SCHERK-SCHWARZ REDUCTION OF EFFECTIVE STRING THEORIES IN EVEN DIMENSIONS

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ABSTRACT

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Scherk-Schwarz reductions are a generalization of Kaluza-Klein reductions in which the higher dimensional fields are allowed to have a dependence on the compactiifed coordinates. This is possible only if the higher dimensional theory has a global symmetry and the dependence is dictated by this symmetry. In this thesis we consider generalised Scherk Schwarz reductions of supergravity and superstring theories with twists by electromagnetic dualities that are symmetries of the equations of motion but not of the action, such as the S-duality of $D = 4, N = 4$ super-Yang-Mills coupled to supergravity. The reduction cannot be done on the action itself, but must be done either on the field equations or on a duality invariant form of the action, such as one in the doubled formalism in which potentials are introduced for both electric and magnetic fields. The resulting theory in odd dimensions has massive form fields satisfying a self-duality condition $dA \sim m * A$. We apply these methods to theories in $D = 4, 6, 8$, and obtain new gauged supergravity theories with massive form fields, with Chern-Simons like couplings and with a scalar potential in $D = 3, 5, 7$.

Keywords: Superstring, Supergravity, Duality, Scherk-Schwarz Reduction

ÖZ

ÇİFT SAYIDA BOYUTA SAHİP UZAYLARDA EFEKTİF SİCİM KURAMLARININ SCHERK-SCHWARZ İNDİRGEMESİ

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Scherk-Schwarz indirgemeleri Kaluza-Klein indirgemelerinin yüksek boyuttaki alanların sıkıştırılmış koordinatlara bağımlı olmasına izin verilen bir genelleştirilmesidir. Bu ancak yüksek boyuttaki kuramın global bir simetrisi varsa mümkündür ve bağımlılık bu simetri tarafından belirlenir. Bu tezde süperkütleçekim ve süpersicim kuramlarının, aksiyonun değil haraket denklemlerinin simetrisi olan elektromanyetik düalitelerle burkulmuş, ki $D = 4, N = 4$ süperkütleçekime eşlenmiş süper-Yang-Mills kuramının S-düalitesi buna bir örnektir, genelleştirilmiş Scherk Schwarz indirgemelerini inceliyoruz. İndirgeme aksiyonun kendisi üzerinde değil, ya alan denklemleri üzerinde ya da hem elektrik hem de manyetik alanlar için potensiyallerin kullanıldığı çiftlenmiş formalizmde olduğu gibi aksiyonun düalite altında değişmeyen bir formu üzerinde yapılabilir. Elde ettiğimiz tek sayıda boyuta sahip uzaydaki kuram, $dA \sim m * A$ kendine düal şartını sağlayan kütleli form alanlarına sahiptir. Bu yöntemleri $D = 4, 6, 8$ boyutlardaki kuramlara uyguluyor ve $D = 3, 5, 7$ boyutlarda ayarlı, kütleli form alanlarına, Chern-Simons tipi etkileşimlere ve skalar potansiyele sahip yeni süperkütleçekim kuramları elde ediyoruz.

Anahtar Sözcükler: Süpersicimler, Süperkütleçekim, Düalite, Scherk-Schwarz İndirgemesi

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CHAPTER 1

INTRODUCTION

One of the biggest challenges in theoretical physics is the unification of the four fundamental forces in nature. Three of them, the electromagnetic, the weak and the strong nuclear forces are successfully described by the Standard Model (SM), a non-abelian gauge theory with $SU(3) \times SU(2) \times U(1)$ gauge group. Although this model passes many experimental tests, it has several shortcomings, which points out to the possibility that there might be a more fundamental theory describing the nature, of which the SM is only a low-energy approximation.

For example, in the SM there are 17 parameters whose values cannot be determined theoretically. Also the gauge group is not determined by the dynamics of the theory. Another point that the SM fails to explain is that there are three fermion generations in nature. There is no theoretical ground for this number in the SM.

SM, being a gauge theory, describes massless particles. However, in particle physics, almost all the particles that we know have mass. Then one has to find a way to break (some of) the non-abelian gauge symmetry so that one can describe the massive particles in a consistent way. The mechanism through which this becomes possible is the Higgs mechanism. Here one introduces a scalar field with a nonzero vacuum expectation value, which is called the Higgs field. Then the ground state of the theory is not invariant under the whole symmetry group, say G but is invariant only under a subgroup of it, say H and this leads to the breaking of the symmetry G down to the subgroup H. This type of symmetry breaking is known as *spontaneous symmetry breaking*. The breakdown of any continuous symmetry gives rise to massless bosons, the so-called Goldstone bosons, which parametrize the coset space G/H. These massless bosons are eaten by the gauge fields, which means that their degrees of freedom are absorbed by the gauge fields so that the gauge fields now have the right degrees of freedom for a massive field. This is how the SM particles acquire their masses and the fact that the masses predicted by the theory are in very good agreement with the values we measure experimentally is a big success for the theory. However, the Higgs sector is introduced in an *ad hoc* way, rather than being determined by the gauge principles alone. Moreover, the Higgs particle, whose existence is required by this mechanism has never been observed and its mass cannot be predicted by the theory.

One of the main reasons that one should seek to go beyond the SM, in spite of its experimental success is that it is not possible to incorporate gravity, the fourth fundamental force in nature into SM. The SM is a quantum field theory and all the attempts that have been made to unify gravity with the other three forces of nature in the framework of quantum field theory have failed since all such theories are non-renormalizable. The short-distance singularities become so severe that the usual perturbative methods of quantum field theory are not valid anymore and calculations of physical quantities, such as scattering amplitudes, based on these methods diverge. In other words, the General Relativity theory of Einstein, which describes the large-scale physics in a successful and beautiful way, and the quantum field theory of the SM, which describes the atomic-scale physics in a satisfactory way, cannot be brought together. The gravitational effects are negligible above the scale of Planck length $\sim 10^{-33}$ cm (or below the energy scale $\sim 10^{19}$GeV)[1], much smaller than the scale of particle physics

[1] The Planck energy M_p is the characteristic scale of any theory which includes gravitation in a relativistic and quantum mechanical setting. It depends on the speed of light c, the Planck's

$\sim 10^{-16}$ cm ($\sim$ 100GeV), and this explains why the SM of particle physics is so successful experimentally. However, the ultimate theory of nature should be capable of explaining all the large and small phenomena in nature in a consistent way, so one should not be satisfied with the SM and General Relativity and seek for a more fundamental theory, which is sometimes referred to as the "Theory of Everything". The appropriate limits of this theory should give us the SM and the General Relativity. Then one would expect from this theory to unify the four fundamental forces and also to cure the problems of the SM that were listed above.

Many attempts have been made in this direction. One attempt is to study Grand Theories(GUTs), where one considers that the gauge group of the SM is in fact embedded in a larger gauge group at higher energy scales, which breaks down to the $SU(3) \times SU(2) \times U(1)$ at the SM scale. In some of these models the 15 fermions in the three generations (5 fermions in each generation) can be put in a single multiplet and some of the arbitrary parameters can be fixed. The possible gauge groups that give the correct particle multiplets at low energies are $SU(5)$, $SO(10)$ and E_6. For such theories to make sense the coupling strengths $\alpha_1, \alpha_2, \alpha_3$ associated with the groups $SU(3), SU(2)$ and $U(1)$ should meet at a single value at the energy scale at which the gauge group is assumed to be a simple group. By studying the renormalization group equations for the coupling strengths and feeding in the measured values of them, it is seen that this occurs only when the theories are supersymmetric (and when supersymmetry is broken at the TeV scale). We will talk about supersymmetry below.

Introduction of extra dimensions to the space-time has been another fruitful approach in the attempt of unifying the fundamental forces of nature and curing the problems of the SM. In this approach one considers a space-time with $4 + d$

constant $\hbar$, and the Newton's constant G as $M_p c^2 = (c\hbar/G)^{1/2} c^2 \sim 10^{19}$ GeV.

dimensions, where the extra d dimensions curl up into a too small volume to be observed. This idea was first introduced by Kaluza [1] and followed up by Klein, who showed that pure gravity theory in five dimensions with a circular dimension of the order of $10^{-33}cm$ gives rise to gravity + electromagnetism in four dimensions [2]. This idea was also used for justifying the unnatural Higgs sector of the SM [3]. In these models one identifies the Higgs field with the components of the $4+d$-dimensional field strength in the compacitifed directions. In this way one can predict the Weinberg angle and the mass of the Higgs particle. The early models in this direction gave contradicting results with the experiments. However, recently, realistic models have been constructed by using the machinery of noncommutative geometry of Connes. In this case the extra dimensions are taken to be coordinates of a non-commutative manifold. By choosing an appropriate noncommutative space-time manifold, Connes successfully reproduced the SM with the right gauge symmetry and the right field content, with no arbitrary free parameters [4]. Also, in such models the Higgs sector arises on the same footing with the Yang-Mills sector and the Higgs mass can be predicted. In [5] we applied these ideas to the electroweak model ($SU(2) \times U(1)$ gauge theory unifying the electromagnetic and the weak nuclear forces). By taking the noncommutative space as the manifold corresponding to the C^* algebra $C^\infty(V) \otimes M_3(C)$, we obtained the 4D electroweak model with the Higgs sector appearing naturally and with a prediction of 130 GeV for the Higgs mass. Here $C^\infty(V)$ is the space of continuous functions on the 4 dimensional Riemannian manifold V and $M_3(C)$ is the space of 3×3 complex matrices.

Supersymmetry, which is a symmetry that relates bosons and fermions, is another suggestion for beyond-the-standard-model physics. According to supersymmetry every boson has a fermionic partner and *vice versa*, that is, they are in the same irreducible multiplet of the supersymmetry. The minimal supersymmetric extension of the SM (the MSSM) allows the prediction of the Higgs mass. Einstein's theory of gravity can be combined with supersymmetry and such theories are called supergravity theories. We will talk about supergravity in the forthcoming paragraphs.

Supersymmetry is not observed in nature (otherwise the fermions and the bosons would have the same masses) so it must be a broken symmetry. However, it is difficult to break global supersymmetry. Fortunately, in supergravity and superstring theories, which are of main interest in this thesis, supersymmetry is a local symmetry and there are various mechanisms through which it can be broken. For example, there is a supersymmetric version of the Higgs mechanism, the superHiggs mechanism, where the spin-3/2 gravitino (the gauge fermion of supergravity) becomes massive by eating the goldstino (the Goldstone fermion). We will mainly be interested in one of these mechanisms, namely the Scherk-Shwarz mechanism, although our main concern will not be the breaking of supersymmetry.

The most successful theory which attempts to unify gravity with the SM is, so far, string theory [6]. It is very remarkable that string theory contains all the important previous ideas that we have described above briefly: grand unification, extra dimensions and supersymmetry. Every string theory has a massless spin-2 particle, which behaves like a graviton, the gauge particle of general relativity, or in other words every string theory contains gravity. The non-renormalizability problems are cured and there are no arbitrary free parameters like in the SM. Also the gauge groups appearing in these theories are determined by the dynamics of the theory, unlike in the SM. These are all very attractive features, all coming from a single theory. So, what is string theory?

The underlying idea in string theory is very simple: the elementary particles are regarded as strings rather than as points. Strings can vibrate and the normal modes are determined by the tension of the string. Each vibrational mode corresponds to an elementary particle whose mass is determined by the vibrational frequency of the mode. These strings are very small, of the order of $10^{-33} cm$ long, 10^{20} times smaller than the diameter of the proton. So at the energy scales that our contemporary accelerators can reach, they look like points, the stringy structure cannot be probed into.

In classical mechanics, the motion of a point particle in Minkowski space is described by the following action

$$S = \frac{1}{2} \int d\tau \sum_{ij} \eta_{ij} \frac{dX^i}{d\tau} \frac{dX^j}{d\tau}. \tag{1.1}$$

Here η_{ij} is the Minkowski metric, τ is an arbitrary parameter along the trajectory (which can be taken to be the proper time) and $X^i(\tau)$ is the position of the particle. The solutions of the equations of motion of (1.1) are straight lines, the geodesics of the Minkowski space.[2] A point particle is a zero dimensional object and its motion is described by its one dimensional trajectory, its *world line.*

The natural analogue of (1.1) for the 1-dimensional string is:

$$S = \frac{1}{2\alpha'} \int d^2\sigma \sum_{ij} \sum_{\alpha} \eta_{ij} \frac{dX^i}{d\sigma^\alpha} \frac{dX^j}{d\sigma^\alpha}. \tag{1.2}$$

The string sweeps out a two dimensional manifold (a surface), the *world sheet*, and the parameters $\sigma^\alpha = (\sigma, \tau)$ parametrize this surface. It is useful to think of τ as a time-like evolution parameter and σ parametrizes the string. Then $X^i(\sigma, \tau)$ specifies the position of the string at given values of σ and τ. The parameter σ is taken to run from 0 to π. If we take periodic boundary conditions, i.e. $X^i(0, \tau) = X^i(\pi, \tau)$ then the string is *closed.* If the endpoints of the string can move freely in spacetime, then it is an *open string.*

[2] This is complemented with the constraint that the geodesics are light-like if the particle is massless.

The constant α' in (1.2) ,with the units of $length^2$ (where one chooses to work in 'natural units' $\hbar = c = 1$) is called the Regge slope ($\alpha' \sim (10^{-32} cm)^2$) . It is related to the string tension T and is a new fundamental constant measuring the "stringiness" of a physical system. Analogous to the fact that the Planck's constant $\hbar$ controls the passage from classical to quantum physics, α' controls the passage to stringy physics.

The action (1.2) describes a non-linear sigma model on the worldsheet. The functions X^i take values in the *target space*, the D dimensional space-time in this case. The metric η_{ij} is the metric on this target space. The symmetries of (1.1) are the diffeomorphism invariance (reparametrization invariance) and the Weyl invariance (a local rescaling) of the world-sheet and the D dimensional Poincaré invariance of the target space.[3] Classical free string theory can be consistently formulated for any space-time dimension D. However, after quantization, the requirement of having a ghost-free spectrum constrains the dimension of the space-time to be $D = 26$. This is the *bosonic string theory* (the coordinates X^i transform as bosonic fields from the target-space point of view). But this theory has several shortcomings. The obvious one is that it does not include the fermions and there are tachyonic states (negative-mass states) in the theory. The resolution is to introduce world-sheet supersymmetry that relates $X^i(\sigma, \tau)$ to fermionic partners $\psi(\sigma, \tau)$. This gives a consistent string theory, free of tachyons, but only when the space-time dimension is $D = 10$. One can also obtain target space supersymmetry by truncating the spectrum with a method called the Gliozzi-Scherk-Olive (GOS) projection.

Let us recall the Feynman diagrams in quantum field theory. In Feynman's sum-over-histories approach one regards a (point) particle moving from space-time event x to the space-time event y as having taken all the possible paths

[3] From the two dimensional world sheet point of view, this is a local symmetry since its action on the fields depend on σ and τ.

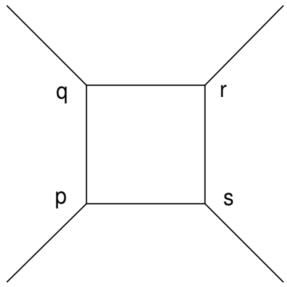

Figure 1.1: A 1-loop Feynman diagram contributing to a 4-particle amplitude, the interactions occurring at p,q,r,s

between the events x and y and the propagator is calculated by integrating over all these paths, with a weight factor derived from the classical action for this path. Similarly, for a system of interacting particles, the interactions are represented in a Feynman diagram by the branching and rejoining of the particles' worldlines and the scattering amplitude is found by summing over all such possible arrangements. For example, the diagram in Figure 1.1 represents two incoming and two outgoing particles interacting at the space-time events p,q,r and s. It is evident from the diagram that such interactions are likely to lead to divergent quantities. The points at which the interactions take place are singularities when the Feynman diagram is regarded as a one dimensional manifold. Indeed when the space-time events p,q,r and s in Figure 1.1 nearly coincide potential infinities arise in the integration. Non-renormalizable theories, such as the quantum theory of gravity, are the theories for which such infinities cannot be renormalized away.

Feynman's approach carries over to string theory with particles being replaced by strings and the wordlines being replaced by worldsheets. Now one sums over all possible surfaces that join the initial and the final state of a string or a set of interacting strings. An h-loop string theory Feynman diagram is represented by a genus h Riemann surface and contains a factor of g_s^{2h}, where g_s is the string coupling constant. Then a diagram of the form Figure 1.1 takes the form in

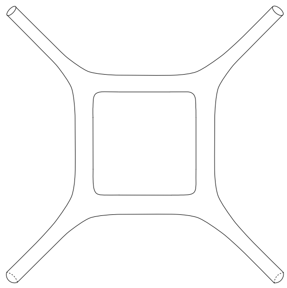

Figure 1.2: Counterpart of figure 1.1 for closed strings

Figure 1.2. This is a smooth manifold, pointing to the possibility that the divergences appearing in quantum field theory of point-like particles can be cured. This does indeed happen most of the time and moreover there is only one single Feynman diagram at each order of perturbation expansion in string theory, unlike in quantum field theory. This, combined with the fact that, every string theory *must* contain a spin-2 particle which behaves like a graviton at long distances[4], explains why string theory is so promising as a quantum theory of gravity. Perhaps, this is the right place to mention that one should not expect to understand string theory merely by perturbative methods as in, for example QED, because the string coupling constant g_s is not necessarily small. In fact g_s is determined dynamically by the vacuum expectation value of a scalar field, called the dilaton field. So one also needs non-perturbative methods which we will discuss when we talk about dualities.

