INERT SUBGROUPS AND CENTRALIZERS OF INVOLUTIONS IN LOCALLY FINITE SIMPLE GROUPS

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INERT SUBGROUPS AND CENTRALIZERS OF INVOLUTIONS IN LOCALLY FINITE SIMPLE GROUPS

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

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A subgroup H of a group G is called inert if $[H : H \cap H^g]$ is finite for all $g \in G$. A group is called totally inert if every subgroup is inert. Among the basic properties of inert subgroups, we prove the following. Let M be a maximal subgroup of a locally finite group G. If M is inert and abelian, then G is soluble with derived length at most 3. In particular, the given properties impose a strong restriction on the derived length of G.

We also prove that, if the centralizer of every involution is inert in an infinite locally finite simple group G, then every finite set of elements of G can not be contained in a finite simple group. In a special case, this generalizes a Theorem of Belyaev–Kuzucuoğlu–Seçkin, which proves that there exists no infinite locally finite totally inert simple group.

Keywords: inert groups, involution, locally finite groups, commensurable property

ÖZ

DİNGİN ALTGRUPLAR VE YEREL SONLU BASİT GRUPLARDA İNVOLUSYONLARIN MERKEZLEYENİ

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H grubu G nin bir altgrubu olsun. Eğer verilen her $g \in G$ için $[H : H \cap H^g]$ sonlu ise H altgrubuna dingin altgrup denir. G grubunun bütün altgrupları dingin ise G ye tam dingin grup denir. Dingin altgrupların temel özelliklerinin yanı sıra şu propozisyonu kanıtladık. M yerel sonlu, sonsuz G grubunun maksimal bir altgrubu olsun. Eğer M dingin ve değişmeli ise G grubu çözülebilirdir ve çözülebilirlik derecesi en fazla 3 tür. Böylece, dinginlik ve değişmelilik özellikleri G nin çözülebilirlik derecesi üzerinde güçlü kısıtlamalar ortaya çıkarmıştır.

Aynı zamanda, eğer yerel sonlu, sonsuz bir basit grup içerisinde her involusyonun merkezleyeni dingin ise bu durumda G grubundan alınan her sonlu küme için bu kümeyi içeren sonlu basit grup olamayacağı kanıtlanmıştır. Bu Belyaev-Kuzucuoğlu-Seçkin'e ait, tam dingin sonsuz basit grup yoktur, teoreminin özel durumda genelleştirilmiş halidir.

Anahtar sözcükler: dingin altgruplar, involusyon, yerel sonlu gruplar, akranlık

To my baby who will born soon

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CHAPTER 1

INTRODUCTION

A natural question in group theory is the following: If subgroups of a group satisfy a certain property, then does this property impose some structural property on the group or not? This question will be our main motivation for our interest in inert subgroups. A subgroup H of a group G is called an inert subgroup of G, if for all $g \in G$, $|H : H \cap H^g|$ is finite . Obviously, the group G itself, finite subgroups of G and normal subgroups of G are inert subgroups of G. These are called the trivial inert subgroups. A group G is called totally inert, if every subgroup of G is inert in G.

In Chapter 2, we obtain basic properties and examples of inert subgroups. A natural example is the following: In a barely transitive group, a stabilizer of a point is inert in G. Recall that a group G is called barely transitive, if it acts on an infinite set Ω faithfully and transitively such that every orbit of every proper subgroup is finite. We also show in detail that the subgroup $SL(n, \mathbb{Z})$ is inert in $GL(n, \mathbb{Q})$. A group G is called an FC-group, if centralizer of every element has finite index in G. There is a connection between totally inert groups and FC-groups. Namely, the class of FC-groups is contained in the class of totally inert groups. But this containment is proper, see Example 2.19.

Two subgroups H and K of a group G are called commensurable, if $H \cap K$ is a subgroup of finite index in both H and K. For a simple totally inert group G, we show in Lemma 2.31 that for any two non identity elements x an y in G the groups $C_G(x)$ and $C_G(y)$ are commensurable. Moreover, for a totally inert simple group, if x is a non-identity torsion element in G, then $C_G(x)$ is finite. Observe that the assumption about centralizers gives a restriction on G.

In Chapter 3, we discuss the commensurable property. A group G satisfies the commensurable property, if any two non-identity subgroups are commensurable. We prove that any group satisfying commensurable property is countable, see Lemma 3.3. In Lemma 3.7, we show that in an infinite locally graded group containing a torsion element, any two proper non-identity subgroups of G are commensurable if and only if G is isomorphic to C_{p^∞} for some prime p. This gives another characterization of the well known group C_{p^∞}. For abelian groups Lemma 3.9 gives a strong restriction on the infinite group G satisfying commensurable property namely such a group is either isomorphic to $\mathbb{Z}$ or C_{p^∞}.

It is well known that a finite group with a maximal abelian subgroup is soluble. In Lemma 3.12, by using this result, we show that if M is a maximal subgroup of a locally finite group G and M is inert and abelian in G, then G is soluble with derived length at most 3.

The second part of the thesis deals with simple groups. In [2], it is proved that there exists no locally finite simple totally inert group. But on the other hand the groups constructed by Ol'sanskii [23] are the examples of infinite simple groups in which every proper subgroup is finite. Therefore these groups are totally inert simple groups, but observe that such groups are non-locally finite groups as they are generated by any two non-commuting elements. So examples of non locally finite simple totally inert groups exist.

The general question we are interested in is the following: Does there exist a locally finite simple group in which centralizer of every involution is inert? By Lemma 4.29, this question can be reduced to the countable case. By [19, Theorem 4.4], a locally finite countable simple group has a local system consisting of finite subgroups G_i and maximal normal subgroups M_i in G_i such that $G_i \cap M_{i+1} = 1$. By this property, G_i can be embedded into a finite simple group G_{i+1}/M_{i+1}. The sequence (G_i, M_i) is called a Kegel sequence. It is well known that there are examples of infinite locally finite simple groups such that we can not choose $M_i = 1$ for all i. Namely, there are countably infinite simple groups, which can not be a union of finite simple groups [32].

If G has a local system consisting of finite simple subgroups G_i, then $C_G(x) = \bigcup_{i=1}^{\infty} C_{G_i}(x)$. Therefore, the structure of the centralizers of elements in finite simple groups may give information about the centralizers of elements in infinite locally finite simple groups. For the structure of centralizer of elements in finite simple groups there is a vast amount of information. Using this information, one can get some information about $C_G(x)$, see [21]. On the other hand, if a Kegel sequence (G_i, N_i) is given, then the groups G_i/N_i are simple for all i. We need information about centralizer of elements in G. But we have some information about the centralizers of elements in G_i/N_i. It is usually difficult to use the information about $C_{G_i/N_i}(x_i N_i)$ to understand the structure of $C_{G_i}(x)$.

Centralizer of elements in simple groups of Lie type has been studied, if the order of the element and characteristic of the field are relatively prime, [21]. In this work, we have involutions in linear groups over fields of characteristic 2 and we need structure of centralizers of involutions in these simple groups. In the finite case, this has been studied in detail by the work of Aschbacher and Seitz, see [1]. By using their results and the above results, we prove the following Theorem: If centralizer of every involution is inert in an infinite locally finite simple group G, then every finite set of elements of G cannot lie in a finite simple group. This theorem generalizes in some special cases the theorem of Belyaev-Kuzucuoğlu-Seçkin. In their theorem, every subgroup is inert, but we assume only that the centralizer of every involution is inert in G.

Unfortunately, but perhaps inevitably the proof of the theorem uses the classification of finite simple groups (CFSG). By using the known results in finite simple groups and extensions of these results to some locally finite simple groups, we prove that, if the rank of the infinite simple group of Lie type is greater than or equal to 3, then it has involutions such that centralizers of these involutions involve an infinite simple group. For the simple groups of Lie type of rank ≤ 3, we show case by case that in these simple groups centralizers of involutions are not inert in G. Finally we show that such a simple group does not exist.

CHAPTER 2

Inert Subgroups

2.1 Some Examples Of Inert Subgroups

Definition 2.1. *A subgroup H of a group G is called an inert subgroup of G, if $[H : H \cap H^g] < \infty$ for all $g \in G$. A proper inert subgroup is one, which is neither G, nor a finite subgroup of G.*

It is clear that every normal subgroup is inert. Also, every finite subgroup is inert. The other trivial example is that the group G is itself inert. We give some examples of proper inert subgroups.

Example 2.2. *If $G = GL(n, \mathbb{Q})$, then $S = SL(n, \mathbb{Z})$ is inert in G.*

For $T \in S$, $[T]_N$ stands for the $n \times n$ matrix whose (i, j) entry is $[T_{ij}]_N$.

$$S_N = \{T \in S \mid [T]_N = [I]_N\}.$$

Also for A, $B \in S$, $A \equiv B \pmod N$ has the obvious meaning.

Lemma 2.3. *Let $b \in \mathbb{Z}^+$. The map*

$$\begin{array}{rcccc} \varphi & : & S = SL(n,\mathbb{Z}) & \to & SL(n,\mathbb{Z}_b) \\ & & a & \mapsto & [a]_b \end{array}$$

is an epimorphism. Moreover, $S/\ker\varphi$ is finite.

Proof. For simplicity, set $\bar{a} = [a]_b$. The map φ is well-defined: $a = b$ implies $a_{ij} = b_{ij}$ for all i, j. This implies $\bar{a}_{ij} = \bar{b}_{ij}$ and that $\bar{a} = \bar{b}$.

The map is a homomorphism: $\varphi(ab) = \overline{ab} = \bar{a}\bar{b} = \varphi(a)\varphi(b)$ The map φ is onto: indeed, given any $(\bar{a}_{ij}) \in SL(n, \mathbb{Z}_b)$ there is $a_{ij} \in S$ such that $\varphi(a_{ij}) = (\bar{a}_{ij})$. Note that

$$\begin{aligned} \ker\varphi &= \{a \in S \mid \varphi(a) = \bar{I}_n\} \\ &= \{a \in S \mid a_{ij} \equiv \delta_{ij} \pmod{b}\} \\ &= \{a \in S \mid a \equiv I_n \pmod{b}\} \\ &= S_b. \end{aligned}$$

By the first isomorphism theorem, we get $S/S_b \cong SL(n, \mathbb{Z}_b)$. Then $[S : S_b] < \infty$. □

Lemma 2.4. *Let $\beta \in M(n, \mathbb{Z})$, where $\det\beta = b \neq 0$. Then $S_{Nb} \subseteq \beta^{-1}S_N\beta \cap \beta S_N\beta^{-1}$ for every nonnegative integer N.*

Proof. Let $S = SL(n, \mathbb{Z})$ and $G = GL(n, \mathbb{Q})$. Let $\beta' = b\beta^{-1} = b.\frac{1}{det\beta}adj(\beta) = b.b^{-1}adj(\beta) = adj(\beta) \in M(n, \mathbb{Z})$. If $T \equiv I_n \pmod{Nb}$, then $\beta'T\beta \equiv \beta'\beta \pmod{Nb} \equiv b\beta^{-1}\beta \pmod{Nb} \equiv b.I_n \pmod{Nb}$. This implies that $b\beta^{-1}T\beta \equiv b.I_n \pmod{Nb}$. Then $\beta^{-1}T\beta \equiv I_n \pmod{N}$. Therefore $\beta^{-1}T\beta \in M(n, \mathbb{Z})$

Let $T \in S_{Nb}$. Then $T \equiv I_n \pmod{Nb}$ and $\det(\beta^{-1}T\beta) = 1$ so $\beta^{-1}T\beta \in S_N$. This implies that $T \in \beta S_N\beta^{-1}$. Similarly, $\det(\beta T\beta^{-1}) = 1$ and $\beta T\beta^{-1} \in S_N$. Then $T \in \beta^{-1}S_N\beta$. Therefore, $T \in \beta S_N\beta^{-1} \cap \beta^{-1}S_N\beta$ which completes the proof of lemma. □

Proof Of Example Now, let us prove that S is inert in G. Let $\alpha \in GL(n, \mathbb{Q})$. We can write $\alpha = c\beta$ where $c \in \mathbb{Q}$ and $\beta \in GL(n, \mathbb{Z})$. We obtain $\alpha S \alpha^{-1} = \beta S \beta^{-1}$. Indeed, let $\alpha a \alpha^{-1} \in \alpha S \alpha^{-1}$. Then substituting $\alpha = c\beta$ we get $c\beta a(\frac{1}{c}\beta^{-1}) = \frac{c}{c}\beta a \beta^{-1} \in \beta S \beta^{-1}$. Conversely, if $\beta d \beta^{-1} \in \beta S \beta^{-1}$. Then $\frac{c}{c}\beta d \beta^{-1} = c\beta d \frac{1}{c}\beta^{-1} \in \alpha S \alpha^{-1}$. Now, let us take $N = 1$, and $b = \det(\beta)$. Then $S_b \subseteq \beta^{-1} S \beta \cap \beta S \beta^{-1} \subseteq S \cap \beta S \beta^{-1} = S \cap \alpha S \alpha^{-1}$. Then by Lemma 2.3 we get $[S : S_b] < \infty$ and thus $[S : S \cap \alpha S \alpha^{-1}] < \infty$. Hence $SL(n, \mathbb{Z})$ is inert in $GL(n, \mathbb{Q})$. □

Definition 2.5. *Let G be a permutation group on an infinite set Ω. We say that G is a barely transitive group if G acts transitively on Ω, and all orbits of any proper subgroup of G are finite. The concept of a barely transitive permutation group was introduced by Hartley in* [13].

Lemma 2.6. [13, Lemma] *An infinite group G can be represented faithfully as a barely transitive group if and only if G possesses a subgroup H such that $\bigcap_{x \in G} H^x = \{1\}$ and $[K : K \cap H] < \infty$ for every proper subgroup $K < G$.*

Example 2.7. (*Barely transitive group*) *Let G be a locally finite barely transitive permutation group on the set Ω and H be some point stabilizer. By Lemma* 2.6 *for every proper subgroup K of G, the intersection $K \cap H$ has finite index in K. In particular, given any $g \in G$ the group $H^{g^{-1}}$ is a proper subgroup of G, then $[H^{g^{-1}} : H^{g^{-1}} \cap H] < \infty$. Then taking conjugate with g we get $[H : H \cap H^g] < \infty$ for all $g \in G$. Thus H is an inert subgroup in G.*

Definition 2.8. *Let X and Y be two subgroups of a group G. We say that, X and Y are commensurable, if $[X : X \cap Y] < \infty$ and $[Y : X \cap Y] < \infty$.*

Lemma 2.9. *A subgroup H is inert in a group G if and only if H^x and H^y are commensurable for all $x, y \in G$.*

Proof. Let H be inert in G. Then we have $[H : H \cap H^{yx^{-1}}] < \infty$. Taking conjugates with x we get $[H^x : H^x \cap H^y] < \infty$. Similarly, $[H : H \cap H^{xy^{-1}}] < \infty$ Taking conjugates with y, we get $[H^y : H^x \cap H^y] < \infty$. Then H^x and H^y are commensurable.

