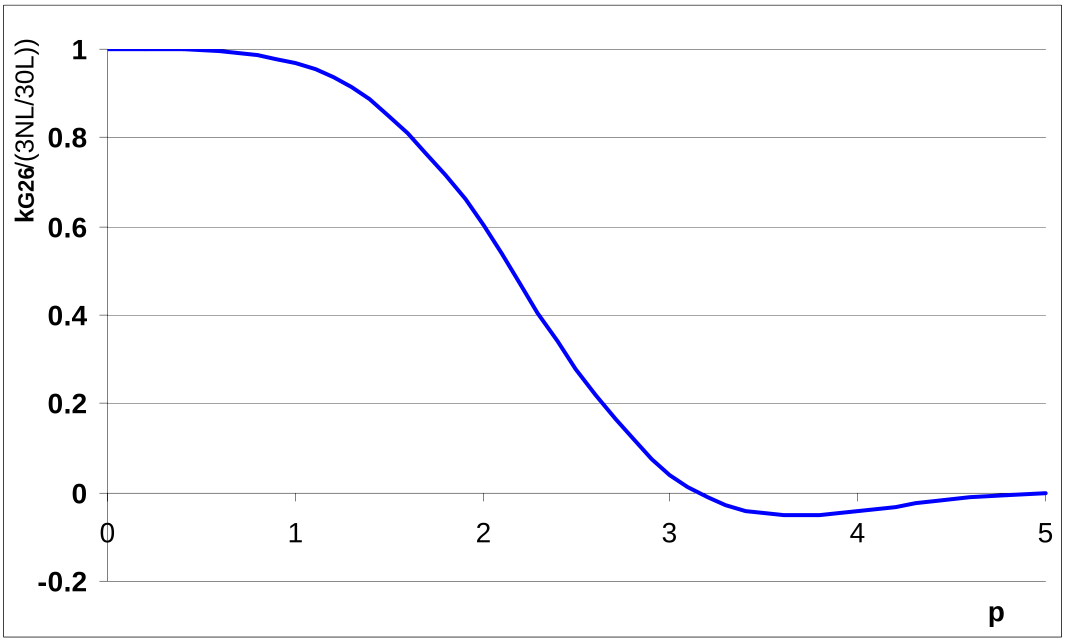


Figure 3.19: The Normalized Consistent Geometric Stiffness Term k_{G26} for Beam Elements Resting on One-Parameter Foundation

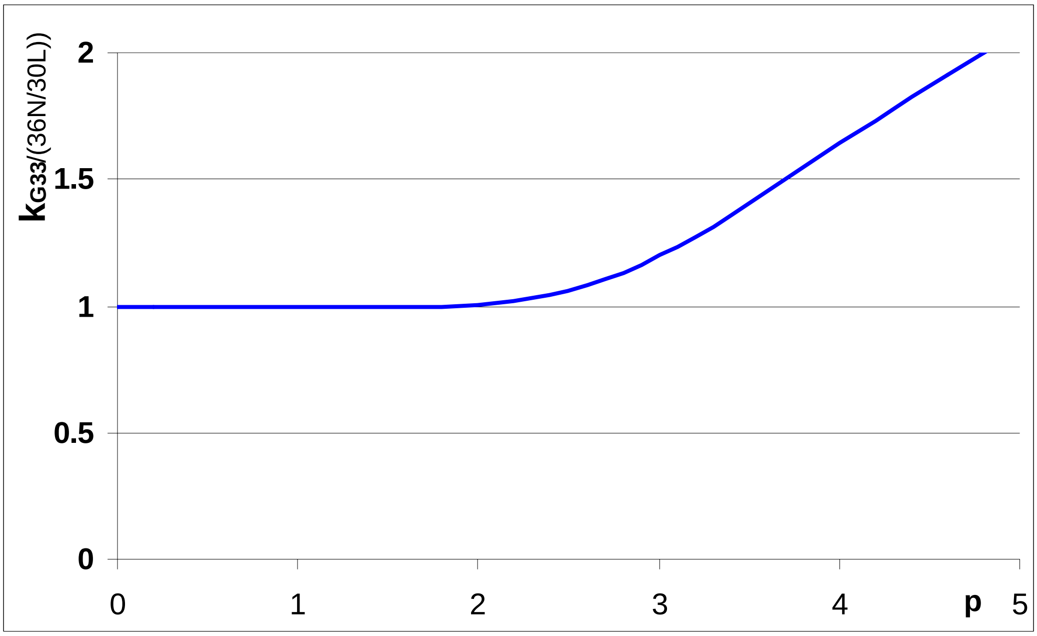


Figure 3.20: The Normalized Consistent Geometric Stiffness Term k_{G33} for Beam Elements Resting on One-Parameter Foundation

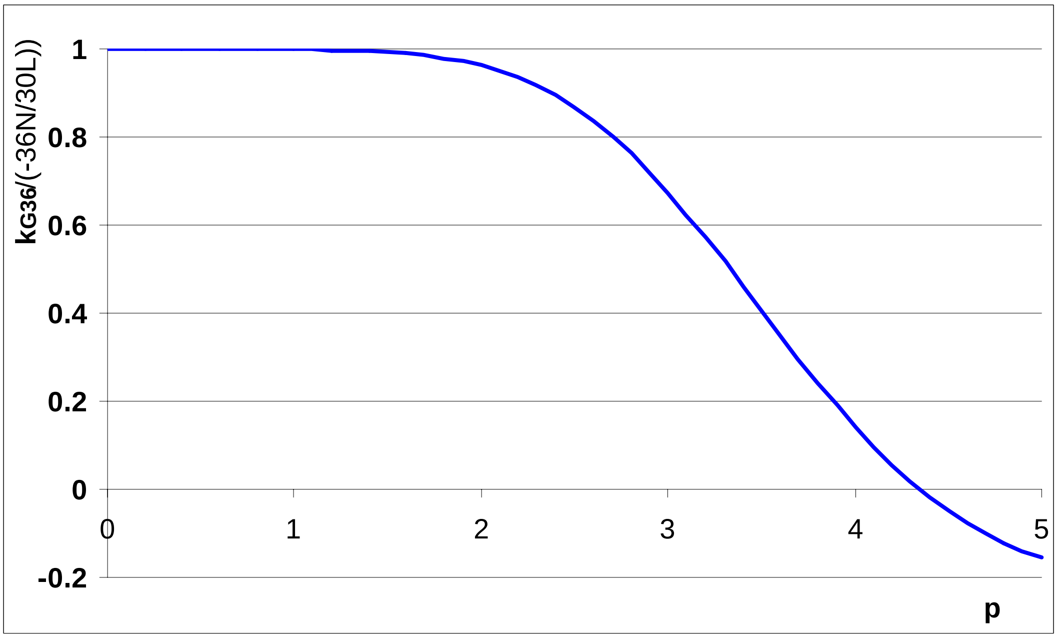


Figure 3.21: The Normalized Consistent Geometric Stiffness Term k_{G36} for Beam Elements Resting on One-Parameter Foundation.

3.3.2 Consistent Geometric Stiffness Matrix for Two-Parameter Foundation

To obtain the geometric stiffness terms, it is necessary to reuse the same shape functions with same procedures again. That is, the corresponding shape functions given in the Equations (2.74) and (2.81) for the beam elements resting on two-parameter elastic foundation can be substituted into Equation (3.20) to evaluate the geometric stiffness terms. After evaluating the necessary integrations, the terms are obtained. Because of long expressions of the terms for the two-parameter cases, they are presented in Appendix B. The terms are verified by letting both of the foundation parameters tend to zero. The same conventional beam geometric stiffness terms are again obtained as given in Equation 3.22.

The influence of the foundation parameters k_1 and k_θ on the consistent mass terms for $A < 2\sqrt{B}$ with corresponding terms of Equation (3.13) can be normalized as shown in Figures 3.22 to 3.27. Note that, as the second parameter tends to zero (t $\rightarrow$0) the same two-dimensional curves of one-parameter case given in Figures 3.16 – 3.21 are obtained. The p and t values given in the following figures, represents the influence of the foundation parameters, are defined in Equations (2.97) and (2.98).

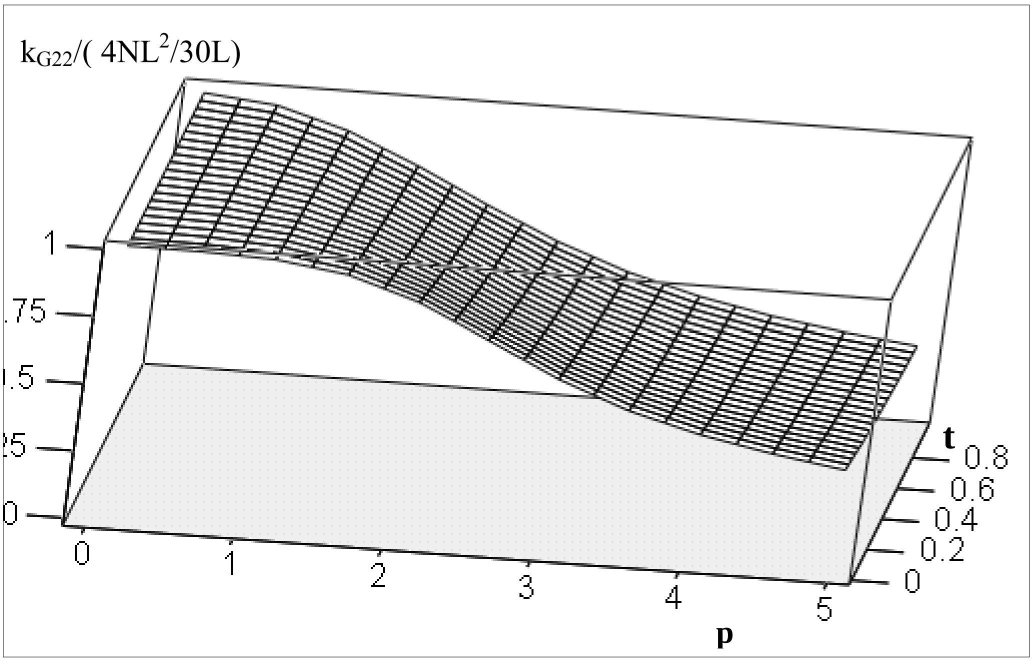


Figure 3.22: The Normalized Consistent Stiffness Term k_{G22} for Beam Elements Resting on Two-Parameter Foundation

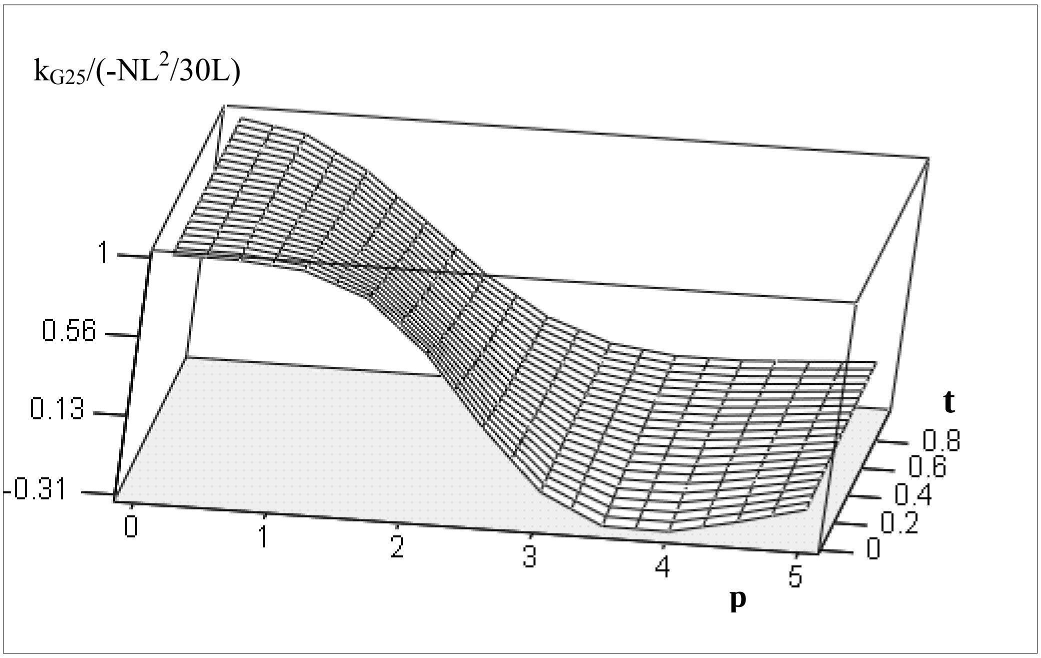


Figure 3.23: The Normalized Consistent Stiffness Term k_{G25} for Beam Elements Resting on Two-Parameter Foundation

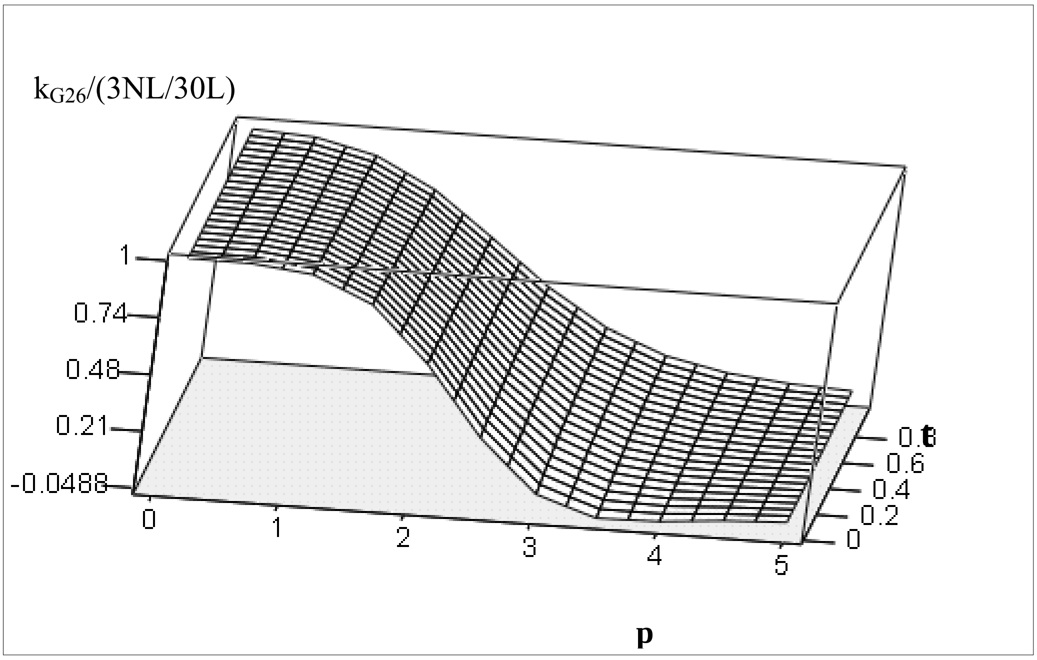


Figure 3.24: The Normalized Consistent Stiffness Term k_{G26} for Beam Elements Resting on Two-Parameter Foundation

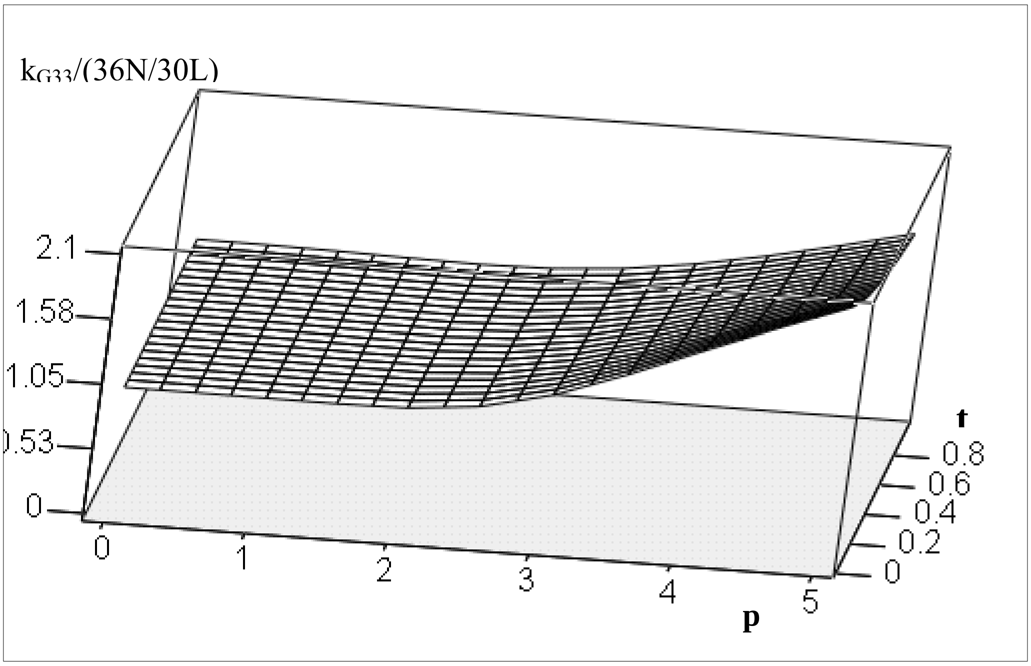


Figure 3.25: The Normalized Consistent Stiffness Term k_{G33} for Beam Elements Resting on Two-Parameter Foundation

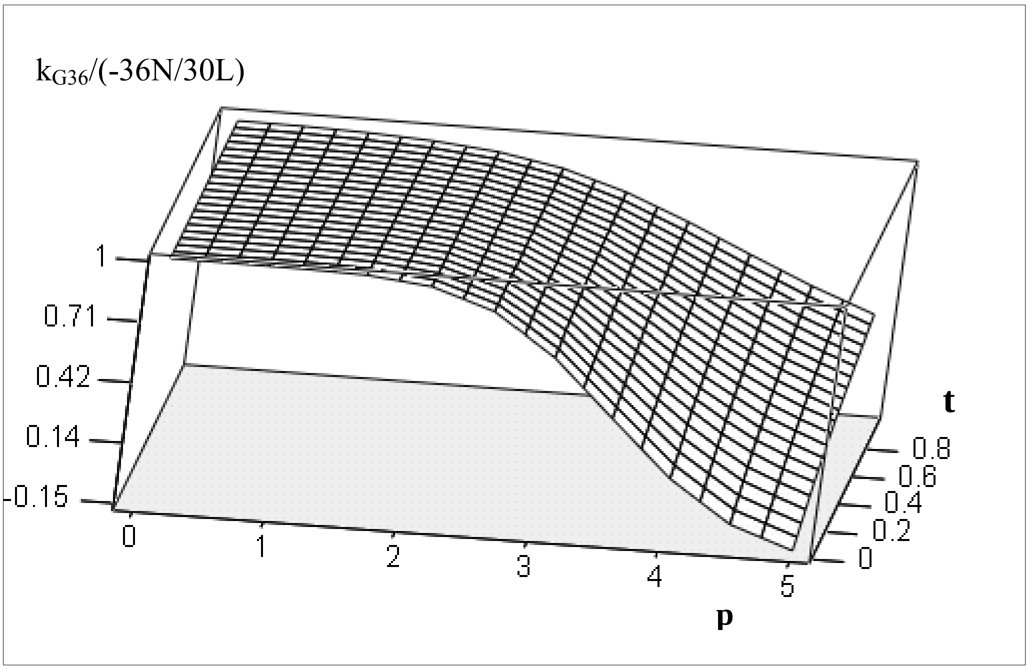


Figure 3.26: The Normalized Consistent Stiffness Term k_{G36} for Beam Elements Resting on Two-Parameter Foundation

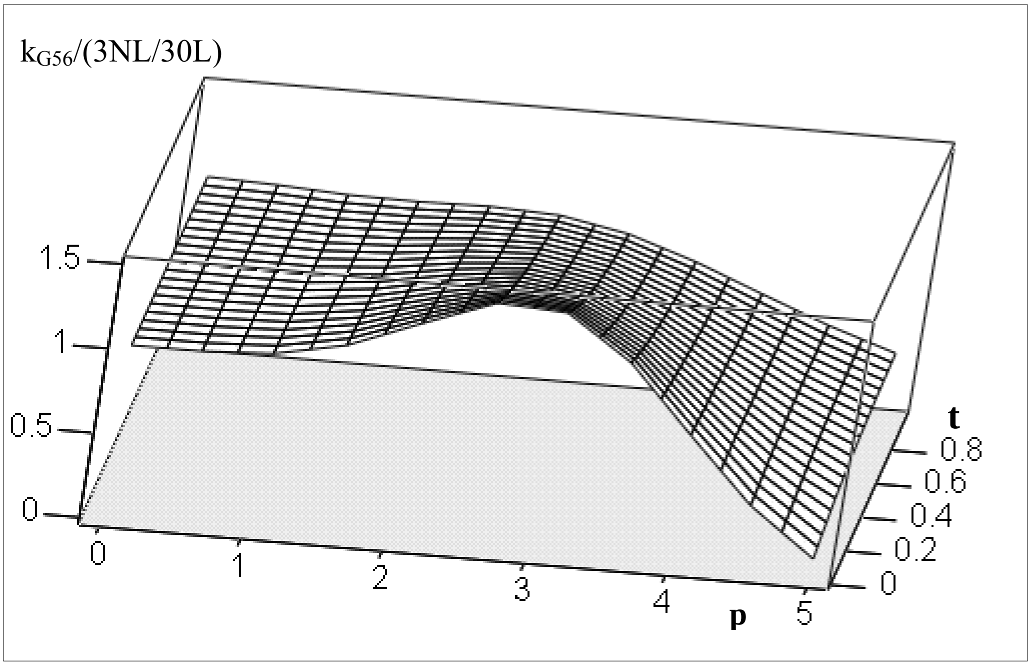


Figure 3.27: The Normalized Consistent Stiffness Term k_{G56} for Beam Elements Resting on Two-Parameter Foundation

CHAPTER 4

DISCRETIZED PLATES ON GENERALIZED FOUNDATIONS

4.1 INTRODUCTION

Since the structural behavior of a beam resembles that of a strip in a plate (Wilson, 2000), the framework method that replaces a continuous surface by an idealized discrete system can represent a two-dimensional plate. The representation of a plate through the lattice analogy at which the discrete elements are connected at finite nodal points is shown in Figure (4.1). The phrase “lattice analogy” has been used among others for this representation. The plate is modeled as an assemblage of individual beam elements interconnected at their neighboring joints. Therefore the exact fixed end forces and stiffness matrices obtained in Chapter 2 and the exact consistent mass and geometric stiffness matrices derived in Chapter 3 for conventional beam elements and beam elements resting on one or two parameter foundation are valuable tools to solve general plate vibration, buckling and bending problems.

By this representation, the plate problems including buckling and free vibration, which have non-uniform thickness and foundation properties, arbitrary boundary and loading conditions and discontinuous surfaces, can be solved in a general form. Of course as Hrennikof (1949) stated the system cannot truly be equal to the continuous structure. However apart from errors associated with torsional and discretization effects sufficient accuracy can be obtained.

4.2 REPRESENTATION OF PLATES BY BEAM ELEMENTS

In order to simplify the problem a rectangular plate can be represented by two sets of intersecting beam elements as a simple version of three dimensional structure connected at finite nodal points as shown in Figure 4.1. It is not necessary to have the elements intersect at right angles. That is the replacement implies that there are rigid intersection joints between all sets of beam elements, ensuring slope continuity. Because of plane rigid intersection, the elements can resist torsion as well as bending moment and shear. Therefore the idealized discrete element as shown in Figure 4.2 can be replaced with a beam element that has 3 DOF at each node.

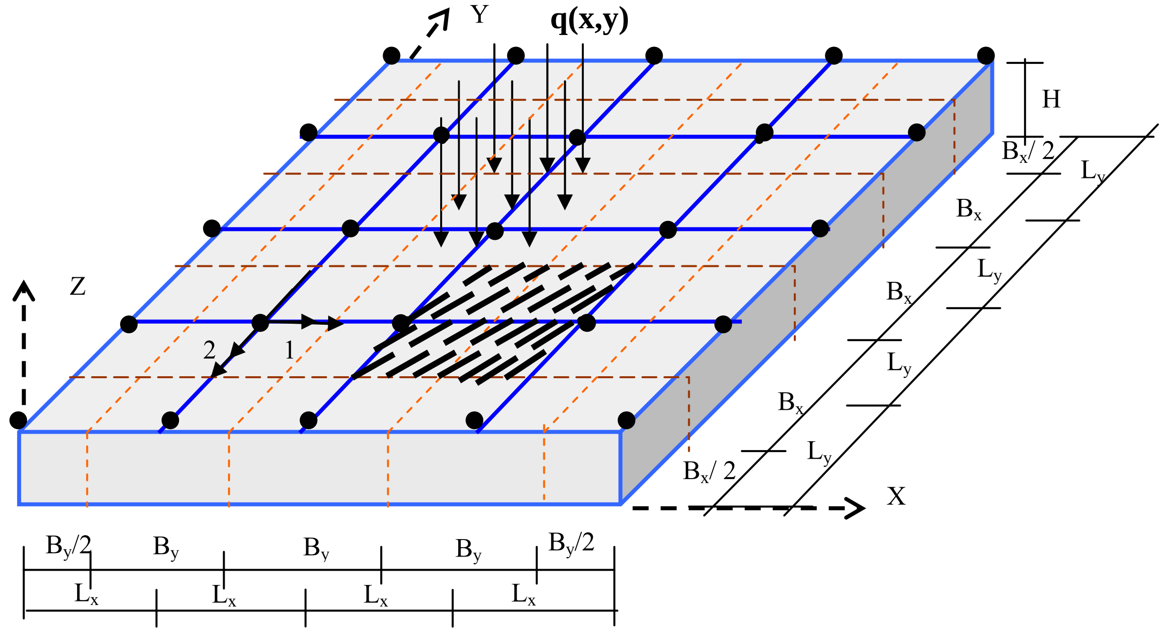


Figure 4.1: Idealized Discrete System at Which the Elements Are Connected at Finite Nodal Points of a Rectangular Thin Plate

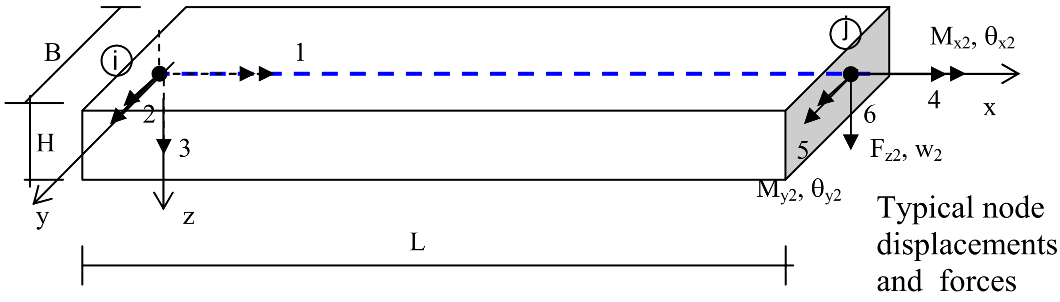


Figure 4.2: Local Coordinates for a Grid Element

The main advantage of this method is that plate problems, which may have complex loading and boundary conditions, can be represented as assemblies of the individual beam elements. If suitable stiffness coefficients can be provided, the accuracy of the method will be high. Since the element stiffness matrices of the discrete beam element resting on one or two parameter elastic foundations have already been determined in Chapter 2, the method can be extended to solve the plates resting on generalized foundation problems.

4.3 ASSEMBLY OF DISCRETIZED PLATE ELEMENTS

In gridwork systems two or three elements are connected along external or internal peripheries. At interior nodes four typical discrete individual beam elements as shown in Figure 4.2 intersect. Matrix displacement method based on stiffness-matrix approach is a very useful tool to solve gridworks with arbitrary load and boundary conditions. It can be defined as a horizontal frame structure with rigid joints whose members and joints lie in a common plane. The applied loads are usually normal to the plane of the structure as limited by the degrees of freedom directions as shown in Figures 4.3 and 4.4.

