ELECTRICALLY CHARGED VORTEX SOLUTIONS IN BORN-INFELD THEORY WITH A CHERN-SIMONS TERM

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ABSTRACT

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In this thesis, we considered electrically charged vortex solutions of Born-Infeld Chern-Simons gauge theory in 2+1 dimensions, with a sixth order charged scalar field potential. For this purpose, first Nielsen-Olesen vortex solutions are extensively reviewed. Then, Born-Infeld and Chern-Simons theories are summarized. Finally, vortex solutions are obtained for the Born-Infeld-Higgs system with a Chern-Simons term. These solutions are analyzed numerically, comparing their properties with Nielsen-Olesen vortices.

Keywords: Vortex solutions, Nielsen-Olesen vortices, Born-Infeld theory, Chern-Simons theory.

ÖZ

CHERN-SIMONS TERİMİ İÇEREN BORN-INFELD TEORİSİNDE ELEKTRİKSEL YÜKLÜ VORTEX ÇÖZÜMLERİ

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Bu çalışmada, yüklü ve altıncı dereceden bir potensiyel içeren, 2+1 boyutta Born-Infeld Chern-Simons ayar teorisinin elektriksel yüklü vorteks çözümleri incelendi. Bu amaçla, ilk önce Nielsen-Olesen vorteks çözümleri etraflıca gözden geçirildi. Daha sonra Born-Infeld ve Chern-Simons teorileri özetlendi. Son olarak Chern-Simons terimi içeren Born-Infeld-Higgs sisteminin vorteks çözümleri elde edildi. Bu çözümlerin, özellikleri Nielsen-Olesen vorteksleriyle karşılaştırılarak, nümerik analizleri yapıldı.

Anahtar Kelimeler: Vorteks Çözümleri, Nielsen-Olesen vorteksleri, Born-Infeld teorisi, Chern-Simons teorisi.

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CHAPTER 1

INTRODUCTION

Born-Infeld electrodynamics [1], which is a non-linear version of Maxwell electrodynamics, has recently received much attention in connection with string theory and brane dynamics [2, 3]. Classical configurations have been studied for different models in which the gauge field dynamics is governed by a Born-Infeld Lagrangian and in this way domain wall, vortex and monopole solutions were constructed [4, 5, 6] and also gravitating solutions relevant in supergravity were found [7]. Supersymmetry and BPS saturated solutions in connection with D-brane dynamics have also been investigated [8, 9].

An important point in the Born-Infeld theory is that it admits finite energy static solutions (originally proposed in [1] to describe the electron) which are not source free like solitons but need point like sources which can be interpreted as the ends of electric-flux carrying strings. In particular, for some solutions discussed in [6] (called there BIons) strings look like tubes joining smoothly onto a p-brane.

The solutions we find in this work are not BIonic but source free regular solitons of a Born-Infeld theory. They are related to Nielsen-Olesen strings [10], which are vortex solutions in Abelian-Higgs model and in this sense they also look like tubes which are in fact electrically charged vortices (in contrast to Nielsen-Olesen vortices which carry magnetic flux but no electric charge).

Since in contrast with Maxwell electrodynamics, Born-Infeld theory allows for finite energy solutions which corresponds to the field of electric charges, one could envisage to construct finite energy electrically charged vortex solutions just by studying a Born-Infeld-Higgs system, the non-linear electrodynamics version of the Nielsen-Olesen model [10] (which does not admit finite energy charged solutions). However, in [11], it is shown that, even allowing singularities as electric sources for the vortex charge, such solutions do not exist. Only if one adds a Chern-Simons term, finite energy Born-Infeld charged vortices can be found, in analogy with what happens in the Maxwell case [12, 13, 14, 15, 16].

Introducing a Chern-Simons term implies that static, z-independent vortices in 3+1 dimensions should be considered as static configurations in 2+1. Hence, they should be thought more as planar configurations than as string-like objects but one should coincide with those arising when instead of a Chern-Simons term (forcing to work in 2+1) one couples fermions to the Born-Infeld model [17].

Our work in this thesis is organized as follows. In the second chapter, we begin by a review of the concept of soliton. The historical development and some main properties of the solitons are discussed. We also construct the vortex, which are soliton solutions in two space dimensions. Then, we review the vortex solutions of Abelian-Higgs model (Nielsen-Olesen model) in detail. The results of this part will be necessary for us, we will compare these results with the solutions of our model.

In chapter three, we review the Chern-Simons theory, which we used in our model. We give the basic properties and definitions of the theory and then focus on the Abelian Relativistic Chern-Simons model.

In chapter four, we review the Born-Infeld theory, the second theory of our model. We emphasize the history and importance of the theory in physics. In addition, we give some basic properties of the theory, which will be useful in the next chapter.

In chapter five, we construct our model, by starting from a Born-Infeld Chern-Simons-Higgs Lagrangian and we try to achieve vortex solutions of the theory. We follow a similar procedure with the one of Nielsen-Olesen vortices. But in our case equations of motion are non-linear because of the Born-Infeld term. To obtain a detailed profile of solutions, we solved numerically the equations of motions and compare the results with the Nielsen-Olesen vortices. This comparison will be also between Maxwell and Born-Infeld theories in a sense.

Finally, in the last chapter, we discussed the results achieved throughout the study.

CHAPTER 2

VORTEX SOLUTIONS

2.1 Solitons

Solitons made their first appearance in the world of science with the beautiful report on waves, presented by J. Scott Russell in 1842 and 1843 at the British Association for the Advancement of Science [18].

Scott Russell conjectured that the propagation of an isolated wave, such as the one he observed, was a consequence of the properties of the medium rather than of the circumstances of the wave's generation. This was not universally accepted and a long time had to elapse before it became established that some special non-linear wave equations admit solutions consisting of isolated waves that can propagate and even undergo collisions without losing their identity.

Zabusky and Kruskal [19] introduced the word soliton to characterize waves that do not disperse and preserve their form during propagation and after a collision. Because of their defining features, the soliton might appear the ideal mathematical structure for the description of a particle. Yet particle physicist did not need, nor had to wait for the emergence of the concept of soliton to find analytical instruments for the study of particle phenomena. In the course of the evolution of field theory and quantum mechanics, the notion of a particle became associated with the elementary excitation of a quantized field. The propagation of a free particle is described by a quantized mode of a linear system, rather than by the solution of a classical non-linear wave equation. Non-linear interactions among the quantized fields can be described in perturbation theory and lead to scattering phenomena: the individuality of particles emerging from the collision is guaranteed by their being quanta of some field; apart from this the collision can change their state of motion and their quantum numbers.

The situation began to change when the difficulties encountered in providing a perturbative description of some particle phenomena, especially, in the domain of strong interactions, induced many theorists to reconsider from a new perspective the consequences of the non-linearities in the equations of motion. Thus it was that in the early and mid-seventies the existence of soliton type solutions was basically rediscovered by several particle physicists. These solitons however were not to be interpreted as the elementary particles of the theory, for which the standard association with the quantized modes of linearized small oscillations was to be maintained, but as new, additional particle-like excitations that the system possessed by virtue of its non-linear nature.

