AN APPROXIMATE MODEL FOR PERFORMANCE MEASUREMENT IN BASE-STOCK CONTROLLED ASSEMBLY SYSTEMS

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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The aim of this thesis is to develop a tractable method for approximating the steady-state behavior of continuous-review base-stock controlled assembly systems with Poisson demand arrivals and manufacturing and assembly facilities modeled as Jackson networks. One class of systems studied is to produce a single type of finished product assembling a number of components and another class is to produce two types of finished products allowing component commonality. The performance measures evaluated are the expected backorders, fill rate and the stockout probability for finished product(s). A partially aggregated but exact model is approximated assuming that the state-dependent transition rates arising as a result of the partial aggregation are constant. This approximation leads to the derivation of a closed-form steady-state probability distribution, which is of product-form. Adequacy of the proposed model in approximating the steady-state performance measures is tested against simulation experiments over a large range of parameters and the approximation turns out to be quite accurate with absolute errors of 10% at most for fill rate and stockout probability, and of less than 1.37 (≈2) requests for expected backorders. A greedy heuristic which is proposed to be employed using approximate steady-state probabilities is devised to optimize base-stock levels while aiming at an overall service level for finished product(s).

Keywords: Assembly Systems, Approximation, Performance Evaluation, Greedy Heuristic, Base-Stock Control, Steady-State Behavior, Jackson Network.

ÖZ

BAZ-STOK DENETİMİNDEKİ MONTAJ SİSTEMLERİNDE PERFORMANS ÖLÇÜMÜ İÇİN BİR YAKLAŞIK MODEL

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Bu çalışmada baz-stok denetim mekanizması altında çalışan, son ürün talebi Poisson süreci, üretim ve montaj atölyeleri ise Jackson ağı olarak modellenmiş bir sistemde kararlı durum davranışını belirleyerek, anında karşılanamayıp ileri tarihte karşılanmak üzere kabul edilen taleplerin beklenen değeri, talebin anında karşılanma olasılığı gibi performans ölçütlerini hesaplamaya yönelik bir yaklaşık modelin geliştirilmesi amaçlanmaktadır. Herhangi bir sayıda alt ürün montajı ile tek tip bir son ürünün üretildiği veya iki farklı tipte son ürünün ortak alt ürünlere izin verilerek üretildiği sistemler üzerinde çalışılmıştır. Kısmen kümüle edilmiş, ancak kesin bir modelde kümülasyon dolayısı ile oluşan ve sistemin durumuna göre değişen geçiş oranları sabit varsayılarak bir yaklaşık modele ulaşılmıştır. Yaklaşık model ile elde edilen sayısal değerler benzetim ile hesaplanan değerlerle kıyaslanarak önerilen yaklaşık modelin yeterliliği sınanmış ve talebin anında karşılanma olasılığı için %10'dan, anında karşılanamayıp ileri tarihte karşılanmak üzere kabul edilen taleplerin beklenen değeri için ise 1.37 (≈2) adet karşılanamayan talepten daha az hata oranları gözlenmiştir. Ayrıca, yaklaşık model ile elde edilen değerler kullanılarak, belirli son ürün servis seviyesi hedefini sağlamak üzere en iyi baz-stok seviyelerini hesaplayan sezgisel bir algoritma tasarlanmıştır.

Anahtar Kelimeler: Montaj Sistemleri, Yaklaştırım, Performans Değerlendirmesi, Açgözlü Sezgisel Algoritma, Baz-Stok Denetimi, Kararlı Durum, Jackson Ağı.

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CHAPTER 1

1. INTRODUCTION

Production/inventory control policies and bill of materials (BOM) of the finished products manufactured are the underlying characteristics to identify the structure of assembly systems. Different combinations of various production/inventory control policies and BOM lead to a high variety in the structure of assembly systems. The systems considered in this study are in the class of pull-type make-to-stock systems under continuous-review base-stock type inventory control policies and with the simplest possible BOM structures (a number of components assembled to make a single type of finished product at least to start with, and an immediate extension is also touched upon), which are already difficult to analyze but form the basic building block for a further study on more complex BOM structures (multiple finished products, component commonalities, closed systems). In fact, contribution of this study to future research along the same direction would be identifying the function of an assembly subsystem within a joint collection of many different subsystems (not only assembly but also serial or disassembly subsystems), most probably while employing a decomposition approach for the analysis of the joint collection. Such a further progress of research would reveal the importance of the investigation in this study to figure out steady-state behavior of the basic assembly models.

Even in the case of the most tractable form of the basic assembly systems (two components manufactured at their respective dedicated exponential single-server manufacturing facilities and stored at the respective base-stock controlled stock points and assembled at an exponential single-server assembly facility, Poisson demand arrivals), exact analytical steady-state probabilities of the corresponding model can not be found, pointing out the requirement to develop approximation approaches. The one proposed in this study is an analytical approximation which first appeared in [2] and [38] for two-echelon and two-indenture systems, respectively. The development of the adaptation of the work in [2] and [38] to two-component assembly systems with single-server facilities is in Chapter 3. This development over the corresponding queuing model is completely analytical unlike its intuitive use in Chapter 4 for further extensions with more than two components to be assembled and also with two different types of finished products having a common component. The approximate steady-state probability distribution proposed is of product-form, which is important to relate this thesis to the work in [30] on exactly the same type of systems except the one with a common component. In [30], the analysis is based on decomposition of the system, which immediately calls product-form solutions. In spite of so many common points of the decomposition in [30] and the development in this thesis, the product-form solutions are not recognized as closed-form solutions in [30] while using matrix-geometric solution algorithms for the decomposed subsystems. [30], being the only work in the literature concentrating on the same systems as the ones in this thesis, is comparable to ours in terms of not only the analytical approach but directly related to this also the approximation performance. This comparison underlines the contribution of this thesis: the closed-form (product-form) solution with its good numerical performance and immediate generalization possible for open Jackson network models of manufacturing and assembly facilities and for different BOMs.

To summarize, this thesis is on the performance analysis and design of assembly systems controlled by continuous-review base-stock inventory policies. Objective of the study is two-staged: to construct a model for approximating the steady-state performance measures of the assembly systems; namely expected backorder level, fill rate and stockout probability of finished products, and to use the approximate values with a greedy approach for finding near-optimal design parameters like base-stock levels at the stock points considering the trade-off between the required stock investment and some target service level to be achieved on the average in the long-run.

CHAPTER 2

2. LITERATURE REVIEW

The manufacturing system models considered in this study include fork-join stations, which bring about the difficulty and so the challenge to analyze them. As the name implies, a customer arrival at a fork-join station starts generation (fork) of a number of different jobs of this customer to be (instantaneously) connected (merged/joined) later for further processes to be carried out on the joined entity. The following overview (classification) of the systems with fork-join stations is by Krishnamurthy et al. [21]. Uses of fork-join stations appear in queuing models of not only manufacturing but also computer systems to analyze parallel processing, database concurrency control and communication protocols. Some related references for the latter are [3], [4], [5], [26], [37]. As for the models of manufacturing systems, the function of fork-join stations can be in one of the following two categories: Queuing model of an assembly station, which is typically a fork-join station, where a number of entities are merged to form a single entity representing an assembly as in [15], [19], [23], [27], [28] and fork-join stations in multi-stage manufacturing systems to model synchronization constraints under inventory control policies (base-stock, kanban) as in [8], [9], [13], [16], [30]. The queuing models of two-stage assembly systems under continuous-review base-stock type inventory control policy in this thesis fall into the last category. In this chapter, previous studies in this category are reviewed revealing how they are related to or different from the work in this thesis.

In Sbiti et al. [30], Di Mascolo and Dallery [13], Hazra et al. [16] and Chaouiya et al. [9], fork-join stations handle production coordination of assembly manufacturing systems under different inventory control policies that are all of pull-type. In Sbiti et al. [30] and Di Mascolo and Dallery [13], Hazra et al. [16], base-stock control and kanban control policies are employed, respectively. The work by Chaouiya et al. [9] is to extend a combination of these two one-parameter policies, which was previously proposed in Dallery and Liberopoulos [10] for serial manufacturing systems to achieve better trade-offs between inventory holding costs and customer service, to assembly systems. This combined policy is called as Extended Kanban Control System. Extended kanban control is a two-parameter (one set of parameters specifying the base-stock levels to provide buffer against stockouts and another set for the number of kanban cards circulating used to limit work-in-process) policy with the advantages of work-in-process (WIP) limitation over the base-stock control and of immediate transfer of demands to all manufacturing facilities over the kanban control. Chaouiya et al. [9] study and compare two variants of this policy: each component of an assembly is released into the assembly facility independently of the other components required for assembly or simultaneously with the other components. Unlike [13] and [30], the work by Chaouiya et al. [9] is just to introduce this new combined policy for assembly systems without any performance evaluation using simulation or some analytical approximate techniques.

On the other hand, Sbiti et al. [30] approximate base-stock controlled two-component assembly system's steady-state performance measures like probability of immediately satisfying demand, probability that demand is backordered, average number of backordered demands, average WIP for each stage of the system, average waiting time per demand, etc., and they compare these with simulation results. They extend their approximation to systems assembling any number of components and containing any number of operations in series after the assembly operation. Their simple two-component assembly system is modeled as a queuing network with three exponential single-server facilities. Two types of components are manufactured at their dedicated facilities and then, are assembled at the assembly facility. Each facility is succeeded by an output buffer where the processed components or finished products are stocked. The output buffers initially contain components and finished products at the levels which we call the respective base-stock levels. The buffers are assumed to be of infinite capacity. There is always available raw material input for the facilities manufacturing components. Arrival process is Poisson from an infinite population and arrivals that cannot be satisfied immediately upon arrival are backordered. Since the model by Sbiti et al. [30] is exactly the same as the one studied in Chapter 3, next their approximation approach is further detailed now. They decompose the system into two, one manufacturing and storing the components to be simultaneously picked up and the other assembling the components and storing the finished assemblies; solution of the former (upstream) subsystem feeding the latter (downstream) subsystem. The model of the former subsystem is truncated and solved for the steady-state probability distribution using matrix-geometric approach. Then, summing up the probabilities over four different regions corresponding to each possible state of the downstream to identify different arrivals (different sequence of operations) at the latter subsystem, another set of steady-state equations is solved using matrix-geometric approach. In case there are some more workstations following assembly, the system is decomposed into more than two subsystems. Due to the curse of dimensionality, the case of more than two components is handled incorporating a further independence assumption for different types of components in their respective queues, which leads to treating each component manufacturing facility as an $M / M / 1$ station. The authors restrict their study to the calculation of the performance measures and do not make any study for optimizing the base-stock levels at the buffers (service level for the finished product) subject to some service level (budget) constraint.

In addition to the classification of the queuing models involving fork-join stations mentioned at the beginning of this chapter, one could think of a further classification based on the type of arrival process (distribution of the inter-arrival times and size of the requests per arrival, unit or batch arrivals, and size of the calling source population) or on the server facilities (service distribution, single server or a network, capacitated or uncapacitated) or on the buffer sizes. Regarding these, two sets of studies on finite-capacity buffers ([1], [11], [12], [18]) and finite calling source population (closed queueing models) ([8], [13], [16]) are reviewed next.

Altıok's work, [1] and a series of studies by Dallery, Liu and Towsley, [11], [12] are on the analysis of fork-join queuing networks with finite-capacity buffers under various operating mechanisms regulating blocking (before service, after service) and loading (independent, simultaneous). Altıok, [1] makes an exact analysis of simple asynchronous assembly systems assembling two components to get a finished assembly and an approximate analysis (using the concept of two-node decomposition) for both synchronous and more complex (more than two components, network of assembly stations instead of single server facilities) asynchronous systems. On the other hand, the primary focus of the authors in [11], [12] is on investigating the behavior of the throughput of these networks through the properties of reversibility, symmetry, monotonicity and concavity of the corresponding queueing models. Applying the results of the studies on these properties to various problems in the design and operation of manufacturing systems, Dallery et al. [12] evaluate the performance measures of simple assembly systems consisting of three servers (two for manufacturing components and one for assembly operation) with finite buffer capacities, serial production systems with finite buffer capacities and kanban controlled production lines. Dallery et al. [12] also consider an optimization problem for achieving a given production capacity at a minimal cost by determining the capacities of the buffers in assembly systems and serial production systems and also for kanban controlled production systems by determining the number of kanbans. The authors present some of their observations on the relation between the number of kanbans and buffer capacities depending on the costs of each, which are towards reducing the complexity of the optimization search procedure drastically.

Hemechandra and Kumar [18] study on a fork-join queuing model to investigate the steady-state behavior of open assembly systems. The model consists of two manufacturing servers; each working on one task after an arriving job is split into two, and an assembly server to join the separately processed subassemblies. Servers are all single operating under first-come-first-served (FCFS) discipline with exponential service times and job arrivals are Poisson. There are four buffers in the system, two of them are before and two are after the parallel manufacturing servers. Since these buffer sizes are all limited, an arriving demand is lost if an input buffer is full and blocking specified as of after-processing-type could occur. The authors numerically solve the steady-state equations to compute mean throughput of the system, fraction of arrivals lost, utilization of the servers, etc. Then, they consider the determination of buffer sizes enumerating all possible configurations for maximizing fraction of customers served or minimizing the average waiting time or minimizing the average number of jobs in the system emphasizing the trade-off between these performance measures.

Fork join stations need to be analyzed within the context of closed queuing networks when, for example, a fixed number of automated guided vehicles circulate in the networks to feed the assembly operations (as in [27], [28]) or kanban control mechanisms are employed in multi-stage manufacturing systems (as in [8], [13], [16]) or resources are shared in parallel or distributed computer systems (as in [17]). Hazra et al. consider multi-stage assembly systems operating under CONWIP (Constant work-in-process) control which is a WIP limiting type of kanban control mechanism, characterized by directed graphs (trees in particular) with one root node (server), a set of two or more leaf nodes, a set of intermediary nodes (not necessarily all of these intermediary ones being at the same level) and directed edges representing the buffers that connect server. Service times of the machines are exponentially distributed having at least one input buffer and, aside from the root node, exactly one output buffer for each machine. Analysis of the authors is a new heuristic version of the exact numerical aggregation-disaggregation procedure by Takahashi [35] to solve continuous-time Markov chains with large state spaces, making [16] the first work on extending the aggregation ideas to fork-join kanban controlled queuing networks. The approximation by Hazra et al. [16] has the novel feature of doing simultaneous multiple partitions of the state space in such a way that these partitions generate mutually consistent estimates of the aggregate transition rates. This consistency leads to a fixed-point problem, which itself is solved by iteration. Good (fast and accurate) approximations of the throughput (with an error of 5% or less in all case, errors not affected by the number of kanbans) and of the expected local buffer occupancy (with an error of 30% or better and of roughly one job in absolute value, errors greater for upstream stages than for downstream stages and not necessarily correlated with the number of kanbans) are obtained for any given topology and number of kanbans. It is numerically observed that increase in the number of kanbans result in a concave increase in the system throughput, which could be compared to almost concave behavior of the fill rate as a function of the base-stock levels in all numerical experiments performed for this thesis and presented in section 3.4 and to the similar analytical results by Dallery et al. in [11] and [12].

Di Mascolo and Dallery [13] study kanban controlled assembly systems under two different release mechanisms (simultaneous and independent release of kanbans attached to components when assembly occurs) as in Chaouiya et al. [9].

Due to the implementation nature of a kanban type control, any production system must be decomposed into several stages (subsystems), each with a manufacturing process and an output buffer for the parts processed at that stage to be stocked. Each stage is associated with a fixed number of kanbans. The authors allow the manufacturing process at any stage to consist of a set of identical machines or a more complex system like a manufacturing flow line and the service time distribution of each server and the arrival process of external demands being general and approximated by phase-type (a mixture of exponential) distributions in the study. The steady-state performance measures they consider to use for resolving the design issue on the determination of the number of kanbans are the average WIP and the average number of finished parts at each stage, the proportion of backordered demands, the average number of backordered demands and the average waiting time of a backordered demand. As for the approximation of these performance measures, Di Mascolo and Dallery [13] extend the analytical method based on the product-form approximation in [6] developed for serial configurations to assembly systems. This method results from viewing the system as a multi-class closed queuing network, each type of kanban representing one class of customers. The idea is to set the load-dependent service rates of the associated stations in the equivalent single-class networks and to come up with the arrival rates as the functions of the service rates using an iterative procedure. The numerical results the authors refer to in [13] for justifying the approximation in terms of accuracy and rapidity as compared to simulation are for service times assumed to be distributed according to coxian 2 distribution.

A line of research by [7], [15], [19], [24], [31], [32] is on Poisson arrivals. There is exact analysis of the cases with not only exponential but also coxian interarrival times in [32] to derive expressions for throughput and mean queue lengths. Analysis of general arrival processes are mostly under the assumption of infinite calling source population as in [33], [34]. In order to develop two-moment approximations for throughput and mean queue lengths at the input buffers when the arrival process is general from a finite population, Krishnamurty et al. [21], [22] work with the assumption that arrival process is a renewal process. As for the use of their approximation in decomposing larger closed queuing networks with fork-join stations, based on their simulation experiments the authors point out the importance of determining variability of the departure process (coefficient of variation of inter-departure times) from the fork-join stations and of the impact of correlations between successive inter-departure times on different performance measures.

Another research stream which to a certain extent could be related to the work in this thesis (especially in Section 4.2) is worth mentioning: commonality and postponement of product differentiation issues in assembly systems drawing great attention during the last few decades with the requirement arising to manage increasing product variety in supply chain excellence. For a neat overview of the literature on these issues' different aspects and impacts on the system performance, the reader is referred to Ma et al. [25]. Related with these issues, [20] is summarized next, pointing out also the difference of the approach taken compared to ours, namely working with estimated lead times unlike the way we proceed to handle lead times implicitly. De Kok and Visschers [20] work on the assembly systems with multiple finished products and component commonality and propose an algorithm to decompose these systems into purely divergent multi-echelon systems with the inspiration from [29] and [36] where it is shown that a pure assembly system, each stage supplying at most one (assembly) stage, is equivalent to a serial multi-echelon system. Since it is possible to calculate near-optimal order-up-to-levels (to minimize the total inventory handling cost) subject to some service-level constraint (fill rate or stockout probability) for the decomposed divergent multi-echelon systems, the authors proceed with these order-up-to-levels in the original assembly system. Throughout their study on the decomposition algorithm, they formulate a constraint imposing any assembly system under this constraint to decompose into a series system only. For their further analysis, de Kok and Visschers concentrate on systems satisfying this constraint. Different from the model studied in this thesis, in [20] there may be subassemblies in addition to components and finished products and periodic review policy is used for the stock points and the lead times of the component and (sub)assembly processes are assumed constant (planned lead times). It is further assumed that these lead times are multiples of the review period. The random demand variables of the finished products are identically and independently distributed (i.i.d.) for all of the time periods. Component commonality is allowed under the restriction that the two subassemblies that have a common component can not be used in the same finished product because that would result in two different cumulative lead times for the same component with respect to the same finished product. The key point in the study is the allocation of the common components to sets of subassemblies and finished products, which reveals why [20] is reviewed under the heading of commonality an product differentiation. Alternative allocation policies such as series system allocation (pre-allocation), fixed order allocation, random order allocation and combination of series system allocation and random order allocation are evaluated simulating the systems. As compared to the series system allocation, others which allocate common components as late as possible perform worse in terms of both costs and service level.

For a further study along the research direction in this thesis to use the basic building block (the simplest assembly system investigated in this thesis) within any joint collection of many other types of subsystems, it is inevitable to cite [14]. Ettl et al. [14] study base-stock controlled supply network of which structure is identified by BOM under consideration. In [14], a supply network is modeled as a collection of sites producing components, subassemblies or finished products. A single-level BOM is associated with each site and with each product produced at this site, containing the components and/or subassemblies making up a unit of the product. For the components, subassemblies and finished products that appear on the single-level BOMs of a site, there exist corresponding stock points at the site to hold inventories of all these items. In general, sites have input and output stores that keep one type of stock keeping unit (sku) and modeled as infinite-server queues, i.e., $M^X / G / \infty$, where batch arrivals of size X are allowed. Ettl et al. [14] work with an approximate analysis of lead times at each store and the associated normal distribution approximation for the demand over lead times. Approximate characterization of the lead time at a store is based on the assumptions that the stockout events at the supply stores of the one under consideration are independent and simultaneous stockout events at the stores are ignored. As an extension, they study the case of non-stationary demands adopting a rolling-horizon point of view. As for the optimization of the base-stock levels, the conjugate gradient routine they propose is to minimize the overall inventory capital for both the expected on-hand inventory (finished products) at the stores and work-in process inventory, applying cost coefficients as a function of the inventory capital per sku at different stores for on-hand (finished) products and the usage counts implied by the BOM to make up the finished products at different levels of BOM, and to guarantee the customer service requirements. The underlying difference between the approach taken by Ettl et al. [14] and the one in this thesis is that the former is based on the detailed lead time analysis unlike the latter. The uncapacitated model by Ettl et al. [14] allows any product structure to be specified by single-level BOM for each site whereas the product structure in the capacitated model we propose is quite specific. The optimization technique being a standard in nonlinear optimization requires the derivation of the gradients in explicit forms as opposed to the immediate usefulness of the greedy heuristic employed in this thesis.

CHAPTER 3

3. TWO-COMPONENT ASSEMBLY SYSTEMS

In this chapter, a simplified base-stock controlled assembly system is studied considering the manufacturing and assembly facilities as single (exponential) servers. Such a system with two components making up an assembly is depicted in Figure 3.1. Two semi-finished products called components 1 and 2 manufactured by the corresponding manufacturing servers are assembled to come up with a finished product (assembly). It is assumed there is no shortage of the raw materials 1 and 2 feeding the manufacturing servers. Upon completion of the process of an item, it is put in the associated stock point controlled by continuous-review base-stock policies with base-stock level S_i for component i, i=1, 2, and S_0 for the finished assemblies. The principle of base-stock policy is to keep the inventory position (total net inventory and amount on-order) at the target stock level specified as base-stock level.

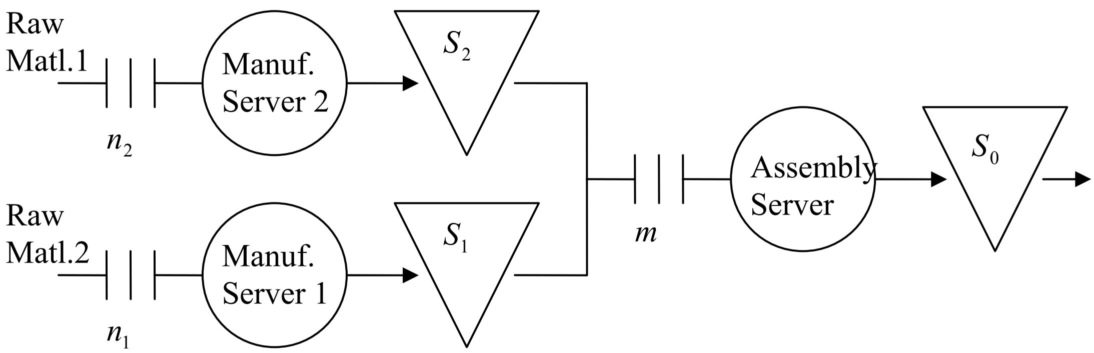


Figure 3.1 Two-Component Assembly System

A queuing model is presented for the two-component assembly systems a sketch of which is given above. A partial aggregation and a further slight modification of the queuing model lead to an approximate model that can be completely investigated to obtain the (closed-form) steady-state distribution which is of near-product-form. The numerical experiments show that the approximation is quite good in terms of the key performance measures and performance of the approximation does not deteriorate as the number of components increases.

The assembly system with two components is a building block to approximate the systems with more than two components in a recursive manner based on the approximations of the systems with lower number of components.

For the purpose of approximating the steady-state behavior of the assembly systems characterized above, the approach for two-echelon systems in Avşar and Zijm [2] and for two-indenture systems in Avşar and Zijm [38] is extended. To put it shortly, this approach is approximating a partially aggregated, but exact, queuing model which in our case is equivalent to an alternative model introduced in the next subsection. Unlike the references [2] and [38] listed above with just one aggregation step, for the two-component assembly system there are two aggregation steps, one corresponding to each of the components picked up sequentially. Approximation is analogous to the ones in these references where there is only one set of transition rates assumed constant, in this study there are three sets of such transition rates treated as constant rates.

3.1 Modeling

A queuing model for the assembly system with two components is given in Figure 3.2. While introducing notation for the parameters and the variables in Figure 3.2 formally, mechanics of the system (model) are explained next. When demand for an assembly arrives according to a Poisson process with rate λ, the demand request is transmitted to all stock points instantaneously due to the employment of continuous-review base-stock policies. Then, the following occurs:

- An assembly in stock, if there is any, i.e., $\overline{m} > 0$, is withdrawn merging it with the request generated by the demand arrival. If there is not any assembly in stock, i.e., $\overline{m} = 0$, the request is backordered, maybe in addition to the ones that are already backordered denoted by k_0.
- A component from each corresponding stock point is picked up, if there are both of the components, i.e., $\overline{n}_i > 0$ for every component $i = 1$, 2, merging them with the request generated by the demand being considered. All those merged are sent to the assembly server with exponential rate μ_0, m denoting the number of merged entities waiting for or being processed at the assembly server. If at least one of the component stock points is empty, the request is backordered increasing the value of k by one.

- Manufacturing one of both components is started instantaneously, i.e., n_1 and n_2, the number of components to be processed by the manufacturing servers with exponential rates μ_1 and μ_2 for components 1 and 2, respectively, increased by one to replenish the stock to be withdrawn with the requests just generated by the demand arrival.

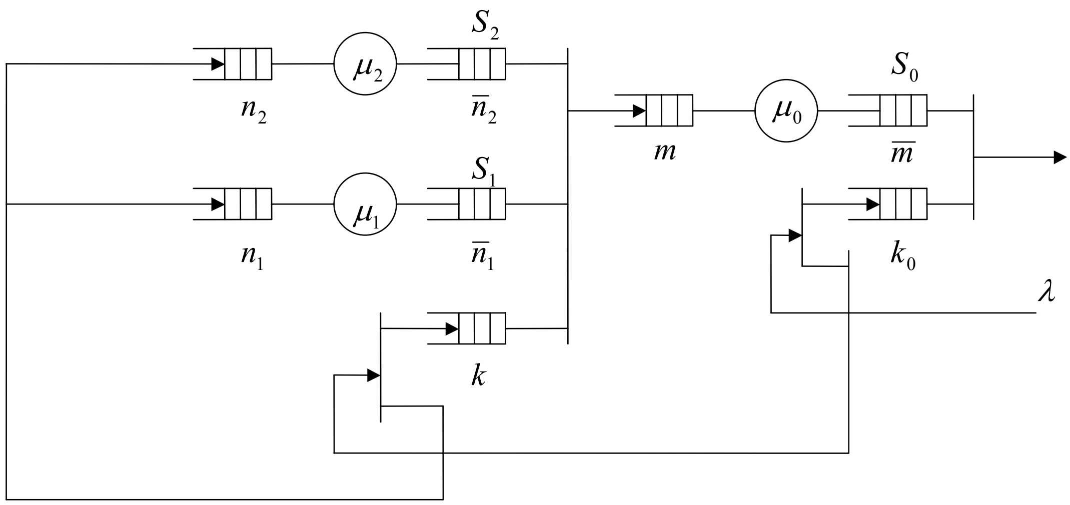


Figure 3.2 Model for the Assembly System with Two Components

The system is analyzed for given parameters λ, μ_0, μ_1, μ_2 such that $\lambda < \mu_i$, $i = 0, 1, 2$, and S_0, S_1, S_2 except the optimization section 3.5 where base-stock levels S_0, S_1, S_2 are optimized. The other entire notation, i.e., n_1, n_2, m, $\overline{n}_1$, $\overline{n}_2$, $\overline{m}$, that appears in Figure 3.2 is to represent some specific values of the random variables to be denoted by the corresponding capital letters.

The following below are the equations implied by the use of base-stock control policies and the synchronization to coordinate materials:

$$n_1 + \overline{n}_1 = S_1 + k, \tag{3.1}$$

$$n_2 + \overline{n}_2 = S_2 + k, \tag{3.2}$$

$$m + \overline{m} + k = S_0 + k_0, \tag{3.3}$$

$$\overline{n}_1 \cdot \overline{n}_2 \cdot k = 0 \text{ and } \overline{m} \cdot k_0 = 0.$$

More precisely, equations above with nonnegative random variables N_i, $\overline{N}_i$, $i = 1$, 2 and K, K_0, M, M_0 imply that

If $n_1 \le S_1$ and $n_2 \le S_2$, then $\overline{n}_1 = S_1 - n_1$, $\overline{n}_2 = S_2 - n_2$ and $k = 0$;

If $n_1 > S_1$ and $n_2 \le S_2$, then $\overline{n}_1 = 0$, $\overline{n}_2 = S_2 - n_2$ and $k = n_1 - S_1$;

If $n_1 \le S_1$ and $n_2 > S_2$, then $\overline{n}_1 = S_1 - n_1$, $\overline{n}_2 = 0$ and $k = n_2 - S_2$;

If $n_1 > S_1$ and $n_2 > S_2$, then $\overline{n}_1 = 0$, $\overline{n}_2 = 0$ and $k = \max\{n_1 - S_1, n_2 - S_2\}$;

If $m + k \le S_0$, then $\overline{m} = S_0 - (m + k)$ and $k_0 = 0$;

If $m + k > S_0$, then $\overline{m} = 0$ and $k_0 = (m + k) - S_0$.

From these relations, it immediately follows that (n_1, n_2, m) is adequate to completely determine state of the system. Thus this base-stock assembly system can be modeled as a continuous time Markov chain with state description (n_1, n_2, m). The corresponding transition diagram is given in Figure 3.4 where plus signs in parentheses beside transition rates denote an increment of m but the decreases in m are not shown not to complicate the figure with the inclusion of the third dimension for m. $Pr(N_1 = n_1, N_2 = n_2, M = m)$ is the steady-state probability of being in state (n_1, n_2, m), to be denoted by $P_{n_1 n_2 m}$ for simplicity of the notation.

A similar model is given in Figure 3.3 as an alternative to the one in Figure 3.2. As it is clarified in the next subsection, alternative model is appropriate to employ the type of approximation proposed by Avşar and Zijm in [2] and [38] although the original model is not. The difference between the original and the alternative models are questioned below.

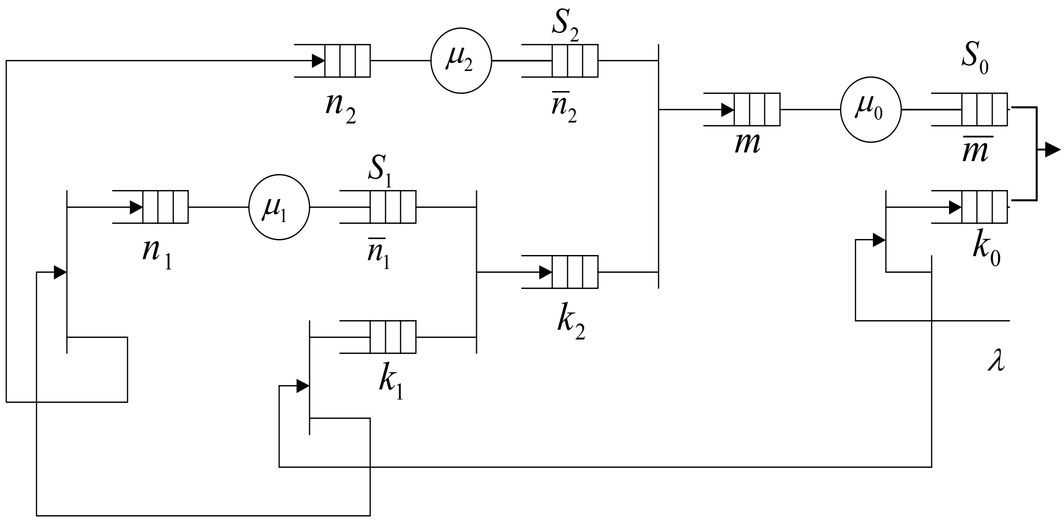


Figure 3.3 Alternative Model for the Assembly System with Two Components

In the original model, a request is sent to the assembly stage only when both of the two components are available. Alternative model, on the other hand, is to pick up the components to be sent to the assembly stage sequentially. Only after the first component becomes available, the request merged with this component proceeds to pick up the second component. So, random variable K in the original model does not appear in the alternative one but instead random variables K_1 and K_2 appear to denote the backordered requests for both components and for just component 2 after being merged with an available component 1, respectively. Since a request cannot be sent to the assembly stage without picking up one of each component, mechanics of the two models are the same regardless of the sequence the components are picked up. As a matter of fact, the transition diagrams of both the original model in Figure 3.2 and the alternative model in Figure 3.3 are as given in Figure 3.4 for the state description (n_1, n_2, m).

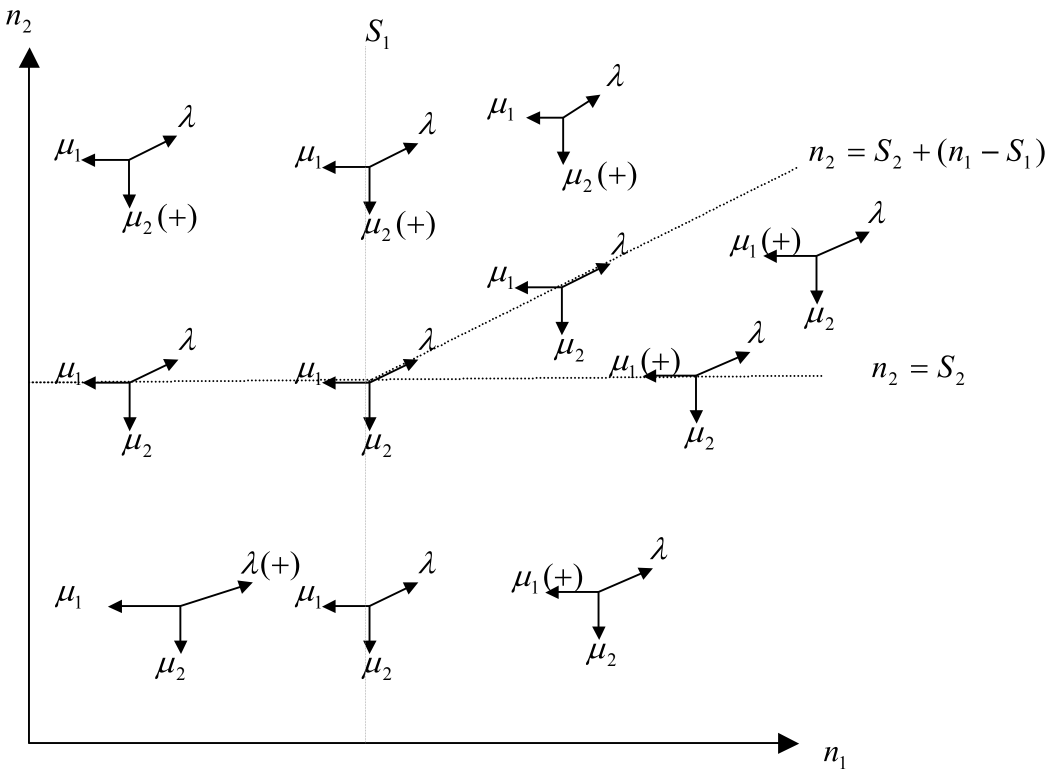


Figure 3.4 Transition Diagram of the Assembly Model for State Description (n_1, n_2, m)

Then, due to the use of base-stock policies the equations below are satisfied.

$$n_1 + \bar{n}_1 = S_1 + k_1, \tag{3.4}$$

$$n_2 + \bar{n}_2 = S_2 + k_1 + k_2, \tag{3.5}$$

$$m + \bar{m} + k_1 + k_2 = S_0 + k_0, \tag{3.6}$$

$$\bar{n}_i \cdot k_i = 0 \quad for\ i = 1,\ 2 \text{ and } \bar{m} \cdot k_0 = 0.$$

One could read equation (3.5) as the use of the base-stock level $S_2 + k_1$ for component 2, which is dependent on the value of r.v. K_1, and also one can argue the validity of $k_1 + k_2 = k$. As already noted, the state of the alternative system can also be determined by (n_1, n_2, m), but this time, according to the following relations:

If $n_1 \le S_1$ and $n_2 \le S_2 + k_1$, then $\overline{n}_1 = S_1 - n_1$, $\overline{n}_2 = S_2 - n_2$ and $k_1 = 0$, $k_2 = 0$;

If $n_1 > S_1$ and $n_2 \le S_2 + k_1$, then $\overline{n}_1 = 0$, $\overline{n}_2 = S_2 + k_1 - n_2$ and $k_1 = n_1 - S_1$, $k_2 = 0$;

If $n_1 \le S_1$ and $n_2 > S_2 + k_1$, then $\overline{n}_1 = S_1 - n_1$, $\overline{n}_2 = 0$ and $k_1 = 0$, $k_2 = n_2 - (S_2 + k_1)$;

If $n_1 > S_1$ and $n_2 > S_2 + k_1$, then $\overline{n}_1 = 0$, $\overline{n}_2 = 0$ and $k_1 = n_1 - S_1$, $k_2 = n_2 - (S_2 + k_1)$;

If $m + k_1 + k_2 \le S_0$, then $\overline{m} = S_0 - (m + k_1 + k_2)$ and $k_0 = 0$;

If $m + k_1 + k_2 > S_0$, then $\overline{m} = 0$ and $k_0 = (m + k_1 + k_2) - S_0$.