Conversely, if H^x and H^y are commensurable for all $x, y \in G$. Then taking $x = 1$ we get $[H : H \cap H^y] < \infty$ for all $y \in G$. So H is inert in G. □

Example 2.10. (*Open compact subgroups*) *Let G be a topological group space(G is a topological space and the group operation and the inverse operation are continuous). Then an open compact subgroup H is inert in G.*

Proof. Since H is open, $g^{-1}Hg$ are open for all $g \in G$, and $H \cap H^g \leq H$. By compactness the following open cover

$$H = \bigcup_{x_i \in X} x_i(H \cap H^g),$$

where X is the left transversal of H in G, has a finite subcover. Hence there exists an element $n \in \mathbb{N}$ such that $H = \bigcup_{i=1}^{n} x_i(H \cap H^g)$. Therefore $[H : H \cap H^g] < \infty$. □

Example 2.11. (*Locally finite graph*) *A graph Γ is a pair (V, E) of sets V (of vertices) and E (of edges) where $E \subset V \times V$. An edge $(\alpha, \beta) \in E$ is said to join α to β, and β is adjacent to α. The degree of α is the number of vertices to which α is adjacent. If α and β are vertices of a graph Γ, then a directed path in Γ from α to β of length d is a list of $d + 1$ vertices*

$$\alpha = \alpha_0, \alpha_1, \ldots, \alpha_d = \beta$$

such that $(\alpha_{i-1}, \alpha_i) \in E$ for $i = 1, \ldots, d$.

Let $\Gamma = (V, E)$ be a connected locally finite graph (i.e. the degree of each vertex is finite). If we consider a group G as

$$G = Aut(\Gamma) = \{g \in Sym(V) \mid (a, b) \in E \text{ iff } (g(a), g(b)) \in E\},$$

then $Stab_G(a) = G_a$ is inert in G for all $a \in V$.

Proof. We need to prove $[G_a : G_a \cap G_a^x] < \infty$ for all $x \in G$. Note $G_a^x = G_{x(a)}$, say G_b. It is enough to prove that G_a and G_b are commensurable for all $a, b \in V$. Let us define a distance between two vertices as length of the shortest path from a to b and denote it by $d(a, b)$. Now, let $\delta = d(a, b) < \infty$. Then $C(a, \delta) = \{v \in V \mid d(a, v) = \delta\}$ is finite since Γ is a connected locally finite graph. So the orbit of b under G_a is $G_a(b) = \{g(b) \mid g \in G_a\} \subseteq C(a, \delta)$, since automorphism preserves distance. Also $(G_a)_b = \{g \in G_a \mid g(b) = b\} = G_a \cap G_b$. Then $|G_a(b)| = [G_a : (G_a)_b] = [G_a : G_a \cap G_b] < \infty$, since $|G_a(b)| \leq |C(a, \delta)| < \infty$. Therefore G_a and G_b are commensurable. □

Example 2.12. (*Totally imprimitive groups*) *Let G be a permutation group which acts transitively on Ω. Note that $\triangle \subseteq \Omega$ is called a "G-block" if $g(\triangle) = \triangle$ or $g(\triangle) \cap \triangle = \emptyset$ for all $g \in G$. Moreover, G is called "totally imprimitive" group, if for any finite subset $\Sigma \subseteq \Omega$, there exists a finite G-block $\triangle$ containing Σ. If G is a totally imprimitive permutation group on Ω, then the stabilizer of any point in Ω is inert in G.*

Proof. Let G_x be a stabilizer of x in G. We have $G_x = \{g \in G \mid g(x) = x\}$. Since $G_x^g = G_{g(x)}$ for all $g \in G$ and for all $x \in \Omega$, it is sufficient to prove that G_α and G_β are commensurable for any $\alpha, \beta \in \Omega$. Since $G_\alpha \cap G_\beta = (G_\alpha)_\beta$, $[G_\alpha : (G_\alpha)_\beta] < \infty$ if and only if the orbit of β with respect to G_α is finite, i.e. $|G_\alpha(\beta)| < \infty$. According to the definition of totally imprimitive group, there exists a finite G-block $\triangle$, such that $\{\alpha, \beta\} \subseteq\triangle$. We have $\alpha = g(\alpha)$ for all $g \in G_\alpha$. Therefore, $g(\triangle) =\triangle$ for all $g \in G_\alpha$. Then $G_\alpha(\beta) = \{g(\beta) \mid g \in G_\alpha\} \subseteq\triangle$ and $\triangle$ is finite. Then $|G_\alpha(\beta)| < \infty$. □

Example 2.13. (*Locally finite groups*) *Let G be a countable locally finite group. Then we can find finite subgroups $G_1 \leq G_2 \leq \ldots$ such that $G = \cup_{i=1}^{\infty} G_i$. If $N_i \trianglelefteq G_i$, then $N := \langle N_i \mid i \in I\rangle$ is inert in G.*

Proof. Let $g \in G$, then $g \in G_i$ for some i. Hence $g \in G_j$ for all $j \geq i$. Note that $N_j \trianglelefteq G_j$ implies $N_j^g = N_j$ for all $j \geq i$. Now, let $L := \langle N_j \mid j \geq i\rangle$ and $K = \langle N_k \mid k < i\rangle$. First of all, we have $N = \langle L, K\rangle$. Since G is countable and locally finite, and all N_k are finite. We have $K = \langle N_k \mid k < i\rangle = \langle N_1, N_2, \ldots, N_{i-1}\rangle$ is finite. We have $L \trianglelefteq N$. Indeed, $L^x = L$ for all $x \in N_k$, where $k < i$ and given $x \in G_j$ for $j \geq i$, $N_j \trianglelefteq G_j$ implies $N_j^x = N_j$. Then $L^x = L$ for all $x \in N$. Therefore, $L \trianglelefteq N$. Clearly, $N = \langle K, L\rangle$. Since K is finite, this implies that $[N : L]$ is finite. Since $N_j^g = N_j \leq N^g$ for all $j \geq i$, then $L \leq N \cap N^g$ and hence $[N : N \cap N^g] \leq [N, L] < \infty$. So N is inert in G. □

We give an example using the following theorem, [7].

Theorem 2.14. [7, Theorem 1] *Let G be a group and H a subgroup of G. Then $[H : H \cap H^g] \leq 2$ for all $g \in G$ if and only if G has a normal subgroup N such that either*

(a) $H \leq N$ and $[N : H] \leq 2$, or

(b) $N \leq H$ and $[H : N] \leq 2$.

Example 2.15. *Let V be a vector space of large (possibly infinite) dimension over the field of p elements. Let H be a subspace of codimension 1 and let A be a group of automorphisms of V large enough so that the intersection in V of the* *orbits of H under A has "large" codimension. Finally let G be the semidirect product determined by the action of A on V. Then H is inert in V.*

Proof. Let

$$V = \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p \ldots = Dr_{i \in \mathbb{N}} (\mathbb{Z}_p)^i$$

Let us define the automorphism

$$\begin{array}{cccccc} L_i^j & : & V & \to & V & \\ & & e_i & \mapsto & e_j & \\ & & e_j & \mapsto & e_i & \\ & & e_k & \mapsto & e_k & \text{where} \quad k \notin \{i, j\}. \end{array}$$

We may assume that $H = \langle \mathcal{B} \backslash \{e_1\} \rangle$. Clearly $V = \langle e_1 \rangle \times H$. We get $H \cap H^{L_1^i}$ has codimension 2 in V. Then $[H : H \cap H^{L_1^i}] = p$.

Now, let

$$A = \langle L_1^i \mid i \in \mathbb{N} \rangle \quad \text{and} \quad G = V \rtimes A.$$

Then $[H : H \cap H^{va}] = [H : H \cap H^a] < \infty$. Note that V is abelian, then $H^{av} = H^a$. So H is inert in G. □

2.2 Properties of Inert Subgroups

Definition 2.16. *An element g of a group G is called an FC-element, if it has only a finite number of conjugates in G, that is to say, $[G : C_G(g)]$ is finite. The subgroup of all FC-elements is called the FC-center. A group is called an FC-group if it equals its FC-center, i.e. every conjugacy class in G is finite.*

Definition 2.17. *A group G is called totally inert group (TIN-group), if every subgroup of G is inert in G.*

Example 2.18. *Every FC-group is TIN-group.*

Proof. Let G be an FC-group. Then we need to show that for any subgroup H in G and any element $x \in G$, $[H : H \cap H^x] < \infty$. Since G is an FC-group, $[G : C_G(x)] < \infty$. Let us show that $C_H(x) \leq H \cap H^x$. Taking intersection with H we get $[H \cap G : H \cap C_G(x)] < \infty$. So $[H : C_H(x)] < \infty$. But $C_H(x) = (C_H(x))^x \leq H^x$. Then $C_H(x) \leq H \cap H^x$. Hence $[H : H \cap H^x] \leq [H : C_H(x)] < \infty$. Then H is inert in G. □

Therefore TIN-groups are generalizations of FC-groups. So every FC-group is a TIN group but the converse is not true. The groups constructed by Ol'sanskii are the examples of non-FC TIN-groups. The following is an easier example of a non-FC TIN-group.

Example 2.19. *Let A be an infinite abelian $2'$-group. Let*

$$\begin{array}{rcrcl} t & : & A & \rightarrow & A \\ & & x & \mapsto & x^{-1}, \end{array}$$

where t is a fixed-point-free automorphism of A of order 2. Indeed, $x^t = x$ which implies $x^{-1} = x$, then $x^2 = 1$, but it is impossible, since A is $2'$ group. Namely there exists no element of order two. Then $x = 1$.

We construct the semidirect product $G = A \rtimes \langle t \rangle$

(1) G is not an FC-group.

(2) G is a TIN-group.

Proof. (1) $C_G(t) = \{g \in G \mid g^t = g\}$. We claim that $C_G(t) = \langle t \rangle$. Let $g \in C_G(t)$, then $g = xt$ for some $x \in A$. Since $g^t = g$, we have $(xt)^t = xt$. But then $x^{-1}t = xt$ which implies $x^2 = 1$. Since A is infinite abelian $2'$-group, we get $x = 1$ and so $[G : C_G(t)]$ is infinite. Hence, G is not an FC-group.

(2) Let H be any subgroup of G. $[H : H \cap A] = |HA/A| \leq |G/A| \leq \infty$. We claim that $H \cap A \trianglelefteq G$. Indeed, $H \cap A \leq A$ and A is abelian, then for $a \in A$, $(H \cap A)^a = H \cap A$. Since t is an involutory automorphism of A, $x^{at} = x^t = x^{-1} \in H \cap A$ for any $x \in H \cap A$ and $at \in G$ where $a \in A$. Thus, we get $H \cap A$ is normal in G. Moreover, $(H \cap A)^g = H \cap A \leq H^g$ and $H \cap A \leq H$ so $H \cap A \leq H \cap H^g$. Therefore $[H : H \cap H^g] \leq [H : H \cap A] < \infty$. Hence G is a TIN-group. □

Remark 2.20. *In the above example G is an FC-by-finite group. i.e. $A \triangleleft G$, A is an FC-group and G/A is finite. One may think that every FC-by-finite group is a TIN group. But this is not true as following example shows:*

Example 2.21. *Let $G = A \rtimes \langle t \rangle$ where $A = \langle a_1 \rangle \times \langle a_1^t \rangle \times \langle a_2 \rangle \times \langle a_2^t \rangle \ldots$, where A is infinite abelian group and t is an involution of G. Therefore $[G : A] = 2$ and so G is an FC-by-finite, since A is abelian group and so it is an FC-group. Let $C = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots$, then $C^t = \langle a_1^t \rangle \times \langle a_2^t \rangle \times \ldots$. Then $C \cap C^t = 1$. Therefore $[C : C \cap C^t]$ is infinite. Hence C is not inert in G.*

Lemma 2.22. *The intersection of two inert subgroups is an inert subgroup.*

Proof. We have $[H : H \cap H^g] < \infty$ and $[K : K \cap K^g] < \infty$. Then $[H \cap K : K \cap H \cap H^g] < \infty$ and $[H \cap K : K \cap H \cap K^g] < \infty$. Then $[H \cap K : (K \cap H \cap H^g) \cap (K \cap H \cap K^g)] < \infty$. So $[H \cap K : (H \cap K) \cap (H \cap K)^g] < \infty$. □

Lemma 2.23. *Let H be an inert subgroup in G and $K \leq G$. Then $H \cap K$ is inert in K.*

Proof. Since H is inert in G, given any $k \in K$, we have $[H : H \cap H^k] < \infty$. In order to show that $H \cap K$ is inert in K, we need to show that for any $k \in K$, $[H \cap K : (H \cap K) \cap (H \cap K)^k] < \infty$. Consider $H \cap H^k \cap K = H \cap H^k \cap K \cap K^k = (H \cap K) \cap (H^k \cap K^k) = (H \cap K) \cap (H \cap K)^k$. Hence, $[H \cap K : (H \cap K) \cap (H \cap K)^k] < \infty$. Therefore, $H \cap K$ is inert in K. □

Lemma 2.24. *If H is an inert subgroup of group G and K is a subgroup of finite index in H, then K is an inert subgroup of G.*

Proof. Since H is inert, $[H : H \cap H^g] < \infty$ and $[H : K] < \infty$. We need to show that $[K : K \cap K^g] < \infty$ for all $g \in G$.

Since $[H : K] < \infty$, taking conjugation with $g \in G$, we get $[H^g : K^g] < \infty$. Since H is inert, taking conjugation with g, we get $[H^g : H^g \cap H^{g^2}] < \infty$. Taking intersection with H, we get $[H \cap H^g : H \cap H^g \cap H^{g^2}] < \infty$ and $[H : H \cap H^g][H \cap H^g : H \cap H^g \cap H^{g^2}] = [H : H \cap H^g \cap H^{g^2}] < \infty$. Taking intersection with K, we get $[K : K \cap H \cap H^g \cap H^{g^2}] < \infty$. Taking conjugation with g, we get $[K^g : K^g \cap H^g \cap H^{g^2} \cap H^{g^3}] < \infty$. Since $[H^g : K^g] < \infty$, then $[H^g : K^g][K^g : K^g \cap H^g \cap H^{g^2} \cap H^{g^3}] = [H^g : K^g \cap H^g \cap H^{g^2} \cap H^{g^3}] < \infty$. Taking intersection with H we get $[H \cap H^g : H \cap H^g \cap K^g \cap H^{g^2} \cap H^{g^3}] < \infty$, so $[H : H \cap H^g][H \cap H^g : H \cap H^g \cap K^g \cap H^{g^2} \cap H^{g^3}] = [H : H \cap H^g \cap K^g \cap H^{g^2} \cap H^{g^3}] < \infty$. Also by the inequalities $[H : K \cap H^g \cap H^{g^2}] < \infty$ and $[H : H \cap H^g \cap K^g \cap H^{g^2} \cap H^{g^3}] < \infty$, we get $[H : H \cap H^g \cap K \cap K^g \cap H^{g^2} \cap H^{g^3}] < \infty$. This implies that $[H : K \cap K^g] < \infty$, then $[K : K \cap K^g] < \infty$, and K is inert in G. □

Lemma 2.25. *Let H be an inert subgroup of G and let K be a subgroup of G such that $[K : H] < \infty$. Then K is inert in G.*

Proof. We have $[H : H \cap H^g] < \infty$ for all $g \in G$ and $[K : H] < \infty$. We need to show that $[K : K \cap K^g] < \infty$.