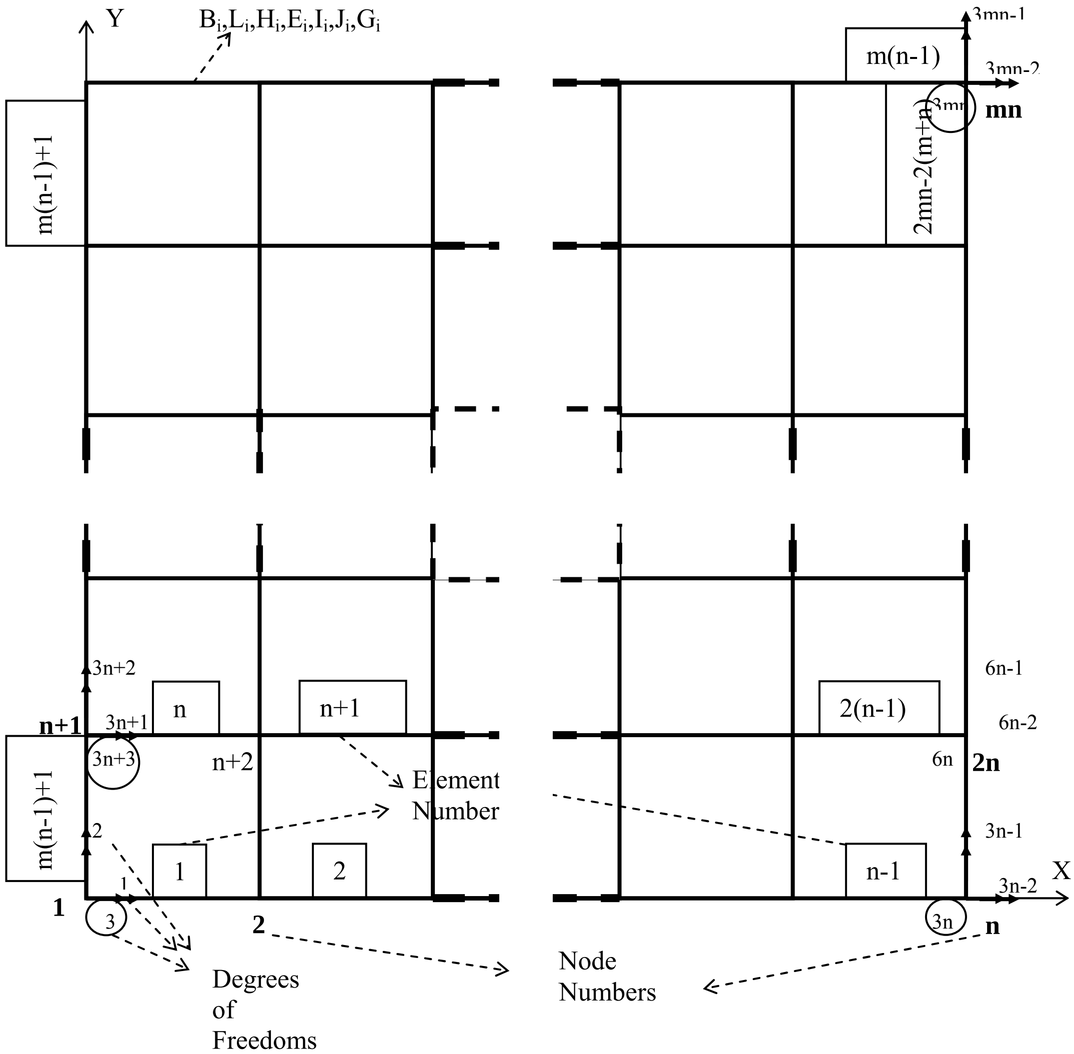


Figure 4.3: Typical Numbering of Nodes, DOF's and Elements of a Rectangular Plate

Consider a typical member from a structural grid as shown in Figure 4.4 with the ends of the member denoted by i and j . The local axes of the member are x, y, z and the global axes (previously defined in Figure 4.3) are X, Y, Z. The possible end deformations of the element are a joint translation in z-direction and the torsional and bending rotations, respectively about x- and y- axes. That is, the degrees of freedoms (possible end deformations) of the element at i are two rotations, 1 and 2, and one translation, 3, at j they are similarly 4 and 5 for rotations and 6 for translation.

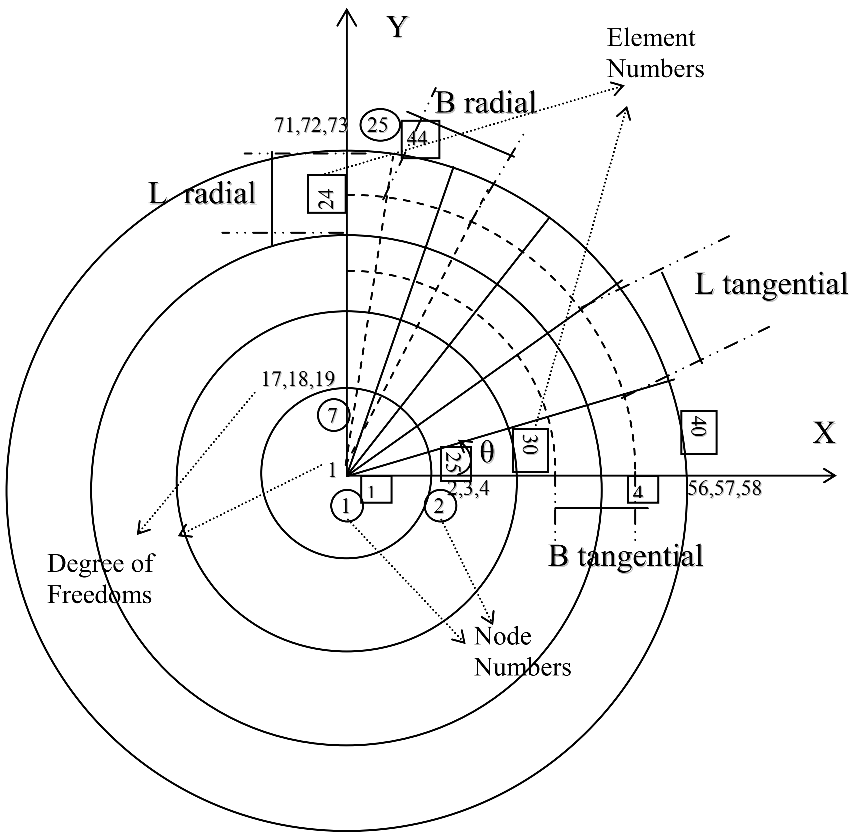


Figure 4.4: Typical Numbering of Nodes, DOF's and Elements of a Quarter of a Circular Plate

By using a proper numbering scheme to collect all displacements for each nodal point in a convenient sequence the stiffness matrix of the system shown in Figures 4.3 for circular grids and Figure 4.4 for rectangular grids can be generated as follows:

$$\underline{k}_{sys} = \sum_{i=1}^{NE} \underline{a}_i^{\,T} \underline{k}_i \ \underline{a}_i \tag{4.1}$$

where i is the individual element number, NE is the number of elements, $\underline{a}_i$ is the individual transformation matrix, $\underline{k}_i$ is the proper element stiffness matrix for a conventional beam element as given by Equation (2.43), for a beam element resting on one-parameter elastic foundation as given by Equation (2.42) and two-parameter elastic foundation as given by Equations (2.89) and (2.89) and $\underline{k}_{sys}$ is the stiffness matrix of the total structure.

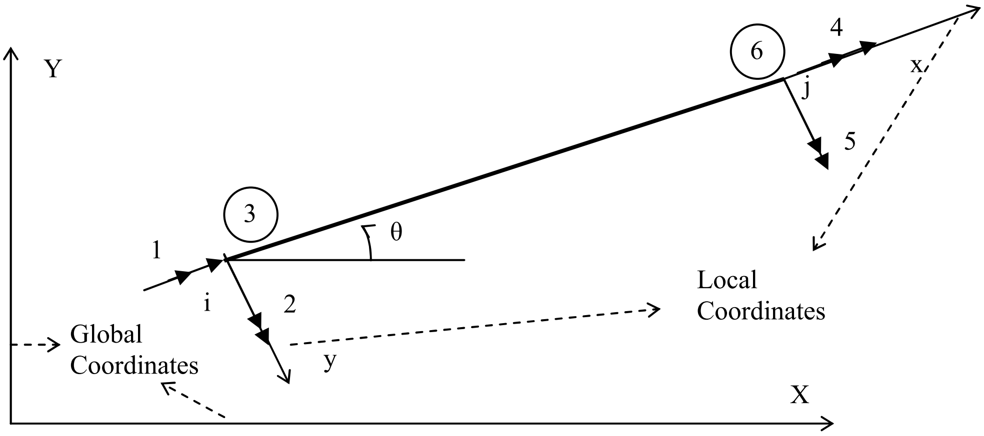


Figure 4.5: Transformation of the Degrees of Freedom of a Typical Plane Element from Local (x, y, z) Coordinates to the Global (X, Y, Z) Coordinates

Since the local positive direction of the beam elements have been defined previously, from Figure 4.5 the transformation matrix of an arbitrary plane element will be

$$\underline{a}_i = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ S & -C & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & S & -C & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.2}$$

where C = Cos(θ) and S = Sin(θ)

Similar to determining the system stiffness matrix, for buckling problems the system geometric stiffness matrix can be obtained as follows:

$$\underline{k}_{Gsys} = \sum_{i=1}^{NE} \underline{a}_i^{\,T} \underline{k}_{Gi} \, \underline{a}_i \tag{4.3}$$

where $\underline{k}_{Gi}$ is the proper element geometric stiffness matrix. For a conventional beam element this is given by Equation (3.22), and for a beam element resting on one-parameter elastic foundation Equation (3.21) and two-parameter elastic foundation is given in Appendix B and $\underline{k}_{Gsys}$ is the geometric stiffness matrix of the total structure.

The compressive axial loads decrease the effective stiffness of the structure. The critical load P, from the well-known equation of eigenvalue analysis must be found from

$$(\underline{k}_{sys} - \lambda \ \underline{k}_{Gsys})\underline{w} = 0 \tag{4.4}$$

The set of λ values that satisfy the above equation to be zero are called the eigenvalues of the problem, while the corresponding displacement vector $\underline{w}$ expresses the corresponding shapes of the buckling system known as the eigenvectors or mode shapes. The lowest eigenvalue can be defined as the first buckling load. For vibration problems the system consistent mass matrix is

$$\underline{M}_{sys} = \sum_{i=1}^{NE} \underline{a}_i^{T} \underline{M}_i \ \underline{a}_i \tag{4.5}$$

where $\underline{M}_i$ is the proper element consistent mass matrix for a beam. Depending on the number of foundation parameters, it is given by Equation 3.12, 3.11, or Appendix A. $\underline{M}_{sys}$ is the consistent mass matrix for the system.

The equations of motion for a system in a free vibration as an eigenvalue problem may be written

$$(\underline{k}_{sys} - \omega^2 \underline{M}_{sys})\underline{w} = 0 \tag{4.6}$$

where the quantities ω^2 are the eigenvalues indicting the square of free vibration frequencies that satisfy the above Equation, while the corresponding displacement vector $\underline{w}$ express the fitting shapes of the vibrating system as the eigenvectors of mode shapes.

CHAPTER 5

CASE STUDIES

5.1 INTRODUCTION

In order to check the validity of the solution techniques an example of a plane grid that consist of rigid attached mutually perpendicular beams without any foundation in a horizontal plane will be shown. After verifying the method widely different plate problems will be examined.

There are examples of a wide range of plates (such as; plate analysis, grid analysis, plates on one-parameter elastic foundation and plates on two- parameter elastic foundation) were solved by the finite grid solution. Comparison with known analytical and other numerical solutions yields accurate results as an approximate method. In addition the method developed for plates (extend beam elements to the plates) on generalized foundation is also applicable to slabs, girders and mat foundations in bridge and building structures.

5.2 SAMPLE PROBLEM FOR PLANE GRID SUBJECTED TO TRANSVERSE LOADS

The first study is to analyze the plane-grid system solved by Wang (1970) shown in Figure 5.1. The system is a monolithic reinforced-concrete plate simply supported on four columns at A, B, C and D. the values of flexural and torsional rigidities for all elements are EI=288000 kip-ft^2 and GJ=79142.4 kip-ft^2 respectively.

Two loading conditions are to be investigated: 1) a 10-kip concentrated load applied at H and (LC1) and 2) a uniform load of 3 klf on the element BF (LC2).

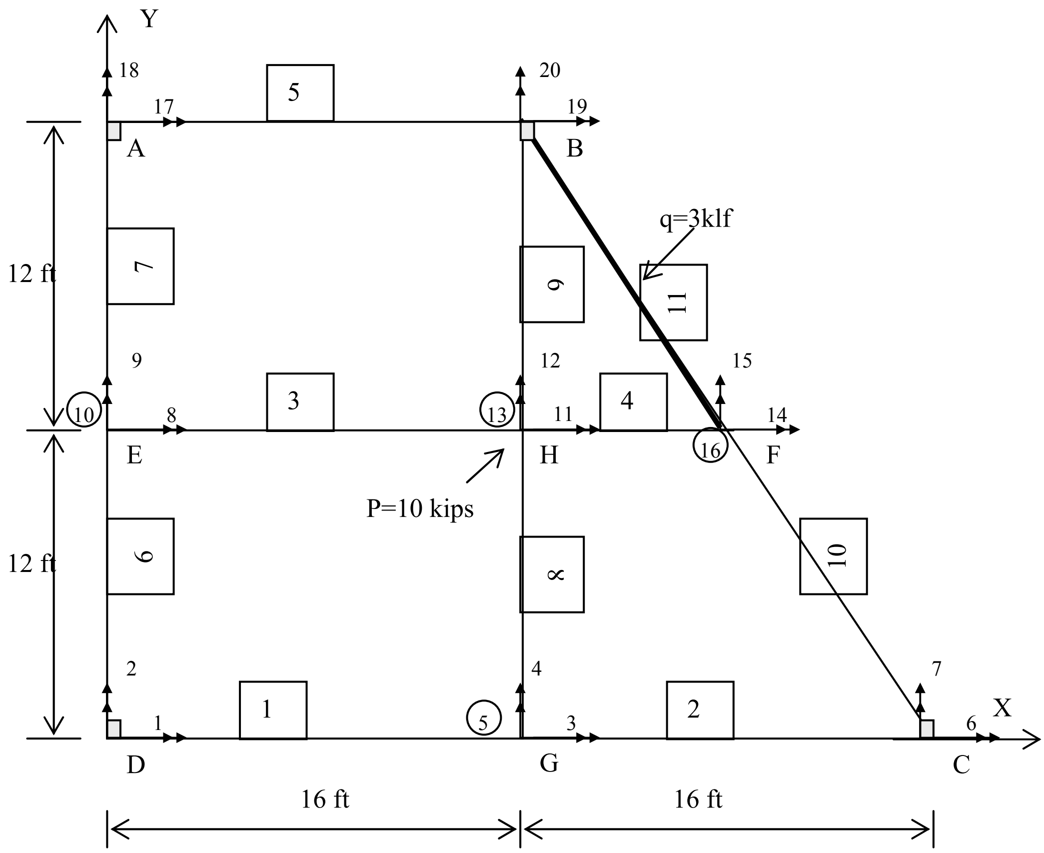


Figure 5.1: The Given Grid System

There are 8 nodes, 11 elements and 20 degrees of freedoms in the Figure. The element internal forces and the displacements values of the reference for both loading conditions are compared with the Finite Grid Solution and they are tabulated in Tables 5.1 and 5.2, respectively.

Table 5.1: The Comparison of the End Forces for the Present Study, Finite Grid Method (FGM) with Wang (1970)

COMPARISON OF INTERNAL FORCES							
El. No		LC1			LC2		
		L. End M.	R. End M.	Tors. M.	L. End M.	R. End M.	Tors. M.
1	Ref.	0.2245	20.1033	0.4221	0.6877	26.6887	3.5376
	FGM	0.22432	20.1	0.42159	0.68671	26.67	3.5349
2	Ref.	-19.7902	0.753	-0.1456	-33.3826	-0.4439	6.6647
	FGM	-19.785	0.75496	-0.14548	-33.357	-0.43532	6.6642
3	Ref.	2.761	23.1228	-1.2437	6.0058	-8.286	-0.8249
	FGM	2.7607	23.128	-1.2435	6.0043	-8.2632	-0.82376
4	Ref.	-23.4002	0.3811	0.2419	16.4602	24.026	10.2151
	FGM	-23.408	0.38659	0.24008	16.424	24	10.206
5	Ref.	-2.9855	1.2746	-2.528	-6.6936	-49.4419	-9.9166
	FGM	-2.985	1.2754	-2.522	-6.691	-49.436	-9.9123
6	Ref.	-0.4221	11.8008	0.22432	-3.5376	6.2845	0.68671
	FGM	-0.42159	11.802	0.22432	-3.5349	6.2884	0.68671
7	Ref.	-10.557	2.5228	2.985	-5.4596	9.9166	6.691
	FGM	-10.558	2.522	2.985	-5.4646	9.9123	6.691
8	Ref.	0.5677	28.9562	0.31442	-3.1271	49.0296	-6.6871
	FGM	0.56707	28.948	0.31442	-3.1293	48.992	-6.6871
9	Ref.	-30.4418	-4.9497	0.034322	-60.0696	43.5324	1.4741
	FGM	-30.432	-4.9445	0.034322	-60.022	43.554	1.4741
10	Ref.	0.2966	22.6675	0.70886	5.2992	92.2964	-4.0588
	FGM	0.29773	22.663	0.70886	5.3034	92.266	-4.0588
11	Ref.	-22.6776	2.7461	0.25402	-174.4697	32.9234	10.249
	FGM	-22.678	2.7422	0.25402	-174.46	32.909	10.249

From Table 5.1, apart from errors that may be associated by rounding the input numbers, error of the forces is less than 0.05% and from Tables 5.2 the displacements are exact. The results obtained are almost the same as the reference values. The results those can be accepted as exact are valuable for checking the correctness of the method.

Table 5.2: The Comparison of the Displacements for the Present Study, Finite Grid Method (FGM) with Wang (1970)

COMPARISON OF THE DISPLACEMENTS				
DOF NO:	LC1		LC2	
	Reference	FGM	Reference	FGM
1	-2.5330E-04	-2.5331E-04	-2.2107E-04	-2.2107E-04
2	5.5371E-04	5.5371E-04	7.8485E-04	7.8485E-04
3	-3.3854E-04	-3.3854E-04	-9.3573E-04	-9.3573E-04
4	1.6224E-06	1.6224E-06	6.3081E-05	6.3081E-05
5	-5.9481E-03	-5.9481E-03	-8.8099E-03	-8.8099E-03
6	-3.0913E-04	-3.0913E-04	-2.2830E-03	-2.2830E-03
7	-5.6894E-04	-5.6894E-04	-8.5142E-04	-8.5142E-04
8	1.3421E-06	1.3421E-06	-1.6423E-05	-1.6423E-05
9	5.1970E-04	5.1970E-04	6.8072E-04	6.8072E-04
10	-1.9860E-03	-1.9860E-03	-1.5397E-03	-1.5397E-03
11	2.5274E-04	2.5274E-04	1.5012E-04	1.5012E-04
12	-4.6053E-05	-4.6053E-05	1.0770E-03	1.0770E-03
13	-7.6928E-03	-7.6928E-03	-1.5435E-02	-1.5435E-02
14	2.2847E-04	2.2847E-04	-8.8162E-04	-8.8162E-04
15	-3.6579E-04	-3.6579E-04	9.7183E-04	9.7183E-04
16	-5.6048E-03	-5.6048E-03	-2.4379E-02	-2.4379E-02
17	2.7385E-04	2.7385E-04	3.0393E-04	3.0393E-04
18	6.7087E-05	6.7087E-05	-3.3383E-04	-3.3383E-04
19	7.8373E-04	7.8373E-04	2.3079E-03	2.3079E-03
20	-5.1258E-05	-5.1258E-05	8.5352E-04	8.5352E-04

5.3 SAMPLE BENDING PROBLEMS FOR RECTANGULAR PLATES

5.3.1 Comparison with Boundary Element Method for Simply Supported Rectangular Plate on Two-Parameter Foundation

A rectangular plate with uniform thickness, h = 0.05 m, and sides of length A = 1 m, B = 0.5 m, was solved first with a concentrated central loading F= 3000 kN, and then with a uniformly-distributed loading of intensity q = 6000 kN/m^2. The material properties are: $E = 2.1x10^8$ kN/m^2, v = 0.3, and the foundation parameters are: $k_1 = 6.48 \times 10^4$ kN/m^3, $k_\theta = 2250.0$ kN/m. The results were evaluated at internal nodes on the longer central line of the rectangle, and non-dimensional parameters w/w_0 were defined such that, for concentrated loading w_0=FAB/D; and for distributed loading $w_0=q(AB)^2/D$.

The distributions of non-dimensional parameter: w/w_0 and M_x/M_0 for the two cases of loading are plotted against the corresponding boundary element solutions solved by EI-Zafrany (1996), as shown in Figures 5.2 - 5.4. However, the three-dimensional view of deflections for concentrated loading at the centre and uniform distributed loading are plotted in Figures 5.5 and 5.8, respectively.

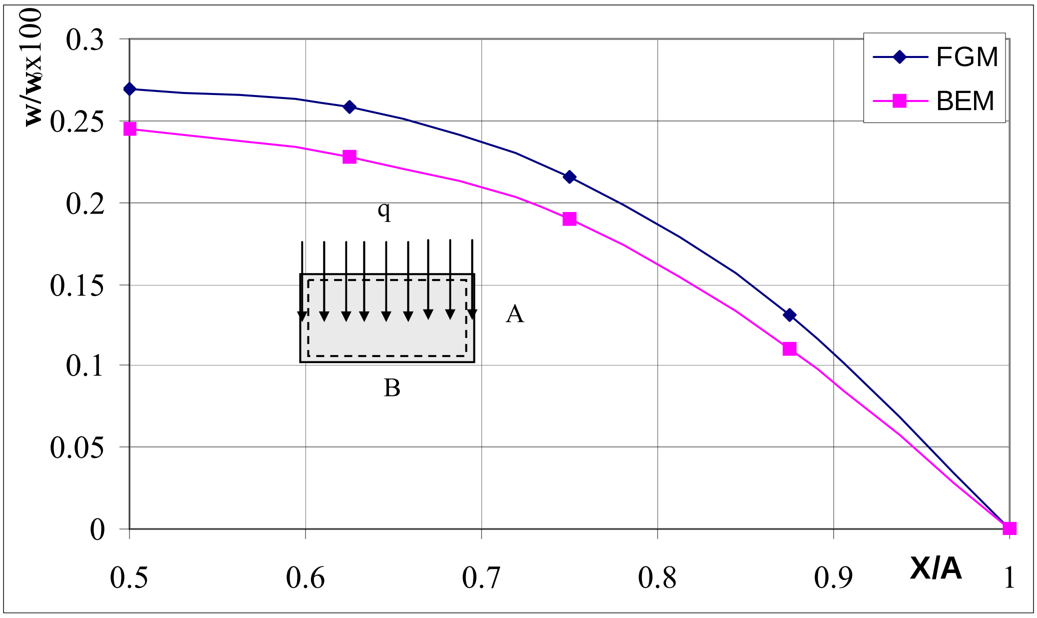


Figure 5.2: Comparison of FGM to BEM for Deflection along the Centerline of the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation under Uniform Distributed Load

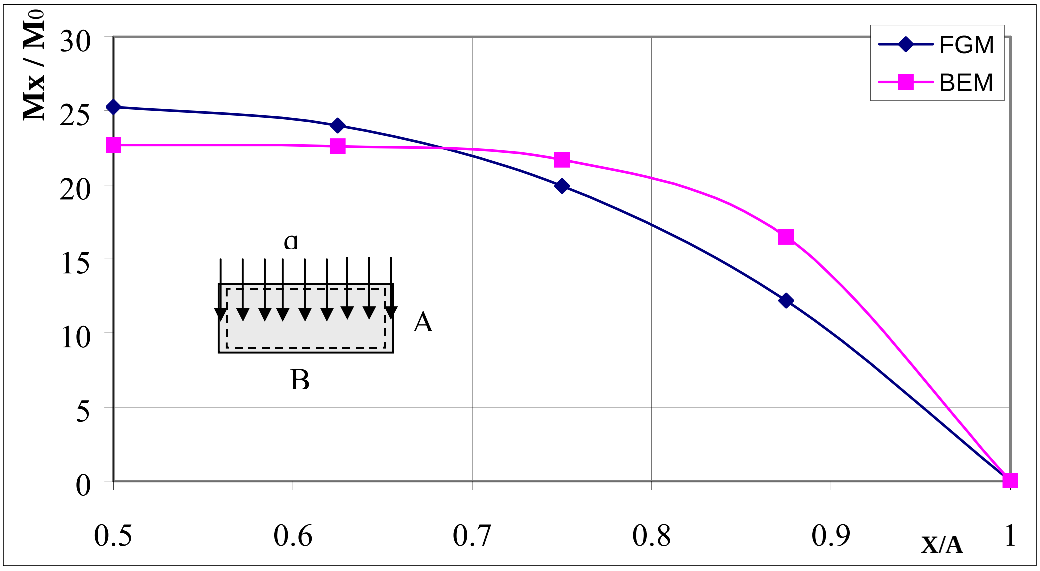


Figure 5.3: Comparison of FGM to BEM for Moment Mx Along the Centerline of the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation under Uniform Distributed Load

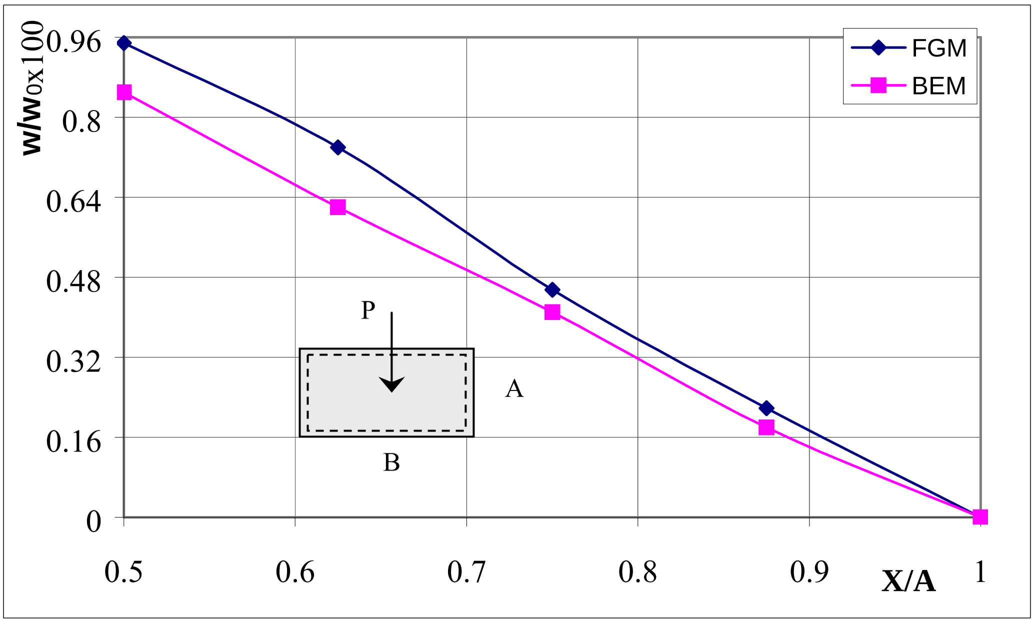
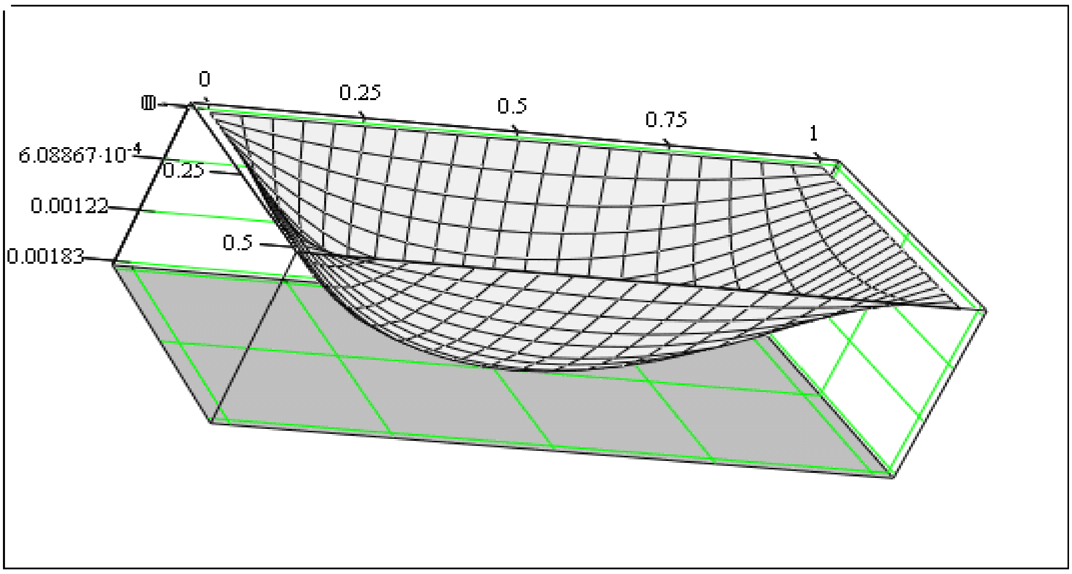
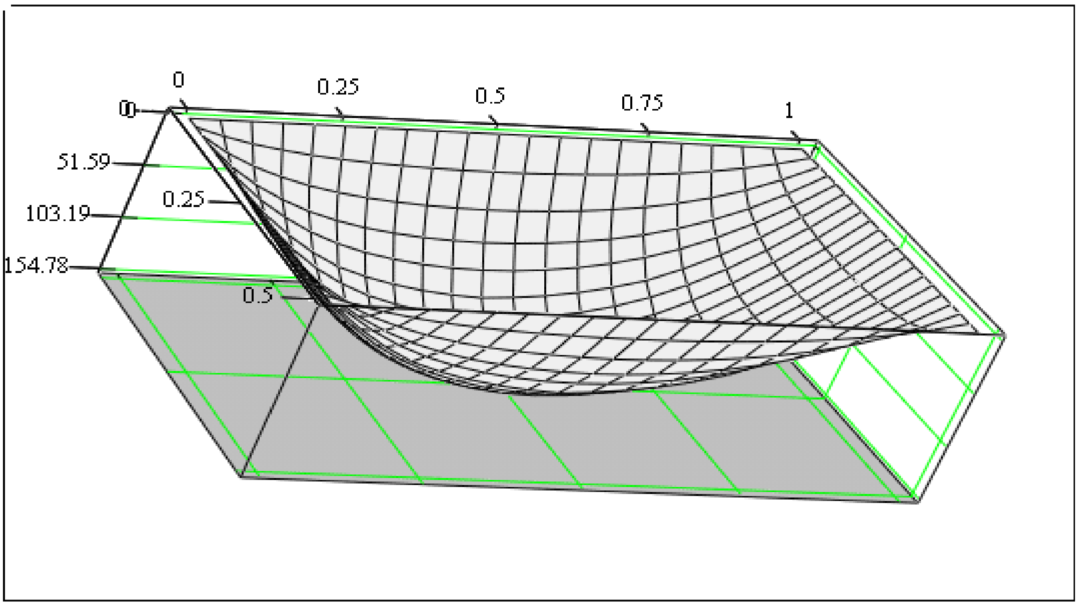