As a candidate theory for being the unified theory of nature, the string theory should contain the Yang-Mills fields and gauge symmetries along with gravity. Indeed the spin-0 gauge particles for the other forces and also spin-0 and spin-1/2 particles are included in the massless spectrum of both the open and closed string theories. The way Yang-Mills gauge symmetries appear are, however, different

[4] Strictly speaking, this is true only for the closed string. Particles associated with the vibrational modes of the open string does not include the graviton. However every theory of open strings contains also closed strings since open strings can join to form closed strings.

for the open and the closed string cases. In the open string case, Yang-Mills group quantum numbers are introduced by attaching charges at the end of the open strings, by what is known as the Chan-Paton method. At the classical level any group is allowed, however after quantization only the gauge group $SO(32)$ is possible. The closed strings have no free ends, therefore some other way must be found to incorporate gauge symmetries. In the closed string, the right-moving modes and the left-moving modes decouple, that is, they move independently. So, it is possible to think that they are of different types. By taking the right-moving modes as those of a superstring theory living in 10 dimensions and the left-moving modes as those of a bosonic string living in 26 dimensions, one can incorporate gauge symmetries. That the right-moving modes are supersymmetic ensures immediately that there are fermions and no tachyons in the theory. The 16 extra dimensions of the left-moving modes are interpreted as internal dimensions and are associated to an even self-dual 16-dimensional lattice. On the other hand, there are two Lie groups, $SO(32)$ and $E_8 \times E_8$ which have rank 16 and whose weight lattices correspond to even self-dual 16 dimensional lattices, so these are the only gauge groups that can appear in this way. Such a type of string is called the *heterotic string*. It is quite remarkable that these two Lie groups had been singled out previously by Green and Schwarz as they showed that any 10 dimensional chiral theory containing gravity should also include either of these groups as Yang-Mills gauge symmetry in order to lead to chiral theories in lower dimensions without any anomalies.

As we have seen, consistent superstring theories live in 10 dimensions which does not agree with our "real world", which is only 4 dimensional. This is where the Kaluza-Klein type of ideas enter string theory. One contemplates that the remaining six dimensions have been curled up into a very small volume to be observed. This is not an unnatural idea for a theory containing gravity, such as string theory, since in general relativity the space-time is a dynamical, geometric object. In string theory, there are severe constraints on the type of the manifold on which the extra dimensions compactify. These constraints come from, for example, supersymmetry and the diffeomorphism invariance of the worldsheet. It turns out that this manifold should be a very specific type of manifold, what is known in mathematics as the Calabi-Yau manifolds. We will have many occasions in this thesis to talk about compactification methods.

We have talked about three different types of strings: the open, closed and heterotic strings. In mid 80s, string theorists believed that there were 5 different consistent superstring theories: type I, type IIA, type IIB, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic. The type I is based on unoriented open and closed strings, whereas all the others are based on oriented closed strings. The type IIA is non-chiral, *i.e.*, parity conserving and the others are chiral, *i.e.* parity violating. This was a quite disturbing situation for a theory claiming to be the fundamental theory of nature. However, later it was realized that all these seemingly distinct superstring theories are actually equivalent under the so-called duality symmetries. The best understood dualities are the *T dualities*, which relates string theories compacitifed on T-dual manifolds. For example, type IIA string theory compacitifed on a circle S^1 of radius R is T-dual to type IIB compactified on a circle with radius $1/R$. In a similar way heterotic $E_8 \times E_8$ (HE) is T-dual to heterotic $SO(32)$ (HO). Also type I theory on a circle of radius R is obtained from type IIA on a line interval I with size proportional to $1/R$, or on a circle of radius $1/R$ by acting with P, where P is the world sheet parity operator $(I = S^1/P)$[5]. So, when T-duality is taken into account the number of distinct superstring theories is now two, not five.

T-dualities are perturbative duality symmetries. In string theory, there are

[5] This, in turn, means that type I is obtained from type IIB by the action of P. This relation is also true in 10 dimensions by taking R to infinity.

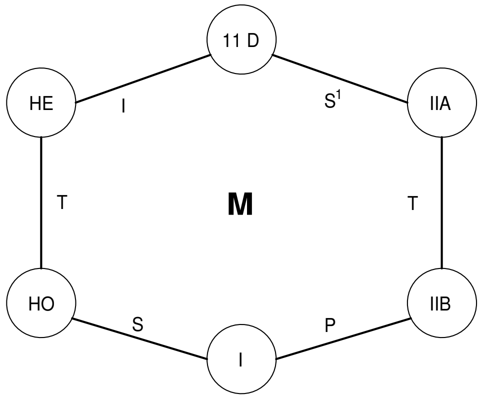

Figure 1.3: The M-theory moduli space

also non-perturbative symmetries, which are the generalizations of the duality symmetry in Maxwell's electromagnetic theory, under which the electric field is exchanged with the magnetic field. This type of duality is called *S-duality* and relates the weak-coupling regime of one superstring theory to the strong coupling regime of another (or the same) superstring theory. S-duality relates type I theory to $SO(32)$ heterotic theory and the type IIB theory is self-dual under S-duality. The strong coupling limit of type IIA theory comes as a big surprise. In this case one obtains a theory in eleven dimensions and the interpretation is that the type IIA is a theory in eleven dimensions with the eleventh dimension being a circle whose radius becomes large at strong coupling, i.e., the eleventh dimension decompactifies at strong coupling. There is a similar story for the $E_8 \times E_8$ heterotic string theory as was shown by Horava and Witten. This 11 dimensional theory is related to all the five superstring theories with various dualities [7]. So the new picture arising is that the five superstring theories, which look different perturbatively are, in fact, different limits of an underlying, non-perturbative theory, which is called the *M theory*[6]. More precisely, the five

[6] Sometimes the 11 dimensional theory discussed above is referred to as the M-theory.

superstring theories and the new eleven dimensional theory are the vacua of M-theory which correspond to consistent perturbative theories that are related by duality symmetries. This is depicted in Figure 1.3. Note that both the radius of the compactification circle R and the string coupling constant g_s arise as vacuum expectation values of some fields, so these duality symmetries correspond to a motion in the moduli space rather than to a change in the parameters of the theories.

We have seen that the correct description of the "Theory of Everything" should be non-perturbative. In the non-perturbative theory, there are new p-dimensional extended objects, called *p-branes*, which are of fundamental importance since they are states in the duality multiplets. However we won't go into that.

Historically, string theory had been first proposed as a theory of strong nuclear interactions but lost its charm soon afterwards as it turned out that the latter is described very successfully by QCD, the quantum chromodynamics. Then in 1974, inspired by the fact that the massless spectrum of every string theory includes a spin-2 particle, Jöel Scherk and John Schwarz proposed string theory as a theory of quantum gravity [8]. However, string theory didn't become popular among theoretical physicists until mid80s, when most of the discoveries that we mentioned above were made. This period is known as the *first superstring revolution.* The developments in non-perturbative string theory started in 1994 and this date is known as the beginning of the *second superstring revolution.*

In the spring of 1976, two years after the proposal of Scherk and Schwarz, the supersymmetric extension of Einstein's theory of gravity, supergravity theory, was formulated by Freedman, van Nieuwenhuizen and Ferrara [9] and a simplified version was presented by Deser and Zumino soon after [10]. The formalism of [10], the so-called "first-order formalism" also implied that in any supersymmetric theory coupled to gravity, supersymmetry must be a local symmetry. Many achievements in supergravity theory had already been made by the end of 1976, such as the matter coupling to Maxwell theory, to Yang-Mills theory, and the first formulation of extended [$N = 2$] supergravity. Although these theories are beautiful, in the sense that all the couplings are dictated almost in a unique way by the symmetries, they cannot be proposed as a quantum theory of gravity since it turns out that they are non-renormalizable already at the one loop level.

String theory contains gravity and it should be supersymmetric for consistency, so one would expect a relation between string theory and supergravity theory. In fact supergravity is the low energy effective field theory of superstring theory and the massless modes of string theories are governed by a 10 dimensional supergravity. As we had mentioned before, each vibrational mode of the string corresponds to a particle whose mass is determined by the frequency of the vibration, which is then determined by the tension of the string. Since string theory is to contain gravity, the tension of the string must be related to the Planck energy, and hence the energy gap between the normal modes of the vibration of the string must be huge. In fact, the lowest energy modes correspond to massless particles and the other vibrational modes are very heavy (the next vibrational mode corresponding to $\sim 10^{-5}$g). When these heavy modes are integrated out, one is left with the massless modes, which include the graviton, the spin-0 gauge particles and the spin-1/2 fermions that are described by supergravity in the limit $\alpha' \to 0$.

In this thesis, our main interest will be in reduction of supergravity and superstring theories. A theory in d dimensions with global symmetry G can be compactified on a circle with fields not periodic but with a G monodromy around the circle, and the monodromy introduces masses into the theory and breaks some of the symmetry. Such type of compactifications were first introduced by Jöel

Scherk and John Schwarz in 1979 and they are called twisted toroidal compactifications or Scherk-Schwarz (SS) reductions [11]. Almost around the same time, Scherk and Schwarz introduced the SS compactification, it was understood that supergravity theories possess non-compact global symmetry groups. We will see that these are inherited from the duality symmetries in string theories, as would be expected. These duality symmetries can be used to give twists in the SS reductions and this has been studied extensively in the literature [11-27]. In almost all of the cases, the symmetry that is used is a symmetry of the action. The purpose here is to generalise such compactifications to the case in which G is a symmetry of the equations of motion only, not of the action. A standard example is S-duality in 4-dimensions. The heterotic string compactified to four dimensions has a classical $SL(2,\mathbb{R})$ symmetry which acts through electromagnetic duality transformations and so is only a symmetry of the equations of motion [28]. In this case, we consider a circle reduction with a monodromy in $SL(2,\mathbb{R})$. In the quantum theory, the symmetry is broken to $SL(2,Z)$ [28] and in that case the monodromy must be in $SL(2,Z)$ [15]. We generalise this to other dimensions, and discuss examples in $d = 4, 6$ and 8 dimensions.

Many supergravity theories in $d = 2n$ dimensions have a set of n form field strengths H^i_n where $i = 1, ..., r$ labels the potentials, which typically satisfy a generalised self-duality equation of the form

$$H^i_n = Q^i{}_j(\phi) * H^j_n, \tag{1.3}$$

where $Q^i{}_j$ is a matrix depending on the scalar fields ϕ and $*$ is the Hodge dual in d dimensions [29]. For any n, consistency requires that $(Q^i{}_j(\phi)*)^2 = 1$, so that if $(*)^2 = -1$, as in Lorentzian space of dimension $4m$, then $Q^2 = -\mathbb{I}$ and Q is a complex structure, while if $(*)^2 = 1$, as in Lorentzian space of dimension $4m+2$, then $Q^2 = \mathbb{I}$ and Q is a product structure. In the theories we will consider, the H^i_n transform in an r-dimensional representation of a rigid duality group G. In

$d = 4$, $N = 8$ supergravity, there are $r = 56$ 2-form field strengths transforming as a **56** of the duality group $G = E_7$ [30]. These split into 28 field strengths $F = dA$ and 28 dual field strengths $\tilde{F} = \hat{*}F + \ldots$, with Q a complex structure on $\mathbb{R}^{56}$. In $d = 6$, $N = 8$ supergravity, there are 5 3-form field strengths which split into 5 self-dual ones and 5 anti-self dual ones, and these 10 transform as a **10** of $G = SO(5,5)$ [31]. The 10 3-form field strengths $\hat{H}_n^i$ with $i = 1, ..., 10$, satisfy (anti) self-duality constraints of the form (1.3) with Q related to the $SO(5,5)$-invariant metric. In $d = 8$ maximal supergravity, there is a 3-form potential, and its field strength and its dual combine into an $SL(2, \mathbb{R})$ doublet, satisfying a constraint of the form (1.3) with $Q = i\sigma_2$.

Our main interest is in reductions in which the monodromy $\mathcal{M} \in G$ is a symmetry of the equations of motion but not of the action, acting on the field strengths $\hat{H}_n^i$ via transformations involving Hodge or electromagnetic dualities, so that they cannot be realised locally on the fundamental $n-1$ form potentials. In all of the cases that we will consider, it will be possible to construct a manifestly G-invariant Lagrangian such that the field equations derived from this Lagrangian are equivalent to those of the original one when a constraint of the type (1.3) is imposed. In this formalism, the dual fields are regarded as new fields and the number of degrees of freedom is conserved by the imposition of the constraint. This formalism is called the doubled formalism.

As the global symmetry G is extended to the level of the action, one can now perform the Scherk-Schwarz reduction of the action on a circle. As mentioned above, the fields are not independent of the circular coordinates in such twisted reductions, since they have a G monodromy around the circle. This dependence is characterized by a Lie algebra element M in the reduction ansatz, which is called the mass matrix. The mass matrix M introduces mass parameters into the theory, and fields in non-trivial representations of the group G typically become massive with masses given in terms of M, or are "eaten" by gauge fields that become massive in a generalised Higgs mechanism. In particular, the scalar fields will obtain a scalar potential given in terms of M. The mass matrix M generates a one dimensional subgroup L of G, which becomes a gauge symmetry of the reduced theory, so that such a reduction of a supergravity gives a gauged supergravity [16, 19, 20, 21].

In the doubled formalism the auxiliary Lagrangian should be supplemented with a constraint as discussed above and so one should also consider the Scherk-Schwarz reduction of this constraint. We find that (in the case in which M is invertible) the field strengths $\hat{H}_n^i$ satisfying the constraint (1.3) give rise to $n-1$ form potentials A_{n-1}^i in $2n-1$ dimensions satisfying massive self-duality constraints of the form

$$DA_{n-1} = \tilde{M} * A_{n-1}, \tag{1.4}$$

where D is a gauge-covariant exterior derivative, $*$ is the Hodge dual in D dimensions and the matrix $\tilde{M} \propto QM$. Such odd-dimensional self-duality conditions were first considered in [32] and often occur in odd-dimensional gauged supergravity theories, and follow from a Chern-Simons action with mass term of the form

$$L = P_{ij} A^i \wedge DA^j + \hat{M}_{ij} A^i \wedge *A^j, \tag{1.5}$$

where $\hat{M} = P\tilde{M}$ and P_{ij} is a suitably chosen constant matrix. In the general case in which M is not invertible, some of the gauge fields remain massless.

As a result, by considering the Scherk-Schwarz reductions of supergravity theories in $4, 6, 8$ dimensions, we obtain new gauged/massive supergravities in $3, 5, 7$ dimensions with massive gauge fields which satisfy a generalized self-duality condition of the type (1.4).

The outline of this thesis will be as follows. In chapter 2 we will give a general review of the Kaluza-Klein and Scherk-Schwarz dimensional reductions. This chapter will introduce the main concepts along with some useful formulas that will be needed subsequently. In chapter 3, we will give a detailed discussion of the origin and structure of the duality symmetries appearing in string and supergravity theories. Chapter 4 will be a preliminary chapter, where we will introduce some useful concepts that will be needed in chapter 5. The doubled formalism, the Stückelberg mechanism and the idea of self-duality in odd dimensions will be discussed in this preliminary chapter. Chapter 5, which is the main chapter of this thesis, will be devoted to the applications of the tools developed in the previous chapters to several supergravity theories. We will give all the calculational techniques and will present the new gauged/massive supergravity theories obtained in dimensions $3, 5$ and 7. We will finish with conclusions and discussions for future directions.

CHAPTER 2

KALUZA-KLEIN AND SCHERK-SCHWARZ COMPACTIFICATION

2.1 The Kaluza-Klein Philosophy

It was Kaluza who first came up with the idea of extra dimensions. He started with pure gravity in five dimensions. Reducing the metric g_{MN}, a degree two symmetric tensor, which he took to be independent of the fifth coordinate, he obtained in 4 dimensions a metric $g_{\mu\nu}$, a one-form $A_\mu = g_{\mu y}$ and a scalar $\phi = g_{yy}$. Here M is the 5 dimensional index and it splits as (μ, y), where μ is the 4 dimensional index and y is the fifth coordinate. In fact, the appearance of a scalar field was first regarded as an unwanted feature and hence ϕ was set to a constant value. This was because the four dimensional theory, without the scalar field, is a pure Einstein-Maxwell theory. But later it was realized that setting ϕ to a constant value was inconsistent with the higher dimensional field equations [33, 34]. Kaluza was able to show that splitting the five-dimensional Einstein equation $R_{MN} = 0$ gives in four dimensions the coupled Einstein-Maxwell equations for the metric $g_{\mu\nu}$, and the one-form A_μ [1]. Hence he identified the one-form A_μ with the photon. Thus starting from pure gravity in five dimensions one obtains gravity plus electromagnetism in four dimensions, which seems like a very appealing mechanism. But of course the question is, if there exists a fifth dimension why can we not observe it? Another question is as to why we should suppress the y dependence of the five dimensional fields.