The fact that the original and the alternative models are equivalent for any sequence to pick up the components is easily observed in Figure 3.4 rewriting line $n_2 = S_2 + (n_1 - S_1)$ (or $n_2 = S_2 + k_1$) as $n_1 = S_1 + (n_2 - S_2)$ (or $n_1 = S_1 + k_2$) for state description (n_1, n_2, m), and then noting the interchange of the roles of the transitions with rates μ_1 and μ_2.

Lemma 3.1*: The original and the alternative assembly models with two components are equivalent for state description (n_1, n_2, m), independent of the sequence the components are picked up in the alternative model.*

Proof: Balance equations below being the same for any m for both the original and the alternative model in Figure 3.3 with state description (n_1, n_2, m) immediately lead to the equivalence of these two models.

1) $n_1 < S_1,\ n_2 < S_2$

$$\left(\lambda + \mu_1 I_{\{n_1>0\}} + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right) P_{n_1 n_2 m}$$
$$= \lambda I_{\{n_1>0\}} I_{\{n_2>0\}} I_{\{m>0\}} P_{n_1-1, n_2-1, m-1} + \mu_1 P_{n_1+1, n_2 m} + \mu_2 P_{n_1, n_2+1, m} + \mu P_{n_1 n_2, m+1}$$

2) $n_1 < S_1,\ n_2 = S_2$

$$\left(\lambda + \mu_1 I_{\{n_1>0\}} + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right) P_{n_1 n_2 m}$$
$$= \lambda I_{\{n_1>0\}} I_{\{n_2>0\}} I_{\{m>0\}} P_{n_1-1, n_2-1, m-1} + \mu_1 P_{n_1+1, n_2 m} + \mu_2 I_{\{m>0\}} P_{n_1, n_2+1, m-1} + \mu P_{n_1 n_2, m+1}$$

3) $n_1 = S_1,\ \ n_2 < S_2$

$$\left(\lambda + \mu_1 I_{\{n_1>0\}} + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m}$$
$$= \lambda I_{\{n_1>0\}} I_{\{n_2>0\}} I_{\{m>0\}} P_{n_1-1,n_2-1,m-1} + \mu_1 I_{\{m>0\}} P_{n_1+1,n_2,m-1} + \mu_2 P_{n_1,n_2+1,m} + \mu P_{n_1 n_2,m+1}$$

4) $n_1 = S_1,\ \ n_2 = S_2$

$$\left(\lambda + \mu_1 I_{\{n_1>0\}} + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m}$$
$$= \lambda I_{\{n_1>0\}} I_{\{n_2>0\}} I_{\{m>0\}} P_{n_1-1,n_2-1,m-1} + \mu_1 I_{\{m>0\}} P_{n_1+1,n_2,m-1} + \mu_2 I_{\{m>0\}} P_{n_1,n_2+1,m-1} + \mu P_{n_1 n_2,m+1}$$

5) $n_1 > S_1,\ \ n_2 = S_2 + (n_1 - S_1)$

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m}$$
$$= \lambda P_{n_1-1,n_2-1,m} + \mu_1 I_{\{m>0\}} P_{n_1+1,n_2,m-1} + \mu_2 I_{\{m>0\}} P_{n_1,n_2+1,m-1} + \mu P_{n_1 n_2,m+1}$$

6) $n_1 < S_1,\ \ n_2 > S_2$

$$\left(\lambda + \mu_1 I_{\{n_1>0\}} + \mu_2 + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m}$$
$$= \lambda I_{\{n_1>0\}} P_{n_1-1,n_2-1,m} + \mu_1 P_{n_1+1,n_2 m} + \mu_2 I_{\{m>0\}} P_{n_1,n_2+1,m-1} + \mu P_{n_1 n_2,m+1}$$

7) $n_1 = S_1,\ \ n_2 > S_2$

$$\left(\lambda + \mu_1 I_{\{n_1>0\}} + \mu_2 + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m}$$
$$= \lambda I_{\{n_1>0\}} P_{n_1-1,n_2-1,m} + \mu_1 P_{n_1+1,n_2 m} + \mu_2 I_{\{m>0\}} P_{n_1,n_2+1,m-1} + \mu P_{n_1 n_2,m+1}$$

8) $n_1 > S_1,\ \ n_2 > S_2 + (n_1 - S_1)$

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m} = \lambda P_{n_1-1,n_2-1,m} + \mu_1 P_{n_1+1,n_2 m} + \mu_2 I_{\{m>0\}} P_{n_1,n_2+1,m-1} + \mu P_{n_1 n_2,m+1}$$

9) $n_1 > S_1,\ \ n_2 < S_2 + (n_1 - S_1)$

$$\left(\lambda + \mu_1 + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)P_{n_1 n_2 m}$$

$$= \lambda I_{\{n_2>0\}} P_{n_1-1,n_2-1,m} + \mu_1 I_{\{m>0\}} P_{n_1+1,n_2,m-1} + \mu_2 P_{n_1,n_2+1,m} + \mu P_{n_1 n_2,m+1}$$

□

As for the independence of the equivalence from the sequence the components are picked up in the alternative model, it will be shown that the alternative model in Figure 3.3 is equivalent to the alternative model where component 2 is picked up first. Keeping the balance equations in cases 1, 4 and 5 (noting the representation of case 5 as $n_2 > S_2,\ \ n_1 = S_1 + (n_2 - S_2)$) as they are, noticing the changes in the roles of n_1 (μ_1) and n_2 (μ_2) in the balance equations of cases 2 and 3, and considering cases 6, 7, 8 altogether as $n_2 > S_2$, $n_1 < S_1 + (n_2 - S_2)$ to be compared to case 9, it is obvious that the balance equations above are also the balance equations of the alternative model picking up component 2 first. □

In spite of proving equivalence of the original and the alternative models for the state description (n_1, n_2, m), one would recognize that $\bar{n}_1$ and k in the original model correspond to $\bar{n}_1 + k_2$ and $k_1 + k_2$ in the alternative model in Figure 3.3. This is because, unlike the original model, in the alternative model in Figure 3.3 component 1 is picked up immediately when there is a request for it even if there is not any available component 2 in stock. That is, in the alternative model in Figure 3.3 $\bar{n}_1 < S_1$ but not in the original model. Requests merged with available components of type 1 are taken into account in the secondary backorder queue the size of which is denoted by k_2. The only difference between the original and the alternative model in Figure 3.3 is that component 1 is stored both in buffer stock (not merged with a request) and second backorder queue (merged with a request) before merging with component 2 to be next sent to the assembly facility. But this difference does not change mechanics of the system since a request needs to be merged with both component 1 and 2 to be assembled at the assembly facility in both models.

3.2 Aggregation of the model

We pursue an aggregation to change the parameters of the state description from (n_1, n_2, m) to (k_1, k_2, m). Consider the transition diagram of the alternative model shaded in Figure 3.5 for the first aggregation step. The part of the state space shaded is aggregated because states with $0 \le n_1 \le S_1$ are precisely those with no backlogged entity for component 1, i.e., with $k_1 = 0$, while any $k_1 > 0$ corresponds to the set of states with $n_1 = (S_1 + k_1)$. Therefore, the description of the system through the state vector (k_1, n_2, m) is the result of a natural aggregation. Denote the steady-state probabilities of the new description by $\hat{P}_{k_1 n_2 m}$ used for $Pr(K_1 = k_1, N_2 = n_2, M = m)$. Then, for any (n_2, m)

$$\hat{P}_{k_1 n_2 m} = \begin{cases} \sum_{n_1=0}^{S_1} P_{n_1 n_2 m} & k_1 = 0, \\ P_{S_1 + k_1, n_2 m} & k_1 > 0. \end{cases} \tag{3.7}$$

The transition diagram for state description (k_1, n_2, m) is given in Figure 3.6 where introduction of the conditional steady-state probabilities $\hat{q}(n_2, m)$ that appear to adjust the transition rates for $k_1 = 0$ is due to the aggregation. $\hat{q}(n_2, m)$ represents the steady-state probability that an arriving request at backorder queue for component 1 has to wait, given that it finds no other waiting requests in front of it and $N_2 = n_2$, $M = m$, i.e.,

$$\begin{aligned} \hat{q}(n_2, m) &= Pr(N_1 = S_1, N_2 = n_2, M = m \mid K_1 = 0, N_2 = n_2, M = m) \\ &= \frac{P_{S_1 n_2 m}}{\sum_{n_1=0}^{S_1} P_{n_1 n_2 m}}. \end{aligned} \tag{3.8}$$

Note that $\bar{\hat{q}}$ in Figure 3.6 denotes (*1* - $\hat{q}$).

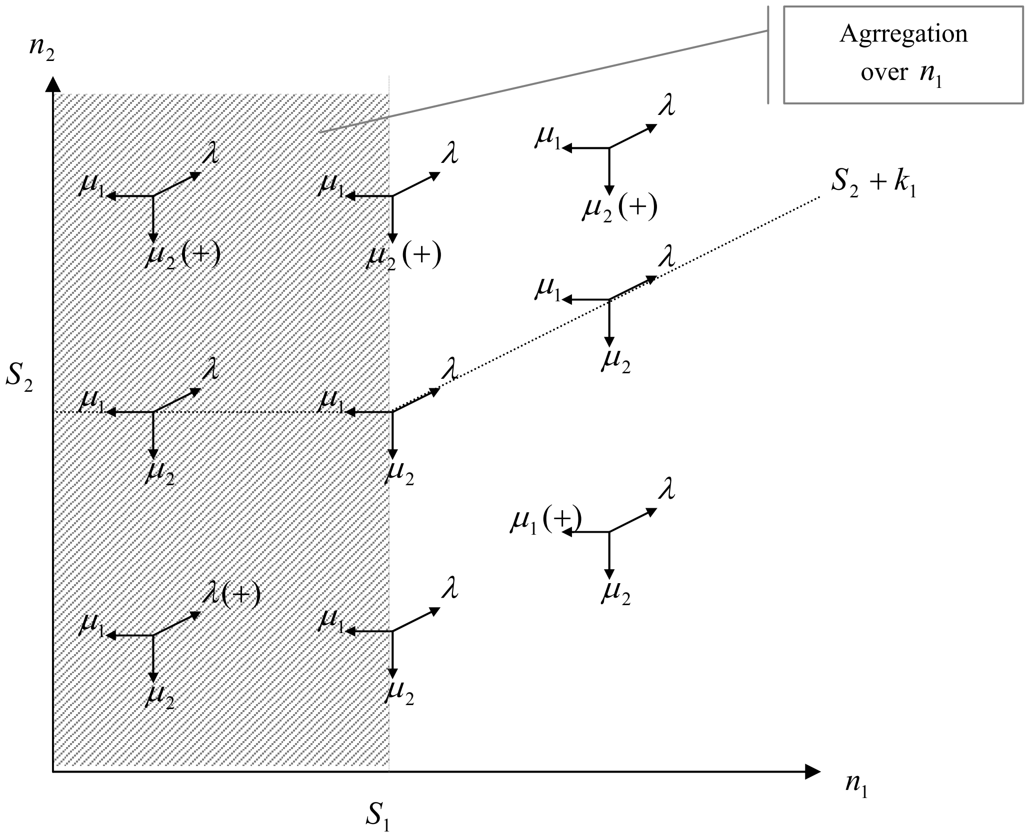


Figure 3.5 Transition Diagram of the Assembly Model with State Description (n_1 , n_2 , m)

Lemma 3.2: *The model with state description (k_1, n_2, m) is an aggregate formulation of the one with state description (n_1 , n_2 , m).*

Proof: The steady-state balance equations for nine cases, a, b, .., i, of Figure 3.6 are obtained from cases 1, 2, .., 9 of Figure 3.5 in the proof of Lemma 3.1.

a) $k_1 = 0,\ n_2 < S_2 + k_1$ Cases 1 and 3

$$\left(\lambda + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)\hat{P}_{0n_2m} + \mu_1 I_{\{s_1>0\}} \sum_{n_1=1}^{s_1} P_{n_1n_2m}$$

$$= \lambda I_{\{n_2>0\}} I_{\{m>0\}} \sum_{n_1=0}^{s_1} P_{n_1-1,n_2-1,m-1} * \frac{\hat{P}_{0n_2,m-1}}{\hat{P}_{0n_2,m-1}} + \mu_1 I_{\{s_1>0\}} \sum_{n_1=1}^{s_1} P_{n_1n_2m} + \mu_1 I_{\{m>0\}} \hat{P}_{1n_2,m-1} + \mu_2 \hat{P}_{0n_2+1,m} +$$

$$\mu \hat{P}_{0n_2,m+1}$$

The second terms on both sides of the equation cancel out and the first term on the right hand side of the equation is rewritten as $\lambda I_{\{n_2>0\}} I_{\{m>0\}} \bar{\hat{q}}_{(n_2-1,m-1)} \hat{P}_{0n_2,m-1}$.

b) $k_1 = 0,\ n_2 = S_2 + k_1$ Cases 2 and 4

$$\left(\lambda + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)\hat{P}_{0n_2m} + \mu_1 I_{\{s_1>0\}} \sum_{n_1=1}^{s_1} P_{n_1n_2m}$$

$$= \lambda I_{\{n_2>0\}} I_{\{m>0\}} \bar{\hat{q}}_{(n_2-1,m-1)} \hat{P}_{0n_2,m-1} + \mu_1 I_{\{s_1>0\}} \sum_{n_1=1}^{s_1} P_{n_1n_2m} + \mu_1 I_{\{m>0\}} \hat{P}_{1n_2,m-1}$$

$$+ \mu_2 I_{\{m>0\}} \hat{P}_{0,n_2+1,m-1} + \mu \hat{P}_{0n_2,m+1}$$

The second terms on both sides of the equation cancel out, and the first term on the right hand side comes up as in case (a).

c) $k_1 = 0,\ n_2 > S_2 + k_1$ Cases 6 and 7

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\hat{P}_{0n_2m} + \mu_1 I_{\{s_1>0\}} \sum_{n_1=1}^{s_1} P_{n_1n_2m}$$

$$= \lambda \bar{\hat{q}}_{(n_2-1,m)} \hat{P}_{0,n_2-1,m} + \mu_1 I_{\{s_1>0\}} \sum_{n_1=1}^{s_1} P_{n_1n_2m} + \mu_1 \hat{P}_{1n_2m} + \mu_2 I_{\{m>0\}} \hat{P}_{0,n_2+1,m-1} + \mu \hat{P}_{0n_2,m+1}$$

Cancellations and the explanation for the $\bar{\hat{q}}$ term are as in cases (a) and (b).

d) $k_1 = 1,\ n_2 < S_2 + k_1$ Case 9

$$\left(\lambda + \mu_1 + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)\hat{P}_{1n_2m}$$

$$= \lambda I_{\{n_2>0\}} P_{n_1=s_1,n_2-1,m} * \frac{\hat{P}_{0,n_2-1,m}}{\hat{P}_{0,n_2-1,m}} + \mu_1 I_{\{m>0\}} \hat{P}_{2n_2,m-1} + \mu_2 \hat{P}_{1,n_2+1,m} + \mu \hat{P}_{1n_2,m+1}$$

The first term on the right hand side of the equation is rewritten as follows:

$$\lambda I_{\{n_2>0\}} \hat{q}_{(n_2-1,m)} \hat{P}_{0,n_2-1,m}.$$

e) $k_1 = 1,\ n_2 = S_2 + k_1$ Case 5

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)\hat{P}_{1n_2m}$$

$$= \lambda \hat{q}_{(n_2-1,m)} \hat{P}_{0,n_2-1,m} + \mu_1 I_{\{m>0\}} \hat{P}_{2n_2,m-1} + \mu_2 I_{\{m>0\}} \hat{P}_{1,n_2+1,m-1} + \mu \hat{P}_{1n_2,m+1}$$

where the first term on the right hand side comes up as in case (d).

f) $k_1 = 1,\ n_2 > S_2 + k_1$ Case 8

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\langle m>0\rangle}\right)\hat{P}_{1n_2m}$$

$$= \lambda \hat{q}_{(n_2-1,m)} \hat{P}_{0,n_2-1,m} + \mu_1 \hat{P}_{2n_2m} + \mu_2 I_{\{m>0\}} \hat{P}_{1,n_2+1,m-1} + \mu \hat{P}_{1n_2,m+1}$$

where the derivation is as in cases (d) and (e).

g) $k_1 > 1,\ n_2 < S_2 + k_1$ Case 9

$$\left(\lambda + \mu_1 + \mu_2 I_{\{n_2>0\}} + \mu I_{\{m>0\}}\right)\hat{P}_{k_1n_2m}$$

$$= \lambda I_{\{n_2>0\}} \hat{P}_{k_1-1,n_2-1,m} + \mu_1 I_{\{m>0\}} \hat{P}_{k_1+1,n_2,m-1} + \mu_2 \hat{P}_{k_1,n_2+1,m} + \mu \hat{P}_{k_1n_2,m+1}$$

h) $k_1 > 1,\ n_2 = S_2 + k_1$ Case 5

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)\hat{P}_{k_1n_2m}$$

$$= \lambda \hat{P}_{k_1-1,n_2-1,m} + \mu_1 I_{\{m>0\}} \hat{P}_{k_1+1,n_2,m-1} + \mu_2 I_{\{m>0\}} \hat{P}_{k_1,n_2+1,m-1} + \mu \hat{P}_{k_1n_2,m+1}$$

i) $k_1 > 1,\ n_2 > S_2 + k_1$ Case 8

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)\hat{P}_{k_1n_2m}$$

$$= \lambda \hat{P}_{k_1-1,n_2-1,m} + \mu_1 \hat{P}_{k_1+1,n_2m} + \mu_2 I_{\{m>0\}} \hat{P}_{k_1,n_2+1,m-1} + \mu \hat{P}_{k_1n_2,m+1}$$

□

The second natural aggregation is over the part of the state space shaded in Figure 3.6 since the states with $0 \le n_2 \le S_2 + k_1$ are exactly the ones with no backlogged entity at the backorder queue for component 2, i.e., with $k_2 = 0$, while any $k_2 > 0$ represents the set of states with $n_2 = (S_2 + k_1 + k_2)$. This further aggregation, then results in the change of the state description from (k_1, n_2, m) to

(k_1, k_2, m). Denote the steady-state probabilities of the new description by $\widetilde{P}_{k_1 k_2 m}$ used for $Pr(K_1 = k_1, K_2 = k_2, M = m)$. Then, for any m

$$\widetilde{P}_{k_1 k_2 m} = \begin{cases} \sum_{n_2=0}^{S_2+k_1} \hat{P}_{k_1 n_2 m} & k_1 \geq 0,\ \ k_2 = 0, \\ \hat{P}_{k_1, S_2+k_2, m} & k_1 \geq 0,\ k_2 > 0. \end{cases} \tag{3.9}$$

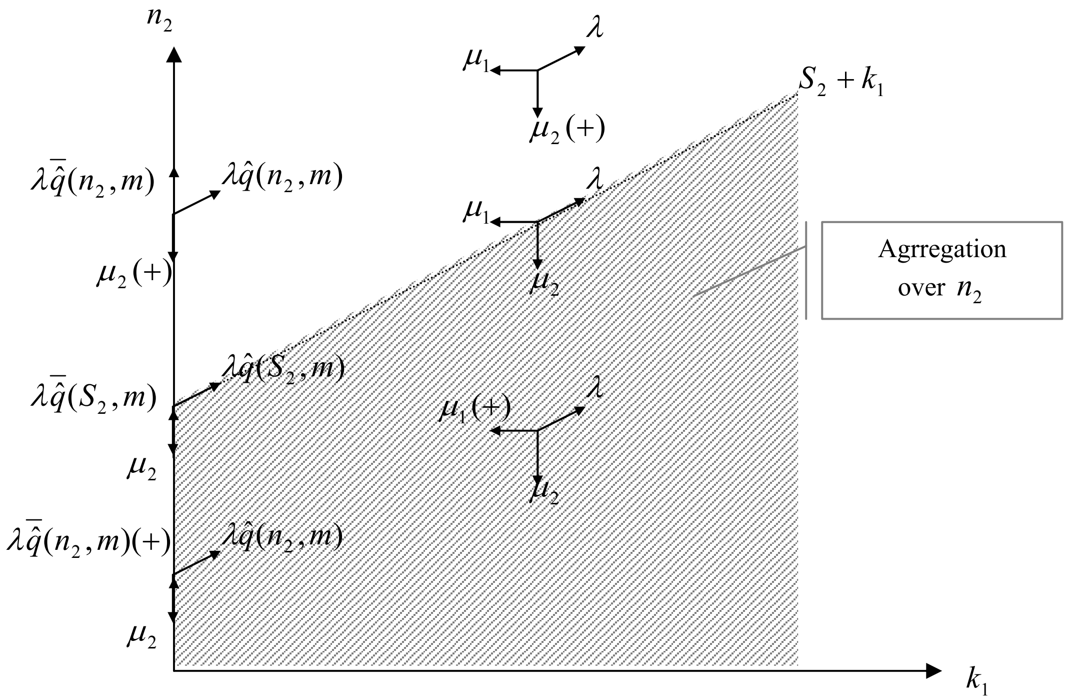


Figure 3.6 Transition Diagram of the Aggregate Model with State Description (k_1, n_2, m)

The transition diagram for the aggregated model turns out to be as in Figure 3.7 and 3.8 where three sets of conditional probabilities, called as q, q', q'', appear this time. Note that $\bar{q}$, $\bar{q}'$, $\bar{q}''$ denote complementary cumulatives of q, q', q'', respectively. $q(k_2, m)$ is the steady-state probability that an arriving request at backorder queue for component 1 has to wait, given that it finds no other waiting requests in front of it when $K_2 = k_2,\ M = m$, i.e.,

$$q(0,m) = \sum_{n_2=0}^{S_2} \hat{q}(n_2,m) Pr(K_1 = 0, N_2 = n_2, M = m \mid K_1 = 0, N_2 \leq S_2 + K_1, M = m)$$

$$= \sum_{n_2=0}^{S_2} \hat{q}(n_2,m) \frac{\hat{P}_{0n_2m}}{\sum_{n_2=0}^{S_2} \hat{P}_{0n_2m}} \qquad \text{for} \qquad k_2 = 0, \tag{3.10}$$

$$q(k_2,m) = \hat{q}(S_2 + k_2, m) \qquad \text{for} \qquad k_2 > 0. \tag{3.11}$$

$q'(k_1,m)$ is the steady-state probability that an arriving request at backorder queue for component 2 has to wait, given that it finds no other waiting requests in front of it when $K_1 = k_1, M = m$, i.e.,

$$q'(k_1,m) = Pr(K_1 = k_1, N_2 = S_2 + K_1, M = m \mid K_1 = k_1, N_2 \leq S_2 + K_1, M = m)$$

$$= \frac{\hat{P}_{k_1,S_2+k_1,m}}{\sum_{n_2=0}^{S_2+k_1} \hat{P}_{k_1n_2m}} \qquad \text{for} \quad k_1 > 0. \tag{3.12}$$

Finally, $q''(m)$ is the steady-state probability that an arriving request at backorder queue for component 1 is merged with an available component at the buffer stock but this merged entity that passes to backorder queue for component 2 has to wait, given that it finds no other waiting requests in front of it when $M = m$, i.e.,

$$q''(m) = \frac{Pr(N_1 < S_1, N_2 = S_2, M = m)}{Pr(K_1 = 0, K_2 = 0, M = m)}$$

$$= \bar{\hat{q}}(s_2,m) \frac{\hat{P}_{0S_2m}}{\sum_{n_2=0}^{S_2} \hat{P}_{0n_2m}}$$

$$= q'(0,m) - \hat{q}(s_2,m) \frac{\hat{P}_{0S_2m}}{\sum_{n_2}^{S_2} \hat{P}_{0n_2m}}. \tag{3.13}$$

Note that

$$q(0,m) = Pr(N_1 = S_1, K_2 = k_2, M = m | N_1 \le S_1, N_2 = k_2, M = m), \qquad (3.14)$$

i.e.,

$$q(k_2,m) = \begin{cases} \dfrac{\sum_{n_2=0}^{S_2} P_{S_1 n_2 m}}{\sum_{n_2=0}^{S_2}\sum_{n_1=0}^{S_1} P_{n_1 n_2 m}} & k_2 = 0, \\ \dfrac{P_{S_1, S_2+k_2, m}}{\sum_{n_1=0}^{S_1} P_{n_1, S_2+k_2, m}} & k_2 > 0, \end{cases}$$

and

$$q'(k_1,m) = Pr(K_1 = k_1, N_2 = S_2 + K_1, M = m | K_1 = k_1, N_2 \le S_2 + K_1, M = m), \qquad (3.15)$$

i.e.,

$$q'(k_1,m) = \begin{cases} \dfrac{\sum_{n_1=0}^{S_1} P_{n_1 S_2 m}}{\sum_{n_1=0}^{S_1}\sum_{n_2=0}^{S_2} P_{n_1 n_2 m}} & k_1 = 0, \\ \dfrac{P_{S_1+k_1, S_2+k_1, m}}{\sum_{n_2=0}^{S_2+k_1} P_{S_1+k_1, n_2, m}} & k_1 > 0, \end{cases}$$

and

$$q''(m) = Pr(N_1 < S_1, N_2 = S_2, M = m | K_1 = 0, K_2 = 0, M = m) \qquad (3.16)$$

$$= \frac{\sum_{n_1=0}^{S_1-1} P_{n_1 S_2 m}}{\sum_{n_1=0}^{S_1}\sum_{n_2=0}^{S_2} P_{n_1 n_2 m}}.$$

Lemma 3.3: *The model with the state description (k_1, k_2, m) is an aggregate formulation of the one with state description (k_1, n_2, m).*

Proof: The steady-state balance equations for state description (k_1, k_2, m) of Figure 3.7 are obtained from cases a, b, .., i of Figure 3.6 in the proof of Lemma 3.2.

$k_1 > 1,\; k_2 = 0$ Cases g and h

$$\sum_{n_2=0}^{S_2+k_1} \left(\lambda + \mu_1 + \mu I_{\{m>0\}}\right)\hat{P}_{k_1 n_2 m} + \sum_{n_2=0}^{S_2+k_1} \mu_2 I_{\{n_2>0\}} \hat{P}_{k_1 n_2 m}$$

$$= \lambda I_{\{n_2>0\}} \sum_{n_2=0}^{S_2+k_1} \hat{P}_{k_1-1,n_2-1,m} + \mu_1 I_{\{m>0\}} \sum_{n_2=0}^{S_2+k_1} \hat{P}_{k_1+1,n_2,m-1}$$

$$+ \mu_2 \sum_{n_2=0}^{S_2+k_1-1} \hat{P}_{k_1,n_2+1,m} + \mu_2 I_{\{m>0\}} \hat{P}_{k_1,s_2+k_1+1,m-1} + \mu \sum_{n_2=0}^{S_2+k_1} \hat{P}_{k_1 n_2,m+1}$$

The second term on the left hand side of the equation and the third term on the right hand side cancel out. Then, rewriting the remaining terms as

$$\left(\lambda + \mu_1 + \mu I_{\{m>0\}}\right)\widetilde{P}_{k_1 0 m} = \lambda \widetilde{P}_{k_1-1,0m} + \mu_1 I_{\{m>0\}} \left(\frac{\sum_{n_2=0}^{S_2+k_1+1} \hat{P}_{k_1+1,n_2,m-1}}{\widetilde{P}_{k_1+1,0,m-1}} - \frac{\hat{P}_{k_1+1,S_2+k_1,m-1}}{\widetilde{P}_{k_1+1,0,m-1}} \right) \widetilde{P}_{k_1+1,0,m-1}$$

$$+ \mu_2 I_{\{m>0\}} \widetilde{P}_{k_1 1,m-1} + \mu \widetilde{P}_{k_1 0,m+1},$$

the second term on the right hand side becomes

$$\mu_1 I_{\{m>0\}} (1 - q'(k_1+1, m-1)) \widetilde{P}_{k_1+1,0,m-1}.$$

$k_1 = 1,\; k_2 = 0$ Cases d and e

$$\sum_{n_2=0}^{S_2+k_1}\left(\lambda+\mu_1+\mu I_{\{m>0\}}\right)\hat{P}_{1n_2m}+\sum_{n_2=0}^{S_2+k_1}\mu_2 I_{\{n_2>0\}}\hat{P}_{1n_2m}$$

$$=\lambda\sum_{n_2=1}^{S_2+k_1}\hat{q}(n_2-1,m)\hat{P}_{0,n_2-1,m}+\mu_1 I_{\{m>0\}}\sum_{n_2=0}^{S_2+k_1}\hat{P}_{2n_2,m-1}+\mu_2\sum_{n_2=0}^{S_2+k_1-1}\hat{P}_{1,n_2+1,m}$$

$$+\mu_2 I_{\{m>0\}}\hat{P}_{2,S_2+2,m-1}+\mu\sum_{n_2=0}^{S_2+k_1}\hat{P}_{1n_2,m+1}$$

The second term on the left hand side of the equation and the third term on the right hand side cancel out. Rewriting the equation as

$$\left(\lambda+\mu_1+\mu I_{\{m>0\}}\right)\widetilde{P}_{1,0m}=\lambda\left(\sum_{n_2=1}^{S_2}\frac{P_{S_1n_2m}}{P_{n_1\le S_1,n_2m}}*\frac{\hat{P}_{0n_2m}}{\widetilde{P}_{00m}}\right)\widetilde{P}_{0,0m}$$

$$+\mu_1 I_{\{m>0\}}\left(\frac{\sum_{n_2=0}^{S_2+2}\hat{P}_{2n_2,m-1}}{\widetilde{P}_{2,0,m-1}}-\frac{\hat{P}_{2,S_2+2,m-1}}{\widetilde{P}_{2,0,m-1}}\right)\widetilde{P}_{2,0,m-1}+\mu_2 I_{\{m>0\}}\widetilde{P}_{1,1,m-1}+\mu\widetilde{P}_{1,0,m+1},$$

the first and the second terms on the right hand side turn out to be $\lambda q(0,m)\widetilde{P}_{00m}$ and $\mu_1 I_{\{m>0\}}(1-q'(2,m-1))\widetilde{P}_{20,m-1}$, respectively.

$k_1=0,\ k_2=0$ Case a and b

$$\sum_{n_2=0}^{S_2}\left(\lambda+\mu I_{\{m>0\}}\right)\hat{P}_{0n_2m}+\sum_{n_2=0}^{S_2}\mu_2 I_{\{n_2>0\}}\hat{P}_{0n_2m}=\lambda I_{\{n_2>0\}}I_{\{m>0\}}\sum_{n_2=0}^{S_2}\bar{q}(n_2-1,m)\hat{P}_{0,n_2-1,m-1}$$

$$+\mu_1 I_{\{m>0\}}\sum_{n_2=0}^{S_2}\hat{P}_{1n_2,m-1}+\mu_2\sum_{n_2=0}^{S_2-1}\hat{P}_{0,n_2+1,m}+\mu_2 I_{\{m>0\}}\hat{P}_{0,S_2+1,m-1}+\mu\sum_{n_2=0}^{S_2}\hat{P}_{0n_2,m+1}$$

The second term on the left hand side and the third term on the right cancel out. Rewriting the terms as

$$\left(\lambda+\mu I_{\{m>0\}}\right)\widetilde{P}_{0,0m}=\lambda I_{\{m>0\}}\widetilde{P}_{0,0,m-1}$$

$$\left(\left(\sum_{n_2=0}^{S_2}\frac{\hat{P}_{0n_2,m-1}}{\widetilde{P}_{0,0,m-1}}-\frac{\hat{P}_{0S_2,m-1}}{\widetilde{P}_{0,0,m-1}}\right)-\left(\sum_{n_2=0}^{S_2}\hat{q}(n_2,m-1)\frac{\hat{P}_{0n_2,m-1}}{\widetilde{P}_{0,0,m-1}}-\hat{q}(S_2,m-1)\frac{\hat{P}_{0S_2,m-1}}{\widetilde{P}_{0,0,m-1}}\right)\right)$$

$$+\ \mu_1 I_{\{m>0\}}\left(\frac{\sum_{n_2=0}^{S_2+1}\hat{P}_{1n_2,m-1}}{\widetilde{P}_{1,0,m-1}} - \frac{\hat{P}_{1,S_2+2,m-1}}{\widetilde{P}_{1,0,m-1}}\right)\widetilde{P}_{1,0,m-1} + \mu_2 I_{\{m>0\}}\widetilde{P}_{0,1,m-1} + \mu\widetilde{P}_{00,m+1,}$$

the equation takes the following form:

$$\left(\lambda + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,0m} = \lambda I_{\{m>0\}}\left((1-q(0,m-1))-(q'(0,m-1)-\hat{q}(S_2,m-1)\frac{\hat{P}_{0S_2,m-1}}{\widetilde{P}_{0,0,m-1}})\right)\widetilde{P}_{0,0,m-1}$$

$$+\ \mu_1 I_{\{m>0\}}(1-q'(1,m-1))\widetilde{P}_{1,0m} + \mu_2 I_{\{m>0\}}\widetilde{P}_{0,1,m-1} + \mu\widetilde{P}_{0,0,m+1}$$

where the first term on the right hand side of the equation is

$$\lambda I_{\{m>0\}}((1-q(0,m-1)) - q''(m-1))\widetilde{P}_{0,0,m-1}.$$

$k_1 = 0,\ k_2 = 1$ Special case of c

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\hat{P}_{0S_2m} = \lambda\hat{\hat{q}}_{(S_2,m)}\hat{P}_{0S_2m} + \mu_1\hat{P}_{1,S_2+1,m} + \mu_2 I_{\{m>0\}}\hat{P}_{0,S_2+2,m-1} + \mu\hat{P}_{0,S_2+1,m+1}$$

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,1m} = \lambda\left(\frac{P_{S_1S_2m}}{P_{n_1\le S_1,S_2m}}\cdot\frac{\hat{P}_{0S_2m}}{\widetilde{P}_{0,0m}}\right)\widetilde{P}_{0,0m} + \mu_1\frac{\hat{P}_{1,S_2+1,m}}{P_{1,0m}}\widetilde{P}_{1,0m}$$

$$+\ \mu_2 I_{\{m>0\}}\widetilde{P}_{0,2,m-1} + \mu\hat{P}_{0,1,m+1}$$

where the first and the second terms on the right side are $\lambda q''(m)\widetilde{P}_{0,0m}$ and $\mu_1 q'(1,m)\widetilde{P}_{10m}$, respectively.