Figure 5.4: Comparison of FGM to BEM for Deflection Along the Centerline of the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation Subjected to a Concentrated Load at the Center



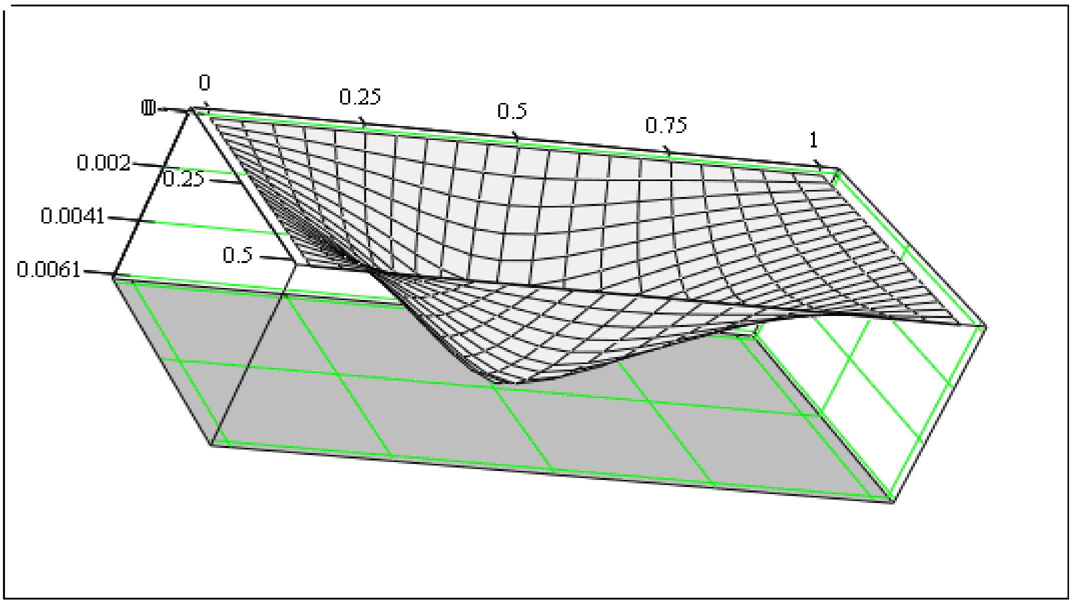
(x, y, z)

Figure 5.5: Three Dimensional Deflection View of the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation Subjected to the Uniform Distributed Load



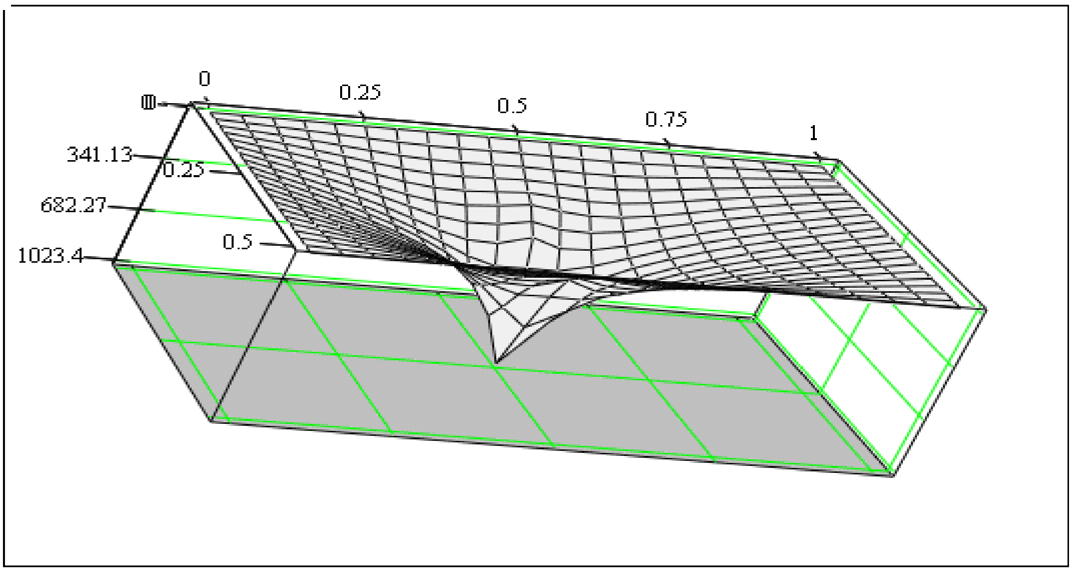
(x, y, z)

Figure 5.6: Three Dimensional View of Moment Mx Values in the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation Subjected to the Uniform Distributed Load



(x, y , z)

Figure 5.7: Three Dimensional Deflection View of the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation Subjected to a Concentrated Load at the Center



(x, y , z)

Figure 5.8: Three Dimensional View of Moment Mx Values in the Simple Supported Rectangular Plate on Two-Parameter Elastic Foundation Subjected to a Concentrated Load at the Center

5.3.2 Comparison with Meshless Local Boundary Integral Equation Method for Simply Supported and Clamped Square Plates on Two-Parameter Foundation

A simply supported and a clamped square plate subjected to a uniformly distributed load will be considered. In the reference (Sladek et al., 2002), the side length a, the flexural rigidity D and Poisson ration v were chosen as 8 m, 1000 Nm and 0.3 respectively. The uniformly distributed load q was taken as 1N/mm2.

Firstly for the simple supported case, Winkler and Pasternak foundations is considered. The comparison of the FGM results with the local boundary integral equation method (LBIE) on the centreline of the plate for three different Winkler coefficients is given in Table 5.3. From the table one can see that the maximum relative error for deflections of points located on the axis passing through the centre of the plate is about less than 1%. This reflects a high degree of accuracy. The deflection of the centreline of the plate for three different Winkler coefficients is shown in Figure 5.9.

Table 5.3: The Comparison of the Deflections at the Centreline for a Simply Supported Plate Resting on a Winkler Foundation with the LBIE

coordinate (m)	k_1=100 N/m^3			k_1=300 N/m^3			k_1=500 N/m^3		
	LBIE (mm)	FGM (mm)	Relative Error %	LBIE (mm)	FGM (mm)	Relative Error %	LBIE (mm)	FGM (mm)	Relative Error %
0	7.925	7.933	0.25	3.751	3.719	0.84	2.399	2.373	1.08
0.8	7.596	7.6138	0.12	3.622	3.596	0.73	2.331	2.309	0.94
1.6	6.604	6.62	0.28	3.211	3.202	0.30	2.103	2.092	0.50
2.4	4.95	4.99	0.80	2.472	2.481	0.36	1.657	1.66	0.19
3.2	2.683	2.727	1.64	1.376	1.392	1.14	0.944	0.952	0.89
4	0	0	-	0	0	-	0	0	-

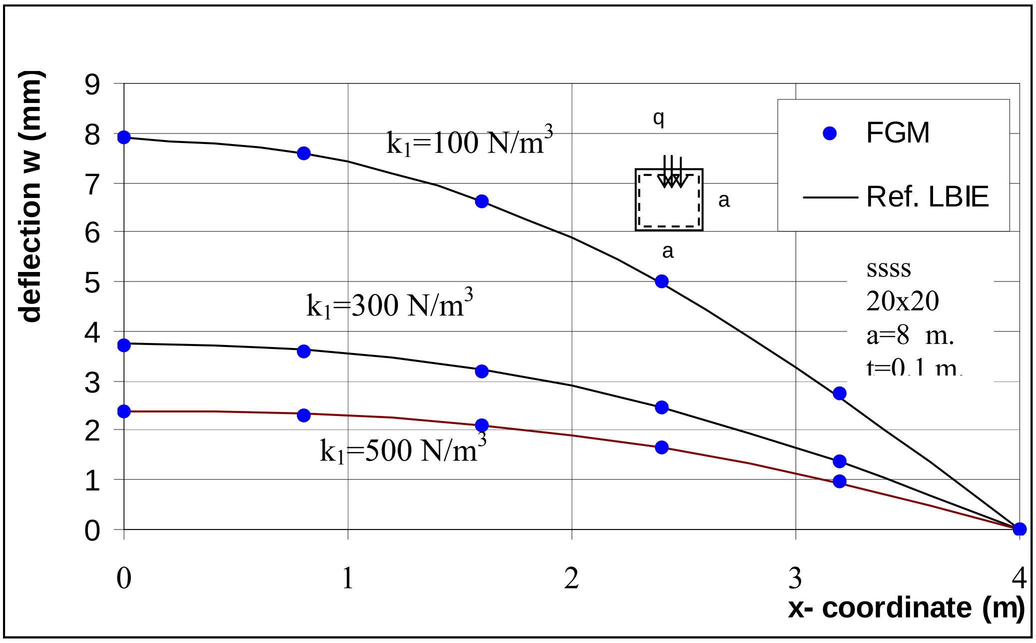


Figure 5.9: Comparison of Deflections at the Centreline of the Simple Supported Square Plate on Winkler Foundation with the LBIE results

One the other hand for the two-parameter foundation case, the numerical results of the maximum deflection w_{max} are given in Tables 5.4. The relative error is also less than 1 % as for the Winkler foundation. Then the accuracy is high and comparable with that for Winkler model. The influence of the variation of foundation parameters on the maximum deflections is shown in Figure 5.10.

Table 5.4: The Comparison of the Maximum Deflections for a Simply Supported Plate Resting on the Two-Parameter Foundation with the LBIE

coefficients		LBIE (mm)	FGM (mm)	Relative Error %
k_1(N/m^3)	k_2 (N/m)			
100	100	6.8147	6.7913	0.34
300	300	3.0276	3.0034	0.80
500	500	1.911	1.8941	0.88

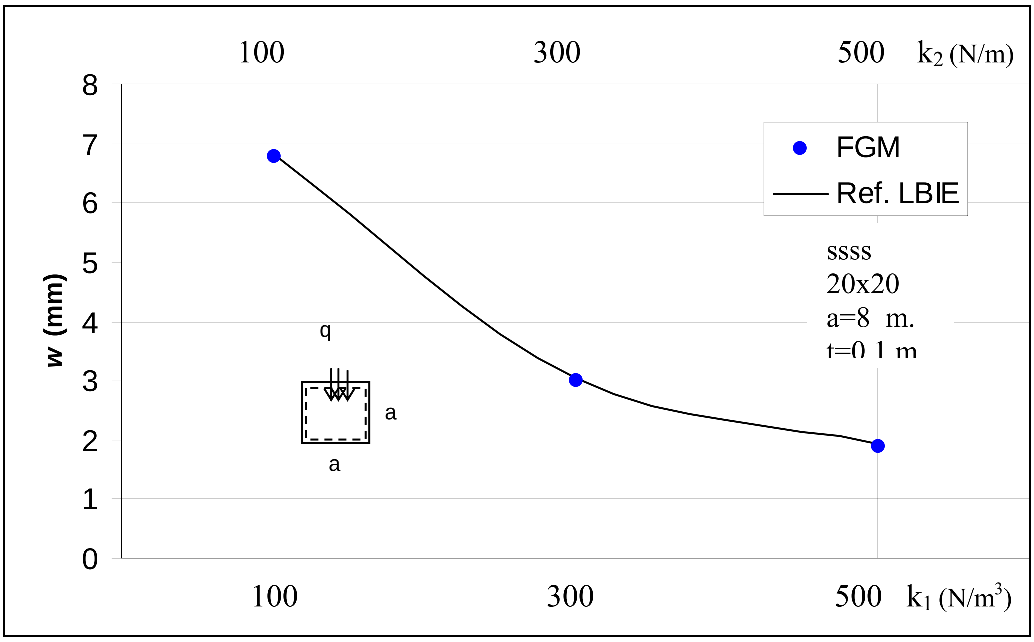


Figure 5.10: Comparison of Maximum Deflections of Simple Supported Plate for the Variation of Foundation Parameters under Uniformly Distributed Load with the LBIE Results

For the clamped case, the plate under the uniformly distributed load is considered to rest on Winkler foundation. The geometric and the material properties are the same as in the simple supported case. The comparison of the FGM results with the local boundary integral equation method (LBIE) at the centre of the plate for three different Winkler coefficients is given in Table 5.5. From the table it can be seen that the maximum relative error of the central deflections is less than 3%.

Table 5.5: The Comparison of the Maximum Deflections for a Clamped Plate Resting on a Winkler Foundation with the LBIE

coefficients k_1(N/m^3)	LBIE (mm)	FGM (mm)	Relative Error %
100	3.872	3.9696	2.52
300	2.5518	2.5826	1.21
500	1.8787	1.8904	0.62

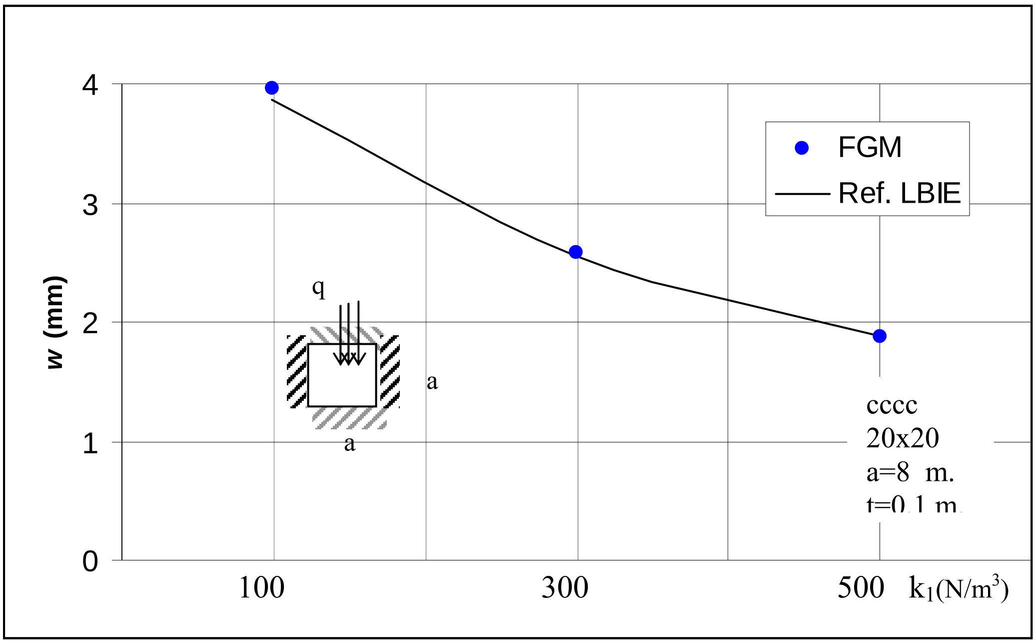


Figure 5.11: Comparison of Maximum Deflections of the Clamped Plate on Winkler Foundation under Uniformly Distributed Load with the LBIE Results

5.3.3 Comparison with Conical Exact Solution for Levy Plates on Two-Parameter Elastic Foundation

A parametric study for uniformly loaded SSSS (all edges of the plate are simple supported), SCSC (opposite two edges are simple supported, the others are clamped) and SFSF (opposite two edges are simple supported the others are free) square plates on two-parameter foundation was studied by Lam (2000). In that study, for Levy plates by using Green's functions a solution method named conical exact solutions have been derived. It is denoted that this solutions can be accepted as benchmark results to check the convergence, validity and accuracy of numerical solutions. We will make further use of this article in the following sections.

For checking the validity of the finite grid method (FGM), a comparison study was carried out for plates resting on two-parameter foundations. For uniformly loaded square SSSS, SCSC and SFSF plates with different values of k_1 and k_2 non-dimensional foundation parameters, the central deflection and bending moment values have been compared with the benchmark results. For convenience and generality the following parameters have been introduced:

$$\mathrm{k}_1 = \frac{k_1 a^4}{D}$$

$$\mathrm{k}_2 = \frac{k_2 a^2}{D}$$

$$w = \frac{D \quad w_{(0.5a,0.5a)}}{qa^4} 10^3 \tag{5.1}$$

$$\mathrm{M_x} = \frac{M_{xx(0.5a,0.5a)}}{qa^2} 10^2$$

$$\mathrm{M_y} = \frac{M_{yy(0.5a,0.5a)}}{qa^2} 10^2$$

where a is the length of the square plate, k_1 and k_2 are the Winkler and the second foundation parameters, D is the flexural rigidity of the plate, t is the thickness of plate, q is the uniform distributed load and $w_{(0.5a,0.5a)}$,. $M_{xx(0.5a,0.5a)}$ and $M_{yy(0.5a,0.5a)}$ are deflection, moment about x-direction and moment about y-direction at the centre of the plate respectively .

The comparison of non-dimensional parameters of the central deflections and bending moments w, $\mathrm{M_x}$ and $\mathrm{M_y}$ for the three cases of boundary conditions are plotted against the corresponding the conical exact solutions by Lam (2000), as shown in Figures 5.12 - 5.17

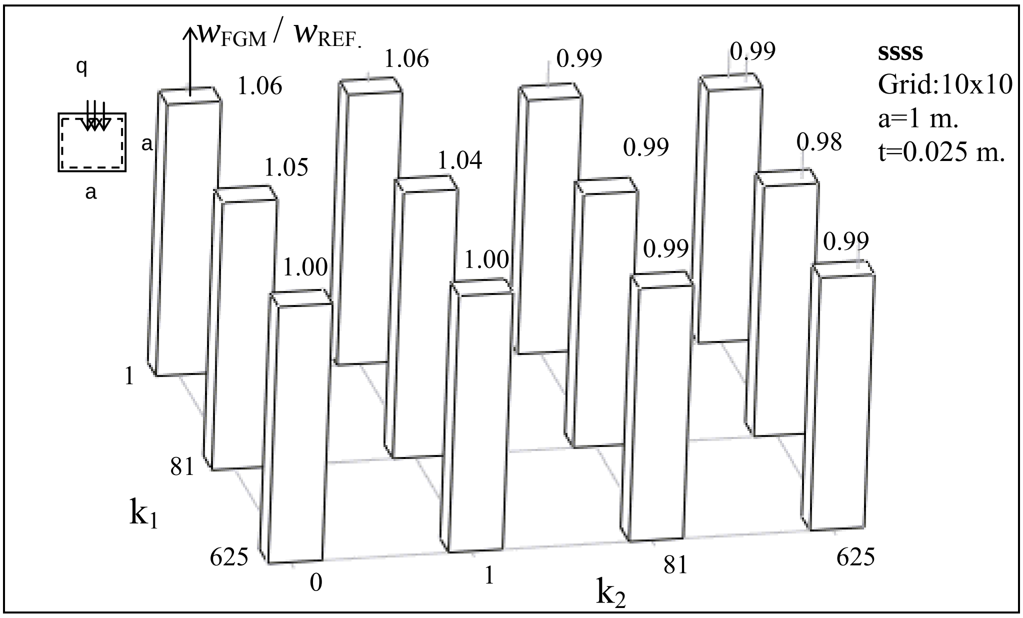


Figure 5.12a: Comparison of Deflection Ratios with Lam et al. (2002) at Midpoint of (SSSS) Rectangular Plate under Uniformly Distributed Load for t=0.025 m

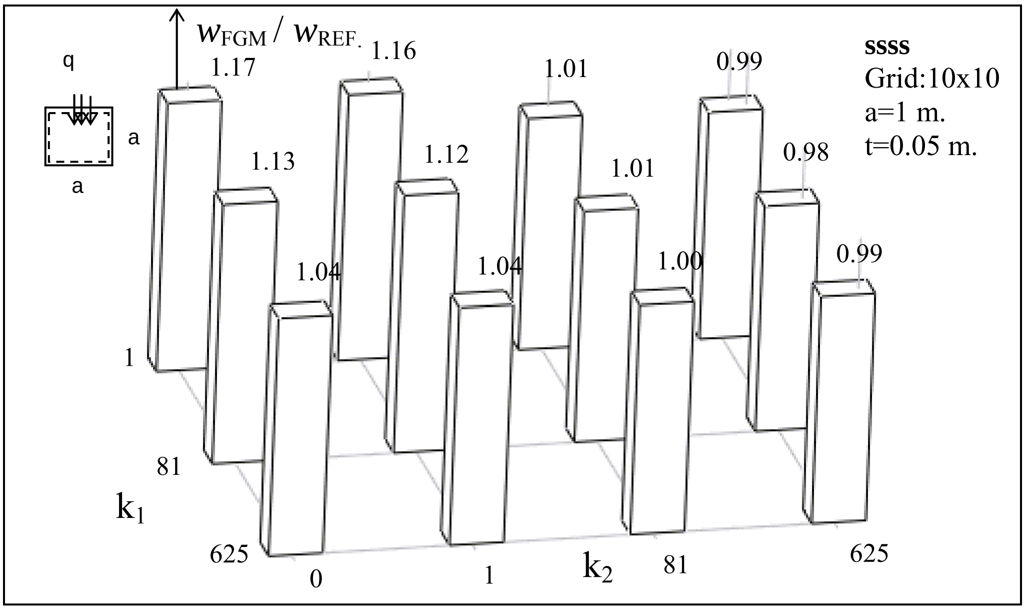


Figure 5.12b: Comparison of Deflection Ratios with Lam et al. (2002) at Midpoint of (SSSS) Rectangular Plate under Uniformly Distributed Load for t=0.05 m

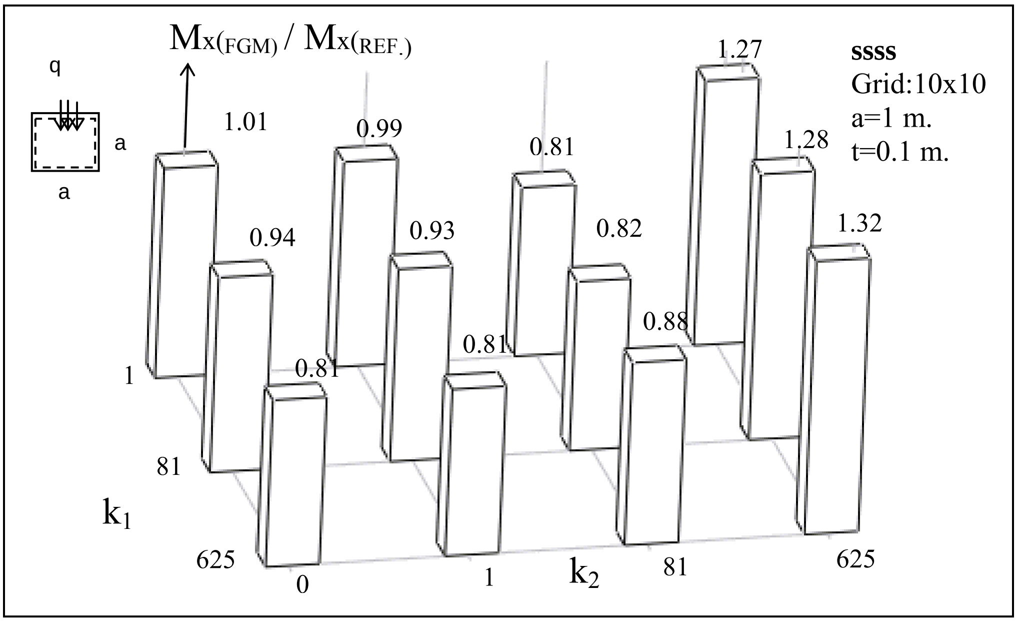


Figure 5.13a: Comparison of Bending Moment M_{xx} Ratios with Lam et al. (2002) at Midpoint of (SSSS) Rectangular Plate under Uniformly Distributed Load for t=0.1 m

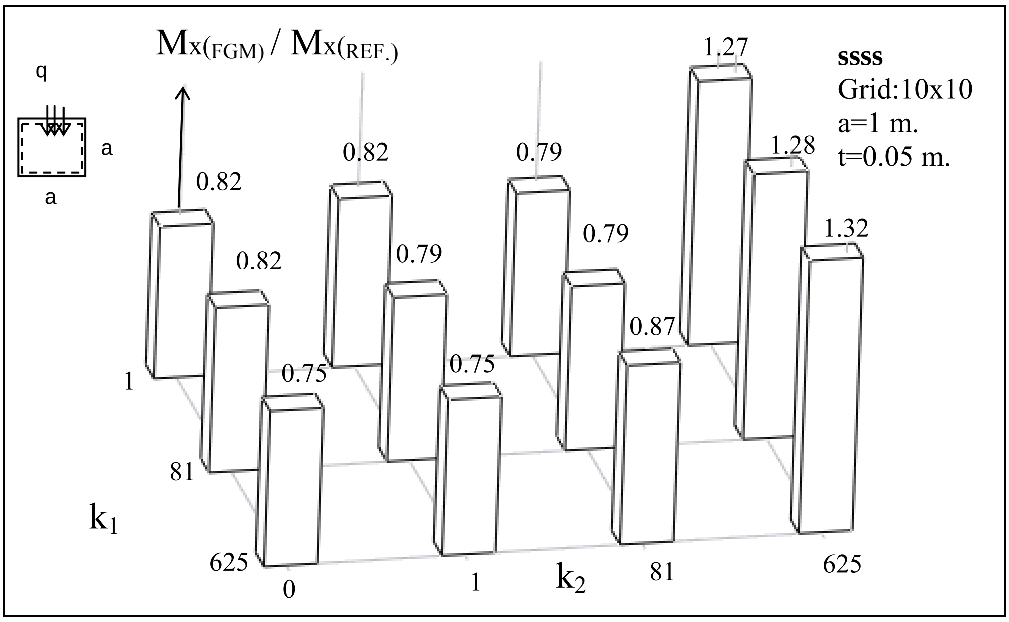


Figure 5.13b: Comparison of Bending Moment M_{xx} Ratios with Lam et al. (2002) at Midpoint of (SSSS) Rectangular Plate under Uniformly Distributed Load for t=.05m

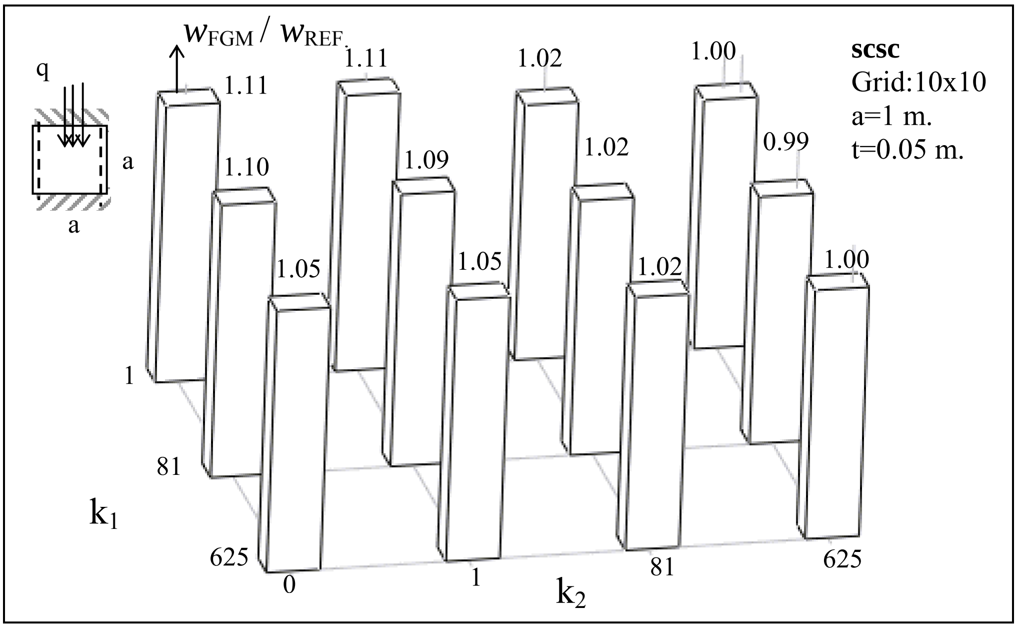


Figure 5.14a: Comparison of Deflection Ratios with Lam et al. (2002) at Midpoint of (SCSC) Rectangular Plate under Uniformly Distributed Load for t=0.05 m