Two features make solitons quantum effects particularly interesting;

i. Most of the soliton solutions are non-perturbative. That is they cannot be obtained by starting from solutions of the corresponding linear part of the field equations and treating the non-linear terms perturbatively. Given that these classical soltions are themselves non-perturbative, the quantum effects obtained from them also turn out to be non-perturbative.

ii. Typically, soliton solutions are characterized by some topological index, related to their behavior at spatial infinity. For solitons, this index turns out to be a conserved quantity which, in the quantized theory, becomes a conserved quantum number characterizing the soliton state. Such a topological quantum number is quite different in origin from the familiar Noether charges associated with continuous symmetries of the Lagrangian.

Soliton solutions can be classified into three subgroups; vortices, magnetic monopoles and instantons, which are soliton solutions in two space dimensions (i.e. a string in 3-dimensional case), three space dimensions (localized in space but not in time) and 4-dimensional space-time (localized in space and time). In the rest of the discussion, two dimensional models will be concerned generally. Therefore in the next section, we will deal with vortices.

2.2 Vortices

2.2.1 Construction

Consider a scalar field in a 2-dimensional space. The boundary of this space is the circle at infinity, denoted S^1. We construct a field whose value on the boundary is

$$\phi = \eta e^{in\varphi} \qquad (\rho \to \infty), \tag{2.1}$$

where ρ and φ are polar coordinates in the plane, η is a constant and to make ϕ single-valued, n is an integer. From (2.1), we have

$$\nabla\phi = \frac{1}{\rho}(in\eta e^{in\eta})\hat{\varphi}. \tag{2.2}$$

The Lagrangian and Hamiltonian functions are

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}|\nabla\phi|^2 - V(\phi), \tag{2.3}$$

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}|\nabla\phi|^2 + V(\phi). \tag{2.4}$$

Now let us consider a static configuration with, for example,

$$V(\phi) = [\eta^2 - |\phi|^2]^2, \tag{2.5}$$

so that $V = 0$ on the boundary. Then as $\rho \to \infty$

$$\mathcal{H} = \frac{1}{2}|\nabla\phi|^2 = \frac{n^2\eta^2}{2\rho^2} \tag{2.6}$$

and the energy of the static configuration is

$$E \approx \int^{\infty} \mathcal{H} \rho d\rho d\varphi = \pi n^2 \eta^2 \int^{\infty} \frac{1}{\rho} d\rho. \tag{2.7}$$

This is logarithmically divergent.

To proceed, we add a gauge field , so that what counts is the covariant derivative

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi. \tag{2.8}$$

By choosing A_μ of the form

$$A = \frac{1}{e} \nabla(n\varphi) \qquad (\rho \to \infty), \tag{2.9}$$

then

$$\begin{aligned} A_\rho &\to 0 \qquad (\rho \to \infty), \\ A_\varphi &\to -\frac{n}{e\rho} \qquad (\rho \to \infty). \end{aligned} \tag{2.10}$$

We find that at $\rho = \infty$

$$\begin{aligned} D_\varphi \phi = \frac{1}{\rho} \left(\frac{\partial \phi}{\partial \varphi} \right) + ieA_\varphi \phi = 0, \\ D_\rho \phi = 0, \end{aligned} \tag{2.11}$$

so $D_\mu \phi \to 0$ on the boundary at infinity. The Lagrangian is now

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - V(\phi), \tag{2.12}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.13}$$

Since (2.10) is a pure gauge,

$$A_\mu \to \partial_\mu \chi \qquad (\rho \to \infty), \tag{2.14}$$

then $F_{\mu\nu} \to 0$. For a static configuration $\mathcal{H} = -\mathcal{L}$ and with $V(\phi)$ given by (2.5). We have $\mathcal{H} \to 0$ as $\rho \to \infty$, making possible a field configuration of finite energy.

Now let us verify that the effect of adding the gauge field is to give the soliton magnetic flux. Consider the integral $\oint A.dl$ round the circle S^1 at infinity. By stokes theorem, this is $\int B.ds = \Phi$, the flux enclosed, hence

$$\Phi = \oint A.dl = \oint A_\theta \rho d\theta = -\frac{2\pi n}{e} \tag{2.15}$$

and the flux is quantized. So we have, after all, constructed a 2-dimensional field configuration consisting of a charged scalar field and a gauge field; and named it as **vortex**. It carries magnetic flux, and since $D_\mu \to 0$ and $F_{\mu\nu} \to 0$ on the boundary at infinity, it appears to have finite energy.

2.2.2 Nielsen-Olesen Vortex Solution

To be a little more systematic, we consider a particular gauge theory, the famous Abelian Higgs model with a complex scalar field ϕ, coupled to a $U(1)$ gauge field A_μ with coupling constant e. The Lagrangian density for the model is [10];

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - \frac{1}{8}\lambda(|\phi|^2 - \eta^2)^2, \tag{2.16}$$

where $D_\mu = (\partial_\mu + ieA_\mu)\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$\mathcal{L}$ is invariant under the $U(1)$ local gauge transformations. Working again in two dimensions, we are going to find static, non-singular, finite energy solutions (vortex solutions) to this theory.

The potential of the theory can be read directly from the Lagrangian to be

$$V(\phi) = \frac{1}{8}\lambda(|\phi|^2 - \eta^2)^2. \tag{2.17}$$

Therefore, V attains its minimum (zero) when

$$\phi(\infty, \varphi) = \eta e^{i\sigma}. \tag{2.18}$$

The Euler-Lagrange equations for the fields ϕ and A_μ from (2.16) are

$$D_\mu D^\mu\phi + \frac{1}{4}\lambda(|\phi|^2 - \eta^2)\phi = 0, \tag{2.19}$$

$$\partial^\nu F_{\mu\nu} - ie(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - 2e^2|\phi|^2 A_\mu = 0. \tag{2.20}$$

ϕ can be written as

$$\phi(\rho, \varphi) = f(\rho)e^{i\sigma}. \tag{2.21}$$

This form of ϕ will be very useful when we come to consider the equations of motion in detail. But before we should settle down the limits for f. As $\rho \to \infty$, in accordance with equation (2.18) we have $f(\rho) \to \eta$. Also, as $\rho \to 0$ we have $f(\rho) \to 0$.

It is also compulsory to remember, that we are working in a gauge in which $A_0 = 0$, $A_\rho = 0$. For simplicity we will write $A_\varphi(\rho) = A(\rho)$

In the static case equation of motion for ϕ becomes

$$(\partial_i + ieA_i)^2\phi - \frac{1}{4}\lambda(|\phi|^2 - \eta^2)\phi = 0, \tag{2.22}$$

which on summing over the ρ and φ components, gives

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \left[\left(\frac{n}{\rho} - eA\right)^2 + \frac{1}{4}\lambda(f^2 - \eta^2)\right] f = 0. \tag{2.23}$$

On the other hand, taking the φ component of (2.20) gives

$$\frac{2en}{\rho}f^2 + 2e^2Af^2 = -\partial iF_{\varphi i} \tag{2.24}$$

or explicitly

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho A)\right] - 2e(\frac{n}{\rho} + eA)f^2 = 0. \tag{2.25}$$

Equations (2.23) and (2.25) constitute a system of coupled non-linear differential equations in f and A. However, no exact analytical solutions to this system has yet been discovered. We should deal with the asymptotic profiles of the fields instead [10].