□

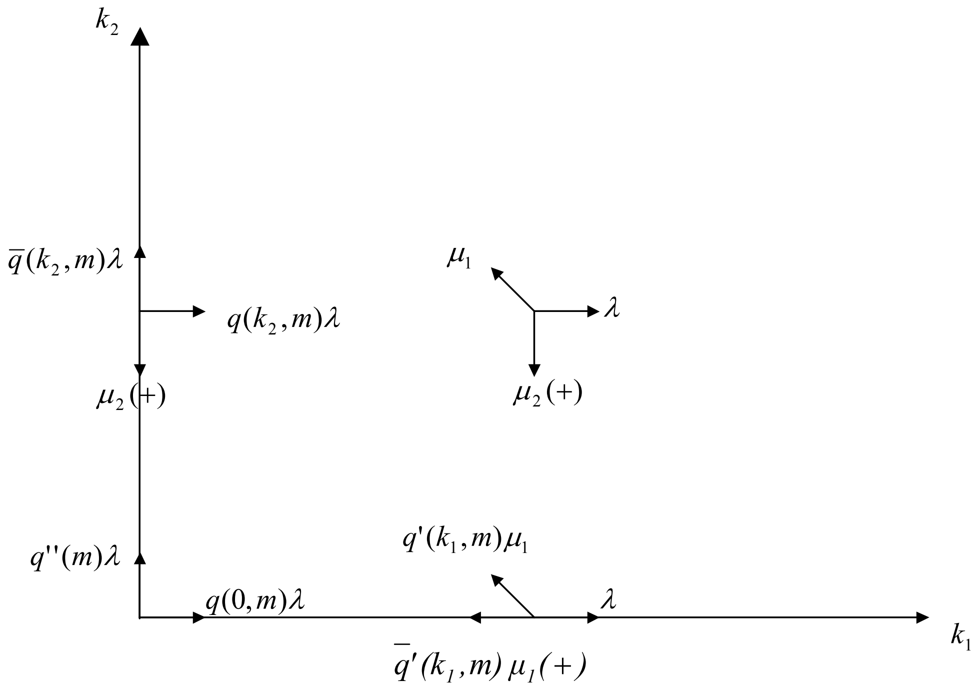


Figure 3.7 Transition Diagram of the Aggregate Model with State Description (k_1, k_2, m)

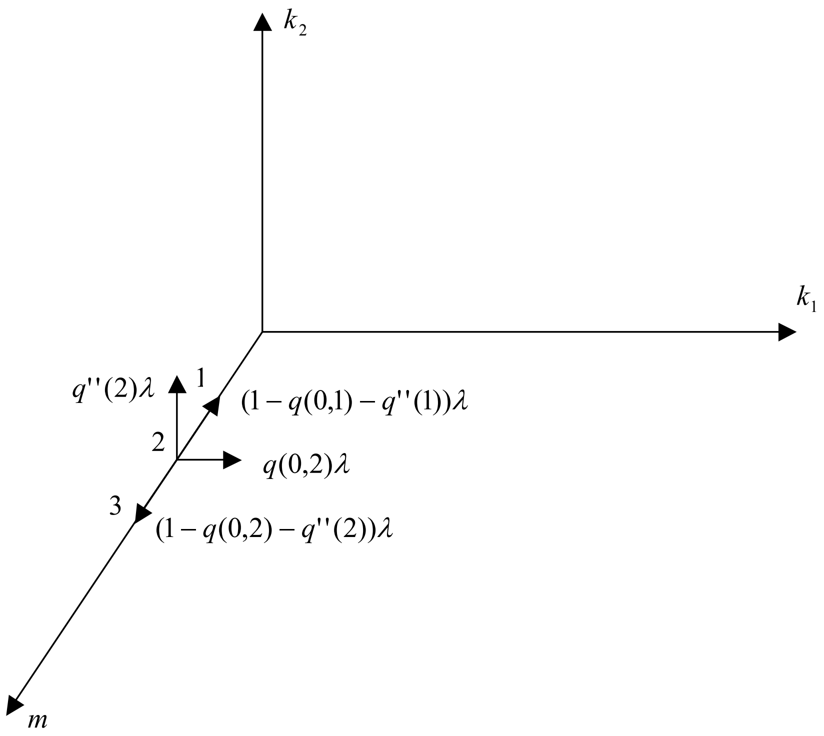


Figure 3.8 A Part of the Transition Diagram of the
Aggregate Model with State Description (k_1, k_2, m)

It is also possible to consider the aggregation of (n_1, n_2, m) to come up with the formulation of (k_1, k_2, m) at one shot. The corresponding proof is given in Appendix A.

Now, we consider the special case $S_1 = 0$. Since there is no stock for component 1 in this case, arriving requests are always backordered and the number of items being processed at manufacturing server 1 (n_1) is always equal to number of requests backordered (k_1). Then, the state descriptions (n_1, n_2, m) and (k_1, n_2, m) are equivalent and the transition diagrams of the aggregate models with state descriptions (k_1, n_2, m) and (k_1, k_2, m) become as in Figure 3.9 and Figure 3.10, respectively.

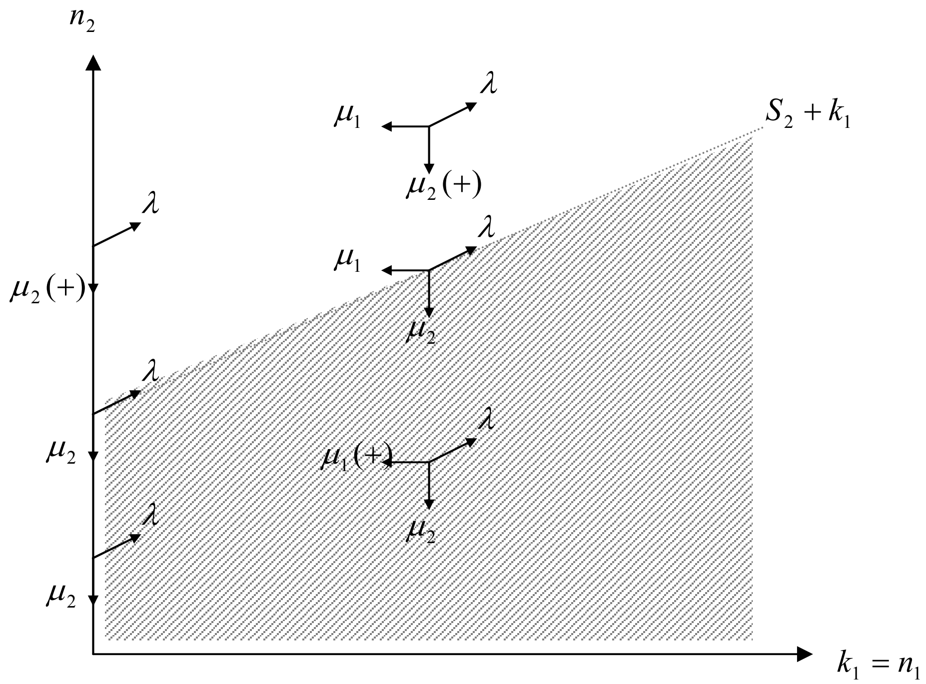


Figure 3.9 Transition Diagram of the Aggregate Model with State Description (k_1, n_2, m) when $S_1 = 0$

Note that there is nothing modified (approximated) while developing the aggregate model. This is formally pointed out with the following remark. To justify the correctness of the remark, one should refer to Lemma 3.1.

Remark 3.1: *The aggregate model is exact (equivalent to the original model) regardless of the sequence the components are picked up.* □

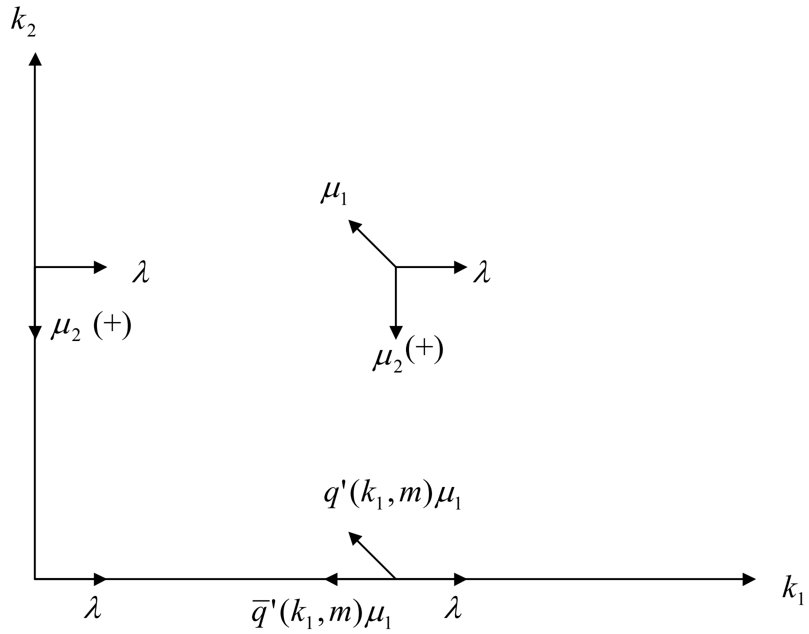


Figure 3.10 Transition Diagram of the Aggregate Model
with State Description (k_1, k_2, m) when $S_1 = 0$

3.3 Approximation of the aggregate model

The difficulty in solving balance equations of the aggregate model with state description (k_1, k_2, m) to find the steady-state probability distribution is due to the dependence of $q(k_2, m)$ on k_2 and m, and of $q'(k_1, m)$ on k_1 and m, and of $q''(m)$ on m. Basically, the approximation discussed below comes down to ignoring this dependence and working with some constant q, q' and q'' values. Throughout this study, the aggregate model modified to include any constant q, q' and q'' values (not necessarily the ones given by Lemma 3.4) is called as the approximate model.

The constant q, q' and q'' values considered in this study are the expected values of $q(k_2, m)$, $q'(k_1, m)$ and $q''(m)$, respectively. According to this specification, formulas for q, q' and q'' given in Lemma 3.4 are derived in Appendix B.

Lemma 3.4: *The expected values of the* $q(k_2, m)$, $q'(k_1, m)$ *and* $q''(m)$ *are*

$$q = \frac{Pr(N_1 = S_1)}{Pr(N_1 \leq S_1)}, \tag{3.17}$$

$$q' = \frac{Pr(N_2 = S_2 + K_1)}{Pr(N_2 \leq S_2 + K_1)}, \tag{3.18}$$

and

$$q'' = \frac{Pr(N_1 < S_1, N_2 = S_2)}{Pr(K_1 = 0, K_2 = 0)}, \tag{3.19}$$

respectively.

q'' in Lemma 3.4 is approximated by $\overline{q} \cdot q'$ rearranging the terms as seen below.

$$q'' = \frac{Pr(N_1 < S_1, N_2 = S_2)}{Pr(K_1 = 0, K_2 = 0)}$$

$$= \frac{Pr(N_1 < S_1)}{Pr(K_1 = 0)} \cdot \frac{Pr(N_2 = S_2 | N_1 < S_1)}{Pr(K_2 = 0 | K_1 = 0)}$$

$$= \frac{Pr(N_1 < S_1)}{Pr(N_1 \leq S_1)} \cdot \frac{Pr(N_2 = S_2 | N_1 < S_1)}{Pr(N_2 \leq S_2 | N_1 \leq S_1)}$$

where in the last equation the first ratio is $\overline{q}$ and the second ratio is approximated by q'.

Next, exactness of the approximate model with the q, q' and q'' values specified by Lemma 3.4 is observed for some extreme values of the base-stock levels. Exactness above means equivalence of the approximate model to the aggregate model which is equivalent to the original one as already noted in Remark 3.1.

Lemma 3.5: *The approximate model with q, q' and q'' given in Lemma 3.4 is exact for (S_1, S_2) being $(0,\infty)$, $(\infty,0)$, (∞,∞).*

Proof: Recalling (3.14), (3.17) and (3.15), (3.18) and (3.16), (3.19),

$$q = q(k_2, m) = \begin{cases} 1 & for\ (S_1, S_2) = (0,\infty), \quad for\ all\ k_2\ and\ m, \\ 0 & for\ (S_1, S_2) \in \{(\infty,0),(\infty,\infty)\} \quad for\ all\ k_2\ and\ m, \end{cases}$$

and

$$q' = q'(k_1, m) = \begin{cases} 1 & for\ (S_1, S_2) = (\infty,0), \quad for\ all\ k_1\ and\ m, \\ 0 & for\ (S_1, S_2) \in \{(0,\infty),(\infty,\infty)\} \quad for\ all\ k_1\ and\ m, \end{cases}$$

and

$$q'' = q''(m) = \begin{cases} 1 & for\ (S_1, S_2) = (\infty,0), \quad for\ all\ m, \\ 0 & for\ (S_1, S_2) \in \{(0,\infty),(\infty,\infty)\} \quad for\ all\ m, \end{cases}$$

respectively. □

Two-step aggregation of the alternative model and then replacement of a part of the transition rates with constant rates lead to the approximate model with a product-form steady-state distribution, which is called a near-product-form due to the partial aggregations, given in Theorem 3.1.

Theorem 3.1: *For the approximate model, the steady-state distribution is*

$$\breve{P}_{k_1k_2m} = \breve{P}_1(K_1 = k_1)\breve{P}_2(K_2 = k_2)\breve{P}_0(M = m)$$

where

$$\breve{P}_1(K_1 = k_1) = \begin{cases} \dfrac{a}{q} & for \quad k_1 = 0, \\ a\rho_1^{k_1} & for \quad k_1 \geq 1, \end{cases} \qquad \breve{P}_2(K_2 = k_2) = \begin{cases} \dfrac{b}{q'} & for \quad k_2 = 0, \\ b\rho_2^{k_2} & for \quad k_2 \geq 1, \end{cases}$$

$\breve{P}_0(M = m) = (1-\rho)\rho^m$ *for* $m \geq 0$

and

$$a = (1-\rho_1)\rho_1^{S_1}, \qquad b = \frac{q'(1-\rho_2)}{1-\overline{q}'\rho_2}, \qquad \rho = \frac{\lambda}{\mu}, \; \rho_1 = \frac{\lambda}{\mu_1}, \; \rho_2 = \frac{\lambda}{\mu_2}.$$

Proof: It is shown that the near-product-form distribution satisfies balance equations of the approximate model, which are as in the proof of Lemma 3.3 except that q, q' and q'' are assumed constant. The balance equations are given below for each case plugging in the near-product-form distribution, then the cancellations are immediate to show that these equations hold true.

$k_1 = 0, \; k_2 = 0$

$$\left(\lambda + \mu I_{\{m>0\}}\right)\frac{a}{q}\frac{b}{q'}(1-\rho)\rho^m = \lambda I_{\{m>0\}}(1\text{-}q\text{-}\overline{q}q')\frac{a}{q}\frac{b}{q'}(1-\rho)\rho^{m-1}$$
$$+ \mu_1 I_{\{m>0\}}(1-q')a\rho_1 \frac{b}{q'}(1-\rho)\rho^m + \mu_2 I_{\{m>0\}}\frac{a}{q}b\rho_2(1-\rho)\rho^{m-1} + \mu\frac{a}{q}\frac{b}{q'}(1-\rho)\rho^{m+1}$$

$k_1 = 0,\ \ k_2 = 1$

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\frac{a}{q}b\rho_2(1-\rho)\rho^m = \lambda\overline{q}q'\frac{a}{q}\frac{b}{q'}(1-\rho)\rho^m + \mu_1 q' a\rho_1 \frac{b}{q'}(1-\rho)\rho^m$$
$$+\mu_2 I_{\{m>0\}}\frac{a}{q}b\rho_2^2(1-\rho)\rho^{m-1} + \mu\frac{a}{q}b\rho_2(1-\rho)\rho^{m+1}$$

$k_1 = 0,\ \ k_2 > 1$

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\frac{a}{q}b\rho_2^{k_2}(1-\rho)\rho^m = \lambda(1-q)\frac{a}{q}b\rho_2^{k_2-1}(1-\rho)\rho^m$$
$$+\mu_1 a\rho_1 b\rho_2^{k_2-1}(1-\rho)\rho^m + \mu_2 I_{\{m>0\}}\frac{a}{q}b\rho_2^{k_2+1}(1-\rho)\rho^{m-1} + \mu\frac{a}{q}b\rho_2^{k_2}(1-\rho)\rho^{m+1}$$

$k_1 = 1,\ \ k_2 = 0$

$$\left(\lambda + \mu_1 + \mu I_{\{m>0\}}\right)a\rho_1 \frac{b}{q'}(1-\rho)\rho^m = \lambda q\frac{a}{q}\frac{b}{q'}(1-\rho)\rho^m$$
$$+\mu_1 I_{m>0}(1-q')a\rho_1 \frac{b}{q'}(1-\rho)\rho^{m-1} + \mu_2 I_{\{m>0\}}a\rho_1 b\rho_2(1-\rho)\rho^{m-1} + \mu a\rho_1 \frac{b}{q'}(1-\rho)\rho^{m+1}$$

$k_1 = 1,\ \ k_2 = 1$

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)a\rho_1 b\rho_2(1-\rho)\rho^m = \lambda q\frac{a}{q}b\rho_2(1-\rho)\rho^m + \mu_1 q' a\rho_1^2 \frac{b}{q'}(1-\rho)\rho^m$$
$$+\mu_2 I_{\{m>0\}}a\rho_1 b\rho_2^2(1-\rho)\rho^{m-1} + \mu a\rho_1 b\rho_2(1-\rho)\rho^{m+1}$$

$k_1 = 1,\ \ k_2 > 1$

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)a\rho_1 b\rho_2^{k_2}(1-\rho)\rho^m = \lambda q\frac{a}{q}b\rho_2^{k_2}(1-\rho)\rho^m$$
$$+\mu_1 a\rho_1^2 b\rho_2^{k_2-1}(1-\rho)\rho^m + \mu_2 I_{\{m>0\}}a\rho_1 b\rho_2^{k_2+1}(1-\rho)\rho^{m-1} + \mu a\rho_1 b\rho_2^{k_2}(1-\rho)\rho^{m+1}$$

$k_1 > 1,\ k_2 = 0$

$$\left(\lambda + \mu_1 + \mu I_{\{m>0\}}\right) a\rho_1^{k_1} \frac{b}{q'}(1-\rho)\rho^m = \lambda a\rho_1^{k_1-1} \frac{b}{q'}(1-\rho)\rho^m + \mu_1 I_{\{m>0\}}(1-q')$$
$$a\rho_1^{k_1+1} \frac{b}{q'}(1-\rho)\rho^{m-1} + \mu_2 I_{\{m>0\}} a\rho_1^{k_1} b\rho_2 (1-\rho)\rho^{m-1} + \mu a\rho_1^{k_1} \frac{b}{q'}(1-\rho)\rho^{m+1}$$

$k_1 > 1,\ k_2 = 1$

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right) a\rho_1^{k_1} b\rho_2 (1-\rho)\rho^m = \lambda a\rho_1^{k_1-1} b\rho_2 (1-\rho)\rho^m +$$
$$\mu_1 q' a\rho_1^{k_1+1} \frac{b}{q'}(1-\rho)\rho^m + \mu_2 I_{\{m>0\}} a\rho_1^{k_1} b\rho_2^2 (1-\rho)\rho^{m-1} + \mu a\rho_1^{k_1} b\rho_2 (1-\rho)\rho^{m+1}$$

$k_1 > 1,\ k_2 > 1$

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right) a\rho_1^{k_1} b\rho_2^{k_2} (1-\rho)\rho^m = \lambda a\rho_1^{k_1-1} b\rho_2^{k_2} (1-\rho)\rho^m +$$
$$\mu_1 a\rho_1^{k_1+1} b\rho_2^{k_2-1} (1-\rho)\rho^m + \mu_2 I_{\{m>0\}} a\rho_1^{k_1} b\rho_2^{k_2+1} (1-\rho)\rho^{m-1} + \mu a\rho_1^{k_1} b\rho_2^{k_2} (1-\rho)\rho^{m+1}$$

□

Remark 3.2: *In this study, equivalence or difference of the approximate models with respect to the sequence the components are picked up is not questioned analytically. For the constant q, q', q'' values giving the best numerical results (presented in subsection 3.4), it is numerically observed that different sequences to pick up the components lead to different approximations. This issue is discussed in detail in Chapter 4.* □

3.4 Performance of the approximation

This section is devoted to the assessment of the proposed approximation. Performance measures that are considered for the assembly system are the stockout probability $Pr(K_0 > 0)$, the fill rate $Pr(\overline{M} > 0)$ and the expected backordered quantity for the assembly $E(K_0)$, to be denoted by SP, FR and EB respectively throughout the thesis. These performance measures are computed by simulation and also by using the analytical near-product-form solution of the approximate model, to be compared to investigate performance of the approximation. Approximation errors calculated are absolute percentage errors for stockout probability and fill rate, and relative errors in percent for expected backorders.

The constant q value given in Lemma 3.4 is $(1-\rho_1)\rho_1^{S_1}/(1-\rho_1^{S_1+1})$ due to the $M/M/1$ nature of the marginal behavior of the manufacturing server 1. As for the constant q', starting point being (3.18), six alternative values given below are tried. The corresponding formulas are derived in Appendix C.

A) $q'_A = \dfrac{\sum_{k_1=0}^{\infty} Pr(N_2 = S_2 + K_1 | K_1 = k_1) Pr(K_1 = k_1)}{\sum_{k_1=0}^{\infty} Pr(N_2 \le S_2 + K_1 | K_1 = k_1) Pr(K_1 = k_1)}$.

B) $q'_B = \sum_{m,k_1} q'([k_1 - 1]^+, m) Pr(K_1 = k_1, M = m | N_2 \le S_2 + K_1)$.

where $[k_1 - 1]^+ = \begin{cases} 0 & for \quad k_1 = 0, \\ k_1 - 1 & for \quad k_1 > 0. \end{cases}$

C) $q'_C = \dfrac{Pr(N_2 = S_2 + E(K_1))}{Pr(N_2 \le S_2 + E(K_1))}$.

D) Ignoring dependence on M, $q'_D = \sum_{k_1=0}^{\infty} q'(k_1) Pr(K_1 = k_1)$.

E) Ignoring dependence on M, $q'_E = \sum_{k_1=0}^{\infty} q'([k_1 - 1]^+) Pr(K_1 = k_1)$.

F) $q_F^{'} = \dfrac{Pr(N_2 = S_2)}{Pr(N_2 \le S_2)} = \dfrac{(1-\rho_2)\rho_2^{S_2}}{(1-\rho_2^{S_2+1})}$.

The use of $[k_1 - 1]^+$ that appears in $q_B^{'}$ and $q_E^{'}$ could be explained by the departure of an entity merged with an available component 1 leaving $[k_1 - 1]^+$ backorders behind to join the queue the size of which is represented by k_2. Since $q^{'}$ is defined for the arrivals a the second request queue to increase k_2 by one while the number of requests in the first request queue is $[k_1 - 1]^+$ at this point in time.

Remark 3.3: *Using $q_F^{'}$ in the approximate model, the near-product-form steady-state distribution presented by Theorem 3.1 becomes*

$$\breve{P}_{k_1 k_2 m} = \breve{P}_1(K_1 = k_1)\breve{P}_2(K_2 = k_2)\breve{P}_0(M = m)$$

where

$$\breve{P}_i(K_i = k_i) = \begin{cases} 1-\rho_i^{S_i+1} & for \quad k_i = 0, \\ (1-\rho_i)\rho_i^{S_i+k_i} & for \quad k_i \ge 1, \end{cases} \qquad \text{for } i = 1,\ 2,$$

$$P_m = (1-\rho)\rho^m \qquad \text{for } m \ge 0.$$

This is the immediate result of complete independence assumption leading to independent marginal M/M/1 behavior of the n_1 and n_2 queues which are, in fact, correlated. Thus, the approximation with $q_F^{'}$ can be thought of the worst one can do.

□

In order to test performance of the approximation as compared to the simulation results, a wide range of parameters μ_0, μ_1, μ_2, λ, S_0, S_1, and S_2 is considered. Arrival rate λ is fixed at 9 entities per time unit and μ_0, μ_1 and μ_2

take the values of 10, 15 and 20 entities per time unit, respectively, to serve the purpose of scanning cases with various traffic intensities. Parameters S_1 and S_2 vary between 0 and 20, and S_0 takes the values of 5, 10 and 15. Different combinations of these parameters give 27 different parameter sets of μ_0, μ_1, μ_2, λ and 1200 different values of S_0, S_1, and S_2 in each set for testing the proposed approximate model.

The approximate performance measures are calculated with the use of a Pascal code. The run time of the code at a Pentium 4 2.0 processor is 57 minutes for a parameter set with 1200 different S_0, S_1 and S_2 combinations and six alternative q' values. For the calculations, the state space is truncated ensuring that 99.9999% of the cases is covered, which seems adequate to justify the truncation levels. The Pascal code is given in Appendix D.

For simulation, Rockwell Arena 5.0 software is used. Since it has an object oriented visual interface, it is easier to model the system and trace the entities to see whether the model works in the right way or not. 15 replications are generated with simulation 15000 time units (meaning 135000 entities on the average for $\lambda = 9$) and warm-up period of 3000 time units (meaning 27000 entities on the average for $\lambda = 9$) for every parameter set. The simulation parameters and the system are initialized at every replication to get independent results. The run time of the code at a Pentium 4 2.0 processor is 19 hours and 20 minutes for calculating values of each parameter set. The object-oriented visualization of the code and the plots used for determining the simulation time can be seen in Appendix D.

Comparison of the approximate performance measures with simulation results in the following observations:

- Models with q'_A, q'_B and q'_C have superior results compared to q'_D, q'_E and q'_F. There are slight differences between the errors for q'_A, q'_B and q'_C, but in almost every case errors for q'_C are smaller than those of others. The approximation errors of the 6 alternative q' values with system parameters

$\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9$, $S_0 = 5$ and $S_1 = 5$ are given in Figure 3.11. For the cases with low traffic intensity, and S_1 and S_2 approaching 20, the difference between the approximate results found by the alternative q' values gets smaller. According to the observations for 27 different parameter sets of μ_0, μ_1, μ_2, λ, we select $q' = q_C^{'}$ and continue our further inquiries with it.

- Two-dimensional graphs like the ones in Figure 3.12, 3.13 and 3.14 and in Appendix G and the summary Table 3.1 show that the approximation and the simulation results are very close. At first glance, we can say that the approximation works better in cases S_1 and S_2 are higher meaning that the system has lower traffic intensity.

Table 3.1 Average and maximum errors for several parameter sets ($\lambda = 9$)

					Average Errors				Maximum Errors			
μ_0	μ_1	μ_2	S_0	S_1-S_2	SP	FR	EB (Rel%)	EB (Abs)	SP	FR	EB (Rel%)	EB (Abs)
10	10	10	5	0-20	2,19	2,12	5,24	0,51	4,92	5,25	16,34	1,27
10	10	10	10	0-20	2,05	2,12	6,76	0,45	5,12	5,16	23,15	1,22
10	10	10	15	0-20	1,6	1,67	8,21	0,38	4,23	4,25	28,25	1,37
20	10	10	5	0-20	1,55	1,75	6,39	0,23	7,83	9,27	28,7	0,69
20	10	10	10	0-20	0,76	0,85	9,26	0,22	3,64	4,42	36,22	0,74
20	10	10	15	0-20	0,72	0,69	13,85	0,21	2,1	2,15	48,08	0,81
10	10	20	5	0-20	1,74	1,71	3,81	0,30	4,19	4,1	12,96	1,07
10	10	20	10	0-20	1,48	1,56	4,81	0,24	5,96	4,55	19,34	0,75
10	10	20	15	0-20	1,08	1,16	5,36	0,18	3,15	3,28	22,99	0,69
10	20	10	5	0-20	1,59	1,57	3,89	0,30	3,68	3,44	14,54	1,11
10	20	10	10	0-20	1,36	1,43	4,45	0,23	3,89	3,96	17,85	0,70
10	20	10	15	0-20	1,06	1,12	5,35	0,18	3,23	3,32	25,03	0,78
10	20	20	5	0-20	0,47	0,45	2,51	0,13	2,43	2,51	9,32	0,46
10	20	20	10	0-20	0,51	0,5	3,78	0,13	1,97	1,93	17,09	0,53
10	20	20	15	0-20	0,53	0,54	5,41	0,10	2,89	3,14	21,49	0,42

- As in Figures 3.12, 3.13 and 3.14 and Appendix G, the errors get smaller at the extreme points $(S_1 = 0,\ S_2 = 20)$, $(S_1 = 20,\ S_2 = 0)$ and

$(S_1 = 20,\ S_2 = 20)$ as expected from the theoretical judgments that the model is exact for cases $(S_1 = 0,\ S_2 = \infty)$, $(S_1 = \infty,\ S_2 = 0)$ and $(S_1 = \infty,\ S_2 = \infty)$ (refer to Lemma 3.5). This results in a conical shape of the three-dimensional drawings of errors as seen in Appendix F.

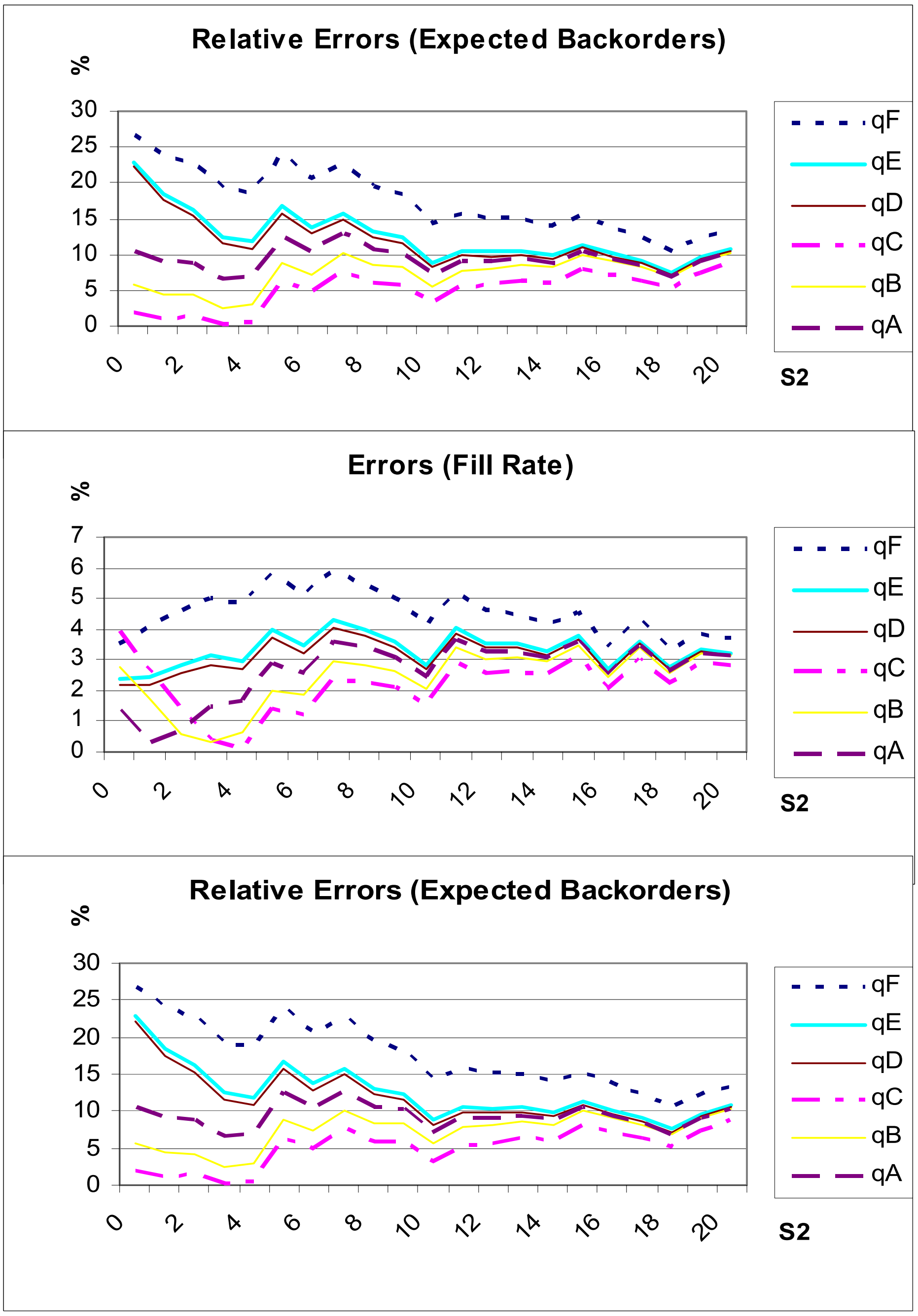


Figure 3.11 Approximation errors for 6 different q' values,

$\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9$, $S_0 = 5$ and $S_1 = 5$

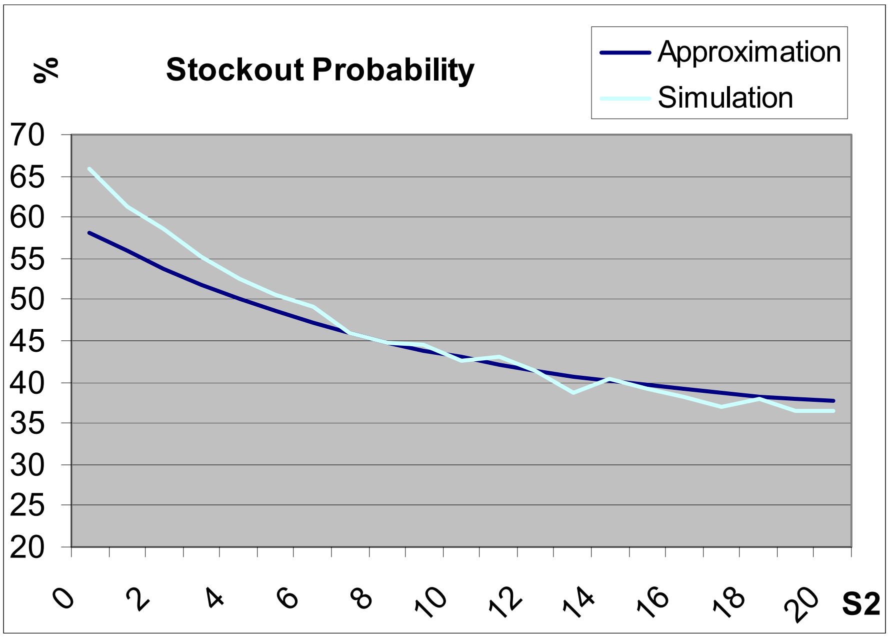


Figure 3.12 Stockout Probability $\mu_0 = \mu_1 = 10$, $\mu_2 = 20$, $\lambda = 9$, $S_0 = 5$ and $S_1 = 5$

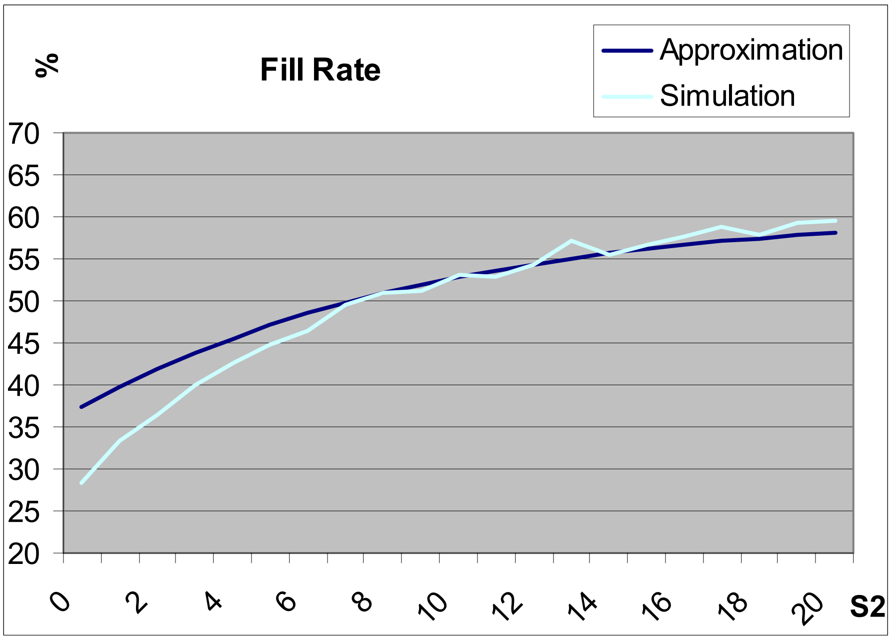


Figure 3.13 Fill Rate $\mu_0 = \mu_1 = 10$, $\mu_2 = 20$, $\lambda = 9$, $S_0 = 5$ and $S_1 = 5$

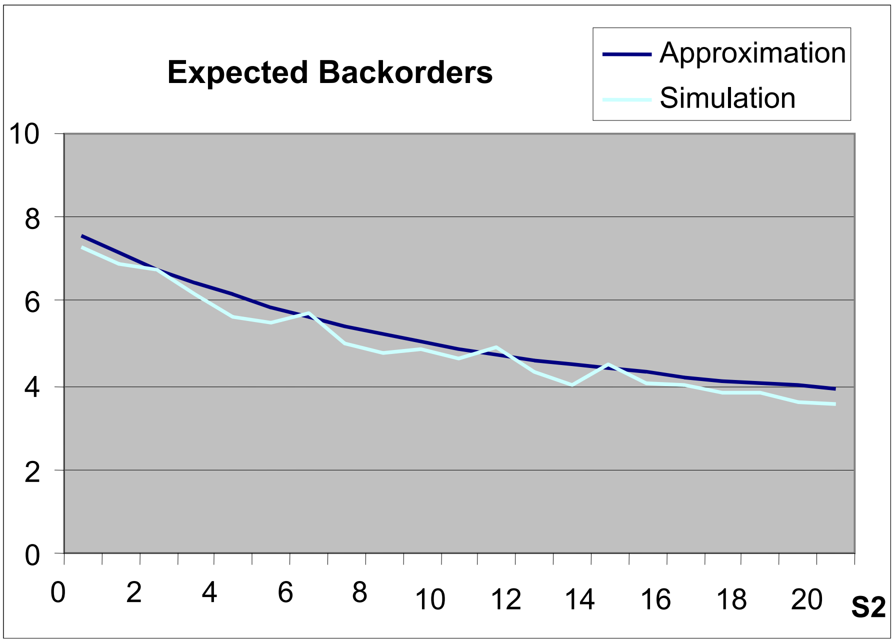


Figure 3.14 Expected Backorders $\mu_0 = \mu_1 = 10,\ \mu_2 = 20$, $\lambda = 9$, $S_0 = 5$ and $S_1 = 5$

- If we take the real errors into account instead of absolute errors, it is observed that when the traffic intensity is high, the expected backorder and stockout probabilities are underestimated and fill rate is overestimated. When the traffic intensity is lower, the errors spread almost equally at both sides of the zero level but on the average there is still the tendency of underestimating and overestimating for SP, EB and FR, respectively, as seen in Table 3.2 and Appendix H.