The answer to these questions came from Oskar Klein in 1926 [2]. He admitted the existence of a fifth dimension but assumed that it had a circular topology. This means that y is periodic, $0 \leq y \leq 2\pi R$, where R is the radius of the circle S^1. Thus the space has topology $\mathbb{R}^4 \times S^1$. The periodicity in y allows a harmonic expansion for the fields $g_{\mu\nu}(x,y), A_\mu(x,y)$ and $\phi(x,y)$ on the circle. Thus for example, the metric $g_{\mu\nu}$ can be expanded in the form

$$g_{\mu\nu} = \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x) e^{iny/R} \tag{2.1}$$

So a Kaluza-Klein theory describes an infinite number of four-dimensional fields. The $n = 0$ modes in the expansions of the fields $g_{\mu\nu}(x,y), A_\mu(x,y)$ and $\phi(x,y)$ are just the graviton, photon and dilaton respectively. The $n \neq 0$ modes correspond to massive fields. An easy way to illustrate this fact is to consider the compactification of a massless scalar field $\hat{\psi}$ in flat $D+1$-space on a circle. It satisfies

$$\Box \hat{\psi} = 0, \tag{2.2}$$

where $\Box = \partial^M \partial_M$. If we compactify the coordinate y and then Fourier expand the field $\hat{\psi}$ so that

$$\hat{\psi} = \sum_{n=-\infty}^{\infty} \psi_n(x) e^{iny/R} \tag{2.3}$$

we see that the lower dimensional field ψ satisfies the wave equation for a scalar field of mass $|n|/R$:

$$\Box \psi_n - \frac{n^2}{R^2} \psi_n = 0. \tag{2.4}$$

What Klein suggested is that the compactification radius R should be as small as the order of the Planck's length ($\sim 10^{-33} cm$) so that the masses of the non-zero modes are of the order of the Planck's mass (10^{19}GeV), way beyond the range of contemporary accelerators. This answers both of the questions Kaluza left unanswered: 1) We cannot see the fifth dimension because it is too small to observe, 2) There is no dependence of the fields on the extra coordinate y because we truncate to the massless $n = 0$ modes which have no y-dependence.

The 5 dimensional Einstein theory is invariant under general coordinate transformations

$$\delta\hat{x}^M = -\hat{\xi}^M, \qquad \delta\hat{g}_{MN} = (\mathcal{L}_{\hat{\xi}}\hat{g})_{MN} \tag{2.5}$$

Here $\xi = \hat{\xi}^M d\hat{x}_M$, $M = 1, \cdots, 5$ are arbitrary functions of all the 5 coordinates. The transformations that preserve the form of the Kaluza-Klein ansatz (see (2.7)) is

$$\hat{\xi}^\mu = \xi^\mu(x), \qquad \hat{\xi}^y = cy + \lambda(x), \tag{2.6}$$

where c is a constant and $\lambda(x)$ is an arbitrary function depending only on the 4 dimensional coordinates x^μ. It can be seen that the parameters $\xi(x)$ describe the general coordinate invariance of the 4-dimensional theory, whereas the local parameter $\lambda(x)$ describes a local $U(1)$ invariance for the Kaluza-Klein vector field. In fact $U(1)$ is the isometry group of S^1 and this is the reason why it arises as the local gauge group of the lower dimensional theory. c is a constant parameter which describes a shift symmetry for the scalar and the vector fields.

It is possible to generalize this mechanism to more general compactification manifolds. The first straightforward generalization is to that of a torus $T^k = S^1 \times S^1 \times \ldots \times S^1$. This might be considered as doing k circular compactifications subsequently. In this case, the gauge group is $G = [U(1)]^k$, which is the isometry group of the torus T^k, instead of the $U(1)$ gauge group of the Maxwell theory. In $d = 4 + k$ dimensions the graviton (spin-2) has $(4 + k)(1 + k)/2$ degrees of freedom and this matches with the number of degrees of freedom of massless modes in dimension four: 1 spin-2 (2 degrees of freedom), k spin-1 ($2k$ DOF) and $k(k+1)/2$ spin-0 ($k(k+1)/2$ DOF). The number of scalars is given by the moduli of T^k and they parametrize the non-linear σ-model $GL(k, R)/SO(k)$.

It is obvious that it is not possible to obtain realistic four dimensional physics by compactifying higher dimensional theories, such as string theory or supergravity theories, on a circle or a torus. One reason is that realistic four dimensional models should have non-abelian Yang-Mills gauge invariance whereas we have seen that the only gauge group we can obtain by compactification on a torus is the abelian $[U(1)]^k$. A second obvious problem is that the spectrum of the particles we observe in four dimensions contain massive particles, such as electron, proton, etc. However in the Kaluza-Klein mechanism, in order to have consistency, we should truncate to the $n = 0$ modes which leave us with massless particles only. Hence if we start with massless theories like string theory and supergravity, we cannot expect to have massive fields in four dimensions. As we will discuss below one resolution to these problems is to consider compactification on more general, abstract manifolds. Another realistic resolution will be discussed later in the context of Scherk-Schwarz mechanism [11].

Consider the Kaluza-Klein compactification of a higher dimensional theory on some general compact manifold M with isometry group G. It turns out that the massless fields in the lower dimensional theory include the Yang-Mills gauge bosons of the isometry group G. One possibility is that M is the group manifold of G which implies that $dimM = dimG$. A more economical choice would be that M is a coset space $M = G/H$ in which case $dimM = dimG/H = dimG - dimH$. However it is not always guaranteed that such compactifications are consistent. Here consistency means that the solutions of the lower dimensional field equations should also be solutions of the higher dimensional theory. One should also ensure that the lower dimensional action and the transformation laws be independent of the extra coordinates. For a discussion of the consistency problem in the compactification of supergravity and string theories, see [35] and the references therein.

One way to remedy the consistency problem is to consider *spontaneous compactifications* [36]. Here one takes the vacuum state to be a product manifold $\mathbb{R}^4 \times M$, that is, one looks for stable ground state solutions of the field equations for which the metric describes a product manifold $\mathbb{R}^4 \times M$. In the case of sphere reductions in supergravities one usually finds a vacuum solution of the form Anti de Sitter space $\times S^n$. The physical fluctuations might not respect this form of the vacuum but studying the vacuum state gives a lot of information on the low energy excitations. It was shown in [37] that it is not enough to start with pure gravity to achieve a satisfactory compactification. The higher dimensional theory should also contain matter fields.

In [38] Freund and Rubin showed that in eleven dimensional supergravity the existence of the 3-form gives a dynamical mechanism by which a spontaneous compactification is possible. Then it was shown by Duff and Pope that eleven dimensional supergravity may yield ground-state solutions of the form $AdS_4 \times S^7$, where AdS_4 is the 4 dimensional Anti de Sitter space [39]. The seven dimensional sphere S^7 has 8 Killing spinors and has isometry group $SO(8)$. Therefore the resulting theory in $D = 4$ has $N = 8$ supersymmetry and enjoys $SO(8)$ gauge invariance. Similarly, by the reduction of the 11 dimensional supergravity theory on S^4, one obtains in $D = 7$ a $SO(5)$-gauged supergravity theory [40].

In order to obtain realistic 4-dimensional theories from string theory compactifications, the internal manifold should be a Calabi-Yau 3-fold, i.e. a compact, complex, Kähler manifold which has $SU(3)$ holonomy (See [41] for a review). In M-theory, this manifold should be a seven dimensional manifold with G_2 holonomy [42].

In the section below we will give the computational details of the Kaluza-Klein compactification on the circle, which we will need in the forthcoming sections (for more about Kaluza-Klein theory see [43, 44]). Then in the next section we will give a detailed description of the Scherk-Schwarz mechanism, which is a generalization of the Kaluza-Klein mechanism. In this formalism, one does not insist on the assumption that the fields do not depend on the extra coordinates but still obtains consistent compactifications with a specific ansatz.

2.2 Kaluza-Klein Compactification on the Circle: Computational Details

2.2.1 Reduction of the Metric

Let $\hat{g}_{MN}(x, y)$ be the metric in $D + 1$ dimensions, a degree two symmetric tensor. Here the index M runs from 1 to $D + 1$. From the D dimensional point of view, this index splits as $M = (\mu, y)$. Thus we denote the components of the metric $\hat{g}_{MN}$ by $\hat{g}_{\mu\nu}, \hat{g}_{\mu y}, \hat{g}_{yy}$ which, from the D dimensional point of view, look like a metric, a 1-form and a scalar field, respectively. Instead of identifying $\hat{g}_{\mu\nu}, \hat{g}_{\mu y}, \hat{g}_{yy}$ with the D-dimensional fields $g_{\mu\nu}, \mathcal{A}_\mu$ and ϕ, we prefer a better parametrisation, namely we prefer to work in the Einstein frame, which make the calculations much easier. We write the $D+1$ dimensional metric in terms of the D dimensional fields as follows:

$$d\hat{s}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi}(dy + \mathcal{A})^2 \tag{2.7}$$

where $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ and α, β are constants. We fix the values of α and β such that the reduced Lagrangian is in the Einstein-Hilbert form, i.e., the term $R * 1$ has constant coefficient 1 and the kinetic term for the scalar field ϕ is in the canonical form, i.e. it appears as $-\frac{1}{2} d\phi \wedge *d\phi$. It turns out that the right choice which ensures the above form of the reduced Lagrangian is

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\alpha. \tag{2.8}$$

All the fields on the right hand side of (2.7) are independent of the extra coordinate y. For the metric components, (2.7) implies that

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} \mathcal{A}_\mu \mathcal{A}_\nu, \qquad \hat{g}_{\mu y} = e^{2\beta\phi} \mathcal{A}_\mu, \qquad \hat{g}_{yy} = e^{2\beta\phi}. \tag{2.9}$$

A convenient choice of the vielbein is

$$\hat{e}^a = e^{\alpha\phi} e^a, \qquad \hat{e}^y = e^{\beta\phi}(dy + \mathcal{A}). \tag{2.10}$$

Here the Latin letters a, b, *etc* denote tangent-space indices in D dimensions. For the spin connection one finds that

$$\begin{aligned} \hat{w}^{ab} &= w^{ab} + \alpha e^{-\alpha\phi}[(\partial^b\phi)\hat{e}^a - (\partial^a\phi)\hat{e}^b] - \frac{1}{2}\mathcal{F}^{ab} e^{(\beta-2\alpha)\phi}\hat{e}^y & (2.11) \\ \hat{w}^{ay} &= -\hat{w}^{ya} = -\beta e^{-\alpha\phi}(\partial^a\phi)\hat{e}^y - \frac{1}{2}\mathcal{F}^a{}_b e^{(\beta-2\alpha)\phi}\hat{e}^b, & (2.12) \end{aligned}$$

where $\partial_a\phi$ means $E^\mu_a\partial_\mu\phi$, and E^μ_a is the inverse of the D dimensional vielbein $e^a = e^a_\mu dx^\mu$. Also, $\mathcal{F} = d\mathcal{A}$. These formulas can be achieved by using (2.10) and the relation

$$2w_{ab} = -i_a de_b + i_b de^a + (i_a i_b de_c)e^c \tag{2.13}$$

Now from the formula

$$\mathcal{R}_{ab} = dw_{ab} + w_{ac}w^c{}_b, \tag{2.14}$$

(2.8) and (2.11), one can calculate the components of the Ricci tensor

$$\begin{aligned} \hat{\mathcal{R}}_{ab} &= e^{-2\alpha\phi}(\mathcal{R}_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi) - \frac{1}{2}e^{-2D\alpha\phi}\mathcal{F}_a{}^c\mathcal{F}_{bc}, \\ \hat{\mathcal{R}}_{ay} &= \hat{\mathcal{R}}_{ya} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla^b(e^{-2(D-1)\alpha\phi}\mathcal{F}_{ab}), \\ \hat{\mathcal{R}}_{yy} &= (D-2)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2, \end{aligned} \tag{2.15}$$

where $\mathcal{F}^2$ means $\mathcal{F}_{ab}\mathcal{F}^{ab}$. Now one can calculate the Ricci scalar $\hat{R} = \eta^{AB}\hat{\mathcal{R}}_{AB} = \eta^{ab}\hat{\mathcal{R}}_{ab} + \hat{\mathcal{R}}_{yy}$:

$$\hat{R} = e^{-2\alpha\phi}(R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi) - \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2. \tag{2.16}$$

The last thing to calculate is the determinant of the metric $\hat{g}$ in terms of the determinant of g, which can be done easily by using (2.9) and the result is

$$\sqrt{-\hat{g}} = e^{(\beta+D\alpha)\phi}\sqrt{-g} = e^{2\alpha\phi}\sqrt{-g}. \tag{2.17}$$

Putting all the results together one finds that the lower-dimensional Lagrangian coming from the reduction of the Einstein-Hilbert term in higher dimensions is

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} = \sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2) \tag{2.18}$$

where we have dropped the $\Box\phi$ term in (2.18) since it's a total derivative.

In coordinate free notation, (2.18) can be written as:

$$\mathcal{L} = R * 1 - \frac{1}{2}d\phi \wedge *d\phi - e^{-2(D-1)\alpha\phi}\frac{1}{2}\mathcal{F} \wedge *\mathcal{F}. \tag{2.19}$$

2.2.2 The Reduction of the n-Form Field Strength

Suppose the higher dimensional theory contains an n-form field strength $\hat{F}_n = d\hat{A}_{n-1}$. In terms of indices we can see that the reduction of the $(n-1)$-form $\hat{A}_{n-1} = \hat{A}_{\hat{\mu}_1\cdots\hat{\mu}_{n-1}} d\hat{x}^{\hat{\mu}_1\cdots\hat{\mu}_{n-1}}$ gives in D dimensions another $(n-1)$-form with indices $A_{\mu_1\cdots\mu_{n-1}}$ and an $(n-2)$-form with indices $A_{\mu_1\cdots\mu_{n-2}y}$. This is expressed as

$$\hat{A}_{n-1}(x,y) = A_{n-1}(x) + A_{n-2}(x) \wedge dy. \tag{2.20}$$

Then for the field strength we have

$$\hat{F}_n = dA_{n-1} + dA_{n-2} \wedge dy.$$

For computational simplicity we again prefer to work in the Einstein-frame and then

$$\begin{aligned} \hat{F}_n &= dA_{n-1} - dA_{n-2} \wedge \mathcal{A} + dA_{n-2} \wedge (dy + \mathcal{A}) \\ &\equiv F_n + F_{n-1} \wedge (dy + \mathcal{A}) \end{aligned} \tag{2.21}$$

where $\mathcal{A}$ is the Kaluza-Klein vector potential that comes from the metric reduction. Thus the D-dimensional field strengths are

$$F_n = dA_{n-1} - dA_{n-2} \wedge \mathcal{A}, \qquad F_{n-1} = dA_{n-2}. \tag{2.22}$$

Now it is easy to find how the kinetic term $-\frac{1}{2}\hat{F}_n \wedge \hat{*}\hat{F}_n$ reduces to D dimensions. Of course one should also calculate the relation between the $D+1$-dimensional Hodge operator $\hat{*}$ and the D-dimensional Hodge operator $*$ which is

$$\begin{aligned} \hat{*}(X_n \wedge (dy + \mathcal{A}_1)) &= e^{2(D-n)\alpha\varphi} * X_n \\ \hat{*}X_n &= e^{-2(n-1)\alpha\varphi} * X_n \wedge (dy + \mathcal{A}_1) \end{aligned} \tag{2.23}$$

Here X_n is an $n-$form living in D dimensions. Then the final result is

$$-\frac{1}{2}\hat{F}_n \wedge \hat{*}\hat{F}_n \rightarrow [-\frac{1}{2}e^{-2(n-1)\alpha\phi}F_n \wedge *F_n - \frac{1}{2}e^{2(D-n)\alpha\phi}F_{n-1} \wedge *F_{n-1}] \wedge dy. \tag{2.24}$$

2.3 Scherk-Schwarz Compactification

Twisted toroidal compactifications or Scherk-Schwarz reductions are a useful way of introducing masses into supergravity and string compactifications, generating a potential for the scalar fields [11-27]. A theory in $D+E$ dimensions with global symmetry G can be compactified on a manifold of dimension E with fields which have a particular dependence on the extra coordinates, determined by the symmetry G. The choice of this dependence must be such that the fields define fiber bundles that are continuous on the compact space. The non-trivial topology of the space-time (e.g. $\mathbb{R}^4 \times S^1$ in the circle reduction) allows one to give a twist on the fibers, that is, the fields need only be continuous on the coordinate patches. Scherk and Schwarz were able to show that with their ansatz the y dependence [1] cancels out of the Lagrangian and the transformation laws of

[1] Here we denote the extra coordinates collectively by y.

the resulting low dimensional theory and the compactification is consistent. The picture that emerges from such a compactification scheme is as follows:

- The y dependence of the fields introduce mass parameters into the theory. The fields in non-trivial representations of the group G typically become massive.

- It leads to a scalar potential in the lower dimensions which, when applied to supergravity, provides a mechanism for the breaking of local supersymmetry. This was actually the main motivation of Scherk and Schwarz.

- With this mechanism one can construct new massive and gauged supergravities and corresponding string compactifications.

In their original work Scherk and Schwarz started with a supergravity theory in $D + E$ dimensions and compactified it to D dimensions on an internal space which they took to be a compact E dimensional manifold on which a special E dimensional noncompact group acted. In fact they identified the y's with the coordinates on the manifold of a Lie group G having E generators. (This does not necessarily mean that the internal manifold is the group manifold of G. Actually the internal manifold is compact whereas the group is usually noncompact.) This was an example of reduction with internal symmetries, symmetries which involved space-time properties of the $D + E$-dimensional manifold.

It is also possible to use external symmetries to give twists in the Scherk-Schwarz reduction. As an example consider the case in which the theory has a $U(1)$ global symmetry under which a scalar field $S(x, y)$ transforms as $S \to e^{i\alpha}S$. Such a scalar field can be expanded in terms of the D dimensional scalar fields S_n as follows

$$S(x, y) = e^{imy/R} \sum_{n=-\infty}^{\infty} e^{iny/R} S_n(x). \qquad (2.25)$$

This field is not single-valued on the $D+1$ dimensional space-time manifold. However the fields define a fiber bundle on the circle and are continuous on the coordinate patches. This makes sure that the lower dimensional theory is independent of the extra coordinate y. The idea in SS mechanism is to keep m fixed as $R \to 0$ and then to truncate to the lightest sector as in standard Kaluza-Klein mechanism. A short-hand description of this procedure is to say that $S(x,y) = e^{imy/R}S(x)$. Note that the D-dimensional field S_0 has mass m.