Let us first consider the case in which $\rho \to \infty$. Then $f(\rho) \to \eta$ and the equation for A (2.25) becomes

$$\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho A)\right) - 2e^2\eta^2 A = 2e\eta^2 n\frac{1}{\rho}, \tag{2.26}$$

this equation has the the solution

$$A = -\frac{n}{e\rho} + \frac{c}{e}K_1(\sqrt{2}e\eta\rho) \to \frac{n}{e\rho} + \frac{c}{e}\left(\frac{\pi}{2\sqrt{2}e\eta\rho}\right)^{\frac{1}{2}} e^{-\sqrt{2}e\eta\rho} \quad (\rho \to \infty) \tag{2.27}$$

with magnetic field

$$B_z = cfK_0(\sqrt{2}e\eta\rho) \to \frac{c}{e}\left(\frac{\pi\eta}{2\sqrt{2}e\rho}\right)^{\frac{1}{2}} e^{-\sqrt{2}e\eta\rho}, \tag{2.28}$$

where K_0, K_1 are the modified Bessel functions of the zeroth and first order and c is a constant of integration.

Taking $A \approx -\frac{n}{e\rho}$, the equation for f is also satisfied. However, we can go one step further and obtain the small oscillations about $f = \eta$.

$$f = \eta + \zeta(\rho) \tag{2.29}$$

(with ζ small compared to η), and replace into the equation for f (2.23), bearing in mind that ρ is large and keeping the terms up to first order in ζ, gives the approximate equation

$$\frac{d^2\zeta}{d\rho^2} - \frac{1}{2}\lambda\eta^2\zeta \approx 0. \tag{2.30}$$

The solution consistent with $\rho \to \infty$ limit is

$$\zeta(\rho) \approx e^{-\sqrt{\frac{1}{2}\lambda\eta^2}\rho} \tag{2.31}$$

and hence

$$f \to \eta - e^{-\sqrt{\frac{1}{2}\lambda\eta^2\rho}} \qquad (\rho \to \infty). \tag{2.32}$$

To get the behaviour of the fields near the origin, we proceed analogously and note that $f \to 0$ as $\rho \to 0$. Equation for A then reduce to

$$\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho A)\right) \approx 0. \tag{2.33}$$

Thus near the origin, we simply get

$$A(\rho) \to \rho \qquad (\rho \to \infty). \tag{2.34}$$

The equation for f comes with a term proportional to $\frac{1}{\rho^2}$, which can not be neglected in this limit. The approximate equation for f is, then

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) \approx \frac{n^2}{\rho^2}f, \tag{2.35}$$

giving the profile for f near the origin as,

$$f(\rho) \approx \rho^{|n|}. \tag{2.36}$$

After studying the vortex solution of the abelian Higgs model in detail, we will try to answer the question-*why are these solutions stable?* The reason is topological. The Lagrangian is invariant under a symmetry group. In this case it is $U(1)$, the electromagnetic gauge group. The field ϕ (with boundary value given by (2.1)) is a representation of $U(1)$. The group space of $U(1)$ may be written $exp(i\varphi) = exp[i(\varphi + 2\pi)]$, so the space of all values of φ is a line with $\varphi = 0$ identified with $\varphi = 2\pi$, and the line becomes a circle S^1. The field φ in (2.1) is a representation basis of $U(1)$, but it is the boundary value of the field in a 2-dimensional space. This boundary is clearly a circle S^1 (the circle $\rho \to \infty$, $\varphi = (0 \to 2\pi)$). Hence φ defines a mapping of the boundary S^1 in physical space onto the group space S^1.

$$\phi : S^1 \to S^1, \tag{2.37}$$

the mapping being specified by the integer n. Now a solution characterized by one value of n is stable since it cannot be continuously deformed into a solution with a different value of n. This is to say that the first homotopy group of S^1, the group space of $U(1)$, is not trivial.

$$\pi_1(S^1) = Z. \tag{2.38}$$

Z is the additive group of integers.

The model considered here is known as the famous Nielsen Olesen model [10], the relativistic analog of the Ginzburg-Landau theory [20]. At the end, we find vortex solutions, carrying magnetic flux but no electric charge. Only If one adds a Chern-Simons term, which we will deal with in the next chapter, finite energy charge vortices can be found.

CHAPTER 3

CHERN-SIMONS THEORY

3.1 Basic Properties

Chern-Simons theories involve charged scalar fields minimally coupled to gauge fields whose dynamics is provided by a Chern-Simons term in 2+1 dimensions (i.e. two spatial dimensions). The physical context in which these Chern-Simons models arise is that of anyonic quantum field theory, with direct applications to such planar models as the quantum hall effect, anyonic superconductivity and Aharanov-Bohm scattering. In addition, there are further interesting connections with the more mathematically inspired theory of integrable models.

A novel feature of these Chern-Simons theories is that they permit a realization with either relativistic or non-relativistic dynamics for the scalar fields. The non-relativistic Chern-Simons equations may be solved completely for all finite charge solutions and the solutions exhibit many interesting relations to two dimensional integrable models. In the relativistic case, while the general exact solutions correspond to topological and non-topological solitons and vortices, many characteristics of which can be deduced from algebraic and asymptotic data.

The Chern-Simons theories describe scalar fields in 2+1 dimensional space-time, minimally coupled to a gauge field whose dynamics is given by a Chern-Simons Lagrangian rather than by a conventional Maxwell Lagrangian. The possibility of describing gauge theories with a Chern-Simons term rather than with a Yang-Mills tern is a special feature of odd-dimensional case is especially distinguished in the sense that the derivative part of the Chern-Simons Lagrangian is quadratic in the gauge fields. To conclude, let us briefly review some of the important properties of the Chern-Simons Lagrange density;

$$\mathcal{L}_{CS} = \epsilon^{\mu\nu\rho} tr \left(\partial_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right). \tag{3.1}$$

The gauge field A_μ takes values in a finite dimensional representation of the gauge Lie algebra $\mathcal{G}$. The totally antisymmetric ϵ-symbol $\epsilon^{012} = 1$. In an abelian theory, the gauge fields A_μ commute, and so the trilinear term in (3.1) vanishes due to the antisymmetry of the ϵ-symbol. The rest of the discussion will be about abelian theory so our Lagrangian density becomes

$$\mathcal{L}_{CS} = \epsilon^{\mu\nu\rho} tr(\partial_\mu A_\nu A_\rho). \tag{3.2}$$

The Euler-Lagrange equations of motion derived from this Lagrange density are simply

$$F_{\mu\nu} = 0, \tag{3.3}$$

which follows from the fact that

$$\frac{\partial \mathcal{L}_{CS}}{\partial A_\mu} = \epsilon^{\mu\nu\rho} F_{\mu\nu}. \tag{3.4}$$

At first sight. the equations of motion seem somewhat trivial. with solutions that are simply pure gauges $A_\mu = g^{-1}\partial_\mu g$. However, with the inclusion of topological effects and external sources these equations are far from trivial.

It is very important to notice that the equations of motion (3.3) are first order in space-time derivatives, in contrast to the Yang-Mills equations of motion which are second-order.