Table 3.2 Errors for high and low traffic intensities

	$\mu_0 = \mu_1 = \mu_2 = 10$, $\lambda = 9$, $S_0 = 5$			$\mu_0 = 10,\ \mu_1 = \mu_2 = 20$, $\lambda = 9$, $S_0 = 5$		
	FR	SP	EB	FR	SP	EB
Min error	-4.4310	-4.9207	-16.3446	-3.2933	-2.4333	-9.3229
Max error	5.2460	4.5867	5.2260	2.5077	3.8017	8.0124
Avg. error	1.4816	-1.5281	-5.0027	0.1748	-0.1589	-0.5297

The 3-dimensional drawings for the high and low traffic intensity cases in Table 3.2 are given in Appendix H, where the observation in this item can be seen clearly over many combinations of S_1 and S_2 values.

- Examining 27 different parameter sets of μ_0, μ_1, μ_2, λ, it can be inferred that as the traffic intensity of the system decreases, the errors also decrease for stockout probability and fill rate. But an interesting observation is that for lower traffic intensities that result in lower expected backorders, decrease in the traffic intensity causes an increase in error of expected backorders. In fact, in such cases, absolute error for expected backorders still decreases but since this decrease is less than the decrease in the value of the expected backorder, the relative error increases. This leads to a misinterpretation that the approximation does not work well for estimating expected backorders in case of lower traffic intensities. Maybe, for some small values of the expected backorder, absolute errors should be taken into consideration for interpreting the accuracy of the approximation. For example, at $\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9$ and $S_0 = 5$ the errors for stockout probability, fill rate and expected backorder are 2.19, 2.12 and 5.24 on the average, respectively, and at $\mu_0 = 20,\ \mu_1 = \mu_2 = 10,\ \lambda = 9$ they are 1.55, 1.75, 6.39 on the average. It is seen that the relative error of the expected backorders increases in the latter case as composed to the former one but the absolute error of the expected backorder falls from 0.51 to 0.23 on the average. Since this decrease (from 0.51 to 0.23) is less than the decrease in the expected backorders (from 10.60 to 4.23 on the average), the relative increases. A table for the errors of expected backorders for relatively lower traffic intensities is given in Appendix E. For cases that the expected backorder value is small, the relative measure of the error is higher even when errors are pretty small in absolute measurement such that, do not exceed even one backordered request.

- Constructing 95% confidence intervals for the performance measures using simulation results and student's t distribution with $\overline{X} = \sum_{i=1}^{15} X_i / 15$ and $S = (\sum_{i=1}^{15} (X_i - \overline{X})^2 / 14$ in $\overline{X} \pm t_{0.025,14} \dfrac{S}{\sqrt{15}}$ it is seen for case $\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9$ in Appendix I that not all of the approximate measures fall into the confidence intervals but number of measures that fall into the confidence intervals is higher for low traffic intensity cases.

Analyzing all the results gained from 32400 (27x1200) different parameter sets, the approximation turns out to be quite accurate with absolute errors of 10% at most for fill rate, stockout probability and of less than 1.37 (≈2) requests for expected backorder. So, we can conclude that the approximation works well and gives satisfying results for the two-component assembly model.

3.5 Optimizing base-stock levels

We have observed that the performance measures calculated for the approximate model serve as good approximations for those of the original model. This suggests the use of the approximate model for optimizing the system parameters like μ_0, μ_1, μ_2 and/or S_0, S_1, S_2. Noting that changing the server capacities (μ_1, μ_2, μ_0) would be an involved task in practice, though not in setting up the optimization model theoretically, compared to changing the stock allocation, in this study we proceed with the latter to numerically justify the use of approximate performance measures for optimization purposes. The stock allocation problem could be posed as determining optimal investment alternative given some target service level like fill rate in the case of the numerical examples presented in this section or optimal service level given the budget restriction for stock investment. Corresponding formulations are

Min $S_0 \cdot c_0 + S_1 \cdot c_1 + S_2 \cdot c_2$
subject to $FR(S_1, S_2, S_0) \geq \alpha$

Max $FR(S_1, S_2, S_0)$
subject to $S_0 \cdot c_0 + S_1 \cdot c_1 + S_2 \cdot c_2 \leq B$

where c_i in the model represents the cost of allocating one stock keeping unit (SKU) for stock point i, $FR(S_1, S_2, S_0)$ is the fill rate for stock allocation (S_1, S_2, S_0), α is a given target fill rate and B is available budget. The optimization of μ_1, μ_2, μ_0 and even overall design of the system as the joint optimization of both server capacities and stock allocation could also be considered in terms of the corresponding investment functions of these design parameters given some service level constraints, even these constraints need not be restricted to just fill rate.

Referring to the smooth almost concave behavior of the approximate fill rate as a function of (S_1, S_2) given some S_0, the following greedy heuristic is proposed to solve the minimization formulation above.

Greedy Heuristic:

Step 0: Assign S_0 to the value that guarantees $FR(S_0, \infty, \infty) \geq \alpha$.

Set $S_1 = S_2 = 0$.

Step 1: Let $FR = FR(S_0, S_1, S_2)$ and $l = \arg\min_{j=0,1,2} \left\{ \frac{c_j}{FR(..., S_j + 1, ...) - FR} \right\}$.

Set $S_l^{'} = S_l + 1$ and $S_j^{'} = S_j$ for $j \neq l$.

Step 2: If $FR(S_0^{'}, S_1^{'}, S_2^{'}) \geq \alpha$ go to step 3,

else set $(S_0, S_1, S_2) = (S_0^{'}, S_1^{'}, S_1^{'}{}')$ and go to step 1

Step 3: Let $l = \arg\min_{j=0,1,2} \left\{ c_j \middle| FR(..., S_j + 1, ...) \geq \alpha \right\}$. Assign $S_l = S_l + 1$ and stop.

We have performed a number of numerical experiments for the two-component assembly system choosing α =0.95 and $\frac{c_0}{2} = c_1 = c_2$. In Table 3.3, iterations of the greedy heuristic can be seen for the case $\mu_0 = 20,\ \mu_1 = \mu_2 = 15,$ $\lambda = 9$. Minimum investment turns out to be 18 ($S_0 = 7,\ S_1 = 2,\ S_2 = 2$). In order to see if the heuristic works well, all possible allocations having total investment less than or equal to the one found by the greedy heuristic are enumerated again using the approximate fill rates. Table 3.4 is to display the enumeration for all possible allocations with FR≥0.95 for the case $\mu_0 = 20,\ \mu_1 = \mu_2 = 15,\ \lambda = 9$. Also, details of the employment of the greedy heuristic for the cases $\mu_0 = 20,\ \mu_1 = 10,\ \mu_2 = 20,$ $\lambda = 9$ and $\mu_0 = 20,\ \mu_1 = \mu_2 = 10,\ \lambda = 9$ can be found in Appendix J.

We should point out that none of the allocations with investment less than the solution found by the greedy heuristic can satisfy the fill rate constraint. The highest fill rates in all cases correspond to the allocations found by the greedy heuristic, showing the power of the heuristic at least for the three cases considered. If the greedy approach is not taken to determine allocations having the minimum investment level, extensive mathematical modeling (nonlinear programming) or enumeration would be required, which seems rather impractical for complex realistic systems.

Table 3.3 Iterations for $\mu_0 = 20,\ \mu_1 = \mu_2 = 15,\ \lambda = 9$, α =0.95, $\frac{c_0}{2} = c_1 = c_2$

S_0	S_1	S_2	FR(S_0, S_1, S_2)	FR(S_0+1, S_1, S_2)	FR(S_0, S_1+1, S_2)	FR(S_0, S_1, S_2+1)
4	999	999	0,95899			
4	0	0	0,66077	**0,76482**	0,7086	0,7021
5	0	0	0,76482	**0,83989**	0,8	0,79932
6	0	0	0,83989	0,89252	0,86479	**0,86686**
6	0	1	0,86686	0,91267	**0,8953**	0,88305
6	1	1	0,8953	0,93151	0,91295	**0,91361**
6	1	2	0,91361	0,94461	**0,9323**	0,92459
6	2	2	0,9323	**0,95706**	0,94373	0,94391
7	2	2	0,95706			

Table 3.4 Enumeration for $\mu_0 = 20,\ \mu_1 = \mu_2 = 15,\ \lambda = 9$ with investment ≤ 18

S_0	S_1	S_2	FR
7	2	2	0,95706
6	3	3	0,95563
8	1	1	0,95552
9	0	0	0,95308
7	1	3	0,95246
8	0	2	0,95196
7	3	1	0,95156
6	2	4	0,95087
6	4	2	0,95069

CHAPTER 4

4. EXTENSIONS

In Chapter 3, we have worked on a two-component assembly system and have shown that the approximate near-product-form steady-state distribution we have proposed performs well. Two extensions of the two-component assembly model are taken into consideration in this chapter without a complete analytical development but with the inspiration from the approximation of the two-component assembly model.

4.1 Generalization for more than two components

This section is to generalize the approximation for assembly systems with more than two components and show numerically how it performs then. The approximate solution for the n-component case is obtained in a recursive manner using the approximate solution for the (n-1)-component case. To present this recursive development, we could first consider the three-component system and show that it is resolved given the approximate near-product-form solution of a part of this system with just two components of the three.

As in the two-component case, an alternative model to pick the components sequentially as in Figure 4.1 could be considered, instead of simultaneously picking them up, in order to handle the difficulties of simultaneously merging the components by approximating some conditional probabilities that appear as a result of (sequential) partial aggregations.

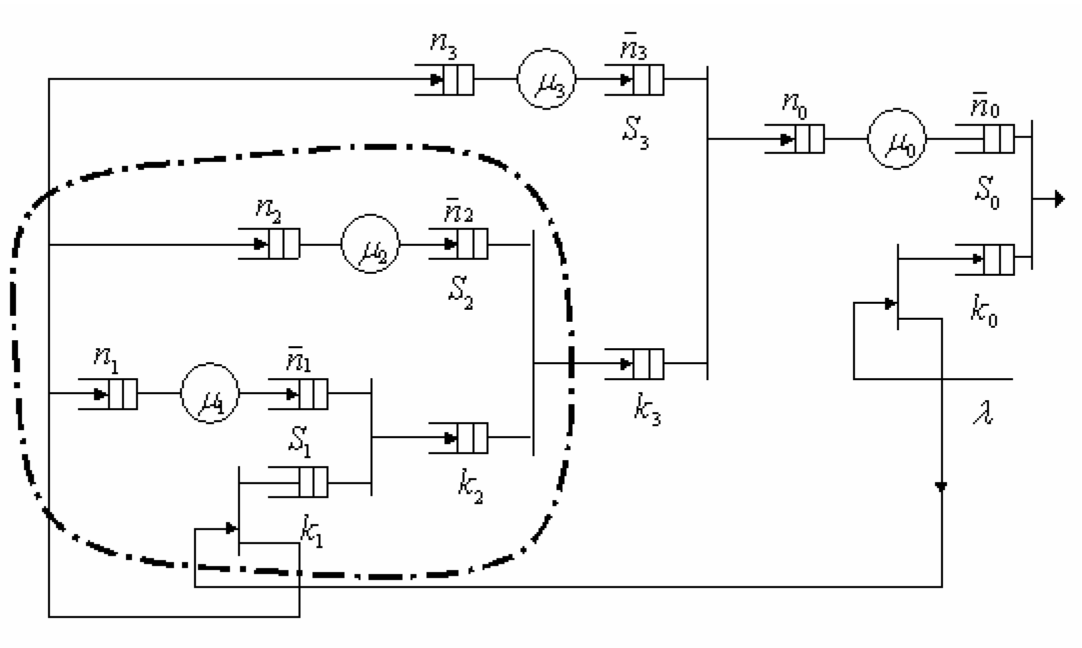


Figure 4.1 Alternative Model for the Assembly System with Three-Components

The explanation on equivalence of the original and the alternative models is in section 3.1. Employment of the base-stock policies leads to the following equations in the alternative model:

$$n_1 + \bar{n}_1 = S_1 + k_1, \tag{4.1}$$

$$n_2 + \bar{n}_2 = S_2 + k_1 + k_2, \tag{4.2}$$

$$n_3 + \bar{n}_3 = S_3 + k_1 + k_2 + k_3, \tag{4.3}$$

$$m + \bar{m} + k_1 + k_2 + k_3 = S_0 + k_0. \tag{4.4}$$

Now, treat the part of the system circled with dashed line in Figure 4.1 as a whole with its steady-state distribution given by the analysis of the two-component system in the previous chapter. Then, view the backorders in this part in total as $k_1 + k_2$ to be denoted by k_{12}. This is to reduce the system to two-component system with backorders k_{12} to be satisfied first and k_3 to be satisfied next with the third component. That is, the relations in (4.1) and (4.2) are reflected by the near-product-form solution of the two-component case, and then (4.3) and (4.4) take the following form

$n_3 + \overline{n}_3 = S_3 + k_{12} + k_3$,

$m + \overline{m} + k_{12} + k_3 = S_0 + k_0$,

to be compared with (3.5) and (3.6), respectively. Representing the overall departure rate from the part within dashed line by μ_{12} as if it is exponential without questioning validity of this, one could draw the transition diagram in Figure 4.2 to be compared with Figure 3.6 of two-component system to understand the assumptions, basically the reduction to two-component case, and the system mechanics under these assumptions. Here, when we compare transition diagrams of the two-component and the three-component diagrams in Figure 3.6 and Figure 4.2, respectively, the definition $Q = q + q''$ is adequate to peer the two models. Also, μ_{12} in the model is defined as the processing rate of the imaginary exponential single-server facility representing the part of the system within the dashed line. In fact, there is no need to know or approximate the value of μ_{12} in order to propose a near-product-form steady-state distribution for the three-component model. Questions about μ_{12} are bypassed by the correspondence between Figure 4.2 and 3.6 (and between Figure 4.3 and 3.7 with further aggregation discussed after Remark 4.1) and the use of near-product-form distribution of two-component case for the part within dashed line.

Remark 4.1: *Overall expected output rates of the system handling components 1 and 2 can be put as*

$$\mu_1 Pr(K_1 > 0) + \lambda(1-q)Pr(K_2 = 0) \qquad (4.5)$$

and

$$\mu_2 Pr(K_2 > 0) + \mu_1(1-q')Pr(K_1 > 0, K_2 = 0) + \lambda(1-q-q'')Pr(K_1 = 0, K_2 = 0), \qquad (4.6)$$

respectively. The last terms with λ and the other terms of (4.5) and (4.6) are comparable. Thinking the system within the dashed line as a single-server facility with processing rate μ_{12}, the first two terms of (4.6) can be equated to something similar to the first term of (4.5) as follows:

$$\mu_{12} Pr(K_{12} > 0) = \mu_2 Pr(K_2 > 0) + \mu_1(1 - q')Pr(K_1 > 0, K_2 = 0).$$

Then, the approximate μ_{12} *turns out to be*

$$\mu_{12} = \frac{\mu_2 \cdot Pr(K_2 = 0) + \mu_1 \cdot (1-q) \cdot Pr(K_1 > 0, K_2 = 0)}{Pr(K_{12} = 0)}$$

$$= \mu_2 \frac{q}{\alpha} + \mu_1 (1-q)(\frac{q}{\alpha} - 1)$$

where the second equality follows using the joint distribution in Theorem 3.1 to compute probabilities $\Pr(K_2 = 0)$, $\Pr(K_1 > 0, K_2 = 0)$ *and* $\Pr(K_{12} = 0)$. *Note that* $\mu_{12} > \lambda$ *since* $\mu_2 \frac{q}{\alpha} > \mu_2$ *and* $\mu_1 (1-q)(\frac{q}{\alpha} - 1) > 0$.

To check how good this definition is to represent the imaginary exponential single-server facility within the dashed line, we compare the marginal distribution of K_{12} *from Figure 4.2 using the approximation above, i.e.,*

$$Pr(K_{12} = k_{12}) = \rho_{12}^{k_{12}} \frac{1 - \rho_{12}}{1 - \rho_{12} + \rho_{12} Q} \qquad \textit{for all } k_{12}$$

where $\rho_{12} = \lambda / \mu_{12}$,
with the one obtained from the joint distribution in Theorem 3.1, i.e.,

$$Pr(K_{12} = k_{12}) = \sum_{k_1=0}^{k_{12}} \breve{P}_1(K_1 = k_1) \breve{P}_2(K_2 = k_{12} - k_1).$$

The results show that the maximum absolute difference between the two solutions is 0.12% (see Appendix K for the case $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 10$, $S_0 = S_1 = S_2 = S_3 = 5$, $\lambda = 9$*), which justifies proceeding with Figure 4.2.* □

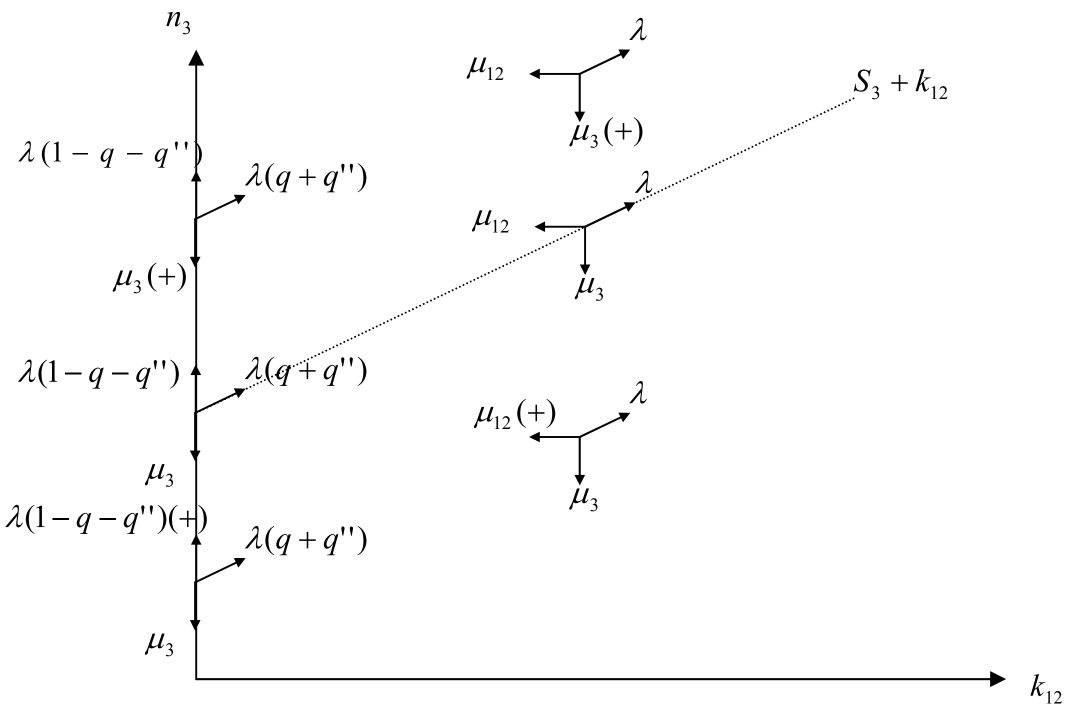


Figure 4.2 Transition Diagram of Three-Component Assembly Model
with State Description (k_{12}, n_3, m)

Then, aggregation (the second aggregation step as explained in section 3.2) of the system in Figure 4.2 with state description (k_{12}, n_3, m) leads to Figure 4.3 with state description (k_{12}, k_3, m) and introduction of

$$Q' = Pr(N_3 = S_3 + E(K_{12}) | N_3 \le S_3 + E(K_{12}))$$

recalling q' from the two-component case. The correspondence between the state-transition diagram in Figure 4.3 for three-component system and the one in Figure 3.7 for two-component system leads to the generalization of the near-product-form solution as in Remark 4.2.

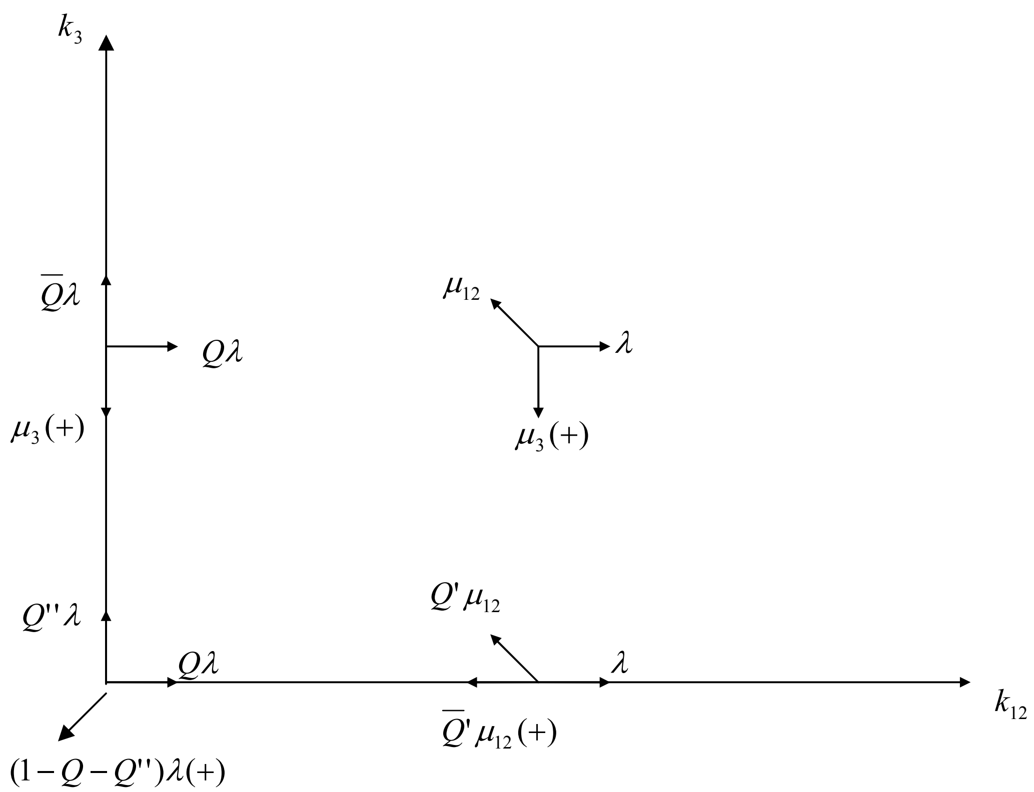


Figure 4.3 Transition Diagram of Three-Component Aggregate Model with State Description (k_{12}, k_3, m)

Remark 4.2: *The near-product-form steady-state distribution proposed for the three-component assembly system is*

$$\breve{P}_{k_{12}k_3m} = \breve{P}_{12}(K_{12} = k_{12})\breve{P}_3(K_3 = k_3)\breve{P}_0(M = m) \qquad (4.7)$$

where

$$\breve{P}_3(K_3 = k_3) = \begin{cases} \dfrac{B}{Q'} & for \quad k_3 = 0, \\ B\rho_3^{k_3} & for \quad k_3 \geq 1, \end{cases}$$

$\breve{P}_{12}(K_{12}=k_{12})=\sum_{k_1=0}^{k_{12}}\breve{P}_1(K_1=k_1)\breve{P}_2(K_2=k_{12}-k_1)$ *referring to Theorem 3.1 for the computation of* $\breve{P}_1$ *and* $\breve{P}_2$,

$\breve{P}_0(M=m)=(1-\rho)\rho^m$ *for* $m \geq 0$,

and

$$B=\frac{Q'(1-\rho_3)}{1-\bar{Q}'\rho_3},\ \rho_3=\frac{\lambda}{\mu_3},\ Q'=\frac{Pr(N_3=S_3+E(K_{12}))}{Pr(N_3\leq S_3+E(K_{12}))}),$$

$Q=q+q''$ *recalling* q *and* q'' *from section 3.1,*
$Q''=(1-Q)Q'$. □

Using the proposed approximate distribution given by (4.7), one can compute the distribution for $K_{12}+K_3$ which would be called as K_{123} for the analysis of a four-component assembly system so that Q and Q'' serve the functions of q and q'', respectively. New Q' is $Pr(N_4=S_4+E(K_{123})|N_4\leq S_4+E(K_{123}))$ defining $\rho_4=\lambda/\mu_4$ and new B is defined in terms of new Q' and ρ_4. Proceeding this way, the recursion would be to obtain the approximate near-product-form distribution of n-component system given that of (n-1)-component system.

In order to test performance of the approximation for systems with more than two components, nine different parameter sets are considered to cover the range from high to low traffic intensities. The parameter sets and the corresponding numerical results of the performance measures are given in Appendix L. As can be seen from the numerical results, the performance of the approximation is still satisfactory. A summary of the average errors of the performance measures for nine different parameter sets is given in Table 4.1.

Table 4.1 Average errors (%) of the nine parameter sets
for systems with more than two components

	Fill Rate	Stockout Probability	Expected Backorder
2 components	0.923	0.906	4.272
3 components	1.108	1.139	6.230
4 components	1.407	1.449	9.169
5 components	1.703	1.598	9.822
6 components	1.803	1.906	9.957
12 components	2.980	3.363	15.443

Regarding the approximation performance, two points to be investigated are raised with the following questions:

- Does the approximation performance deteriorate as the number of components increase? If it does, how much is this deterioration?
- How does the sequence the components are picked up affect the approximation performance?

Graphs of the approximation errors in Appendix M show that performance of the approximation deteriorate with increasing number of components. The deterioration is more apparent for expected backorders such that the maximum increase in the error between two-component model and twelve-component model is 19.54% for the expected backorder while the maximum difference between the errors is only 5.91% and 6.74% for fill rate and stockout probability, respectively, for the two-component and twelve-component models. Also, as seen from the graphs the increase in errors seems to decrease as the number of components increase, but it is not possible to claim convergence of the errors for sufficiently large number of components based on only the representative numerical experiments in this thesis.

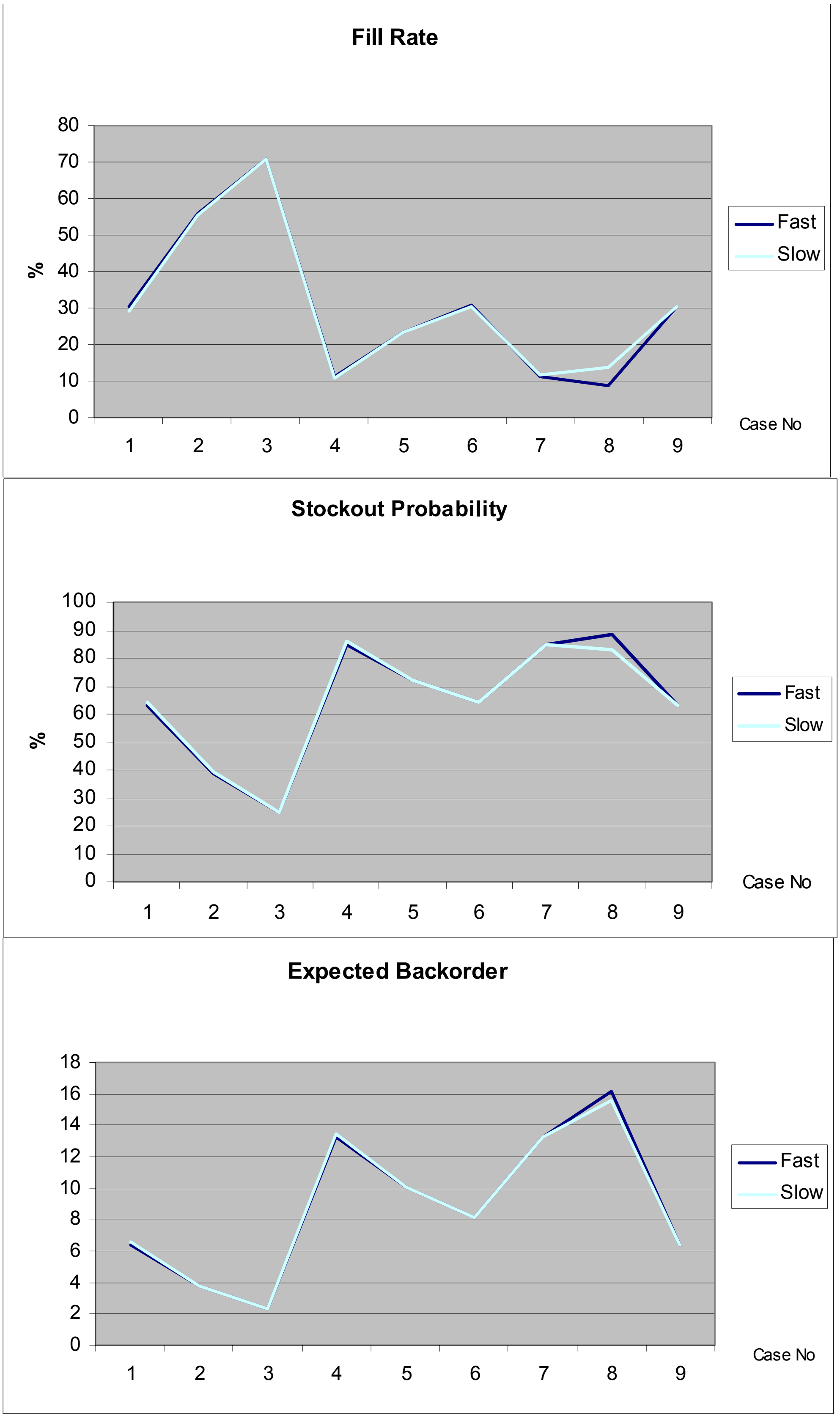


Figure 4.4 Effect of the sequence the components are picked up
(Numerical Results in Appendix N)

The impact of the component sequence (to pick them up in the alternative model) on the approximation performance is questioned by the numerical experiment results in Appendix N. The only and immediate way of revealing such an impact, if there is any, is to consider parameter sets with high manufacturing capacity (high service rate and high base-stock level combination) for some components and low manufacturing capacity for others so that the sequences to pick up components could be differentiated and then compared. But, it should be noted that we could have such service rate and base-stock level combinations that it may not be possible to differentiate the components to identify the sequence to pick them up for having small approximation errors. In fact, even in the case of apparent differentiation, numerical experiments (in Appendix N) do not favor a sequence from faster to slower (referred as fast) or slower to faster (referred as slow) as seen in Figure 4.4.

Next, the greedy heuristic is employed for optimization of three-component systems with parameters $\mu_0 = 20,\ \ \mu_1 = \mu_2 = 15,\ \ \mu_2 = 20,\ \ \lambda = 9$. The heuristic finds the best possible allocation ($S_0 = 6,\ \ S_1 = 3$, $S_2 = 4$, $S_3 = 1$) with minimum investment level 26 satisfying FR$\geq$0.95. Iterations of the greedy heuristic and the enumeration over all possible allocations satisfying FR$\geq$0.95 can be found in Appendix O.

4.2. Component Commonality

In the assembly system considered in this section, two types of finished assemblies are manufactured. Each finished assembly is composed of two components, one of the components being common. That is, there is manufacture of three components, each at its own dedicated facility, feeding the assembly operations of two different types, say 0 and $\overline{0}$, at the single assembly facility. A sketch of such a system with single exponential servers at each facility is in Figure 4.5. Component 3 is the common component and its demand is the sum of the two assemblies' demands, i.e., Poisson with rate $\lambda_0 + \lambda_{\overline{0}}$, while the demands of components 1 and 2 are determined by the Poisson demand arrival process with rates λ_0 and $\lambda_{\bar{0}}$, respectively, of the corresponding assemblies in which they function.

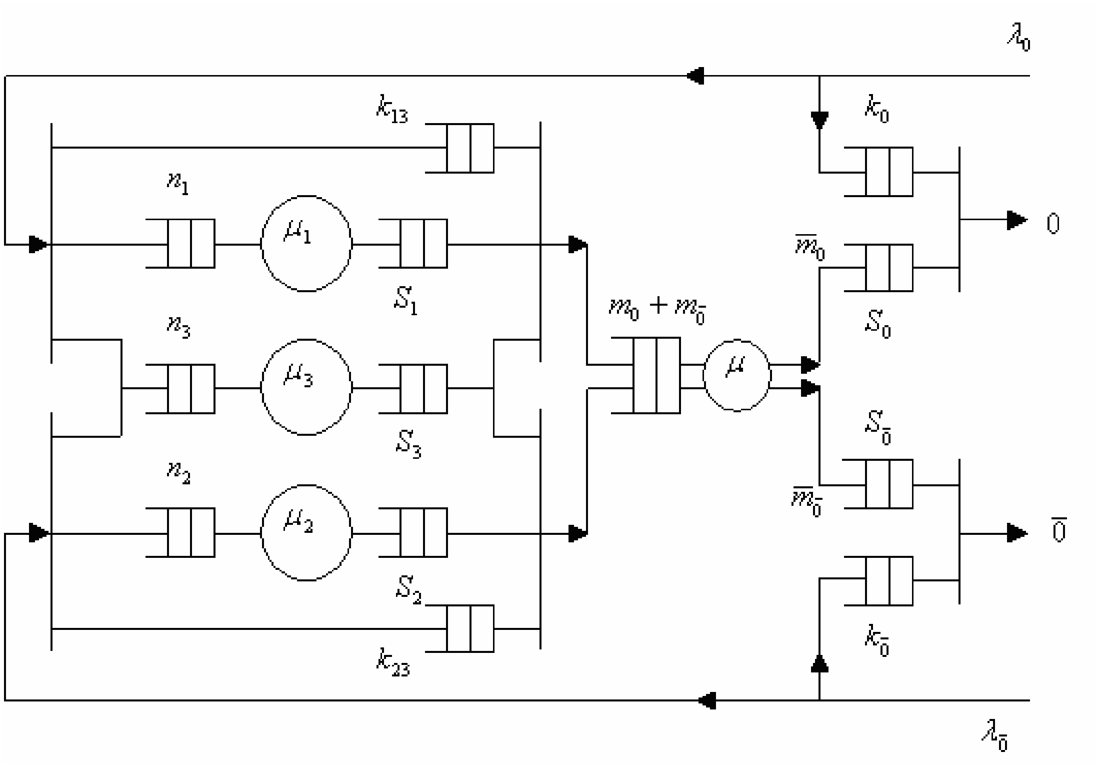


Figure 4.5 Assembly Model with Component Commonality

Employment of the base-stock policies leads to the following equations for the (original) model in Figure 4.5:

$n_i + \overline{n}_i = S_i + k_{i3} \quad for\ i = 1,\ 2,$

$n_3 + \overline{n}_3 = S_3 + k_{13} + k_{23},$

$m_0 + \overline{m}_0 + k_{13} = S_0 + k_0,$

$m_{\bar{0}} + \overline{m}_{\bar{0}} + k_{23} = S_{\bar{0}} + k_{\bar{0}}.$

Random variable K_{13} (K_{23}) represents the number of backordered requests for both component 1 and 3 (2 and 3) to feed the assembly of type 0 ($\bar{0}$). M is the summation of joined entities for assembly of type 0, M_0, and $\bar{0}$, $M_{\bar{0}}$. Service rate of the assembly facility is, type dependent, taking the value of either μ_0 or $\mu_{\bar{0}}$. In this thesis, the analysis is presented for the case $\mu_0 = \mu_{\bar{0}}$ and further extension will be done for $\mu_0 \neq \mu_{\bar{0}}$. The service discipline to match an available component 3 with component 1 or 2 can be thought of as first-come-first-served (FCFS), but there would be a need to further specify this discipline just regarding the coordination of component 3 to resolve cases like the following: k_{13} and k_{23} are both equal to one, suppose request of 1 and 3 has been generated before that of 2 and 3. There are available components of type 2 but there is no component of type 1 in stock. If a component of type 3 becomes available in this state of the system, should we use this component to match it with component 2 available (although its request is new compared to the request of 1 and 3) or should we wait for a component of type 1 to become available? The distinction between these two service disciplines is made by giving two alternative models to pick up components sequentially as for the employment of the approximation approach proposed in this thesis. Alternative model in Figure 4.6 is equivalent to the original one in Figure 4.5 if first-come-first-served discipline is employed strictly without paying attention to the resolution of the cases like the one mentioned above. On the other hand, alternative model in Figure 4.7 allows reasonable resolution (matching component 2 with component 3 as soon as it becomes available in the example case above) of such cases. The former alternative model allocates common components first and the latter allocates them last.