Since $H \leq K$ then $H \cap H^g \leq K \cap K^g$. Then $[K : H][H : H \cap H^g] = [K : H \cap H^g] < \infty$. Then $[K : K \cap K^g] < [K : H \cap H^g] < \infty$. So we are done. □

Lemma 2.26. *If X is an inert subgroup of G and X is commensurable with the subgroup Y of G, then Y is an inert subgroup of G.*

Proof. Let $g \in G$. By assumption $[X : X \cap X^g] < \infty$, $[X : X \cap Y] < \infty$ and $[Y : X \cap Y] < \infty$. We want to show that $[Y : Y \cap Y^g] < \infty$ for all $g \in G$. Taking intersection with Y we get $[X \cap Y : X \cap Y \cap X^g] < \infty$. So $[Y : X \cap Y][X \cap Y : X \cap Y \cap X^g] = [Y : X \cap Y \cap X^g] < \infty$. Since taking conjugation preserves index, $[X : X \cap Y] < \infty$ implies that $[X^g : X^g \cap Y^g] < \infty$. Taking intersection with X, we get $[X \cap X^g : X \cap X^g \cap Y^g] < \infty$. Then $[X : X \cap X^g][X \cap X^g : X \cap X^g \cap Y^g] = [X : X \cap X^g \cap Y^g] < \infty$. This implies that $[X : X^g \cap Y^g] < \infty$. Using the inequalities $[Y : X \cap Y] < \infty$ and $[Y \cap X : Y \cap X \cap X^g \cap Y^g] < \infty$, then we obtain $[Y : Y \cap X][Y \cap X : Y \cap X \cap X^g \cap Y^g] = [Y : Y \cap X \cap X^g \cap Y^g] < \infty$, and $[Y : Y \cap Y^g] < \infty$ for all $g \in G$. □

Definition 2.27. *H is called residually finite group, if for any $1 \neq h \in H$ there exists a normal subgroup $N_h \lhd H$ such that $h \notin N_h$ and $[H : N_h] < \infty$.*

Lemma 2.28. (i) *If H is an infinite simple inert subgroup in G, then $H \trianglelefteq G$.*

(ii) *If G is simple and H is a proper inert subgroup of G, then H is residually finite.*

(iii) *If H is an inert subgroup of G and $N \lhd G$, then HN is inert in G.*

Proof. (i) Since H is inert, then $[H : H \cap H^g] = n < \infty$. Say $K = H \cap H^g$. Let $S = \{Kx_1, \ldots, Kx_n\}$ be a set of cosets of K in H. Then H acts on S by right multiplication, i.e for all $x \in H$, the map

$$\begin{aligned} \varphi_x \ : \ S &\to S \\ Kx_i &\mapsto Kx_i x \end{aligned}$$

is a bijection. Then there exists a homomorphism

$$\begin{array}{rcl}\varphi : H & \to & Sym(S) \simeq Sym(n) \\ x & \mapsto & \varphi_x\end{array}$$

Since H is simple, then $N := \ker\varphi$ is either $\{1\}$ or H and $N := \ker\varphi = \bigcap_{x\in H}(H \cap H^g)^x \le H \cap H^g$. If $n > 1$, then $N \le H \cap H^g < H$. Since H is simple we get, $N = 1$. Hence, H is isomorphic to a subgroup of $Sym(n)$. This is a contradiction since order of H is infinite. This implies that $[H : H \cap H^g] = 1$ for all $g \in G$. Therefore $H \trianglelefteq G$.

(ii) We know that $\bigcap_{g\in G} H^g \trianglelefteq G$. But G is simple, so $\bigcap_{g\in G} H^g = 1$ i.e for any $h \in H$ there exists $g \in G$ such that $h \notin H \cap H^g$. H is inert implies $[H : H \cap H^g] < \infty$. Then, there exists $N_h \triangleleft H$ such that $N_h \le H \cap H^g$, $[H : N_h] < \infty$ and $h \notin N_h \le H \cap H^g$. That is H is residually finite.

(iii) We have to prove that $[HN : HN \cap (HN)^g] < \infty$. Now $(HN)^g = g^{-1}Hg\underbrace{g^{-1}Ng}_{N} = H^gN$. Moreover, $[H : H \cap H^g] < \infty$ implies that there exists $M \trianglelefteq H$ such that $M \le H \cap H^g$ and $[H : M] < \infty$. So $[HN : HN \cap H^gN] < [HN : MN \cap MN] = [HN : MN] = [HMN : MN] = [H : H \cap MN] = [H : M(H \cap N)] < [H : M] < \infty$. Hence $[HN : HN \cap (HN)^g] < \infty$ as we required. □

Lemma 2.29. *A homomorphic image of an inert subgroup is inert.*

Proof. Let, H be an inert subgroup of G and let N be a normal subgroup of G. Then it is enough to show that $[HN/N : (HN/N) \cap (HN/N)^{gN}] < \infty$ for all $gN \in G/N$. Indeed, $[HN/N : (HN/N) \cap (HN/N)^{gN}] = [HN/N : (HN/N) \cap (H^gN/N)] \le [HN/N : (HN \cap H^gN)/N] = [HN : HN \cap H^gN]$. Since by Lemma 2.28$(iii)$ we have $[HN : HN \cap H^gN] < \infty$. Therefore, $[HN/N : (HN/N) \cap (HN/N)^{gN}] < \infty$. □

Lemma 2.30. *A homomorphic image of a TIN group is a TIN group.*

Proof. Let G be a totally inert group. Let N be a normal subgroup of G. We know that $[G/N : K/N] = [G : K]$. Indeed, let $\bar{K} = K/N \leq G/N = \bar{G}$. Let $\Sigma = \{\bar{x}_i\bar{K} \mid \bar{x}_i \in \bar{G}\}$. Then $|\Sigma| = [G/N : K/N]$. So if we define $\Gamma = \{x_iK \mid x_i \in G\}$, then $|\Gamma| = [G : K]$: Consider the map

$$\begin{array}{rcl} \alpha : \Sigma & \rightarrow & \Gamma \\ \bar{x}_i\bar{K} & \mapsto & x_iK \end{array}$$

The map is well-defined and one-to-one.
Let $\bar{x}_i\bar{K} = \bar{y}_i\bar{K}$. Then $x_iN(K/N) = y_iN(K/N)$ implies $(y_i^{-1}x_iN)K/N = K/N$. Then $y_i^{-1}x_iN \in K/N \Rightarrow y_i^{-1}x_i \in K \Rightarrow y_iK = x_iK$. So $\alpha(\bar{x}_i\bar{K}) = \alpha(\bar{y}_i\bar{K})$. Clearly α is onto, since for any $x_iK \in G/K$, we have $\alpha(\bar{x}_i\bar{K}) = x_iK$. Now, let $K/N \leq G/N$, then we want to show that $[K/N : K/N \cap (K/N)^{gN}] < \infty$ for all $gN \in G/N$. Since $(K/N)^{gN} = K^g/N$ and $K/N \cap K^g/N \geq (K \cap K^g)/N$, by the first paragraph we get $[K/N : (K^g/N) \cap (K/N)] \leq [K/N : (K \cap K^g)/N] = [K : K \cap K^g] < \infty$. □

Lemma 2.31. *Let G be a simple TIN-group. Then the following holds.*

(i) For all non-identity elements x and y in G, the groups $C_G(x)$ and $C_G(y)$ are commensurable.

(ii) If For a non-identity torsion element x in G, $C_G(x)$ is finite.

Proof. (i) Let $1 \neq x$ be an element of G. Then $N = \{y \in G \mid [C_G(x) : C_G(x) \cap C_G(y)] < \infty\}$ is a normal subgroup of G:

Let $y_1, y_2 \in N$. Then $[C_G(x) : C_G(x) \cap C_G(y_1)] < \infty$ and $[C_G(x) : C_G(x) \cap C_G(y_2)] < \infty$. Since $C_G(y_1) \cap C_G(y_2) \leq C_G(y_1y_2)$, we get $[C_G(x) : C_G(x) \cap C_G(y_1y_2)] < \infty$, which implies that $y_1y_2 \in N$ and $C_G(y) = C_G(y^{-1})$ implies

$y^{-1} \in N$. To see that $N \trianglelefteq G$, let $g \in G$ and $y \in N$. We have $[C_G(x) : C_G(x) \cap C_G(y)] < \infty$ and $[C_G(x)^g : C_G(x)^g \cap C_G(y)^g] < \infty$. Taking intersection we get $[C_G(x) \cap C_G(x)^g : C_G(x) \cap C_G(x)^g \cap C_G(y)^g] < \infty$. Since every subgroup is inert, then $C_G(x)$ is inert, i.e. $[C_G(x) : C_G(x) \cap C_G(x)^g] < \infty$. Hence $[C_G(x) : C_G(x) \cap C_G(x)^g][C_G(x) \cap C_G(x)^g : C_G(x) \cap C_G(x)^g \cap C_G(y)^g] = [C_G(x) : C_G(x) \cap C_G(x)^g \cap C_G(y)^g] < \infty$, which implies that $[C_G(x) : C_G(x) \cap C_G(y)^g] < \infty$. Then $y^g \in N$, as $C_G(y)^g = C_G(y^g)$.

We know that $1 \neq x \in N$, so N is not trivial. Since G is simple, we get $N = G$. Then for all x and y in G, the group $C_G(x)$ and $C_G(y)$ are commensurable. It follows that for any $x \in G$ the group $C_G(x)$ is an FC-group.

(ii) Let $1 \neq x$ be a fixed torsion element in G such that $C_G(x)$ is infinite. Let $T(G) = \{g \in G \mid |g| < \infty\}$. Since given $x \in G$ and $g \in T(G)$ we have $|x^{-1}gx| = n$ if $g^n = 1$. So $T(G) \trianglelefteq G$. Let $K = \{k_1, k_2, \ldots, k_n\}$ be a finite subset of $T(G)$. Then by (i) given any $k_i \in K$ we have $[C_G(x) : C_G(x) \cap C_G(k_i)] < \infty$. So $[C_G(x) : C_G(x) \cap C_G(k_1) \cap \ldots \cap C_G(k_n)] = [C_G(x) : C_G(K)] < \infty$. This implies that $C_G(K)$ is an infinite group. Let $a \in C_G(K)$. Then $K \leq C_G(a)$ and $C_G(a)$ is an FC-group. Then by Dietzman lemma [24, 14.5.7 page 442], $\langle K \rangle \leq \langle K^{C_G(a)} \rangle$ is finite. Hence $\langle K \rangle$ is finite group. Since G is simple and $T(G) \neq 1$, so $T(G) = G$ and G is locally finite. But by [2] locally finite simple TIN-group does not exist. Hence $|C_G(x)|$ is finite for all non-identity torsion element x in G. $\square$

CHAPTER 3

The Commensurable Property

Subgroups H and K of a group G are called commensurable, if $H \cap K$ is a subgroup of finite index in H and K respectively. We give an equivalent definition for inert subgroups as H is inert in G, if a subgroup H is commensurable with each of its conjugate subgroups in G.

Definition 3.1. *A group G is said to satisfy commensurable property if any two nonidentity subgroups are commensurable.*

Observe that if G satisfies the commensurable property, then G is a totally inert group.

3.1 Some Observations

Lemma 3.2. *If G satisfies the commensurable property, so does any homomorphic image of G.*

Proof. Let G satisfy the commensurable property. Let $N \lhd G$, consider the subgroup K/N and H/N, then $[K/N : H/N \cap K/N] \leq [K/N : (H \cap K)/N] = [K : H \cap K] < \infty$. □

Lemma 3.3. *Every group satisfying commensurable property is countable.*

Proof. If $G = \langle x_1 \rangle$, then G is countable. Suppose $\langle x_1 \rangle \lneqq G$. Then there exists $x_2 \in G \backslash \langle x_1 \rangle$. By commensurable property $[\langle x_1, x_2 \rangle : \langle x_1 \rangle] < \infty$. Now, if $G = \langle x_1, x_2 \rangle$, then G is again countable. Suppose $\langle x_1, x_2 \rangle \lneqq G$, then there exists $x_3 \in G \backslash \langle x_1, x_2 \rangle$ and so $[\langle x_1, x_2, x_3 \rangle : \langle x_1, x_2 \rangle][\langle x_1, x_2 \rangle : \langle x_1 \rangle] = [\langle x_1, x_2, x_3 \rangle : \langle x_1 \rangle] < \infty$. Continuing in same manner, we get $[\cup_{i=1}^{\infty} \langle x_1, x_2, \ldots, x_i \rangle : \langle x_1 \rangle]$ is infinite and so $G = \cup_{i=1}^{\infty} \langle x_1, x_2, \ldots, x_i \rangle$ is countable. □

Lemma 3.4. [12] (*P. Hall-Kulatilaka*) *Every infinite locally finite group has an infinite abelian subgroup.*

Definition 3.5. *A group G is called locally graded, if any finitely generated subgroup in G contains a proper subgroup of finite index.*

Lemma 3.6. *Let G be a locally graded group containing a non-trivial torsion element such that any two proper non identity subgroups are commensurable. Then G is locally finite.*

Proof. Let x be a nontrivial element of finite order and H be a proper subgroup of G. Then H and $\langle x \rangle$ are commensurable implies that H is finite. Hence every proper subgroup of G is finite. Let Y be a finitely generated subgroup of G. Then by the property of being locally graded, Y contains a proper subgroup of finite index i.e. $[Y : M] < \infty$, and M is finite implies that Y is finite. Hence G is locally finite. □

Note that the condition that there exists a torsion element is not superfluous. For $\mathbb{Z}$, any two proper subgroup is commensurable and locally graded, but $\mathbb{Z}$ is not locally finite.

Lemma 3.7. *Let G be an infinite locally graded group containing a torsion element $x \neq 1$. Then any two proper nonidentity subgroups of G are commensurable if and only if G is isomorphic to C_{p^∞} for some prime p.*

Proof. Assume that G is isomorphic to C_{p^∞} for some prime p. Since every proper subgroup of G is finite, we get any two proper subgroups are commensurable.

Conversely, let x be an element of finite order. Then for any proper subgroup H of G we have $[H : H \cap \langle x \rangle] < \infty$. It follows that every proper subgroup of G is finite. By Lemma 3.6 G is locally finite. Then by Hall-Kulatilaka Theorem every locally finite group contains an infinite abelian subgroup. So we get G is abelian and every proper subgroup is finite. We may write G as a direct product of maximal p-subgroups G_p. If infinitely many primes divide the orders of elements of G, then we may obtain an infinite proper subgroup. Hence only finitely many primes divide the order of the group G. Since each proper subgroup is finite there can be only one prime dividing the order of the elements of G. Hence G is an abelian p-group. Consider the map

$$\begin{array}{rcccc} \varphi & : & G & \longrightarrow & G^p \\ & & x & \longmapsto & x^p \end{array}$$

If G^p is a proper subgroup then G^p is finite. $G/Ker\varphi \cong G^p$ and G is infinite imply that $Ker\varphi = G$. Hence G becomes an infinite elementary abelian p-group. This is impossible as these groups have infinite proper subgroups. Hence $G^p = G$, which implies G is a divisible abelian p-group. Hence $G \cong C_{p^\infty}$. □

Theorem 3.8. [24, page 117](*Pontryagin*) *Let G be a countable torsion-free abelian group. Then G is free abelian if and only if every subgroup with finite rank is free abelian.*

Lemma 3.9. *Let G be an infinite abelian group. If G satisfies commensurable property, then $G \cong \mathbb{Z}$ or C_{p^∞}.*

Proof. If G is infinite abelian, then $G \cong D \oplus R$ where D is a divisible group and

R is a reduced group. Also every divisible group D is a direct sum of copies of $\mathbb{Q}$ and of copies of C_{p^∞} for various primes p.

Case 1: If G is a torsion-free group, then G does not contain any copies of C_{p^∞}. So divisible part is a direct sum of copies of $\mathbb{Q}$. But we may have only one copy by commensurable property. On the other hand $\mathbb{Q}$ does not satisfies commensurable property as $\mathbb{Q}_p/\mathbb{Z}$ is infinite where $\mathbb{Q}_p = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z}\}$. Then $G \cong R$ is reduced. We also know that G is countable by Lemma 3.3. Now consider G^n. Since G is reduced, there exists n such that $G \neq G^n$. By Lemma 3.2, any homomorphic image of a group satisfying commensurable property satisfies commensurable property. We get G/G^n satisfies commensurable property. Also it has a torsion element. Then since G/G^n is periodic abelian, and it is locally finite by Lemma 3.7, we get $G/G^n \cong C_{p^\infty}$. Since G/G^n has finite exponent n, but C_{p^∞} has infinite exponent, we get a contradiction with $G/G^n \cong C_{p^\infty}$. Since G is torsion-free then $G^n \neq 1$. So G/G^n is a nontrivial group. Let $H = \langle x_1, x_2, \dots, x_n \rangle$ be a finitely generated subgroup of G. Since G is torsion-free, then $H \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$. By the commensurable property, $H \cong \mathbb{Z}$. Then every finitely generated subgroup of G is cyclic. Rank of G is 1. Then, by Theorem 3.8, G is free abelian. Therefore $G \cong \mathbb{Z}$.