The equations of motion (3.3) are gauge invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu^g g^{-1} A_\mu g + g^{-1}\partial_\mu g \tag{3.5}$$

and so the Lagrange density (3.2) defines a sensible gauge theory even though the Lagrange density itself is not invariant under the gauge transformations (3.5). Indeed under the gauge transformation $\mathcal{L}_{CS}$ transforms as

$$\mathcal{L}_{CS}(A) \rightarrow \mathcal{L}_{CS}(A) - \epsilon^{\mu\nu\rho}\partial_\mu tr(\partial_\nu g g^{-1} A_\rho). \tag{3.6}$$

The change in $\mathcal{L}_{CS}$ is a total space-time derivative. Hence the action, $S = \int d^3x \mathcal{L}_{CS}$, is gauge invariant and so we expect that a sensible quantum gauge theory may be formulated.

Another important feature of the Chern-Simons theories is that the Chern-Simons term describes a topological gauge field theory [21] in the sense that there is no explicit dependence on the space-time metric. This follows because the Lagrange density (3.1) can be written directly as a 3-form; $\mathcal{L}_{CS} = tr(AdA + A^3)$.

Thus the action is independent of the space-time metric, and so the Chern-Simons Lagrange density $\mathcal{L}_{CS}$ does not contribute to the energy momentum tensor. This may be understood by noting that $\mathcal{L}_{CS}$ (3.1) is first order in space-time derivatives;

$$\mathcal{L}_{CS} = \epsilon^{ij} tr(A_i \dot{A}_j + tr(A_0 F_{12}). \tag{3.7}$$

The time derivative part of $\mathcal{L}_{CS}$ indicates the canonical structure of the theory, with A_1 and A_2 being canonically conjugate fields. This is radically different from conventional (Yang-Mills) gauge theory in which the gauge field components A_i may be regarded as coordinate field, canonically conjugate to the electric field $E_i = F_{0i}$. The A_0 part of the Lagrange density produces the Gauss law constraint, and there is no contribution to the Hamiltonian. This implies that the Chern-Simons gauge field does not have any real dynamics of its own. It is a non-propagating field whose dynamics come from the fields to which it is minimally coupled.

The Chern-Simons Lagrange density (3.2) may be coupled to an external matter current J^μ as

$$\mathcal{L} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\mu A_\nu A_\rho - A_\mu J^\mu. \tag{3.8}$$

This leads to the equation of motion

$$F_{\mu\nu} = -\frac{1}{\kappa} \epsilon_{\mu\nu\rho} J_\rho, \tag{3.9}$$

involving the covariantly conserved ($D_{\mu\nu}J^{\mu} = 0$) current. Thus the time component, J^0, of the matter current is proportional to the magnetic field

$$\begin{aligned} J^0 &= \kappa F_{12}, \\ J^0 &= \kappa \vec{B}, \end{aligned} \tag{3.10}$$

which is the Chern-Simons Gauss law constraint, and which is important for the interpretation of Chern-Simons theories as field theories for anyons [22]. The spatial components J^i of the current are everywhere perpendicular to the electric field

$$\begin{aligned} J^i &= \kappa \epsilon^{ij} F_{j0}, \\ J^i &= \kappa \epsilon^{ij} E_j. \end{aligned} \tag{3.11}$$

With this coupling to external matter, the equations of motion (3.9) are still first order in space-time derivatives acting on the fields. The relations (3.10) and (3.11) are fundamental to the application of Chern-Simons theories to condensed matter systems such as the quantum Hall effect [23].

The last property of Chern-Simons theories that we want to mention here is that Higgs mechanism behaves very differently when the gauge fields are Chern-Simons fiels. In a conventional gauge theory, the Higgs mechanism produces a massive mode in a broken vacuum in which the gauge , to which the gauge fields are coupled, possesses a non-vanishing vacuum expectation value. Formally, this mechanism is independent of the dimension of space-time, but in 2+1 dimensions the possibility of including a Chern-Simons term for the gauge field leads to a richer variety of mass generation effects.

3.2 Abelian Relativistic Model

Since our main concern is the Abelian relativistic Chern-Simons theory, in this part, we will deal with its properties. First consider the Lagrange density

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - (D_\mu\phi)^* D^\mu\phi - V(|\phi|), \tag{3.12}$$

where this model is defined in 2+1 dimensional Minkowski space-time, with metric $g_{\mu\nu} = diag(-1, 1, 1)$.

This theory has a local $U(1)$ gauge symmetry and the Euler-Lagrange equations of motion are

$$D_\mu D^\mu\phi = \frac{\partial V}{\partial\phi^*}, \tag{3.13}$$

$$F_{\mu\nu} = -\frac{1}{\kappa}\epsilon_{\mu\nu\rho}J^\rho, \tag{3.14}$$

where J^μ is the relativistic matter current defined as

$$J_\mu = -i(\phi^* D^\mu\phi - (D^\mu\phi)^*\phi), \tag{3.15}$$

which is conserved

$$\partial_\mu J^\mu = 0. \tag{3.16}$$

The energy density corresponding to the Lagrange density (3.12) is

$$\epsilon = |D_0\phi|^2 + |\overrightarrow{D}\phi|^2 + V(|\phi|), \tag{3.17}$$

supplemented by the Gauss law constraint

$$F_{12} = \frac{1}{\kappa} J^0 = -\frac{i}{\kappa}(\phi^* D^0 \phi - (D^0 \phi)^* \phi). \tag{3.18}$$

As is familiar for Chern-Simons theories (in which the gauge field action consists of only a Chern-Simons term, with no Maxwell term), the Chern-Simons term does not contribute to the energy density ϵ since $\mathcal{L}_{CS}$ is first order in space-time derivatives.

CHAPTER 4

BORN-INFELD THEORY

Born-Infeld theory [1] was proposed in the thirties as an alternative for Maxwell theory. The main motivation of Born and Infeld was to construct a theory which has classical solutions representing electrically charged particles with finite self-energy (Their theory is a non-linear generalization of Maxwell theory). Born-Infeld theory indeed admits such solutions. However after Dirac's paper on a classical electron [24] and the birth of the quantum electrodynamics in the forties, Born-Infeld theory was almost totally forgotten for a long time. Because the theory can not be quantized in contrast to Maxwell theory.

Recently there is a new interest in this theory due to the investigations in string theory. It turns out that some very natural objects in this theory, so called D-branes, are described by a kind of non-linear Born-Infeld action [25]. However, in this work, we analyze this theory in view of quantum field theory for point particles, instead of string theory. There are following reasons to consider non-linear electromagnetism: it is well known that Maxwell electrodynamics when applied to point-like objects is inconsistent [26]. This inconsistency originates in the infinite self-energy of the point charge. In the Born-Infeld theory, this self-energy is already finite. Therefore, one may hope that in the theory, which gives finite value of this quantity, it would be possible to describe the particle's self-interaction in a consistent way. Moreover, the assumption, that the theory is effectively non-linear in the vicinity of the charged particle is very natural from the physical point of view.

We consider very specific model of non-linear theory because, among other non-linear theories of electromagnetism, Born-Infeld theory possesses very distinguished physical properties. For example, it is the only causal spin-1 theory (apart from the Maxwell one) [27]. It was shown that the electrodynamical part of the Einstein's unified field model with non-symmetrical metric is equivalent to Born-Infeld electrodynamics [28] . The characteristic equation for Born-Infeld electrodynamics has a very notable form [29].

Recently, Born-Infeld electrodynamics was successfully applied as a model for generation for multipole moments of charged particles [30].

At present Born-Infeld model is considered as a possible vacuum electrodynamics and future experiments must have solved what a model is more appropriate [31].