For the alternative model in Figure 4.6, inventory balance equations implied by the base-stock policies are as follows:

$$n_1 + \overline{n}_1 = S_1 + k_{30} + k_1 \,,$$

$$n_2 + \overline{n}_2 = S_2 + k_{3\bar{0}} + k_2 \,,$$

$$n_3 + \overline{n}_3 = S_3 + k_3 \,,$$

$$m_0 + \overline{m}_0 + k_{30} + k_1 = S_0 + k_0 \,, \quad (4.8)$$

$$m_{\bar{0}} + \overline{m}_{\bar{0}} + k_{3\bar{0}} + k_2 = S_{\bar{0}} + k_{\bar{0}} \,, \quad (4.9)$$

where $k_3 = k_{30} + k_{3\bar{0}}$.

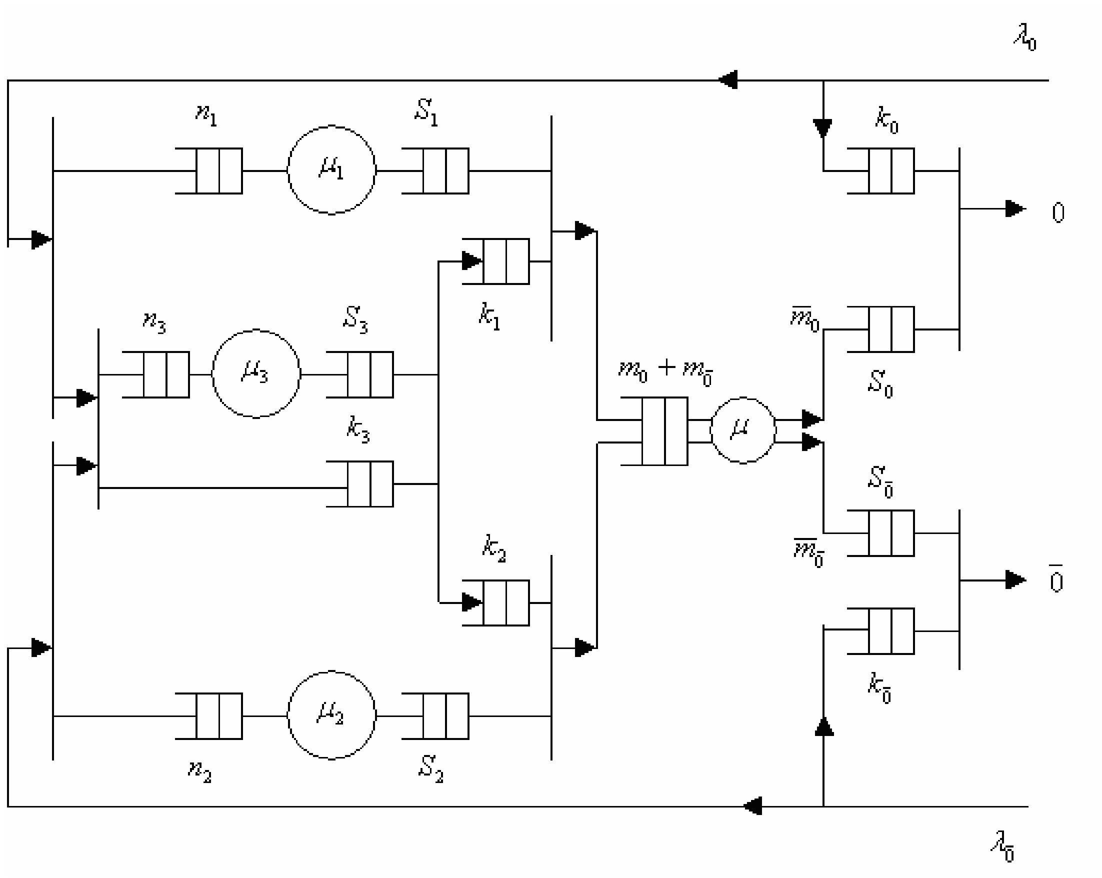


Figure 4.6 Alternative Assembly Model where Common Component is Picked up First

Equivalence of Figure 4.6 and Figure 4.5 comes up with the recognition that $\overline{n}_3$ ($\overline{n}_1$ and $\overline{n}_2$) and k_{13}, k_{23} in the latter model correspond to $\overline{n}_3 + k_1 + k_2$ ($\overline{n}_1 + k_{30}$ and $\overline{n}_2 + k_{3\bar{0}}$) and $k_{30} + k_1$, $k_{3\bar{0}} + k_2$ in the former model, respectively.

Recalling the type of aggregation in section 3.2, the state description of this alternative model can be transformed from (n_1, n_2, n_3, m) to (k_1, k_2, k_3, m). Then, analogous to Theorem 3.1, the approximate near-product-form distribution in Remark 4.3 can be proposed without any formal proof but by just an intuitive analogy.

Remark 4.3: *For the case* $\mu_0 = \mu_{\bar{0}} = \mu$ *of Figure 4.6, the near-product-form steady-state distribution proposed for the three-component assembly system with two finished products and one common component is*

$$\breve{P}_{k_1k_2k_3m} = \breve{P}_1(K_1 = k_1)\breve{P}_2(K_2 = k_2)\breve{P}_3(K_3 = k_3)\breve{P}_{0\bar{0}}(M = m) \qquad (4.10)$$

where

$$\breve{P}_i(K_i = k_i) = \begin{cases} \dfrac{G_i}{q_i^{'}} & for \quad k_i = 0, \\ G_i \rho_i^{k_i} & for \quad k_i \geq 1, \end{cases} \qquad \textit{for } i = 1, 2,$$

$$\breve{P}_3(K_3 = k_3) = \begin{cases} \dfrac{H}{q_3} & for \quad k_3 = 0, \\ H\rho_3^{k_3} & for \quad k_3 \geq 1, \end{cases}$$

$$\breve{P}_{0\bar{0}}(M = m) = (1-\rho)\rho^m ,$$

and

$$q_1^{'} = \frac{(1-\rho_1)\rho_1^{S_1+E(K_{30})}}{1-\rho_1^{S_1+E(K_{30})+1}}, \quad q_2^{'} = \frac{(1-\rho_2)\rho_2^{S_2+E(K_{3\bar{0}})}}{1-\rho_2^{S_2+E(K_{3\bar{0}})+1}}, \quad q_3 = \frac{(1-\rho_3)\rho_3^{S_3}}{1-\rho_3^{S_3+1}}$$

$$G_i = \frac{q_i^{'}(1-\rho_i)}{1-\bar{q}_i^{'}\rho_i} \quad \textit{for } i = 1, 2, \quad H = (1-\rho_3)\rho_3^{S_3}. \qquad \square$$

For the purpose of computing $q_i^{'}$, $i = 1, 2$, and using balance equations (4.8) and (4.9) to evaluate the performance measures of both assemblies at the last stage where customer demand arises, the product-form distribution above is detailed by

$$\Pr(K_{30} = k_{30}, K_{3\bar{0}} = k_{3\bar{0}} | K_3 = k_3) = \binom{k_3}{k_{30}} \left(\frac{\lambda_0}{\lambda_0 + \lambda_{\bar{0}}}\right)^{k_{30}} \left(\frac{\lambda_0}{\lambda_0 + \lambda_{\bar{0}}}\right)^{k_{3\bar{0}}} \text{ for } k_3 = k_{30} + k_{3\bar{0}},$$

$$\Pr(M_0 = m_0, M_{\bar{0}} = m_{\bar{0}} | M = m) = \binom{m}{m_0} \left(\frac{\lambda_0}{\lambda_0 + \lambda_{\bar{0}}} \right)^{m_0} \left(\frac{\lambda_0}{\lambda_0 + \lambda_{\bar{0}}} \right)^{m_{\bar{0}}} \text{ for } m = m_0 + m_{\bar{0}}.$$

In order to test performance of the approximation for the assembly system with common component, 10 different parameter sets to cover the range from high traffic intensities to low traffic intensities are considered. The parameter sets and the corresponding simulation and approximation results of the performance measures for the system in Figure 4.6 are given in Appendix P. As can be seen from these numerical results, the performance of the approximation is similar to the results in sections 3.4 and 4.1 and still satisfactory.

For the alternative model where common component is picked up at last, inventory balance equations are:

$$n_i + \overline{n}_i = S_i + k_i \qquad \textit{for } i = 1,\ 2,$$

$$n_3 + \overline{n}_3 = S_3 + k_1 + k_2 + k_{30} + k_{3\bar{0}},$$

$$m_0 + \overline{m}_0 + k_{30} + k_1 = S_0 + k_0,$$

$$m_{\bar{0}} + \overline{m}_{\bar{0}} + k_{3\bar{0}} + k_2 = S_{\bar{0}} + k_{\bar{0}},$$

where $k_3 = k_{30} + k_{3\bar{0}}$.

The alternative model in Figure 4.7 is equivalent to the original one in Figure 4.5 with the resolution for the coordination of component 3 noting that $\overline{n}_1 + k_{30}$, $\overline{n}_2 + k_{3\bar{0}}$ and $k_1 + k_{30}$, $k_2 + k_{3\bar{0}}$ in the latter model correspond to $\overline{n}_1$, $\overline{n}_2$ and k_{13}, k_{23} in the former model, respectively.

For the solution of the alternative model in Figure 4.7, the type of aggregation in section 3.2 would transform the state description of this alternative model from (n_1, n_2, n_3, m) to (k_1, k_2, k_3, m). Then, as in Remark 4.3, the approximate near-product-form distribution is proposed in Remark 4.4 for the alternative model in Figure 4.7 without any formal analytical development.

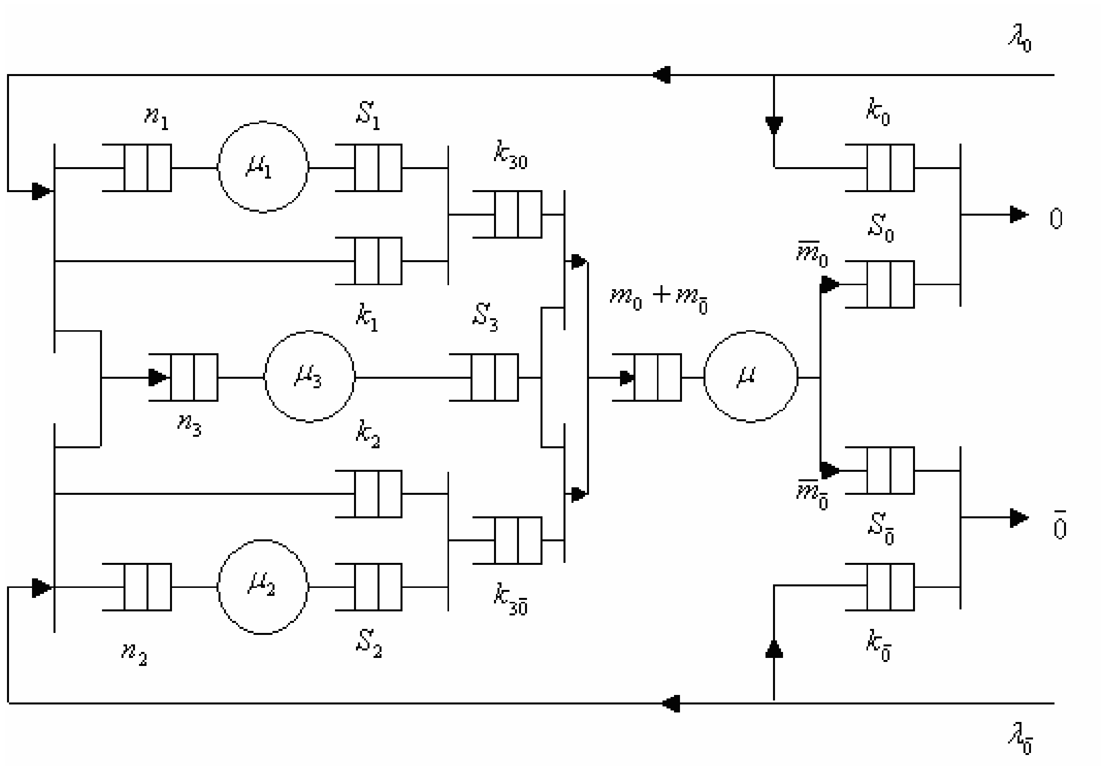


Figure 4.7 Alternative Assembly Model Where Common Component is Picked up at Last

Remark 4.4: *For the case* $\mu_0 = \mu_{\bar{0}} = \mu$ *of Figure 4.7, the near-product-form steady-state distribution proposed for the three-component assembly system where one component is common and there are two finished products is*

$$\breve{P}_{k_1k_2k_3m} = \breve{P}_1(K_1 = k_1)\breve{P}_2(K_2 = k_2)\breve{P}_3(K_3 = k_3)\breve{P}_{0\bar{0}}(M = m)$$

where

$$\breve{P}_i(K_i = k_i) = \begin{cases} \dfrac{H_i}{q_i} & for \quad k_i = 0, \\ H_i \rho_i^{k_i} & for \quad k_i \geq 1, \end{cases} \qquad \text{for } i = 1, 2,$$

$$\breve{P}_3(K_3 = k_3) = \begin{cases} \dfrac{G}{q_3^{'}} & for \quad k_3 = 0, \\ G \rho_3^{k_3} & for \quad k_3 \geq 1, \end{cases}$$

$$\breve{P}_{0\bar{0}}(M = m) = (1-\rho\ )\rho^{m},$$

and

$$q_i = \frac{(1-\rho_i)\rho_i^{S_i}}{1-\rho_i^{S_i+1}} \quad \textit{for } i = 1,\ 2, \qquad q_3^{'} = \frac{(1-\rho_3)\rho_3^{S_3+E(K_1+K_2)}}{1-\rho_3^{S_3+E(K_1+K_2)+1}},$$

$$H_i = (1-\rho_i)\rho_i^{S_i} \quad \textit{for } i = 1,\ 2, \qquad G = \frac{q_3^{'}(1-\rho_3)}{1-\overline{q}'_3\ \rho_3}. \qquad \square$$

The same parameter sets as the ones used for testing performance of the approximation in Remark 4.3 are used also for testing performance of the approximation in Remark 4.4. As can be seen in Appendix P, the results are still good and very similar to the ones of the system in Figure 4.6.

These two alternative models of the original model in Figure 4.5 can be selected according to several different performance criteria which are not considered in this thesis. For example, the model in Figure 4.6 would be selected to minimize the maximum waiting time of a demand arriving or the model in Figure 4.7 would be used to minimize WIP. In these respects, the analysis could be extended and the performance measures can be compared within the context of common component allocation recalling the related work in the literature like the one by de Kok and Visschers [20] reviewed in Chapter 2.

CHAPTER 5

5. CONCLUSION

In this thesis, the basic assembly systems are studied to investigate the steady-state probability distribution that would serve the purpose of performance analysis for given configurations and of system design based on the performance analysis. Although the setting of assembly operation is presented in the context of manufacturing in this thesis, validity of the same setting to resolve computer systems and telecommunication issues is underlined in the literature. As for the manufacturing setting, the system configuration is characterized by the BOM of the finished products manufactured and the production/inventory policies employed. The former characteristic considered in this thesis is to assemble two components to come up with a single type of finished product as in Chapter 3, others in Chapter 4 are extensions to assemble more than two components of again a single type of finished product and to produce two types of finished products assembling two components for each, one of the three components under consideration being common. Concentration on only these simple BOM structures can be justified because analysis of them would be sufficient to investigate many complex BOMs as a collection of these basics. The latter characteristic identified by the use of continuous-review base-stock inventory control policies places the system setting within the class of pull-type make-to-stock systems. The systems studied are further specified with Poisson demand arrivals and exponential servers, which allow generalization for general distributions due to the capability of the phase-type distributions to approximate general ones with mixtures of exponentials. It is assumed that there is no shortage of raw materials feeding the manufacturing servers and that unsatisfied customers due to stockout are backordered.

For the systems outlined above, an approximate near-product-form steady-state distribution is proposed based on the approximation approach introduced for the two-echelon systems in [2] and extended for two-indenture systems in [38]. This approach is approximating an exact partially aggregated queuing model which in our case is equivalent to what we call an alternative model (proposed to take the type of approach mentioned) allowing the components to be picked up sequentially before the assembly operation. The approximation is treating state-dependent transition rates that result from partial aggregation as constant rates, the immediate implication of which is the near-product-form steady-state distribution. Based on the numerical results gathered by examining 27 different parameter sets of the demand rate and manufacturing and assembly server rates and 1200 different base-stock level combinations, the proposed method is shown to perform well in approximating the performance measures (fill rate, stockout probability, expected backorder at the downstream stage) of the two-component system with single server manufacturing and assembly facilities. Generally, the approximation seems to be more precise for the systems with lower traffic intensities than for systems with higher traffic intensities. A greedy heuristic run using approximate distribution is observed to be good to determine design parameters, namely base-stock levels, in order to meet a given target fill rate.

The approximation method introduced for the simplest two-component systems is then extended intuitively to systems with more than two components assembled to produce a single type finished product. The approximate solution proposed for the n-component system is obtained by applying the approximate solution for the two-component system in a recursive manner. Numbering the components from 1 to n in the order they are picked up in the alternative model, the part of the system including manufacturing servers associated with components from the 1^{st} to the $(n-1)^{st}$ is assumed as a black-box and the two-component results are applied to the collection of the black-box and the servers associated with the n^{th} component. The marginal steady-state probabilities of the black-box representing manufacturing servers of the first (n-1) components is found by applying the two-component results to the system with the $(n-1)^{st}$ component and the servers of components from the 1^{st} to the $(n-2)^{nd}$, this time servers of components from 1^{st} to the $(n-2)^{nd}$ forming the next black-box. This recursion continues until the steady-state probabilities of a pure two-component system can be applied to determine the steady-state probabilities of a black-box. The results gathered for these systems have shown that the approach still serves well for approximating the performance measures of the n-component systems. Questioning the impact of component sequence on the performance of the approximation, parameter sets are selected in such a way that the components can be picked up in decreasing or increasing order of their dedicated server rates and base-stock levels. Note that ordering components for both the server rates and the base-stock levels to change in the same direction in the same order restricts the possible test configurations considerably. Even over such a specific restricted test set of configurations we are unable to determine impact of the component sequence, meaning although it is obvious that approximation results are dependent on the sequence, the specific behavior of this dependency can not be identified.

One other extension of the approximation proposed in this study is to handle component commonality. The “alternative” queuing models generated in the case of commonality turn out to be of two different forms; one to allocate common components before the other components and one to allocate them after the others. A near-product-form solution is proposed for each of these alternative models based on the intuitive use of the observations about the solution of two-component systems. The numerical experiments show that the two alternative models are comparable in terms of the performance measures considered in this thesis, putting the implications of immediate or latest common component allocation aside.

The approach presented in this study can easily be extended to more complicated cases. Results remain valid as long as the manufacturing and assembly facilities are of product-form, namely Jackson, networks. General service and interarrival time distributions can be handled approximating these distributions with a mixture of exponentials which maybe to recall Jackson network generalizations. Another extension could be thought of concerning more complex BOM structures for the finished products. Taking the n-component model and the common component model as modules of the complex system and applying the proposed solutions recursively, various BOM structures could be resolved.

Regarding the commonalities and product differentiation issues in general multi-stage assembly systems (the related work in literature like [20] and many others as summarized in [25]), points to investigate the connection of these issues to processing times, procurement lead-times from external suppliers, performance of the system in satisfying customer requests, e.g., under various allocation policies for the common components, risk-pooling, lead time reduction, (safety) stock reduction etc. may be of great benefit in foreseeing the performances of different “alternative” models constructed to employ the approximation proposed in this thesis.

In conclusion, we believe that the analytical framework presented in this study provides a powerful tool for approximating the steady-state performance of fairly general assembly manufacturing systems, and subsequently to design these systems, while various extensions seem to be possible for future research.

REFERENCES

[1] Altıok, T., (1997) *Performance Analysis of Manufacturing Systems*, Springer Series in Operations Research, New York.

[2] Avşar, Z.M., Zijm, W.H.M., (2003). "Capacitated Two-Echelon Inventory Models For Repairable Item Systems", S.B. Gershwin, Y. Dallery, C.T. Papadopoulos, J. McGregor-Smith (eds.): *Kluwer Academic Publishers Special Issue on Analysis and Modeling of Manufacturing Systems*, Chapter 1.

[3] Baccelli, F. and Makowski, A.M., (1989). "Queuing Models for Systems with Synchronization Constraints", *Preceedings of the IEEE*, Vol. 77, pp. 138-161.

[4] Baccelli, F., Makowski, A.M., Scwartz. A., (1989). "The Fork-Join Queue and Related Systems With Synchronization Constraints: Stochastic Ordering and Computable Bounds", *Applied Probability Trust*, Vol. 21, pp. 629-660.

[5] Baccelli, F., Massey, W.A. and Towsley, D., (1989). "Acyclic Fork/Join Queuing Networks", *Journal of ACM*, Vol. 36 (1), pp.615-642.

[6] Baynat, B., Dallery, Y., Di Mascolo, M. and Frein, Y., (2001). "A Multi-Class Approximation Technique for the Analysis of Kanban-Like Control Systems", *International Journal of Production Research, Special Issue on Modeling, Specification and Analysis of Manufacturing Systems*, Vol. 39 (2), pp. 307-328.

[7] Bhat, U.N., (1986). "Finite Capacity Assembly Like Queues", *Queueing Systems,* Vol. 1, pp.85-101.

[8] Buzacott, J.A. and Shanthikumar, J.G., (1993). *Stochatic Models of Manufacturing Systems*, Prentice Hall, New Jersey.

[9] Chaouiya, C., Liberopoulos, G., Dallery, Y., (2000). "The Extended Kanban Control System for Production Coordination of Assembly Manufacturing Systems", *IIE Transactions*, Vol. 32, pp.999-1012.

[10] Dallery, Y., Liberopoulos, G., (2000). “Extended Kanban Control System: Combining Kanban and Base-Stock”, IIE Transactions on Design and Manufacturing, Vol. 32, pp. 369-386.

[11] Dallery, Y., Liu, Z., Towsley, D., (1994). “Equivalence, Reversibility, Symmetry and Concavity Properties in Fork-Join Queuing Networks with Blocking”, *Journal of Association for Computing Machinery*, Vol. 41, pp. 903-942.

[12] Dallery, Y., Liu, Z., Towsley, D., (1997). “Properties of Fork/Join Queueing Networks with Blocking Under Various Operating Mechanisms”, *IEEE Transaction on Robotics and Automation*, Vol. 13 (4), pp. 503-518.

[13] Di Mascolo, M., Dallery, Y., (1996). “Performance Evaluation of Kanban Controlled Assembly Systems”, Symposium on Discrete Events and Manufacturing Systems of the Multiconference IMACS-IEEE/SMC CESA’96, Lille, France.

[14] Ettl, M., Feigin, G.E., Lin, G.Y., Yao, D., (2000). “A Supply Network Model with Base-Stock Control and Service Requirements”, *Operations Research*, Vol. 48, No 2, pp. 216-232.

[15] Harrison, J.M., (1973). “Assembly-like Queues”, *Journal of Applied Probability*, Vol. 10 (2), pp. 354-367.

[16] Hazra, J., Schweitzer, P.J., Seidmann, A., (1999). “Analyzing Closed Kanban-Controlled Assembly Systems by Iterative Aggregation-Disaggregation”, *Computers and Operations Research*, Vol. 26, pp. 1015-1039.

[17] Heidelberger, P. and Triverdi, K.S., (1983). “Queueing Network Models for Parallel Processing with Asynchronous Tasks”, *IEEE Transactions on Computers*, Vol. C-31(11), pp.1099-1109.

[18] Hemachandra, N., Edupuganti, S.K., (2003). “Performance Analysis and Buffer Allocations in Some Open Assembly Systems”, *Computers and Operations Research 30*, pp. 695-704.

[19] Hopp, W. and Simon, J.T., (1989). “Bounds and Heuristics for Assembly-Like Queues”, *Queueing Systems*, Vol. 4, pp. 137-156.

[20] Kok, T.G., Visschers, J.W.C.H., (1999). “Analysis of Assembly Systems with Service Level Constraints”, *International Journal of Production Economics*, Vol. 59, pp. 313-326.

[21] Krishnamurthy, A., Suri, R. and Vernon, M., (2003). “Two-Moment Approximations for Throughput and Mean Queue Length of a Fork/Join Station with General Input Processes from Finite Populations”, J.G. Shanthikumar, D.D. Yao and W.H.M. Zijm (eds.): *Kluwer International Series in Operations Research and Management Science, Stochastic Modeling and Optimization of Manufacturing Systems and Supply Chains,* Chapter 5.

[22] Krishnamurthy, A., Suri, R. and Vernon, M., (2001). “Analytical Performance Analysis of Kanban Systems Using a New Approximation for Fork / Join Stations”, *Proceedings of the Industrial Engineering Research Conference* Dallas, Texas.

[23] Latouche, G., (1981). “Queues with Paired Customers”, *Journal of Applied Probability,* Vol. 18, pp. 684-696.

[24] Lipper, E.H. and Sengupta, B., (1986). “Assembly Like Queues with Finite Capacity: Bounds, Asymptotic and Approximations”, *Queueing Systems*, Vol. 1, pp. 67-83.

[25] Ma, S., Wang, W., Liu, L., (2002). “Commonality and Postponement in Multi-stage Assembly Systems”, *European Journal of Operational Research*, Vol. 142, pp. 523-538.

[26] Prabhakar, B., Bambos, N. and Mountford, T.S., (2000). “The Synchronization of Poisson Processes and Queuing Networks with Service and Synchronization Nodes”, *Advances in Applied Probability,* Vol. 32, pp. 824-843.

[27] Rao, P.C. and Suri, R., (1994). “Approximate Queuing Network Models of Fabrication/Assembly Systems: Part I – Single Level Systems”, *Production and Operations Management*, Vol. 3 (4), pp.244-275.

[28] Rao, P.C. and Suri, R., (2000). “Performance Analysis of an Assembly Station With Input From Multiple Fabrication Lines”, *Production and Operations Management*, Vol. 9 (3) , pp.283-302.

[29] Rosling, K., (1989). “Optimal Inventory Policies for Assembly Systems under Random Demands”, *Operations Research,* Vol. 37, pp. 365–579.

[30] Sbiti, N., Di Mascolo, M., Bennani, T., Amghar, M., (2002). “Modeling and Performance Evaluation of Base-Stock Controlled Assembly Systems”, S.B. Gershwin, Y. Dallery, C.T. Papadopoulos, J. McGregor-Smith (eds.): *Kluwer Academic Publishers Special Issue on Analysis and Modeling of Manufacturing Systems*, Chapter 13.

[31] Som, P., Wilhelm, P.E. and Disney, R.L., (1994). "Kitting Process in a Stochastic Assembly System", *Queueing Systems*, Vol. 17, pp. 471-490.

[32] Takahashi, M. Osawa, H. and Fujisawa, T., (1996). "A Stochastic Assembly System With Resume Levels", *Asia-Pacific Journal of Operations Research*, Vol. 15, pp. 127-146.

[33] Takahashi, M. Osawa, H. and Fujisawa, T., (2000). "On a Synchronization Queue With Two Finite Buffers", *Queueing Systems*, Vol. 36, pp. 107-123.

[34] Takahashi, M. and Takahashi, Y., (2000). "Synchronization Queue With Two MAP Inputs and Finite Buffers", *Proceedings of the Third International Conference on Matrix Analytical Methods in Stochastic Models*, Belgium.

[35] Takahashi, Y.,(1975). "A Lumping Method of Numerical Calculation of Stationary Distributions of Markow Chains", *Research Report No. B-18, Department of Information Sciences, Tokyo Institute of Technology,* Tokyo, Japan.

[36] Van Houtum, G.J., Zijm, W.H., (1991). "Computational Procedures for Stochastic Multi-Echelon Production Systems", *International Journal of Production Economics*, Vol. 23, pp. 223–237.

[37] Varki, E., (1999). "Mean Value Technique For Closed Fork-Join Networks", *Proceedings of ACM SIGMETRICS Conference on Meaurement and Modeling of Computer Systems*, Atlanta, Georgia.

[38] Zijm, W.H., Avşar, Z.M., (2003). "Capacitated Two-Indenture Models For Repairable Item Systems", *International Journal of Production Economics*, Vol. 81-82, pp. 573-588.

APPENDIX A

ALTERNATIVE PROOF FOR THE AGGREGATE FORMULATION

The model with state description (k_1, k_2, m) *is an aggregate formulation of the one with state description* (n_1, n_2, m).

Proof:

$k_1 = 0,\ k_2 = 0$ Cases 1, 2, 4, 5

$$\left(\lambda + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,0m} + \mu_1 I_{S_1>0} \sum_{n_1=1}^{S_1} \sum_{n_2=0}^{S_2} P_{n_1 n_2 m} + \mu_2 I_{S_2>0} \sum_{n_1=0}^{S_1} \sum_{n_2=1}^{S_2} P_{n_1 n_2 m}$$

$$= \lambda I_{\{n_1>0\}} I_{\{n_2>0\}} I_{\{m>0\}} \sum_{n_1=0}^{S_1} \sum_{n_2=0}^{S_2} P_{n_1-1,n_2-1,m-1} + \mu_1 I_{\{S_1>0\}} \sum_{n_1=0}^{S_1-1} \sum_{n_2=0}^{S_2} P_{n_1+1,n_2 m} + \mu_1 I_{\{m>0\}} \sum_{n_2=0}^{S_2} P_{S_1+1,n_2,m-1} +$$

$$\mu_2 I_{\{S_2>0\}} \sum_{n_1=0}^{S_1} \sum_{n_2=0}^{S_2-1} P_{n_1,n_2+1,m} + \mu_2 I_{\{m>0\}} \sum_{n_1=0}^{S_1} P_{n_1,S_2+1,m-1} + \mu \widetilde{P}_{0,0,m+1}$$

The following terms cancel out in the equation above; the second terms on both sides, the third term on the left hand side and the fourth term on the right. Rewriting the remaining terms as

$$\left(\lambda + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,0m}$$

$$= \lambda I_{\{m>0\}} \left(\frac{\sum_{n_1=0}^{S_1} \sum_{n_2=0}^{S_2} P_{n_1 n_2,m-1}}{\widetilde{P}_{0,0,m-1}} - \frac{\sum_{n_2=0}^{S_2} P_{S_1 n_2,m-1}}{\widetilde{P}_{0,0,m-1}} - \frac{\sum_{n_1=0}^{S_1-1} P_{n_1 S_2,m-1}}{\widetilde{P}_{0,0,m-1}} \right) * \widetilde{P}_{0,0,m-1}$$

$$+ \mu_1 I_{\{m>0\}} \widetilde{P}_{1,0m} \left(\frac{\sum_{n_2=0}^{S_2+1} P_{S_1+1,n_2 m}}{\widetilde{P}_{0,0m}} - \frac{P_{S_1+1,S_2+1,m}}{\widetilde{P}_{0,0m}} \right) + \mu_2 I_{\{m>0\}} \widetilde{P}_{0,1,m-1} + \mu \widetilde{P}_{0,0,m+1},$$

one comes up with

$$\left(\lambda + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,0m} = \lambda I_{\{m>0\}}(1 - q(0,m-1) - q''(m-1))\,\widetilde{P}_{0,0,m-1}$$
$$+ \mu_1 I_{\{m>0\}}(1-q'(1,m-1))\widetilde{P}_{1,0m} + \mu_2 I_{\{m>0\}}\widetilde{P}_{0,1,m-1} + \mu\widetilde{P}_{0,0,m+1}$$

$k_1 = 0,\ \ k_2 = 1$ Cases 3, 6

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,1m} + \mu_1 I_{\{S_1>0\}}\sum_{n_1=1}^{S_1} P_{n_1 n_2 m} = \lambda I_{\{n_1>0\}}\sum_{n_1=0}^{S_1} P_{n_1-1,n_2-1,m} + \mu_1 I_{\{S_1>0\}}\sum_{n_1=0}^{S_1-1} P_{n_1+1,n_2,m}$$
$$+ \mu_1 P_{S_1+1,n_2 m} + \mu_2 I_{\{m>0\}}\widetilde{P}_{0,2,m-1} + \mu\widetilde{P}_{0,0,m+1}$$

The last term on the left hand side of the equation and the second term on the right cancel out. Rewriting

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\widetilde{P}_{0,1m} = \lambda I_{\{S_1>0\}}\frac{\sum_{n_1=0}^{S_1-1} P_{n_1 S_2 m}}{\widetilde{P}_{0,0m}}\widetilde{P}_{0,0m} + \mu_1\frac{P_{S_1+1,S_2+1,m}}{\widetilde{P}_{1,0m}} * \widetilde{P}_{1,0m} + \mu_2 I_{\{m>0\}}\widetilde{P}_{0,2,m-1}$$

$$+ \mu\widetilde{P}_{0,0,m+1},$$

where the first and the second terms on the right side are $\lambda I_{\{n_1>0\}}q''(m)\widetilde{P}_{0,0m}$ and $\mu_1 q'(1,m)\widetilde{P}_{1,0m}$.

$k_1 = 0,\ \ k_2 > 1$ Cases 3, 6

$$\left(\lambda + \mu_2 + \mu I_{\{m>0\}}\right)\widetilde{P}_{0k_2 m} + \mu_1 I_{S_1>0}\sum_{n_1=1}^{S_1} P_{n_1 n_2 m} = \lambda I_{\{n_1>0\}}\sum_{n_1=0}^{S_1} P_{n_1-1,n_2-1,m} + \mu_1 I_{S_1>0}\sum_{n_1=0}^{S_1-1} P_{n_1+1,n_2 m}$$
$$+ \mu_1\widetilde{P}_{S_1+1,k_2-1,m} + \mu_2 I_{\{m>0\}}\widetilde{P}_{0,k_2+1,m-1} + \mu\widetilde{P}_{0,k_2,m+1}$$

The last term on the left hand side of the equation and the second term on the right cancel out. Then,

$$\left(\lambda+\mu_2+\mu I_{\{m>0\}}\right)\widetilde{P}_{0k_2m}=\lambda I_{\{S_1>0\}}\frac{\sum_{n_1=0}^{S_1}P_{n_1,S_2+(k_2-1),m}-P_{S_1,S_2+(k_2-1),m}}{\widetilde{P}_{0,k_2-1,m}}\widetilde{P}_{0,k_2-1,m}+\mu_1\widetilde{P}_{S_1+1,k_2-1,m}$$

$$+\mu_2 I_{\{m>0\}}\widetilde{P}_{0,k_2+1,m-1}+\mu\widetilde{P}_{0,k_2,m+1}$$

where the first term on the right hand side is $\lambda I_{\{S_1>0\}}(1-q(k_2-1,m))\widetilde{P}_{0,k_2-1,m}$.

$k_1=1,\ k_2=0$ Cases 7, 8

$$\left(\lambda+\mu_1+\mu I_{\{m>0\}}\right)\widetilde{P}_{1,0m}+\mu_2\sum_{n_2=1}^{S_2+1}P_{n_1n_2m}=\lambda\sum_{n_2=1}^{S_2+1}P_{n_1-1,n_2-1,m}+\mu_1 I_{\{m>0\}}\sum_{n_2=0}^{S_2+1}P_{n_1+1,n_2,m-1}$$

$$+\mu_2\sum_{n_2=0}^{S_2}P_{n_1,n_2+1,m}+\mu_2 I_{\{m>0\}}\widetilde{P}_{1,1,m-1}+\mu\widetilde{P}_{1,0,m+1}$$

The last term on the left hand side and the third term on the right cancel out. Rewriting,

$$\left(\lambda+\mu_1+\mu I_{\{m>0\}}\right)\widetilde{P}_{1,0m}=\lambda\frac{\sum_{n_2=0}^{S_2}P_{s_1n_2m}}{\widetilde{P}_{0,0m}}\widetilde{P}_{0,0m}$$

$$+\mu_1 I_{\{m>0\}}\frac{(\sum_{n_2=0}^{S_2+2}P_{S_1+2,n_2,m-1}-P_{S_1+2,S_2,m-1})}{\widetilde{P}_{2,0,m-1}}\widetilde{P}_{2,0,m-1}+\mu_2 I_{\{m>0\}}\widetilde{P}_{1,1,m-1}+\mu\widetilde{P}_{1,0,m+1}$$

The first and the second terms on the right hand side are rewritten as $\lambda q(0,m)\widetilde{P}_{0,0m}$ and $\mu_1 I_{\{m>0\}}1-q'(2,m-1)\widetilde{P}_{2,0,m-1}$, respectively.

$k_1 = 1,\ \ k_2 = 1$ Case 9

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)\widetilde{P}_{1,1m} = \lambda I_{\{n_1>0\}} \frac{P_{S_1,S_2+1,m}}{\widetilde{P}_{0,1m}} \widetilde{P}_{0,1m} + \mu_1 \frac{P_{S_1+2,S_2+2,m}}{\widetilde{P}_{2,0m}} \widetilde{P}_{2,0m}$$

$$+ \mu_2 I_{\{m>0\}} \widetilde{P}_{1,2,m-1} + \mu \widetilde{P}_{1,1,m+1}$$

The first and the second terms on the right hand side are rewritten as $\lambda I_{\{n_1>0\}} q(1,m)\widetilde{P}_{0,1m}$ and $\mu_1 q'(2,m)\widetilde{P}_{2,0m}$, respectively.