Case 2: If G has a torsion element, then by Lemma 3.7, $G \cong C_{p^\infty}$. □

Theorem 3.10. [16, Theorem] *Let G be a finite group, A an abelian subgroup of G. If A is a maximal subgroup of G, then G is soluble.*

Lemma 3.11. *Let G be a finite group with trivial center. If G has a non-normal abelian maximal subgroup A, then $G = AN$ and $A \cap N = 1$ for some elementary abelian p-subgroup N, which is minimal normal in G. Also A must be cyclic of order prime to p. Moreover, G is soluble of derived length at most two.*

Proof. Let A be an abelian maximal subgroup of G such that A is not normal.

Then for any $x \in G\backslash A$, we get $\langle A, x\rangle = G$. Therefore for any $x \in G\backslash A$, we have $A^x \neq A$. Otherwise A would be normal in G. But then consider $A \cap A^x$. Since $A^x \neq A$ and A is maximal, $\langle A, A^x\rangle = G$. If $w \in A \cap A^x$, then $C_G(w) \geq \langle A, A^x\rangle = G$. Since A is abelian and A^x is isomorphic to A, A^x is also maximal and abelian in G. But $C_G(w) = G$ implies $w \in Z(G) = 1$. Hence $A \cap A^x = 1$. This shows that A is Frobenius complement in G. Hence there exists a Frobenius kernel N such that $G = AN$ and $A \cap N = 1$. By Frobenius Theorem, Frobenius kernel is a normal and nilpotent subgroup of G. So $G = AN$ soluble since $G/N = AN/N = A/A \cap N$ is abelian. Hence G/N and N are soluble. It follows that G is soluble. Also it follows from the fact that minimal normal subgroup of a soluble group G is elementary abelian p-group for some prime p. Therefore N is an elementary abelian p-group. Indeed, If there exists a normal subgroup $1 \neq M \leq N$ in G such that $G = AM$ and $A \cap M \leq A \cap N = 1$. Moreover $|G| = \frac{|A||M|}{|A \cap M|} = \frac{|A||N|}{|A \cap N|} = |A||M| = |A||N|$. Hence $|M| = |N|$, this implies $M = N$. Hence, N is minimal normal subgroup of G.

Since N is an elementary abelian p-group if A contains an element g of order power of p, then the group $H = N\langle g\rangle$ is a p-group. Hence $Z(H) \neq 1$. Let $1 \neq x \in Z(H)$. If $x \in A$, then $C_G(x) \geq \langle A, N\rangle = G$. This implies that $x \in Z(G) = 1$, which is impossible. So $x \in G\backslash A$. Then $\langle g\rangle \cap \langle g\rangle^x \leq A \cap A^x = 1$. But $\langle g\rangle \cap \langle g\rangle^x = \langle g\rangle$ implies that $g = 1$. Hence, $(|A|, p) = 1$, i.e. $p \nmid |A|$.

Claim: A is cyclic: By Frobenius Theorem, Sylow p-subgroups of Frobenius complement A are cyclic, if $p > 2$ and cyclic or generalized quaternion, if $p = 2$ [24, 10.5.6]. Since A is abelian, Sylow subgroup can not be generalized quaternion group. Hence all Sylow subgroups of A are cyclic. This implies that A is cyclic.

□

Proposition 3.12. *Let M be a maximal subgroup of a locally finite group G. If*

M is inert and abelian, then G is soluble with derived length at most 3.

Proof. If M is normal, then for any $x \in G\backslash M$, we have $\langle M, x\rangle = G$ implies that $G/M = \langle x\rangle M/M \cong \underbrace{\langle x\rangle/(\langle x\rangle \cap M)}_{abelian}$ Then $[G,G] \leq M$. So $[G,G]$ is abelian. Therefore, $G \geq [G,G] \geq 1$. So that G is soluble of derived length at most 2.

Assume M is not normal in G. Then $N_G(M) = M$ as M is maximal. Then for any $x \in G\backslash M$ we have $M^x \neq M$. Hence $\langle M, M^x\rangle = G$. By inertness, we have $[M : M \cap M^x] < \infty$ and $[M^x : M \cap M^x] < \infty$. Then by [5, Lemma 5], $[G : M \cap M^x] = [\langle M, M^x\rangle : M \cap M^x] < \infty$. The group $M \cap M^x \trianglelefteq G$. Since $N_G(M \cap M^x) \geq \langle M, M^x\rangle = G$, $\bar{G} = G/(M \cap M^x)$ is a finite group with abelian maximal subgroup $\bar{M} = M/(M \cap M^x)$.

Consider $Z(\bar{G})$. If $Z(\bar{G})$ is not contained in $\bar{M}$, then $\bar{M}Z(\bar{G}) = \bar{G}$. This implies that $\bar{G}$ is abelian, since if we extend any abelian group with center, we get again abelian group, and so G is soluble of derived length at most 2. Hence we may assume that $Z(\bar{G})$ is contained in $\bar{M}$. Also $Z(\bar{G})^x = Z(\bar{G}) \in \bar{M}^x$. Then $Z(\bar{G}) \leq \bar{M} \cap \bar{M}^x$,

Claim: $Z(\bar{G}) = \bar{M} \cap \bar{M}^x$. If $\bar{w} \in \bar{M} \cap \bar{M}^x$, then $C_{\bar{G}}(\bar{w}) \geq \langle \bar{M}, \bar{M}^x\rangle = \bar{G}$. So $\bar{w} \in Z(\bar{G})$, which implies $\bar{M} \cap \bar{M}^x \subseteq Z(\bar{G})$. But on the other hand, $\bar{M} \cap \bar{M}^x = \bar{1}$. Indeed, if $a(M \cap M^x) \in M/(M \cap M^x) \cap M^x/(M \cap M^x)$, then

$$\begin{aligned} a &= a_1 m_1, \quad \text{where} \quad a_1 \in M,\ m_1 \in M \cap M^x \quad \text{and} \\ a &= a_2 m_2, \quad \text{where} \quad a_2 \in M^x,\ m_2 \in M \cap M^x \end{aligned}$$

Then, $a_1 m_1 = a_2 m_2$, and $a_1 = a_2 m_2 m_1^{-1} \in M^x$. So $a \in M^x$ implies that $a \in M \cap M^x$, i.e. $a(M \cap M^x) = \bar{1}$. So $\bar{M} \cap \bar{M}^x = \bar{1}$. Hence $Z(\bar{G}) = \bar{1}$. Then by Lemma 3.11, $\bar{G}$ is a Frobenius group with Frobenius complement $\bar{M}$ and Frobenius kernel $\bar{N}$ and $\bar{G} = \bar{M}\,\bar{N}$. It follows that $\bar{G}$ is soluble. The group $\bar{G}/\bar{N}$ is abelian implies

$\bar{G}' \leq \bar{N}$, i.e. $(G/(M \cap M^x))' \leq N/(M \cap M^x)$. Then $G''(M \cap M^x) \leq M \cap M^x$ implies that $G'' \leq M \cap M^x$. Then $G^{(3)} = 1$, as M is abelian. □

Definition 3.13. *Let us consider a partially ordered set $\mathcal{M}$ with partial order $\leq$. We say that $\mathcal{M}$ satisfies the minimal condition if each nonempty subset $\mathcal{M}_0$ contains at least one minimal element.*

Example 3.14. *There exists a locally finite group G such that G satisfies the minimal condition, but it need not be TIN.*

Proof. Let A be an infinite locally finite abelian group satisfying minimal condition and t be an element of order 2. Consider $T = A \times A^t$. Let t act on T by $(a, b^t)^t = (b, a^t)$. One may consider this action in the following way. Let

$$\begin{array}{rcccc} \varphi & : & A \times A & \longrightarrow & A \times A \\ & & (a, b) & \longmapsto & (b, a) \end{array}$$

where φ is an automorphism of $A \times A$ of order 2. Let $G = (A \times A) \rtimes \langle \varphi \rangle$. Consider the group $A \times \{1\}$. Then $[A \times A : (A \times \{1\})^t \cap (A \times \{1\})]$ is infinite. So $A \times A$ is not inert in G. The group G satisfies min, it is locally finite, but G is not a TIN group.

For Example, $A = C_{p^\infty}$ satisfies min or any Chernikov group. □

Example 3.15. *Let V be an infinite dimensional vector space over a field $\mathbb{F}_p$ and T be a finitary linear transformation of order p. Let $G = V \rtimes \langle T \rangle$. Then $|G/V| = p$ and V is maximal and abelian in G and G is soluble of derived length 2.*

The following example shows that maximal subgroups could be inert but need not be normal.

Example 3.16. *Let $G = A \times Sym(3)$, where A is an infinite abelian group. Then G has a maximal subgroup M such that M is not normal in G, but inert in G. If M is a maximal subgroup of an infinite group G and M is inert in G, then M need not be normal in G.*

Proof. Let $M = A \times \langle(1,2)\rangle$. Clearly M is maximal and abelian in G. Now let us show that it is inert: Given any $x \in G$, we have $(A \times \langle(1,2)\rangle)^x \geq (A \times \{1\})^x \cong A$. Then $[A \times \langle(1,2)\rangle : A \times \langle(1,2)\rangle \cap (A \times \langle(1,2)\rangle)^x] \leq [A \times \langle(1,2)\rangle : (A \times \langle(1,2)\rangle) \cap A \times \{1\}] \leq [A \times \langle(1,2)\rangle : A \times \{1\}] \leq 2$. Therefore, M is inert in G. □

Lemma 3.17. *Let M be an infinite inert and maximal subgroup of a locally finite group G, then G is not simple.*

Proof. Let G be an infinite simple locally finite group and M be an infinite maximal and inert subgroup of G. Then by a fact [3, Lemma 5], we have $[\langle M, M^x\rangle : M \cap M^x] < \infty$. Then there exists a subgroup N in $M \cap M^x$ such that $N \trianglelefteq G$. It is a contradiction, as G is infinite simple. □

CHAPTER 4

Centralizer Of Involutions

We are interested in whether the centralizers of involutions in infinite locally finite simple groups are inert or not. We study the structure of the centralizers of involutions in alternating and Lie type groups.

Definition 4.1. *Let $\mathfrak{F}_{n,r}$ consist of all groups (not necessarily locally finite) having a series of finite length, in which at most n factors are non-abelian simple, and the rest are soluble groups the sum of whose derived lengths is at most r. Also let $\mathfrak{F}_n$ consist of all locally finite groups having a series of finite length, in which there are at most n non-abelian simple factors and the rest are locally soluble.*

Lemma 4.2. [20, Lemma 2.1] *i) The classes $\mathfrak{F}_{n,r}$ and $\mathfrak{F}_n$ are closed under taking normal subgroups and quotients.*

ii) Let $N \lhd M \lhd G$. If $G \in \mathfrak{F}_{n,r}$ and M/N is soluble, then the derived length of M/N is at most r.

iii) If $M \lhd G$, $M \in \mathfrak{F}_n$ and $G/M \in \mathfrak{F}_m$, then $G \in \mathfrak{F}_{m+n}$

Lemma 4.3. [21, Lemma 2.4] *Let G be the symmetric group of degree m, and let x be an element of order n in G. Then $C_G(x) \in \mathfrak{F}_{[n/2]+1}$.*

Lemma 4.4. *Let G be a locally finite group and i be an involution of G. If $C_G(i) \in \mathfrak{F}_n$, then $C_{G/Z}(i) \in \mathfrak{F}_n$ where Z is the center of G.*

Proof. Let $C/Z = C_{G/Z}(i)$ and $C_G(i) = C_1 \rhd C_2 \rhd \ldots \rhd C_m = 1$ be a series of $C_G(i)$ such that each factor is either nonabelian simple or locally soluble. Then $C/Z = C_1Z/Z \rhd C_2Z/Z \rhd \ldots \rhd C_mZ/Z = Z$ is a series of C/Z. Recall that the map

$$\begin{aligned} \varphi_i \ : \ C &\to Z \\ g &\mapsto [g, i] \end{aligned}$$

is a group homomorphism with kernel $C_G(i)$. Hence $C_G(i) \lhd C$ and $C/C_G(i)$ is abelian. If C_i/C_{i+1} is nonabelian simple, then $(C_iZ/Z)/(C_{i+1}Z/Z) \cong C_iZ/C_{i+1}Z = C_iC_{i+1}Z/C_{i+1}Z \cong C_i/(C_i \cap C_{i+1}Z) = C_i/C_{i+1}(C_i \cap Z)$. Now $C_{i+1}(C_i \cap Z)/C_{i+1} \trianglelefteq C_i/C_{i+1}$ is nonabelian simple, but $C_{i+1}(C_i \cap Z)/C_{i+1}$ is an abelian normal subgroup. Hence, $C_i \cap Z \leq C_{i+1}$. Moreover, the factor $(C_iZ/Z)/(C_{i+1}Z/Z)$ is non abelian simple. On the other hand, if C_i/C_{i+1} is locally soluble, then as above $(C_iZ/Z)/(C_{i+1}Z/Z) \cong C_i/C_{i+1}(C_i \cap Z)$. As homomorphic image of a locally soluble group is locally soluble, $C_i/C_{i+1}(C_i \cap Z)$ is locally soluble. It follows that $C/Z \in \mathfrak{F}_n$. □

Lemma 4.5. [20, Lemma 4.1] *Let G be a locally finite group in $\mathfrak{F}_n$. If G is residually finite, then G is almost locally soluble. In particular, G does not involve infinite simple groups.*

Lemma 4.6. [21, Lemma 2.2] *Let G be a group and α be an automorphism of G. Let N be an α-invariant subgroup of G and $C/N = C_{G/N}(\alpha)$.*

(i) If $N \leq Z(G)$, then $C_G(\alpha) \lhd C$ and $C/C_G(\alpha)$ is isomorphic to a subgroup of N.

(ii) If $[N, G, \ldots, G] = 1$ with a finite number of terms G, and $C \in \mathfrak{F}_n$, then $C_G(\alpha) \in \mathfrak{F}_n$.

Proof. (*i*) Consider the Frobenius map,

$$\begin{array}{rcccc} \varphi_\alpha & : & C & \to & N \\ & & g & \mapsto & [g, \alpha] \end{array}$$

The map is homomorphism if and only if $\varphi_\alpha(C) \subset Z(G)$. Therefore, by assumption, the map is a homomorphism and so $\ker \varphi_\alpha = C_G(\alpha)$. By the First homomorphism theorem, we get $C_G(\alpha) \lhd C$ and $C/C_G(\alpha)$ is isomorphic to a subgroup of N.

(*ii*) Let us define $N_0 = N$ and $N_{i+1} = [N_i, G]$. Then, by assumption $N_r = 1$. Also $N_{i+1} = [N_i, G] \lhd G$. So that we get the normal series $N = N_0 \rhd N_1 \rhd \ldots \rhd N_r = 1$. Let $C_i/N_i = C_{G/N_i}(\alpha)$. By (*i*) $C_{i+1} \lhd C_i$, and so C_i is subnormal in C and the result follows from Lemma 4.2 □

Lemma 4.7. [21, Lemma 2.3] (*i*) *If $G \in L\mathfrak{F}_n$ where $L\mathfrak{X}$ denotes the class of all groups in which every finite set of elements belongs to some $\mathfrak{X}$-subgroup, then G has a finite series of length at most $2n + 1$, the factors of which comprise at most n non-abelian simple groups, at most $n + 1$ soluble groups of derived length at most r, and no others.*

(*ii*) $L\mathfrak{F}_n = \mathfrak{F}_n$.