Thus any results which may help the mathematical investigation of the theory will be useful.

The Born-Infeld Lagrangian is defined as;

$$\mathcal{L}_{BI} = -\beta^2 \left(\sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}} - 1 \right). \tag{4.1}$$

The parameter β has a dimension of a field strength (Born and Infeld called it the absolute field) and it measures the non-linearity of the theory. In the limit $\beta \to \infty$, the Lagrangian $\mathcal{L}_{BI}$ (4.1) tends to the standard Maxwell Lagrangian

$$\mathcal{L}_{BI} \to L_{MAXWELL} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (\beta \to \infty). \tag{4.2}$$

The equations of motion derived from (4.1) are

$$\partial_\mu \left(\frac{F^{\mu\nu}}{\sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}}} \right) = 0. \tag{4.3}$$

In Born-Infeld theory, the electric field E^i and magnetic field B^i are defined as usual as

$$E^i = F^{oi}, \qquad B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}, \tag{4.4}$$

while electric induction D^i and the magnetic intensity H^i, take the form

$$D^i = f^{oi}, \qquad H^i = \frac{1}{2} \epsilon^{ijk} f_{jk}, \tag{4.5}$$

with [1]

$$f_{ij} = \frac{F_{ij}}{R}. \tag{4.6}$$

In the limit $\beta^2 \to \infty$ the Born-Infeld action coincides with the Maxwell action so in this limit $D_i \to E^i$ and $H_i \to B^i$.

[1] where $R = \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}}$

CHAPTER 5

VORTEX SOLUTION IN BORN-INFELD CHERN-SIMONS THEORY

In this chapter, we will analyze the vortex solution in Born-Infeld Chern-Simons Theory.

5.1 Model

To start with, we shall consider the 2+1 dimensional Lagrangian density for a complex scalar field ϕ minimally coupled to a $U(1)$ gauge field A_μ with dynamics governed by a Born-Infeld action plus a Chern-Simons action

$$L = -\beta^2 \left(\sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}} - 1 \right) + \frac{\kappa}{4\pi} \epsilon^{\mu\nu\alpha} A_\mu F_{\nu\alpha} + \frac{1}{2} D_\mu \phi^* D^\mu \phi - V[\phi], \quad (5.1)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - A_\nu \partial_\mu, \quad (5.2)$$

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\phi, \quad (5.3)$$

and $V[\phi]$ is a symmetry breaking scalar field potential such that

$$\left. \frac{dV[\phi]}{d\phi} \right|_{\phi=\eta} = 0. \quad (5.4)$$

The constant β (Born-Infeld parameter) has the dimension of the electromagnetic field, $[\beta] = m^{\frac{3}{2}}$, while $[e] = m^{\frac{1}{2}}$ (gauge coupling term) and $[\kappa] = m$ (strength of the Chern-Simons term). For fields much weaker than the critical field value β, Lagrangian (5.1) reduces to Maxwell-Chern-Simons-Higgs Lagrangian.

The equations of motion resulting from (5.1) are

$$\partial_\mu \left(\frac{F^{\mu\nu}}{R} \right) + \frac{\kappa}{2\pi} \epsilon^{\mu\nu\alpha} F_{\nu\alpha} = J^\nu \tag{5.5}$$

and

$$\Box \phi + 2ieA_\mu \partial^\mu \phi - e^2 A_\mu A^\mu \phi = -\frac{\partial V}{\partial \phi^*}, \tag{5.6}$$

with

$$R = \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}} \tag{5.7}$$

and

$$J^\nu = \frac{ie}{2}(\phi^* \partial^\nu \phi - \phi \partial^\nu \phi^*) - e^2 |\phi|^2 A^\nu, \tag{5.8}$$

where J^ν is the conserved matter current. $\Box$ is defined as; $\Box = \partial_\mu \partial^\mu$.

5.2 Rotationally Symmetric Solutions

We now specialize to rotationally symmetric solutions. In polar coordinates the adequate axially symmetric ansatz [32] reads

$$\phi(\overrightarrow{x}) = f(\rho) exp(-in\varphi), \quad A_\varphi(\overrightarrow{x}) = -\frac{1}{\rho} A(\rho), \quad A_0(\overrightarrow{x}) = A_0(\rho), \tag{5.9}$$

where ρ and φ are the radial and angular polar coordinates and to make ϕ single valued, n is an integer. By using this ansatz; the equations of motion (5.5) and (5.6) reduce to

$$\rho \frac{d}{d\rho}\left(\frac{1}{R}\frac{1}{\rho}\frac{dA}{d\rho}\right) - f^2 e(n+eA) = \frac{\kappa}{\pi}\frac{dA_0}{d\rho}\rho, \tag{5.10}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{1}{R}\rho\frac{dA_0}{d\rho}\right) - e^2 f^2 A_0 = \frac{\kappa}{\pi}\frac{1}{\rho}\frac{dA}{d\rho}, \tag{5.11}$$

$$\frac{d^2 f}{d\rho} + \frac{1}{\rho}\frac{df}{d\rho} - \frac{e}{\rho^2} f(n+eA)^2 + e^2 f A_0 = \frac{\partial V}{\partial f}. \tag{5.12}$$

Now R is defined as

$$R = \sqrt{1 + \frac{1}{\beta^2}\left[\frac{1}{\rho^2}\left(\frac{dA}{d\rho}\right)^2 - \left(\frac{dA_0}{d\rho}\right)^2\right]}. \tag{5.13}$$

5.3 Scalar Field Potential

In this section, we will describe the potential of the model and discuss some its properties. As we mentioned before, the potential of the model has a symmetry breaking potential at $|\phi| = \nu$.

In the model our particular choice of potential is

$$V(|\phi|) = \frac{\lambda}{8}|\phi|^2(|\phi|^2 - \eta^2)^2, \tag{5.14}$$

which is at sixth order. This potential has interesting properties in contrast to the usual fourth order potential (2.17), which is used in the original Nielsen-Olesen vortex. In order to obtain finite energy solutions to the equations of motion (5.10), (5.11), (5.12), the scalar field ϕ must tend at spatial infinity to a minimum of the potential $V(|\phi|)$. For the potential which is used in the original Nielsen-Olesen vortex, there is one minimum of the potential. However, in the potential (5.14), there are two inequivalent minima and correspondingly there are two different types of solutions

$$|\phi|^2 \to \eta^2 \qquad\qquad \rho \to \infty, \tag{5.15}$$

$$|\phi|^2 \to 0 \qquad\qquad \rho \to \infty. \tag{5.16}$$

The former case corresponds to topologically stable solutions with broken symmetry at large distances, while the latter case corresponds to solutions with unbroken symmetry at large distances, and for which there is therefore no topological stability argument. In the rest of the discussion we will deal with the topological solutions only.

5.4 Topological Solutions

We start to the discussion of topological solutions with establishing the boundary conditions for the equations (5.10), (5.11) and (5.12) for the case ($\phi \to \eta$ when $\rho \to \infty$).

The boundary conditions at the origin for $f(\rho)$, $A(\rho)$ follow from the requirement that the fields be nonsingular. This implies that

$$f(0) = 0, \tag{5.17}$$

and

$$A(0) = 0. \tag{5.18}$$

However this requirement leave $A_0(0)$ undetermined. So

$$A_0(0) = c, \tag{5.19}$$

with c is a constant to be determined by numerical solution.