In this work we will study supergravity theories in $D+1$ dimensions and consider SS compactifications of them on a circle to D dimensions. The theories in consideration have a global, external symmetry group G, which is usually inherited from the compactification of a higher dimensional theory in $D+E$ dimensions to $D+1$ dimensions. Our main interest will be in constructing new massive and gauged supergravities.

Massive supergravity theories appear basically in three different ways in the compactification of ungauged supergravities. The first way is the Scherk-Schwarz compactification that we discuss in detail here. A well-known example is the Salam and Sezgin's $D=8$ $SU(2)$-gauged supergravity which is obtained by compactifying the 11 dimensional supergravity theory by using the $SU(2)$ group as the global symmetry group [45].

The second way is the compactification in non-trivial manifolds, such as spontaneous compactification on spheres. We have already given the example of $N=8, D=4,$ $SO(8)$-gauged supergravity obtained by compactifying 11 dimensional supergravity on S^7 [39]. Other examples are $N=4, D=7, SO(5)$-gauged supergravity and $N=8, D=5, SO(6)$-gauged supergravity which are obtained by compactifying the 11d supergravity on S^4 and $N=IIB, D=10$ supergravity on S^5, respectively [40, 46]. This is related with the Scherk-Schwarz compactification. For example, Salam and Sezgin's eight dimensional theory can be considered as having been obtained on the sphere S^3 which is the group manifold of $SO(3)$. Such theories are not only gauged, but also massive, because one has to introduce mass parameters along with the gauge parameters. We will discuss this in detail in section 4.2.

Thirdly, massive supergravity theories appear in compactifications with non-trivial p-form fluxes (see e.g. [21, 22, 47, 48]). This is also related with the SS compactification, as we will see in subsection 2.3.3.

In the subsection below we will describe the method of Scherk-Schwarz generalized dimensional reduction, leaving the explicit calculations to chapter 5. For simplicity we will consider reductions on a circle but the results can be extended to more general compactification manifolds.

2.3.1 The Formalism

Consider a $D + 1$ dimensional supergravity with a global symmetry G. An element g of the symmetry group acts on a generic field ψ as $\psi \to g[\psi]$. Consider now a dimensional reduction of the theory to D dimensions on a circle of radius R with a periodic coordinate $y \sim y + 1$. In the twisted reduction, the fields are not independent of the internal coordinate but are chosen to have a specific dependence on the circle coordinate y through the ansatz

$$\psi(x^\mu, y) = g(y)\,[\psi(x^\mu)] \tag{2.26}$$

for some y-dependent group element $g(y)$ [15]. An important restriction on $g(y)$ is that the reduced theory in D dimensions should be independent of y. This is achieved by choosing

$$g(y) = \exp(My) \tag{2.27}$$

for some Lie-algebra element M. The map $g(y)$ is not periodic around the circle, but has a *monodromy*

$$\mathcal{M}(g) = g(1)g(0)^{-1}. \tag{2.28}$$

For the maps of the form (2.27), the monodromy is

$$\mathcal{M}(g) = \exp M. \tag{2.29}$$

The Lie algebra element M in (2.27) is called the *mass matrix* and it generates a one dimensional subgroup L of G, which becomes a gauge symmetry of the reduced theory, so that such a reduction of a supergravity gives a gauged supergravity [16, 20, 21]. We will see that, for a field ψ transforming in some representation of G such that $\delta\psi = \lambda\bar{M}\psi$, with λ being an infinitesimal parameter and $\bar{M}$ being the matrix through which M acts on ψ, the derivative becomes a covariant derivative $D\psi = d\psi + \mathcal{A}\bar{M}\psi$. Thus L is indeed the local symmetry group and the gauge field is the graviphoton $\mathcal{A}$. The mass matrix M determines not only the gauge couplings but also the mass parameters (as the name would suggest) and the scalar potential, as we will discuss shortly. So the lower dimensional theory is determined completely by the mass matrix M.

Now the question arises as to how many distinct theories we can obtain in lower dimensions. Suppose that we use another element $g'(y)$ in the same conjugacy class with $g(y)$ to give a twist in the SS reduction. Then $g'(y) = h^{-1}g(y)h$ for some group element h. This means that the ansatz (2.26) changes as $h\psi(x,y) = g(y)h\psi(x)$. But this is just a field redefinition for the field ψ as $\psi \to h\psi$. Hence two group elements in the same conjugacy class give equivalent reductions up to field redefinitions [15].

In quantum string theory, a global group of the classical theory typically becomes a discrete gauge symmetry $G(Z)$ [50] and for such theories the monodromy must be in $G(Z)$, giving quantization conditions on the mass parameters, and the distinct theories are determined by the monodromy $\mathcal{M} \in G(Z)$ up to $G(Z)$ conjugation.

The geometrical picture that arises is as follows. The fields ψ are sections of a principal fiber bundle $P(M^{D+1}, G)$, where the base manifold M^{D+1} is the $D+1$-dimensional Minkowski space and the structure group is the global symmetry group G. Before compactification this bundle is trivial because the Minkowski space can be covered with one coordinate chart. After the compactification the base space is $M^D \times S^1$ (this discussion can be readily extended to more general compactification manifolds), so the bundle is not trivial any more. Thus the sections, *i.e.* the fields, need not be globally defined over the circle S^1 and it is enough that they are related by some element of G as they go from one chart to another. $\mathcal{M}$ is this group element. We can see this as follows. Suppose that we parametrize the circle with $y \sim y+1$. We can cover the circle with two coordinate charts A and B and $0 \sim 1$ should be in $A \cap B$. Hence $\psi(1)$ and $\psi(0)$ should be related by a group element, which is in fact the monodromy $\mathcal{M} = g(1)g(0)^{-1} = \psi(x,1)\psi(x,0)^{-1}$. Recall that the monodromy is the holonomy of a connection around a loop on the base space in the fiber bundle language. Roughly speaking, it measures how much a loop on the base space deviates from being closed when horizantally lifted to the bundle. A non-trivial monodromy means a non-trivial twist in the fibers. The question of how many distinct reductions there are can also be answered easily in this geometric framework. Two such bundles with monodromy in the same conjugacy class are equivalent.

We have just discussed that the distinct lower dimensional theories are classified by the conjugacy classes of the monodromy matrix $\mathcal{M}$. We had also discussed that the mass matrix M determines completely the couplings, mass parameters and the scalar potential in the lower dimensions. This seems to give rise to a paradox because for a given monodromy matrix $\mathcal{M}$ there exists infinitely many mass matrices M. As an example consider the case of trivial twist $\mathcal{M} = \mathbb{I}$ which corresponds to the standard KK compactification. It is obvious that there are infinitely many M satisfying $e^M = \mathbb{I}$ and each M is supposed to give a different supergravity action in lower dimensions. To resolve this paradox consider the example in (2.25). Note that the mass matrix is the 1×1 matrix $M = im/R$ and the monodromy matrix is[2] $\mathcal{M} = e^{2\pi im}$. For the trivial reduction the mode expansion is

$$S(x, y) = \sum_n e^{iny/R} S_n(x) \tag{2.30}$$

whereas the twisted mode expansion is

$$S(x, y) = \sum_n e^{i(n+m)y/R} \tilde{S}_n(x) \tag{2.31}$$

In the trivial reduction case the modes $S_n(x)$ have mass $m_n \propto n/R$ and the zero mode S_0 is a massless field as usual. In the twisted case the modes $\tilde{S}_n(x)$ have mass $\tilde{m}_n \propto (n+m)/R$ and the zero mode is a massive field of mass m/R. If m is an integer, one would expect the two mode sums in (2.30) and (2.31) to be equivalent since they both correspond to the monodromy matrix $\mathcal{M} = \mathbb{I}$. In fact they are equivalent when all of the massive Kaluza-Klein states are kept. If one truncates to the mode $\tilde{S}_{-m} = S_0$ instead of $\tilde{S}_0$ one again obtains a massless field. Similarly, two non-integral choices of mass $m = m_1, m = m_2$ which differ by an integer would give equivalent Kaluza-Klein spectra. The reduced theory is different only after the truncation to the $n = 0$ mode is made. In general, for a given monodromy matrix $\mathcal{M}$ the corresponding mass matrices M all give the same KK spectra but can give distinct truncations to the "zero-mode" sector. However the zero-mode does not give the truncation to the lightest fields any more. In supergravity theories it is important to truncate to the mode which gives the lightest states because the effective theory is to define the lightest states.

[2] Note that here the identification is as $y \sim y + 2\pi R$.

Hence the correct method should be to truncate to the mode for which $|m+n|$ is minimum, i.e. for which the field $\tilde{S}_n$ has the minimum mass [21].

2.3.2 The Scherk-Schwarz Potential

In this section we will describe the form of the scalar potential that appears in lower dimensions, again leaving the explicit calculations to the main chapter. The existence of such a potential is perhaps the most interesting feature in SS compactifications. Suppose that one starts with a theory in $D+1$ dimensions, performs a standard KK compactification on the circle and then gauges the global symmetry that appears in D dimensions. The theory that is obtained is the same with the theory that would be obtained if one performed SS reduction on the $D+1$-dimensional theory[3] except for one difference. In the first case the scalar potential does not appear naturally and is introduced in an *ad hoc* way due to, for example, requirements coming from supersymmetry. In the second case a non-trivial SS potential appears naturally which gives mass to some of the scalar fields, which provides a mechanism for supersymmetry breaking and which can be used to fix the moduli space [21].

Our main interest in this thesis will be in the reduction of supergravity and superstring theories. Extended supergravity theories typically have a global symmetry G and the scalars take values in the coset space G/H where G is a non-compact group and H is the maximal compact subgroup of G. The theory can be formulated with a local H symmetry as well as a global G symmetry. The scalars in the coset space G/H can be represented by a vielbein $\mathcal{V}(x) \in G$ which transforms under global G and local H transformations as

$$\mathcal{V} \rightarrow h(x)\mathcal{V}, \qquad h(x) \in H,$$

[3] However, note that one cannot gauge the theory with any given group G, see [49] for a discussion.

$$\mathcal{V} \quad \rightarrow \quad \mathcal{V}\, g, \qquad\qquad g \in G. \tag{2.32}$$

The Lagrangian is

$$L = -\frac{1}{2}\mathrm{tr}[d\mathcal{V}\mathcal{V}^{-1} \wedge *d\mathcal{V}\mathcal{V}^{-1}]. \tag{2.33}$$

In this formulation there are an extra $dim(H)$ non-physical scalars which can be gauged away using the local H symmetry. Here $\mathcal{V}, g, h$ can be taken to be matrices in some representation of G. We will present our results for real representations of G such that the representatives of H are orthogonal matrices $h^T h = \mathbb{I}$ so that δ_{ab} is an invariant, but the generalisation to other representations is straightforward.

An alternative formulation that does not involve extra scalars is to use a metric $\mathcal{K}$ on G/H instead of a vielbein, transforming as (for a real representation of G)

$$\mathcal{K} \rightarrow g^T \mathcal{K} g. \tag{2.34}$$

Such a metric can be constructed from the vielbein as $\mathcal{K}_{ij} = \delta_{ab}\mathcal{V}^a{}_i\mathcal{V}^b{}_j$, where i and a are the curved and flat indices, respectively. $\mathcal{K}$ is invariant under local H transformations as $h^T h = \mathbb{I}$. This means that the non-physical scalars drop out in this formulation, without any need for gauge fixing. (For complex representations with $h^\dagger h = \mathbb{I}$, we would use the hermitian metric $\mathcal{K} = \mathcal{V}^\dagger\mathcal{V}$ transforming as $\mathcal{K} \rightarrow g^\dagger \mathcal{K} g$.) The Lagrangian can be written in terms of $\mathcal{K}$ as

$$L = \frac{1}{4}\mathrm{tr}[d\mathcal{K}^{-1} \wedge *d\mathcal{K}]. \tag{2.35}$$

As will be seen in chapter 5, the y-dependent ansatz (2.26) on the scalar fields gives the following scalar potential in D dimensions:

$$V(\Phi) = e^{\alpha\phi}\mathrm{tr}[M^2 + M^T\mathcal{K}(\Phi)M\mathcal{K}^{-1}(\Phi)]. \tag{2.36}$$

Now the question is whether the potential has any stable minima and which moduli acquire mass at these minima. Define $\tilde{M} = \mathcal{V}M\mathcal{V}^{-1}$. The potential takes the form

$$V(\Phi) = e^{\alpha\phi}\mathrm{tr}[\tilde{M}^2 + \tilde{M}^T\eta\tilde{M}\eta^{-1}]. \tag{2.37}$$

Now it is easy to see that the scalar potential (2.37) has stable minima for the values $\tilde{M} = \tilde{M}_0$ for which

$$\mathrm{tr}[\tilde{M}_0(\tilde{M}_0 + \eta^{-1}\tilde{M}_0^T\eta)] = 0. \tag{2.38}$$

We mentioned above that when K is compact, the case of our interest, $\eta = \mathbb{I}$. Then the scalar potential can be written in the form

$$V(\Phi) = \frac{1}{2}e^{\alpha\phi}\mathrm{tr}(Y^2) \tag{2.39}$$

where Y is the real symmetric matrix, $Y \equiv [\tilde{M} + \tilde{M}^T]$. Y is a diagonalizable matrix with real eigenvalues, so $\mathrm{tr}(Y^2)$ is the sum of the squares of the eigenvalues. Thus the potential (2.39) is manifestly positive. Suppose that $Y = 0$ for some value of the moduli $\Phi = \Phi_0$. Then it is obvious that the scalar potential vanishes at this point. At such a point $\tilde{M}(\Phi_0) = -\tilde{M}^T(\Phi_0)$ so $\tilde{M}(\Phi_0) = \tilde{M}_0$ equals a rotation generator. Also since $V(\Phi) \geq 0$, the point Φ_0 is a global minimum that is stable or at least marginally stable. The corresponding value $\mathcal{V}_0$ of the vielbein $\mathcal{V}$ at the point $\Phi = \Phi_0$ is determined by the relation $\tilde{M}_0 = \mathcal{V}_0 M \mathcal{V}_0^{-1}$. As a result, the only critical points of the potential for finite ϕ are the stable minima where the potential vanishes and where $\tilde{M}(\Phi_0)$ is a rotation generator.

2.3.3 Twisted Tori and Flux Compactification

Scherk-Schwarz compactifications or twisted toroidal compactifications are related with flux compactifications. The low energy limit of a string theory compactification, with some flux turned on, is the Scherk-Schwarz reduction of the corresponding supergravity theory [15, 21, 22, 47, 48]. We will demonstrate this with a simple example.

In this thesis we are mainly interested in compactifications with $SL(2,\mathbb{R})$ twists. $SL(2,\mathbb{R})$ is the mapping class group of the torus T^2 and such a compactifications can be regarded as a compactification on T^2 followed by a twisted reduction on S^1. This is equivalent to reducing on a fiber bundle with T^2 fiber over S^1, where the T^2 geometry varies as it traverses around the circle and returns to its original form after one cycle, up to an $SL(2,\mathbb{R})$ transformation [15]. Let E be the 3 dimensional manifold which is the total space of this torus bundle. Then its metric is

$$ds_B^2 = (2\pi R)^2 dy^2 + \frac{A}{\tau_2}|dx_1 + \tau_1(y)dx_2|^2 \tag{2.40}$$

where A is the constant area modulus and $\tau(y) = \tau_1(y) + i\tau_2(y)$ is the complex structure of the torus, which depends on the circular coordinate y. The fiber T^2 has periodic coordinates $x_1, x_2,\ x_i \sim x_i + 1$. This space E is an example of a *twisted torus.* The y dependence of $\tau(y)$ is given by (2.26), where $g(y)$ is as in (2.27). Now choosing an appropriate Lie algebra element M, we have $\tau(y) = \tau_1 + i\tau_2 + my$ where m is the mass parameter in the mass matrix M [4]. Then the metric is

$$ds_B^2 = (2\pi R)^2 dy^2 + \frac{A}{\tau_2}(dx_1 + \omega)^2 + A\tau_2 dx_2^2 \tag{2.41}$$

where $\omega = (\tau_1 + my)dx_2$. This space can also be regarded as a circle bundle over a 2-torus with fiber coordinate x_1, connection 1-form ω and first Chern number m. Now we T-dualize along the fiber direction x_1. We use the Buscher rules [51]

$$\begin{aligned}
g_{x_1x_1} &= \frac{1}{\jmath_{x_1x_1}}, \\
g_{x_1\alpha} &= -\frac{\mathcal{B}_{x_1\alpha}}{\jmath_{x_1x_1}}, \\
g_{\alpha\beta} &= \jmath_{\alpha\beta} - \frac{1}{\jmath_{x_1x_1}}(\jmath_{x_1\alpha}\jmath_{x_1\beta} - \mathcal{B}_{x_1\alpha}\mathcal{B}_{x_1\beta}), \\
B_{x_1\alpha} &= -\frac{\jmath_{x_1\alpha}}{\jmath_{x_1x_1}}, \\
B_{\alpha\beta} &= \mathcal{B}_{\alpha\beta} - \frac{1}{\jmath_{x_1x_1}}(\jmath_{x_1\alpha}B_{x_1\beta} - \mathcal{B}_{x_1\alpha}\jmath_{x_1\beta}).
\end{aligned} \tag{2.42}$$

[4] In fact this is equivalent to choosing M as the representative of the parabolic conjugacy class of $SL(2,\mathbb{R})$. See (5.6) for the explicit form of M.