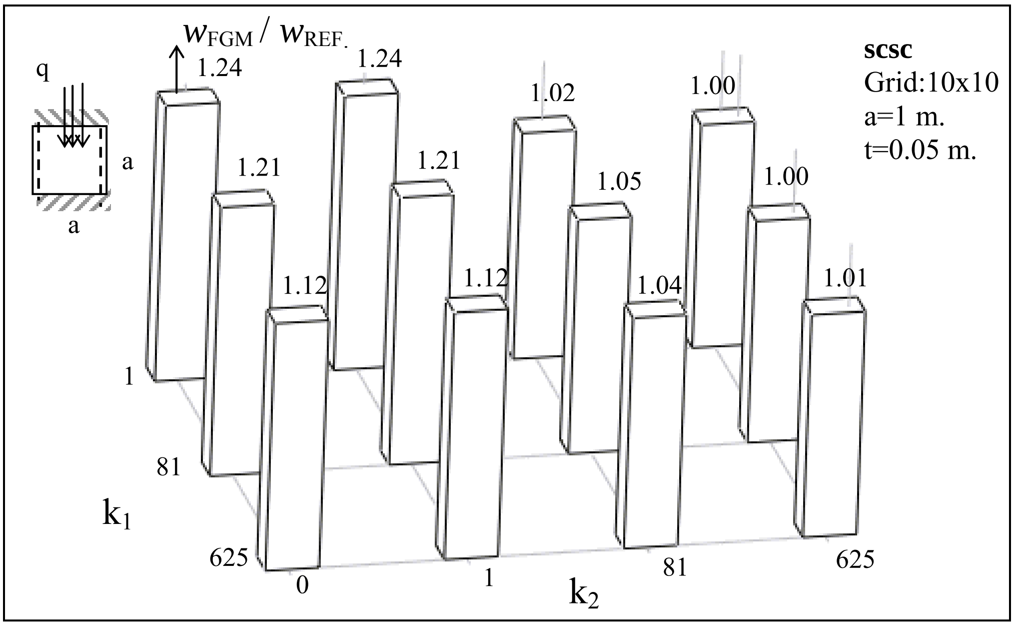


Figure 5.14b: Comparison of Deflection Ratios with Lam et al. (2002) at Midpoint of (SCSC) Rectangular Plate under Uniformly Distributed Load for t=0.1 m

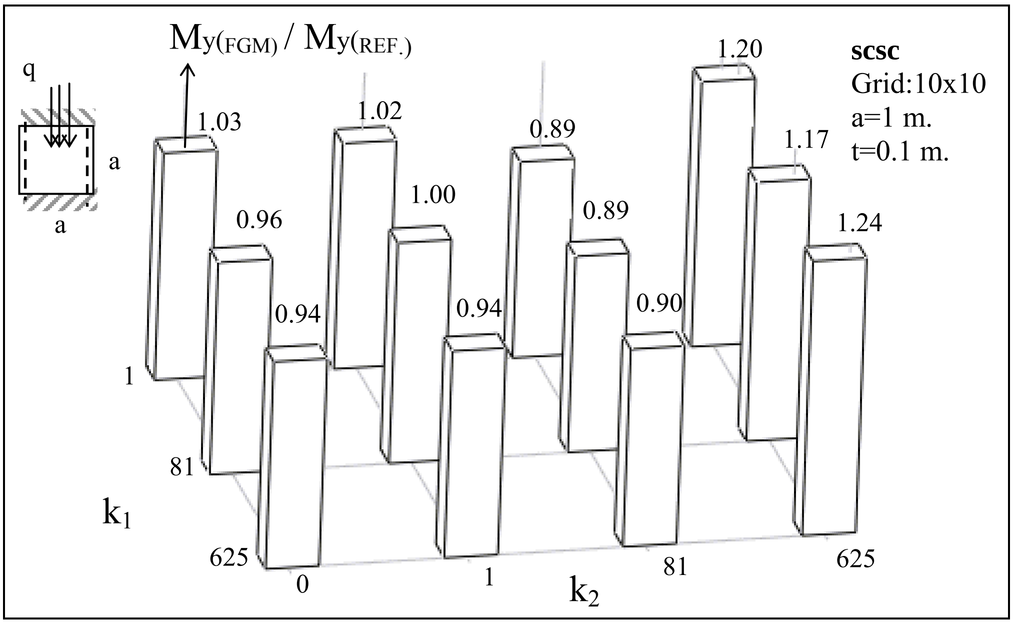


Figure 5.15a: Comparison of Bending Moment M_{yy} Ratios with Lam et al. (2002) at Midpoint of (SCSC) Rectangular Plate under Uniformly Distributed Load for t=.1 m

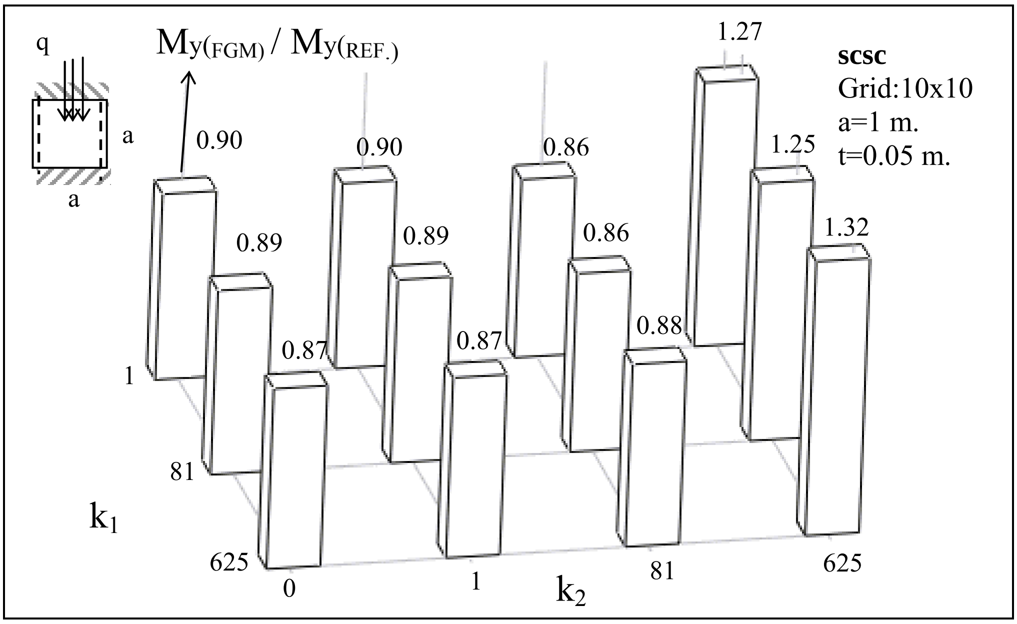


Figure 5.15b: Comparison of Bending Moment M_{yy} Ratios with Lam et al. (2002) at Midpoint of (SCSC) Rectangular Plate under Uniformly Distributed Load for t=.05m

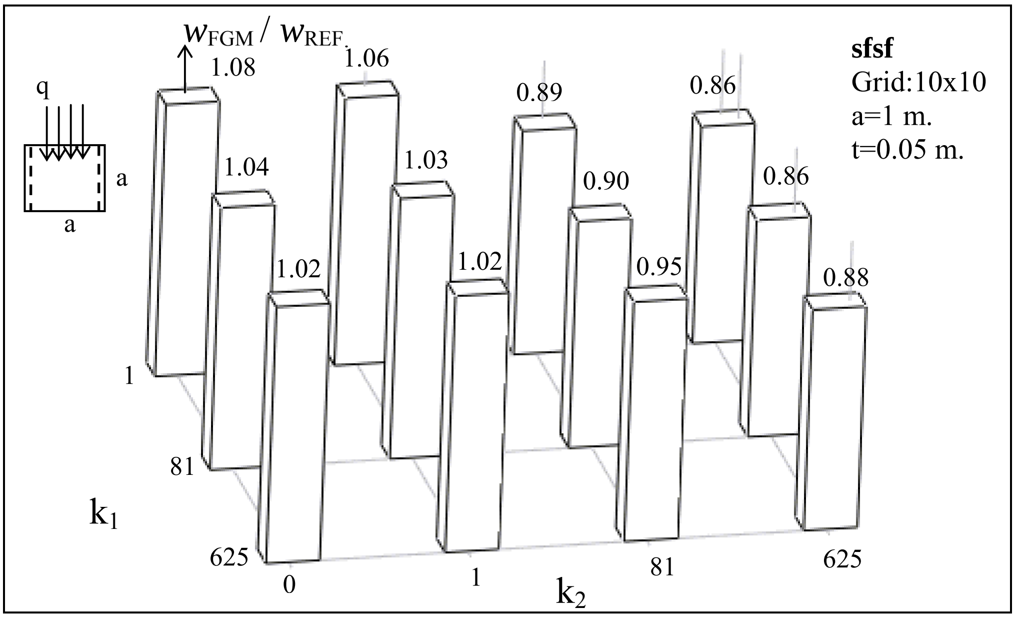


Figure 5.16a: Comparison of Deflection Ratios with Lam et al. (2002) at Midpoint of (SFSF) Rectangular Plate under Uniformly Distributed Load for t=0.05 m

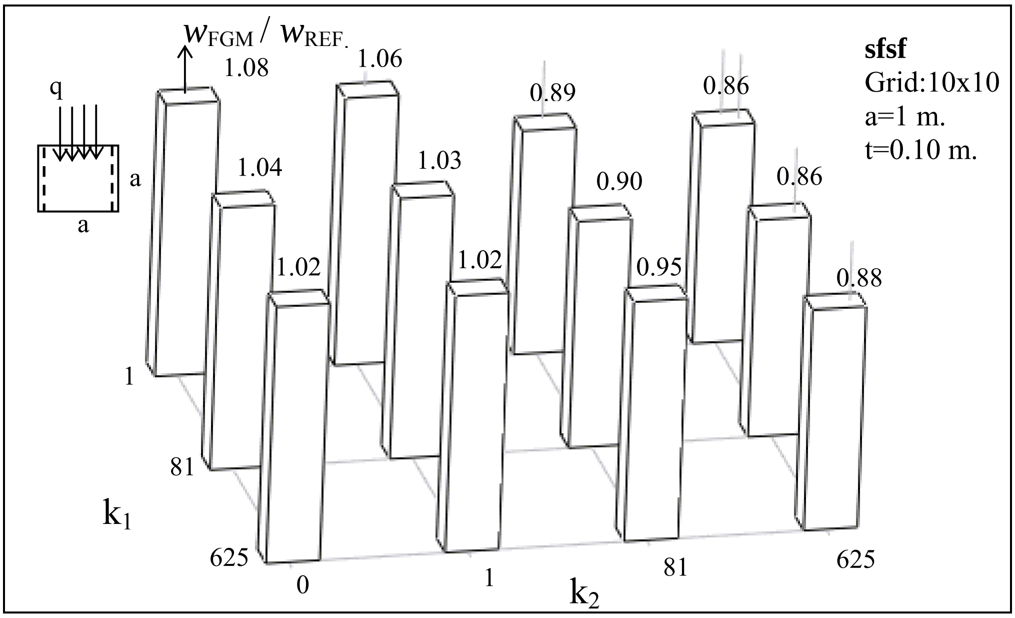


Figure 5.16b: Comparison of Deflection Ratios with Lam et al. (2002) at Midpoint of (SFSF) Rectangular Plate under Uniformly Distributed Load for t=0.1 m

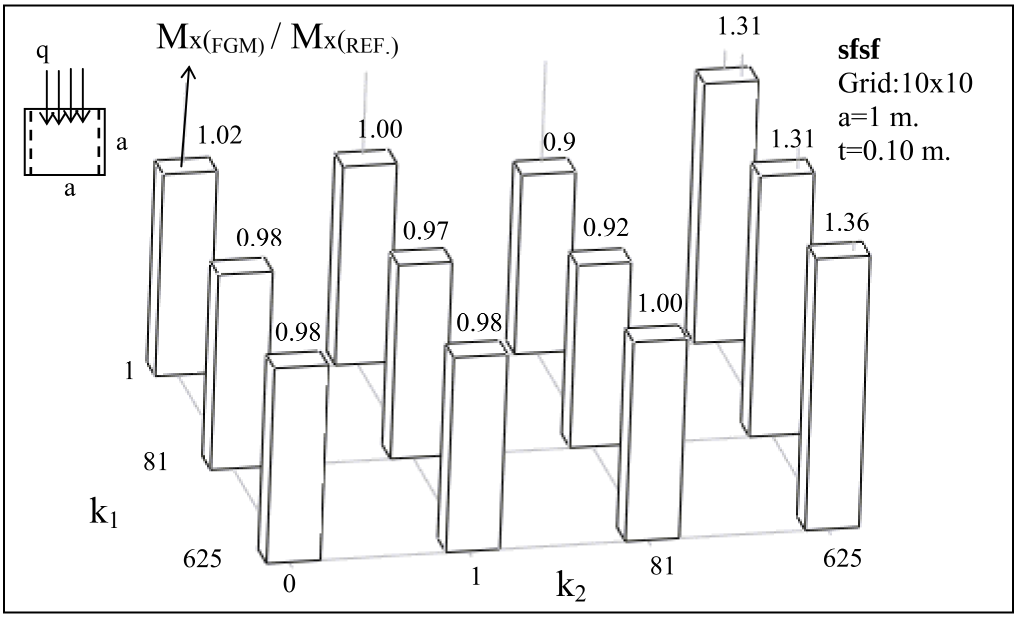


Figure 5.17a: Comparison of Bending Moment M_{xx} Ratios with Lam et al. (2002) at Midpoint of (SFSF) Rectangular Plate under Uniformly Distributed Load for t=.1 m

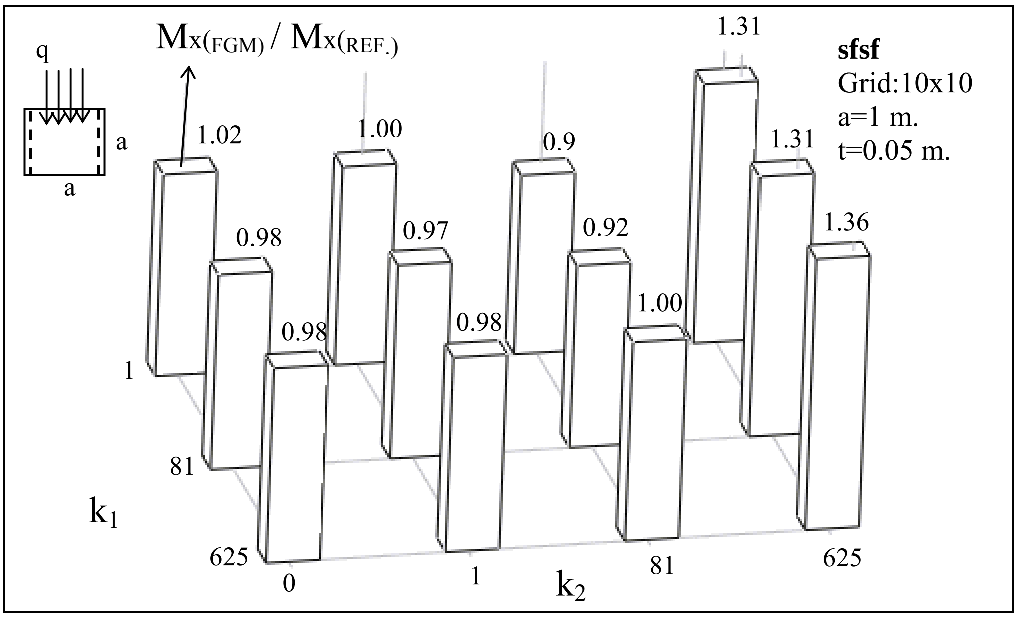


Figure 5.17b: Comparison of Bending Moment M_{xx} Ratios with Lam et al. (2002) at Midpoint of (SFSF) Rectangular Plate under Uniformly Distributed Load for t=.05m

5.4 BENDING PROBLEMS OF CIRCULAR AND ANNULAR PLATES

5.4.1 Simple Support Annular Plate Under Distributed Loading On One-Parameter Elastic Foundation

The annular plate shown in Figure 5.18 is supported on an elastic foundation with Winkler parameter, k_1 = 10000 kN/m3, has a uniform thickness, h = 0.25 m, and radiuses a = 2.5 m., b = 5 m., was attempted a uniformly distributed loading of intensity q = 200 kN/m^2. The material properties are: E = 2.7x10E7 kN/m^2 and v = 0.2.

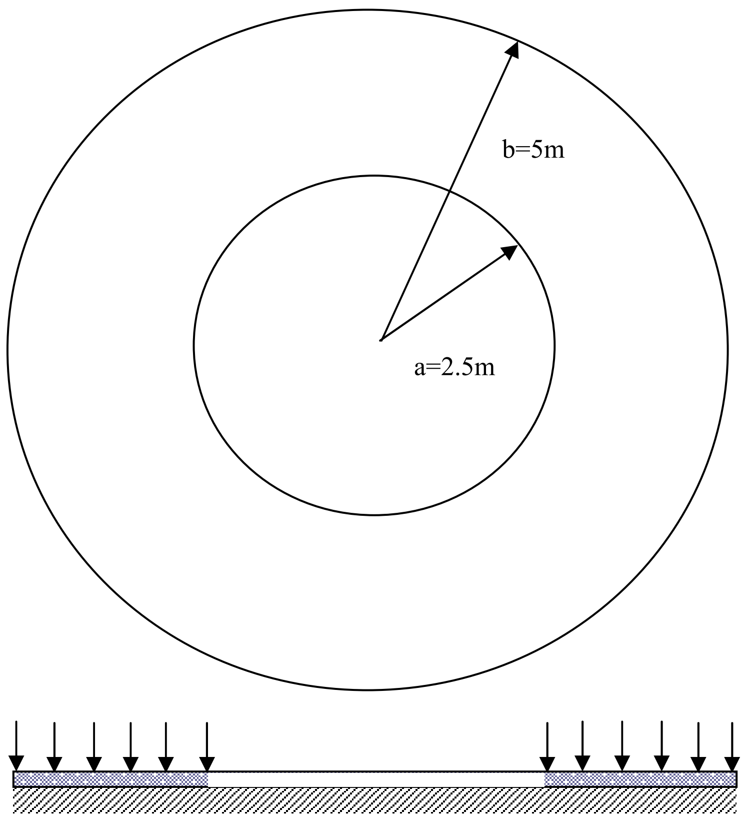


Figure 5.18: Uniformly Distributed Loaded Annular Plate Resting on Elastic Foundation

The plate, with simple support boundary conditions, results was evaluated at span of the plate. The comparison of the FGM solution with the reference (Utku and İnceleme, 2000) is shown in Table 5.6 and Figure 5.19. The results with respect to the reference can be accepted as accurate.

Table 5.6: The Comparison of the Deflections in Radial Direction and Maximum Moment for a Simply Supported Annular Plate Resting on Winkler Foundation with the Reference Values

Radius	2750	3000	3250	3500	3750	4000	4250	4500	4750	Mmax
REF. (mm)	0.81	1.51	2.04	2.35	2.43	2.28	1.92	1.39	0.73	134.5
FGM (mm)	0.85	1.59	2.16	2.49	2.58	2.43	2.05	1.49	0.78	140.5
Relative Error %	4.35	4.98	5.42	5.75	6	6.21	6.4	6.59	6.79	4.291

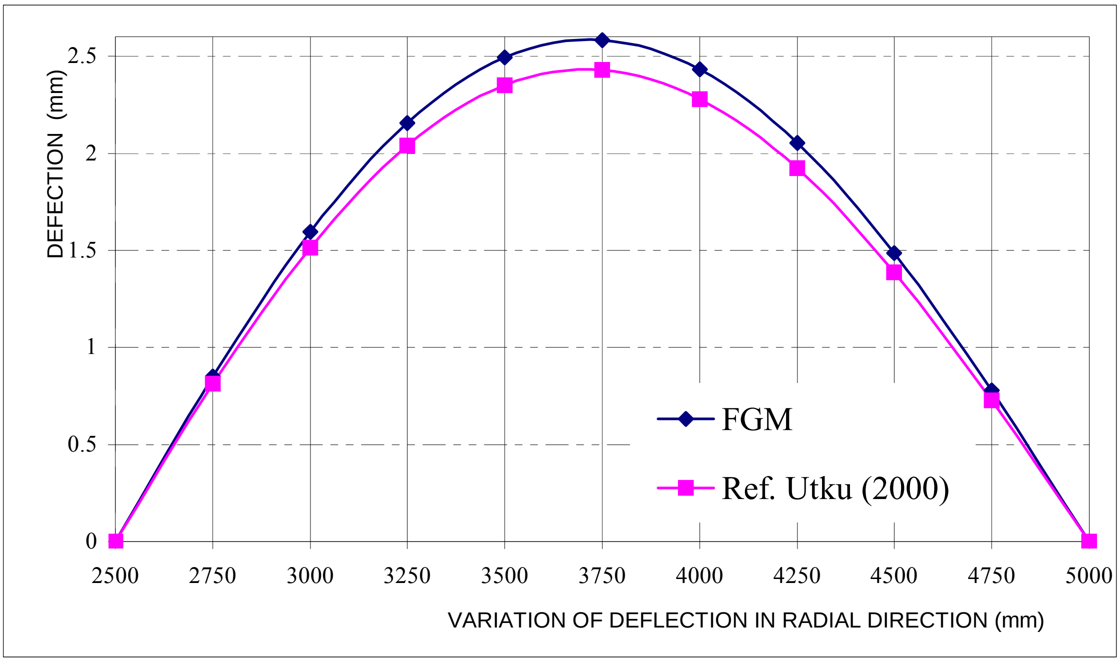


Figure 5.19: Comparison of the Results with Utku (2000) for an Annular Plate on Elastic Foundation under Uniform Distributed Loading Condition

5.4.2 Ring Foundation on One-Parameter Elastic Foundation

The ring foundation example solved by Bowles (1996) is shown in Figure 5.20. From the Figure there are three equally spaced (120^o) concentreted loads of 675 kN applied at points A,B and C and a 200 kNm tangential moment applied at A.

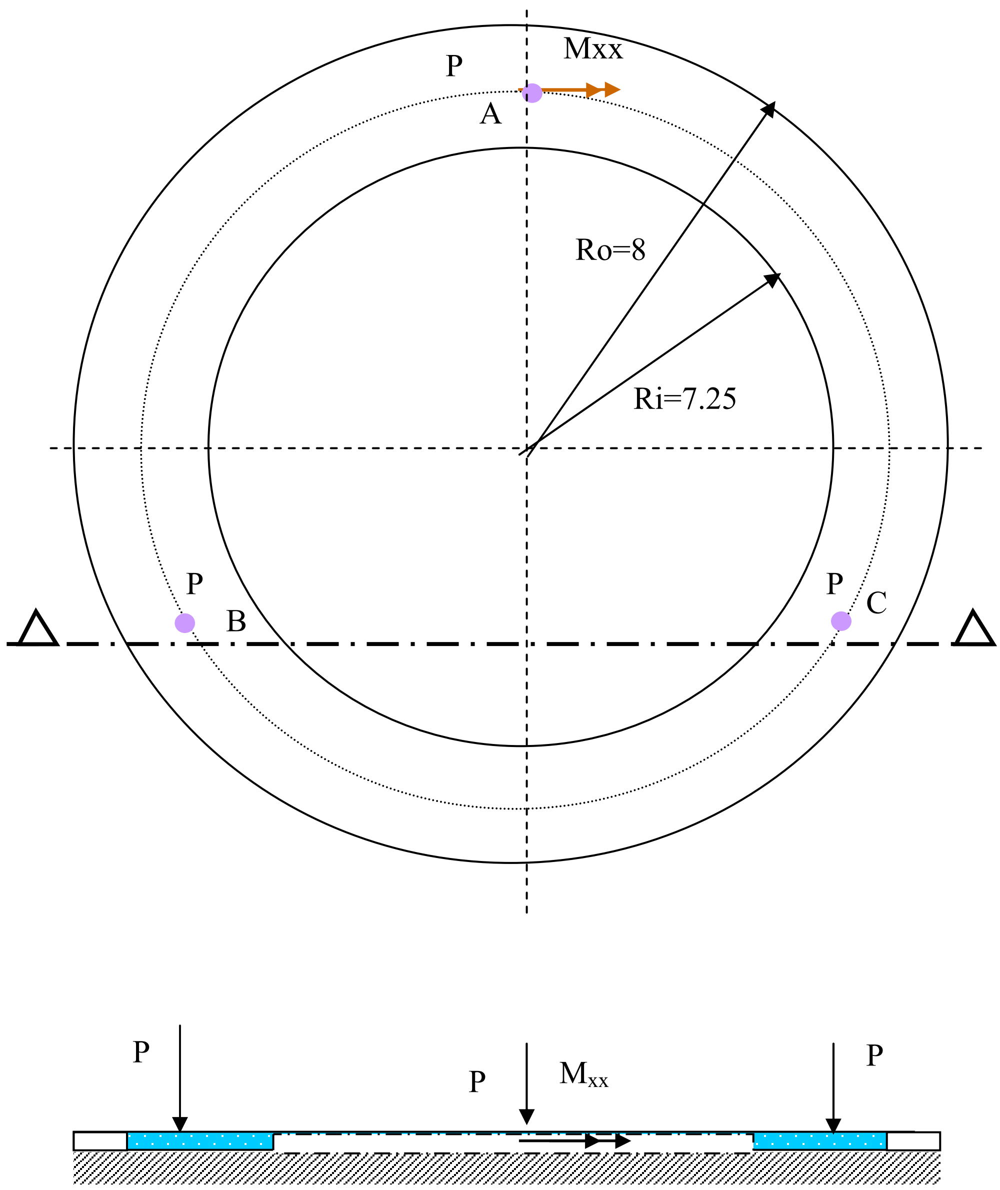


Figure 5.20: The Representation of the Forces Applied at the Ring Foundation

The plate properties, modulus of elastisity E, Poisson ratio *v*, and thickness t are given as 22400 Mpa,0.15 and 0.76 m. respectivelly and the foundation parameter is given as 13600 kN/m^3. The results are tabulated in Table 5.6. The maximum relative error for deflections is obtained about 1.5 %. This reflects a high degree of accuracy with respect to the reference. The deflections along the ring are shown in Figure 5.21.

Table 5.7: The Comparison of the Deflections along the Ring Foundation with the Reference Values

Locations in Degrees	Reference (mm)	FGM (mm)	Relative Error %
0	7.93	7.995	0.82
18	5.95	5.960	0.16
36	2.92	2.913	0.23
54	1.27	1.252	1.45
72	1.64	1.638	0.13
90	3.85	3.857	0.18
108	6.66	6.623	0.56
126	7.38	7.316	0.87
144	4.91	4.920	0.19
162	2.26	2.285	1.09
180	1.24	1.259	1.55
198	2.26	2.285	1.09
216	4.91	4.920	0.19
234	7.38	7.316	0.87
252	6.66	6.623	0.56
270	3.85	3.857	0.18
288	1.64	1.638	0.13
306	1.27	1.252	1.45
324	2.92	2.913	0.23
342	5.95	5.960	0.16
360	7.93	7.995	0.82

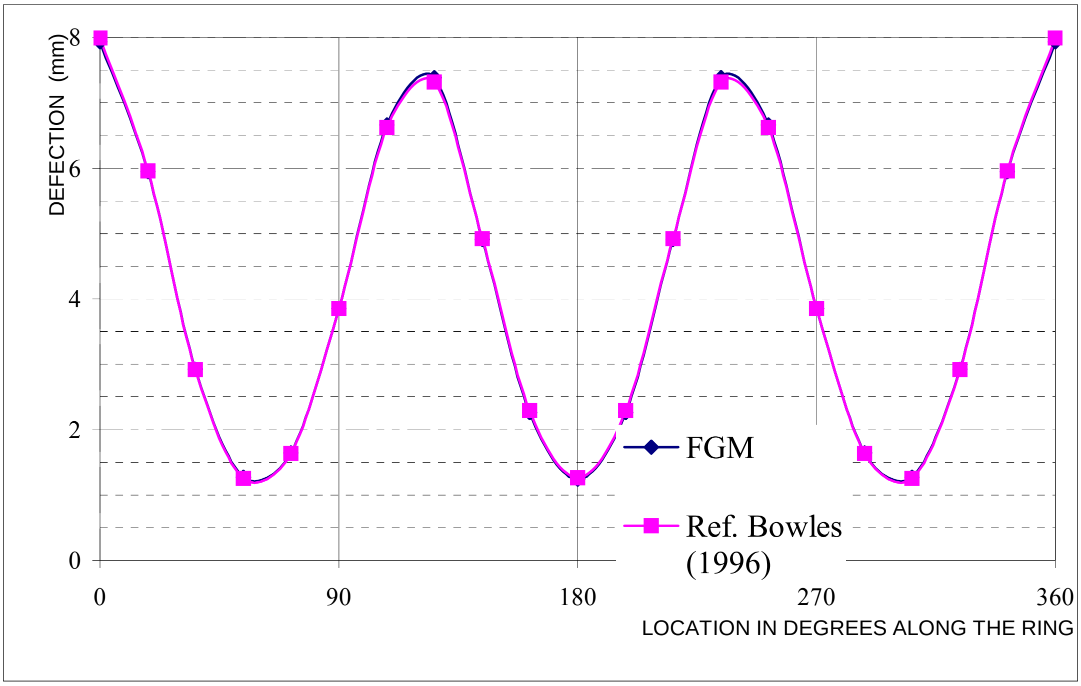


Figure 5.21: Comparison of the Results with Bowles (1996) for the Ring Foundation

5.4.3 Circular Plate with Variable Thickness under Non-Uniform Loading Conditions on One-Parameter Elastic Foundation

An industrial tower footing with various thickness and loading condition on elastic foundation with free end boundary conditions is considered. Wind moments of the tower idealized to puling and pushing vertical forces applied at convenient nodes as shown Figure 5.22. The details of the problem can be found in Bowles (1996).