On the other hand, the boundary conditions at infinity follow from the finiteness of energy. Finiteness of energy will require that the scalar field to approach to its value at one of the vacua. This condition for the topological solutions is

$$f(\infty) = \eta. \tag{5.20}$$

It will also be required that [1]

$$A(\infty) = -\frac{n}{e}, \tag{5.21}$$

and

$$A_0(\infty) = 0. \tag{5.22}$$

After determining the boundary conditions, we can calculate the magnetic flux of the vortex. It is given by

$$\Phi = \int d^2x H. \tag{5.23}$$

[1] Finiteness of energy also require that $D_\mu\phi \rightarrow 0$ and $F_{\mu\nu} \rightarrow 0$ when $\rho \rightarrow \infty$. The boundary conditions for $A(\rho)$ and $A_0(\rho)$ comes from this facts.

From the stokes theorem and using the boundary conditions and equations (4.5), (4.6), one can easily see that the magnetic flux is quantized

$$\Phi = -\frac{2\pi}{e}n \tag{5.24}$$

As it is well known, the Abelian Higgs model with the usual Maxwell action (and no Chern-Simons term) has finite energy axially symmetric static solutions, the well honored Nielsen-Olesen vortices [10]. Vortex solutions carry quantized magnetic flux but, they are necessarily neutral since the existence of an electric charges implies infinite energy. The only way to get a charged vortex solution is to make dynamics governed by something else than the Maxwell action. In particular, one add a Chern-Simons term endowing the vortex with charge through its well-known relation with magnetic flux forced by equations of motion [33]. Another possibility is related to the Born-Infeld elecromagnetism which allows singular electric induction $\overrightarrow{D}$ but with a regular electric field $\overrightarrow{E}$ and a finite energy. One may expect that regular charged vortex solutions could be constructed starting with a Born-Infeld Lagrangian instead of a Maxwell Lagrangian, even in the absence of Chern-Simons term. But in [11], it is shown that this is not true and Born-Infeld system also requires a Chern-Simons term in order to exhibit finite charged energy solutions.

To calculate the electric charge of the vortex, we use the formula

$$Q = \int d^2xj^0 = \frac{\kappa}{2\pi}\int d^2x\epsilon^{ij}F_{ij} - \int d^2x\partial_i D^i = \frac{\kappa}{2\pi}\Phi - 2\pi\rho D_\rho|_0^\infty . \tag{5.25}$$

Although, the D_ρ field has a long distance damping effected by the photon mass, one could in principle have a behaviour at the origin of the form

$$D_\rho(0) \rightarrow \frac{q}{2\pi}\frac{1}{\rho} \qquad \rho \rightarrow 0, \tag{5.26}$$

contributing to the electric charge,

$$Q = -\frac{\kappa}{n}e + q. \tag{5.27}$$

5.5 Numerical Solution

Now we return to the analysis of equations of motion. It will be convenient to define dimensionless quantities

$$\begin{aligned} x(\tau) &= n + eA(\tau), \\ y(\tau) &= \frac{A_0(\tau)}{\eta}, \\ z(\tau) &= \frac{f(\tau)}{\eta}, \end{aligned} \tag{5.28}$$

with

$$\tau = e\eta\rho. \tag{5.29}$$

In terms of new variables, equations (5.10), (5.11) and (5.12) become

$$\tau\frac{d}{d\tau}\left(\frac{1}{R}\frac{\dot{x}}{\tau}\right) - z^2 x = \delta\dot{y}\tau, \tag{5.30}$$

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\frac{\tau}{R}\dot{y}\right) - z^2 y = \delta\frac{\dot{x}}{\tau}, \tag{5.31}$$

$$\frac{1}{\tau}\frac{d}{d\tau}(\tau\dot{z}) + z\left(y^2 - \frac{x^2}{\tau^2}\right) - \frac{\lambda}{8}\frac{1}{e^2}(6z^5 - 8z^3 + 2z) = 0. \tag{5.32}$$

With

$$R = \sqrt{1 + \frac{1}{\overline{\beta}^2}\left(\frac{\dot{x}^2}{\tau^2} - \dot{y}^2\right)} \tag{5.33}$$

and

$$\overline{\beta} = \frac{\beta}{e\eta^2}, \qquad \delta = \frac{\kappa}{\pi e \eta}, \tag{5.34}$$

where dot denotes differentiation with respect to τ. In terms of these new quantities our boundary conditions become;

$$\begin{aligned} x(\infty) &= 0, & x(0) &= n \\ y(\infty) &= 0, & y(0) &= \frac{c}{\eta} \\ z(\infty) &= 1, & z(0) &= 0 \end{aligned} \tag{5.35}$$

It is also necessary to find the expressions for the action S and the energy E in terms of these new quantities.

$$\begin{aligned} S &= 2\pi\eta^2\overline{\beta}^2 \int \tau d\tau \left[1 - R + \frac{1}{2\overline{\beta}^2}^2 \left(-\frac{x^2 z^2}{\tau^2} - \dot{z}^2 + y^2 z^2\right)\right] \\ &\quad + 2\pi\eta^2\overline{\beta}^2 \int \tau d\tau \left[\delta\frac{\dot{x}}{\tau}y - \frac{1}{\overline{\beta}^2}\frac{\lambda\eta^2}{8e^2}z^2(z^2-1)^2\right] \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} E &= \int d^2x T_{00} \\ &= 2\pi\eta^2\overline{\beta}^2 \int \tau d\tau \left(\frac{1}{R}(1 + \frac{1}{\overline{\beta}^2}\frac{\dot{x}^2}{\tau^2}) - 1 + \frac{1}{2\overline{\beta}^2}(\dot{z}^2 + y^2 z^2 + z^2\frac{x^2}{\tau^2})\right) \\ &\quad + 2\pi\eta^2\overline{\beta}^2 \int \tau d\tau \frac{1}{\overline{\beta}^2}\frac{\lambda\eta^2}{8e^2}z^2(z^2-1)^2. \end{aligned} \tag{5.37}$$

Here $T_{\mu\nu}$ is the energy momentum tensor. the Chern-Simons coefficient δ enters in the action but not in the energy of the vortex.

To obtain a detailed profile of the vortex solution, we solved numerically the differential equations (5.30), (5.31) and (5.32) using shooting method for boundary values problem [34].