Here g and j are the metrics on E and the T-dual of E, respectively; α, β refers to any coordinates except x_1 along which the dualization is being performed, $B_{\alpha\beta}$ is the B-field on the space E and $\mathcal{B}$ is the B-field on the T-dual space. As a result we find that the metric of the T-dual space is

$$ds^2 = (2\pi R)^2 dy^2 + \frac{\tau_2}{A} dx_1^2 + A\tau_2 dx_2^2. \tag{2.43}$$

This is a torus metric on T^3, which means that the bundle E has been untwisted. However now a B-field has been turned on with field strength $H = m dx_1 \wedge dx_2 \wedge dy$ corresponding to a constant H-flux over T^3. This generalises to the reductions with $SL(n, \mathbb{R})$ twists. A twisted reduction on a p torus in which all monodromies are in $SL(n, \mathbb{R})$ corresponds to a compactification on a bundle space for which the base is T^p and the fibers are T^n.

We have discussed that in a SS reduction, the extra coordinates y^i can be regarded as a system of coordinates on the manifold of a Lie group G. This manifold is called the twisted torus, although in general it doesn't have the topology of a fibered torus as above. G is usually a non-compact group, so this manifold is not a group manifold. This also means that one cannot use the usual Cartan metric as a metric on this manifold, since it is degenerate. However, the invariant matrix of G can be used as a metric to lower and raise indices.

From this point of view a twisted torus can be defined as a parallelizable manifold with a well defined nowhere vanishing basis of vielbein one-forms as $\eta^a, a = 1, \cdots, n$ such that it is related to the holonomic basis as

$$\eta^a = (g(y))^a_\alpha dx^\alpha, \tag{2.44}$$

where $g(y)$ is as in (2.26). Then the vector fields L_a which are dual to the 1-forms η^a, i.e., $L_a(\eta^b) = \delta^b_a$ are related to the holonomic coordinate vector fields as

$$L_a = (g^{-1})^\alpha_a \partial_\alpha. \tag{2.45}$$

They satisfy the following commutation relations

$$[L_a, L_b] = f^c{}_{ab} L_c, \tag{2.46}$$

where

$$f^c{}_{ab} = (g^{-1})^\alpha_a (g^{-1})^\beta_b (\partial_\beta g^c_\alpha - \partial_\alpha g^c_\beta). \tag{2.47}$$

An important criterion for the twisted torus is that the structure constants $f^c{}_{ab}$ are constants and they correspond to flux parameters.

It is easily seen that (2.46) implies that the vielbein one forms η^a satisfy

$$d\eta^a = -\frac{1}{2} f^a{}_{bc} \eta^b \wedge \eta^c. \tag{2.48}$$

The Bianchi identity $d^2\eta^a = 0$ implies

$$f^a{}_{b[c} f^b{}_{de]} = 0, \tag{2.49}$$

where the square bracket denotes antisymmetrization of the enclosed indices.

Now when $g(y)$ is of the form (2.27), using (2.47) we find that

$$f^c{}_{ab} = M^c_a \delta^1_b - M^c_b \delta^1_a. \tag{2.50}$$

Here one thinks that the $n \times n$ matrix M has been embedded in a $(n+1) \times (n+1)$ matrix whose first row and column are zero. $f^c{}_{ab}$ are indeed constants, as intended.

Now we go back to the example we considered at the beginning. There the mass matrix M is in the parabolic class, so $g(y)$ is of the form

$$g(y) = \frac{1}{\sqrt{A\tau_2}} \begin{pmatrix} \frac{\sqrt{A\tau_2}}{2\pi R} & 0 & 0 \\ 0 & 1 & -(\tau_1 + my) \\ 0 & \tau_2 & 0 \end{pmatrix}. \tag{2.51}$$

Then we see that

$$L_1 = \frac{1}{2\pi R}\frac{\partial}{\partial y}, \qquad L_2 = \frac{1}{\sqrt{A\tau_2}}\left(\frac{\partial}{\partial x_2} - (\tau_1 + my)\frac{\partial}{\partial x_1}\right), \qquad L_3 = \frac{1}{\sqrt{A/\tau_2}}\frac{\partial}{\partial x_1}, \tag{2.52}$$

and the vielbein one-forms are

$$\eta^1 = (2\pi R)dy, \quad \eta^2 = \sqrt{A\tau_2}dx_2, \quad \eta^3 = \frac{1}{\sqrt{A/\tau_2}}(dx_1 + (\tau_1 + my)dx_2). \quad (2.53)$$

We see that these vielbein one-forms indeed give the metric in (2.41).

Note that L_3 in (2.52) is an isometry since it satisfies $\mathcal{L}_{L_3} g = 0$, but L_1 and L_2 are not isometries. The L_a satisfy the commutation rules

$$[L_1, L_2] \sim -mL_3, \quad [L_1, L_3] = 0, \quad [L_2, L_3] = 0. \quad (2.54)$$

Note that these are the commutation rules for the Heisenberg algebra.

CHAPTER 3

DUALITY

Supergravity and string theories possess global, (mostly) non-compact symmetries which can be used in employing SS compactification. In this chapter we will discuss the structure and the origin of these symmetries.

In the first section we will describe electric-magnetic duality and its generalizations. This will serve several purposes. Firstly, the idea of S-duality, which is a conjectured non-perturbative symmetry of string theory has its origins in electric-magnetic duality. Secondly, it was argued in [52, 53] that the non-compact symmetries in string theory are inherited from the duality rotations on the two dimensional world sheet of the string. These duality rotations are similar in structure to electric-magnetic duality rotations. In the second, third and fourth sections we will describe the perturbative T-duality and the non-perturbative S and U dualities of string theories. In the last section we will discuss how these symmetries can be understood by studying the duality symmetries of supergravity theories.

3.1 Electric-Magnetic Duality and Its Generalizations

3.1.1 Electric-Magnetic Duality

Consider the Maxwell equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho_e \qquad \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{J}_e \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \end{aligned} \tag{3.1}$$

When $\rho_e = \vec{J}_e = 0$ these equations are invariant under the duality transformations

$$\vec{E} \to \vec{B}, \qquad \vec{B} \to -\vec{E}. \tag{3.2}$$

This can be generalized to U(1) duality rotations parametrized by an arbitrary angle ϕ:

$$(\vec{E} + i\vec{B}) \to e^{i\phi}(\vec{E} + i\vec{B}). \tag{3.3}$$

The duality symmetry in (3.2) is a Z_2 subgroup of the U(1) in (3.3). If we write the Maxwell equations in the covariant form

$$dF = 0 \qquad d * F = 0, \tag{3.4}$$

where $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk}B^k$, the duality transformation (3.2) takes the form $F \to *F$. The duality symmetry is broken by the presence of the electric source terms. In this case (3.4) take the form

$$dF = 0, \qquad d * F = j, \tag{3.5}$$

which is obviously not symmetric. If we want to restore the symmetry we should include magnetic source terms so that $dF = k$. However this requires the existence of a magnetic monopole. If the existence of a magnetic charge is assumed then the duality transformations (3.2) take the form

$$\begin{aligned} \vec{E} &\to \vec{B}, \qquad \vec{B} \to -\vec{E} \\ Q_e &\to Q_m, \quad Q_m \to -Q_e, \end{aligned} \tag{3.6}$$

where Q_e and Q_m are the electric and magnetic charges, respectively. Because of the Dirac charge quantization condition, the form of these charges should be as

$$Q_e = n_e g, \qquad Q_m = \frac{n_m}{g}, \tag{3.7}$$

where n_e and n_m are integers and g is the electric coupling constant $g = e^2/4\pi$, e is the unit electric charge. The transformation of Q_e and Q_m then implies

$$n_e \to n_m, \qquad n_m \to n_e, \qquad g \to \frac{1}{g}. \tag{3.8}$$

The last transformation shows that we have a strong-weak coupling duality.

Note that this symmetry is a symmetry of the field equations. However it is not a symmetry of the Lagrangian

$$L = Re(\vec{E} + i\vec{B}) \cdot (\vec{E} + i\vec{B}) = E^2 - B^2. \tag{3.9}$$

It is possible to include a topological term $\frac{\theta g}{2\pi}\vec{E} \cdot \vec{B}$ in the action (4.48), which does not alter the field equations. Now the action itself is invariant under the duality transformation $SL(2, Z)$ which is generated by the two transformations

$$Z_2 \;:\; \tau \to -\frac{1}{\tau}, \qquad n_e \to n_m, \quad n_m \to n_e, \tag{3.10}$$

$$\theta\text{−shift} \;:\; \tau \to \tau + 1, \quad n_e \to n_e + n_m, \quad n_m \to n_m. \tag{3.11}$$

Here τ is a complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \tag{3.12}$$

The two transformations (3.10) and (3.11) generate the group $SL(2, Z)$ under which we have the following transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{3.13}$$

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}, \tag{3.14}$$

where $a, b, c, d \in Z$ and $ad - bc = 1$.

When $\theta = 0$ we have only the Z_2 symmetry (4.49) which corresponds to the strong-weak coupling duality

$$\frac{e^2}{4\pi} \to \frac{4\pi}{e^2}. \tag{3.15}$$

Hence this is a non-perturbative symmetry which cannot be checked order-by-order in the coupling parameter e. This extension of electric-magnetic duality to $SL(2,Z)$, which maps the strong coupling regime of the theory to the weak coupling regime, is usually referred to as S-duality.

This strong coupling-weak coupling duality, which was first conjectured by Montonen and Olive [54] is realized by $N=4$ supersymmetric Yang-Mills theory [55]. One special feature of this theory is charge non-renormalization (the β function vanishes), i.e. the coupling constant e does not run with the energy scale of the theory and hence there is no asymmetry between e and g. Another feature is that a topological θ term can be added without spoiling the renormalizability of the theory and without changing the field equations [56]. Seiberg and Witten generalized the duality also to $N=2$ SYM, for which $\beta \neq 0$ [57].

3.1.2 Duality Rotations for Interacting Fields

The duality rotations $F \to *F$ of the free Maxwell theory that we considered above can be extended to the case when the electromagnetic fields interact with other fields [58](See also [59] for a recent review).

In this section we will consider the duality symmetries in even dimensional theories, including M $(n-1)$-form gauge fields A^I $(D=2n)$ interacting with other fields $\phi_i(x)$. We will consider Lagrangians of the form

$$\mathcal{L} = \mathcal{L}(F^I, \phi_i, \partial_\mu \phi_i). \tag{3.16}$$

Here $F^I = dA^I$. The gauge potentials A^I appear only through their field strengths. Lagrangians of supergravity theories are also of this form. Note that in even dimensions we have

$$**F = \epsilon F, \qquad \epsilon = \begin{cases} +1 & \text{for D = 4k+2,} \\ -1 & \text{for D = 4k.} \end{cases} \tag{3.17}$$

For A^I the equations of motion and the Bianchi identities are

$$dG^I = 0 \qquad\qquad dF^I = 0 \tag{3.18}$$

where

$$G^I = \frac{\partial \mathcal{L}}{\partial F^I}. \tag{3.19}$$

For the free Maxwell theory we have $G = *F$ and more generally we have $G = f(\phi) * F + g(\phi)F$, for some functions f and g depending on the fields ϕ_i. (3.18) are invariant under the transformations

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \qquad \delta\phi^i = \xi^i(\phi), \tag{3.20}$$

where a, b, c, d are constant $n \times n$ matrices and $\xi^i(\phi)$ are functions of ϕ^i.

One can find the conditions on these matrices from the covariance of the definition (3.19) and from the covariance of the equations of motion for ϕ^i under the duality transformation (3.20). It turns out that the duality group is $Sp(2M, \mathbb{R})$ in the $D = 4k, \epsilon = -1$ case and $SO(M, M)$ in the $D = 4k+2, \epsilon = +1$ case [58]. In the case when there are no scalar fields in the theory only compact subgroups are possible as duality symmetries without introducing ghosts. The maximal compact subgroup of $Sp(2M, \mathbb{R})$ and $SO(M, M)$ are $U(M)$ and $SO(M) \times SO(M)$ respectively.

An interesting case is when $D = 2$. Strings are one dimensional objects which sweep out two dimensional world sheets and hence can be described by two dimensional σ-models with fields taking values in the d-dimensional target space for a string theory in d dimensions. Evidence is given in [52] and [53] that the non-compact symmetries of string theories, in particular the T-duality, are inherited from the duality symmetries of the kind discussed above, realized on the world sheet of the string.

3.2 T-duality

Target Space Duality Symmetry, T-duality, is a perturbative symmetry appearing in string theory compactifications [60]. It is the geometrical symmetry group of the manifold of compactification, plus some discrete translational symmetries. The simplest case is when the compactification manifold is a circle. In this case T-duality implies that compactification on a circle with radius R is the same with compactification on a circle with radius $1/R$. So, a small compactification radius is equivalent to a large compactification radius to the string, and this points out to the existence of a minimal length in string theory.

In order to see how this $R \leftrightarrow 1/R$ duality appears in circle compactifications first consider the string world sheet action:

$$S = \frac{1}{4}\pi\alpha' \int d\tau d\sigma (\partial_i X^\mu)(\partial_i X^\nu) G_{\mu\nu}(X). \tag{3.21}$$

Here X^μ are the parameters of the space in which the string propagates, (σ, τ) are the world sheet parameters such that each $\tau =$ constant denotes the string at a given time. $G_{\mu\nu}$ is the world-sheet metric. i runs over the σ and τ directions. The compactification is defined by the periodic identification

$$X \approx X + 2\pi R m, \tag{3.22}$$

where m is an arbitrary integer.

When $G_{\mu\nu} = \eta_{\mu\nu}$, that is, when the string propagates in flat space-time, the fields X^μ on the world-sheet satisfy free wave equations and admit a decomposition in terms of left- and right-movers as

$$X(\sigma, \tau) = X_R(\sigma - \tau) + X_L(\sigma + \tau). \tag{3.23}$$

Solving the free field equations for X we have

$$\partial_L X^\mu = \sum_n \alpha^\mu_{-n} e^{-in(\tau+\sigma)},$$

$$\partial_R X^\mu = \sum_n \tilde{\alpha}^\mu_{-n} e^{-in(\tau-\sigma)}. \tag{3.24}$$

The zero modes of the oscillators correspond to the center of mass motion and thus α_0 ($\tilde{\alpha}_0$) gets identified with the left-moving (right-moving) momentum of the center of mass. In particular for the center of mass we have

$$X = \alpha_0(\tau+\sigma) + \tilde{\alpha}_0(\tau-\sigma), \tag{3.25}$$

where we identify

$$(\alpha_0, \tilde{\alpha}_0) = (p_L, p_R). \tag{3.26}$$

So the mode expansion of the world-sheet fields X is

$$\begin{aligned} X_R(\sigma-\tau) &= x_R - \sqrt{\frac{\alpha'}{2}} p_R(\sigma-\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{l\neq 0} \frac{1}{l} \alpha_l e^{+il(\sigma-\tau)} \\ X_L(\sigma+\tau) &= x_L - \sqrt{\frac{\alpha'}{2}} p_L(\sigma+\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{l\neq 0} \frac{1}{l} \alpha_l e^{-il(\sigma+\tau)} \end{aligned} \tag{3.27}$$

The normal ordered Hamiltonian reads

$$H = L_{0L} + L_{0R} \tag{3.28}$$

where

$$\begin{aligned} L_{0R} &= \frac{1}{2} p_R^2 + \sum_{l=1}^{\infty} \alpha_{-l}\alpha_l \\ L_{0L} &= \frac{1}{2} p_L^2 + \sum_{l=1}^{\infty} \tilde{\alpha}_{-l}\tilde{\alpha}_l \end{aligned} \tag{3.29}$$

These results are irrespective of whether one of the dimensions is compactified or not. Compactifying on a circle and thus imposing the periodicity condition (3.22) has two effects. First, string states must be invariant under this identification. This means, the operator $\exp(2\pi i R p)$ which translates strings once around the periodic dimension must leave states invariant, so the center of mass momentum is quantized

$$k = \frac{n}{R}, \qquad n \in Z. \tag{3.30}$$

This is what would happen in field theory as well. The second effect is completely a stringy effect. A closed string may now wind around the compact direction,

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w, \qquad w \in Z. \tag{3.31}$$

Here the integer w is the *winding number* and it counts the number of times the string wraps around the circle.

For the closed string we have $p_L = p_R = k$, where p_L, p_R are as in (3.26), and k is the center of mass momentum of the string because X is periodic in σ. This is true for the non-compact dimensions. For compact dimensions we have

$$\begin{aligned} p_L &\equiv \sqrt{\frac{2}{\alpha'}}\alpha_0 = \frac{n}{R} + \frac{wR}{\alpha'} \\ p_R &\equiv \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0 = \frac{n}{R} - \frac{wR}{\alpha'} \end{aligned} \tag{3.32}$$

Notice that in this case L_{0L} and L_{0R} in (3.29) are invariant under the transformation

$$\frac{R}{\sqrt{\alpha'}} \to \frac{\sqrt{\alpha'}}{R}, \qquad w \leftrightarrow n. \tag{3.33}$$

Under (3.33), p_R transforms into $-p_R$, whereas p_L is invariant. The oscillators $\alpha_n, \tilde{\alpha}_n$ also transform in a simple way:

$$\alpha_n \to -\alpha_n, \qquad \tilde{\alpha}_n \to \tilde{\alpha}_n. \tag{3.34}$$

Hence, there is a $R \leftrightarrow 1/R$ symmetry of the closed string theory when compactified on a circle. This is the simplest example of T-duality. This is a stringy property which is due to the fact that the string can wrap around the circle.