The deflections along the diameter of the footing compared with the reference values can be seen in Figure 5.23.

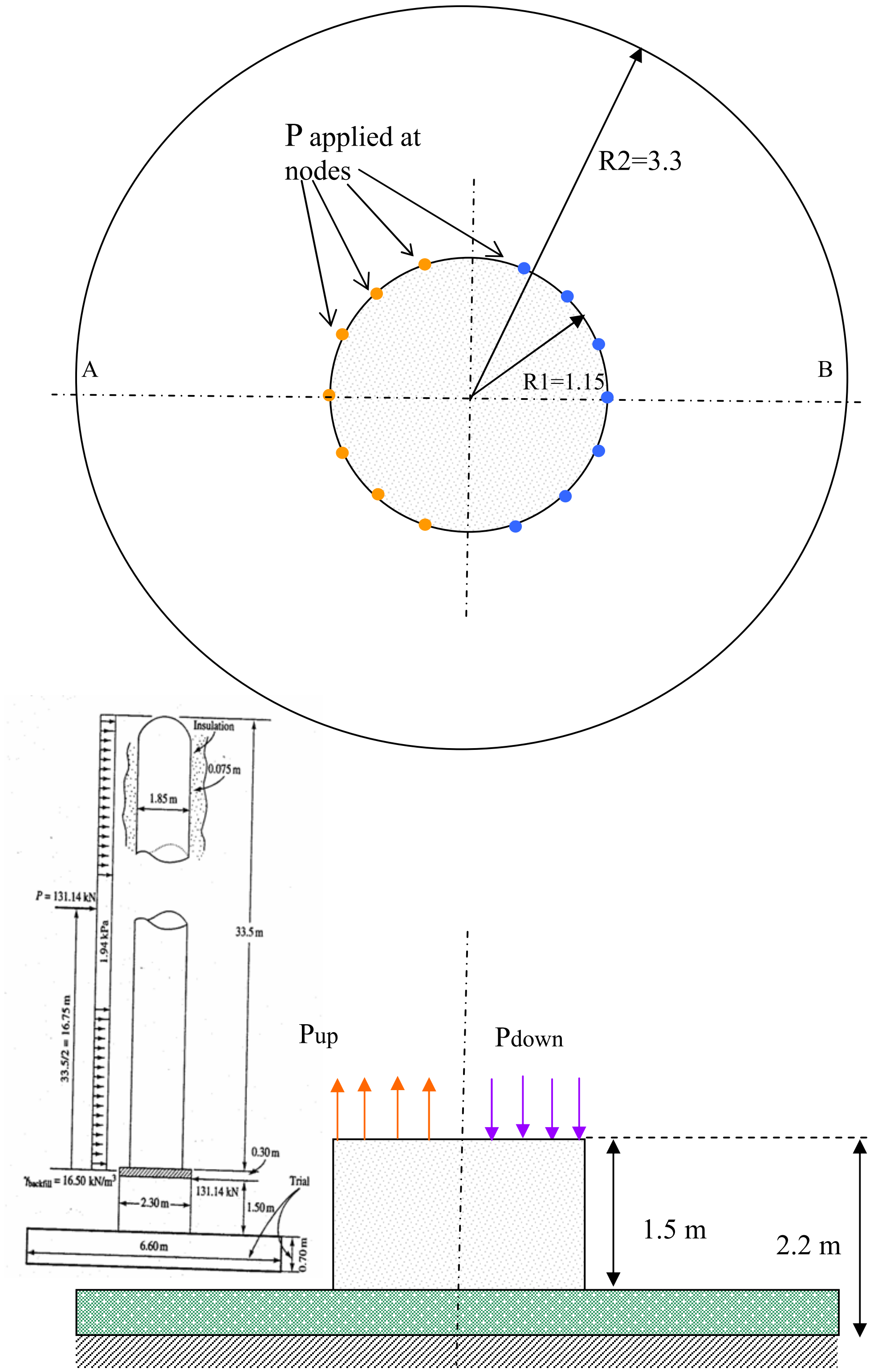


Figure 5.22: Wind Moments of the Refining Vessel Idealized to Puling and Pushing Vertical Forces Applied at Convenient Nodes

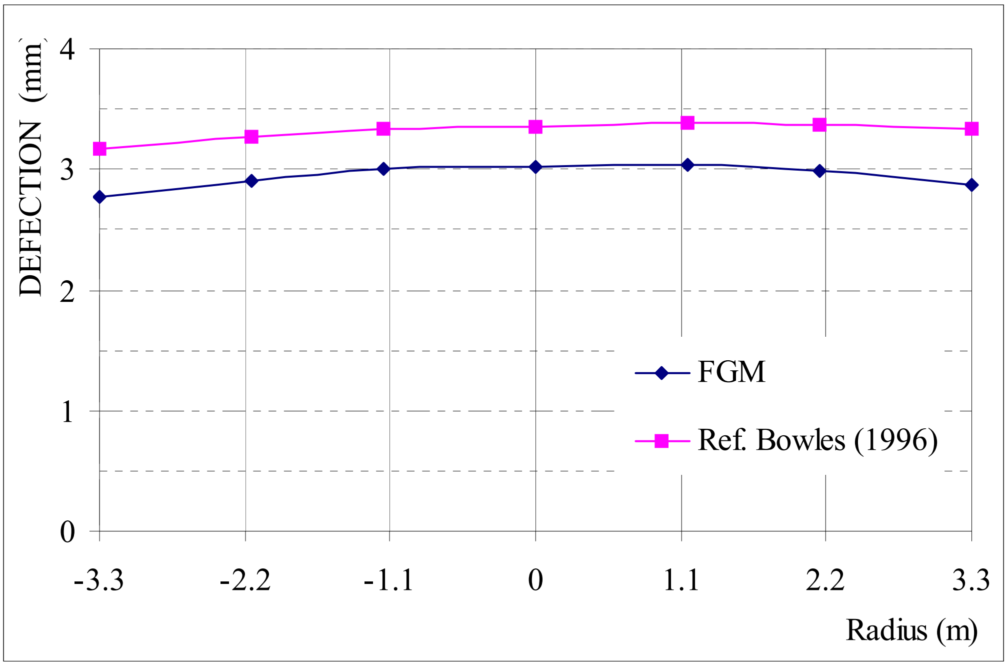


Figure 5.23: Comparison of Deflection Results along A to B Direction with Bowles (1996)

5.4.4 Clamped Circular Plate Under Concentrated Loading on Two-Parameter Foundation.

A solid circular plate with a uniform thickness 0.05 m and an outer radius R_0=0.5 m, subjected to a concentrated force F = 3000 kN at its centre is considered. The plate is resting on a two-parameter elastic foundation, and its material and foundation properties are

$k_1 = 6.48 \times 10^4$ kN/m^3

$k_\theta = 2250.0$ kN/m

E = 2.1x10^8 kN/m^2 and $v = 0.3$

The parameter w_o employed for disc cases is $w_0 = F(R_0)^2/D$. The radial distributions of the non-dimensional parameter w/w_0 is plotted against the corresponding boundary element solutions (EI-Zafrany and Fadhil, 1996), as shown in Figure 5.24. Also the three-dimensional view of deflection (w) is plotted in Figures 5.25.

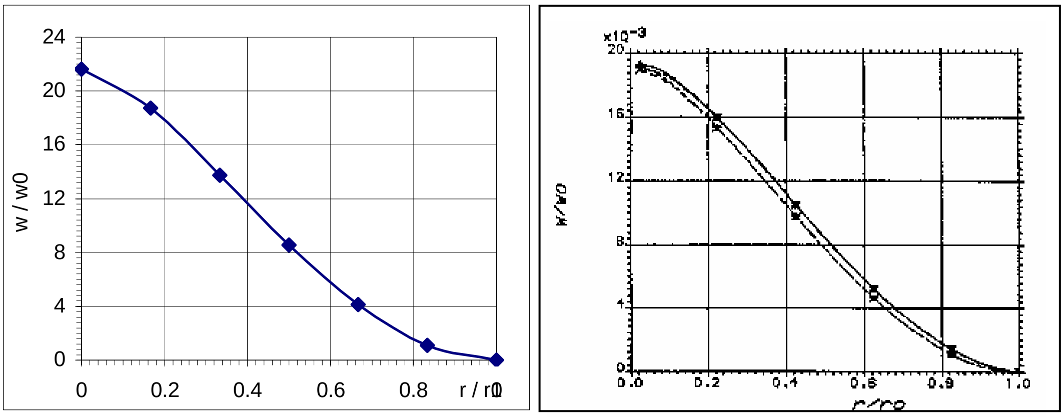


Figure 5.24: Comparison of the Finite Grid Solution to The Boundary Element Solution for Deflection of Clamped Circular Plates under Concentrated Loading on 2-Parameter Elastic Foundation

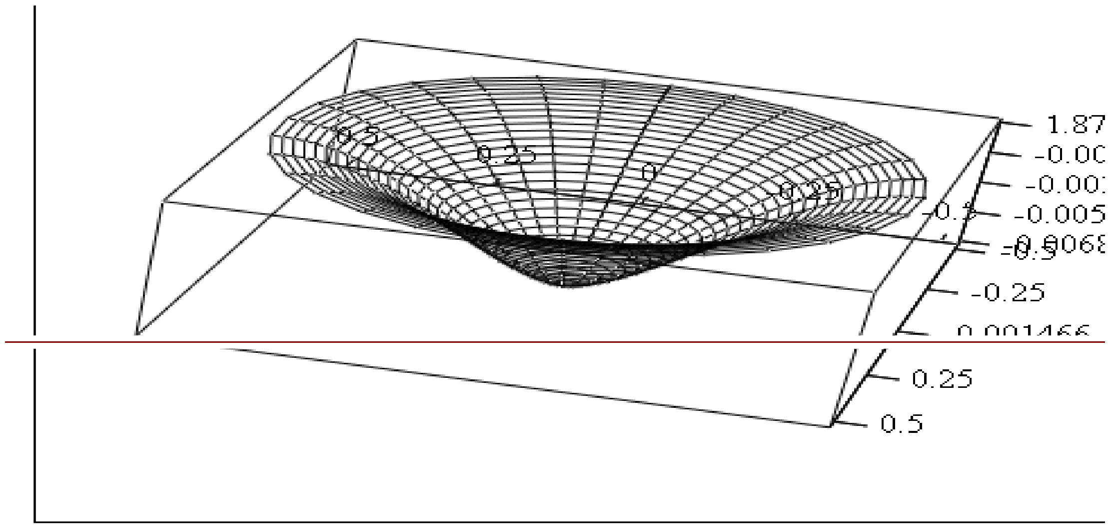


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Figure 5.25: Three-Dimensional View of Deflection of Clamped Circular Plates under Concentrated Loading on 2-Parameter Elastic Foundation

5.5 BUCKLING AND FREE VIBRATION PROBLEMS

5.5.1 Plates with Abrupt Changes in Thickness

Chung et al. (2000) studied a bi-directionally stepped square plate with simply supported edges shown in Figure 5.26. In this study, the variation of free vibration and buckling loads with the thickness ratio h_2/h_1 will be compared. the fundamental angular frequencies and the critical buckling loads are compared. The results are portrayed graphically in Figures 5.27 and 5.28.

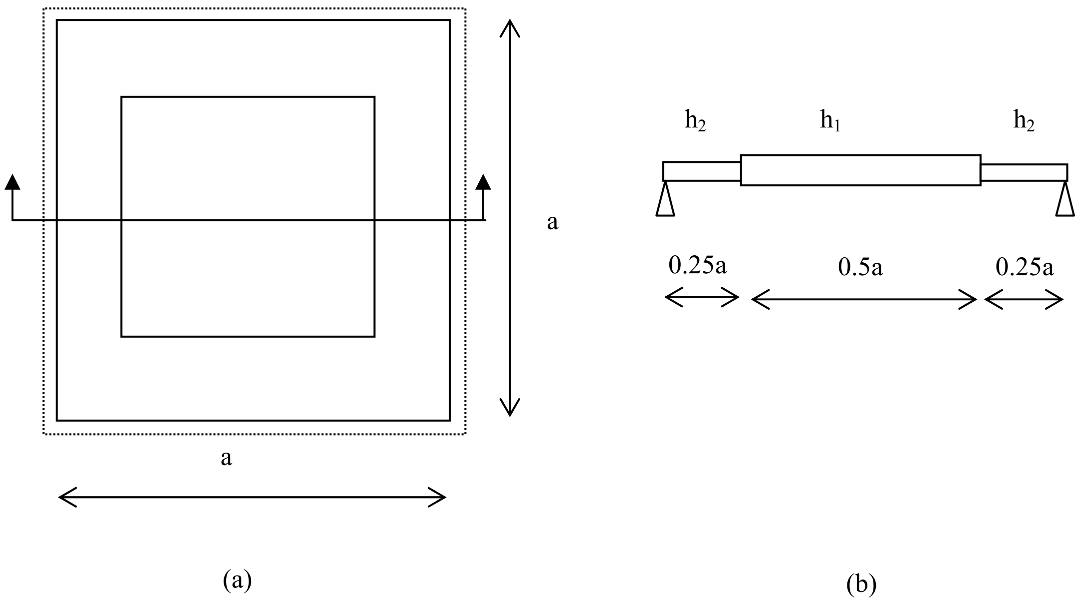

Figure 5.26: Bi-Directionally Stepped Square Plate with all Edges Simply Supported (a) Plan; (b) Section

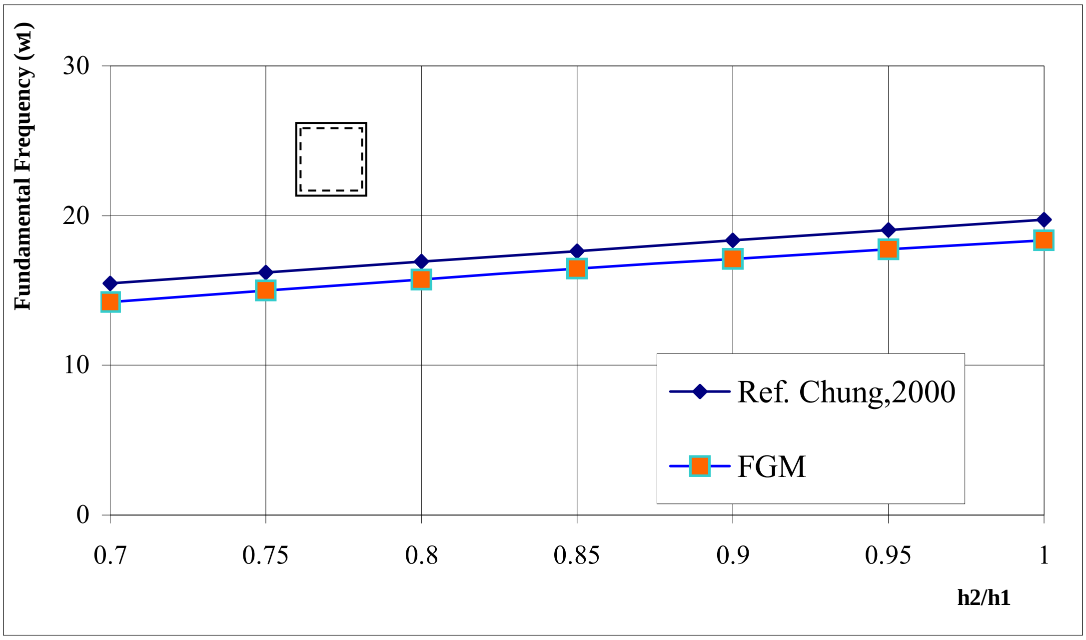

Figure 5.27: Comparison of Fundamental Frequencies with Chung (2000) for Bi-Directionally Stepped and Simply Supported (SSSS) Square Plate

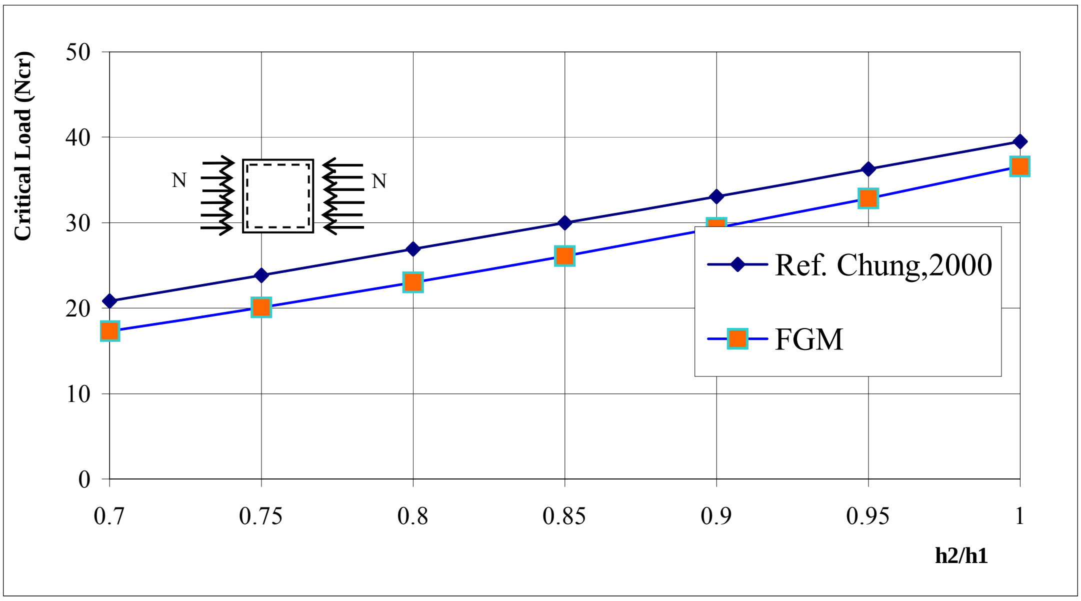

Figure 5.28: Comparison of Buckling Loads with Chung (2000) for Bi-Directionally Stepped and Simply Supported (SSSS) Square Plate under Uniaxial Compression

5.5.2 Uniform Plates on Non-Homogeneous Foundation

In this case another study of Chung et al. (2000), a uniform square plate resting on a non-homogeneous foundation shown in Figure 5.28 is considered. The plate is supported on elastic foundation of modulus K2 within the central square region of size 1.2ax1.2a, and elsewhere the foundation modulus is K1. The fundamental angular frequencies are compared for all of the edges simple supported (SSSS) and clamped (CCCC) boundary conditions. The results for both simple supported and clamped edges are portrayed graphically in Figures 5.29 and 5.30 respectively.

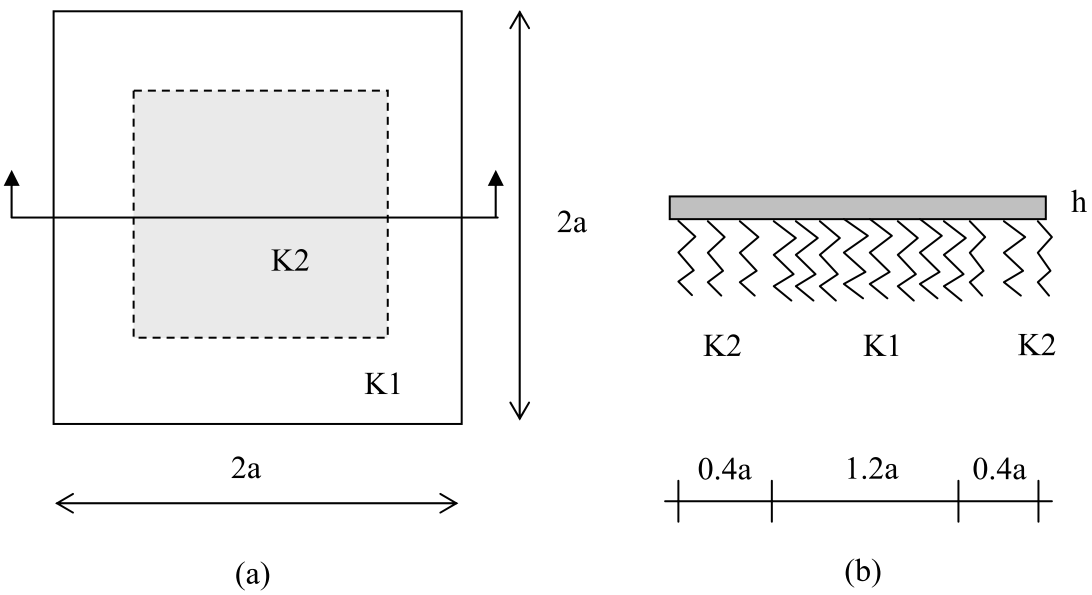

Figure 5.29: A Uniform Square Plate on Non-Homogeneous Elastic Foundation (a) Plan, (b) Section

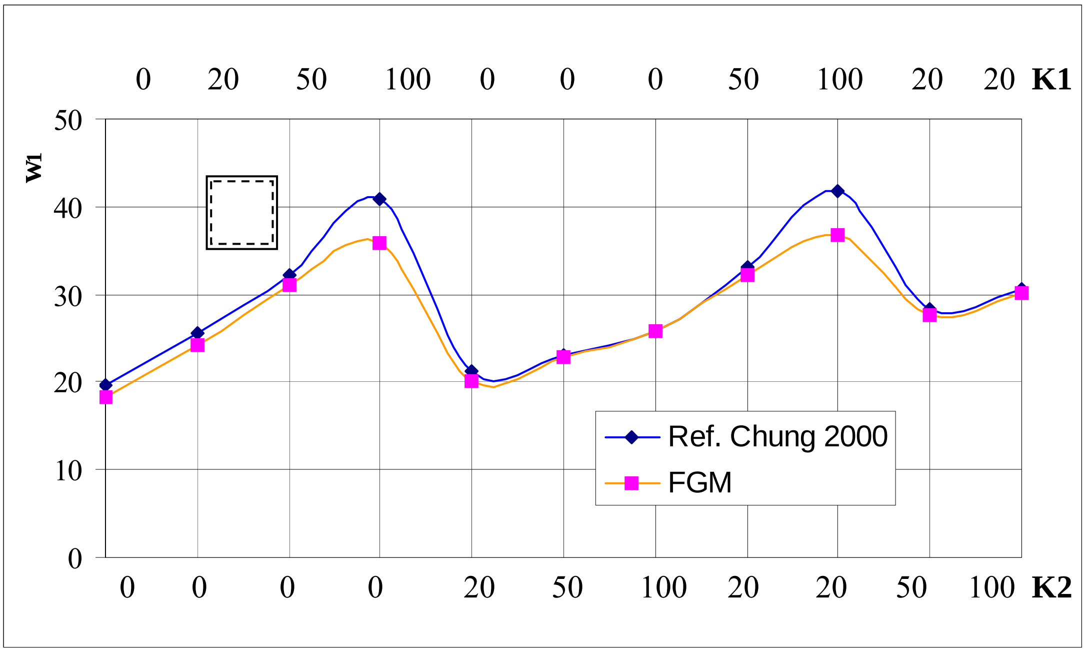

Figure 5.30: Fundamental Frequency Coefficients of Square Plate on Non-Homogeneous Elastic Foundation (SSSS)

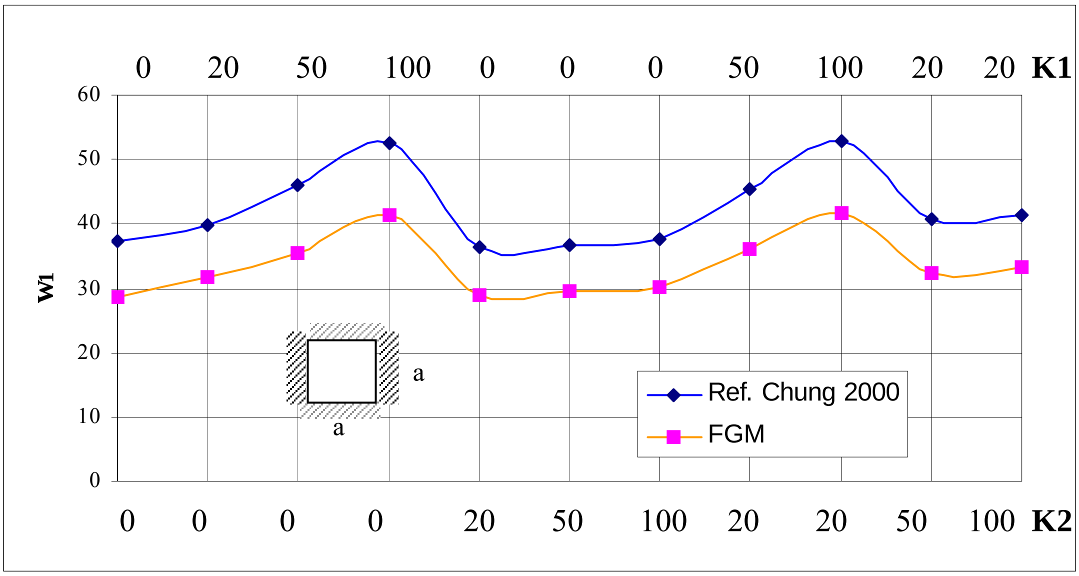

Figure 5.31: Fundamental Frequency Coefficients of Square Plate on Non-Homogeneous Elastic Foundation (CCCC)

5.5.3 Free Vibration Problems of Levy Plates on Two-Parameter Elastic Foundation

Free vibration analysis of SSSS (all edges of the plate are simple supported), SCSC (opposite two edges are simple supported, the others are clamped) and SFSF (opposite two edges are simple supported the others are free) square plates on two-parameter foundation was studied by Lam (2000). In this study, for Levy plates conical exact solutions have been derived by Green's functions. It is denoted that these solutions can be accepted as benchmark results to check the convergence, validity and accuracy of numerical solutions.

For checking the validity of the finite grid method (FGM), a comparison study was carried out for plates resting on two-parameter foundations. For the square plates with different values of k_1 and k_2 non-dimensional foundation parameters defined in Equation 5.1, the fundamental frequency values have been compared with the reference results. The results for SSSS and SCSC cases are tabulated in Table 5.7. The comparisons of Fundamental frequencies for the two cases of boundary conditions are plotted against the corresponding the conical exact solutions by Lam (2000), as shown in Figures 5.31 and 5.32.