In order to use the shooting method, we must write our coupled second-order ordinary differential equations as a set of N coupled first order ordinary differential equations satisfying n_1 boundary conditions at the starting point x_1, and a remaining set of $n_2 = N - n_1$ boundary conditions at the final point x_2. All differential equations of order higher than first can be written as coupled sets of first-order equations. By introducing variables W, S, T such that

$$W = \frac{dx}{d\tau}, \qquad S = \frac{dy}{d\tau}, \qquad T = \frac{dz}{d\tau}, \tag{5.38}$$

we achieve a set of 6 coupled first-order differential equations to be solved for 6 variables:

$$\frac{dx}{d\tau} = W, \tag{5.39}$$

$$\frac{dy}{d\tau} = S, \tag{5.40}$$

$$\frac{dz}{d\tau} = T, \tag{5.41}$$

$$\begin{aligned} \frac{dW}{d\tau} &= \frac{W^3\tau - S^2W\tau^2 - \overline{\beta}^2 W - xz^2RS^2\tau^2 - \overline{\beta}^2 xz^2R\tau^2}{W^2S^2 - \tau W^2S^2 - \overline{\beta}^2\tau^2 - W^2 + S^2\tau^2} \\ &\quad + \frac{z^2xRW^2 - \overline{\beta}^2 z^2xR\tau^2S^2 + \overline{\beta}^2\delta\tau^3R^5}{W^2S^2 - \tau W^2S^2 - \overline{\beta}^2\tau^2 - W^2 + S^2\tau^2}, \end{aligned} \tag{5.42}$$

$$\begin{aligned} \frac{dS}{d\tau} &= \frac{\tau^2S^2 + z^2yW^2 - \overline{\beta}^2 S\tau^2 - xz^2\tau RWS}{W^2S^2 - \tau W^2S^2 - \overline{\beta}^2\tau^2 - W^2 + S^2\tau^2} \\ &\quad + \frac{-\overline{\beta}^2 Rz^2y\tau^2 - Rz^2yW^2 + \overline{\beta}^2 Rz^2y\tau^2S^2 - \overline{\beta}^2\delta RW\tau}{W^2S^2 - \tau W^2S^2 - \overline{\beta}^2\tau^2 - W^2 + S^2\tau^2}, \end{aligned} \tag{5.43}$$

$$\frac{dT}{d\tau} = \frac{\lambda}{8}\frac{1}{e^2}(6z^5 - 8z^3 - 2z) + z(\frac{x^2}{\tau^2} - y^2) - \frac{T}{\tau}. \tag{5.44}$$

Now R is defined as

$$R = \sqrt{1 + \frac{1}{\beta^2}\left(\frac{W^2}{\tau^2} - S^2\right)}. \tag{5.45}$$

For simplicity, we will concentrate in the case $n = \eta = e = 1$. With the boundary conditions given in (5.35), we are ready to use the method. The profile of vortex solutions for different values of β, for $\delta = 0.1$ and $\lambda = 1$ are shown in (figure 5.1, 5.2, 5.3).

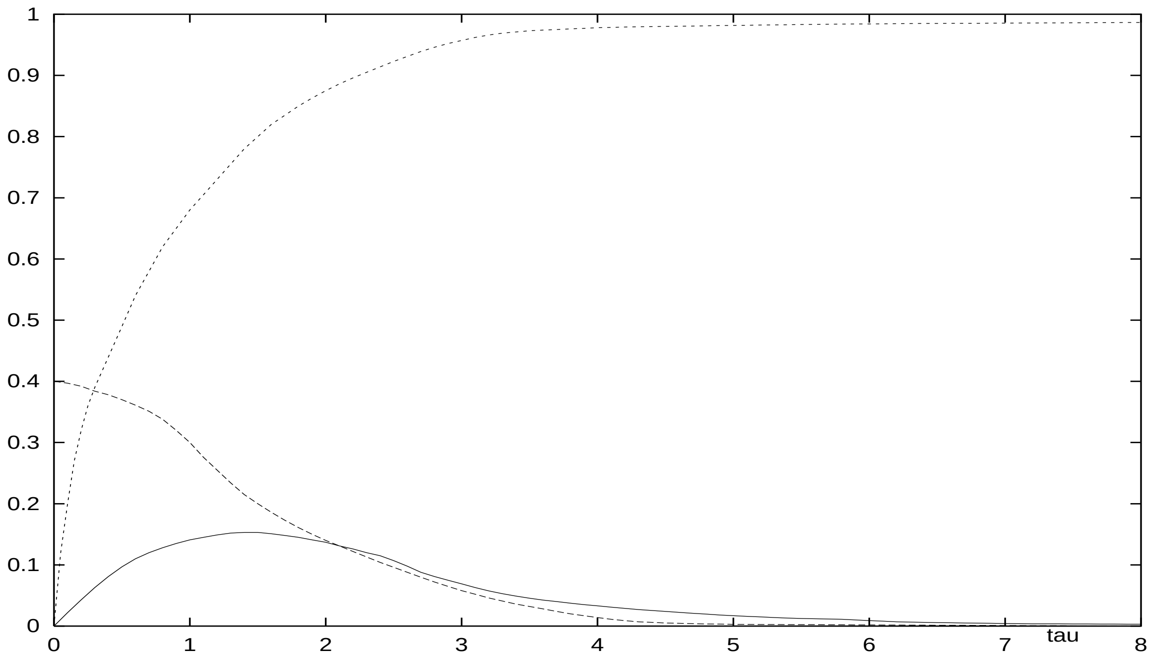

Figure 5.1: Vortex solution for $\beta = 20, \delta = 0.1, \lambda = 1$. The solid line corresponds to the 10 times electric field $10E(\tau)$, the dashed line correspond to the magnetic field $H(\tau)$ and the dot line to the Higgs field $|\phi(\tau)|$.

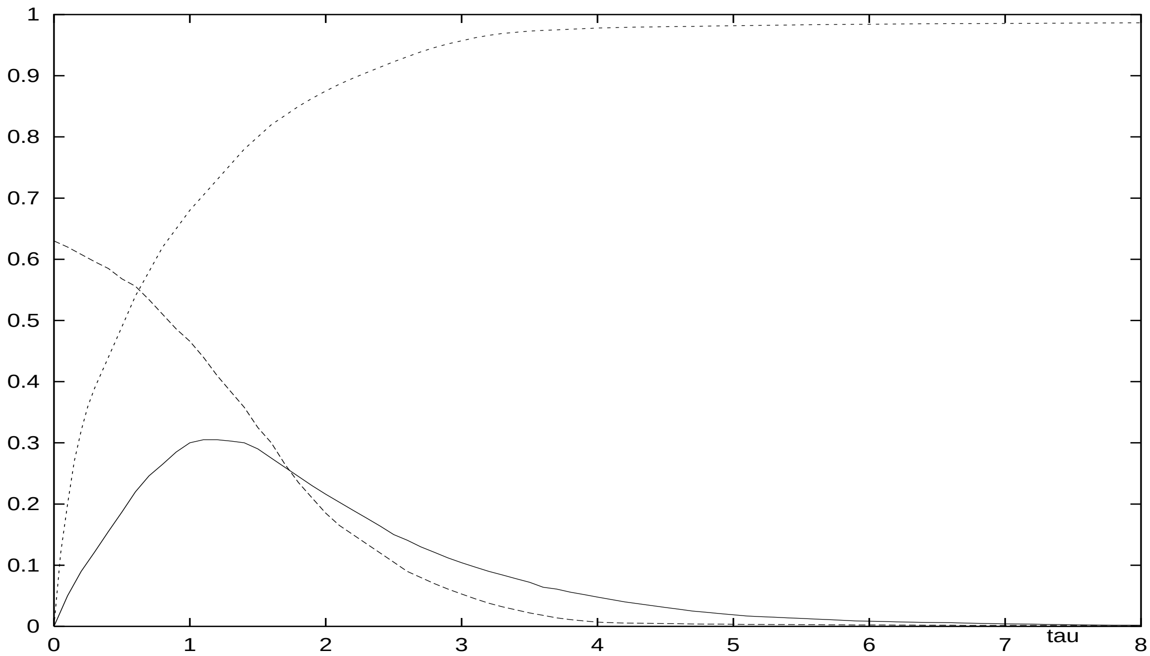

Figure 5.2: Vortex solution for $\beta = 5, \delta = 0.1, \lambda = 1$. The solid line corresponds to the 10 times electric field $10E(\tau)$, the dashed line correspond to the magnetic field $H(\tau)$ and the dot line to the Higgs field $|\phi(\tau)|$.