What happens when we compactify more than one dimensions? Consider the case when there are d periodic dimensions

$$X^m \approx X^m + 2\pi R, \qquad 26 - d \leq m \leq 25. \tag{3.35}$$

Spacetime is now $M^k \times T^d$ where $k = 26 - d$. The actual geometry of the d-torus depends on the internal metric G_{mn}. With more than one dimension the antisymmetric tensor also has scalar components B_{mn}. The total number of scalars from both sources is then d^2 ($d(d+1)/2$ degrees of freedom for the metric and $d(d-1)/2$ degrees of freedom for B_{mn}).

In this case one can show that (p_L, p_R) belong to a $2d$ dimensional lattice with signature (d, d). This lattice should be integral, self-dual and even. Self-dual means that any vector which has integral product with all the vectors in the lattice sits in the lattice as well. Even means that $p_L^2 - p_R^2$ is even for each lattice vector. All the torus compactifications can be obtained by doing an $SO(d, d)$ Lorentz boost on (p_L, p_R) vectors of a given torus compactification. Rotating (p_L, p_R) by an $O(d) \times O(d)$ transformation does not change the spectrum of the string states, so the totality of such vectors is given by

$$\frac{SO(d,d)}{SO(d) \times SO(d)}. \tag{3.36}$$

This is the moduli space of the compactification torus and the d^2 scalars that we discussed above parametrize this coset space [61]. The action of $O(d, d)$ on this moduli space is such that it takes one string theory to another one. A discrete subgroup $O(d, d; Z)$ takes a string theory into an equivalent one because the Lorentz boosts sitting in this group do not change the lattice of the torus. So the space of inequivalent compactifications is given by

$$\frac{SO(d,d)}{SO(d) \times SO(d) \times O(d,d;Z)}. \tag{3.37}$$

The $O(d, d; Z)$ generalizes the T-duality considered in the 1-dimensional case.

In the discussion above we considered only the bosonic fields. When fermionic fields are taken into account the discussion divides into two cases. In the first case of $N = 2$ supersymmetric theories (IIA and IIB), the moduli space of the toroidal compactifications is as in (3.37) and the T-duality group is again $O(d,d;Z)$[1]. In the second case of $N=1$ supersymmetric theories (type I and heterotic) the moduli space of toroidal compactifications is

$$\frac{SO(16+d,d)}{SO(16+d)\times SO(d)\times SO(16+d,d;Z)} \tag{3.38}$$

and the T-duality group is $SO(d+16,d;Z)$.

3.3 S-duality

Two theories A and B are called S-dual if theory A at strong coupling is equivalent to theory B at weak coupling and vice versa. This means that for each physical observable $f_A(g)$ in theory A there is a corresponding observable $f_B(g)$ in theory B such that $f_A(g)=f_B(1/g)$, where g is the coupling constant. Thus it is a symmetry which cannot be checked order-by-order in the coupling constant, i.e. it is a non-perturbative symmetry.

In the string theory setting the first proposal of S-duality was for heterotic string toroidally compactified to four dimensons, by Font *et al*[62]. It was shown by Cremmer, Ferrara and Scherk that $N=4$, $D=4$ supergravity theory has a global $SL(2,\mathbb{R})$ symmetry [63]. Font *et al* conjectured that the $SL(2,Z)$ subgroup of this symmetry group should be an exact symmetry of the full 4d heterotic string theory [62]. Strong evidence for this conjecture was given in [64]. A Z_2 subgroup is an electric-magnetic duality in which the string coupling constant is inverted and hence it relates the strong coupling limit to a weakly coupled description. Thus this proposal extends the Montonen-Olive duality conjecture for supersymmetric gauge theories.

In string theory S-duality is a transformation which acts on the complex scalar

[1] Type IIA and Type IIB are equivalent below ten dimensions. In fact, e.g., IIA compactified on a circle of radius R is T-dual to IIB compactified on a circle of radius $1/R$.

field S:

$$S = S_1 + S_2 = \chi + ie^{-\phi} \tag{3.39}$$

where χ is the axion field and ϕ is the dilaton field. In this case vacuum expectation values of χ and ϕ play the roles of the vacuum angle θ and the coupling constant e respectively:

$$< \chi >= \frac{\theta}{2\pi} \tag{3.40}$$

$$\frac{e^2}{4\pi} =< e^{\phi} >= \frac{8G}{\alpha'} \tag{3.41}$$

Here G is the Newton's constant and $2\pi\alpha'$ is the inverse string tension.

As mentioned above, the heterotic string theory compactified to four dimensions on a flat torus T^6 is conjectured to have an $SL(2, Z)$ symmetry. Another case in which the group $SL(2, Z)$ appears as a duality symmetry is the type IIB string theory in 10 dimensions. This is a self-duality of the theory which takes it to itself, at another coupling. The conjectured self-duality of IIB string implies the conjectured duality of Montonen and Olive within $D = 4,\ N = 4$ gauge theory. SO(32) type I and SO(32) heterotic at $D = 10$ are also interchanged by weak-strong coupling duality. (See [65, 66] and references therein.)

Since S-duality is a strong-weak coupling duality it cannot be checked in perturbative string theory. However there is strong evidence for it coming from the analysis of BPS states and studying the constraints of supersymmetry on the low energy effective action. (See e.g. [66].)

3.4 U-duality

So far we have seen two types of string dualities: the perturbative T-duality and the non-perturbative S-duality. Although these dualities look like very different ideas, in fact they are essentially tightly related. Both the coupling constant of the string theory and the radius of the compactification circle (in the S^1 reduction case) are determined by the expectation value of some scalar field (it is the dilaton for the coupling constant). Hence the definitions of T-duality ($R \to 1/R$) and S-duality ($g \to 1/g$) depend on the choice of these scalar fields among the many scalars parametrizing the moduli space. Pursuing this analogy, Hull and Townsend conjectured in 1995 the existence of a symmetry group which combines these two symmetries and gave it the name U-duality [50]. (See also [67].)

In the late 70s it was realized that compactified supergravity theories possess non-compact global symmetries, say G [30, 68, 69, 70]. In 1990 it was conjectured that some discrete subgroups of these symmetries should be promoted to T-duality or S-duality symmetries of corresponding string theories [53]. (See also [71].) The U-duality group conjectured by Hull and Townsend is the maximal integer subgroup of G. The conjecture is that this subgroup extends as the symmetry of the full string theory, just as the extension of some smaller subgroups to the T- and S-duality symmetries of the full string theory.

All these conjectures rely on the idea that the duality symmetries of the string theories should be understood by studying the corresponding supergravity theories. In fact this is a very natural idea since supergravity is the low energy effective action of string theory and the form of the low energy effective action is completely determined by supersymmetry and the spectrum of massless states in the theory. Hence it does not receive any quantum corrections. Thus a symmetry of the string theory should also be a symmetry of the corresponding supergravity theory.

So, before giving the list of the U-duality groups appearing in type II and heterotic string compactifications, we will first discuss how duality symmetries appear in compactified supergravity theories.

3.5 Duality Symmetries from Compactification of D=11 SUGRA

In this section we will consider the classical, internal, global symmetries of the bosonic sector of various maximal supergravities in dimensions $D \leq 11$ which are obtained from eleven dimensional supergravity by toroidal compactification. The Lagrangian for the 11d supergravity is

$$\mathcal{L}_{11} = R * 1 - \frac{1}{2} F_4 \wedge * F_4 + \frac{1}{6} F_4 \wedge F_4 \wedge A_3. \tag{3.42}$$

Here $F_4 = dA_3$. Suppose now that we compactify on a torus T^n with coordinates y^i. This can be regarded as doing n subsequent circle compactifications. At each step one obtains Kaluza-Klein vectors $\mathcal{A}_i$ and Kaluza-Klein scalars (dilatons) ϕ_i coming from the reduction of the metric and also 0-form potentials, or axions $\mathcal{A}^i_{(0)j}$ coming from the reduction of the Kaluza-Klein vectors, obtained in a previous step. It is obvious that one should have $i < j$. In addition one obtains the potentials $A_{(3)}, A_{(2)i}, A_{(1)ij}, A_{(0)ijk}$ coming from the reduction of the 3-form potential A_3 in 11 dimensions. The indices $i, j, k, \cdots$ correspond to torus directions and they are antisymmetrised. The Lagrangian for the resulting supergravity theory can be found in [44].

The resulting supergravity theory in D dimensions has $GL(N, \mathbb{R}) \times \mathbb{R}^q$ symmetry in its scalar sector. Here $N = 11 - D$ and $q = \frac{1}{6}(11 - D)(10 - D)(9 - D)$. The $GL(N, \mathbb{R})$ part of the symmetry is due to the compactification of the gravitational sector of the theory. In fact we can only be sure of the $SL(N, \mathbb{R})$ part and the last generator of the $GL(N, \mathbb{R})$ gives an internal R symmetry in dimensions $D > 2$. The $\mathbb{R}^q$ part comes from the local abelian gauge symmetry of the antisymmetric tensor field strength in the original 11 dimensions.

If one first reduces to D dimensions on a torus and then dualises all the field strengths whose degree exceeds $\frac{1}{2}D$, then the $SL(N, \mathbb{R}) \times \mathbb{R}^q$ symmetry above extends to a global $E_{(11-D)(11-D)}$ internal symmetry [30, 68, 69, 70]. Here $E_{n(n)}$

is the maximal noncompact form of the exceptional group E_n and for brevity we shall write them simply as E_n.

It is important to note that the global symmetry can depend on the choice of dualisation and it might not be possible to dualise all the fields in a given theory. A sufficient condition is that the fields appear in the action purely through their field strengths. In some cases it might still be possible to dualise a field which appears in the field equations and Bianchi identities through its field strength even if this is not the case in the action. The symmetry E_{11-D} appears only when we dualise all the fields. If, for example, we dualise the R-R fields only and keep the NS-NS gauge potentials undualised, the symmetry group is $O(10-D, 10-D) \times \mathbb{R}^q$ for $D \leq 6$ where $q = 2^{9-D}$. When $D \geq 7$, the duality group is still E_{11-D} since it is only the R-R potential that suffers dualisation in these dimensions.

In the case when we dualise all the fields in the theory the scalar sector of the Lagrangians are sigma models on the symmetric spaces $E_{11-D}/K(E_{11-D})$ where $K(G)$ is the maximal compact subgroup of G. The axionic scalars in the fully dualised theory are in one to one correspondence with the positive roots of the E_{11-D} algebra. Using this fact one can give a simple parametrisation of the $E_{11-D}/K(E_{11-D})$ scalar manifold, which is the Borel or upper triangular parametrisation. In this parametrisation, the axionic scalars are the parameters in the exponentiation of the positive roots and the dilatonic scalars are the parameters in the exponentiation of the Cartan generators.

Can this symmetry of the scalar sector be extended to the full theory? In other words, do the other terms in the Lagrangian involving the higher degree fields also share this symmetry? The answer is affirmative in the odd dimensional case, the full Lagrangian can be shown to be invariant under the symmetry of its scalar sector. In even dimensions the symmetry can be extended to the level of the field equations only, involving an electric-magnetic type duality. However, by

Table 3.1: Symmetries and matter content of (the bosonic sector of) maximal supergravities

D	Global Duality Group	Local Symmetry	Matter Content
9	$GL(2,\mathbb{R})$	$SO(2)$	$1e^r_\mu, 1A_3, 2A_2,$ $3A_1,\ 3\ A_0$
8	$E_{3(3)} = SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$1e^r_\mu, 1A_3, 3A_2,$ $6A_1,\ 7A_0$
7	$E_{4(4)} = SL(5,\mathbb{R})$	$SO(5)$	$1e^r_\mu, 5A_2, 10A_1,$ $14A_0$
6	$E_{5(5)} = SO(5,5)$	$SO(5) \times SO(5)$	$1e^r_\mu, 5A_2, 16A_1,$ $25A_0$
5	$E_{6(6)}$	$USp(8)$	$1e^r_\mu, 27A_1,\ 42A_0$
4	$E_{7(7)}$	$SU(8)$	$1e^r_\mu, 28A_1,\ 70A_0$
3	$E_{8(8)}$	$SO(16)$	$1e^r_\mu,\ 128A_0$

using a formalism one can implement this symmetry at the level of an auxiliary Lagrangian which is equivalent to the original Lagrangian, as it yields the same field equations. The name of this formalism is "Doubled Formalism" and we will describe it in chapter 5.

The matter content and the symmetries of maximal supergravities in dimension $D \leq 9$ after full dualisations are given in Table 3.1. In [50] it was conjectured that after quantisation a discrete subgroup becomes the U-duality symmetry group of the toroidally compactified type II quantum string theories.

3.6 Duality Symmetries from Type II and Heterotic Compactifications

Consider type II string theory compactified on a torus. It doesn't matter whether we start with the IIA or IIB since their compactifications are the same. The resulting low energy field theory is a D-dimensional supergravity theory with global symmetry group G. This global symmetry G is a symmetry of the action if the theory is in odd dimensions and in even dimensions it is a symmetry of the field equations only. The group G has a $O(10-D, 10-D)$ subgroup which is broken to the discrete T-duality group $O(10-D, 10-D, Z)$. Also, as mentioned in the previous section, the type IIB string theory in ten dimensions has a conjectured $SL(2, Z)$ symmetry. This $SL(2, Z)$ symmetry commutes with the T-duality symmetry of the lower dimensional type IIB string theory and together they generate the U-duality group conjectured by Hull and Townsend [50].

Consider the interesting and well understood case of $D = 4$. When compactified on the torus T^6, the type II string theory gives the $N = 8$, $D = 4$ supergravity effective action. This theory has E_7 symmetry as was noticed by Cremmer and Julia [30]. This symmetry is at the level of field equations as in other even dimensions. The 70 scalars in the theory parametrize the coset space $E_{7(7)}/[SU(8)/Z_2]$. The symmetry group $E_{7(7)}$ contains $SL(2, R) \times O(6, 6)$ as a maximal subgroup. In [50], it was shown that certain quantum mechanical effects break $E_{7(7)}$ to a discrete subgroup which they called $E_7(Z)$. This implies the breaking of the maximal $SL(2, \mathbb{R}) \times O(6, 6)$ subgroup to $SL(2, Z) \times O(6, 6, Z)$. The $O(6, 6, Z)$ factor extends to the full string theory as the T-duality group and it was conjectured in [62] that the $SL(2, Z)$ extends to the full string theory as the S-duality group. In [50] it was conjectured that the full $E_7(Z)$ group also extends to the full string theory as a new unified duality group, namely the U-duality.

The low energy effective field theory of heterotic string theory is 10 dimensional supergravity theory coupled to 16 Yang-Mills vector fields. When compactified on a torus T^n, it has a duality group G which contains $O(10-d, 26-d)$ as a subgroup, where $d = 10 - n$. The discrete subgroup $O(10-d, 26-d, Z)$ is the T-duality group of the corresponding string theory. The conjecture of [50] is that a discrete subgroup of G extends to the corresponding string theory as a unified symmetry.

Again consider the $D = 4$ case. In this dimension the theory has a 2-form field which can be dualised to a scalar field. Together with the dilaton field they parametrize a $SL(2, \mathbb{R})/SO(2)$ coset and after this dualisation the theory has a $SL(2, \mathbb{R})$ symmetry realized at the level of field equations. The discrete subgroup $SL(2, Z)$ is conjectured to be a symmetry of the corresponding string theory [72, 73]. So the conjectured U-duality group, $SO(6, 22, Z) \times SL(2, Z)$ of the 4 dimensional heterotic string theory unifies the T and S dualities.

Let G and G' be the supergravity symmetry groups in d and d' dimensions respectively, where $d' < d$. The dimensional reduction from d to d' dimensions gives an embedding of G in G' and $G(Z)$ is a subgroup of $G'(Z)$. The reason for this is that G does not act on the d-dimensional space-time and hence survives dimensional reduction. So one can see that, for dimensions $d > 4$, the U-duality group G must be a subgroup of $E_7(Z)$ in the type II case and a subgroup of $SO(6, 22, Z) \times SL(2, Z)$ in the heterotic case. So, in these dimensions, one can define the U-duality group as $E_7(Z) \cap G$ for type II and as $G \cap [O(6, 22, Z) \times SL(2, Z)]$ for heterotic string theory.

Below we present the tables for the duality symmetries for the type II and the heterotic string theories compactified to d dimensions [50].