Table 5.8: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Fundamental Frequencies of the S.S.S.S Square Plate in Case of Different Foundation Parameters

H=0.01, D=1, (10x10)			SSSS			SCSC		
Case	k_1	k_2	FGM	REF	Err %	FGM	REF	Err %
1	0	0	20.761	19.74	5.2	27.182	28.95	6.1
2	0	100	48.275	48.62	0.7	49.875	54.68	8.8
3	100	0	23.252	22.13	5.1	28.702	30.63	6.3
4	100	100	49.109	49.63	1.0	50.567	55.59	9.0

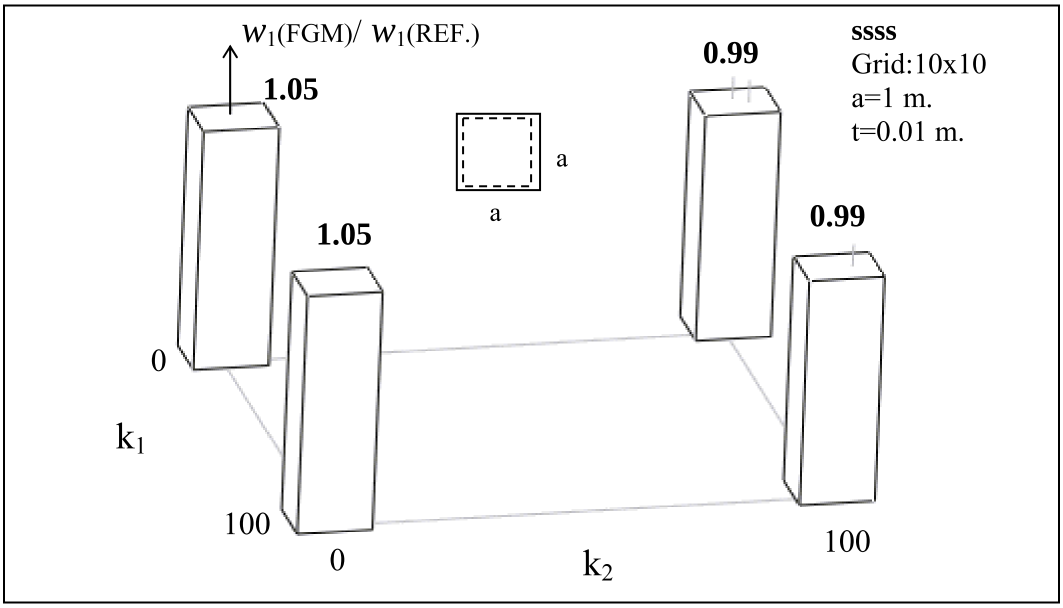

Figure 5.32: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Fundamental Frequencies of the SSSS Square Plate for Different Foundation Parameters

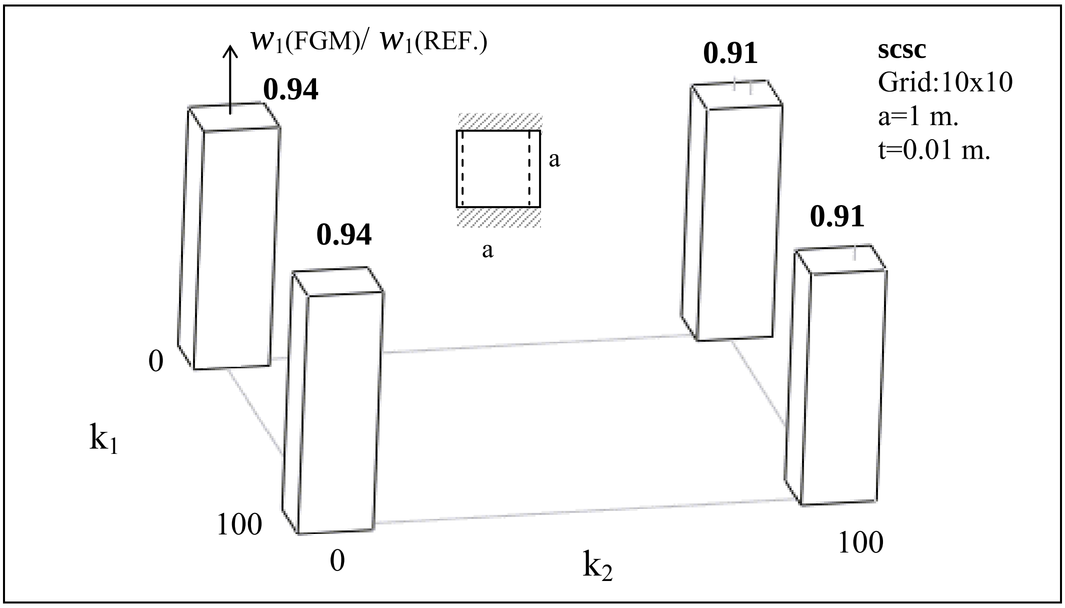

Figure 5.33: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Fundamental Frequencies of the SCSC Square Plate for Different Foundation Parameters

5.5.4 Buckling Problems of Levy Plates on Two-Parameters Elastic Foundation

For the square plates which is defined in Section 5.4 with different values of k_1 and k_2 the buckling load parameters (Nc) have been compared with the benchmark (Lam et al., 2000) values. Tables 5.7, 5.8 and 5.9 represent the non-dimensional buckling load parameters due to uniaxial and biaxial inplane loads for SSSS, SCSC and SFSF square plates on two-parameter elastic foundations. For each case of boundary conditions and inplane loads the results are plotted against the corresponding the conical exact solutions by Lam (2000), as shown in Figures 5.33 - 5.41.

Table 5.9: Comparison of the Finite Grid Solution with the Exact Solutions for Buckling Load Cases of the SSSS Square Plate in Case of Different Foundation Parameters

a=1,h=.01.D=1 E=10920000			ssss (Nc/π^2)								
			Nx=1,Ny=0			Nx=0,Ny=1			Nx=1,Ny=1		
Case	k_1	k_2	FG	Ref.	Error %	FG	Ref.	Error %	FG	Ref.	Error %
1	0	0	3.855	4	3.63	3.855	4	3.63	1.924	2	3.8
2	100	0	4.87	5.027	3.12	4.87	5.027	3.12	2.436	2.513	3.06
3	0	100	18.51	18.92	2.18	18.51	18.92	2.18	12.06	12.13	0.6
4	100	100	18.76	19.17	2.12	18.76	19.17	2.12	12.57	12.65	0.7

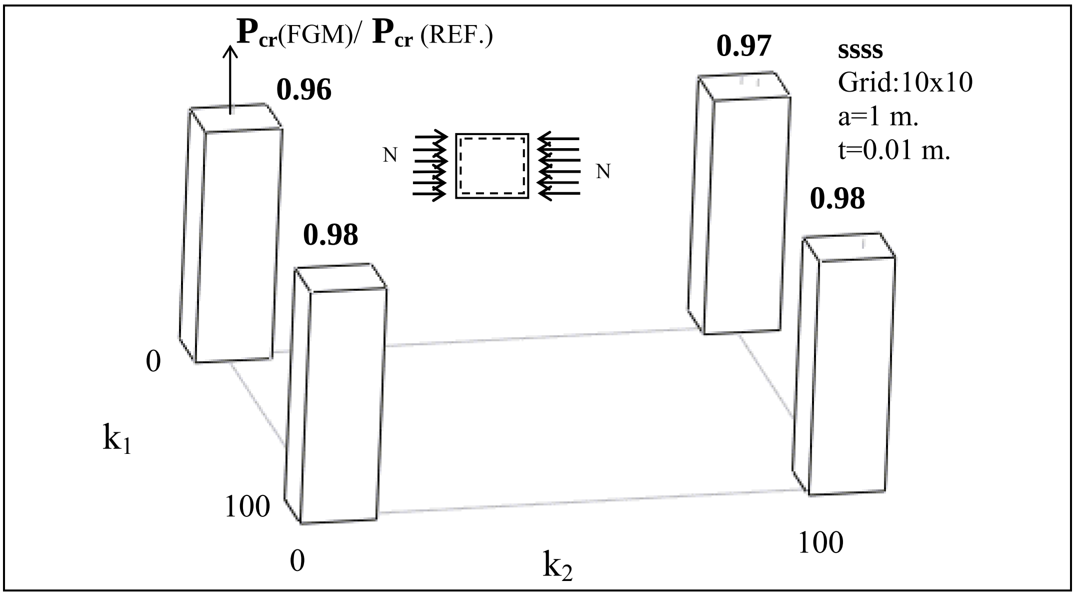

Figure 5.34: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SSSS) Plate under Uniaxial Compressive Loading (Nx=1, Ny=0) on Two-Parameter Foundation

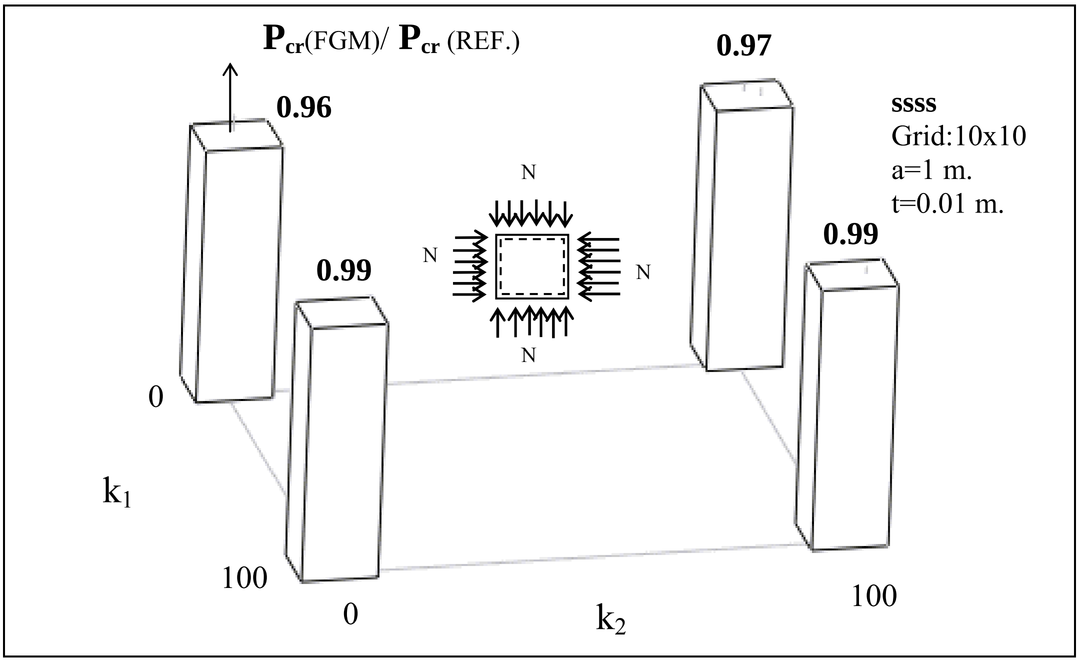

Figure 5.35: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SSSS) Plate under Biaxial Compressive Loading (Nx=1, Ny=1) on Two-Parameter Foundation

Table 5.10: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load Cases of the SCSC Square Plate in Case of Different Foundation Parameters

a=1,h=.01.D=1 E=10920000			SCSC (Nc/π^2)								
			Nx=1,Ny=0			Nx=0,Ny=1			Nx=1,Ny=1		
	k1	k2	FG	Ref.	Error %	FG	Ref.	Error %	FG	Ref.	Error %
1	0	0	7.4147	7.691	3.59	6.5548	4	63.87	3.7219	3.83	2.82
2	100	0	7.6712	7.948	3.48	7.304	7.491	2.50	4.1715	4.28	2.54
3	0	100	20.446	20.74	1.42	21.893	18.81	16.39	13.861	13.96	0.71
4	100	100	20.701	20.99	1.38	22.219	19.11	16.27	14.308	14.41	0.71

The error percentage for k_1=0 and k_2=0 values shown in Table 5.10 and Figure 5.37 suggests that this may have been a printing error in Lam et al. (2000).

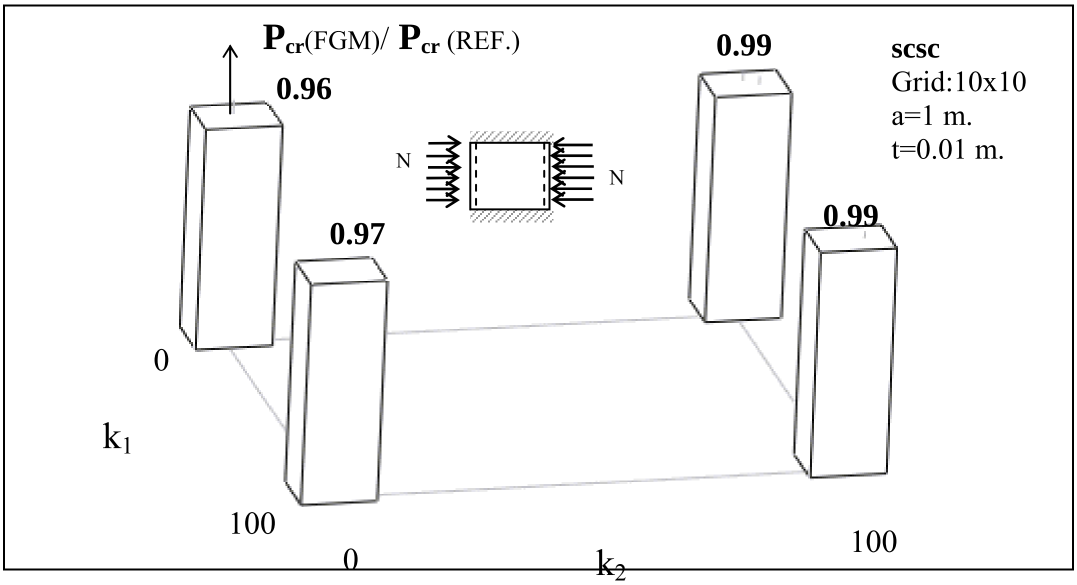

Figure 5.36: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SCSC) Plate under Uniaxial Compressive Loading (Nx=1, Ny=0) on Two-Parameter Foundation

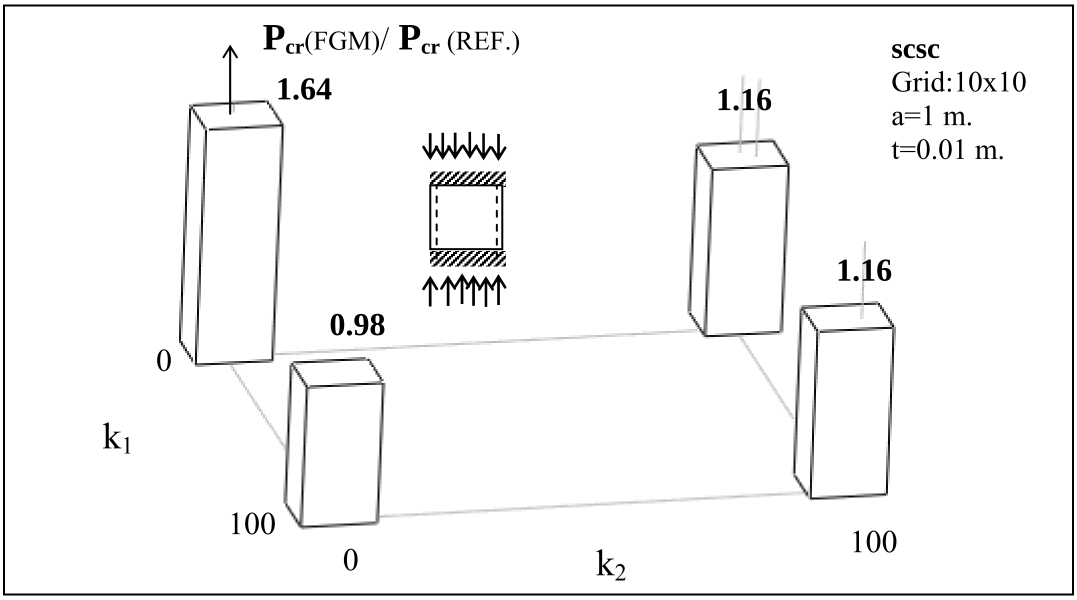

Figure 5.37: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SCSC) Plate under Uniaxial Compressive Loading (Nx=0, Ny=1) on Two-Parameter Foundation

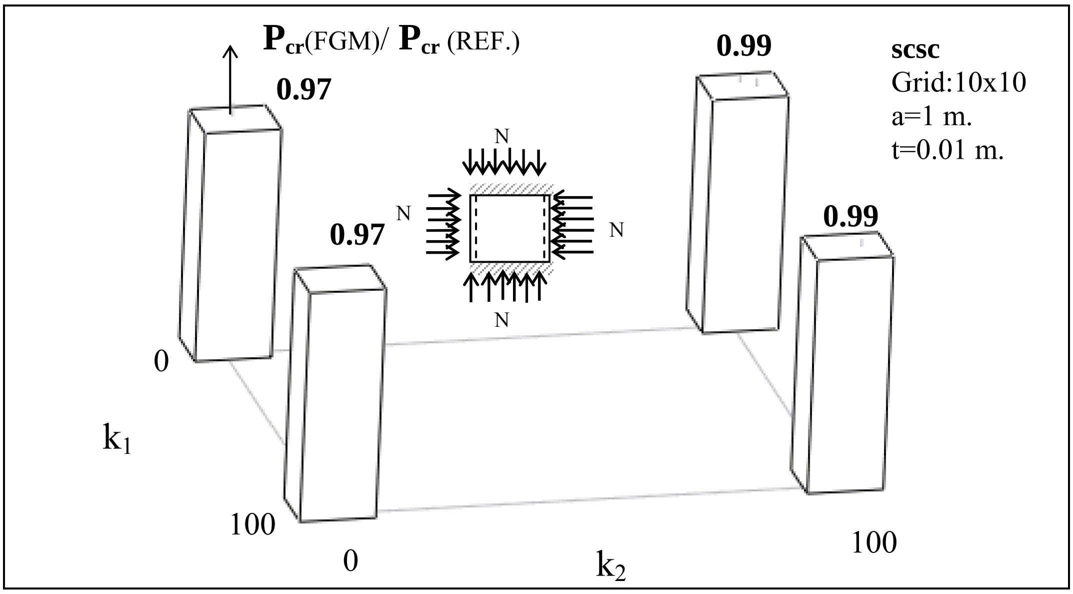

Figure 5.38: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SCSC) Plate under Biaxial Compressive Loading (Nx=1, Ny=1) on Two-Parameter Foundation

Table 5.11: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load Cases of the SFSF Square Plate in Case of Different Foundation Parameters

a=1,h=.01.D=1 E=10920000			SFSF (Nc/π^2)								
			Nx=1,Ny=0			Nx=0,Ny=1			Nx=1,Ny=1		
Case	k1	k2	FG	Ref.	Error %	FG	Ref.	Error %	FG	Ref.	Error %
1	0	0	0.9098	0.952	4.43	2.9702	2.593	14.55	0.9098	0.93	2.38
2	100	0	1.9362	1.979	2.16	3.454	2.82	22.48	1.9362	1.63	19.08
3	0	100	11.043	14.07	21.51	15.291	15.28	0.07	11.795	11.06	6.65
4	100	100	12.068	14.33	15.79	15.459	15.4	0.38	12.068	11.76	2.62

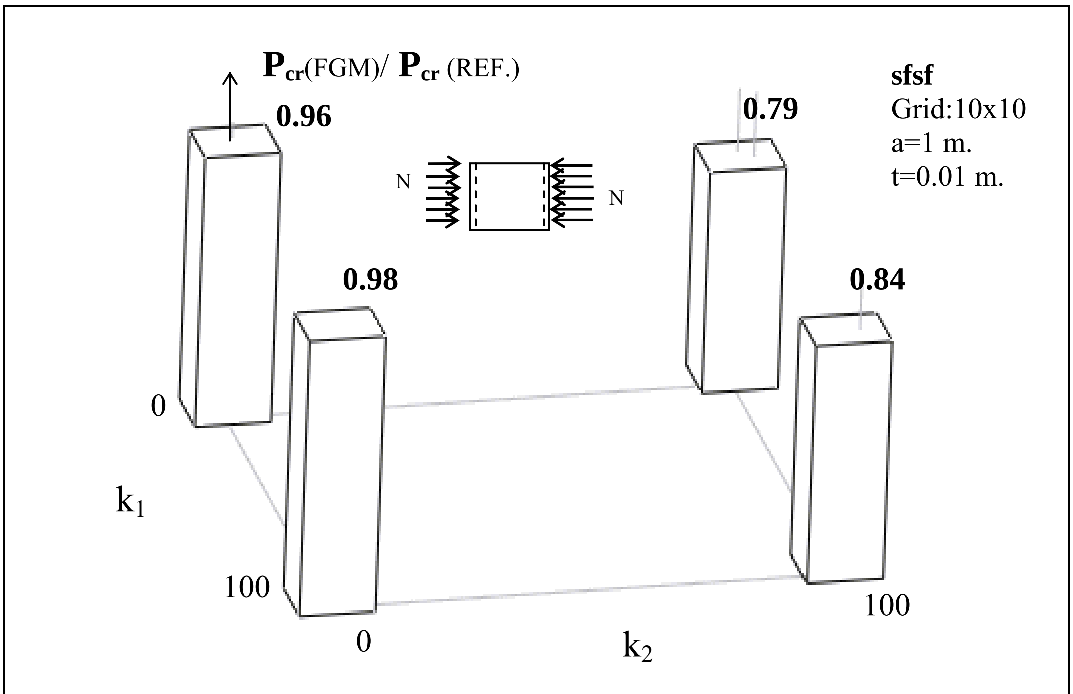

Figure 5.39: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SFSF) Plate under Uniaxial Compressive Loading (Nx=1, Ny=0) on Two-Parameter Foundation

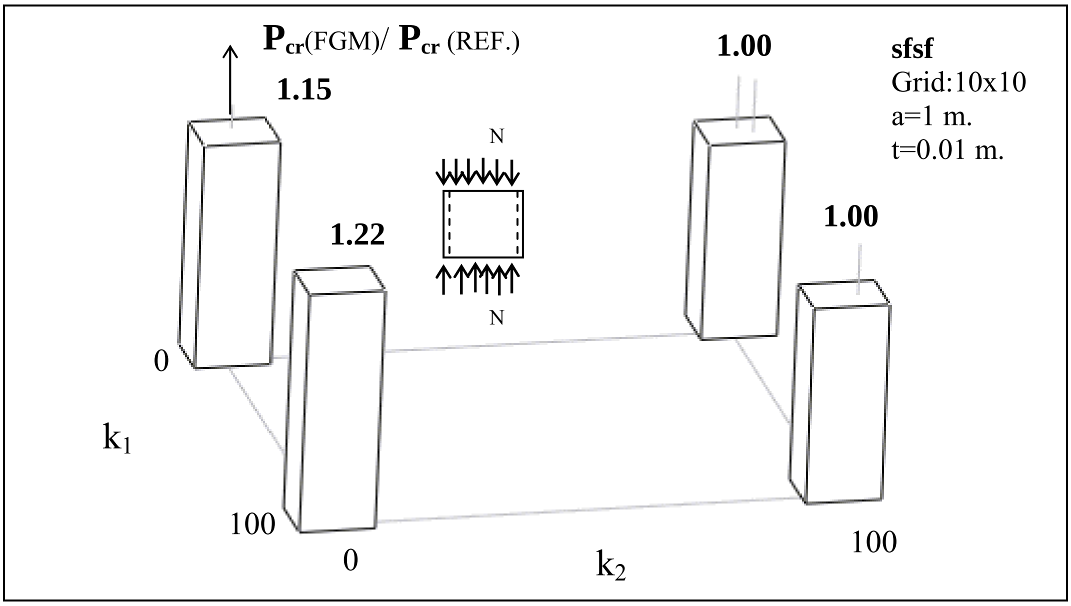

Figure 5.40: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SFSF) Plate under Uniaxial Compressive Loading (Nx=0, Ny=1) on Two-Parameter Foundation

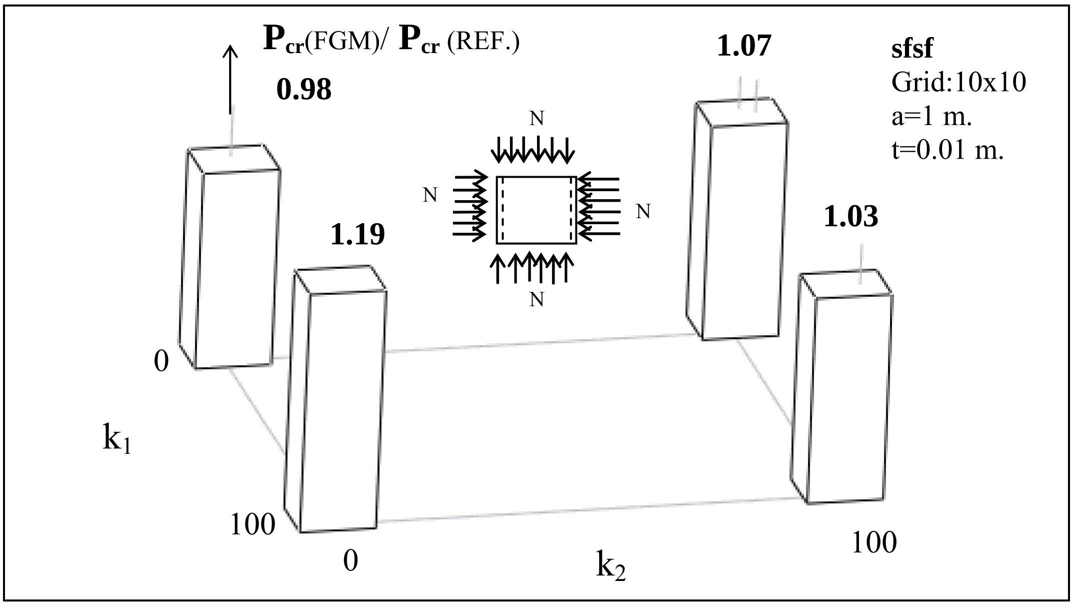

Figure 5.41: Comparison of the Finite Grid Solution with the Conical Exact Solutions for Buckling Load of a Square (SFSF) Plate under Biaxial Compressive Loading (Nx=1, Ny=1) on Two-Parameter Foundation

CHAPTER 6

CONCLUSIONS

6.1 SUMMARY

Research on easily understood engineering approaches for analysis of plates resting on elastic foundations has not been covered sufficiently in the literature. For particular plate problems, closed form solutions have been obtained. However, even for conventional plate analysis these solutions can only be applied to the problems with simple geometry, load and boundary conditions. Of course for the two-parameter elastic foundation soil model underneath plate problems the solution will be too much complex and there is no analytical solution other than simple cases. Therefore the objective of the present study has been to develop a quite general numerical solution for plates on elastic foundations.

In this study a combination of finite element method, Lattice analogy and matrix displacement analysis of gridworks used to obtain a finite grid solution. In this method the plate is modeled as an assemblage of individual beam elements interconnected at their neighboring joints. Therefore the exact fixed end forces and the exact stiffness, consistent mass and geometric stiffness matrices for conventional beam elements and beam elements resting on one or two parameter foundation are valuable tools to solve plate vibration, buckling and bending problems. By this representation, also the plate problems which have non-uniform thickness and foundation properties, arbitrary boundary and loading conditions and discontinuous surfaces, can be solved in a general form.

The first aim has been to review the governing differential equations of beam elements. After obtaining solutions of the governing differential equations, exact shape functions (interpolation functions) have been derived by imposing boundary conditions. This study is extended to derive exact stiffness matrix, consistent mass and geometric stiffness matrices and work equivalent load vectors by finite element method. Then the discretized plate element reassembled by the matrix displacement method. That is, the stiffness, consistent mass and geometric stiffness matrices of the total structure is generated by using a proper numbering shame to collect all displacements for each nodal point in a convenient sequence.

A wide range of complicated circular, annular and rectangular plate problems (such as; plate analysis, plates on one-parameter elastic foundation and plates on two- parameter elastic foundation) were solved by this solution technique called finite grid solution. It has been verified the validity of our solution with a broad range of applications such as bending, buckling and free vibration analysis of plates on either one or two parameter elastic foundation. In addition the method is also applicable to slabs, girders and mat foundations of structures.

6.2 DISCUSSION OF RESULTS AND CONCLUSIONS

From the derivations and applications of this dissertation some general points can be underlined as follows:

The shape functions related to beams on elastic foundations are very sensitive to variation of foundation parameters after some limits. There are significant differences between fixed end loads obtained from Hermitian polynomials rather than exact shape functions. Comparisons between the two cases have been shown graphically. It can be recommended that for consistency one must use fixed end loads to represent load vector by using the results obtained from exact shape functions.

A different solution method has been proposed by researchers for the problem of beams resting on Winkler type foundation. They have inserted Hermitian polynomials into strain energy functional that has been derived in this study. In order to converge to the exact solution, the beam needs to be divided into smaller segments. The solution method is acceptable from two points of view. First, they use the same strain energy functional that is a correct treatment. In the second point, they divide the beam into smaller elements. Shape functions converge towards Hermitian polynomials when the parameter λL becomes smaller. This trend is portrayed graphically in Chapter 2. Therefore the solution is acceptable only at the expense of additional calculation.

In this dissertation the response of plates underlain by a Winkler foundation and two-parameter foundation for the same problem was compared in many applications. From these results, it is inferred that presence of second foundation parameter k_θ in the analysis is remarkably dominant. For instance, it gives smaller displacements so that smaller internal stresses, larger buckling loads and larger free vibration frequencies This might have been anticipated because strain energy density functional includes one more term in the case of two parameter foundation than in the case of the Winkler foundation.

It is observed that the buckling load parameters increase as the foundation parameters increase. However the second foundation parameter exerts a greater influence on buckling loads and the fundamental frequencies when compared to the first foundation parameter.

6.3 SUGGESTED FURTHER STUDIES

Some assumptions have been used throughout the present study. Under these assumptions, our solution strategy can be extended to many applications. The following assumptions are drawn. Violation of these assumptions can lead other researches to continue the studies in this subject:

1. In this solution, nonlinear effects of both plate and foundation are not included. One can apply the solutions to analyze plates with elastic foundation only if displacement field consists of small deflections.

2. Depth effect of foundation is ignored in response. Since properties of continuum may change significantly at a considerable depth, depth effect may be needed for certain types of problems.

3. Friction, shear deformation and torsional changes due to foundation parameters are not included.

4. The foundation parameters are assumed to be constant and equal in compression and tension. In the general case, the foundation behaves differently, cannot take any tension. An iterative technique can be adapted to this solution for plates on tensionless foundations.

5. In this study, the foundation parameters are assumed to be constants and their properties are not considered. Both experimental and theoretical studies must be performed.

6. The solution technique presented in this study could be extended to three-dimensional structures by discretizing vertical elements such as columns and shear walls.