Theorem 4.8. [21, Theorem D] *Let $G = \mathfrak{G}(\mathbb{F})$, where $\mathbb{F}$ is an infinite locally finite field of characteristic p, and let n be a positive integer not divisible by p. Let $\prod = \{r_1, \ldots , r_l\}$ be a set of fundamental roots of the Lie algebra from which G was constructed, and let $r = m_1r_1 + \ldots + m_lr_l$ be the highest root.*

(*i*) *if $n \leq \sum_{i=1}^{l} m_i$, then the centralizer in G of every element of order n (if these exist) involves an infinite simple group.*

(*ii*) *Suppose that $n > \sum_{i=1}^{l} m_i$, and that $\mathbb{F}$ contains a primitive n-th root of 1 if n is odd, or a primitive $2n$-th root of 1 if n is even. Then G contains an* *element of order n whose centralizer is an extension of an abelian p'-group by a subgroup of the fundamental group.*

Theorem 4.9. [21, Theorem B] *Suppose that every finite set of elements of G lies in a finite simple subgroup, and suppose that G is not linear. Then there exists a prime p with the following property.*

Let n be any natural number not divisible by p, let g be any element of order n in G, and let $r(n) = n + [4/n]$. Then $C_G(g)$ has a finite series of length at most $2r(n)+1$, in which each factor is either non-abelian simple or soluble. The number of non-abelian simple factors is at most $r(n)$, and at least one of them is non-linear. The derived length of each soluble factor is at most 6, and there are at most $r(n)+1$ of them.

Theorem 4.10. (The classification Theorem). *Let G be a finite simple group. Then G is either*

(*a*) *a cyclic group of prime order,*

(*b*) *an alternating group of degree $n \geq 5$,*

(*c*) *a finite simple group of Lie type,*

(*d*) *one of 26 sporadic finite simple groups.*

4.1 Centralizer Of Involutions in Alternating Groups

The following lemma shows that not only centralizers of elements, but also centralizers of finite subgroups in infinite alternating groups are not inert in infinite alternating groups.

Lemma 4.11. *Let $G = Alt(\Omega)$ be the group of even finitary permutations in $Sym(\Omega)$ for an infinite set Ω. Let $\mathbb{F}$ be a finite subgroup. Then*

$$C_G(\mathbb{F}) = C_{Alt(\Omega_1)}(\mathbb{F}) \times Alt(\Omega_2),$$

where $\Omega_1 = supp(\mathbb{F}) = \{\alpha \in \Omega \mid \alpha^x \neq \alpha \text{ for some } x \in \mathbb{F}\}$ and $\Omega_2 = \Omega \backslash \Omega_1$.

Proof. First, let us show that $C_{Alt(\Omega_1)}(\mathbb{F})Alt(\Omega_2) \leq C_G(\mathbb{F})$:

If $g \in Alt(\Omega_2)$ and $f \in \mathbb{F}$, then for all $\alpha \in \Omega_1$ we have $\alpha^{gf} = \alpha^f = \alpha^{fg}$ since $\Omega_1 \cap \Omega_2 = \emptyset$. Similarly, for $\alpha \in \Omega_2$ we have $\alpha^{gf} = \alpha^g = \alpha^{fg}$. Hence any $\alpha \in \Omega$ satisfies $\alpha^{gf} = \alpha^{fg}$. Therefore $g \in C_G(\mathbb{F})$. Clearly, $C_{Alt(\Omega_1)}(\mathbb{F}) \leq C_G(\mathbb{F})$. So, $C_{Alt(\Omega_1)}(\mathbb{F})Alt(\Omega_2) \leq C_{Alt(\Omega)}(\mathbb{F})$. Since $\Omega_1 \cap \Omega_2 = \emptyset$, we have $C_{Alt(\Omega_1)}(\mathbb{F}) \cap Alt(\Omega_2) = 1$ and hence $C_{Alt(\Omega_1)}(\mathbb{F})Alt(\Omega_2) \leq C_G(\mathbb{F})$.

For the converse, let $g \in C_G(\mathbb{F})$. Then, g can be written as $g = g_1 g_2 \ldots g_s \in C_G(\mathbb{F})$ where g_i's are disjoint cycles. We claim that given any g_i, all entries of g_i are either in Ω_1 or in Ω_2. Suppose, this is not true. Then there exists a $g_i = (\gamma_1 \gamma_2 \ldots \gamma_{j-1} \beta_j \gamma_{j+1} \ldots \gamma_k)$ such that $\gamma_1, \gamma_2, \ldots, \gamma_{j-1} \in \Omega_1$ and $\beta_j \in \Omega_2$. Now, choose $x \in \mathbb{F}$ containing γ_{j-1} as one of its entries. We know that every element of $\mathbb{F}$ is of the form: $x = (\alpha_{11}\alpha_{12} \ldots \alpha_{1k_1})(\alpha_{21}\alpha_{22} \ldots \alpha_{2k_2}) \ldots (\alpha_{n1}\alpha_{n2} \ldots \alpha_{nk_n})$, where $\alpha_{st} \in supp(\mathbb{F}) = \Omega_1$. Now if we take conjugate of x with g then we get $x^g = (g(\alpha_{11}) \ldots g(\alpha_{1k_1})) \ldots (g(\alpha_{n1}) \ldots g(\alpha_{nk_n}))$. Without lost of generality, we can assume that $\gamma_{j-1} = \alpha_{1i_1}$. Then it follows that $g_i(\gamma_{j-1}) = g(\alpha_{1i_1}) = \beta_j$ and $\beta_j \notin supp(\mathbb{F})$. But then $x^g \neq x$. It contradicts with $g \in C_G(\mathbb{F})$. So all entries of g_i are in Ω_1 or Ω_2.

If the entries are in Ω_2, then $g \in Alt(\Omega_2)$. Otherwise, the entries are in Ω_1 and $g \in C_{Alt(\Omega_1)}(\mathbb{F})$. Then, we get $C_G(\mathbb{F}) \leq C_{Alt(\Omega_1)}(\mathbb{F})Alt(\Omega_2)$. We get the equality. □

Corollary 4.12. *The centralizer of involutions in $Alt(\Omega)$ are not inert in $Alt(\Omega)$ where Ω is an infinite set.*

Proof. Let i be an involution in $Alt(\Omega)$. By Lemma 4.11 $C_{Alt(\Omega)}(i) = C_{Alt(\Omega_1)}(i) \times Alt(\Omega_2)$ where $\Omega_1 = Supp(i)$ and $\Omega_2 = \Omega \backslash \Omega_1$. then $Alt(\Omega_2)$ is simple since Ω_2 is infinite. We get $C_{Alt(\Omega)}(i)$ involves infinite simple subgroup. Since every inert subgroup is residually finite by Lemma 2.28(ii). This is a contradiction since the group is simple. □

4.2 Centralizer of involutions in simple groups of Lie type

4.2.1 Centralizer of involutions in simple groups of Lie type of rank 1 where $\mathbb{F}$ is a locally finite field of odd characteristic

In this section we will examine the centralizer of an arbitrary involution in each simple group of Lie type and show that it is not inert in the ambient group.

Centralizer of involutions in $PSL(2, \mathbb{F})$ where $\mathbb{F}$ is field of odd characteristic

Lemma 4.13. *Let $\mathbb{F}$ be a field of odd characteristic and $G = PSL(2, \mathbb{F})$. Then $[C_G(i) : C_G(i) \cap (C_G(i))^g] \geq |\mathbb{F}^*|$ for any involution $i \in G$ and any element $g \in G$. In particular, if $\mathbb{F}$ is infinite then $C_G(i)$ is not inert in G.*

Proof. Note $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Z$ is an involution in G. Let $g = \begin{pmatrix} p & r \\ s & t \end{pmatrix} Z \in G$. Let us find $C_G(i)$. It must satisfy the equality $ig = gi$.

Then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p & r \\ s & t \end{pmatrix} Z = \begin{pmatrix} p & r \\ s & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Z$$

It is equivalent to

$$\begin{pmatrix} s & t \\ -p & -r \end{pmatrix} = \begin{pmatrix} -r & p \\ -t & s \end{pmatrix} \lambda I$$

where $\lambda = 1$, or $\lambda = -1$. If $\lambda = 1$ we get

$$C_1 = \{ \begin{pmatrix} p_1 & r_1 \\ -r_1 & p_1 \end{pmatrix} Z \mid p_1^2 + r_1^2 = 1,\ p_1, r_1 \in \mathbb{F} \}$$

and if $\lambda = -1$ we get

$$C_2 = \{ \begin{pmatrix} p_2 & r_2 \\ r_2 & -p_2 \end{pmatrix} Z \mid p_2^2 + r_2^2 = -1,\ p_2, r_2 \in \mathbb{F} \}$$

Since $\begin{pmatrix} p_2 & r_2 \\ r_2 & -p_2 \end{pmatrix} Z \begin{pmatrix} p_1 & r_1 \\ -r_1 & p_1 \end{pmatrix} Z = \begin{pmatrix} p_1p_2 - r_1r_2 & r_1p_2 + p_1r_2 \\ r_1p_2 + p_1r_2 & -(p_1p_2 - r_1r_2) \end{pmatrix} Z \in C_2$

it follows that $C_G(i) = \langle C_1 \cup C_2 \rangle = C_1 \cup xC_1$ where $x = \begin{pmatrix} p_2 & r_2 \\ r_2 & -p_2 \end{pmatrix} Z \in C_2$.

If we choose $y = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) Z$, we get

$$C_1^y = \{ \begin{pmatrix} p_1 + r_1 & r_1 \\ -2r_1 & p_1 - r_1 \end{pmatrix} Z \mid p_1^2 + r_1^2 = 1,\, p_1, r_1 \in \mathbb{F} \}.$$

Note that $[C_1 : C_1 \cap C_1^y] = |\mathbb{F}^*|$. Indeed, $r_1 = -2r_1$ implies $r_1 = 0$.

Then $C_1 \cap C_1^y = \{ \begin{pmatrix} p_1 & r_1 \\ -r_1 & p_1 \end{pmatrix} Z \} \cap \{ \begin{pmatrix} p_1 + r_1 & r_1 \\ -2r_1 & p_1 - r_1 \end{pmatrix} Z \} = \{ \begin{pmatrix} p_1 & 0 \\ 0 & p_1 \end{pmatrix} Z \}$.

Therefore if $\mathbb{F}$ is infinite, then C_1 is not inert in G. So $C_G(i)$ is not inert in G. Since $[C_G(i) : C_1]$ finite using contrapositive of Lemma 2.25 we get that $C_G(i)$ is also not inert in G. $\square$

Centralizer of involutions in $SU(3, q^2)$ and $PSU(3, q^2)$ where q is odd

Let T be a nontrivial involution in $SU(3, q^2)$. Then $T^2 = I$ implies that T satisfies the polynomial $x^2 - 1$. The minimal polynomial of T is $x - 1$, $x + 1$ or $(x - 1)(x + 1)$. The minimal polynomial can not be $x - 1$ as $T \neq I$ and it can not be $x + 1$ as $T = -I$ implies $\det T = \det\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) = -1$. Hence the minimal polynomial of T is $(x - 1)(x + 1)$. Then T is a diagonalizable matrix and the eigenvalues of T are 1 and -1. Hence T is similar to the diagonal matrix $A = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. Let u be an eigenvector corresponding to the eigenvalue 1 and v_1, v_2 be the eigenvectors corresponding to the eigenvalue -1. Let $V = \langle u \rangle \oplus \langle v_1, v_2 \rangle$ and $A \in C_{SU(3,q^2)}(T)$. Then $AT = TA$ and moreover for any λu, we have $TA(\lambda u) = AT(\lambda u) = A\lambda T(u) = A\lambda u = \lambda Au$ so $A(\lambda u) = T(A(\lambda u))$; i.e. $A(\lambda u)$ is in the eigenspace corresponding to the eigenvalue 1.

Recall that in general if X and Y are commuting matrices then X leaves the eigenspaces of Y invariant. Indeed, let α be an eigenvector of Y corresponding to an eigenvalue, say c. Then $Y(X(\alpha)) = X(Y(\alpha)) = X(c\alpha) = c(X\alpha)$. Hence $Y(X(\alpha)) = c(X(\alpha))$ implies that $X(\alpha)$ is an eigenvector of Y corresponding to the eigenvalue c.

Therefore A has the representation $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$ with respect to the ordered basis $\{u, v_1, v_2\}$ since

$$\begin{aligned} Au &= \lambda u \\ Av_1 &= a_{11}v_1 + a_{21}v_2 \\ Av_2 &= a_{12}v_1 + a_{22}v_2 \end{aligned}$$

Moreover every matrix of this form commutes with T, i.e.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We take the matrix A from $SU(3, q^2)$ therefore A is a unitary matrix and $\lambda(a_{11}a_{22} - a_{21}a_{12}) = 1$.

$$C_{SU(3,q^2)}(T) = \{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{12} & a_{22} \end{pmatrix} \mid \lambda(a_{11}a_{22} - a_{21}a_{12}) = 1\}$$

Now define a map

$$\begin{array}{cccc} \varphi & : & C_{SU(3,q^2)}(T) & \longrightarrow & \mathbb{F}^* \\ & & \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} & \longmapsto & \lambda \end{array}$$

φ is a group homomorphism and

$$\ker\varphi = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} \in SU(3,q^2) \mid a_{ij} \in \mathbb{F}\} \cong SU(2,q^2).$$

Then we have a series

$$C_{SU(3,q^2)}(T) \rhd \ker\varphi \rhd Z(C_{SU(3,q^2)}(T)) \rhd 1.$$

Since $SU(2,q^2) \cong SL(2,q)$ [28, Vol.I, 6.22] and $Z(SU(2,q^2)) \cong Z(SL(2,q))$, we have $PSL(2,q) \cong SU(2,q^2)/Z(SU(2,q^2))$. Recall that $PSU(3,q^2) = SU(3,q^2)/Z(SU(3,q^2))$, in particular, $|Z(SU(3,q^2))| = (3, q^2 - 1)$. Then $|Z(SU(3,q^2))|$ is either 1 or 3. Recall that if s is an element of order n and M is a normal subgroup such that $(|M|, n) = 1$ then $C_{G/M}(s) = C_G(s)M/M$ (see [10, 1.7.7. Lemma, pg. 49]). Using this and taking M as the center of $SU(3,q^2)$ we get

$$C_{PSU(3,q^2)}(T) \cong C_{SU(3,q^2)}(T)Z/Z = C_{SU(3,q^2)}(T)/(C_{SU(3,q^2)}(T) \cap Z).$$

In particular, the above series

$$C_{PSU(3,q^2)}(T) \cong C_{SU(3,q^2)}(T)Z/Z \rhd \ker\varphi Z/Z \rhd \bar{1}$$

is a series of $C_{PSU(3,q^2)}(T)$.

We have $\ker\varphi Z/Z \cong \ker\varphi$ since $Z(SU(3,q^2)) \cap C_{SU(3,q^2)}(T) = 1$.

$\ker\varphi \cong SL(2,q)$ is isomorphic to a subgroup of $C_{PSU(3,q^2)}(T)$. So for an infinite locally finite field $\mathbb{F}$ the group $C_{PSU(3,\mathbb{F})}(T)$ involves an infinite simple group $SL(2,\mathbb{F})$. Since every proper inert subgroup is residually finite. But by Lemma 4.5 residually finite group does not involve infinite simple groups.