Table 3.2: Duality symmetries for type II string compactified to d dimensions

Space-time Dimension d	Supergravity Duality Group G	String T-duality	Conjectured Full String Duality
10A	$SO(1,1)/Z_2$	1	1
10B	$SL(2,\mathbb{R})$	1	$SL(2,Z)$
9	$SL(2,\mathbb{R}) \times O(1,1)$	Z_2	$SL(2,Z) \times Z_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$O(2,2,Z)$	$SL(3,Z) \times SL(2,Z)$
7	$SL(5,\mathbb{R})$	$O(3,3,Z)$	$SL(5,Z)$
6	$O(5,5)$	$O(4,4,Z)$	$O(5,5,Z)$
5	$E_{6(6)}$	$O(5,5,Z)$	$E_{6(6)}(Z)$
4	$E_{7(7)}$	$O(6,6,Z)$	$E_{7(7)}(Z)$
3	$E_{8(8)}$	$O(7,7,Z)$	$E_{8(8)}(Z)$
2	$E_{9(9)}$	$O(8,8,Z)$	$E_{9(9)}(Z)$
1	$E_{10(10)}$	$O(9,9,Z)$	$E_{10(10)}(Z)$

Table 3.3: Duality symmetries for heterotic string compactified to d dimensions

Space-time Dimension d	Supergravity Duality Group G	String T-duality	Conjectured Full String Duality
10	$O(16) \times SO(1,1)$	O(16,Z)	$O(16,Z) \times Z_2$
9	$O(1,17) \times SO(1,1)$	O(1,17,Z)	$O(1,17,Z) \times Z_2$
8	$O(2,18) \times SO(1,1)$	$O(2,18,Z)$	$O(2,18,Z) \times Z_2$
7	$O(3,19) \times SO(1,1)$	$O(3,19,Z)$	$O(3,19,Z) \times Z_2$
6	$O(4,20) \times SO(1,1)$	$O(4,20,Z)$	$O(4,20,Z) \times Z_2$
5	$O(5,21) \times SO(1,1)$	$O(5,21,Z)$	$O(5,21,Z) \times Z_2$
4	$O(6,22) \times SL(2,\mathbb{R})$	$O(6,22,Z)$	$O(6,22,Z) \times SL(2,Z)$
3	$O(8,24)$	$O(7,23,Z)$	$O(8,24,Z)$
2	$O(8,24)^{(1)}$	$O(8,24,Z)$	$O(8,24)^{(1)}(Z)$

CHAPTER 4

A PRELIMINARY CHAPTER

In the previous chapters we gave a description of the Scherk-Schwarz dimensional reduction mechanism and also discussed the symmetries present in supergravity and string theories, which we can employ in this mechanism. In the next chapter, where we will introduce the original part of this thesis work, we apply these methods to specific models. The purpose of the present chapter is to introduce some preliminary material which will be needed for the next chapter.

The specific models that we study are such that the global symmetries that we employ are realized only at the level of field equations rather than the action itself. In the next section, we will describe a formalism, called the "Doubled Formalism" [29], which enables us to realize the symmetry at the level of an auxiliary Lagrangian that yields the same field equations as those of the original Lagrangian.

In order to construct such an auxiliary Lagrangian one has to impose a constraint in the higher dimensional theory. This constraint, when reduced with the SS method, implies a self-duality condition for some of the gauge fields in lower dimensions. In each case one starts with an even dimensional theory, reduces on a circle, and hence ends up with an odd dimensional theory. In the second section of this chapter we discuss what it means to be "self-dual" in odd dimensions [32].

In the last section we the Stückelberg Mechanism will be described. This is a mechanism that appears during the dimensional reduction process and it is this very mechanism that lets some of the gauge fields acquire mass.

4.1 Doubled Formalism

Typically a $D = 2n$ dimensional supergravity theory has a global symmetry group G which can be realised at the level of field equations but not the action, as G acts on n-form field strengths $H = dA$ through electric-magnetic duality transformations. In such cases it is possible to construct a manifestly G-invariant Lagrangian that depends on the potentials A and dual potentials $\tilde{A}$. The dual fields are regarded as independent fields, but the field equations are supplemented with a G-covariant constraint relating the n-form field strengths $d\tilde{A}$ to dA, keeping the number of independent degrees of freedom correct. The new Lagrangian is equivalent to the original one as the two yield equivalent field equations when the constraint is taken into account.

In this section we will describe this formalism which was introduced in [29] where it was called the 'doubled formalism'. We will first consider the case $G = SL(2, \mathbb{R})$ and then give the general case in the following subsection.

4.1.1 G = SL(2, $\mathbb{R}$) Case

Consider the following Lagrangian in $D + 1 = 2n$ dimensions

$$\mathcal{L} = -\frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}e^{2\phi}d\chi \wedge *d\chi - \frac{1}{2}e^{-\phi}F_n \wedge *F_n - \frac{1}{2}\chi F_n \wedge F_n \tag{4.1}$$

Here $F_n = dA_{n-1}$ and ϕ and χ are scalar fields. The scalar fields parametrize the coset space $SL(2, \mathbb{R})/SO(2)$. (Recall the discussion in chapter 2.) For a theory in which the scalar fields parametrize a coset space G/H, where H is the maximal compact subgroup of G, the kinetic term for the scalars can be written in the form (2.33) or (2.35). In the present case $G = SL(2, \mathbb{R})$, $H = SO(2)$ and we have

$$\mathcal{V} = he^{\phi/2}\begin{pmatrix} e^{-\phi} & 0 \\ -\chi & 1 \end{pmatrix} \tag{4.2}$$

where h is an $SO(2)$ matrix

$$h = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.3}$$

Then

$$\mathcal{K} = e^{\phi} \begin{pmatrix} e^{-2\phi} + \chi^2 & -\chi \\ -\chi & 1 \end{pmatrix} \tag{4.4}$$

and is independent of θ.

The field equations of this Lagrangian have an $SL(2, \mathbb{R})$ S-duality invariance (for even n) acting on F through electromagnetic duality transformations, as we now discuss.

Defining a new n-form G_n by

$$G_n = \frac{\delta \mathcal{L}}{\delta F_n} = -e^{-\phi} * F_n - \chi F_n \tag{4.5}$$

the lagrangian (4.1) can be written as

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}(d\mathcal{K} \wedge * d\mathcal{K}^{-1}) + \frac{1}{2} F_n \wedge G_n \tag{4.6}$$

The Bianchi identity and the equation of motion for the n-form field strength F_n are

$$\begin{aligned} dF_n &= 0 \\ dG_n = d(e^{-\phi} * F_n + \chi F_n) &= 0 \end{aligned} \tag{4.7}$$

which can be combined as

$$d\mathcal{H}_n = 0 \tag{4.8}$$

where $\mathcal{H}_n$ is the $SL(2, \mathbb{R})$ doublet

$$\mathcal{H}_n = \begin{pmatrix} F_n \\ G_n \end{pmatrix}. \tag{4.9}$$

The field equations are manifestly $SL(2,\mathbb{R})$ invariant but the $F_n \wedge G_n$ term in the Lagrangian (4.6) is not invariant. However, an invariant Lagrangian can be constructed as in [29] if the field equation $dG_n = 0$ is solved by introducing a dual potential $\tilde{A}_n$ so that $G_n = d\tilde{A}_n$, which can be combined with A_n to form an $SL(2,\mathbb{R})$ doublet, with field strengths $H_n^{(i)}$ given by

$$H_n = \begin{pmatrix} dA_n \\ d\tilde{A}_n \end{pmatrix}. \tag{4.10}$$

It is useful to define the field strength

$$\tilde{F}_n = d\tilde{A}_{n-1} - \chi dA_{n-1}. \tag{4.11}$$

Now consider the following Lagrangian which includes both the original field strength F_n and the new one $\tilde{F}_n$:

$$\mathcal{L}' = -\frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}e^{2\phi}d\chi \wedge *d\chi - \frac{1}{2}e^{-\phi}F_n \wedge *F_n - \frac{1}{2}e^{\phi}\tilde{F}_n \wedge *\tilde{F}_n \tag{4.12}$$

The Bianchi identities and the field equation for F_n and $\tilde{F}_n$ are as below:

$$\begin{aligned} dF_n &= 0, \qquad d(\tilde{F}_n + \chi F_n) = 0 \\ d*(e^{\phi}\tilde{F}_n) &= 0, \qquad d*(e^{-\phi}F_n - \chi e^{\phi}\tilde{F}_n) = 0. \end{aligned} \tag{4.13}$$

The Lagrangian $\mathcal{L}'$ includes two potential fields A_{n-1} and $\tilde{A}_{n-1}$ whereas the Lagrangian $\mathcal{L}$ in (4.1) has only one potential field A_{n-1}. If we want these two Lagrangians to be equivalent we should impose a constraint on the potential fields in $\mathcal{L}'$ so that the number of degrees of freedom are halved. The following constraint, which we will call a "twisted duality condition" is consistent with the field equations (4.13):

$$\tilde{F}_n = e^{-\phi} * F_n. \tag{4.14}$$

Note that this constraint implies

$$d\tilde{A}_{n-1} = -e^{-\phi} * F_n - \chi F_n, \tag{4.15}$$

which, according to (4.5), is equivalent to imposing $d\tilde{A}_{n-1} = G_n$.

Now it is easy to see that the Lagrangian $\mathcal{L}'$ has the same set of equations of motion and Bianchi identities with those of (4.1) after imposing the relation (4.14). In fact there is an interchange between the Bianchi identities and equations of motion. So we are on half way to our goal. We have been able to write down an auxiliary Lagrangian which, after imposing some constraint, yields the same field equations as those of the original one. Now the question is whether this new Lagrangian $\mathcal{L}'$ is manifestly $SL(2,\mathbb{R})$ invariant or not. In fact it is, as can be seen by writing it in the form

$$\mathcal{L}' = \frac{1}{4}\mathrm{tr}(d\mathcal{K} \wedge *d\mathcal{K}^{-1}) - \frac{1}{4}H_n^i \mathcal{K}_{ij} * H_n^j. \tag{4.16}$$

Here $i, j = 1, 2$ are $SL(2,\mathbb{R})$ indices. It is crucial to note that $H_n^{(2)} = G_n$ only after we impose the constraint (4.14). One first varies the action, then imposes the constraint (4.14) and only then the field equations are equivalent with those of (4.1). If, instead, the constraint equation is imposed on the Lagrangian (4.16) before varying the action, then the $H^T\mathcal{M} * H$ part of the Lagrangian vanishes. This observation will be crucial in the next section where we introduce the general formalism.

The invariance of the Lagrangian (4.16) under the following $SL(2,\mathbb{R})$ transformation can be seen manifestly

$$H_n \to \Lambda^{-1} H_n \qquad \mathcal{K} \to \Lambda^T \mathcal{K} \Lambda, \qquad \Lambda \in SL(2,\mathbb{R}). \tag{4.17}$$

The relation (4.14) can also be written in a manifestly $SL(2,\mathbb{R})$ covariant form:

$$H_n^{(i)} = J^i{}_j * H_n^{(j)}, \tag{4.18}$$

where J is the $SL(2,\mathbb{R})$ matrix

$$J = \Omega^{ik}\mathcal{K}_{kj}. \tag{4.19}$$

Here Ω is the $SL(2,\mathbb{R})$ invariant matrix

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.20}$$

Note that the matrix J in (4.19) satisfies $J^2 = -\mathbb{I}$, so that this constraint is consistent in $2n$ dimensions with even n for which $(*)^2 = -1$. The general case including odd n will be discussed in the next subsection.

Of course one should also study the other sectors of $\mathcal{L}$ and $\mathcal{L}'$ and be able to show that the field equations for the other fields (the scalar fields and the metric in this case) are also equivalent. Rather than doing the explicit calculations here we will give a general proof that this holds in the next section where we discuss the general formalism.

4.1.2 The General Formalism

The doubled formalism of the last section can be generalised [29]. Consider the following Lagrangian in $2n$ dimensions

$$\mathcal{L} = -\frac{1}{2} R_{IJ} F_n^I \wedge * F_n^J - \frac{1}{2} S_{IJ} F_n^I \wedge F_n^J + \mathrm{L}(\Phi) \tag{4.21}$$

where $F_n^I = dA_{n-1}^I$ with $I = 1, ..., k$ are k field strengths and Φ denotes all the remaining fields, including the scalars. The matrices R_{IJ}, S_{IJ} are functions of the scalar fields and they satisfy $R_{IJ} = R_{JI}$ and $S_{IJ} = (-1)^{n-1} S_{JI}$. It is useful to define G_n^I as

$$G_n^I = \frac{\delta \mathcal{L}}{\delta F_n^I} \tag{4.22}$$

so that the action can be written as

$$\mathcal{L} = \frac{1}{2} F_n^I \wedge G_n^I + \mathrm{L}(\Phi) \tag{4.23}$$

The field equations and the Bianchi identities can be combined as

$$d\mathcal{H}_n = 0 \tag{4.24}$$

where $\mathcal{H}_n$ is

$$\mathcal{H}_n = \begin{pmatrix} F_n^I \\ G_n^I \end{pmatrix}. \tag{4.25}$$

with $r = 2k$ components.

Such systems arise in supergravity theories, and typically the field equations and Bianchi identities have a global symmetry G under which $\mathcal{H}_n$ transforms as a $2k$-dimensional representation of G, and $\mathrm{L}(\Phi)$ is G-invariant. In $2n$ dimensions, the group G will have an invariant Ω^{ij}, where $i, j = 1, ..., 2k$ are indices for the **2k** representation of G. This is symmetric if n is odd and anti-symmetric if n is even

$$\Omega^{ij} = (-1)^{n-1}\Omega^{ji} \tag{4.26}$$

As before we introduce potential fields $\tilde{A}^I_{n-1}$ with $G^I = d\tilde{A}^I$ to form

$$H_n = \begin{pmatrix} dA_n^I \\ d\tilde{A}_n^I \end{pmatrix} \tag{4.27}$$

transforming in the **2k** representation of G. Then the system can be described by the G-invariant Lagrangian

$$\mathcal{L}' = -\frac{1}{4} H_n^T \mathcal{K} \wedge *H_n + \mathrm{L}(\Phi), \tag{4.28}$$

together with a constraint

$$H_n = Q * H_n \tag{4.29}$$

where $Q^i{}_j$ is a $2k \times 2k$ matrix given in terms of the scalar fields by

$$Q^i{}_j = \Omega^{ik}\mathcal{K}_{kj}. \tag{4.30}$$

Here $\mathcal{K}_{ij}$ is given in terms of R_{IJ}, S_{IJ} by

$$\mathcal{K} = \begin{pmatrix} R + SR^{-1}S^T & -SR^{-1} \\ -R^{-1}S^T & R^{-1} \end{pmatrix}. \tag{4.31}$$

In the supergravity applications we will be considering, the scalars take values in a coset G/H and $\mathcal{K}_{ij}$ is the symmetric matrix representing the scalar fields. Note that

$$Q^2 = (\Omega\mathcal{K})^2 = (-1)^{n-1}1\!\!1 \tag{4.32}$$

so that the constraint (4.29) is consistent as for $2n$-dimensional Lorentzian spacetime $**H_n = (-1)^{n-1}H_n$.

The field equations that are derived from the Lagrangian (4.21) are equivalent to those derived from the Lagrangian (4.28) when supplemented with the constraint (4.29). This is easy to verify for the potential fields. Their field equations and Bianchi identities obtained by varying the Lagrangian (4.21) are

$$dF_n^I = 0, \qquad dG_n^I = 0 \tag{4.33}$$

Now look at the field equations of the Lagrangian (4.28) which are

$$d(\mathcal{K} * H_n) = 0, \qquad dH_n = 0. \tag{4.34}$$

Now we impose the constraint (4.29) and use (4.32) to obtain $\mathcal{K} * H = \Omega H$; so after we impose the constraint, the equations (4.34) are truncated to

$$dH_n = 0. \tag{4.35}$$

In order for these equations to be equivalent to those in (4.33), we should have $d\tilde{A}_{n-1}^I = G_n^I$. In order to check whether this really holds, we should study in more detail what the constraint (4.29) means. For simplicity let's call $d\tilde{A}_{n-1}^I = \tilde{F}_n^I$ for the time being. Then the constraint imposes the following relation between F_n^I and $\tilde{F}_n^I$:

$$\begin{pmatrix} F_n^I \\ \tilde{F}_n^I \end{pmatrix} = \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix} \begin{pmatrix} *F_n^I \\ \tilde{F}_n^I \end{pmatrix}. \tag{4.36}$$

Here

$$Q = \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix} \tag{4.37}$$

and hence A, B, C, D are $k \times k$ matrices depending on the scalar fields. Solving this equation and using (4.32) one obtains

$$\tilde{F}^I_n = B^{-1} * F^I_n + DB^{-1} F^I_n. \tag{4.38}$$

Now using (4.31), it is easy to show that (4.38) is equivalent to saying $G^I = \tilde{F}^I = d\tilde{A}^I$. Hence when one varies the Lagrangian $\mathcal{L}'$ with respect to the potential fields and then imposes the twisted-duality constraint, the field equations that are obtained are equivalent to those of the Lagrangian $\mathcal{L}$, as claimed.

Now we will study the field equations for the other fields which we collectively denote by Φ. First we make the observation that if the solution (4.38) for $d\tilde{A}^I_{n-1}$ is substituted into H_n, we have

$$H^T_n \mathcal{K} \wedge *H_n = 0. \tag{4.39}$$

This can be seen easily by observing that when one substitutes the constraint (4.29) in $H^T_n \mathcal{K} * H_n$, one obtains $H^T_n \wedge \Omega H$ which vanishes in $D = 2n$ dimensions. Varying (4.39) with respect to Φ we get

$$\frac{1}{2} H^T_n \frac{\delta \mathcal{K}}{\delta \Phi} \wedge *H + \frac{\delta H^T}{\delta \Phi} \mathcal{K} \wedge *H = 0. \tag{4.40}$$

Then we have

$$\frac{1}{2} H^T_n \frac{\delta \mathcal{K}}{\delta \Phi} \wedge *H = -\frac{\delta H^T}{\delta \Phi} \mathcal{K} \wedge *H = +\frac{\delta H^T}{\delta \Phi} \Omega \wedge H = -\frac{\delta}{\delta \Phi} (F^I_n \wedge G^I_n). \tag{4.41}$$

where we have used (4.29) in writing the second equality and the form of Ω and the fact that $\delta F / \delta \Phi = 0$ in the last equality.

Thus it is indeed true that the Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ are equivalent because they yield the same field equations after imposing the constraint (4.29) on the field equations of $\mathcal{L}'$.