$k_1 = 1,\ \ k_2 > 1$ Case 9

$$\left(\lambda + \mu_1 + \mu_2 + \mu I_{\{m>0\}}\right)\widetilde{P}_{1k_2m} = \lambda I_{\{n_1>0\}} \frac{P_{S_1,n_2-1,m}}{\widetilde{P}_{0k_2m}} \widetilde{P}_{0k_2m} + \mu_1 P_{2,k_2-1,m} + \mu_2 I_{\{m>0\}} \widetilde{P}_{1,k_2+1,m-1}$$

$$+ \mu \widetilde{P}_{1,k_2,m+1}$$

The first term on the right hand side is rewritten as $\lambda I_{\{n_1>0\}} q(k_2,m)\widetilde{P}_{0k_2m}$.

$k_1 > 1,\ \ k_2 = 0$ Cases 7, 8

$$\left(\lambda + \mu_1 + \mu I_{\{m>0\}}\right)\widetilde{P}_{k_1 0m} + \mu_2 \sum_{n_2=1}^{S_2+k_1} P_{n_1 n_2 m} = \lambda \sum_{n_2=1}^{S_2+k_1} P_{n_1-1,n_2-1,m} + \mu_1 I_{\{m>0\}} \sum_{n_2=0}^{S_2+k_1} P_{n_1+1,n_2,m-1}$$

$$+ \mu_2 \sum_{n_2=0}^{S_2+k_1-1} P_{n_1,n_2+1,m} + \mu_2 I_{\{m>0\}} \widetilde{P}_{k_1 1,m-1} + \mu \widetilde{P}_{k_1 0,m+1}$$

The last term on the left hand side and the third term on the right cancel out. Then,

$$\left(\lambda+\mu_1+\mu I_{\{m>0\}}\right)\widetilde{P}_{k_1 0m} = \lambda\widetilde{P}_{k_1-1,0m} + \mu_1 I_{\{m>0\}}\left(\frac{\sum_{n_2=0}^{S_2+k_1+1} P_{n_1+1,n_2,m-1}}{\widetilde{P}_{k_1+1,0,m-1}} - \frac{P_{n_1+1,S_2+k_1,m-1}}{\widetilde{P}_{k_1+1,0,m-1}}\right)\widetilde{P}_{k_1+1,0,m-1}$$

$$+\mu_2 I_{m>0}\widetilde{P}_{k_1 1,m-1} + \mu\widetilde{P}_{k_1 0,m+1}$$

where the first term on the right hand side is $\mu_1 I_{\{m>0\}}(1-q'(k_1+1,m-1))\widetilde{P}_{k_1+1,0,m-1}$.

$k_1 > 1,\ \ k_2 = 1$ Case 9

$$\left(\lambda+\mu_1+\mu_2+\mu I_{\{m>0\}}\right)\widetilde{P}_{k_1 1m} = \lambda I_{\{n_1>0\}}\widetilde{P}_{k_1-1,1m} + \mu_1 \frac{P_{n_1+1,n_2=S_2+k_2+1,m}}{\widetilde{P}_{k_1+1,0m}}\widetilde{P}_{k_1+1,0m} +$$

$$\mu_2 I_{\{m>0\}}\widetilde{P}_{k_1 2,m-1} + \mu\widetilde{P}_{k_1 1,m+1}$$

The first term on the right hand side is $\mu_1 q'(k_1+1,m)\widetilde{P}_{k_1+1,0m}$.

$k_1 > 1,\ \ k_2 > 1$ Case 9

$$\left(\lambda+\mu_1+\mu_2+\mu I_{\{m>0\}}\right)\widetilde{P}_{k_1 k_2 m} = \lambda\widetilde{P}_{k_1-1,k_2 m} + \mu_1\widetilde{P}_{k_1+1,k_2-1,m} + \mu_2 I_{\{m>0\}}\widetilde{P}_{k_1,k_2+1,m-1} + \mu\widetilde{P}_{k_1 k_2,m+1}$$

□

APPENDIX B

PROOF OF LEMMA 3.4

Rewriting (3.10) and (3.11) as

$$q(k_2,m)=\begin{cases}\dfrac{P_{S_1,S_2+k_2,m}}{\widetilde{P}_{0k_2m}} & k_2>0,\\[2ex] \dfrac{\sum_{n_2=0}^{S_2} P_{S_1n_2m}}{\widetilde{P}_{0,0m}} & k_2=0,\end{cases} \qquad \textit{for any m,}$$

q is derived as follows:

$$q=\sum_{m,k_2} q(k_2,m)Pr(K_2=k_2,M=m\,|\,N_1\le S_1)$$

$$=\sum_m q(0,m)Pr(K_2=0,M=m\,|\,N_1\le S_1)+\sum_{m,k_2\ge1} q(k_2,m)Pr(K_2=k_2,M=m\,|\,N_1\le S_1)$$

$$=\sum_m \frac{\sum_{n_2=0}^{S_2} P_{s_1s_2m}}{\widetilde{P}_{0,0m}}\cdot\frac{Pr(N_1\le S_1,K_2=0,M=m)}{Pr(N_1\le S_1)}+\sum_{m,k_2\ge1}\frac{P_{S_1,S_2+k_2,m}}{\widetilde{P}_{0k_2m}}\cdot\frac{Pr(N_1\le S_1,K_2=k_2,M=m)}{Pr(N_1\le S_1)}$$

$$=\sum_m \frac{Pr(N_1=S_1,K_2=0,M=m)}{Pr(N_1\le S_1)}+\sum_{k_2\ge1}\sum_m\frac{Pr(N_1=S_1,K_2=k_2,M=m)}{Pr(N_1\le S_1)}$$

$$=\frac{Pr(N_1=S_1,K_2=0)}{Pr(N_1\le S_1)}+\sum_{k_2\ge1}\frac{Pr(N_1=S_1,K_2=k_2)}{Pr(N_1\le S_1)}.$$

Then, $q = \dfrac{Pr(N_1 = S_1)}{Pr(N_1 \le S_1)}$.

Rewriting equation (3.12) as

$$q'(k_1, m) = \begin{cases} \dfrac{P_{S_1+k_1, S_2+k_1, m}}{\widetilde{P}_{k_1 0m}} & k_1 > 0, \\ \dfrac{\sum_{n_1=0}^{S_1} P_{n_1 S_2 m}}{\widetilde{P}_{0,0m}} & k_1 = 0, \end{cases} \qquad \textit{for any m,}$$

q' is derived as follows:

$$q' = \sum_{m,k_1} q'(k_1, m) Pr(K_1 = k_1, M = m \mid N_2 \le S_2 + K_1)$$

$$= \sum_m \frac{\sum_{n_1=0}^{S_1} P_{n_1 s_2 m}}{\widetilde{P}_{0,0m}} \cdot \frac{Pr(K_1 = 0, K_2 = 0, M = m)}{Pr(N_2 \le S_2 + K_1)} + \sum_{m,k_1 \ge 1} \frac{P_{S_1+k_1, S_2+k_1, m}}{\widetilde{P}_{k_1 0m}} \cdot \frac{Pr(K_1 = k_1, K_2 = 0, M = m)}{Pr(N_2 \le S_2 + K_1)}$$

$$= \sum_m \frac{Pr(N_1 \le S_1, N_2 = S_2), M = m)}{Pr(N_2 \le S_2 + K_1)} + \sum_{k_2 \ge 1} \sum_m \frac{Pr(N_1 = S_1 + k_1, N_2 = S_2 + k_1, M = m)}{Pr(N_2 \le S_2 + K_1)}$$

$$= \frac{Pr(N_1 \le S_1, N_2 = S_2)}{Pr(N_2 \le S_2 + K_1)} + \sum_{k_1 \ge 1} \frac{Pr(N_1 = S_1 + k_1, N_2 = S_2 + k_1)}{Pr(N_2 \le S_2 + K_1)}$$

$$= \frac{1}{Pr(N_2 \le S_2 + K_1)} \sum_{k_1 \ge 0} Pr(K_1 = k_1, N_2 = S_2 + k_1)$$

$$= \frac{Pr(N_2 = S_2 + K_1)}{Pr(N_2 \le S_2 + K_1)}.$$

Rewriting (3.13) as

$$q''(m) = \begin{cases} 0 & S_1 = 0 \\ \dfrac{\sum_{n_1=0}^{S_1-1} P_{n_1 S_2 m}}{\widetilde{P}_{0,0m}} & S_1 > 0 \end{cases} \qquad \text{for any } m,$$

q'' is derived as follows:

$$q'' = \sum_m q''(m) Pr(M = m \mid K_1 = 0, K_2 = 0)$$

$$= \sum_m \frac{\sum_{n_1=0}^{S_1-1} P_{n_1 S_2 m}}{\widetilde{P}_{00m}} \frac{Pr(K_1 = 0, K_2 = 0, M = m)}{Pr(K_1 = 0, K_2 = 0)}$$

$$= \frac{Pr(N_1 < S_1, N_2 = S_2)}{Pr(K_1 = 0, K_2 = 0)} . \qquad \square$$

APPENDIX C

ALTERNATIVE APPROXIMATE q' VALUES

A) $q_A^{'} = \dfrac{\sum_{k_1=0}^{\infty} Pr(N_2 = S_2 + K_1 | K_1 = k_1) Pr(K_1 = k_1)}{\sum_{k_1=0}^{\infty} Pr(N_2 \le S_2 + K_1 | K_1 = k_1) Pr(K_1 = k_1)}$ from Lemma 3.4.

Assuming that K_1 and N_2 are independent,

$$q_A^{'} = \frac{\sum_{k_1=0}^{\infty} Pr(N_2 = S_2 + k_1) Pr(K_1 = k_1)}{\sum_{k_1=0}^{\infty} Pr(N_2 \le S_2 + k_1) Pr(K_1 = k_1)}$$

$$= \frac{((1-\rho_2)\rho_2^{S_2})(1-\rho_1^{S_1+1}) + \sum_{k_1=1}^{\infty}((1-\rho_2)\rho_2^{S_2+k_1})((1-\rho_1)\rho_1^{S_1+k_1}))}{(1-\rho_2^{S_2+1})(1-\rho_1^{S_1+1}) + \sum_{k_1=1}^{\infty}(1-\rho_2^{S_2+k_1+1})((1-\rho_1)\rho_1^{S_1+k_1}))}.$$

B) $q_B^{'} = \sum_{m,k_1} q'([k_1-1]^+, m) Pr(K_1 = k_1, M = m | N_2 \le S_2 + K_1)$

where

$$[k_1-1]^+ = \begin{cases} 0 & for \quad k_1 = 0, \\ k_1 - 1 & for \quad k_1 > 0. \end{cases}$$

Rewriting

$$= \sum_{m,k_1} \frac{Pr(K_1 = [k_1 - 1]^+, N_2 = S_2 + K_1, M = m)}{Pr(K_1 = [k_1 - 1]^+, N_2 \le S_2 + K_1, M = m)} \cdot \frac{Pr(K_1 = k_1, K_2 = 0, M = m)}{Pr(N_2 \le S_2 + K_1)}.$$

Assuming that *M* and (K_1, N_2) are independent,

$$q_B^{'} = \frac{1}{Pr(N_2 \le S_2 + K_1)} \sum_{m,k_1} \frac{Pr(K_1 = [k_1 - 1]^+, N_2 = S_2 + K_1) Pr(M = m)}{Pr(K_1 = [k_1 - 1]^+, N_2 \le S_2 + K_1) Pr(M = m)} Pr(K_1 = k_1, K_2 = 0) Pr(M = m)$$

$$= \frac{1}{Pr(N_2 \le S_2 + K_1)} \sum_{k_1} \frac{Pr(N_2 = S_2 + K_1 | K_1 = [k_1 - 1]^+) Pr(K_1 = [k_1 - 1]^+)}{Pr(N_2 \le S_2 + K_1 | K_1 = [k_1 - 1]^+) Pr(K_1 = [k_1 - 1]^+)} Pr(K_1 = k_1, N_2 \le S_2 + K_1)$$

$$= \frac{1}{Pr(N_2 \le S_2 + K_1)} \sum_{k_1} \frac{Pr(N_2 = S_2 + K_1 | K_1 = [k_1 - 1]^+)}{Pr(N_2 \le S_2 + K_1 | K_1 = [k_1 - 1]^+)} Pr(N_2 = S_2 + K_1 | K_1 = k_1) \cdot Pr(K_1 = k_1)$$

Next assuming that K_1 and N_2 are independent,

$$q_B^{'} = \frac{1}{Pr(N_2 \le S_2 + K_1)} \sum_{k_1} \frac{Pr(N_2 = S_2 + [k_1 - 1]^+)}{Pr(N_2 \le S_2 + [k_1 - 1]^+)} Pr(N_2 \le S_2 + k_1) Pr(K_1 = k_1)$$

$$= \frac{Pr(N_2 = S_2) \frac{Pr(N_2 \le S_2)}{Pr(N_2 \le S_2)} Pr(K_1 = 0) + \sum_{k_1=1}^{\infty} Pr(N_2 = S_2 + k_1 - 1) \frac{Pr(N_2 \le S_2 + k_1)}{Pr(N_2 \le S_2 + k_1 - 1)} Pr(K_1 = k_1)}{Pr(N_2 \le S_2 + K_1)}$$

$$= \frac{((1-\rho_2)\rho_2^{S_2})(1-\rho_1^{S_1+1}) + \sum_{k_1=1}^{\infty} ((1-\rho_2)\rho_2^{S_2+k_1-1})(\frac{1-\rho_2^{S_2+k_1+1}}{1-\rho_2^{S_2+k_1}})((1-\rho_1)\rho_1^{S_1+k_1}))}{(1-\rho_2^{S_2+1})(1-\rho_1^{S_1+1}) + \sum_{k_1=1}^{\infty} (1-\rho_2^{S_2+k_1+1})((1-\rho_1)\rho_1^{S_1+k_1}))}.$$

C) $q_C^{'} = \dfrac{Pr(N_2 = S_2 + E(K_1))}{Pr(N_2 \le S_2 + E(K_1))}$

where

$$E(K_1) = \sum_{k_1=0}^{\infty} k_1 Pr(K_1 = k_1) = \frac{(1-\rho_1)\rho_1^{S_1}}{(1-\rho_1^{S_1+1})}$$ using $M/M/1$ formulas.

D) Ignoring dependence on M ,

$$q_D^{'} = \sum_{k_1=0}^{\infty} q'(k_1) Pr(K_1 = k_1)$$

$$= \sum_{k_1=0}^{\infty} \frac{Pr(N_2 = S_2 + k_1)}{Pr(N_2 \leq S_2 + k_1)} Pr(K_1 = k_1)$$

$$= \frac{(1-\rho_2)\rho_2^{S_2}}{(1-\rho_2^{S_2+1})}(1-\rho_1^{S_1+1}) + \sum_{k_1=1}^{\infty} \frac{(1-\rho_2)\rho_2^{S_2+k_1}}{(1-\rho_2^{S_2+k_1+1})}(1-\rho_1)\rho_1^{S_1+k_1} .$$

E) Ignoring dependence on M ,

$$q_E^{'} = \sum_{k_1=0}^{\infty} q'([k_1 - 1]^+) Pr(K_1 = k_1)$$

$$= q'(0) Pr(K_1 = 0) + \sum_{k_1=1}^{\infty} q'(k_1 - 1) Pr(K_1 = k_1)$$

$$= \frac{Pr(N_2 = S_2)}{Pr(N_2 \leq S_2)} + \sum_{k_1=1}^{\infty} \frac{Pr(N_2 = S_2 + k_1 - 1)}{Pr(N_2 \leq S_2 + k_1 - 1)} Pr(K_1 = k_1)$$

$$= \frac{(1-\rho_2)\rho_2^{S_2}}{(1-\rho_2^{S_2+1})}(1-\rho_1^{S_1+1}) + \sum_{k_1=1}^{\infty} \frac{(1-\rho_2)\rho_2^{S_2+k_1-1}}{(1-\rho_2^{S_2+k_1})}(1-\rho_1)\rho_1^{S_1+k_1} .$$

F) $q_F^{'} = \dfrac{Pr(N_2 = S_2)}{Pr(N_2 \leq S_2)} = \dfrac{(1-\rho_2)\rho_2^{S_2}}{(1-\rho_2^{S_2+1})}$.

APPENDIX D

CODES FOR THE APPROXIMATION AND SIMULATION

Code for calculating the approximation results of n component assembly model:

```
uses crt;
type n=0..6;
var d5: text;
   S: array [n] of integer;
   Mu: array [n] of Double;
   Lamda: Double;



{----------Approximate Values-----------}

Procedure App_Values;
        var SPapp, FRapp, ESapp: Double;
           Ro: array [n] of Double;
           Ek : array[n] of Double;
           q_Ek : array [n] of Double;
           q_Ek_bar: array [n] of Double;
           Pk : array [n,0..120] of Double;
           P2k: array [n,0..120] of Double;
           Pm: array [0..120] of Double;
           SumEk, SumP2k:Double;
           l,t:integer;

Function Power(Say:Double; U:integer): Double;
       Var i:integer; us:Double;
       Begin
        us:=1;
        if U>0 then
        For i:=1 to U Do
           us:= us * Say;
```

```
          Power:=us;
        End;

Procedure bir;
    var i: integer;
    begin
      for i:=0 to 120 do
      begin
      if i=0 then Pk[1,0]:=1-Power(Ro[1],(S[1]+1))
      else
        begin
        Pk[1,i]:=(1-Ro[1])*Power(Ro[1],(S[1]+i));
        end;
      P2k[1,i]:=Pk[1,i];
      end;
      Ek[1]:=Power(Ro[1],(S[1]+1))/(1-Ro[1]);
      q_Ek[1]:=(1-Ro[2])*EXP((S[2]+Ek[1])*LN(Ro[2]))/(1- EXP((S[2]+Ek[1]+1)*LN(Ro[2])) );
    end;

Procedure ikin;
    var i,j,comp:integer;
    begin
     for comp:=2 to 6 do
     begin
      for i:=0 to 120 do
      begin
        q_Ek_bar[comp-1]:=1-q_Ek[comp-1];
        if i=0 then Pk[comp,0]:=(1-Ro[comp])/(1-q_Ek_bar[comp-1]*Ro[comp])
        else Pk[comp,i]:= q_Ek[comp-1] * (1-Ro[comp]) * Power(Ro[comp], i) / (1-
q_Ek_bar[comp-1] * Ro[comp]);
        SumP2k:=0;
        For j:=0 to i do
          Begin
          SumP2k:=SumP2k+P2k[comp-1,j]*Pk[comp,i-j];
          End;
        P2k[comp,i]:=SumP2k;
      end;
      SumEk:=0;
      for j:=0 to 120 do
```

```
         begin
         SumEk:=SumEk + j*P2k[comp,j];
         End;
       Ek[comp]:=SumEk;
       q_Ek[comp]:=(1-Ro[comp+1])*EXP((S[comp+1]+Ek[comp])*LN(Ro[comp+1]))/(1-
EXP((S[comp+1]+Ek[comp]+1)*LN(Ro[comp+1])) );
     end;
   end;

Procedure assembly;
   Var i: integer;
   Begin
   For i:=0 to 120 do
   Pm[i]:=(1-Ro[0])*Power(Ro[0],i);
   end;

Procedure SP_approximate; {p(k>0)=P(m>S-kab-kc)=1-P(m=<S-kab-kc)}
           Var i,j,r : integer;
              prob,prob2 : double;
        Begin
            prob2:=0;
            For i:=0 to S[0] Do
             begin
                prob:=0;
                for r:=0 to (S[0]-i) do prob:= prob + Pm[r];
                for j:=0 to i do prob2:= prob2 + prob*P2k[5,(i-j)]*Pk[6,j];
             end;
            SPapp:= 1 - prob2;

        End;

Procedure FR_approximate; {p(m_bar > 0)= (1-FR)- P(m_bar=)}
           Var i,j,r : integer;
              prob,prob2 : double;
        Begin
            prob2:=0;
            For i:=0 to (S[0]-1) Do
             begin
                prob:=0;
```

```
            for r:=0 to (S[0]-i-1) do prob:= prob + Pm[r];
            for j:=0 to i do prob2:= prob2 + prob*P2k[5,(i-j)]*Pk[6,j];
          end;
        FRapp:= prob2;
    End;

Procedure Backorder;
    Var i,j,m : integer;
          prob,prob2 : double;
     Begin

        For m:=0 to 100 do
        Begin
        prob:=0;
         For i:=0 to (S[0]+m) Do
         begin
            for j:=0 to i do
            prob:= prob + Pm[(S[0]+m-i)]*P2k[(5),(i-j)]*Pk[6,j];
         end;
        PK[0,m] := prob;
        end;
     End;

Procedure ES_approximate; { Sum of "K * P(K=k)"}
     var i:integer; sonuc:double;
     Begin
        for i:= 0 to 100 do
            sonuc:=sonuc+ i*Pk[0,i];
            ESapp:=sonuc;

     End;

begin {main app procedure}

  For t:=0 to 6 do
   begin
   Ro[t]:= Lamda/Mu[t];
   end;
  bir;
```

```
    ikin;
    assembly;
    SP_approximate;
    FR_approximate;
    Backorder;
    ES_approximate;
    Append(d5);
    writeln(d5,'SP:',SPapp:5:5, '  FR:',FRapp:5:5, '  EB:', ESapp:5:5);


End;

{----------Main Program--------------}

Procedure bilgi_al;

Var i: integer;

    Begin
    Write('Lamda...................:'); Readln(Lamda);
    For i:=0 to 6 do
      begin
      write('S[',i,']................:');Readln(S[i]);
      write('Mu[',i,']................:');Readln(Mu[i]);
      end;
    Append(d5);
    Write(d5,'Lamda:',Lamda:3:0);
    For i:=0 to 6 do
      begin
      write(d5,'  S[',i,']:',S[i]:3,'  Mu[',i,']:',Mu[i]:3:0);
      end;

   end;

begin {main program}
Assign(d5,'qEk.dat');
bilgi_al;
App_Values;
Close(d5);
```

```
writeln('THE END :))');
readln;
End.
```

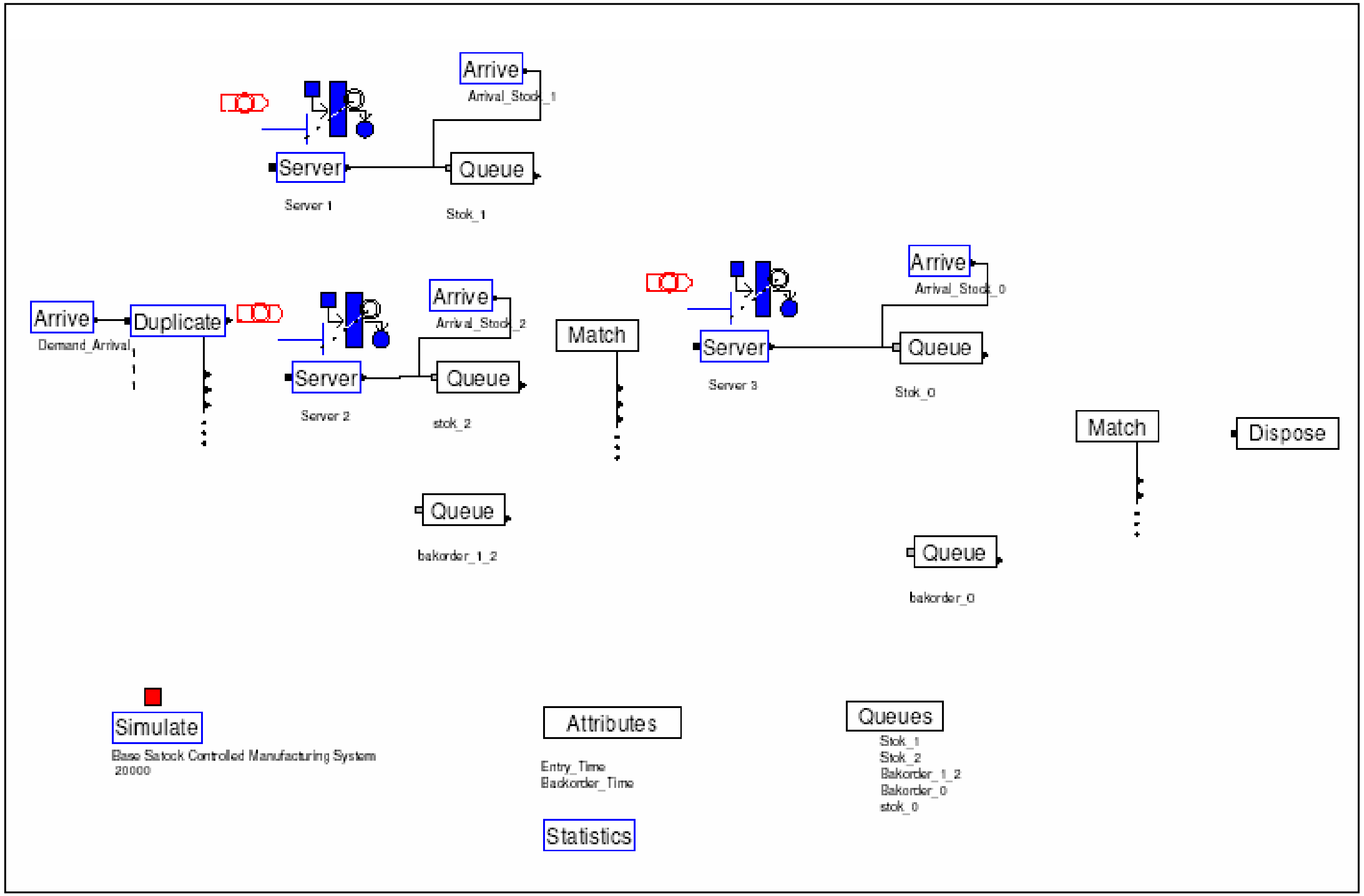


Figure A.1 Object-Oriented Visualization of the Simulation Code

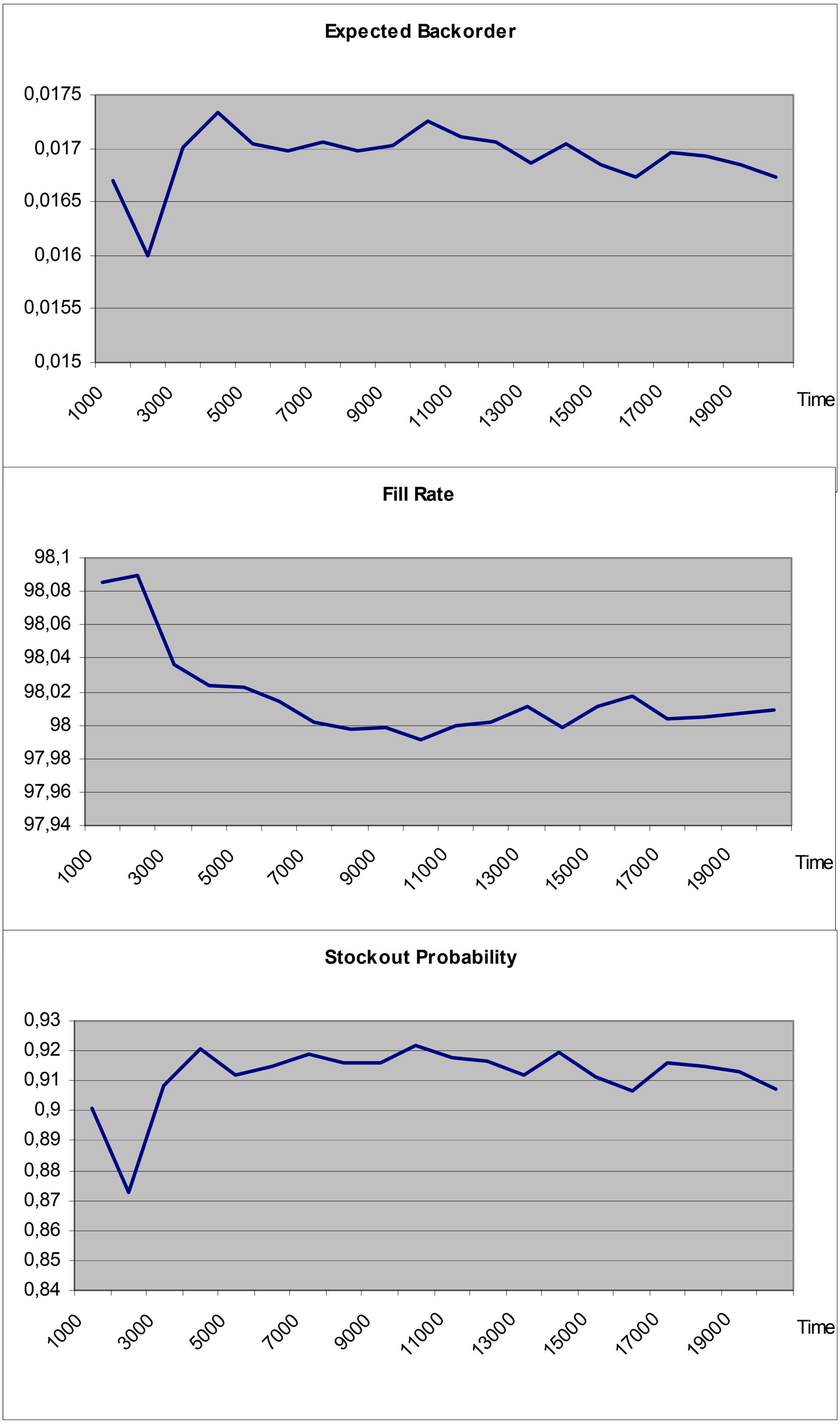


Figure A.2 Results of Performance Measures Versus Simulation Time for Case with Parameters $\mu_0 = \mu_1 = \mu_2 = 20,\ \lambda = 9,\ S_0 = S_1 = S_2 = 5$

APPENDIX E

APPROXIMATION PERFORMANCE FOR SEVERAL PARAMETER SETS: TWO-COMPONENT CASE

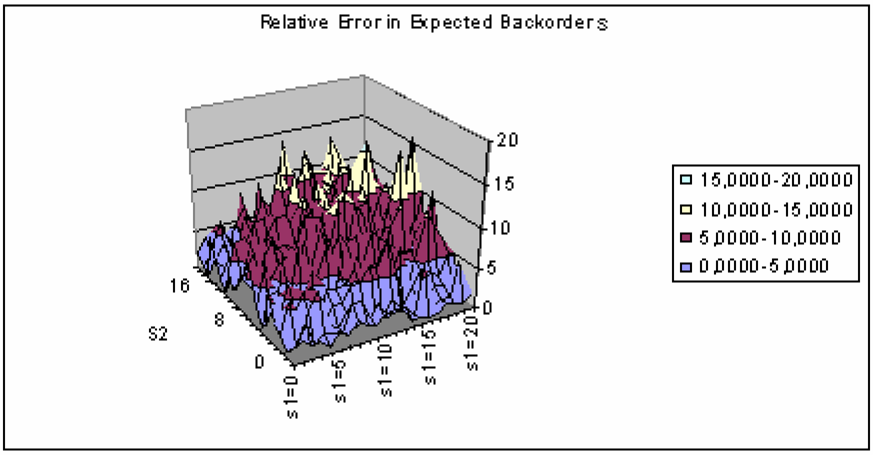
Table A.1 Errors of Expected Backorders for Several Parameter Sets

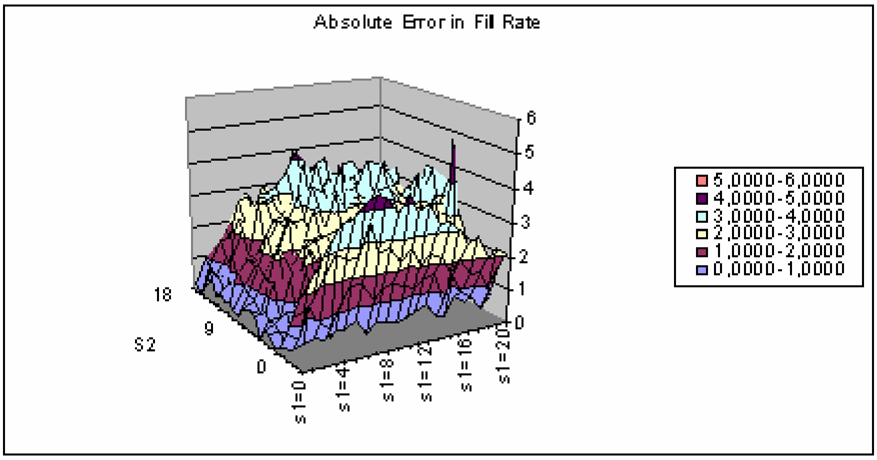
							VALUE		ERROR	
μ_0	μ_1	μ_2	S_0	λ	S_1	S_2	EB (App)	EB (Sim)	EB (Rel)%	EB (Abs)
10	20	10	15	9	0	0	3,930	3,649	7,701	0,281
10	20	10	15	9	0	5	3,155	2,867	10,066	0,289
10	20	10	15	9	0	10	2,698	2,538	6,308	0,160
10	20	10	15	9	0	15	2,427	2,378	2,084	0,050
10	20	10	15	9	0	20	2,268	2,061	10,035	0,207
10	20	10	15	9	5	0	3,328	2,998	11,018	0,330
10	20	10	15	9	5	5	2,458	2,198	11,803	0,259
10	20	10	15	9	5	10	1,944	1,810	7,424	0,134
10	20	10	15	9	5	15	1,641	1,673	1,932	0,032
10	20	10	15	9	5	20	1,462	1,575	7,197	0,113
10	20	10	15	9	10	0	2,888	2,611	10,597	0,277
10	20	10	15	9	10	5	1,996	1,845	8,186	0,151
10	20	10	15	9	10	10	1,470	1,133	29,761	0,337
10	20	10	15	9	10	15	1,159	1,066	8,756	0,093
10	20	10	15	9	10	20	0,975	0,942	3,535	0,033
10	20	10	15	9	20	0	2,372	2,003	18,410	0,369
10	20	10	15	9	20	5	1,502	1,324	13,434	0,178
10	20	10	15	9	20	10	0,988	0,832	18,788	0,156
10	20	10	15	9	20	15	0,685	0,600	14,125	0,085
10	20	10	15	9	20	20	0,506	0,530	4,457	0,024
20	10	10	15	9	0	0	6,479	6,376	1,614	0,103
20	10	10	15	9	0	5	6,479	6,145	5,439	0,334
20	10	10	15	9	0	10	6,479	6,457	0,332	0,021
20	10	10	15	9	0	15	6,479	6,798	4,702	0,320
20	10	10	15	9	0	20	6,479	6,320	2,503	0,158
20	10	10	15	9	5	0	4,589	4,743	3,236	0,153
20	10	10	15	9	5	5	4,584	4,458	2,842	0,127
20	10	10	15	9	5	10	4,584	4,546	0,835	0,038
20	10	10	15	9	5	15	4,584	4,431	3,457	0,153
20	10	10	15	9	5	20	4,584	4,293	6,780	0,291

Table A.1 Errors of Expected Backorder for Several Parameter Sets (Continued)

20	10	10	15	9	10	0	3,488	3,190	9,358	0,298
20	10	10	15	9	10	5	3,466	3,515	1,386	0,049
20	10	10	15	9	10	10	3,466	3,090	12,142	0,375
20	10	10	15	9	10	15	3,466	3,463	0,075	0,003
20	10	10	15	9	10	20	3,466	3,143	10,257	0,322
20	10	10	15	9	20	0	2,505	2,420	3,531	0,085
20	10	10	15	9	20	5	2,417	2,221	8,808	0,196
20	10	10	15	9	20	10	2,415	2,421	0,252	0,006
20	10	10	15	9	20	15	2,415	2,420	0,195	0,005
20	10	10	15	9	20	20	2,415	2,074	16,457	0,341

APPENDIX F
GRAPHS FOR APPROXIMATION ERRORS %





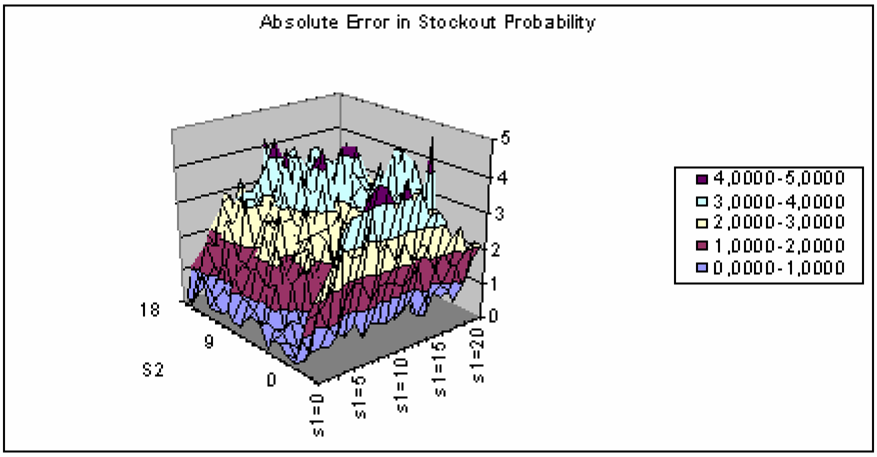


Figure A.3 $\mu_0 = 10,\ \ \mu_1 = 10,\ \ \mu_2 = 10,\ \ \lambda = 9,\ \ S_0 = 5$