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APPENDIX A

Geometric stiffness terms of beam elements resting on two - parameter elastic foundation:

a) For **A<2*Sqrt(B)**

where $A= k_{\theta}/(E*I)$ and $B= k_1/(E*I)$ see page 44

```
XL=L
p=λ*XL
λ= (k1/(4*E*I))**0.25
δ= kθ/(4*E*I)
t =δ/λ**2
        t1=(1-t)**(0.5)
        t2=(1+t)**(0.5)
        z1=t1
        z2=t2
        z=t
C
   D=(p**2*(-2 + Cos(2*p*z1) + z*Cos(2*p* z1) +
   Cosh(2*p*z2) - z*Cosh(2*p*z2)))/XL**2
C
kG33=  kG66 =  (p*(-16*p*z1*z*z2 +8*p*z1*z2*Cos(2*p*z1) +
     8*p*z1*z*z2*Cos(2*p*z1) -8*p*z1*z2*Cosh(2*p*z2) +
     8*p*z1*z*z2*Cosh(2*p*z2) -4*z2*Sin(2*p*z1) +
     12*z*z2*Sin(2*p*z1) +6*z2*Cosh(2*p*z2)*Sin(2*p*z1) -
     8*z*z2*Cosh(2*p*z2)*Sin(2*p*z1) +2*z**2*z2*Cosh(2*p*z2)*
     Sin(2*p*z1) -z2*Sin(4*p*z1) -2*z*z2*Sin(4*p*z1) -
     z**2*z2*Sin(4*p*z1) +4*z1*Sinh(2*p*z2) +
     12*z1*z*Sinh(2*p*z2) -6*z1*Cos(2*p*z1)*Sinh(2*p*z2) -
     8*z1*z*Cos(2*p*z1)*Sinh(2*p*z2) -2*z1*z**2*Cos(2*p*z1)*
     Sinh(2*p*z2) -8*p*Sin(2*p*z1)*Sinh(2*p*z2) +
     8*p*z**2*Sin(2*p*z1)*Sinh(2*p*z2) +
     z1*Sinh(4*p*z2) -2*z1*z*Sinh(4*p*z2) +z1*z**2*Sinh(4*p*z2)))/
     (4*XL*z1*z2*(-2 + Cos(2*p*z1) + z*Cos(2*p*z1) +
      Cosh(2*p*z2) - z*Cosh(2*p*z2))**2)
C
```

$\mathbf{k}_{G22}$= $\mathbf{k}_{G55}$ = (p**3*(8*p*z1*z2 +8*p*z1*z**2*z2 -8*p*z1*z2*Cos(2*p*z1) -
8*p*z1*z*z2*Cos(2*p*z1) -8*p*z1*z2*Cosh(2*p*z2) +
8*p*z1*z*z2*Cosh(2*p*z2) +8*p*z1*z2*Cos(2*p*z1)*
Cosh(2*p*z2) -8*p*z1*z**2*z2*Cos(2*p*z1)*
Cosh(2*p*z2) + 4*z2*Sin(2*p*z1) -12*z*z2*Sin(2*p*z1) -
6*z2*Cosh(2*p*z2)*Sin(2*p*z1) +8*z*z2*Cosh(2*p*z2)*
Sin(2*p*z1) -2*z**2*z2*Cosh(2*p*z2)*Sin(2*p*z1) +
z2*Sin(4*p*z1) +2*z*z2*Sin(4*p*z1) +z**2*z2*Sin(4*p*z1) +
4*z1*Sinh(2*p*z2) +12*z1*z*Sinh(2*p*z2) -
6*z1*Cos(2*p*z1)*Sinh(2*p*z2) -
8*z1*z*Cos(2*p*z1)*Sinh(2*p*z2) -
2*z1*z**2*Cos(2*p*z1)*Sinh(2*p*z2) -
8*p*z*Sin(2*p*z1)*Sinh(2*p*z2) +
8*p*z**3*Sin(2*p*z1)*Sinh(2*p*z2) +
z1*Sinh(4*p*z2) -2*z1*z*Sinh(4*p*z2) +
z1*z**2*Sinh(4*p*z2)))/(8*D**2*XL**3*z1*z2)
C
$\mathbf{k}_{G32}$= $\mathbf{k}_{G23}$ = -(-4*p**4*Sin(p*z1)*Sinh(p*z2)*(p*z2*Cosh(p*z2)*Sin(p*z1)-
p*z*z2*Cosh(p*z2)*Sin(p*z1) +p*z1*Cos(p*z1)*Sinh(p*z2) +
p*z1*z*Cos(p*z1)*Sinh(p*z2) -2*Sin(p*z1)*Sinh(p*z2)))/
(D**2*XL**4)
C
$\mathbf{k}_{G56}$= $\mathbf{k}_{G65}$ = (-4*p**4*Sin(p*z1)*Sinh(p*z2)*(p*z2*Cosh(p*z2)*Sin(p*z1)-
p*z*z2*Cosh(p*z2)*Sin(p*z1) +p*z1*Cos(p*z1)*Sinh(p*z2) +
p*z1*z*Cos(p*z1)*Sinh(p*z2) -2*Sin(p*z1)*Sinh(p*z2)))/
(D**2*XL**4)
C
$\mathbf{k}_{G52}$= $\mathbf{k}_{G25}$= (p**3*(-4*p*z1*z**2*z2*Cos(p*z1)*Cosh(p*z2) +
2*p*z1*z*z2*Cos(3*p*z1)*Cosh(p*z2) +
2*p*z1*z**2*z2*Cos(3*p*z1)*Cosh(p*z2) -
2*p*z1*z*z2*Cos(p*z1)*Cosh(3*p*z2) +2*p*z1*z**2*z2*Cos(p*z1)*
Cosh(3*p*z2) +10*z*z2*Cosh(p*z2)*Sin(p*z1) +
2*z**2*z2*Cosh(p*z2)*Sin(p*z1) +3*z2*Cosh(3*p*z2)*Sin(p*z1) -
4*z*z2*Cosh(3*p*z2)*Sin(p*z1) +z**2*z2*Cosh(3*p*z2)*
Sin(p*z1) -z2*Cosh(p*z2)*Sin(3*p*z1) -
2*z*z2*Cosh(p*z2)*Sin(3*p*z1)-z**2*z2*Cosh(p*z2)*Sin(3*p*z1) -
10*z1*z*Cos(p*z1)*Sinh(p*z2)+2*z1*z**2*Cos(p*z1)*Sinh(p*z2) +
3*z1*Cos(3*p*z1)*Sinh(p*z2)+4*z1*z*Cos(3*p*z1)*Sinh(p*z2) +
z1*z**2*Cos(3*p*z1)*Sinh(p*z2)+4*p*z*Sin(p*z1)*Sinh(p*z2) -
4*p*z**3*Sin(p*z1)*Sinh(p*z2)+2*p*Sin(3*p*z1)*Sinh(p*z2) +
2*p*z*Sin(3*p*z1)*Sinh(p*z2)-2*p*z**2*Sin(3*p*z1)*Sinh(p*z2) -
2*p*z**3*Sin(3*p*z1)*Sinh(p*z2)-z1*Cos(p*z1)*Sinh(3*p*z2) +
2*z1*z*Cos(p*z1)*Sinh(3*p*z2)-z1*z**2*Cos(p*z1)*Sinh(3*p*z2) -
2*p*Sin(p*z1)*Sinh(3*p*z2)+2*p*z*Sin(p*z1)*Sinh(3*p*z2) +
2*p*z**2*Sin(p*z1)*Sinh(3*p*z2) -
2*p*z**3*Sin(p*z1)*Sinh(3*p*z2)))/(4*D**2*XL**3*z1*z2)

C
$\mathbf{k}_{G35}$= $\mathbf{k}_{G53}$= -(p**4*(-2*z + Cos(2*p*z1) + z*Cos(2*p*z1) -
Cosh(2*p*z2) + z*Cosh(2*p*z2))*
(p*z2*Cosh(p*z2)*Sin(p*z1) -
p*z*z2*Cosh(p*z2)*Sin(p*z1) +
p*z1*Cos(p*z1)*Sinh(p*z2) +
p*z1*z*Cos(p*z1)*Sinh(p*z2) -
2*Sin(p*z1)*Sinh(p*z2)))/(D**2*XL**4*z1*z2)
C
$\mathbf{k}_{G26}$= $\mathbf{k}_{G62}$ =(p**4*(-2*z + Cos(2*p*z1) + z*Cos(2*p*z1) -
Cosh(2*p*z2) + z*Cosh(2*p*z2))*
(p*z2*Cosh(p*z2)*Sin(p*z1) -
p*z*z2*Cosh(p*z2)*Sin(p*z1) +
p*z1*Cos(p*z1)*Sinh(p*z2) +
p*z1*z*Cos(p*z1)*Sinh(p*z2) -
2*Sin(p*z1)*Sinh(p*z2)))/(D**2*XL**4*z1*z2)

$\mathbf{k}_{G36}$=$\mathbf{k}_{G63}$= (p**5*(4*p*z1*z*z2*Cos(p*z1)*
Cosh(p*z2) -2*p*z1*z2*Cos(3*p*z1)*Cosh(p*z2) -
2*p*z1*z*z2*Cos(3*p*z1)*Cosh(p*z2) +
2*p*z1*z2*Cos(p*z1)*Cosh(3*p*z2) -
2*p*z1*z*z2*Cos(p*z1)*Cosh(3*p*z2) -
10*z*z2*Cosh(p*z2)*Sin(p*z1)-2*z**2*z2*Cosh(p*z2)*Sin(p*z1) -
3*z2*Cosh(3*p*z2)*Sin(p*z1)+4*z*z2*Cosh(3*p*z2)*Sin(p*z1) -
z**2*z2*Cosh(3*p*z2)*Sin(p*z1) +
z2*Cosh(p*z2)*Sin(3*p*z1)+2*z*z2*Cosh(p*z2)*Sin(3*p*z1) +
z**2*z2*Cosh(p*z2)*Sin(3*p*z1)-10*z1*z*Cos(p*z1)*Sinh(p*z2) +
2*z1*z**2*Cos(p*z1)*Sinh(p*z2) +
3*z1*Cos(3*p*z1)*Sinh(p*z2)+4*z1*z*Cos(3*p*z1)*Sinh(p*z2) +
z1*z**2*Cos(3*p*z1)*Sinh(p*z2) +
16*p*Sin(p*z1)*Sinh(p*z2)-16*p*z**2*Sin(p*z1)*Sinh(p*z2) -
z1*Cos(p*z1)*Sinh(3*p*z2)+2*z1*z*Cos(p*z1)*Sinh(3*p*z2) -
z1*z**2*Cos(p*z1)*Sinh(3*p*z2)))/(2*D**2*XL**5*z1*z2)
C
$\mathbf{k}_{G11}$= $\mathbf{k}_{G44}$=1
$\mathbf{k}_{G14}$= $\mathbf{k}_{G41}$= -1
$\mathbf{k}_{G12}$= $\mathbf{k}_{G13}$ =$\mathbf{k}_{G15}$= $\mathbf{k}_{G16}$ =$\mathbf{k}_{G42}$= $\mathbf{k}_{G43}$ =$\mathbf{k}_{45}$= $\mathbf{k}_{G46}$=0

b) For **A>2*Sqrt(B)**

Geometric stiffness terms of beam elements resting on two - parameter elastic foundation in exponential form;

where A= k_θ/(E*I) and B= k_1/(E*I) see page 44.

XL=L
λ= (k_1/(4*E*I))**0.25
δ= k_θ/(4*E*I)
C
a=sqrt(λ**2+δ)
b=sqrt(δ-λ**2)
C
y1=Exp(2*b*XL)
y2=Exp(2*a*XL)
y3=Exp((a-b)*XL)
y4=Exp((a+b)*XL)
y5=Exp((-a+b)*XL)
y6=Exp(2*(a-b)*XL)
y7=Exp(2*(a+b)*XL)
y8=8*a**2-8*b**2-(4*a**2)/y1-4*a**2*y1+(4*b**2)/y2+4*b**2*y2
y9=2*(-a+b)*(-a-b+a*y1+b*y2)
y10=b+a*y2-b*y2-a*y1*y2
y11=-a+a*y1-b*y1+b*y1*y2
y12=-(b*y1)-a*y2+a*y1*y2+b*y1*y2
y13=2*(b**2*y1-a**2*y2+2*a**2*y1*y2-
2*b**2*y1*y2-a**2*(y1)**2*y2+b**2*y1*(y2)**2)
y14=2*a+2*b-2*a*y1-(2*b)/y2
y15=2*a-2*b-(2*a)/y1+(2*b)/y2
y16=2*(-a+b+a*y1-b*y2)
y17=-2*a-2*b+(2*a)/y1+2*b*y2
y18=-a-b+b*y3*y4+a*y4*y5
y19=b-a*y2-b*y2+a*y6
y20=b+a*y2-b*y2-a*y7
y21=-a+a*y1-b*y1+b*y7
y22=-2*a*y3-2*b*y3+(2*b)/y4+2*a*y4
y23=2*(a*y3-a*y4+b*y4-b*y5)
y24=(-2*a)/y4-2*b*y4+2*a*y5+2*b*y5
y25=a-b+b*y3*y4-a*y4*y5
C
k_{G33}= **k_{G66}**=-((a-b)**2*(a+b)*y12**2)/(2*y13**2)+
(a-b)**2*(a+b)*Exp(2*(a+b)*XL)*y12**2/(2*y13**2)+
Exp((a+b)*XL)*(-2*(-(a**2)+b**2)**2*Exp((-a+b)*XL)*y11*y12/
(b*y1*y13*y8)+2*(a**2-b**2)**2*y10*y12/

(a*Exp((-a+b)*XL)*y13*y2*y8))-
2*(-a+b)*(a+b)**2*y11**2/(y1**2*y8**2)+
2*(-a+b)*(a+b)**2*Exp(2*(-a+b)*XL)*y11**2/(y1**2*y8**2)-
2*(a-b)*(a+b)**2*y10**2/(y2**2*y8**2)+
2*(a-b)*(a+b)**2*y10**2/(Exp(2*(-a+b)*XL)*y2**2*y8**2)+
2*(-(a**2)+b**2)**2*y11*y12/(b*y1*y13*y8)-
2*(a**2-b**2)**2*y10*y12/(a*y13*y2*y8)-
2*(-a+b)*(a+b)**2*y11*y9/(a*y1*y8**2)-
2*(a-b)*(a+b)**2*y10*y9/(b*y2*y8**2)+(a+b)*y9**2/(2*y8**2)-
(a+b)*y9**2/(2*Exp(2*(a+b)*XL)*y8**2)+
(2*(-a+b)*(a+b)**2*Exp((-a+b)*XL)*y11*y9/(a*y1*y8**2)+
2*(a-b)*(a+b)**2*y10*y9/(b*Exp((-a+b)*XL)*y2*y8**2))/
Exp((a+b)*XL)+2*(-a+b)*(a+b)**2*XL*
(4*a*y10*y11*y13-4*b*y10*y11*y13+y1*y12*y2*y8*y9)/
(y1*y13*y2*y8**2)
C
TKG122=((a-b)*(a+b)*y10*y14/(y2*y8**2)+
a**2*(a+b)*y10*y16/(b*Exp(2*a*XL)*y2*y8**2))/Exp(2*(-a+b)*XL)-
2*(-a+b)**2*(a+b)*XL*y11*y14/(y1*y8**2)-
(-a+b)*(a+b)**2*y11*y15/(b*y1*y8**2)+
(-a+b)*(a+b)**2*Exp(2*b*XL)*y11*y15/(b*y1*y8**2)-
(-a+b)*(a+b)**2*y11*y16/(a*y1*y8**2)+
(-a+b)*(a+b)**2*y11*y16/(a*Exp(2*a*XL)*y1*y8**2)-
(-a+b)*(a+b)*y11*y17/(y1*y8**2)+
(-a+b)*(a+b)*Exp(2*(-a+b)*XL)*y11*y17/(y1*y8**2)-
(a-b)*(a+b)*y10*y14/(y2*y8**2)-
(a-b)*(a+b)**2*y10*y15/(a*y2*y8**2)+
(a-b)*(a+b)**2*Exp(2*a*XL)*y10*y15/(a*y2*y8**2)-
a**2*(a+b)*y10*y16/(b*y2*y8**2)+b*(a+b)*y10*y16/(y2*y8**2)-
b*(a+b)*y10*y16/(Exp(2*b*XL)*y2*y8**2)-
(a-b)**2*(a+b)*y12*y14/(2*a*y13*y8)+
(a-b)**2*(a+b)*Exp(2*a*XL)*y12*y14/(2*a*y13*y8)-
(a-b)*(a+b)*y12*y15/(2*y13*y8)
$\mathbf{k}_{\mathbf{G}56}$= $\mathbf{k}_{\mathbf{G}65}$= -$\mathbf{k}_{\mathbf{G}23}$= -$\mathbf{k}_{\mathbf{G}32}$= **TKG**122+
(a-b)*(a+b)*Exp(2*a*XL+2*b*XL)*y12*y15/(2*y13*y8)+
(-a+b)**2*(a+b)*y12*y17/(2*b*y13*y8)-
(-a+b)**2*(a+b)*Exp(2*b*XL)*y12*y17/(2*b*y13*y8)+
XL*(-2*a**3*y10*y13*y17+2*a**2*b*y10*y13*y17+
2*a*b**2*y10*y13*y17-
2*b**3*y10*y13*y17-a**3*y12*y16*y2*y8-a**2*b*y12*y16*y2*y8+
a*b**2*y12*y16*y2*y8)/(y13*y2*y8**2)-
(a**2-b**2)*y14*y9/(2*b*y8**2)+
(a**2-b**2)*y14*y9/(2*b*Exp(2*b*XL)*y8**2)+
(a+b)*y16*y9/(2*y8**2)-
(-(a**2)+b**2)*y17*y9/(2*a*y8**2)+
XL*(b**3*y12*y16*y8-a**2*y13*y15*y9-2*a*b*y13*y15*y9-

```
  b**2*y13*y15*y9)/(y13*y8**2)+
  (-((a+b)*y16*y9)/(2*y8**2)+
  (-(a**2)+b**2)*Exp(2*b*XL)*y17*y9/(2*a*y8**2))/Exp(2*(a+b)*XL)
C
  w1=(a+b)
  w2=(a-b)
  w3=(-a+b)
C
  TKG133=-2*(a**2-b**2)**2*Exp(w3*XL)*y11*y19/(a*y1*y8**2)+
   2*(a**2-b**2)**2*Exp(-2*a*XL+w3*XL)*y11*y19/(a*y1*y8**2)+
   2*(-(a**2)+b**2)**2*Exp(w3*XL)*y10*y19/(b*y2*y8**2)-
   2*(-(a**2)+b**2)**2*Exp(-2*b*XL+w3*XL)*y10*y19/(b*y2*y8**2)-
   2*b**4*y11*y20/(w3*Exp(w1*XL)*y1*y8**2)-
   2*a**2*(-(a**2)+2*b**2)*y11*y20/(w2*Exp(w1*XL)*y1*y8**2)+
   2*b**4*Exp(2*w3*XL-w1*XL)*y11*y20/(w3*y1*y8**2)+
   2*a**2*(-(a**2)+2*b**2)*Exp(2*w3*XL-w1*XL)*y11*y20/
   (w2*y1*y8**2)+2*b**4*y10*y21/(w3*Exp(w1*XL)*y2*y8**2)-
   2*a**2*(-(a**2)+2*b**2)*y10*y21/(w3*Exp(w1*XL)*y2*y8**2)-
   2*b**4*Exp(2*w2*XL-w1*XL)*y10*y21/(w3*y2*y8**2)+
   2*a**2*(-(a**2)+2*b**2)*Exp(-2*w3*XL-w1*XL)*y10*y21/
   (w3*y2*y8**2)-4*(a**2-b**2)**2*XL*(y1*y10*y20+y11*y2*y21)/
   (Exp(w1*XL)*y1*y2*y8**2)+2*(a**2-b**2)**2*y10*y18/
   (a*y2*y4*y8**2)-2*(a**2-b**2)**2*Exp(2*a*XL)*y10*y18/
   (a*y2*y4*y8**2)+2*(a**2-b**2)**2*Exp(-(a*XL)+b*XL)*XL*y12*y19/
   (y13*y8)-(a**2-b**2)**2*y12*y21/(a*Exp(w1*XL)*y13*y8)
kG36=kG63=TKG133+(a**2-b**2)**2*Exp(2*a*XL-w1*XL)*y12*y21/(a*y13*y8)+
   w2**2*w1*y12*y18/(y13*y4*y8)-w2**2*w1*Exp(2*w1*XL)*y12*y18/
   (y13*y4*y8)-(-(a**2)+b**2)**2*(2*y11*y13*y18-y1*y12*
   y20*y4*y8/Exp(w1*XL))/(b*y1*y13*y4*y8**2)+(-(a**2)+b**2)**2*
   Exp(2*b*XL)*(2*y11*y13*y18-y1*y12*y20*y4*y8/Exp(w1*XL))/
   (b*y1*y13*y4*y8**2)-w2*w1*Exp(w3*XL)*y19*y9/y8**2+
   w2*w1*Exp(w3*XL-2*w1*XL)*y19*y9/y8**2-w3*w1**2*y20*y9/
   (a*Exp(w1*XL)*y8**2)+w3*w1**2*Exp(-2*a*XL-w1*XL)*y20*y9/
   (a*y8**2)-w2*w1**2*y21*y9/(b*Exp(w1*XL)*y8**2)+
   w2*w1**2*Exp(-2*b*XL-w1*XL)*y21*y9/(b*y8**2)+
   2*w2*w1**2*XL*y18*y9/(y4*y8**2)
C
  TKG144=Exp(2*b*XL)*(w2*b*y12*y22/(2*y13*y8)+w2*w1*Exp(2*a*XL)*
   y12*y25/(y13*y4*y8))-w3*w1*y11*y22/(y1*y8**2)+w3*w1*
   Exp(2*w3*XL)*y11*y22/(y1*y8**2)-2*w3**2*w1*XL*y10*y22/
   (y2*y8**2)-w3*w1**2*y11*y23/(a*y1*y8**2)+w3*w1**2*y11*y23/
   (a*Exp(2*a*XL)*y1*y8**2)-a**2*w1*y10*y23/(b*y2*y8**2)+
   b*w1*y10*y23/(y2*y8**2)-b*w1*y10*y23/(Exp(2*b*XL)*y2*y8**2)+
   a**2*w1*Exp(-2*a*XL-2*w3*XL)*y10*y23/(b*y2*y8**2)-
   2*w3**2*w1*XL*y11*y24/(y1*y8**2)-w2*w1*y10*y24/(y2*y8**2)+
   w2*w1*Exp(2*w2*XL)*y10*y24/(y2*y8**2)-2*w2*w1**2*y10*y25/
```

(a*y2*y4*y8**2)-w2*b*y12*y22/(2*y13*y8)+w3*w1**2*XL*y12*y23/
(y13*y8)-w2**2*w1*y12*y24/(2*a*y13*y8)+w2**2*w1*Exp(2*a*XL)*
y12*y24/(2*a*y13*y8)-w2*w1*y12*y25/(y13*y4*y8)-w3*(4*a**2*y11*
y13*y25+8*a*b*y11*y13*y25+4*b**2*y11*y13*y25+a**2*y1*y12*y22*
y4*y8)/(2*b*y1*y13*y4*y8**2)+Exp(2*b*XL)*(2*w2*w1**2*
Exp(2*w2*XL)*y10*y25/(a*y2*y4*y8**2)+w3*
(4*a**2*y11*y13*y25+8*a*b*y11*y13*y25+4*b**2*y11*y13*y25+
a**2*y1*y12*y22*y4*y8)/(2*b*y1*y13*y4*y8**2))

$\mathbf{k}_{\mathbf{G}26}$= $\mathbf{k}_{\mathbf{G}62}$= $-\mathbf{k}_{\mathbf{G}35}$= $-\mathbf{k}_{\mathbf{G}53}$= **TKG**144-(-(a**2)+b**2)*y22*y9/(2*a*y8**2)+
(-(a**2)+b**2)*y22*y9/(2*a*Exp(2*a*XL)*y8**2)+
w1*y23*y9/(2*y8**2)-a**2*y24*y9/(2*b*y8**2)+
b*y24*y9/(2*y8**2)-b*y24*y9/(2*Exp(2*b*XL)*y8**2)-
2*w1**2*XL*y25*y9/(y4*y8**2)+(-(w1*y23*y9)/
(2*Exp(2*a*XL)*y8**2)+
a**2*y24*y9/(2*b*y8**2))/Exp(2*b*XL)

C

$\mathbf{k}_{\mathbf{G}22}$= $\mathbf{k}_{\mathbf{G}55}$= Exp(w1*XL)*((a**2-b**2)*Exp(w2*XL)*y14*y15/(a*y8**2)+
(-(a**2)+b**2)*y15*y17/(b*Exp(w2*XL)*y8**2))+
((a**2-b**2)*Exp(w2*XL)*y14*y16/(b*y8**2)+
(-(a**2)+b**2)*y16*y17/(a*Exp(w2*XL)*y8**2))/Exp(w1*XL)-
w2*y14**2/(2*y8**2)+w2*Exp(2*w2*XL)*y14**2/(2*y8**2)-
(a**2-b**2)*y14*y15/(a*y8**2)-w1*y15**2/(2*y8**2)+
w1*Exp(2*w1*XL)*y15**2/(2*y8**2)-
(a**2-b**2)*y14*y16/(b*y8**2)+w1*y16**2/(2*y8**2)-
w1*y16**2/(2*Exp(2*w1*XL)*y8**2)-
(-(a**2)+b**2)*y15*y17/(b*y8**2)-
(-(a**2)+b**2)*y16*y17/(a*y8**2)-
w3*y17**2/(2*y8**2)+w3*y17**2/
(2*Exp(2*w2*XL)*y8**2)+
2*XL*(-(a**2*y15*y16)-2*a*b*y15*y16-b**2*y15*y16-a**2*y14*y17+
2*a*b*y14*y17-b**2*y14*y17)/y8**2

C

$\mathbf{k}_{\mathbf{G}25}$= $\mathbf{k}_{\mathbf{G}52}$= ((-(a**2)+b**2)*(y16*y22+y17*y23)/
(2*a*Exp(w2*XL)*y8**2)+(a**2-b**2)*Exp(w2*XL)*(y14*y23+y16*y24)/
(2*b*y8**2))/Exp(w1*XL)+Exp(w1*XL)*((-(a**2)+b**2)*(2*y17*y25+
y15*y22*y4)/(2*b*Exp(w2*XL)*y4*y8**2)+(a**2-b**2)*Exp(w2*XL)*
(2*y14*y25+y15*y24*y4)/(2*a*y4*y8**2))-w3*y17*y22/(2*y8**2)+
w3*y17*y22/(2*Exp(2*w2*XL)*y8**2)+w1*y16*y23/(2*y8**2)-
w1*y16*y23/(2*Exp(2*w1*XL)*y8**2)-
(-(a**2)+b**2)*(y16*y22+y17*y23)/(2*a*y8**2)-
w2*y14*y24/(2*y8**2)+w2*Exp(2*w2*XL)*y14*y24/(2*y8**2)-
(a**2-b**2)*(y14*y23+y16*y24)/(2*b*y8**2)-w1*y15*y25/(y4*y8**2)+
w1*Exp(2*w1*XL)*y15*y25/(y4*y8**2)-(-(a**2)+b**2)*
(2*y17*y25+y15*y22*y4)/(2*b*y4*y8**2)-(a**2-b**2)*
(2*y14*y25+y15*y24*y4)/(2*a*y4*y8**2)+XL*
(-2*a**2*y16*y25-4*a*b*y16*y25-2*b**2*y16*y25-a**2*y14*y22*y4+

2*a*b*y14*y22*y4-b**2*y14*y22*y4-a**2*y15*y23*y4-
2*a*b*y15*y23*y4-b**2*y15*y23*y4-a**2*y17*y24*y4+
2*a*b*y17*y24*y4-b**2*y17*y24*y4)/(y4*y8**2)

$\mathbf{k}_{G11} = \mathbf{k}_{G44} = 1$

$\mathbf{k}_{G14} = \mathbf{k}_{G41} = -1$

$\mathbf{k}_{G12} = \mathbf{k}_{G13} = \mathbf{k}_{G15} = \mathbf{k}_{G16} = \mathbf{k}_{G42} = \mathbf{k}_{G43} = \mathbf{k}_{G45} = \mathbf{k}_{G46} = 0$

APPENDIX B

Consistent mass matrix terms of beam elements resting on two - parameter elastic foundation;

a) For **A<2*Sqrt(B)**

where A= k_θ/(E*I) and B= k_1/(E*I) see page 44

XL=L
p=$\lambda*$XL
λ= (k_1/(4*E*I))**0.25
δ= k_θ/(4*E*I)
t =$\delta/\lambda**$2
t1=(1-t)**(0.5)
t2=(1+t)**(0.5)
z1=t1
z2=t2
z =t
C