Centralizer of involutions in Ree groups $Re(\mathbb{F})$ where $\mathbb{F}$ is a locally finite field of characteristic 3

For a finite field of order q where $q = 3^{2n+1}$, the group $Re(q) = {}^2G_2(q)$ is studied in [22]. It is proved that 2-groups of equal orders in $Re(q)$ are conjugate in $Re(q)$ (see [18]). In particular, all involutions in $Re(q)$ are conjugate ([31, p. 63] and [18, Lemma 2.1]). Let $\mathbb{F}$ be a locally finite infinite field, then $Re(\mathbb{F})$ can be written as a union of $Re(\mathbb{F}_i)$'s where $\mathbb{F}_i$'s are finite fields. Then all involutions in $Re(\mathbb{F})$ are conjugate. Moreover centralizer of an involution $i \in Re(q)$ is

$$C_{Re(q)}(i) \cong \langle i \rangle \times PSL(2,q) \qquad \text{[22, sec. 1, pg 16-19]}$$

and for a locally finite field $\mathbb{F}$

$$C_{Re(\mathbb{F})}(i) \cong \langle i \rangle \times PSL(2,\mathbb{F}).$$

But by Lemma 2.28(ii) in infinite simple groups, inert subgroups are residually finite. But inert subgroup of a residually finite group is also residually finite. This implies that $PSL(2, \mathbb{F})$ is residually finite. This contradicts the fact that $PSL(2, \mathbb{F})$ is a simple group. Hence centralizer of involutions are not inert in Ree groups.

4.2.2 Centralizer Of Involutions in $SL(n, \mathbb{F})$ where $n = 2, 3$ and $\mathbb{F}$ is a locally finite field of even characteristic

Centralizer of involutions in $PSL(2, \mathbb{F})$

Lemma 4.14. *Let $\mathbb{F}$ be an infinite field of characteristic 2. Then $SL(2, \mathbb{F})$ is isomorphic to $PSL(2, \mathbb{F})$.*

Proof. We know that center of $SL(2, \mathbb{F})$ is

$$Z = \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda^2 = 1 \}.$$

Since characteristic of $\mathbb{F}$ is 2, $\lambda^2 = 1$ implies that $\lambda^2 - 1 = (\lambda - 1)^2 = 0$. Then we get $\lambda = 1$. So the center becomes identity. Therefore $SL(2, \mathbb{F}) \cong PSL(2, \mathbb{F})$. □

Lemma 4.15. *Let $\mathbb{F}$ be a field of characteristic 2. Then for any involution i in $SL(2, \mathbb{F})$, i is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and there exists an element $g \in SL(2, \mathbb{F})$ such that $[C_{SL(2,\mathbb{F})}(i) : C_{SL(2,\mathbb{F})}(i) \cap (C_{SL(2,\mathbb{F})}(i))^g] \geq |\mathbb{F}|$. In particular, if $\mathbb{F}$ is infinite then centralizers of involutions are not inert in $SL(2, \mathbb{F}) \cong PSL(2, \mathbb{F})$.*

Proof. Let i be an involution in $SL(2, \mathbb{F})$. Then i satisfies the polynomial $x^2 - 1$. Since $x^2 - 1 = (x - 1)^2$, the minimal polynomial of i is $(x - 1)^2$. Hence i has the Jordan form $i = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then every involution is conjugate to i in $SL(2, \mathbb{F})$.

Let us find $C_{SL(2,\mathbb{F})}(i)$. Let $y \in C_{SL(2,\mathbb{F})}(i)$. It must satisfy the equality $iy = yi$. Let $y = \left(\begin{smallmatrix} p & r \\ s & t \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p & r \\ s & t \end{pmatrix} = \begin{pmatrix} p & r \\ s & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

It is equivalent to

$$\begin{pmatrix} p & r \\ p+s & r+t \end{pmatrix} = \begin{pmatrix} p+r & r \\ s+t & t \end{pmatrix}$$

$p + r = r$ implies that $r = 0$ and $p + s = s + t$ implies that $p = t$. Since the determinant is 1, we get $pt + rs = 1$. Then $pt = 1$ implies $p^2 = 1$, in characteristic 2 case this implies that $p = 1$. Therefore

$C_{SL(2,\mathbb{F})}(i) = \{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{F}\}$. Now, choose $g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$(C_{SL(2,\mathbb{F})}(i))^g = \{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{F}\}$$

Then

$$C_{SL(2,\mathbb{F})}(i) \cap (C_{SL(2,\mathbb{F})}(i))^g = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

Hence $[C_{SL(2,\mathbb{F})}(i) : C_{SL(2,\mathbb{F})}(i) \cap (C_{SL(2,\mathbb{F})}(i))^g] \geq |\mathbb{F}|$.

In particular, if $|\mathbb{F}|$ is infinite then $C_{SL(2,\mathbb{F})}(i)$ is not inert in $SL(2,\mathbb{F})$ $\square$

Lemma 4.16. *Let V be a vector space of dimension 2 over a field $\mathbb{F}$ of characteristic 2. If involutions i and j are conjugate in $GL(V)$ then they are conjugate in $SL(V)$.*

Proof. Let $i = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ be an involution in $SL(2,\mathbb{F})$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(n,\mathbb{F})$. Then an arbitrary conjugate of i is of the form

$$\begin{aligned} i^g &= \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} ad-bc+bd & -b^2 \\ d^2 & ad-bc-bd \end{pmatrix} \end{aligned}$$

Let us find all conjugates of i in $SL(2,\mathbb{F})$. Let $s = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in SL(2,\mathbb{F})$.

$$\begin{aligned} i^s &= \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} t & -y \\ -z & x \end{pmatrix} \\ &= \begin{pmatrix} x+y & y \\ z+t & t \end{pmatrix}\begin{pmatrix} t & -y \\ -z & x \end{pmatrix} \\ &= \begin{pmatrix} 1+yt & -y^2 \\ t^2 & 1-yt \end{pmatrix} \end{aligned}$$

So

$$\begin{aligned} 1 + \frac{bd}{ad-bc} &= 1 + yt \\ \frac{b^2}{ad-bc} &= -y^2 \\ \frac{d^2}{ad-bc} &= t^2 \end{aligned}$$

Given $a, b, c, d \in \mathbb{F}$ we can find $x, y, z, t \in \mathbb{F}$ satisfying the equations. Since every element over a field of characteristic 2 is a square as

$$\begin{aligned} \varphi : \mathbb{F} &\rightarrow \mathbb{F} \\ x &\mapsto x^2 \end{aligned}$$

φ is a field automorphism. Hence we can find t and y. Then we substitute these values with a, b, c, d in the equations and can get x and z. □

Centralizer Of Involutions in $PSL(3, \mathbb{F})$

Lemma 4.17. *Let $\mathbb{F}$ be a field of characteristic 2. Then there exists an involution $i \in PSL(3, \mathbb{F})$ and an element $y \in PSL(3, \mathbb{F})$ such that $|C_{PSL(3,\mathbb{F})}(i) : C_{PSL(3,\mathbb{F})}(i) \cap C_{PSL(3,\mathbb{F})}(i)^y| \geq |\mathbb{F}|$. In particular, if $\mathbb{F}$ is infinite then $C_{PSL(3,\mathbb{F})}(i)$ is not inert in $PSL(3, \mathbb{F})$.*

Proof. In $PSL(3, \mathbb{F})$ let $i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} Z$. Choose any element $y = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} Z \in PSL(3, \mathbb{F})$. Then from the equality $iy = yi$ we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} Z = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} Z$$

by calculation we get

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+a_{11} & a_{32}+a_{12} & a_{33}+a_{13} \end{pmatrix} Z = \begin{pmatrix} a_{11}+a_{13} & a_{12} & a_{13} \\ a_{21}+a_{23} & a_{22} & a_{23} \\ a_{31}+a_{33} & a_{32} & a_{33} \end{pmatrix} Z$$

Note that in $PSL(3, \mathbb{F})$, if $aZ = bZ$ then $a = b\lambda I = \lambda b$ for some $\lambda \in \mathbb{F}$. Therefore the above equality becomes

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+a_{11} & a_{32}+a_{12} & a_{33}+a_{13} \end{pmatrix} = \lambda \begin{pmatrix} a_{11}+a_{13} & a_{12} & a_{13} \\ a_{21}+a_{23} & a_{22} & a_{23} \\ a_{31}+a_{33} & a_{32} & a_{33} \end{pmatrix}$$

Consider the equalities $a_{12} = \lambda a_{12}$, $a_{13} = \lambda a_{13}$, $a_{23} = \lambda a_{23}$, $a_{22} = \lambda a_{22}$. Then $\lambda = 1$ or $\lambda = 0$.

If $\lambda = 0$ then $a_{12} = a_{13} = a_{23} = a_{22} = a_{32} = a_{11} = 0$. This implies determinant is zero, which is impossible. So we get $\lambda = 1$.

$$\begin{aligned} a_{11}+a_{13} &= a_{11} & \text{implies} \quad a_{13} &= 0 \\ a_{21}+a_{23} &= a_{21} & \text{implies} \quad a_{23} &= 0 \\ a_{32}+a_{12} &= a_{32} & \text{implies} \quad a_{12} &= 0 \\ a_{31}+a_{33} &= a_{31}+a_{11} & \text{implies} \quad a_{11} &= a_{33} \end{aligned}$$

Therefore,

$$C = C_{PSL(3,\mathbb{F})}(i) = \{\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix} Z \mid a_{ij} \in \mathbb{F},\ a_{11}^2 a_{22} = 1\}$$

Now, choose $y = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) Z$ and find C^y:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} Z = \begin{pmatrix} a_{11} & a_{32} & a_{31} \\ 0 & a_{22} & a_{21} \\ 0 & 0 & a_{11} \end{pmatrix} Z$$

So $C \cap C^y = \{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{11} \end{pmatrix} Z \mid a_{ii} \in \mathbb{F},\ a_{11}^2 a_{22} = 1\}$.

Therefore $[C : C \cap C^y] \geq |\mathbb{F}^*|$, since a_{11} can be chosen arbitrarily in $\mathbb{F}^*$. In particular, if $\mathbb{F}$ is infinite then C is not inert in $PSL(3, \mathbb{F})$. □

4.2.3 Centralizer of involutions in $PSU(3, \mathbb{F})$ where $\mathbb{F}$ is locally finite field of characteristic 2

Let $q = 2^k$ and let $GU(3, q^2)$ denote the group of all 3×3 invertible matrices in $GF(q^2)$ that preserve the hermitian form

$$w := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

i.e. $GU(3,q^2) = \{U \in GL(3,q^2) \mid \bar{U}^t\omega U = \omega\}$. Here $\bar{U}$ denotes the matrix obtained from U by raising each entry to the q^{th} power. Let $SU(3,q^2) = \{U \in GU(3,q^2) \mid \det U = 1\}$. We will describe elements of $G = PSU(3,q^2)$ via matrix representatives of $SU(3,q^2)$ modulo scalars.

Lemma 4.18. *Let $\mathbb{F}$ be a field of char 2. Then there exists an involution $i \in PSU(3,\mathbb{F})$ and an element $y \in PSU(3,\mathbb{F})$ such that $|C_{PSU(3,\mathbb{F})}(i) : C_{PSU(3,\mathbb{F})}(i) \cap C_{PSU(3,\mathbb{F})}(i)^y| \geq |\mathbb{F}|$. In particular, if $\mathbb{F}$ is infinite then $C_{PSU(3,\mathbb{F})}(i)$ is not inert in $PSU(3,\mathbb{F})$.*

Proof. In $PSU(3,\mathbb{F})$ let $i = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)Z$. Then taking any element $y = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right)Z \in PSL(3,\mathbb{F})$. Then from the equality $iy = yi$ we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} Z = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} Z$$

by calculation we get

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+a_{11} & a_{32}+a_{12} & a_{33}+a_{13} \end{pmatrix} Z = \begin{pmatrix} a_{11}+a_{13} & a_{12} & a_{13} \\ a_{21}+a_{23} & a_{22} & a_{23} \\ a_{31}+a_{33} & a_{32} & a_{33} \end{pmatrix} Z$$

Note that in $PSL(3,\mathbb{F})$, if $aZ = bZ$ then $a = b\lambda I = \lambda b$ for some $\lambda \in \mathbb{F}$. Therefore, the above equality becomes

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+a_{11} & a_{32}+a_{12} & a_{33}+a_{13} \end{pmatrix} = \lambda \begin{pmatrix} a_{11}+a_{13} & a_{12} & a_{13} \\ a_{21}+a_{23} & a_{22} & a_{23} \\ a_{31}+a_{33} & a_{32} & a_{33} \end{pmatrix}$$

Consider the equalities $a_{12} = \lambda a_{12}$, $a_{13} = \lambda a_{13}$, $a_{23} = \lambda a_{23}$, $a_{22} = \lambda a_{22}$. Then $\lambda = 1$ or $\lambda = 0$.

If $\lambda = 0$, then $a_{12} = a_{13} = a_{23} = a_{22} = a_{32} = a_{11} = 0$. This implies determinant is zero and so which is impossible. So we get $\lambda = 1$.

$$\begin{array}{rclcrcl} a_{11} + a_{13} & = & a_{11} & \text{implies} & a_{13} & = & 0 \\ a_{21} + a_{23} & = & a_{21} & \text{implies} & a_{23} & = & 0 \\ a_{32} + a_{12} & = & a_{32} & \text{implies} & a_{12} & = & 0 \\ a_{31} + a_{33} & = & a_{31} + a_{11} & \text{implies} & a_{11} & = & a_{33} \end{array}$$

Therefore,

$C = C_{PSL(3,\mathbb{F})}(i) = \{\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix} Z \mid a_{ij} \in \mathbb{F},\ a_{11}^2 a_{22} = 1\}$ We are looking for $C_{PSU(3,\mathbb{F})}(i)$. So given a fixed element P in $C_{PSL(3,\mathbb{F})}(i)$ is in $C_{PSU(3,\mathbb{F})}(i)$ if P satisfies the above condition. i.e., $\bar{P}^t \omega P = \omega$. Then

$$\begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ 0 & \bar{a}_{22} & \bar{a}_{32} \\ 0 & 0 & \bar{a}_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \bar{a}_{31} & \bar{a}_{21} & \bar{a}_{11} \\ \bar{a}_{32} & \bar{a}_{22} & 0 \\ \bar{a}_{11} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \bar{a}_{31}a_{11} + \bar{a}_{21}a_{21} + \bar{a}_{11}a_{31} & \bar{a}_{21}a_{22} + \bar{a}_{11}a_{32} & \bar{a}_{11}a_{11} \\ \bar{a}_{32}a_{11} + \bar{a}_{22}a_{21} & \bar{a}_{22}a_{22} & 0 \\ \bar{a}_{11}a_{11} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

By the above equality from the entries $(3,3)$ and $(2,2)$ we get $\bar{a}_{11}a_{11} = \bar{a}_{22}a_{22} = 1$. So $\bar{a}_{11} = {a_{11}}^{-1}$ and $\bar{a}_{22} = {a_{22}}^{-1}$. Since $\bar{a}_{11} = {a_{11}}^q$ we get ${a_{11}}^{-1} = {a_{11}}^q$ and ${a_{22}}^{-1} = {a_{22}}^q$. We also have the determinant is 1. So ${a_{11}}^2 a_{22} = 1$. From the entry $(2,3)$ we have $\bar{a}_{32}a_{11} + \bar{a}_{22}a_{21} = 0$. Multiplying each side by $\bar{a}_{11}$ we get $\bar{a}_{32}a_{11}\bar{a}_{11} + \bar{a}_{11}\bar{a}_{22}a_{21} = 0$. Then since $\bar{a}_{11}a_{11} = 1$ and $\bar{a}_{11}a_{22} = {a_{11}}^{-1}$ we get $\bar{a}_{32} = {a_{11}}^{-1}a_{21}$ or $a_{21} = \bar{a}_{32}a_{11}$. Moreover, from the entry $(1,1)$ we get $\bar{a}_{31}a_{11} + \bar{a}_{21}a_{21} + \bar{a}_{11}a_{31} = 0$.