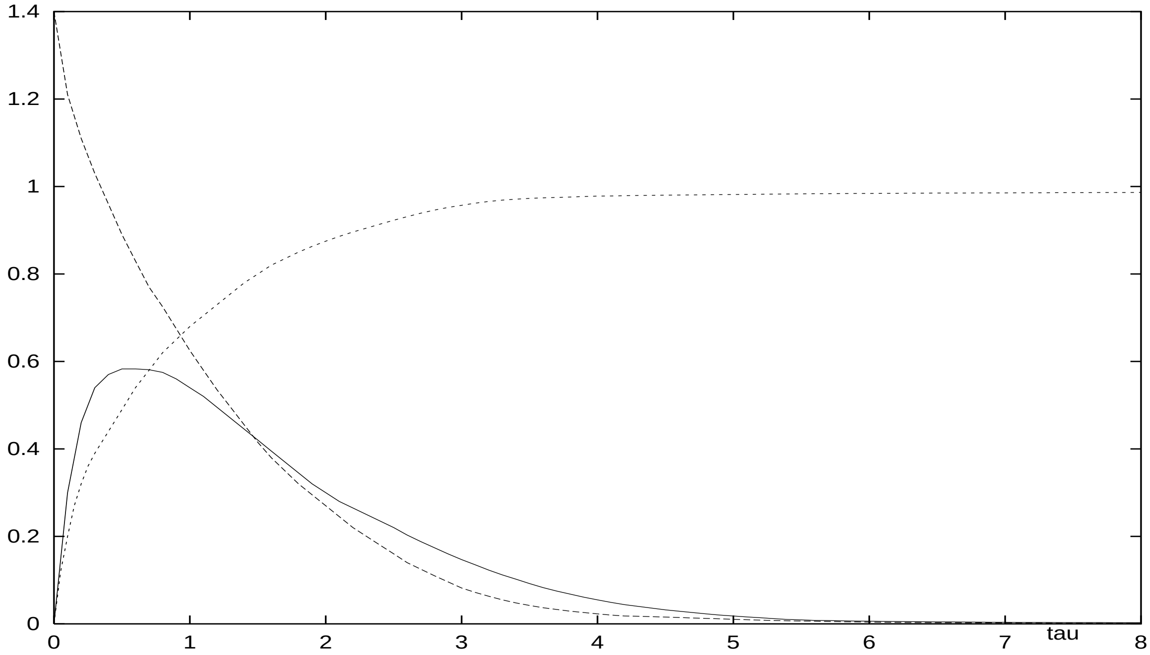

Figure 5.3: Vortex solution for $\beta = 0.5, \delta = 0.1, \lambda = 1$. The solid line corresponds to the 10 times electric field $10E(\tau)$, the dashed line correspond to the magnetic field $H(\tau)$ and the dot line to the Higgs field $|\phi(\tau)|$.

It can be seen from numerical solutions that as expected, for large values of the Born-Infeld parameter β and for small values of the Chern-Simons parameter δ, the solution differs very little from the Maxwell-Higgs vortices (Nielsen-Olesen vortices) [10] (figure 5.1). For large values of β, the Born-Infeld term gets closer to the Maxwell term and as δ goes to zero, the Chern-Simons term goes also to zero. Therefore, our Born-Infeld-Chern-Simons-Higgs system gets closer to the Maxwell-Higgs system when β is large and δ is small. This result can be considered as the evidence of truth of our calculation.

As β decreases the vortex profile changes notably (figure 5.2), (figure 5.3). The magnetic field at the origin tends to infinity, as β approaches to some critical value and the numerical solution ceases to exist if β is smaller than its critical value. This critical value depends on the remaining parameters δ and λ. Although we were unable to find an analytical argument to account for the existence of this singularity, we have enough numerical evidence that supports our claim.

CHAPTER 6

CONCLUSION

In this final chapter, we would like to briefly summarize the discussions of the preceding chapters and conclude the results we have reached throughout the study.

In order to follow a systematic way, we have started our work from the general concepts and reach, through the ends, the specific system, we have dealt with. Therefore, we begin chapter 2 by the discussion of the concept of solitons. The history and importance in physics of the solitons are discussed. Although, solitons have been appeared in the world of science since 1842, their importance in physics has been understood in 1970's. Until that date, soliton solutions have been studied in great detail. In the next sections of chapter 2, we focus on the vortex solutions, which are soliton solutions in 2-dimensional models. Firstly, we construct the vortex, a 2-dimensional field configuration consisting of a charged scalar field and a gauge field. Then, we review the famous Nielsen-olesen vortices (vortex solution of Abelian-Higgs model). The starting point of the model is a

Maxwell-Higgs Lagrangian. However, in our system, we used Born-Infeld-Chern-Simons-Higgs Lagrangian. So, we also review the Born-Infeld and Chern-Simons theories in the following chapters.

In chapter 3, we review the Chern-Simons theory briefly. Chern-Simons theories involve charged scalar field minimally coupled to gauge fields whose dynamics is provided by a Chern-Simons term in 2+1 dimensions. We give the basic properties of the theory, and also give the basic description of the Abelian relativistic Chern-Simons model which is exactly the one we use in our model. The contribution of Chern-Simons theory to our model is that: only if one adds a Chern-Simons term, finite energy charged vortices can be found. Since Nielsen-Olesen vortices does not include Chern-Simons term, they are neutral.

In chapter 4, Born-Infeld theory is briefly reviewed. The theory was first introduced by Born and Infeld as an alternative for Maxwell theory. It is non-linear version of Maxwell theory and it has classical solutions representing electrically charged particles with finite self-energy. From this point of view, it is a successful theory but it can not be quantized. So after the birth of quantum electrodynamics, the theory was forgotten by physicists for a long time. However after solitons became an important subjects in physics, physicists again began to work on Born-Infeld theory. In soliton theory, one obtains information about relativistic quantum field theories starting from classical solutions of the corresponding field equations. Born-Infeld theory is also a classical theory, so it is convenient to use the theory in soliton solutions. By doing this, one can obtain more information about the theory.

In chapter 5, we first construct the model which includes a Born-Infeld-Chern-Simons-Higgs Lagrangian in 2+1 dimensions and we look for magnetic vortex solutions carrying also electric charge. We first find the equations of motion from the Lagrangian. Then we focus on the rotationally symmetric solutions and choose the specific potential of the model. The potential gives rise to topological and non-topological solutions. We are interested only in finding topological solutions. The boundary conditions for the case of topological solutions is derived and finally we solved the equations of motion numerically by using shooting algorithm in order to obtain a detailed profile of the vortex. We have seen numerically that for β (Born-Infeld parameter) sufficiently large, the corresponding Born-Infeld solution differs very little from the Maxwell-Chern-Simons vortices. When β decreases the vortex profile changes notably and for very small values of β the magnetic field H at the origin diverges.

We conclude that vortex solutions in Born-Infeld theories present many interesting features which deserve a thoroughfull study.

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