$\mathbf{m}_{33}$= $\mathbf{m}_{66}$= (XL*(8*p*z1*z2 +8*p*z1*z**2*z2 -8*p*z1*z2*Cos(2*p*z1) -
8*p*z1*z*z2*Cos(2*p*z1) -8*p*z1*z2*Cosh(2*p*z2) +
8*p*z1*z*z2*Cosh(2*p*z2) +8*p*z1*z2*Cos(2*p*z1)*
Cosh(2*p*z2) -8*p*z1*z**2*z2*Cos(2*p*z1)*Cosh(2*p*z2) -
12*z2*Sin(2*p*z1) +4*z*z2*Sin(2*p*z1) +6*z2*Cosh(2*p*z2)*
Sin(2*p*z1) -12*z*z2*Cosh(2*p*z2)*Sin(2*p*z1) +2*z**2*z2*
Cosh(2*p*z2)*Sin(2*p*z1) +4*z**3*z2*Cosh(2*p*z2)*Sin(2*p*z1) +
3*z2*Sin(4*p*z1) +4*z*z2*Sin(4*p*z1) -z**2*z2*Sin(4*p*z1) -
2*z**3*z2*Sin(4*p*z1) -12*z1*Sinh(2*p*z2) -4*z1*z*Sinh(2*p*z2) +
6*z1*Cos(2*p*z1)*Sinh(2*p*z2) +12*z1*z*Cos(2*p*z1)*Sinh(2*p*z2) +
2*z1*z**2*Cos(2*p*z1)*Sinh(2*p*z2) -4*z1*z**3*Cos(2*p*z1)*
Sinh(2*p*z2) +8*p*z*Sin(2*p*z1)*Sinh(2*p*z2) -8*p*z**3*
Sin(2*p*z1)*Sinh(2*p*z2) +3*z1*Sinh(4*p*z2) -4*z1*z*Sinh(4*p*z2) -
z1*z**2*Sinh(4*p*z2) +2*z1*z**3*Sinh(4*p*z2)))/(8*p*z1*z2*
(-2 + Cos(2*p*z1) + z*Cos(2*p*z1) +
Cosh(2*p*z2) - z*Cosh(2*p*z2))**2)
C
$\mathbf{m}_{56}$= $\mathbf{m}_{65}$= -$\mathbf{m}_{23}$= -$\mathbf{m}_{32}$= (XL**2*(-4*z + 4*Cos(2*p*z1) + 4*z**2*Cos(2*p*z1) -
Cos(4*p*z1)-2*z*Cos(4*p*z1)-z**2*Cos(4*p*z1)-4*Cosh(2*p*z2)-

4*z**2*Cosh(2*p*z2) +8*z*Cos(2*p*z1)*Cosh(2*p*z2) +
Cosh(4*p*z2) - 2*z*Cosh(4*p*z2) +z**2*Cosh(4*p*z2) -
4*p*z1*Sin(2*p*z1) -4*p*z1*z*Sin(2*p*z1) +
4*p*z1*Cosh(2*p*z2)*Sin(2*p*z1)+
4*p*z1*z*Cosh(2*p*z2)*Sin(2*p*z1) -4*p*z2*Sinh(2*p*z2) +
4*p*z*z2*Sinh(2*p*z2) +4*p*z2*Cos(2*p*z1)*Sinh(2*p*z2) -
4*p*z*z2*Cos(2*p*z1)*Sinh(2*p*z2)))/
(8*p**2*(-2 + Cos(2*p*z1) + z*Cos(2*p*z1) +
Cosh(2*p*z2) - z*Cosh(2*p*z2))**2)
C
C
$\mathbf{m}_{26}$= $\mathbf{m}_{62}$= -$\mathbf{m}_{35}$= -$\mathbf{m}_{53}$=
(XL**2*(-2*p*z2*Cosh(p*z2)*Sin(p*z1)-2*p*z*z2*Cosh(p*z2)*
Sin(p*z1)+4*p*z**2*z2*Cosh(p*z2)*Sin(p*z1)-p*z2*Cosh(3*p*z2)*
Sin(p*z1)+2*p*z*z2*Cosh(3*p*z2)*Sin(p*z1)-p*z**2*z2*Cosh(3*p*z2)*
Sin(p*z1)+p*z2*Cosh(p*z2)*Sin(3*p*z1)-p*z**2*z2*Cosh(p*z2)*
Sin(3*p*z1)-2*p*z1*Cos(p*z1)*Sinh(p*z2)+2*p*z1*z*Cos(p*z1)*
Sinh(p*z2)+4*p*z1*z**2*Cos(p*z1)*Sinh(p*z2)-p*z1*Cos(3*p*z1)*
Sinh(p*z2)-2*p*z1*z*Cos(3*p*z1)*Sinh(p*z2)-p*z1*z**2*Cos(3*p*z1)*
Sinh(p*z2)-12*Sin(p*z1)*Sinh(p*z2)+2*Sin(3*p*z1)*
Sinh(p*z2)+4*z*Sin(3*p*z1)*Sinh(p*z2)-2*z**3*
Sin(3*p*z1)*Sinh(p*z2)+ p*z1*
Cos(p*z1)*Sinh(3*p*z2)-p*z1*z**2*Cos(p*z1)*Sinh(3*p*z2)+
2*Sin(p*z1)*Sinh(3*p*z2)-4*z*Sin(p*z1)*Sinh(3*p*z2)+2*z**3*
Sin(p*z1)*Sinh(3*p*z2)))/(4*p**2*z1*z2*(-2+Cos(2*p*z1)+z*
Cos(2*p*z1) +Cosh(2*p*z2) - z*Cosh(2*p*z2))**2)
C
$\mathbf{m}_{22}$= $\mathbf{m}_{55}$= (XL**3*(4*p*(1 - z)**(2.5)*z2 - 4*p*z1*(1+z)**(2.5) +
4*p*(1 - z)**(1.5)*(1 + z)**(1.5)*Cos(2*p*z1)+4*p*z1*
(1 + z)**(2.5)*Cos(2*p*z1)-4*p*(1-z)**(2.5)*z2*Cosh(2*p*z2) -
4*p*(1 - z)**(1.5)*(1 + z)**(1.5)*Cosh(2*p*z2) +
4*z2*Sin(2*p*z1) -8*z*z2*Sin(2*p*z1) +4*z**2*z2*Sin(2*p*z1) -
2*(1 + z)**(1.5)*Sin(2*p*z1) +2*z*(1 + z)**(1.5)*Sin(2*p*z1) +
2*(1 + z)**(2.5)*Sin(2*p*z1) -4*z2*Cosh(2*p*z2)*Sin(2*p*z1) +
8*z*z2*Cosh(2*p*z2)*Sin(2*p*z1) -4*z**2*z2*Cosh(2*p*z2)*
Sin(2*p*z1) +2*(1 + z)**(1.5)*Cosh(2*p*z2)*Sin(2*p*z1) -
2*z*(1 + z)**(1.5)*Cosh(2*p*z2)*Sin(2*p*z1)-(1 + z)**(2.5)*
Sin(4*p*z1) -4*z1*Sinh(2*p*z2) +2*(1 - z)**(1.5)*Sinh(2*p*z2) -
2*(1 - z)**(2.5)*Sinh(2*p*z2) -8*z1*z*Sinh(2*p*z2) +
2*(1 - z)**(1.5)*z*Sinh(2*p*z2) -4*z1*z**2*Sinh(2*p*z2) +
4*z1*Cos(2*p*z1)*Sinh(2*p*z2)-2*(1-z)**(1.5)*Cos(2*p*z1)*
Sinh(2*p*z2)+8*z1*z*Cos(2*p*z1)*Sinh(2*p*z2) -2*(1 - z)**(1.5)*z*
Cos(2*p*z1)*Sinh(2*p*z2) +4*z1*z**2*Cos(2*p*z1)*Sinh(2*p*z2) +
8*p*Sin(2*p*z1)*Sinh(2*p*z2)-8*p*z**2*Sin(2*p*z1)*Sinh(2*p*z2) +
(1 - z)**(2.5)*Sinh(4*p*z2)))/(16*p**3*z1*z2*(-2 + Cos(2*p*z1) +
z*Cos(2*p*z1) +Cosh(2*p*z2) - z*Cosh(2*p*z2))**2)

C
W1=Cos(p*z1)
W2=Cosh(p*z2)
W3=Cos(3*p*z1)
W4=Cosh(3*p*z2)
C
W5=Sin(p*z1)
W6=Sinh(p*z2)
W7=Sin(3*p*z1)
W8=Sinh(3*p*z2)
C
W9=Cos(2*p*z1)
W10=Cosh(2*p*z2)
Y1=(1+z)
Y2=(1-z)
C
TM133 = -(p*(Y2)**(3.5)*z2*W1*W2)+p*(Y2)**(2.5)*(Y1)**(1.5)*
W1*W2+p*(Y2)**(1.5)*(Y1)**(2.5)*W1*W2-p*z1*(Y1)**(3.5)*W1*W2-p*
(Y2)**(2.5)*(Y1)**(1.5)*W3*W2+p*z1*(Y1)**(3.5)*W3*W2+p*
(Y2)**(3.5)*z2*W1*W4-p*(Y2)**(1.5)*(Y1)**(2.5)*W1*W4+2*z2*W2*
W5-6*z*z2*W2*W5+6*z**2*z2*W2*W5-
2*z**3*z2*W2*W5+5*(Y1)**(1.5)*W2*
W5-10*z*(Y1)**(1.5)*W2*W5+5*z**2*(Y1)**(1.5)*W2*W5+14*
(Y1)**(2.5)*W2*W5-14*z*(Y1)**(2.5)*W2*W5+3*(Y1)**(3.5)*W2*W5-2*
z2*W4*W5+6*z*z2*W4*W5-6*z**2*z2*W4*W5+2*z**3*z2*W4*W5-5*
(Y1)**(1.5)*W4*W5+10*z*(Y1)**(1.5)*W4*W5-5*z**2*(Y1)**(1.5)*W4*
W5+(Y1)**(2.5)*W4*W5-z*(Y1)**(2.5)*W4*W5 -5*(Y1)**(2.5)*W2*W7+5*
z*(Y1)**(2.5)*W2*W7-(Y1)**(3.5)*W2*W7 +2*z1*W1*W6+5*(Y2)**(1.5)*
W1*W6+14*(Y2)**(2.5)*W1*W6+3*(Y2)**(3.5)*W1*W6+6*z1*z*W1*W6+10*
(Y2)**(1.5)*z*W1*W6+14*(Y2)**(2.5)*z*W1*W6+6*z1*z**2*W1*W6+5*
(Y2)**(1.5)*z**2*W1*W6+2*z1*z**3*W1*W6-2*z1*W3*W6-
5*(Y2)**(1.5)*
W3*W6+(Y2)**(2.5)*W3*W6-6*z1*z*W3*W6-10*(Y2)**(1.5)*z*W3*W6
TM134=(**TM**133+(Y2)**(2.5)*z*W3*W6-6*z1*z**2*W3*W6-
5*(Y2)**(1.5)*z**2*W3*W6-2*
z1*z**3*W3*W6-8*p*z*W5*W6+8*p*z**3*W5*W6-4*p*W7*W6-4*p*z*
W7*W6+4*p*z**2*W7*W6+4*p*z**3*W7*W6-5*(Y2)**(2.5)*W1*W8 -
(Y2)**(3.5)*W1*W8-5*(Y2)**(2.5)*z*W1*W8+4*p*W5*W8-4*p*z*W5*
W8-4*p*z**2*W5*W8+4*p*z**3*W5*W8)
$\mathbf{m}_{36}$= $\mathbf{m}_{63}$=XL***TM**134/(8*p*z1*z2*(-2+W9+z*W9+W10-z*W10)**2)
C
$\mathbf{m}_{25}$= $\mathbf{m}_{52}$= (XL**3*(-(p*(Y2)**(2.5)*z2*W1*W2) +p*z1*(Y1)**(2.5)*W1*
W2-p*(Y2)**(1.5)*(Y1)**(1.5)*W3*W2-p*z1*(Y1)**(2.5)*W3*W2 +
p*(Y2)**(2.5)*z2*W1*W4+p*(Y2)**(1.5)*(Y1)**(1.5)*W1*W4-2*z2*
W2*W5+4*z*z2*W2*W5-2*z**2*z2*W2*W5+(Y1)**(1.5)*W2*W5-z*

(Y1)**(1.5)*W2*W5-3*(Y1)**(2.5)*W2*W5+2*z2*W4*W5-
4*z*z2*W4*W5+
2*z**2*z2*W4*W5-(Y1)**(1.5)*W4*W5+z*(Y1)**(1.5)*W4*W5+
(Y1)**(2.5)*W2*W7+2*z1*W1*W6-(Y2)**(1.5)*W1*W6+3*(Y2)**(2.5)*
W1*W6+4*z1*z*W1*W6-(Y2)**(1.5)*z*W1*W6+2*z1*z**2*W1*W6 -
2*z1*W3*W6+(Y2)**(1.5)*W3*W6-4*z1*z*W3*W6+(Y2)**(1.5)*z*W3*W6-
2*z1*z**2*W3*W6-16*p*W5*W6+16*p*z**2*W5*W6-
(Y2)**(2.5)*W1*W8))/(8*p**3*z1*z2*(-2+W9+z*W9+W10-z*W10)**2)
C

C
$\mathbf{m}_{11} = \mathbf{m}_{44} = 1/3$
$\mathbf{m}_{14} = \mathbf{m}_{41} = 1/6$
$\mathbf{m}_{2} = \mathbf{m}_{\mathbf{13}} = \mathbf{m}_{15} = \mathbf{m}_{\mathbf{16}} = \mathbf{m}_{42} = \mathbf{m}_{43} = \mathbf{m}_{45} = \mathbf{m}_{46} = 0$

b) For **A>2*Sqrt(B)**

Consistent mass matrix terms of beam elements resting on two - parameter elastic foundation in exponential form;

where A= k_θ/(E*I) and B= k_1/(E*I) see page 44
XL=L
p=λ*XL
λ= (k_1/(4*E*I))**0.25
δ= k_θ/(4*E*I)

```
      a=sqrt(λ**2+δ)
      b=sqrt(δ-λ**2)
C
      y1=Exp(2*b*XL)
      y2=Exp(2*a*XL)
      y3=Exp((a-b)*XL)
      y4=Exp((a+b)*XL)
      y5=Exp((-a+b)*XL)
      y6=Exp(2*(a-b)*XL)
      y7=Exp(2*(a+b)*XL)
      y8=8*a**2-8*b**2-(4*a**2)/y1-4*a**2*y1+(4*b**2)/y2+4*b**2*y2
      y9=2*(-a+b)*(-a-b+a*y1+b*y2)
      y10=b+a*y2-b*y2-a*y1*y2
      y11=-a+a*y1-b*y1+b*y1*y2
      y12=-(b*y1)-a*y2+a*y1*y2+b*y1*y2
      y13=2*(b**2*y1-a**2*y2+2*a**2*y1*y2-
       2*b**2*y1*y2-a**2*(y1)**2*y2+b**2*y1*(y2)**2)
      y14=2*a+2*b-2*a*y1-(2*b)/y2
      y15=2*a-2*b-(2*a)/y1+(2*b)/y2
      y16=2*(-a+b+a*y1-b*y2)
      y17=-2*a-2*b+(2*a)/y1+2*b*y2
      y18=-a-b+b*y3*y4+a*y4*y5
      y19=b-a*y2-b*y2+a*y6
      y20=b+a*y2-b*y2-a*y7
      y21=-a+a*y1-b*y1+b*y7
      y22=-2*a*y3-2*b*y3+(2*b)/y4+2*a*y4
      y23=2*(a*y3-a*y4+b*y4-b*y5)
      y24=(-2*a)/y4-2*b*y4+2*a*y5+2*b*y5
      y25=a-b+b*y3*y4-a*y4*y5
C
```

m$_{33}$= **m**$_{66}$=-((a-b)**2*y12**2)/(2*(a+b)*y13**2)+

```
       (a-b)**2*Exp(2*(a+b)*XL)*y12**2/(2*(a+b)*y13**2)+
       Exp((a+b)*XL)*(2*(a**2-b**2)*Exp((-a+b)*XL)*y11*y12/
       (b*y1*y13*y8)+2*(a**2-b**2)*y10*y12/(a*Exp((-a+b)*XL)*
       y13*y2*y8))+2*(a+b)**2*y11**2/((a-b)*y1**2*y8**2)-
       2*(a+b)**2*Exp(2*(-a+b)*XL)*y11**2/((a-b)*y1**2*y8**2)-
```

```
2*(a+b)**2*y10**2/((a-b)*y2**2*y8**2)+
2*(a+b)**2*y10**2/((a-b)*Exp(2*(-a+b)*XL)*y2**2*y8**2)-
2*(a**2-b**2)*y11*y12/(b*y1*y13*y8)-
2*(a**2-b**2)*y10*y12/(a*y13*y2*y8)+
2*(a+b)*y11*y9/(a*y1*y8**2)+
2*(a+b)*y10*y9/(b*y2*y8**2)+y9**2/(2*(a+b)*y8**2)-
y9**2/(2*(a+b)*Exp(2*(a+b)*XL)*y8**2)+
(-2*(a+b)*Exp((-a+b)*XL)*y11*y9/(a*y1*y8**2)-
2*(a+b)*y10*y9/(b*Exp((-a+b)*XL)*y2*y8**2))/Exp((a+b)*XL)+
2*XL*(4*a**2*y10*y11*y13+8*a*b*y10*y11*y13+4*b**2*y10*y11*y13+
a*y1*y12*y2*y8*y9-b*y1*y12*y2*y8*y9)/(y1*y13*y2*y8**2)
C
TM122=2*(a+b)*XL*y11*y14/(y1*y8**2)-(a+b)*y11*y15/(b*y1*y8**2)+
(a+b)*Exp(2*b*XL)*y11*y15/(b*y1*y8**2)+(a+b)*
y11*y16/(a*y1*y8**2)-
(a+b)*y11*y16/(a*Exp(2*a*XL)*y1*y8**2)+
(a+b)*y11*y17/((a-b)*y1*y8**2)-
(a+b)*Exp(2*(-a+b)*XL)*y11*y17/((a-b)*y1*y8**2)-
(a+b)*y10*y14/((a-b)*y2*y8**2)+
(a+b)*y10*y14/((a-b)*Exp(2*(-a+b)*XL)*y2*y8**2)+
(a+b)*y10*y16/(b*y2*y8**2)-(a+b)*y10*y16/
(b*Exp(2*b*XL)*y2*y8**2)+
2*(a+b)*XL*y10*y17/(y2*y8**2)-(a-b)*y12*y15/(2*(a+b)*y13*y8)+
(a-b)*Exp(2*(a+b)*XL)*y12*y15/(2*(a+b)*y13*y8)+
(a-b)*XL*y12*y16/(y13*y8)-(a-b)*y12*y17/(2*b*y13*y8)+
(a-b)*Exp(2*b*XL)*y12*y17/(2*b*y13*y8)
```

$\mathbf{m}_{56}= \mathbf{m}_{65}= -\mathbf{m}_{23}= -\mathbf{m}_{32}=$**TM**122-

```
(2*a*y10*y13*y15+2*b*y10*y13*y15+a*y12*y14*y2*y8-
b*y12*y14*y2*y8)/(2*a*y13*y2*y8**2)+Exp(2*a*XL)*
(2*a*y10*y13*y15+2*b*y10*y13*y15+a*y12*y14*y2*y8-
b*y12*y14*y2*y8)/(2*a*y13*y2*y8**2)+y14*y9/(2*b*y8**2)-
y14*y9/(2*b*Exp(2*b*XL)*y8**2)+
XL*y15*y9/y8**2+y16*y9/(2*(a+b)*y8**2)-
y16*y9/(2*(a+b)*Exp(2*(a+b)*XL)*y8**2)+y17*y9/(2*a*y8**2)-
y17*y9/(2*a*Exp(2*a*XL)*y8**2)
C
TM133=-2*(a**2-b**2)*Exp((-a+b)*XL)*y11*y19/(a*y1*y8**2)+
2*(a**2-b**2)*Exp(-2*a*XL+(-a+b)*XL)*y11*y19/(a*y1*y8**2)-
2*(a**2-b**2)*Exp((-a+b)*XL)*y10*y19/(b*y2*y8**2)+
2*(a**2-b**2)*Exp(-2*b*XL+(-a+b)*XL)*y10*y19/(b*y2*y8**2)+
2*a**2*y11*y20/((a-b)*Exp((a+b)*XL)*y1*y8**2)-
2*b*(2*a+b)*y11*y20/((-a+b)*Exp((a+b)*XL)*y1*y8**2)-
2*a**2*Exp(2*(-a+b)*XL-(a+b)*XL)*y11*y20/((a-b)*y1*y8**2)+
2*b*(2*a+b)*Exp(2*(-a+b)*XL-
(a+b)*XL)*y11*y20/((-a+b)*y1*y8**2)-
2*(a+b)**2*y10*y21/((a-b)*Exp((a+b)*XL)*y2*y8**2)+
```

2*(a+b)**2*Exp(2*(a-b)*XL-(a+b)*XL)*y10*y21/((a-b)*y2*y8**2)+
4*(a+b)**2*XL*(y1*y10*y20+y11*y2*y21)/
(Exp((a+b)*XL)*y1*y2*y8**2)-
2*(-(a**2)+b**2)*y11*y18/(b*y1*y4*y8**2)+
2*(-(a**2)+b**2)*Exp(2*b*XL)*y11*y18/(b*y1*y4*y8**2)-
2*(-(a**2)+b**2)*y10*y18/(a*y2*y4*y8**2)+
2*(-(a**2)+b**2)*Exp(2*a*XL)*y10*y18/(a*y2*y4*y8**2)

$\mathbf{m}_{36}$= $\mathbf{m}_{63}$=**TM**133-2*a**2*Exp(-(a*XL)+b*XL)*XL*y12*y19/(y13*y8)+
2*(2*a-b)*b*Exp(-(a*XL)+b*XL)*XL*y12*y19/(y13*y8)-
(a**2-b**2)*y12*y20/(b*Exp((a+b)*XL)*y13*y8)+
(a**2-b**2)*Exp(2*b*XL-(a+b)*XL)*y12*y20/(b*y13*y8)-
(a**2-b**2)*y12*y21/(a*Exp((a+b)*XL)*y13*y8)+
(a**2-b**2)*Exp(2*a*XL-(a+b)*XL)*y12*y21/(a*y13*y8)+
a**2*y12*y18/((a+b)*y13*y4*y8)-
(2*a-b)*b*y12*y18/((a+b)*y13*y4*y8)-
a**2*Exp(2*(a+b)*XL)*y12*y18/((a+b)*y13*y4*y8)+
(2*a-b)*b*Exp(2*(a+b)*XL)*y12*y18/((a+b)*y13*y4*y8)-
(a-b)*Exp((-a+b)*XL)*y19*y9/((a+b)*y8**2)+
(a-b)*Exp((-a+b)*XL-2*(a+b)*XL)*y19*y9/((a+b)*y8**2)+
(a+b)*y20*y9/(a*Exp((a+b)*XL)*y8**2)-
(a+b)*Exp(-2*a*XL-(a+b)*XL)*y20*y9/(a*y8**2)+
(a+b)*y21*y9/(b*Exp((a+b)*XL)*y8**2)-
(a+b)*Exp(-2*b*XL-(a+b)*XL)*y21*y9/(b*y8**2)+
2*(-a+b)*XL*y18*y9/(y4*y8**2)

C

TM144=(-((a+b)*y10*y23/(b*Exp(2*a*XL)*y2*y8**2))+
a*y10*y24/((a-b)*y2*y8**2))/Exp(2*(-a+b)*XL)+
(a+b)*y11*y22/((a-b)*y1*y8**2)-
(a+b)*Exp(2*(-a+b)*XL)*y11*y22/((a-b)*y1*y8**2)+
(a+b)*y11*y23/(a*y1*y8**2)-(a+b)*y11*y23/
(a*Exp(2*a*XL)*y1*y8**2)+
(a+b)*y10*y23/(b*y2*y8**2)-a*y10*y24/((a-b)*y2*y8**2)+
b*y10*y24/((-a+b)*y2*y8**2)+
2*(a+b)*XL*(y1*y10*y22+y11*y2*y24)/(y1*y2*y8**2)+
(a-b)*XL*y12*y23/(y13*y8)-(a-b)*y12*y25/((a+b)*y13*y4*y8)+
(a-b)*Exp(2*(a+b)*XL)*y12*y25/((a+b)*y13*y4*y8)-
(4*a*y11*y13*y25+4*b*y11*y13*y25+a*y1*y12*y22*y4*y8-
b*y1*y12*y22*y4*y8)/(2*b*y1*y13*y4*y8**2)+
Exp(2*b*XL)*(4*a*y11*y13*y25+4*b*y11*y13*y25+
a*y1*y12*y22*y4*y8-
b*y1*y12*y22*y4*y8)/(2*b*y1*y13*y4*y8**2)

$\mathbf{m}_{26}$= $\mathbf{m}_{62}$= -$\mathbf{m}_{35}$= -$\mathbf{m}_{53}$=**TM**144-
(4*a*y10*y13*y25+4*b*y10*y13*y25+a*y12*y2*y24*y4*y8-
b*y12*y2*y24*y4*y8)/(2*a*y13*y2*y4*y8**2)+
Exp(2*(a-b)*XL)*(-(b*y10*y24/((-a+b)*y2*y8**2))+
Exp(2*b*XL)*(4*a*y10*y13*y25+4*b*y10*y13*y25+

a*y12*y2*y24*y4*y8-
b*y12*y2*y24*y4*y8)/(2*a*y13*y2*y4*y8**2))+y22*y9/(2*a*y8**2)-
y22*y9/(2*a*Exp(2*a*XL)*y8**2)+y23*y9/(2*(a+b)*y8**2)+
y24*y9/(2*b*y8**2)+2*XL*y25*y9/(y4*y8**2)+
(-(y23*y9)/(2*(a+b)*y8**2)-Exp(2*a*XL)*y24*y9/(2*b*y8**2))/
Exp(2*(a+b)*XL)

C

$\mathbf{M}_{22}$= $\mathbf{m}_{55}$=Exp((a+b)*XL)*(Exp((a-b)*XL)*y14*y15/(a*y8**2)+
y15*y17/(b*Exp((a-b)*XL)*y8**2))+
(-(Exp((a-b)*XL)*y14*y16/(b*y8**2))-y16*y17/
(a*Exp((a-b)*XL)*y8**2))/
Exp((a+b)*XL)-y14**2/(2*(a-b)*y8**2)+
Exp(2*(a-b)*XL)*y14**2/(2*(a-b)*y8**2)-y14*y15/(a*y8**2)-
y15**2/(2*(a+b)*y8**2)+Exp(2*(a+b)*XL)*y15**2/(2*(a+b)*y8**2)+
y14*y16/(b*y8**2)+y16**2/(2*(a+b)*y8**2)-
y16**2/(2*(a+b)*Exp(2*(a+b)*XL)*y8**2)-y15*y17/(b*y8**2)+
y16*y17/(a*y8**2)+y17**2/(2*(a-b)*y8**2)-
y17**2/(2*(a-b)*Exp(2*(a-b)*XL)*y8**2)+
2*XL*(y15*y16+y14*y17)/y8**2

C

$\mathbf{m}_{25}$= $\mathbf{m}_{52}$=(-(y16*y22+y17*y23)/(2*a*Exp((a-b)*XL)*y8**2)-
Exp((a-b)*XL)*(y14*y23+y16*y24)/(2*b*y8**2))/Exp((a+b)*XL)+
Exp((a+b)*XL)*((2*y17*y25+y15*y22*y4)/
(2*b*Exp((a-b)*XL)*y4*y8**2)+
Exp((a-b)*XL)*(2*y14*y25+y15*y24*y4)/(2*a*y4*y8**2))+
y17*y22/(2*(a-b)*y8**2)-y17*y22/(2*(a-b)*
Exp(2*(a-b)*XL)*y8**2)+
y16*y23/(2*(a+b)*y8**2)-y16*y23/(2*(a+b)*
Exp(2*(a+b)*XL)*y8**2)+
(y16*y22+y17*y23)/(2*a*y8**2)-y14*y24/(2*(a-b)*y8**2)+
Exp(2*(a-b)*XL)*y14*y24/(2*(a-b)*y8**2)+
(y14*y23+y16*y24)/(2*b*y8**2)-y15*y25/((a+b)*y4*y8**2)+
Exp(2*(a+b)*XL)*y15*y25/((a+b)*y4*y8**2)-
(2*y17*y25+y15*y22*y4)/(2*b*y4*y8**2)-
(2*y14*y25+y15*y24*y4)/(2*a*y4*y8**2)+
XL*(2*y16*y25+y14*y22*y4+y15*y23*y4+y17*y24*y4)/(y4*y8**2)

C

$\mathbf{m}_{11}$= $\mathbf{m}_{44}$= 1/3

$\mathbf{m}_{14}$= $\mathbf{m}_{41}$= 1/6

$\mathbf{m}_{2}$= $\mathbf{m}_{13}$ =$\mathbf{m}_{15}$= $\mathbf{m}_{16}$ =$\mathbf{m}_{42}$= $\mathbf{m}_{43}$ =$\mathbf{m}_{45}$= $\mathbf{m}_{46}$=0

VITA

Abdulhalim Karaşin was born in Derik, on January 1,1965.He received his B.C. degree in Civil Engineering from Middle East Technical University Faculty of Engineering of Gaziantep Campus in June 1989. Then he received his M.S. degree in Civil Engineering From Fırat University. He was employed by Dicle University as a research assistant until 1998. Since then he has been working as a research assistant in Structural Mechanic Division of the Civil Engineering Department in Middle East Technical University. His main areas of interest are theory and analysis of plates, seismic rehabilitation of damaged reinforced concrete structures and design of earthquake-resisting buildings.