Therefore, we obtain that $C = C_{PSU(3,q^2)}(i)$ is of the form $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{11}{a_{32}}^q & a_{11}^{-2} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix}$ where

(i) $\bar{a}_{11}a_{11} = {a_{11}}^{q+1} = 1$

(ii) $\bar{a}_{22}a_{22} = {a_{22}}^{q+1} = 1$

(iii) $a_{32}a_{11} + {a_{22}}^q a_{21} = 0$

(iv) ${a_{31}}^q a_{11} + a_{32}{a_{11}}^q a_{21} + {a_{11}}^q a_{31} = 0$.

Now, choose $y = \omega = \left(\begin{smallmatrix} 0&0&1\\0&1&0\\1&0&0 \end{smallmatrix}\right)Z$ and find C^y:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & 0 & 0 \\ a_{11}{a_{32}}^q & {a_{11}}^{-2} & 0 \\ a_{31} & a_{32} & a_{11} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}Z = \begin{pmatrix} a_{11} & a_{32} & a_{31} \\ 0 & a_{11}^{-2} & a_{11}{a_{32}}^q \\ 0 & 0 & a_{11} \end{pmatrix}Z$$

So $C \cap C^y = \{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{pmatrix} Z \mid a_{11} \in \mathbb{F}^*\}$.

Therefore, $[C : C \cap C^y] \geq |\mathbb{F}^*|$, since a_{31} can be chosen arbitrarily in $\mathbb{F}^*$. In particular, if $\mathbb{F}$ is infinite then C is not inert in $PSU(3, \mathbb{F})$. $\square$

4.2.4 Centralizer of involutions in classical simple groups of Lie type where $n \geq 3$ and $char\mathbb{F} = 2$

Centralizer of involutions in $SL(n, \mathbb{F})$.

Given an involution a in $SL(V)$, define the rank of a as the dimension of the commutator space $[V, a]$ of a. The rank of a is also the number of Jordan blocks of (a_{ij}) of size 2 with respect to a basis of V in which a is in Jordan form [27]. Hence we have the following:

Theorem 4.19. [1, see 4.1] *Let a and b be involutions in $SL(V)$. Then a and b are conjugate in $SL(V)$ if and only if they have the same rank.*

Let $t \in SL(V)$ be an involution and V be n-dimensional vector space over the field $\mathbb{F}$. Given any basis, t is represented by a matrix T such that $T^2 = I_n$. We can find a basis so that the matrix of T is of the form

$$j_l = \begin{pmatrix} I_l & & \\ & I_{n-2l} & \\ I_l & & I_l \end{pmatrix}$$

where $1 \leq l \leq n/2$. Here I_m is the $m \times m$ identity matrix and I_0 is taken to be void. j_l has rank l and is referred to as the $Suzuki\ form$ of this class. We conclude

Theorem 4.20. [27, see pages 1048 and 1049] *The centralizer $C \in SL(n, q)$ of the involution T consists of those matrices of the form*

$$X = \begin{pmatrix} R & & \\ L & S & \\ M & N & R \end{pmatrix} \quad \textit{such that} \quad (\det(R))^2 \det(S) = 1,$$

where R and M are of size $l \times l$, S has size $(n-2l) \times (n-2l)$, L has size $(n-2l) \times l$, and N has size $l \times (n-2l)$. Furthermore, the map $X \mapsto (R, S)$ is a homomorphism of C into $GL(l, q) \times GL(n-2l, q)$ with the image containing $SL(l, q) \times SL(n-2l, q)$, and covering both factors if $n \neq 2l$. The kernel of this homomorphism is $T_l = O_2(C)$.

4.2.5 Centralizer of involutions in symplectic, unitary and orthogonal groups

Throughout this section, $\mathbb{F}$ will denote a locally finite field or finite field $GF(q)$ where q is power of a prime p. V will denote a vector space over $\mathbb{F}$ of dimension n.

We generally adhere to the notation of [27] but specialize our definition of classical group to cover only those groups of interest to us here.

Let θ be an automorphism of order 2 of $GF(q)$.

Centralizers of unitary groups

The central factor group of $SU(n, q^2)$ is denoted by $PSU(n, q^2)$ and it is simple group except when $n = 3$ and $q = 2$. The group $SU(3, 2)$ of exceptional case is a soluble group of order $3^3.2^3$.

Lemma 4.21. [30, page 34] *Let a and b be involutions in $SU(V)$. Then the following are equivalent.*

i) a is conjugate to b in $SU(V)$

ii) a is conjugate to b in $SL(V)$

iii) a and b have the same rank.

Theorem 4.22. [1, (6.2) page 13] *Let t be an involution in $SU(V)$ of rank l. Then there exists a basis for V in which $t = j_l$ is in Suzuki form*

$$J(V) = \begin{pmatrix} & & I_l \\ & I_{n-2l} & \\ I_l & & \end{pmatrix}$$

Further $g \in C_l$ is in $SU(V)$ if and only if

$$X^{\pi} = X^{-1},\ Y^{\pi} = Y^{-1},\ YR^{\pi} = PX^{\pi},\ XQ^{\pi} + RR^{\pi} + QX^{\pi} = 0$$

The map

$$\begin{array}{rccc} \varphi & : C = C_l \cap SU(V) & \longrightarrow & GU(l,q) \times GU(n-2l,q) \\ & g & \longmapsto & (X(g), Y(g)). \end{array}$$

is homomorphism with the image containing $SU(l,q) \times SU(n-2l,q)$ and covering both factors if $n \neq 2l$. The kernel is $T_l \cap C = O_2(C)$.

Centralizer of symplectic groups

The central factor group of $Sp(2m,q)$ is denoted by $PSp(2m,q)$ and it is simple group except when $(2m,q) = (2,2)$, or $(4,2)$. If the characteristic of $\mathbb{F}$ is 2 [28, (5.14)] then the center of the symplectic group is $\{1\}$. We have $Sp(4,2) \cong Sym(6)$. Hence we can work in $SP(2m,q)$.

Lemma 4.23. [1, (7.8) page 17] *Let $t = a_l$ be in Suzuki form, let $g \in C_l$, and $E = E_l$ or E_{n-2l}. Then $g \in Sp(V)$ if and only if*

$$XEX^* = F,\ YEY^* = E,\ YER^* = PEX^*,\ XEQ^* + RER^* + QEX^* = 0$$

The map

$$\begin{array}{rcl} \varphi \quad : C = C_l \cap Sp(V) & \longrightarrow & Sp(l,q) \times Sp(n-2l,q) \\ g & \longmapsto & (X(g), Y(g)). \end{array}$$

is onto homomorphism with kernel $T_l \cap Sp(V) = O_2(C)$.

Centralizers of orthogonal groups

In this section we assume that V is an orthogonal space of sign ϵ

Lemma 4.24. (1) $O^{\epsilon}(2,q)$ *is dihedral group of order* $2(q-\epsilon)$.

(2) $P\Omega^{+1}(4,q) \cong PSL(2,q) \times PSL(2,q)$

(3) $P\Omega^{-1}(4,q) \cong PSL(2,q^2)$

(4) $P\Omega^{-1}(6,q) \cong PSU(4,q)$

(5) $P\Omega^{-1}(6,q) \cong PSL(2,q^2)$

(6) $\Omega(3,q) \cong PSL(2,q)$

(7) $\Omega(5,q) \cong PSp(4,q)$

(8) $P\Omega(7,q) \cong B(3,q)$

Proof. The standard reference for these results is Dieudonné [11] $\square$

For the lemmas below, we assume $n \geq 8$.

Lemma 4.25. [1, (8.6) page 21] *Let* $t = a_l$ *be in orthogonal Suzuki form and* $g \in C_l \cap Sp(V)$. *Then*

(1) $g \in O^\epsilon(V)$ *if and only if* $Y(g) \in O^\epsilon(n-2l, q)$ *and*

$$\sum_{j=1}^{l/2} g_{i(2j-1)} g_{i(n-l+2j)} + g_{i(2j)} g_{i(n-l+2j-1)} = \sum_{j=1}^{(n-2l)/2} g_{i(l+2j-1)} g_{i(l+2j)}.$$

(2) *The map*

$$\begin{aligned} \varphi \quad : C = C_l \cap O^\epsilon(V) \quad &\longrightarrow \quad Sp(l,q) \times O^\epsilon(n-2l,q) \\ g \quad &\longmapsto \quad (X(g), Y(g)). \end{aligned}$$

is a homomorphism with kernel $T_l \cap C = O_2(C)$.

4.2.6 Centralizer of involutions in Suzuki groups $Sz(4, \mathbb{F})$ where $\mathbb{F}$ is a locally finite field of char 2

Let $\mathbb{F}$ be an infinite locally finite field of characteristic 2. As in the finite case (see [26]) the group $Sz(4, \mathbb{F})$ is a subgroup of $SL(4, \mathbb{F})$ defined in terms of an automorphism θ of $\mathbb{F}$ satisfying $a^{\theta^2} = a^2$, $a \in \mathbb{F}$, and generated by the union of the following:

(i) a group Q of matrices of the form:

$$(a, b) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^{1+\theta} + b & a^\theta & 1 & 0 \\ a^{2+\theta} + ab + b^\theta & b & a & 1 \end{pmatrix} \qquad a, b \in \mathbb{F}.$$

(ii) the group D of diagonal matrices of the form

$$\bar{f} := diag[f^{1+\theta^{-1}}, f^{\theta^{-1}}, f^{-\theta^{-1}}, f^{-1-\theta^{-1}}] \qquad f \in \mathbb{F}^*.$$

(iii) the permutation matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 4.26. [26, Proposition 1] *If x is a non identity element of Q then the centralizer $C_{Sz(4,\mathbb{F})}(x)$ is contained in Q.*

Lemma 4.27. *Let $\mathbb{F}$ be an infinite field of characteristic 2. Then there is an involution $x \in Sz(4,\mathbb{F})$ such that $C_{Sz(4,\mathbb{F})}(x)$ is not inert in $Sz(4,\mathbb{F})$*

Proof. In Suzuki group let $x = (0,1)$ be an element of order 2 in Q. We want to find $C_{Sz(4,\mathbb{F})}(0,1)$. Consider $(a_{ij})_{4\times 4} \in SL(4,\mathbb{F})$. Then by the equality:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11}+a_{31} & a_{12}+a_{32} & a_{13}+a_{33} & a_{14}+a_{34} \\ a_{11}+a_{21}+a_{41} & a_{12}+a_{22}+a_{42} & a_{13}+a_{23} & a_{14}+a_{24}+a_{44} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+a_{13}+a_{14} & a_{12}+a_{14} & a_{13} & a_{14} \\ a_{21}+a_{23}+a_{24} & a_{22}+a_{24} & a_{23} & a_{24} \\ a_{31}+a_{33}+a_{34} & a_{32}+a_{34} & a_{33} & a_{34} \\ a_{41}+a_{43}+a_{44} & a_{42}+a_{44} & a_{43} & a_{44} \end{pmatrix}$$

From the above equality of two matrices, we get equalities for entries. Let (i,j) denote the corresponding entry of the matrix.

From $(1,2)$: $a_{12} = a_{12} + a_{14}$, then $a_{14} = 0$

From $(1,1)$: $a_{11} = a_{11} + a_{13} + a_{14}$, then $a_{13} = a_{14}$, then $a_{14} = 0$.

From $(2,2)$: $a_{22} = a_{22} + a_{24}$, then $a_{24} = 0$

From $(2,1)$: $a_{21} = a_{21} + a_{23} + a_{24}$, then $a_{23} = a_{24}$, then $a_{24} = 0$

From $(2,3)$: $a_{13} + a_{23} = a_{23} + a_{24}$, then $a_{13} = 0$

From $(3,2)$: $a_{12} + a_{32} = a_{32} + a_{34}$, then $a_{12} = a_{34}$

From $(3,1)$: $a_{11} + a_{31} = a_{31} + a_{33} + a_{34}$, since $a_{12} = a_{34}$, then $a_{12} = a_{33} + a_{11}$

From $(4,1)$: $a_{11} + a_{21} + a_{41} = a_{41} + a_{43} + a_{44}$, then $a_{21} + a_{43} = a_{11} + a_{44}$

From $(4,2)$: $a_{42} + a_{44} = a_{12} + a_{22} + a_{42}$, then $a_{12} = a_{22} + a_{44}$

So we get $a_{12} = a_{34} = a_{22} + a_{44} = a_{11} + a_{33}$ and $a_{21} + a_{43} = a_{11} + a_{44}$.

Therefore $C_{Sz(4,\mathbb{F})}(x) \leq$

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{12} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \mid a_{12} = a_{34} = a_{22} + a_{44} = a_{11} + a_{33}, \ a_{21} + a_{43} = a_{11} + a_{44} \text{ and } a_{ij} \in \mathbb{F} \right\}.$$

If we take the centralizer in $Sz(4,\mathbb{F})$, then by Proposition 4.26 we get $C_{Sz(4,\mathbb{F})}(x) \leq Q$. Hence $a_{12} = 0$ and

$$a_{11} = a_{22} = a_{33} = a_{44} = 1 \; C_{Sz(4,\mathbb{F})}(x) \leq \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31} & b_{32} & 1 & 0 \\ b_{41} & b_{42} & b_{21} & 1 \end{pmatrix} \mid b_{ij} \in \mathbb{F} \right\} \leq$$

Q

$$\text{Now, let us choose } y = \tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \text{ Then}$$

Given any element $A \in C_{Sz(4,\mathbb{F})}(x)$. Let us compute

$$\begin{aligned} A^{\tau} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 \\ b_{31} & b_{32} & 1 & 0 \\ b_{41} & b_{42} & b_{21} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} b_{41} & b_{42} & b_{21} & 1 \\ b_{31} & b_{32} & 1 & 0 \\ b_{21} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b_{21} & b_{42} & b_{41} \\ 0 & 1 & b_{32} & b_{31} \\ 0 & 0 & 1 & b_{21} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Then $(C_{Sz(4,\mathbb{F})}(x))^{\tau} = \{ \begin{pmatrix} 1 & b_{21} & b_{42} & b_{41} \\ 0 & 1 & b_{32} & b_{31} \\ 0 & 0 & 1 & b_{21} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid b_{ij} \in \mathbb{F}\}$

So $C_{Sz(4,\mathbb{F})}(x) \cap (C_{Sz(4,\mathbb{F})}(x))^{\tau} = \Big\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\}$.

Therefore

$$[C_{Sz(4,\mathbb{F})}(x) : C_{Sz(4,\mathbb{F})}(x) \cap (C_{Sz(4,\mathbb{F})}(x))^{\tau}] \geq |\mathbb{F}|$$

□

4.3 MAIN THEOREM

The following lemma reduces the existence of a locally finite simple group G in which centralizer of every involution is inert to the countable case.