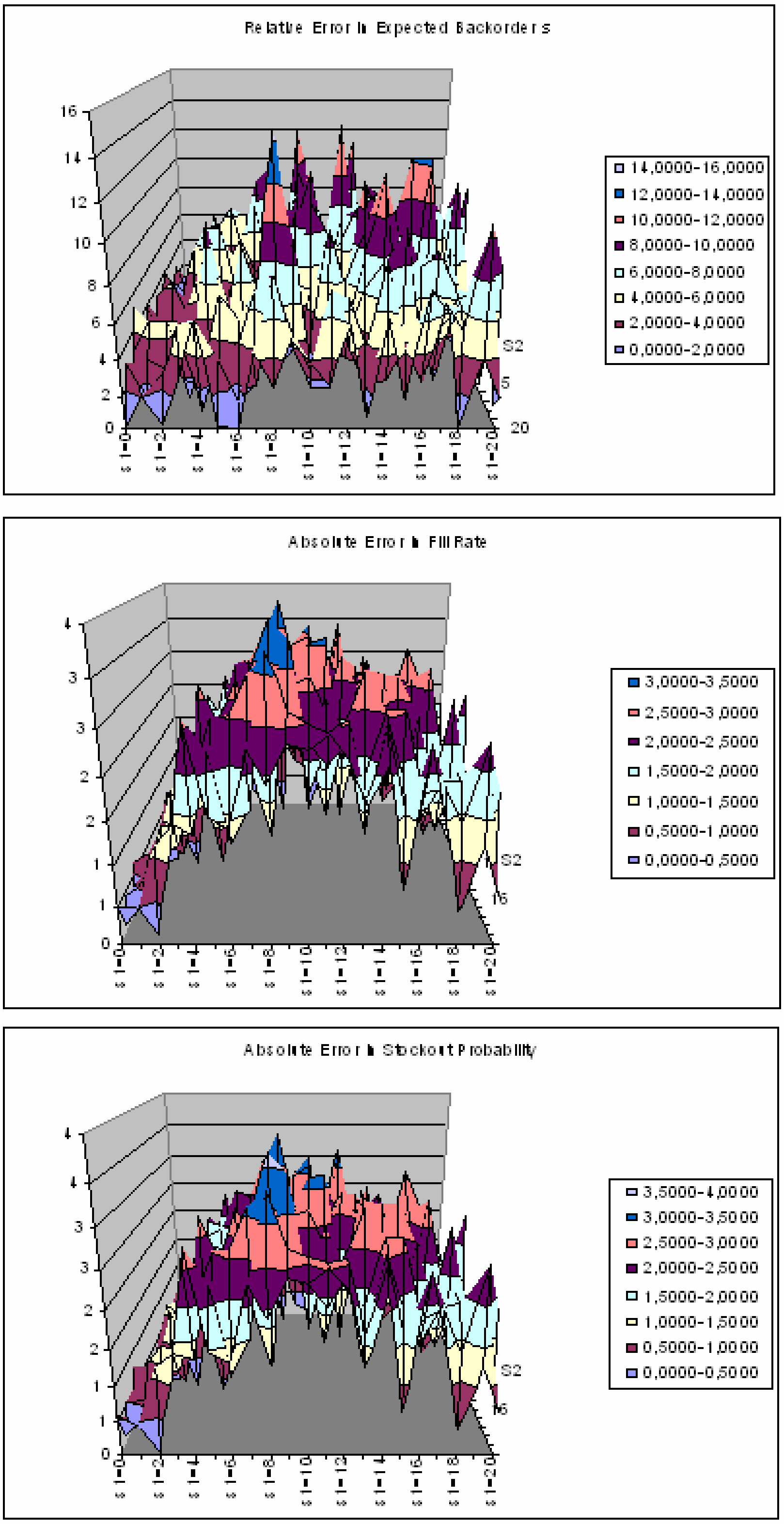


Figure A.4 $\mu_0 = 10,\ \ \mu_1 = 10,\ \ \mu_2 = 20,\ \ \lambda = 9,\ \ S_0 = 5$

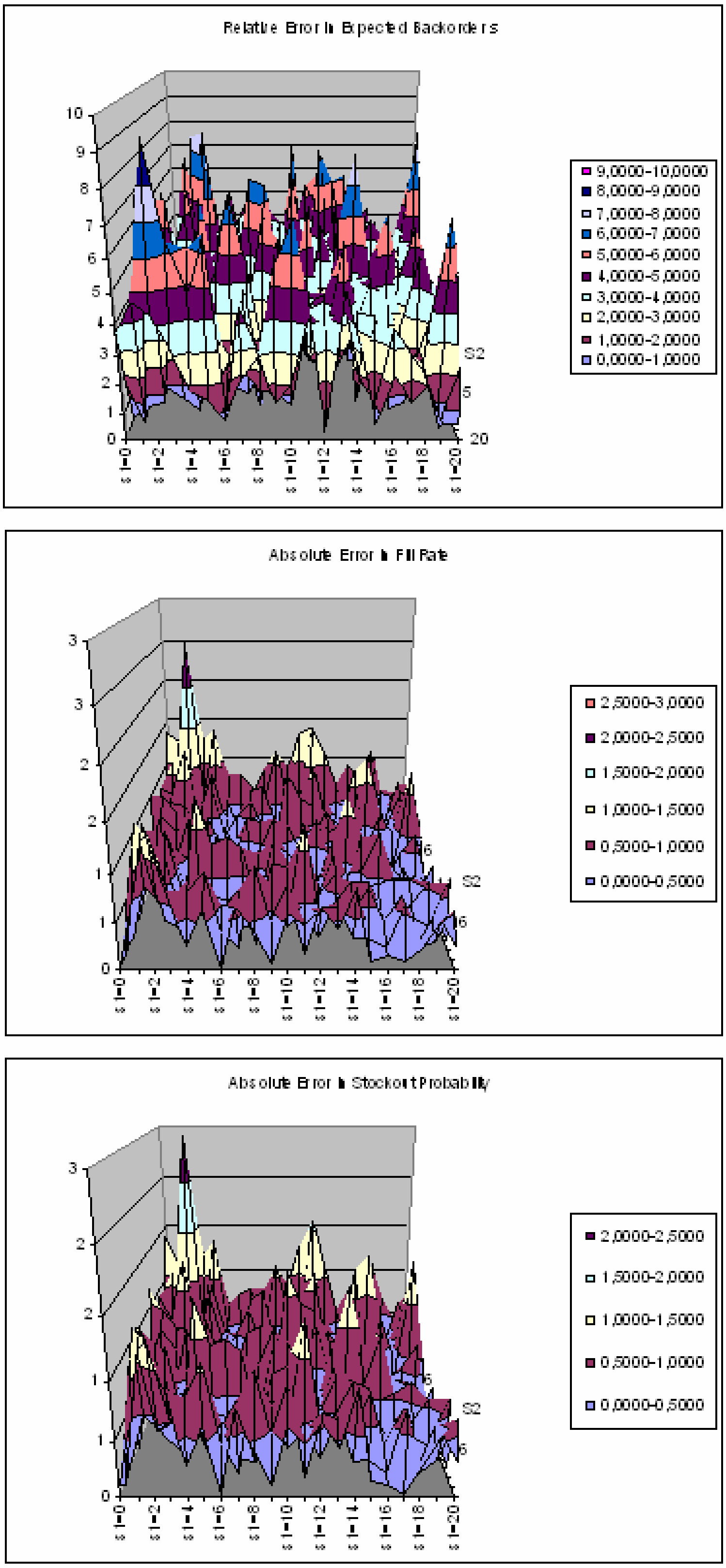


Figure A.5 $\mu_0 = 10, \ \ \mu_1 = 20, \ \ \mu_2 = 20, \ \ \lambda = 9, \ \ S_0 = 5$

APPENDIX G

GRAPHS FOR PERFORMANCE MEASURRES

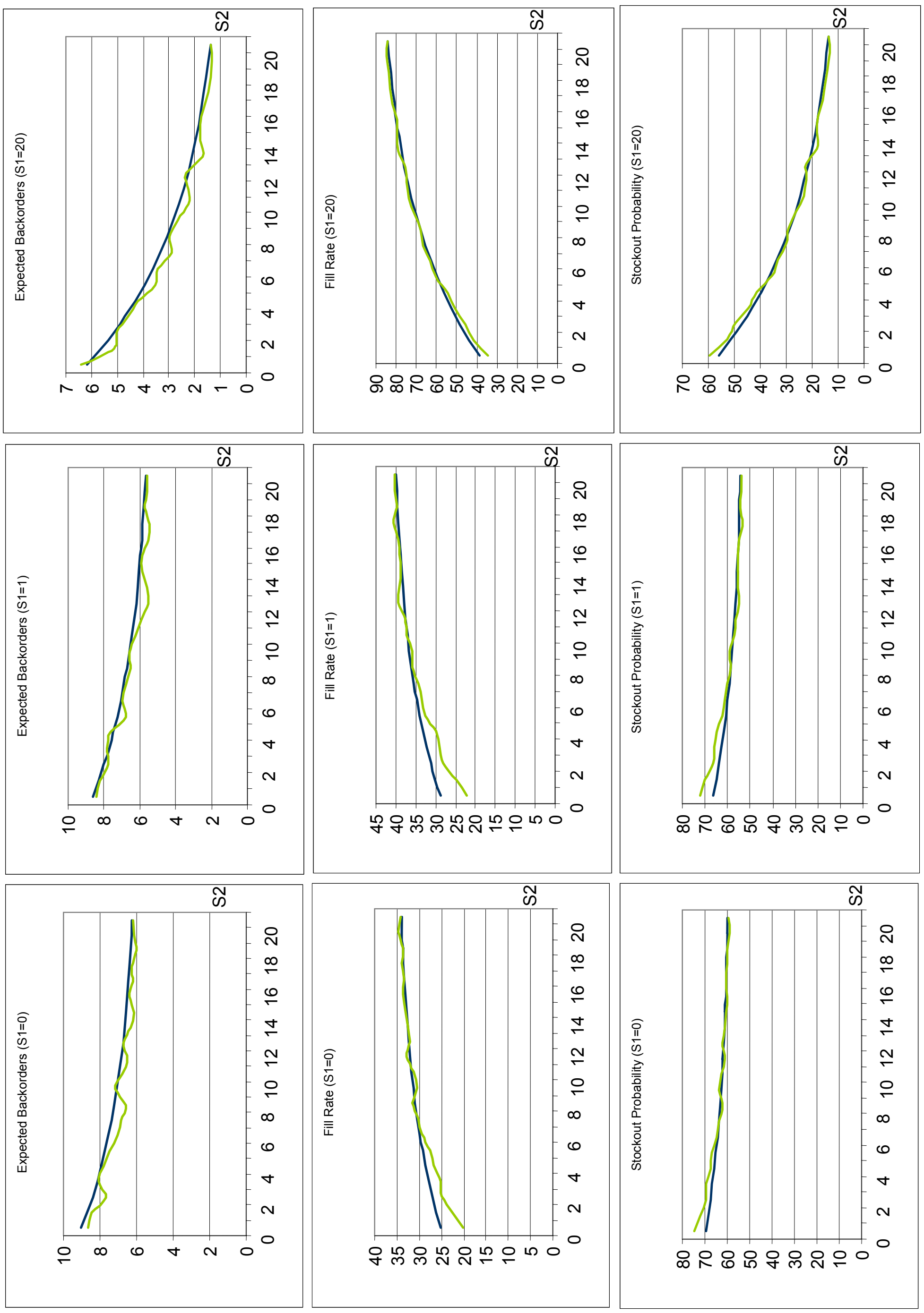
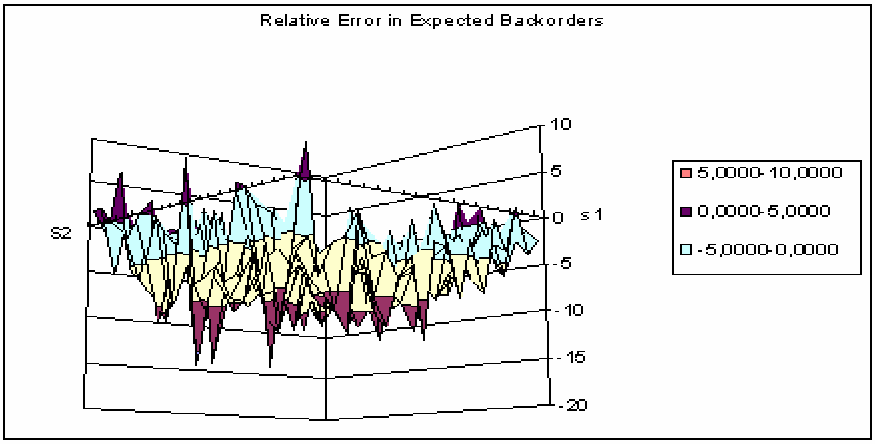
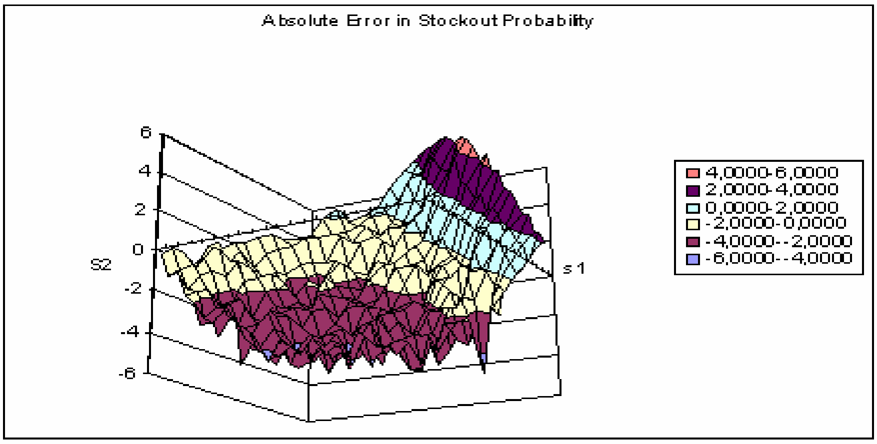


Figure A.6 Values of performance measures for $\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9,\ S_0 = 5$

APPENDIX H

GRAPHS FOR APPROXIMATION ERRORS %





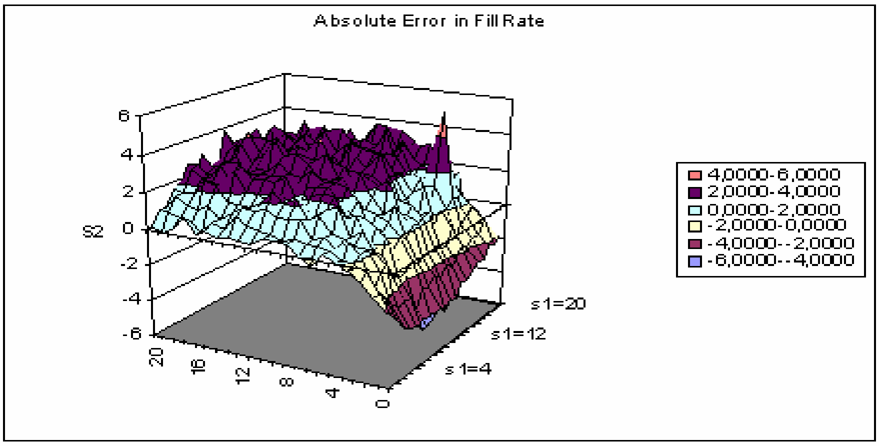


Figure A.7 $\mu_0 = \mu_1 = \mu_2 = 10, \ \lambda = 9, \ S_0 = 5$

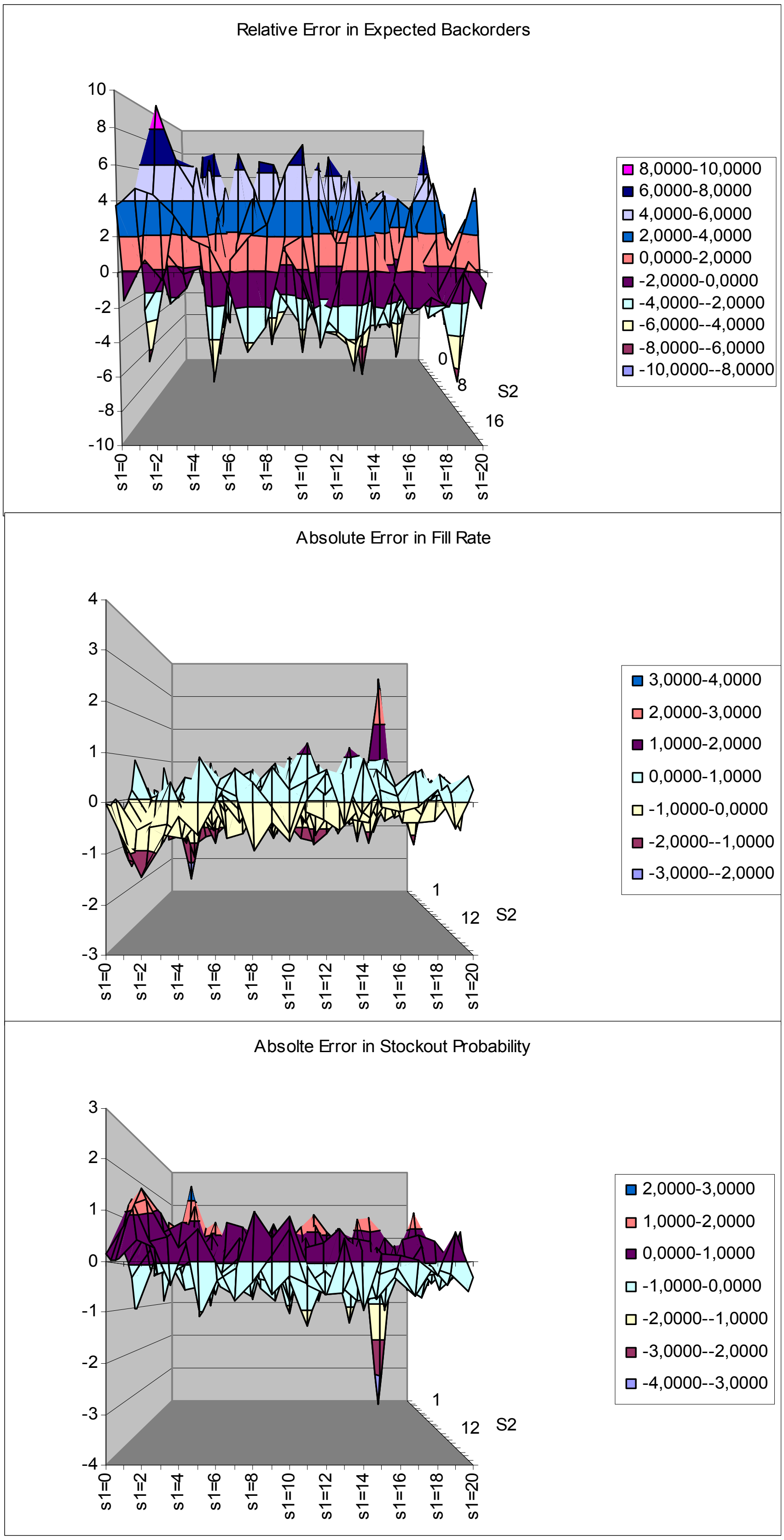


Figure A.8 $\mu_0 = 10,\ \mu_1 = 20,\ \mu_2 = 20,\ \lambda = 9,\ S_0 = 5$

APPENDIX I

95% CONFIDENCE INTERVALS FOR PERFORMANCE MEASURES

Table A.2.A Confidence Intervals for SP, $\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9\ ,\ S_0 = 5$

S1	S2	mean	stdev	Lower Limit	UpperLimit	Approximate	Check
0	0	60,701	1,68	59,771	61,632	61,227	1
0	1	58,253	1,92	57,190	59,317	59,659	0
0	2	58,375	1,298	57,657	59,094	58,953	1
0	3	59,531	2,525	58,132	60,929	58,635	1
0	4	58,676	1,954	57,594	59,758	58,492	1
0	5	57,495	1,077	56,898	58,091	58,428	0
0	6	58,543	1,801	57,545	59,540	58,399	1
0	7	58,605	1,555	57,744	59,467	58,386	1
0	8	57,234	1,717	56,283	58,185	58,380	0
0	9	58,623	1,92	57,560	59,687	58,378	1
0	10	57,856	1,419	57,070	58,642	58,376	1
0	11	58,155	1,447	57,354	58,957	58,376	1
0	12	58,178	1,175	57,527	58,829	58,376	1
0	13	59,4	2,042	58,269	60,531	58,376	1
0	14	58,659	1,631	57,755	59,562	58,376	1
0	15	58,36	2,036	57,232	59,488	58,375	1
0	16	58,104	1,932	57,034	59,174	58,375	1
0	17	58,288	2,114	57,117	59,459	58,375	1
0	18	58,34	1,765	57,363	59,317	58,375	1
0	19	58,355	2,566	56,934	59,776	58,375	1
0	20	58,248	1,497	57,419	59,077	58,375	1
1	0	58,833	1,671	57,908	59,759	59,481	1
1	1	56,159	2,236	54,921	57,398	57,290	1
1	2	55,416	1,422	54,628	56,204	56,305	0
1	3	55,183	2,07	54,037	56,330	55,861	1
1	4	54,522	1,522	53,679	55,365	55,662	0
1	5	54,885	2,056	53,747	56,024	55,572	1
1	6	54,873	2,016	53,757	55,990	55,531	1
1	7	55,263	2,013	54,148	56,377	55,513	1
1	8	55,313	1,872	54,276	56,350	55,505	1
1	9	54,517	1,755	53,545	55,490	55,501	0
1	10	54,661	2,123	53,486	55,837	55,500	1
1	11	55,349	1,401	54,573	56,125	55,499	1

Table A.2.A Confidence Intervals for SP, $\mu_0 = \mu_1 = \mu_2 = 10$, $\lambda = 9$, $S_0 = 5$ (Continued)

1	12	54,673	1,293	53,957	55,389	55,498	0
1	13	55,618	1,941	54,543	56,693	55,498	1
1	14	55,233	1,964	54,145	56,320	55,498	1
1	15	54,89	1,557	54,028	55,752	55,498	1
1	16	54,435	2,515	53,042	55,828	55,498	1
1	17	55,005	1,753	54,034	55,976	55,498	1
1	18	54,197	2,017	53,080	55,314	55,498	0
1	19	54,375	1,623	53,477	55,274	55,498	0
1	20	54,811	2,007	53,699	55,922	55,498	1
2	0	58,507	1,511	57,671	59,344	58,831	1
2	1	54,691	1,573	53,820	55,563	56,286	0
2	2	52,707	1,645	51,796	53,617	55,140	0
2	3	54,076	1,448	53,274	54,878	54,625	1
2	4	53,429	1,512	52,592	54,267	54,393	0
2	5	52,989	1,58	52,114	53,863	54,289	0
2	6	53,595	1,697	52,655	54,535	54,242	1
2	7	54,193	2,578	52,766	55,621	54,221	1
2	8	53,655	1,921	52,591	54,719	54,211	1
2	9	53,415	1,04	52,839	53,991	54,207	0
2	10	53,546	2,447	52,191	54,901	54,205	1
2	11	53,855	1,889	52,808	54,901	54,204	1
2	12	54,097	2,238	52,857	55,336	54,204	1
2	13	53,538	1,974	52,445	54,631	54,204	1
2	14	53,3	1,604	52,412	54,188	54,204	0
2	15	53,507	1,816	52,501	54,512	54,203	1
2	16	54,417	2,021	53,297	55,536	54,203	1
2	17	53,861	1,751	52,891	54,830	54,203	1
2	18	53,769	1,125	53,146	54,393	54,203	1
2	19	52,948	1,689	52,013	53,883	54,203	0
2	20	52,792	2,265	51,538	54,046	54,203	0
3	0	58,952	1,383	58,186	59,718	58,571	1
3	1	54,632	2,358	53,326	55,938	55,848	1
3	2	54,093	1,711	53,146	55,041	54,623	1
3	3	53,683	1,796	52,688	54,678	54,072	1

Table A.2.B Confidence Intervals for FR, $\mu_0 = \mu_1 = \mu_2 = 10$, $\lambda = 9$, $S_0 = 5$

S1	S2	mean	stdev	Lower Limit	UpperLimit	Approximate	Check
0	0	32,724	1,457	31,917	33,531	32,221	1
0	1	35,257	1,656	34,340	36,175	33,876	0
0	2	35,123	1,124	34,500	35,745	34,621	1
0	3	34,187	2,145	32,998	35,375	34,956	1
0	4	34,951	1,7	34,010	35,893	35,107	1
0	5	36,026	0,928	35,512	36,540	35,175	0
0	6	35,076	1,642	34,167	35,985	35,206	1
0	7	35,046	1,373	34,285	35,807	35,219	1
0	8	36,205	1,539	35,353	37,058	35,226	0
0	9	35,024	1,732	34,065	35,983	35,228	1
0	10	35,643	1,231	34,961	36,324	35,230	1
0	11	35,451	1,253	34,757	36,145	35,230	1
0	12	35,409	0,982	34,865	35,953	35,230	1
0	13	34,341	1,767	33,363	35,320	35,231	1
0	14	35,031	1,466	34,219	35,843	35,231	1
0	15	35,266	1,851	34,241	36,291	35,231	1
0	16	35,436	1,693	34,499	36,373	35,231	1
0	17	35,267	1,863	34,236	36,299	35,231	1
0	18	35,195	1,531	34,348	36,043	35,231	1
0	19	35,284	2,348	33,984	36,584	35,231	1
0	20	35,259	1,298	34,540	35,978	35,231	1
1	0	34,649	1,473	33,833	35,465	34,092	1
1	1	37,693	1,917	36,632	38,755	36,449	0
1	2	38,514	1,335	37,774	39,254	37,509	0
1	3	38,768	1,872	37,731	39,805	37,986	1
1	4	39,363	1,393	38,591	40,134	38,201	0
1	5	39,081	1,838	38,063	40,098	38,298	1
1	6	39,09	1,819	38,083	40,097	38,341	1
1	7	38,765	1,758	37,791	39,739	38,361	1
1	8	38,656	1,707	37,710	39,602	38,370	1
1	9	39,379	1,55	38,521	40,238	38,374	0
1	10	39,229	1,905	38,174	40,284	38,375	1
1	11	38,645	1,245	37,956	39,335	38,376	1
1	12	39,215	1,117	38,597	39,834	38,377	0
1	13	38,401	1,729	37,444	39,359	38,377	1
1	14	38,729	1,751	37,759	39,698	38,377	1
1	15	38,987	1,347	38,242	39,733	38,377	1
1	16	39,482	2,31	38,203	40,761	38,377	1
1	17	38,987	1,569	38,118	39,856	38,377	1
1	18	39,713	1,82	38,705	40,721	38,377	0
1	19	39,55	1,484	38,728	40,372	38,377	0
1	20	39,117	1,806	38,117	40,117	38,377	1
2	0	35,091	1,333	34,353	35,829	34,770	1

Table A.2.B Confidence Intervals for FR, $\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9\ ,\ S_0 = 5$ (Continued)

2	1	39,093	1,479	38,274	39,912	37,532	0
2	2	41,283	1,486	40,460	42,106	38,775	0
2	3	40,001	1,356	39,250	40,752	39,335	1
2	4	40,583	1,337	39,843	41,324	39,587	0
2	5	41,037	1,43	40,245	41,829	39,700	0
2	6	40,45	1,557	39,588	41,312	39,751	1
2	7	39,899	2,324	38,612	41,187	39,774	1
2	8	40,405	1,692	39,467	41,342	39,784	1
2	9	40,603	1,041	40,027	41,180	39,789	0
2	10	40,453	2,186	39,242	41,663	39,791	1
2	11	40,197	1,724	39,242	41,152	39,792	1
2	12	39,941	2,038	38,812	41,069	39,792	1
2	13	40,522	1,854	39,495	41,549	39,792	1
2	14	40,745	1,483	39,924	41,567	39,793	0
2	15	40,535	1,609	39,644	41,426	39,793	1
2	16	39,665	1,828	38,652	40,677	39,793	1
2	17	40,207	1,55	39,349	41,066	39,793	1
2	18	40,336	1,066	39,746	40,926	39,793	1
2	19	41,086	1,606	40,197	41,975	39,793	0
2	20	41,259	2,007	40,147	42,370	39,793	0
3	0	34,685	1,231	34,003	35,367	35,034	1
3	1	39,277	2,187	38,066	40,488	38,002	0
3	2	39,97	1,465	39,159	40,781	39,337	1
3	3	40,403	1,616	39,508	41,298	39,938	1

Table A.2.C Confidence Intervals for EB, $\mu_0 = \mu_1 = \mu_2 = 10$, $\lambda = 9$, $S_0 = 5$

S1	S2	mean	stdev	Lower Limit	UpperLimit	Approximate	Check
0	0	6,104	0,757	5,685	6,523	6,143	1
0	1	5,851	0,68	5,475	6,228	5,978	1
0	2	5,743	0,407	5,517	5,969	5,903	1
0	3	6,277	0,792	5,839	6,716	5,870	1
0	4	6,067	0,88	5,579	6,554	5,855	1
0	5	5,491	0,436	5,249	5,733	5,848	0
0	6	5,949	0,633	5,599	6,300	5,845	1
0	7	6,032	0,723	5,632	6,432	5,844	1
0	8	5,482	0,449	5,233	5,730	5,843	0
0	9	5,916	0,733	5,510	6,321	5,843	1
0	10	5,604	0,566	5,291	5,918	5,843	1
0	11	5,539	0,592	5,211	5,867	5,843	1
0	12	5,798	0,58	5,476	6,119	5,843	1
0	13	6,223	0,804	5,778	6,669	5,843	1
0	14	6,027	0,536	5,730	6,324	5,843	1
0	15	5,755	0,587	5,430	6,081	5,843	1
0	16	5,644	0,69	5,262	6,027	5,843	1
0	17	5,843	0,433	5,603	6,083	5,843	1
0	18	5,839	0,757	5,420	6,259	5,843	1
0	19	5,953	0,844	5,485	6,420	5,843	1
0	20	5,666	0,528	5,374	5,958	5,843	1
1	0	5,616	0,477	5,352	5,880	5,962	0
1	1	5,672	0,982	5,128	6,216	5,736	1
1	2	5,626	0,48	5,360	5,892	5,634	1
1	3	5,604	0,846	5,136	6,073	5,589	1
1	4	5,345	0,458	5,092	5,599	5,568	1
1	5	5,524	0,781	5,091	5,957	5,559	1
1	6	5,45	0,612	5,110	5,789	5,555	1
1	7	5,537	0,739	5,127	5,946	5,553	1
1	8	5,517	0,558	5,208	5,826	5,552	1
1	9	5,251	0,594	4,922	5,580	5,552	1
1	10	5,392	0,535	5,096	5,689	5,551	1
1	11	5,631	0,512	5,348	5,915	5,551	1
1	12	5,299	0,534	5,004	5,595	5,551	1
1	13	5,68	0,526	5,389	5,971	5,551	1
1	14	5,682	0,727	5,280	6,085	5,551	1
1	15	5,364	0,54	5,065	5,664	5,551	1
1	16	5,386	0,861	4,909	5,863	5,551	1
1	17	5,467	0,526	5,176	5,759	5,551	1
1	18	5,604	0,771	5,177	6,031	5,551	1
1	19	5,441	0,521	5,153	5,730	5,551	1
1	20	5,247	0,515	4,962	5,532	5,551	0
2	0	5,903	0,625	5,557	6,249	5,893	1

Table A.2.C Confidence Intervals for EB, $\mu_0 = \mu_1 = \mu_2 = 10,\ \lambda = 9,\ S_0 = 5$ (Continued)

2	1	5,348	0,428	5,111	5,585	5,633	0
2	2	5,128	0,408	4,903	5,354	5,516	0
2	3	5,421	0,571	5,105	5,737	5,463	1
2	4	5,25	0,502	4,972	5,528	5,440	1
2	5	5,083	0,407	4,857	5,308	5,429	0
2	6	5,178	0,511	4,895	5,461	5,424	1
2	7	5,486	0,746	5,073	5,900	5,422	1
2	8	5,368	0,615	5,027	5,708	5,421	1
2	9	5,313	0,359	5,115	5,512	5,420	1
2	10	5,472	0,875	4,987	5,957	5,420	1
2	11	5,405	0,669	5,034	5,775	5,420	1
2	12	5,542	0,774	5,113	5,970	5,420	1
2	13	5,41	0,62	5,067	5,754	5,420	1
2	14	5,274	0,449	5,025	5,523	5,420	1
2	15	5,371	0,657	5,007	5,734	5,420	1
2	16	5,575	0,796	5,134	6,016	5,420	1
2	17	5,373	0,552	5,067	5,678	5,420	1
2	18	5,489	0,432	5,249	5,728	5,420	1
2	19	4,976	0,534	4,680	5,271	5,420	0
2	20	5,1	0,632	4,750	5,450	5,420	1
3	0	5,95	0,43	5,712	6,189	5,864	1
3	1	5,428	0,682	5,050	5,806	5,588	1
3	2	5,364	0,811	4,915	5,813	5,463	1
3	3	5,335	0,606	4,999	5,671	5,407	1

APPENDIX J

OPTIMIZATION WITH GREEDY HEURISTIC: TWO-COMPONENT CASE

Table A.4 Iterations for $\mu_0 = 20,\ \mu_1 = 10,\ \mu_2 = 20,\ \lambda = 9$, α =0.95, $\frac{c_0}{2} = c_1 = c_2$

S_0	S_1	S_2	FR(S_0, S_1, S_2)	FR(S_0+1, S_1, S_2)	FR(S_0, S_1+1, S_2)	FR(S_0, S_1, S_2+1)
4	999	999	0,95899			
4	0	0	0,28235	0,35226	**0,34996**	0,28237
4	1	0	0,34996	0,41514	**0,41075**	0,35001
4	2	0	0,41075	0,47169	**0,46539**	0,41086
4	3	0	0,46539	0,52253	**0,51445**	0,46559
4	4	0	0,51445	0,56821	**0,55844**	0,51479
4	5	0	0,55844	0,6092	**0,59781**	0,559
4	6	0	0,59781	0,64595	**0,63296**	0,59869
4	7	0	0,63296	0,67884	**0,66426**	0,63429
4	8	0	0,66426	0,70824	**0,69205**	0,66618
4	9	0	0,69205	0,73446	**0,71661**	0,6947
4	10	0	0,71661	0,7578	**0,73825**	0,72017
4	11	0	0,73825	**0,77853**	0,75722	0,74288
5	11	0	0,77853	0,80734	**0,7969**	0,78159
5	12	0	0,7969	0,82456	**0,81315**	0,80071
5	13	0	0,81315	0,83989	**0,82748**	0,8178
5	14	0	0,82748	**0,85355**	0,8401	0,83304
6	14	0	0,85355	0,87246	**0,86569**	0,85712
6	15	0	0,86569	0,8838	**0,87649**	0,86983
6	16	0	0,87649	0,89395	**0,88608**	0,8812
6	17	0	0,88608	0,90304	**0,89461**	0,89139
6	18	0	0,89461	**0,91116**	0,90218	0,9005
7	18	0	0,91116	0,92302	**0,91844**	0,91487
7	19	0	0,91844	0,9298	**0,92495**	0,92246
7	20	0	0,92495	0,93591	**0,93078**	0,92927
7	21	0	0,93078	**0,94139**	0,936	0,93538
8	21	0	0,94139	0,94908	**0,94633**	0,94432
8	22	0	0,94633	0,95365	**0,95077**	0,94938
8	23	0	0,95077			

Table A.5 Iterations for $\mu_0 = 20,\ \mu_1 = \mu_2 = 10,\ \lambda = 9$, α =0.95, $\frac{c_0}{2} = c_1 = c_2$