Definition 4.28. *The set Σ of subgroups of G is called a local system of G if $G = \bigcup_{S \in \Sigma} S$ and for any pair of $S, T \in \Sigma$, there exists $U \in \Sigma$ such that $S \leq U$ and $T \in U$.*

Lemma 4.29. *If there exists a locally finite simple group G and an involution x in G such that $C_G(x)$ is inert in G, then there exists a collection Σ of countably infinite simple subgroups H of G containing x such that $C_H(x)$ is inert in H for every H in Σ.*

Proof. By [19, Theorem 4.4] a locally finite simple group has a local system Σ_1 which consists of countably infinite simple subgroups of G. Let x be the given involution in G. We can form a subsystem Σ of Σ_1 such that every $H \in \Sigma$ contains the given involution. Indeed, one may write $\Sigma_+ = \{G_k \in \Sigma_1 \mid x \in G_k\}$ and $\Sigma_- = \{G_k \in \Sigma_1 \mid x \notin G_k\}$. Clearly $\Sigma_+ \cup \Sigma_- = \Sigma_1$. By Lemma [19, 1.A.10] either Σ_+ or Σ_- forms a local system for G. But Σ_- can not be a local system since $\bigcup_{G_j \in \Sigma_-} G_j$ does not contain x. Hence Σ_+ is a local system for G. Let $\Sigma = \Sigma_+$. Now for every $H \in \Sigma$, $C_H(x)$ is inert in H by Lemma 2.23. □

Lemma 4.30. *If there exists a locally finite simple group G in which centralizer of every involution is inert in G, then there exists a collection Ω of countably infinite simple subgroups H of G such that $C_H(x)$ is inert in H for all $H \in \Omega$.*

Proof. By [19, Theorem 4.4] a locally finite simple group has a local system Ω which consists of countably infinite simple subgroups of G. Let $H \in \Omega$. Then by Lemma 2.23 for every involution x in H, the group $C_H(x)$ is inert in H. □

Theorem 4.31. [29, Theorem 1] *Let* $G = \bigcup_{i \in \omega} G_i$*, where* ω *is infinite partially ordered set and each* G_i *is isomorphic to a Chavelley group or twisted Chavelley group of Lie type L over a finite field. Then G is isomorphic to a Chavelley group or a twisted Chevalley group of Lie type L over a locally finite field.*

Theorem 4.32. [15, Theorem B] *A group G is an infinite locally finite simple group satisfying Min-p for some prime p if and only if G is a group of Lie type over an infinite locally finite field.*

Remark 4.33. *If G is infinite locally finite simple group, then given an inert subgroup* $C_G(i)$ *is residually finite by Lemma* 2.28(*ii*).

Theorem 4.34. *If centralizer of every involution is inert in an infinite locally finite simple group G, then every finite set of elements of G can not lie in a finite simple group.*

Proof. Assume that G is a locally finite group such that every finite set of elements lies in a finite simple subgroup. Then G has a local system consisting of simple subgroups. Suppose that centralizer of every involution in G is inert in G. Since the group G is infinite, we may discard all sporadic simple groups from the local system. Since every finite set of elements of G lies in a finite simple group by using the classification of finite simple groups see Theorem 4.10 we may assume that:

(**1**) every finite set of elements lies in a finite alternating group,

(**2**) every finite set of elements of G lies in a finite simple group of Lie type.

Case (**1**). Let i be an involution in G. Then there exists a local system Σ of G consisting of alternating groups in which each group contains i. Centralizer of elements in alternating groups are well known and in particular, centralizers of involutions are in $\mathfrak{F}_2$ by Lemma 4.3. Then by Lemma 4.7, $C_G(i) \in \mathfrak{F}_2$. Hence

$C_G(i)$ has a series of finite length in which every factor is either non-abelian simple or locally soluble and involves an infinite simple group. But this is impossible by Lemma 4.5

Case (2). If G has a local system consisting of simple groups of Lie type, then as there are only finitely many types of simple groups of Lie type, we may assume again by Lemma [19, 1.A.10] that G has a local system consisting of finite simple groups of fixed Lie type. In this case again we may reduce to the case that either all of these simple groups are

(**a**) of bounded rank

(**b**) of unbounded rank.

2(**a**) If there is a bound on the rank of finite simple groups of fixed Lie type, then by theorem which is proved independently by Hartley-Shute, Borovik, Belyaev, Thomas ([6], [8], [15], [29]), we get G is a simple linear group of Lie type over an infinite locally finite field $\mathbb{F}$. Then $\mathbb{F}$ has either odd characteristic or even characteristic.

2(**a**)(**i**) *odd characteristic.* If the rank of the Lie group is greater than or equal to 2, then by Theorem 4.8 centralizer of every involution involves an infinite simple linear group, but this is impossible by Lemma 4.5. So we may assume that G has Lie rank 1.

The groups of Lie type over a field of odd characteristic and of Lie rank 1 are $PSL(2,\mathbb{F})$, $PSU(3,\mathbb{F})$, ${}^2G_2(\mathbb{F})$. By Lemma 4.13 centralizers of involutions are not inert in $PSL(2,\mathbb{F})$. For the Ree groups ${}^2G_2(\mathbb{F})$ where $\mathbb{F}$ is a locally finite field of characteristic 3 centralizers of involutions are not inert by Section 4.2.1. Finally for the groups $PSU(3,\mathbb{F})$ where $\mathbb{F}$ is a field of odd characteristic centralizers of involutions are not inert by Section 4.2.1

So we are done for infinite simple locally finite groups of Lie type over a field of odd characteristic.

2(**a**)(**ii**) *even characteristic*: Let G be an infinite simple group of Lie type over a field of characteristic 2. For these type of groups since characteristic of the field and the order of the element is 2, for the centralizers of involutions we use the well known article of Aschbacher and Seitz [1].

PSL(n,$\mathbb{F}$): The conjugacy classes of involutions in $SL(n,q)$ is given in [1, 4.1]. The involution denoted by J_1 in [1, 4.1] is noncentral element in $SL(n,q)$ and by Theorem 4.20 the centralizer $C_{SL(n,q)}(J_1) \in \mathfrak{F}_1$ and involves the simple group $PSL(n-2,q)$ provided that $n \geq 4$. Now by Lemma 4.4 the group $C_{PSL(n,\mathbb{F})}(J_1) \in \mathfrak{F}_1$. Then by Lemma 4.7 for an infinite locally finite field $\mathbb{F}$ of char 2 the group $C_{PSL(n,\mathbb{F})}(J_1)$ involves an infinite simple group of type $PSL(n-2,\mathbb{F})$. But this is impossible by Lemma 4.5.

Now for $n = 2$ by Lemma 4.15 and for $n = 3$ by Lemma 4.17 centralizers of involutions in $PSL(n,\mathbb{F})$ are not inert where $\mathbb{F}$ is an infinite locally finite field of characteristic 2.

Hence we are done for projective special linear groups $PSL(n,\mathbb{F})$ for an infinite locally finite field $\mathbb{F}$ of characteristic 2.

***PSU(n,$\mathbb{F}$)*:** In Theorem 4.22 the representatives of each conjugacy class of involutions in $SU(n,q)$ is given as in the notation of Theorem 4.22. Let $J(V) = \begin{pmatrix} & & 1 \\ & I_{n-2} & \\ 1 & & \end{pmatrix}$. Then by Theorem 4.22 the group $C_{PSU(n,q)}(J_1) \in \mathfrak{F}_2$. For $n \geq 4$ the group $C_{SU(n,q)}(J_1)$ involves the simple group $PSU(n-2,q)$. Then by Lemma 4.7 the group $C_{PSU(n,q)}(J_1) \in \mathfrak{F}_1$ for an infinite locally field $\mathbb{F}$ and involves infinite simple group. But this is impossible by Lemma 4.5. For $n = 2$ we have$PSU(2,q^2) \cong PSL(2,q) \cong SL(2,q)$ [28, Vol.I, 6.22] and this is discussed in Lemma 4.14. For $n = 3$ we have in $PSU(3,\mathbb{F})$ centralizers of involutions are not inert by Section 4.2.3

***PSp(n,$\mathbb{F}$)*:** The conjugacy classes of involutions in $Sp(n,q)$ is given [1, 7.7]. The involution denoted by a_1 of rank 1 in [1, 7.7], the centralizer of a_1 in $Sp(n,q) \in \mathfrak{F}_1$ by Theorem 4.23. Hence for $n \geq 4$ the group $Sp(n,q)$ involves a simple group $Sp(n,q)$ as $Z(Sp(n,q)) = 1$ by [28, 5.14]. Now for an infinite locally finite field $\mathbb{F}$ of char 2, $C_{Sp(n,q)}(a_1) \in \mathfrak{F}_1$ and involves an infinite simple group. But this is impossible by Lemma 4.5. For $n = 2$, we have $PSp(2,\mathbb{F}) \cong PSL(2,\mathbb{F}) \cong SL(2,\mathbb{F})$ see [9, Proposition 4.5]. This is done in Lemma 4.15.

Orthogonal groups: Let $O^{\epsilon}(n,q)$ be orthogonal simple group over a field of characteristic 2. By Lemma 4.24 we may assume that $n \geq 8$. The conjugacy classes of involutions in orthogonal groups are given in [1, 8.2, 8.3, and 8.4]. Choose the involution of type a_1. Then $C_{O^{\epsilon}(n,q)}(a_1) \in \mathfrak{F}_1$ by [1, 8.6] and Theorem 4.25. Hence for $n \geq 8$ the orthogonal group $C_{O^{\epsilon}(n,q)}(a_1)$ involves simple orthogonal group $O^{\epsilon}(n-2,q)$. By using Lemma 4.24(4,5) for $n = 8$, we obtain for an infinite locally finite field $\mathbb{F}$ by Theorem 4.25 the group $C_{O^{\epsilon}(n,\mathbb{F})}(a_1) \in \mathfrak{F}_1$ and involves an infinite simple group. But this is impossible by Lemma 4.5.

Exceptional Chevalley groups: For ${}^2E_6(q)$, q even. There are three conjugacy classes of involutions in this group. This is given in [1, 12.7]. The group $C_G(t)$ involves $PSU(6,q)$, the group $C_G(u)$ involves $SO(7,q)$ and the group $C_G(v)$ involves $SL(2,q)$ [1, see 14.3]. Hence as before centralizer of involutions involves infinite simple groups when the field is infinite locally finite. But this is impossible by Lemma 4.5

For $E_6(q)$, q even. There are three conjugacy classes of involutions in this group. This is given in [1, 12.8]. The group $C_G(x)$ involves $SL(6,q)$, the group $C_G(y)$ involves $Sp(6,q) \cong SO(7,q)$ and the group $C_G(z)$ involves two simple groups $SL(2,q)$ and $SL(2,q)$ [1, see 15.5]. Hence centralizer of every involution in

$E_6(q)$ involves simple groups. If $\mathbb{F}$ is an infinite locally finite field of characteristic 2, then the centralizer of every involution $E_6(\mathbb{F})$ involves an infinite simple group. But this is impossible by Lemma 4.5.

For $F_4(q)$, q even. There are four conjugacy classes of involutions in this group. This is given in [1, 12.6]. The groups $C_G(t)$ and $C_G(u)$ involves $Sp(6, q)$. Moreover $C_G(tu)$ involves $Sp(4, q)$ and $C_G(v)$ involves $SL(2, q)$. Hence for an infinite locally finite field $\mathbb{F}$ of characteristic 2 centralizer of every involution involves an infinite simple group in $F_4(\mathbb{F})$. But this is impossible by Lemma 4.5.

For $E_7(q)$, q even. There are five conjugacy classes of involutions in this group. This is given in [1, 12.9]. The group $C_G(x)$ involves $SO^+(12, q)$. The group $C_G(y)$ involves $Sp(8, q)$ and $SL(2, q)$. The group $C_G(z)$ involves $SL(2, q)$ and $Sp(6, q)$. Moreover, the groups $C_G(u)$ and $C_G(v)$ involves $F_4(q)$ and $Sp(6, q)$ respectively. If $\mathbb{F}$ is an infinite locally finite field, then the centralizer of every involution in $E_7(\mathbb{F})$ involves an infinite simple group. But this is impossible by Lemma 4.5.

For $E_8(q)$, q even. There are four conjugacy classes of involutions in this group. This is given in [1, 12.11]. The group $C_G(x)$ involves $E_7(q)$. The group $C_G(y)$ involves $Sp(12, q)$. The group $C_G(z)$ involves $F_4(q)$ and $SL(2, q)$. Moreover, the group $C_G(u)$ involves $Sp(8, q)$ [1, 17.5]. Hence every involution in $E_8(q)$ involves simple groups. If $\mathbb{F}$ is an infinite locally finite field of characteristic 2, then the centralizer of every involution $E_8(\mathbb{F})$ involves an infinite simple group. But this is impossible by Lemma 4.5.

Exceptional rank* 2 *groups : For $G_2(q)$, q even. Then by [1, 18.2] there are two conjugacy classes of involutions with representatives z and t. Then by [1, 18.4] the groups $C_G(z)$ and $C_G(t)$ involves simple groups $SL(2, q)$. So if $\mathbb{F}$ is infinite locally finite field of characteristic 2, then centralizer of every involution in $G_2(\mathbb{F})$ involves infinite simple group. But this contradicts to Lemma 4.5.

For ${}^3D_4(q)$, q even. Then by [1, 18.2] there are two conjugacy classes of involutions with representatives z and t. Then by [1, 18.5]. The group $C_G(z)$ involves $SL(2, q^3)$ and the group $C_G(t)$ involves simple group $SL(2, q)$. So if $\mathbb{F}$ is an infinite locally finite field of characteristic 2, then centralizer of every involution in ${}^3D_4(\mathbb{F})$ involves infinite simple subgroups. But this is contradicts to Lemma 4.5.

For ${}^2F_4(q)$, q even. Then by [1, 18.2] there are two conjugacy classes of involutions with representatives z and t. Then by [1, 18.6]. The group $C_G(z)$ involves the simple group $Sz(q)$ and the group $C_G(t)$ involves simple group $SL(2, q)$. So if $\mathbb{F}$ is an infinite locally finite field of char 2, then centralizer of every involution in ${}^2F_4(\mathbb{F})$ involves infinite simple subgroups. But this contradicts to Lemma 4.5.

For $Sz(q)$, q even. We show in Section 4.27 that centralizer of an involution involves an infinite simple group if $\mathbb{F}$ is an infinite locally finite field. But this is impossible by Lemma 4.5.

For $PSU(2, q^2)$, q even. Since $SU(2, q^2) \cong SL(2, q)$ [28, Vol.I, 6.22]. Then by Lemma 4.15 centralizer of an involution in $PSU(2, \mathbb{F})$ is not inert in $PSU(2, \mathbb{F})$.

2(**b**) If there is no bound on the rank of the Lie type of finite simple groups, clearly this arises in classical groups. By using the classification of finite simple groups we may assume that G is a nonlinear locally finite simple group in which every finite set of elements lies in a fixed type of Lie group but the rank of these groups are not bounded.

2(**b**)(**i**) If the Lie groups in $2(b)$ over fields of odd characteristic.

By Lemma 4.30 we may assume that G is a countable simple group which has a local system consisting of classical Lie type groups. Then $G = \bigcup_{m=1}^{\infty} G_m$ where G_m are all belong to fixed classical family with unbounded rank parameters. If $i \in G$ then $C_G(i) = \bigcup C_{G_m}(i)$. The groups G_m are all in $\mathfrak{F}_4$ by Theorem 4.9 and

$L\mathfrak{F}_4 = \mathfrak{F}_4$ by Theorem 4.7(ii) . Then centralizers of involutions in finite simple groups of classical Lie type over a field of odd characteristic case are in $\mathfrak{F}_4$ and by Theorem 4.8 they involve an infinite simple group. Hence again by Lemma 4.5 this case is impossible.

2(**b**)(**ii**) Centralizers of involutions in classical groups over even characteristic can be observed by from the $2(a)(ii)$ part of the proof of this theorem. Since rank of the Lie group is unbounded one can see that in this case centralizer of every involution is in $\mathfrak{F}_2$ and involves an infinite simple group. Then by using Lemma 4.7(ii) we have $L\mathfrak{F}_2 = \mathfrak{F}_2$. We obtain centralizers of every involution, involves infinite nonlinear locally finite simple groups. Hence they can not be inert in G by Lemma 4.5. This proves the theorem. □

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