S_0	S_1	S_2	FR(S_0, S_1, S_2)	FR(S_0+1, S_1, S_2)	FR(S_0, S_1+1, S_2)	FR(S_0, S_1, S_2+1)
4	999	999	0,95899			
4	0	0	0,19676	0,25086	**0,2365**	0,20532
4	1	0	0,2365	0,28805	**0,26826**	0,24785
4	2	0	0,26826	0,31801	**0,29318**	0,28252
4	3	0	0,29318	**0,34179**	0,31233	0,31044
5	3	0	0,34179	**0,387**	0,36032	0,35989
6	3	0	0,387	**0,42942**	0,40465	0,40552
7	3	0	0,42942	**0,46933**	0,44608	0,44805
8	3	0	0,46933	**0,5069**	0,48499	0,48785
9	3	0	0,5069	0,54224	0,52157	**0,52513**
9	3	1	0,52513	**0,56005**	0,54143	0,54154
10	3	1	0,56005	**0,59271**	0,57517	0,57607
11	3	1	0,59271	0,62324	0,60673	**0,60827**
11	3	2	0,60827	0,63826	**0,62341**	0,62228
11	4	2	0,62341	0,65221	0,63642	**0,63841**
11	4	3	0,63841	**0,66661**	0,65239	0,65192
12	4	3	0,66661	**0,69277**	0,67952	0,67956
13	4	3	0,69277	0,71704	0,7047	**0,70515**
13	4	4	0,70515	0,72882	**0,71772**	0,71629
13	5	4	0,71772	0,74038	0,72879	**0,72943**
13	5	5	0,72943	0,75148	**0,74105**	0,73998
13	6	5	0,74105	0,76218	0,75132	**0,75208**
13	6	6	0,75208	0,77259	**0,76281**	0,76201
13	7	6	0,76281	0,78247	0,77233	**0,77316**
13	7	7	0,77316	0,79219	**0,78306**	0,78247
13	8	7	0,78306	0,80131	0,79187	**0,79272**
13	8	8	0,79272	0,81035	**0,80185**	0,80141
13	9	8	0,80185	0,81876	0,80999	**0,81083**
13	9	9	0,81083	0,82714	**0,81924**	0,81891
13	10	9	0,81924	0,83488	0,82675	**0,82756**
13	10	10	0,82756	0,84262	**0,8353**	0,83506
13	11	10	0,8353	0,84973	0,84222	**0,84299**
13	11	11	0,84299	0,85686	**0,8501**	0,84992
13	12	11	0,8501	0,86339	0,85646	**0,85719**
13	12	12	0,85719	0,86994	**0,8637**	0,86357
13	13	12	0,8637	0,87592	0,86954	**0,87022**
13	13	13	0,87022	0,88193	**0,87619**	0,87609
13	14	13	0,87619	0,8874	0,88154	**0,88217**
13	14	14	0,88217	0,8929	**0,88763**	0,88755
13	15	14	0,88763	0,8979	0,89252	**0,8931**
13	15	15	0,8931	0,90292	**0,89809**	0,89803
13	16	15	0,89809	0,90748	0,90256	**0,90309**
13	16	16	0,90309	0,91207	**0,90764**	0,9076
13	17	16	0,90764	0,91622	0,91172	**0,91221**
13	17	17	0,91221	0,9204	**0,91635**	0,91632

Table A.5 Iterations for $\mu_0 = 20,\ \mu_1 = \mu_2 = 10,\ \lambda = 9$, α=0.95, $\frac{c_0}{2} = c_1 = c_2$ (Continued)

13	18	17	0,91635	0,92418	0,92008	**0,92052**
13	18	18	0,92052	0,92798	**0,92429**	0,92427
13	19	18	0,92429	0,93142	0,92768	**0,92808**
13	19	19	0,92808	0,93488	**0,93151**	0,93149
13	20	19	0,93151	0,938	0,93459	**0,93495**
13	20	20	0,93495	0,94114	**0,93807**	0,93806
13	21	20	0,93807	0,94397	0,94087	**0,9412**
13	21	21	0,9412	0,94682	**0,94403**	0,94402
13	22	21	0,94403	0,94939	0,94657	**0,94687**
13	22	22	0,94687	0,95197	**0,94943**	0,94942
13	23	22	0,94943	0,9543	0,95174	**0,95201**
13	23	23	0,95201			

Table A.6 Enumeration for $\mu_0 = 20,\ \mu_1 = 10,\ \mu_2 = 20,\ \lambda = 9$ with investment $\leq$39 and FR $\geq$0.95

S_0	S_1	S_2	FR	Total Cost
8	23	0	0,95077	39
7	24	1	0,95024	39

Table A.7 Enumeration for $\mu_0 = 20,\ \mu_1 = 10,\ \mu_2 = 10,\ \lambda = 9$

with investment ≤ 72 and FR ≥ 0.95

S_0	S_1	S_2	FR	Total Cost
13	23	23	0,95201	72
12	24	24	0,952	72
14	22	22	0,95197	72
11	25	25	0,95194	72
15	21	21	0,9519	72
16	20	20	0,95179	72
10	26	26	0,95177	72
13	24	22	0,95174	72
13	22	24	0,95173	72
12	25	23	0,95173	72
12	23	25	0,95172	72
14	23	21	0,9517	72
14	21	23	0,95169	72
11	26	24	0,95167	72
11	24	26	0,95166	72
17	19	19	0,95165	72
15	20	22	0,95162	72
15	22	20	0,95162	72
16	19	21	0,95152	72
16	21	19	0,95152	72
10	27	25	0,9515	72
10	25	27	0,95148	72
18	18	18	0,95147	72
17	18	20	0,95138	72
17	20	18	0,95137	72
9	27	27	0,95137	72
19	17	17	0,95124	72
18	17	19	0,9512	72
18	19	17	0,95118	72
9	28	26	0,9511	72
9	26	28	0,95109	72
19	16	18	0,95098	72
20	16	16	0,95096	72
19	18	16	0,95095	72
13	25	21	0,95091	72
12	26	22	0,95091	72
13	21	25	0,95089	72
12	22	26	0,95089	72
14	24	20	0,95087	72
14	20	24	0,95086	72
11	27	23	0,95084	72
11	23	27	0,95082	72

Table A.7 Enumeration for $\mu_0 = 20, \ \mu_1 = 10, \ \mu_2 = 10, \ \lambda = 9$

with investment ≤ 72 and FR ≥ 0.95 (Continued)

15	23	19	0,9508	72
15	19	23	0,95079	72
20	15	17	0,95072	72
16	18	22	0,95069	72
16	22	18	0,95069	72
10	28	24	0,95067	72
20	17	15	0,95066	72
10	24	28	0,95065	72
21	15	15	0,95064	72
17	17	21	0,95055	72
17	21	17	0,95054	72
8	28	28	0,95051	72
21	14	16	0,9504	72
18	16	20	0,95038	72
18	20	16	0,95034	72
21	16	14	0,95032	72
9	29	25	0,95028	72
9	25	29	0,95026	72
22	14	14	0,95025	72
8	29	27	0,95024	72
8	27	29	0,95023	72
19	15	19	0,95017	72
19	19	15	0,9501	72
22	13	15	0,95004	72

APPENDIX K
NUMERICAL RESULTS FOR REMARK 4.1

Table A.8 Numerical Results for Remark 4.1, case $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 10$, $S_0 = S_1 = S_2 = S_3 = 5$, $\lambda = 9$.

	$\Pr(K_{12} = k_{12})$		
K_{12}	Results From Figure 4.2	Results From Theorem 3.1	Error
0	0,32624	0,32631	0,00007
1	0,05078	0,05124	0,00045
2	0,04696	0,04772	0,00077
3	0,04342	0,04440	0,00099
4	0,04014	0,04127	0,00113
5	0,03712	0,03832	0,00120
6	0,03432	0,03555	0,00123
7	0,03173	0,03294	0,00121
8	0,02934	0,03051	0,00117
9	0,02713	0,02823	0,00110
10	0,02509	0,02610	0,00101
11	0,02319	0,02412	0,00092
12	0,02145	0,02227	0,00082
13	0,01983	0,02055	0,00072
14	0,01834	0,01895	0,00061
15	0,01695	0,01746	0,00051
16	0,01568	0,01609	0,00041
17	0,01449	0,01481	0,00031
18	0,01340	0,01363	0,00023
19	0,01239	0,01253	0,00014
20	0,01146	0,01152	0,00006
21	0,01059	0,01059	-0,00001
22	0,00980	0,00973	-0,00007
23	0,00906	0,00893	-0,00013
24	0,00837	0,00820	-0,00018
25	0,00774	0,00752	-0,00022
26	0,00716	0,00690	-0,00026
27	0,00662	0,00632	-0,00030
28	0,00612	0,00579	-0,00033
29	0,00566	0,00531	-0,00035
30	0,00523	0,00486	-0,00037
31	0,00484	0,00445	-0,00039
32	0,00447	0,00407	-0,00040
33	0,00414	0,00373	-0,00041

Table A.8 Numerical Results for Remark 4.1 (Continued)

34	0,00382	0,00341	-0,00041
35	0,00354	0,00312	-0,00042
36	0,00327	0,00285	-0,00042
37	0,00302	0,00261	-0,00042
38	0,00280	0,00238	-0,00041
39	0,00258	0,00218	-0,00041
40	0,00239	0,00199	-0,00040
41	0,00221	0,00182	-0,00039
42	0,00204	0,00166	-0,00038
43	0,00189	0,00151	-0,00037
44	0,00175	0,00138	-0,00036
45	0,00162	0,00126	-0,00035
46	0,00149	0,00115	-0,00034
47	0,00138	0,00105	-0,00033
48	0,00128	0,00096	-0,00032
49	0,00118	0,00087	-0,00031
50	0,00109	0,00080	-0,00030
51	0,00101	0,00073	-0,00028
52	0,00093	0,00066	-0,00027
53	0,00086	0,00060	-0,00026
54	0,00080	0,00055	-0,00025
55	0,00074	0,00050	-0,00024
56	0,00068	0,00046	-0,00023
57	0,00063	0,00042	-0,00022
58	0,00058	0,00038	-0,00021
59	0,00054	0,00034	-0,00019
60	0,00050	0,00031	-0,00019
61	0,00046	0,00029	-0,00018
62	0,00043	0,00026	-0,00017
63	0,00039	0,00024	-0,00016
64	0,00036	0,00022	-0,00015
65	0,00034	0,00020	-0,00014
66	0,00031	0,00018	-0,00013
67	0,00029	0,00016	-0,00013
68	0,00027	0,00015	-0,00012
69	0,00025	0,00013	-0,00011
70	0,00023	0,00012	-0,00011
71	0,00021	0,00011	-0,00010
72	0,00019	0,00010	-0,00009
73	0,00018	0,00009	-0,00009
74	0,00017	0,00008	-0,00008
75	0,00015	0,00008	-0,00008
76	0,00014	0,00007	-0,00007
77	0,00013	0,00006	-0,00007
78	0,00012	0,00006	-0,00006
79	0,00011	0,00005	-0,00006
80	0,00010	0,00005	-0,00006
81	0,00010	0,00004	-0,00005

Table A.8 Numerical Results for Remark 4.1 (Continued)

82	0,00009	0,00004	-0,00005
83	0,00008	0,00004	-0,00005
84	0,00008	0,00003	-0,00004
85	0,00007	0,00003	-0,00004
86	0,00006	0,00003	-0,00004
87	0,00006	0,00002	-0,00004
88	0,00006	0,00002	-0,00003
89	0,00005	0,00002	-0,00003
90	0,00005	0,00002	-0,00003
91	0,00004	0,00002	-0,00003
92	0,00004	0,00001	-0,00003
93	0,00004	0,00001	-0,00002
94	0,00003	0,00001	-0,00002
95	0,00003	0,00001	-0,00002
96	0,00003	0,00001	-0,00002
97	0,00003	0,00001	-0,00002
98	0,00003	0,00001	-0,00002
99	0,00002	0,00001	-0,00002
100	0,00002	0,00001	-0,00001

APPENDIX L
APPROXIMATE PERFORMANCE MEASURES FOR INCREASING NUMBER OF COMPONENTS

Table A.9 Performance measures of the two-component assembly system

							Simulation			Approximation			Error (%)		
μ_0	μ_1	μ_2	S_0	S_1	S_2	λ	FR	SP	EB	FR	SP	EB	FR	SP	EB
10	10	10	5	0	0	9	6,091	91,503	16,867	7,995	89,354	16,630	1,904	2,149	1,401
10	10	10	5	5	5	9	19,059	77,203	12,069	17,674	78,600	12,838	1,385	1,397	6,371
10	15	15	5	0	0	9	26,812	66,672	6,771	26,423	67,206	6,899	0,389	0,534	1,890
10	15	15	5	5	5	9	40,493	53,606	5,308	39,408	54,607	5,471	1,085	1,001	3,063
15	15	15	5	0	0	9	69,296	21,661	0,665	67,999	23,149	0,759	1,297	1,488	14,179
15	15	15	5	5	5	9	90,511	5,949	0,160	89,858	6,382	0,171	0,653	0,433	6,925
15	10	15	5	5	5	9	56,097	38,956	3,878	55,832	39,069	3,803	0,265	0,113	1,936
15	10	15	5	0	0	9	29,405	63,876	6,308	30,604	62,934	6,364	1,199	0,942	0,888
15	20	20	5	5	5	9	92,090	4,740	0,119	91,962	4,835	0,121	0,128	0,095	1,798
												AVG	**0,923**	**0,906**	**4,272**

Table A.10 Performance measures of the three-component assembly system

									Simulation			Approximation			Error (%)		
μ_0	μ_1	μ_2	μ_3	S_0	S_1	S_2	S_3	λ	FR	SP	EB	FR	SP	EB	FR	SP	EB
10	10	10	10	5	0	0	0	9	4,302	93,785	18,325	6,322	91,501	18,957	2,020	2,284	3,450
10	10	10	10	5	5	5	5	9	15,053	81,715	14,066	14,460	82,316	14,915	0,593	0,601	6,039
10	15	15	15	5	0	0	0	9	24,631	68,836	7,027	23,716	70,075	7,299	0,915	1,239	3,870
10	15	15	15	5	5	5	5	9	40,137	53,898	5,388	38,710	55,275	5,544	1,427	1,377	2,902
15	15	15	15	5	0	0	0	9	64,278	25,724	0,820	62,112	28,410	1,004	2,166	2,686	22,464
15	15	15	15	5	5	5	5	9	89,867	6,542	0,175	88,750	7,203	0,198	1,117	0,661	13,540
15	10	15	20	5	5	5	5	9	55,865	39,159	3,784	55,830	39,070	3,803	0,035	0,089	0,507
15	10	15	20	5	0	0	0	9	29,066	64,135	6,345	30,600	62,938	6,364	1,534	1,197	0,304
15	20	20	20	5	5	5	5	9	91,995	4,807	0,120	91,832	4,920	0,123	0,163	0,113	2,997
														AVG	**1,108**	**1,139**	**6,230**

Table A.11 Performance measures of the four-component assembly system

											Simulation			Approximation			Error (%)		
μ_0	μ_1	μ_2	μ_3	μ_4	S_0	S_1	S_2	S_3	S_4	λ	FR	SP	EB	FR	SP	EB	FR	SP	EB
10	10	10	10	10	5	0	0	0	0	9	3,147	95,333	19,743	5,292	92,837	20,750	2,145	2,496	5,104
10	10	10	10	10	5	5	5	5	5	9	12,536	84,628	15,046	12,363	84,767	16,566	0,173	0,139	10,101
10	15	15	15	15	5	0	0	0	0	9	23,073	70,371	7,083	21,768	72,188	7,618	1,305	1,817	7,557
10	15	15	15	15	5	5	5	5	5	9	39,783	54,341	5,435	38,052	55,906	5,614	1,731	1,565	3,303
15	15	15	15	15	5	0	0	0	0	9	60,766	28,665	0,934	57,713	32,491	1,213	3,053	3,826	30,045
15	15	15	15	15	5	5	5	5	5	9	89,295	6,850	0,188	87,688	8	0,225	1,607	1,150	19,846
15	10	15	20	20	5	5	5	5	5	9	56,661	38,349	3,701	55,829	39,072	3,803	0,832	0,723	2,760
15	10	15	20	20	5	0	0	0	0	9	29,023	64,107	6,352	30,596	62,943	6,364	1,573	1,164	0,203
15	20	20	20	20	5	5	5	5	5	9	91,944	4,845	0,121	91,703	5,004	0,125	0,241	0,159	3,602
																AVG	**1,407**	**1,449**	**9,169**

Table A.12 Performance measures of the five-component assembly system

													Simulation			Approximation			Error (%)		
μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	S_0	S_1	S_2	S_3	S_4	S_5	λ	SP	FR	EB	SP	FR	EB	SP	FR	EB
10	10	10	10	10	10	5	0	0	0	0	0	9	93,757	4,588	22,200	96,208	2,480	21,203	2,451	2,108	4,702
10	10	10	10	10	10	5	5	5	5	5	5	9	86,524	10,873	17,232	86,243	11,135	15,918	0,281	0,262	8,255
10	15	15	15	15	15	5	0	0	0	0	0	9	73,846	20,267	7,886	71,637	21,891	7,423	2,209	1,624	6,234
10	15	15	15	15	15	5	5	5	5	5	5	9	56,505	37,432	5,682	54,270	39,820	5,526	2,235	2,388	2,815
15	15	15	15	15	15	5	0	0	0	0	0	9	35,805	54,237	1,369	31,017	57,972	1,024	4,788	3,735	33,718
15	15	15	15	15	15	5	5	5	5	5	5	9	8,773	86,669	0,241	7,408	88,549	0,207	1,365	1,880	16,459
15	10	10	15	15	20	5	5	5	5	5	5	9	52,376	42,766	6,315	53,062	41,935	5,786	0,686	0,831	9,146
15	10	10	15	15	20	5	0	0	0	0	0	9	77,181	17,715	9,572	78,297	16,443	9,327	1,116	1,272	2,623
15	20	20	20	20	20	5	5	5	5	5	5	9	5,088	91,575	0,128	4,893	91,857	0,122	0,195	0,282	4,446
																		AVG	**1,703**	**1,598**	**9,822**

Table A.13 Performance measures of the six-component assembly system

															Simulation			Approximation			Error (%)		
μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	S_0	S_1	S_2	S_3	S_4	S_5	S_6	λ	FR	SP	EB	FR	SP	EB	FR	SP	EB
10	10	10	10	10	10	10	5	0	0	0	0	0	0	9	1,876	97,082	21,644	4,065	94,445	23,427	2,189	2,637	8,238
10	10	10	10	10	10	10	5	5	5	5	5	5	5	9	9,252	88,391	17,065	9,743	87,866	19,103	0,491	0,525	11,944
10	15	15	15	15	15	15	5	0	0	0	0	0	0	9	20,519	73,111	7,594	19,058	75,201	8,117	1,461	2,090	6,882
10	15	15	15	15	15	15	5	5	5	5	5	5	5	9	39,643	54,407	5,426	36,846	57,073	5,746	2,797	2,666	5,901
15	15	15	15	15	15	15	5	0	0	0	0	0	0	9	55,152	33,577	1,233	51,385	38,579	1,564	3,767	5,002	26,808
15	15	15	15	15	15	15	5	5	5	5	5	5	5	9	88,053	7,858	0,234	85,691	9,523	0,279	2,362	1,665	19,081
15	10	10	15	15	20	20	5	5	5	5	5	5	5	9	40,911	54,154	6,027	42,766	52,376	6,315	1,855	1,778	4,781
15	10	10	15	15	20	20	5	0	0	0	0	0	0	9	20,861	73,682	9,590	21,715	73,181	9,572	0,854	0,501	0,191
15	20	20	20	20	20	20	5	5	5	5	5	5	5	9	91,900	4,885	0,123	91,447	5,171	0,130	0,453	0,286	5,789
																				AVG	**1,803**	**1,906**	**9,957**

APPENDIX M

APPROXIMATION PERFORMANCE

FOR INCREASING NUMBER OF COMPONENTS

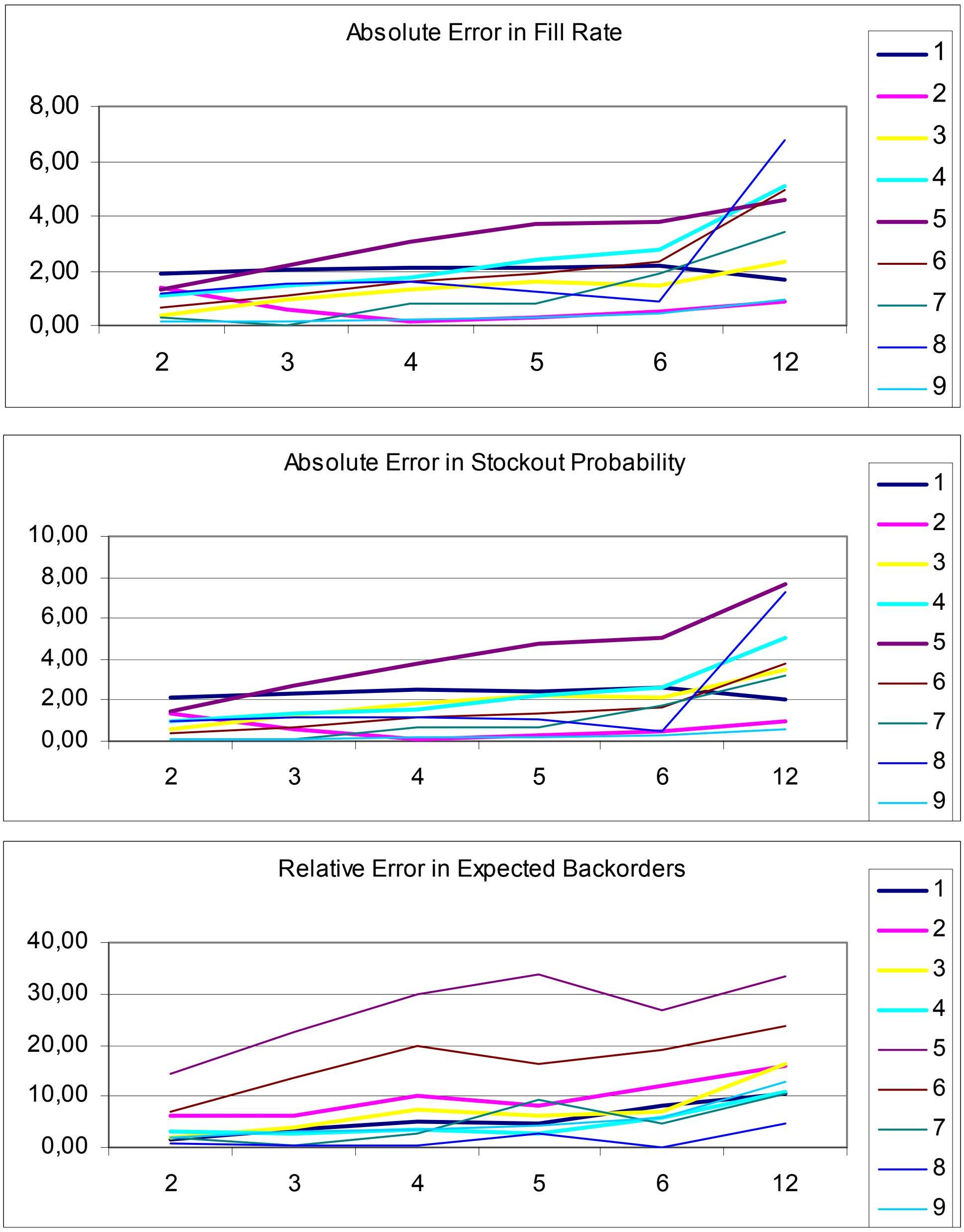


Figure A.9 Graphs of Approximation Errors (%)

for Increasing Number of Components

APPENDIX N

APPROXIMATION PERFORMANCE WITH DIFFERENT SEQUENCES TO PICK UP THE COMPONENTS

Table A.14 Effect of the sequence the components are picked up

										Simulation			Approximation			Error (%)		
	μ_0	μ_1	μ_2	μ_3	S_0	S_1	S_2	S_3	λ	FR	SP	EB	FR	SP	EB	FR	SP	EB
1	15	10	15	20	5	0	0	0	9	28,928	64,373	6,558	30,600	62,938	6,364	1,672	1,435	2,958
	15	20	15	10	5	0	0	0	9	28,928	64,373	6,558	29,013	64,478	6,533	0,085	0,105	0,381
2	15	10	15	20	5	5	5	5	9	55,557	39,462	3,901	55,830	39,070	3,803	0,273	0,392	2,511
	15	20	15	10	5	5	5	5	9	55,557	39,462	3,901	55,361	39,424	3,820	0,196	0,038	2,076
3	15	10	15	20	5	10	10	10	9	70,197	25,648	2,382	70,754	24,963	2,292	0,557	0,685	3,795
	15	20	15	10	5	10	10	10	9	70,197	25,648	2,382	70,709	24,995	2,293	0,512	0,653	3,753
4	10	10	15	20	5	0	0	0	9	10,503	85,973	13,871	11,386	85,077	13,282	0,883	0,896	4,246
	10	20	15	10	5	0	0	0	9	10,503	85,973	13,871	10,680	85,859	13,490	0,177	0,114	2,747
5	10	10	15	20	5	5	5	5	9	24,887	70,655	9,887	23,483	72,008	10,017	1,404	1,353	1,315
	10	20	15	10	5	5	5	5	9	24,887	70,655	9,887	23,183	72,295	10,054	1,704	1,640	1,689
6	10	10	15	20	5	10	10	10	9	32,680	62,229	7,805	30,647	64,272	8,088	2,033	2,043	3,626
	10	20	15	10	5	10	10	10	9	32,680	62,229	7,805	30,617	64,300	8,091	2,063	2,071	3,664
7	10	10	15	20	5	0	5	10	9	11,435	84,983	13,109	11,423	85,034	13,269	0,012	0,051	1,221
	10	20	15	10	5	10	5	0	9	11,435	84,983	13,109	11,481	84,991	13,278	0,046	0,008	1,289
8	10	10	10	10	5	0	5	10	9	9,054	88,018	16,375	8,606	88,594	16,196	0,448	0,576	1,093
	10	10	10	10	5	10	5	0	9	9,054	88,018	16,375	13,690	83,240	15,605	4,636	4,778	4,702
9	10	15	15	15	5	0	5	10	9	30,661	62,891	6,445	30,339	63,205	6,398	0,322	0,314	0,729
	10	15	15	15	5	10	5	0	9	30,661	62,891	6,445	30,560	63,024	6,385	0,101	0,133	0,931

APPENDIX O

OPTIMIZATION WITH GREEDY HEURISTIC: THREE-COMPONENT CASE

Table A.15 Iterations for $\mu_0 = 20,\ \mu_1 = \mu_2 = 15,\ \mu_3 = 20,\ \lambda = 9,\ \alpha = 0.95,\ \frac{c_0}{3} = c_1 = c_2 = c_3$

S_0	S_1	S_2	S_3	FR (S_0, S_1, S_2, S_3)	FR (S_0+1, S_1, S_2, S_3)	FR (S_0, S_1+1, S_2, S_3)	FR (S_0, S_1, S_2+1, S_3)	FR (S_0, S_1, S_2, S_3+1)
4	999	999	999	0,95899				
4	0	0	0	0,63981	0,74729	**0,68188**	0,67552	0,65134
4	1	0	0	0,68188	0,77857	0,70265	**0,72648**	0,69658
4	1	1	0	0,72648	**0,81428**	0,75352	0,75192	0,74601
5	1	1	0	0,81428	0,87562	**0,83524**	0,83511	0,82914
5	2	1	0	0,83524	0,89111	0,8475	**0,85792**	0,85223
5	2	2	0	0,85792	0,90811	0,87135	0,87131	**0,87749**
5	2	2	1	0,87749	0,92141	**0,89257**	0,89255	0,8863
5	3	2	1	0,89257	0,93225	0,90157	**0,9085**	0,90212
5	3	3	1	0,9085	**0,94361**	0,91805	0,91805	0,91884
6	3	3	1	0,94361	0,96523	0,95044	**0,95048**	0,95022
6	3	4	1	0,95048				

Table A.16 Enumeration for $\mu_0 = 20,\ \mu_1 = \mu_2 = 15,\ \mu_3 = 20,\ \lambda = 9$ with investment ≤ 28 and FR ≥ 0.95.

S_0	S_1	S_2	S_3	FR	Total Cost
6	3	4	1	0,95048	26
6	4	3	1	0,95044	26
6	3	3	2	0,95022	26
7	2	3	1	0,9576	27
6	4	4	1	0,95746	27
7	3	2	1	0,95737	27
6	3	4	2	0,95723	27
6	4	3	2	0,95719	27
7	3	3	0	0,95599	27
6	3	5	1	0,95462	27
6	5	3	1	0,95459	27
8	1	2	0	0,95455	27
7	2	2	2	0,9538	27
8	2	1	0	0,95364	27
6	3	3	3	0,95319	27
7	2	4	0	0,95308	27
7	4	2	0	0,95287	27
5	5	5	2	0,95184	27
8	1	1	1	0,9512	27
5	4	5	3	0,95069	27
5	5	4	3	0,95069	27
7	3	3	1	0,96523	28
6	4	4	2	0,96432	28
7	2	4	1	0,96231	28
8	2	2	0	0,96226	28
7	4	2	1	0,96209	28
6	4	5	1	0,96172	28
6	5	4	1	0,96171	28
7	2	3	2	0,9617	28
7	3	2	2	0,96147	28
6	3	5	2	0,96145	28
6	5	3	2	0,96142	28
7	3	4	0	0,96077	28
7	4	3	0	0,96072	28
6	3	4	3	0,96028	28
6	4	3	3	0,96023	28
8	1	2	1	0,96011	28
8	1	3	0	0,95981	28
8	2	1	1	0,95924	28

Table A.16 Enumeration for $\mu_0 = 20, \ \mu_1 = \mu_2 = 15, \ \mu_3 = 20,$ $\lambda = 9$ with investment $\leq$28 and FR $\geq$0.95. (Continued)

8	3	1	0	0,95888	28
9	0	1	0	0,95767	28
5	5	5	3	0,95717	28
6	3	6	1	0,95712	28
6	6	3	1	0,95709	28
7	2	5	0	0,95588	28
7	5	2	0	0,95572	28
5	5	6	2	0,95566	28
5	6	5	2	0,95566	28
7	2	2	3	0,95559	28
9	1	0	0	0,95489	28
6	3	3	4	0,95453	28
5	4	6	3	0,9545	28
5	6	4	3	0,9545	28
7	1	4	2	0,95393	28
8	1	1	2	0,95357	28
7	4	1	2	0,95324	28
8	0	3	1	0,95315	28
5	4	5	4	0,95303	28
5	5	4	4	0,95303	28
7	1	5	1	0,95271	28
6	2	5	3	0,95261	28
6	5	2	3	0,95248	28
7	5	1	1	0,95224	28
6	2	6	2	0,95209	28
6	6	2	2	0,952	28
5	4	7	2	0,95146	28
5	7	4	2	0,95146	28
8	0	4	0	0,9512	28
7	1	3	3	0,95104	28
9	0	0	1	0,95072	28
6	5	5	0	0,95066	28
8	3	0	1	0,95056	28
8	0	2	2	0,95019	28
7	3	1	3	0,95013	28

APPENDIX P

APPROXIMATION PERFORMANCE FOR COMPONENT COMMONALITY

Table A.17 The Model where Common Component is Picked up First

												APPROXIMATION					
	λ_1	λ_2	μ_1	μ_2	μ_3	μ	S_1	S_2	S_3	S_0	$S_{\bar{o}}$	SP_0	FR_0	EB_0	$SP_{\bar{o}}$	$FR_{\bar{o}}$	$EB_{\bar{o}}$
1	8	9	10	10	20	20	5	5	5	5	5	33,70	58,94	1,78	53,77	40,02	5,20
2	8	9	15	10	20	25	5	5	5	5	5	11,13	84,42	0,40	41,26	53,31	3,93
3	8	9	10	10	25	20	5	5	5	3	3	42,03	47,52	2,09	62,53	29,72	6,00
4	9	9	10	10	20	20	5	5	5	3	3	76,41	17,60	8,52	76,41	17,60	8,52
5	9	9	10	15	25	20	5	5	5	3	3	70,40	22,76	7,38	48,57	41,29	2,71
6	9	9	15	15	25	20	0	0	0	3	3	66,01	23,92	4,00	66,01	23,92	4,00
7	9	9	15	10	20	20	5	0	3	3	3	68,72	23,31	5,26	85,84	9,91	11,33
8	8	9	10	15	20	20	0	0	0	3	3	72,45	18,93	4,88	66,08	23,78	3,76
9	9	9	15	15	25	20	3	3	3	5	5	36,20	56,28	2,03	36,20	56,28	2,03
10	9	9	15	15	25	25	5	5	5	5	5	5,56	90,91	0,14	5,56	90,91	0,14

	SIMULATION						ERROR (%)					
	SP_0	FR_0	EB_0	$SP_{\bar{o}}$	$FR_{\bar{o}}$	$EB_{\bar{o}}$	SP_0	FR_0	EB_0	$SP_{\bar{o}}$	$FR_{\bar{o}}$	$EB_{\bar{o}}$
1	38,13	62,10	1,68	58,83	43,10	4,87	4,43	3,16	5,78	5,06	3,08	6,75
2	17,03	75,68	0,55	43,52	50,17	4,41	5,90	8,74	26,76	2,26	3,14	10,95
3	47,53	42,87	1,98	58,72	34,29	5,70	5,51	4,65	5,71	3,81	4,57	5,22
4	66,52	28,05	7,85	66,13	27,98	7,40	9,89	10,45	8,49	10,28	10,38	15,09
5	73,68	23,85	7,64	55,34	37,18	2,89	3,28	1,09	3,38	6,77	4,11	6,16
6	69,45	19,68	4,21	69,12	19,80	3,65	3,44	4,24	5,11	3,11	4,12	9,38
7	63,52	25,54	5,50	78,25	12,69	10,53	5,20	2,23	4,28	7,59	2,78	7,65
8	68,48	20,96	4,71	68,31	22,43	3,88	3,97	2,03	3,50	2,23	1,35	3,12
9	38,81	52,46	2,09	38,75	52,56	2,13	2,61	3,82	3,24	2,55	3,72	4,86
10	6,57	87,18	0,16	6,58	87,16	0,17	1,01	3,73	13,94	1,02	3,75	16,09

Table A.18 The Model where Common Component is Picked at Last

	λ_1	λ_2	μ_1	μ_2	μ_3	μ	S_1	S_2	S_3	S_0	$S_{\bar{o}}$	APPROXIMATION					
												SP_0	FR_0	EB_0	$SP_{\bar{o}}$	$FR_{\bar{o}}$	$EB_{\bar{o}}$
1	8	9	10	10	20	20	5	5	5	5	5	29,48	63,68	1,53	51,93	42,11	5,12
2	8	9	15	10	20	25	5	5	5	5	5	9,79	83,32	0,45	40,30	54,70	3,95
3	8	9	10	10	25	20	5	5	5	3	3	40,87	48,87	2,04	61,83	30,51	5,96
4	9	9	10	10	20	20	5	5	5	3	3	72,97	20,74	8,10	72,97	20,74	8,10
5	9	9	10	15	25	20	5	5	5	3	3	69,65	23,55	7,34	46,50	43,47	2,58
6	9	9	15	15	25	20	0	0	0	3	3	65,23	24,62	3,93	65,23	24,62	3,93
7	9	9	15	10	20	20	5	0	3	3	3	54,98	35,82	4,62	88,92	7,19	11,88
8	8	9	10	15	20	20	0	0	0	3	3	73,83	17,83	5,08	60,93	28,18	3,25
9	9	9	15	15	25	20	3	3	3	5	5	36,03	56,49	2,02	36,03	56,49	2,02
10	9	9	15	15	25	25	5	5	5	5	5	5,54	90,94	0,14	5,54	90,94	0,14

	SIMULATION						ERROR (%)					
	SP_0	FR_0	EB_0	$SP_{\bar{o}}$	$FR_{\bar{o}}$	$EB_{\bar{o}}$	SP_0	FR_0	EB_0	$SP_{\bar{o}}$	$FR_{\bar{o}}$	$EB_{\bar{o}}$
1	37,51	60,25	1,69	58,45	42,25	4,90	8,03	3,43	9,34	6,52	0,14	4,58
2	15,13	78,25	0,52	46,35	48,24	4,34	5,34	5,07	13,26	6,05	6,46	9,01
3	47,68	43,06	1,95	59,19	33,84	5,71	6,81	5,81	4,47	2,64	3,34	4,46
4	65,21	28,35	7,82	65,17	28,38	7,35	7,76	7,62	3,54	7,80	7,65	10,17
5	75,63	19,90	7,83	52,19	38,79	2,69	5,98	3,65	6,28	5,69	4,68	4,12
6	72,33	19,71	4,19	72,38	19,73	3,68	7,10	4,91	6,18	7,15	4,89	6,74
7	60,81	28,15	5,04	80,03	10,54	10,93	5,83	7,67	8,41	8,89	3,35	8,67
8	70,02	19,44	4,83	65,38	24,82	3,65	3,81	1,61	5,06	4,46	3,36	10,96
9	38,73	52,65	2,14	38,71	52,59	2,13	2,70	3,84	5,76	2,68	3,90	5,41
10	6,55	87,23	0,17	6,55	87,19	0,17	1,01	3,71	16,29	1,01	3,75	17,12