4.2 Self-duality in Odd Dimensions

In $D = 2n$ dimensions it is possible to define self-duality via the Hodge duality operator $*$. In such dimensions an n-form F_n is called self-dual if it satisfies

$$F_n = *F_n. \tag{4.42}$$

Since the degree of the form $*F_n$ is equal to $2n - n = n$, it is consistent to talk about such an equality. However, in odd dimensions $D = 2n - 1$, the Hodge dual of an n-form is of degree $(2n - 1) - n = n - 1$ and hence an equation like (4.42) does not make sense in odd dimensions.

The question as to how self-duality might be defined in odd dimensions was first investigated in [32]. The original motivation for this investigation was from $D = 7, N = 4$ $SO(5)$-gauged supergravity. The ungauged theory has a rigid $G = USp(4)$ invariance. The abelian gauge fields form the $\underline{10}$ representation of G, whereas the two-form fields form the $\underline{5}$ representation. In the gauged theory the 2-form fields should couple to the 1-form fields but this violates the antisymmetric tensor gauge invariance. They cannot transform simultaneously under their massless p-form gauge transformations and the new gauge transformations. This problem can be resolved if an explicit antisymmetric tensor mass term is added, since this mass term eliminates the requirement of massless p-form gauge invariance. However, incorporating a mass term raises a new problem– additional modes are propagated and this destroys the balance of bosons and fermions, thus violating supersymmetry. The problem is resolved by dualising the 2-form fields to 3-form fields and then a mass term for the 3-form fields is incorporated by the self-duality mechanism. A similar situation occurs also in $D = 5, N = 8$ $SO(6)$-gauged supergravity. So, what is this mechanism?

Consider first the case $D = 3$ for simplicity. The Proca equation for a massive vector field in this dimension is

$$d * F_{(2)} + m^2 * A_{(1)} = 0. \tag{4.43}$$

The subscript refers to the degree of the form. Here $F_{(2)}$ is the field strength of the vector field $A_{(1)}$, $F_{(2)} = dA_{(1)}$. (4.43) implies the Lorentz condition

$$d * A_{(1)} = 0, \tag{4.44}$$

which means that only two of the three components of $A_{(1)}$ are independent. (4.43) and (4.44) imply that these two components satisfy a free field wave equation with wave operator $(\Box - m^2)$.

It is possible to find a kind of "square root" of (4.43) for which only one mode is propagated:

$$F_{(2)} = m * A_{(1)}. \tag{4.45}$$

This equation implies both (4.43) and (4.44) . It can be derived from the following Lagrangian

$$\mathcal{L} = -m^2 A_{(1)} \wedge *A_{(1)} + mA_{(1)} \wedge dA_{(1)}. \tag{4.46}$$

This Lagrangian propagates only one degree of freedom. We can see this as follows. We will restrict ourselves to the case when $A_{(1)}$ is a homogenous field, that is we will assume that the space derivatives ∂_1 and ∂_2 vanish. The independent components will be taken to be A_1 and A_2; we will eliminate the A_0 component (we have dropped the subindex (1) referring to the degree of the form to avoid complication). Then, in components, the Lagrangian (4.46) takes the form

$$\mathcal{L}(\partial_i = 0) = -m^2(A_i)^2 - m\epsilon^{ij} A_i \dot{A}_j \tag{4.47}$$

where $i, j = 1, 2$ and a dot refers to time derivative. Now we will break the manifest $SO(2)$ rotational symmetry and rewrite (4.47) so that only A_2 occurs through its time derivative[1]:

$$\mathcal{L}(\partial_i = 0) = -m^2(A_1^2 + A_2^2) - 2mA_1\dot{A}_2 + \text{total time derivative.} \tag{4.48}$$

Since A_1 does not appear through its time derivative it is an auxiliary field and hence we can eliminate it using its Euler-Lagrange equations [2] and obtain

$$\mathcal{L}(\partial_i = 0) = \dot{A}_2^2 - m^2 A_2^2, \tag{4.49}$$

which is of canonical form for a single mode of mass m. So the Lagrangian (4.46), which yields the massive self-duality condition (4.45), has indeed halved the degrees of freedom being propagated. In the two next subsections, we will give generalizations of this, first for the $D = 4k - 1$ case and then for the $D = 4k + 1$ case.

4.2.1 Generalizations to $D = 4k - 1$

There is a direct generalization of the discussion above to $D = 4k - 1$ dimensions. Now the field $A_{(2k-1)}$ is a $2k - 1$-form with a $2k$-form field strength $F_{(2k)} = dA_{(2k-1)}$. The massive self duality condition is again

$$F_{(2k)} = m * A_{(2k-1)}. \tag{4.50}$$

Since $A_{(2k-1)}$ is an odd form as before, all the arguments in the previous subsection are valid here too.

Note that the number of modes propagated by a massless $(2k - 1)$-form in $D = 4k - 1$ dimensions is

$$\begin{pmatrix} (4k-1)-2 \\ 2k-1 \end{pmatrix} = \frac{(4k-3)!}{(2k-1)!(2k-2)!} \tag{4.51}$$

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[1] Here one uses $mA_2\dot{A}_1 = \partial_t(mA_2A_1) - mA_1\dot{A}_2$.

[2] $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{A}_1} - \frac{\partial\mathcal{L}}{\partial A_1} = 0 \Rightarrow -m\dot{A}_2 - m^2A_1 = 0 \Rightarrow A_1 = -\frac{1}{m}\dot{A}_2$.

whereas for a massive $(2k-1)$-form it is

$$\begin{pmatrix} (4k-1)-1 \\ 2k-1 \end{pmatrix} = \frac{(4k-2)!}{[(2k-1)!]^2} \tag{4.52}$$

which is exactly twice the number of the modes in (4.51). Hence the self-duality condition reduces the number of massive modes propagated by A to exactly the number of massless modes propagated by the same field. Thus with this mechanism we can incorporate a mass term without changing the number of degrees of freedom.

4.2.2 Generalizations to $D = 4k + 1$

In this dimension, in order for the self-duality equation $F = m * A$ to make sense we should have A a $2k$-form field and $F = dA$ a $2k+1$-form field. However the formalism that has been described above cannot be generalized to dimensions $D = 4k+1$. To understand why, first have a closer look at what sort of an equation the self-duality condition implies for the field $A_{(2k)}$. First note that[3]

$$\begin{aligned} dA = F = m * A \Rightarrow *dA = -mA \quad &\Rightarrow \quad d * dA = -mF = -m^2 * A \\ &\Rightarrow \quad *d * dA = +m^2 A. \end{aligned} \tag{4.53}$$

Now remember the form of the Laplace operator, which is $\triangle = (dd^\dagger + d^\dagger d)$ where $d^\dagger A = *d * A$ when A is of odd degree and $d^\dagger A = - * d * A$ when A is of even degree[4]. In a Lorentzian space with metric convention $(-, +, \cdots, +)$, we have $\Box = +\triangle$[5]. Then $\Box A = +d * d * A + *d * dA$ when A is of odd degree and $\Box A = -d * d * A - *d * dA$ when A is of even degree. The first term vanishes

[3] Note that $* * \omega_n = (-1)^{n(D-n)+t}$ where t is the number of time-like coordinates which is 1 since we are in Lorentzian space-time. So when D is odd $* * \omega_n = -\omega_n$ for all n-forms ω_n.

[4] Recall $d^\dagger \omega_n = (-1)^{D(n+1)+1+t} * d*$, where t is as before.

[5] We can understand this by considering the action of $\triangle$ on a zero-form f: $\triangle f = -\frac{\partial^2}{\partial t^2} + \nabla^2$.

because of the Lorentzian condition (4.44) and the second term gives simply $\pm m^2 A$ according to (4.53). As a result we have

$$(\Box - m^2)A = 0 \tag{4.54}$$

for $D = 4k - 1$ and

$$(\Box + m^2)A = 0 \tag{4.55}$$

for $D = 4k + 1$. In the second case we have a tachyonic wave operator. This problem may be avoided by inserting a factor of i into the self-duality equation:

$$F = im * A \tag{4.56}$$

which requires that the field A is a complex field. This equation can be derived from the Lagrangian

$$\mathcal{L} = m^2 A^* \wedge *A + imA^* \wedge dA. \tag{4.57}$$

Here * refers to complex conjugation. Since A is a complex field it can be written in the form $A = A^1 + iA^2$. When we insert this into (4.57) we obtain

$$\begin{aligned} \mathcal{L} &= m^2(A^1 \wedge *A^1 + A^2 \wedge *A^2) + m(A^2 \wedge dA^1 - A^1 \wedge dA^2) \\ &\quad +im(A^1 \wedge dA^1 + A^2 \wedge dA^2). \end{aligned} \tag{4.58}$$

The term in the second line is a total derivative since A^i are even forms. Also note that $A^1 \wedge dA^2 = -dA^1 \wedge A^2 +$ total derivative. Thus the Lagrangian (4.57) is equivalent to

$$\mathcal{L} = m^2(A^1 \wedge *A^1 + A^2 \wedge *A^2) + 2mA^2 \wedge dA^1. \tag{4.59}$$

There are no derivative terms for the field A^2 so it can be eliminated through its Euler-Lagrange equation which gives

$$A^2 = \frac{1}{m} * dA^1. \tag{4.60}$$

Inserting (4.60) in (4.59) we see that the Lagrangian $\mathcal{L}$ takes the form

$$\mathcal{L} = F^1 \wedge *F^1 + m^2 A^1 \wedge *A^1, \tag{4.61}$$

where $F^1 = dA^1$. Varying this Lagrangian gives the Proca equation $d * F - m^2 * A = 0$. So in $D = 4k + 1$ dimensions the self-duality condition (4.56) is merely a rewriting of the Proca equation and it does not halve the number of degrees of freedom.

4.3 Stückelberg Mechanism

Massive spin-1 particles obey equations which generalise Maxwell equations, known as Proca equations. A massive theory cannot be gauge invariant so neither can be the Proca equation. In this section we will describe a mechanism introduced by Stückelberg, which restores the gauge invariance broken by the mass term (See [74] for a recent review and references). The idea is to introduce an auxiliary field which couples to the theory in such a way that the Lagrangian is now manifestly gauge invariant and also is equivalent to the Proca Lagrangian in a fixed gauge, when the auxiliary field has been gauged to zero.

Consider the Proca equation

$$(\Box - m^2) A_{n-1} = 0. \tag{4.62}$$

This equation is derived from the Lagrangian

$$\mathcal{L} = F_n \wedge *F_n + m^2 A_{n-1} \wedge *A_{n-1}, \tag{4.63}$$

This Lagrangian is not gauge invariant under the antisymmetric tensor gauge invariance of the field A_{n-1}:

$$\delta A_{n-1} = d\lambda_{n-2}. \tag{4.64}$$

To recover the gauge invariance we introduce an auxiliary field B_{n-2} in the action such that the Lagrangian is now

$$\mathcal{L} = F_n \wedge *F_n + H_{n-1} \wedge *H_{n-1} \tag{4.65}$$

where

$$\begin{aligned} F_n &= dA_{n-1} \\ H_{n-1} &= dB_{n-2} - mA_{n-1}. \end{aligned} \tag{4.66}$$

Now the action is invariant under the following "massive gauge transformations":

$$\delta A_{n-1} = d\lambda_{n-2}, \qquad \delta B_{n-2} = m\lambda_{n-2}. \tag{4.67}$$

Note that the field B_{n-2} is a pure gauge and it can be gauged away. In this case the Lagrangian (4.65) is equivalent to the Lagrangian (4.63).

Now consider the field equations of (4.65). They are

$$\begin{aligned} d * F + m * H &= 0, \qquad d * H = 0, \quad \text{If n is odd} \\ d * F - m * H &= 0, \qquad d * H = 0, \quad \text{If n is even.} \end{aligned} \tag{4.68}$$

When we fix the gauge to $B_{n-1} = 0$, they give the following equations:

$$\begin{aligned} d * F_n - m^2 * A_{n-1} &= 0, \quad \text{If n is odd} \\ d * F_n + m^2 * A_{n-1} &= 0, \quad \text{If n is even.} \end{aligned} \tag{4.69}$$

Note that these equations yield the Proca equation (4.62).

As a result, we have rewritten the Lagrangians in a gauge invariant way and in the fixed gauge for which $B_{n-2} = 0$, it is equivalent to the original Proca Lagrangian. The field B_{n-1} is called the Stückelberg field.

CHAPTER 5

COMPACTIFICATIONS WITH S-DUALITY TWISTS

Our main interest in this thesis is in reductions in which the monodromy $\mathcal{M} \in G$ is a symmetry of the equations of motion but not the action, acting on the field strengths $\hat{H}_n^i$ via transformations involving Hodge dualities, so that they cannot be realised locally on the fundamental $n-1$ form potentials. In dimensionally reducing a theory with a twist that is a symmetry of the equations of motion and not of the action, one needs to reduce the equations of motion, not the action in general. However, for the cases of interest here one can use the doubled formalism of [29], which we described in the previous chapter. The doubled action can then be dimensionally reduced in the standard way with a twist by the duality symmetry. This greatly simplifies the calculations.

In this chapter, we apply these results to the reduction of supergravity theories in $4, 6, 8$ dimensions, giving rise to supergravity theories in $3, 5, 7$ dimensions with massive self-dual forms [75]. This constructs new supergravity theories in these dimensions and gives a higher-dimensional origin for theories in $3, 5, 7$ dimensions with Chern-Simons actions. In particular, for $D = 3$, A is a vector field and this gives a higher dimensional origin for 3-dimensional gauged supergravity theories, of the type discussed in [76] with Chern-Simons actions for some of the gauge fields.

The plan of this chapter is as follows. In the first section we will consider the twisted reduction of gravity coupled to scalars and gauge potentials which are used in later sections. In section 2 we perform a twisted dimensional reduction in the doubled formalism, and hence obtain the Lagrangian for dimensional reductions with S-duality twists. Finally, in section 5, we apply our results to the reduction of supergravity theories in $4, 6, 8$ dimensions.

5.1 Scherk-Schwarz Reduction

An example which will play a central role in what follows is a theory of gravity coupled to scalars in the coset G/H and a set of r $n-1$ form gauge potentials A^i_{n-1} with n-form field strengths $H^i_n = dA^i_{n-1}$ (where $i = 1, ..., r$) transforming in a real r-dimensional representation of the symmetry group G. We take the vielbein $\mathcal{V}$ on the coset space G/H to be an $r \times r$ matrix acting in the r-dimensional representation of G and consider the theory in $D+1$ dimensions and work with the metric $\mathcal{K}_{ij} = \mathcal{V}_i{}^a \mathcal{V}_j{}^b \delta_{ab}$. The Lagrangian is

$$\mathcal{L} = R * 1 + \frac{1}{4}\text{tr}(d\mathcal{K} \wedge *d\mathcal{K}^{-1}) - \frac{1}{2} H_n^T \mathcal{K} * H_n \tag{5.1}$$

The action is invariant under the rigid G symmetry

$$\delta A \to L^{-1} A, \qquad \delta\mathcal{K} \to L^T \mathcal{K} L \tag{5.2}$$

where $L^i{}_j$ is a G-transformation in the $\mathbf{r}$ representation, and the spacetime metric is invariant. In later sections, we will be particularly interested in the case in which $D + 1 = 2n$, but for now we will keep D, n arbitrary.

We now reduce the Lagrangian (5.1) on a circle with a twist given by a monodromy $\mathcal{M} = e^M \in G$ with the ansatz (2.26). The metric is invariant under the global symmetry group so we use the standard Kaluza-Klein ansatz (2.7) so that the Einstein-Hilbert term in (5.1) reduces to (2.19).

From (2.26) and (5.2) the ansatz for the scalar fields and the 3-form fields is

$$\hat{\mathcal{K}}(x, y) = \lambda^T(y)\mathcal{K}(x)\lambda(y) \tag{5.3}$$

$$\hat{A}_{n-1}(x, y) = \lambda^{-1}(y)[A_{n-1}(x) + A_{n-2}(x) \wedge dy]. \tag{5.4}$$

We have distinguished the $D+1$-dimensional fields from the D-dimensional ones by a hat. Here $\lambda(y) = e^{My}$, where M is the mass matrix in (2.27).

In chapter 2 we argued that the lower dimensional theory is completely determined by the mass matrix M and the distinct reductions are classified by the conjugacy classes of the monodromy matrix $\mathcal{M} = e^M$ [15]. So one has to know the conjugacy classes of the duality group G, used to give twist in the reduction. For example in the case $G = SL(2,\mathbb{R})$ there are three conjugacy classes, namely the hyperbolic, elliptic and parabolic conjugacy classes. So one can employ three distinct reductions by choosing three different mass matrices M corresponding to the three conjugacy classes of $SL(2,\mathbb{R})$. All the other choices of M give rise to equivalent theories up to field redefinitions. The hyperbolic, elliptic and parabolic monodromy matrices are given below together with the corresponding mass matrices:

$$\mathcal{M}_h = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}, \qquad \mathcal{M}_e = \begin{pmatrix} \cos m & \sin m \\ -\sin m & \cos m \end{pmatrix}, \qquad \mathcal{M}_p = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}. \tag{5.5}$$

$$M_h = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \qquad M_e = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \qquad M_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}. \tag{5.6}$$

As we discussed in chapter 2 these matrices generate one-parameter subgroups of $SL(2,\mathbb{R})$ and this subgroup will be the gauge group in the lower dimensional theory. Thus compactification with M_e will give the compact gauging $SO(2)$ whereas compactification with the M_h and M_p will give rise to $SO(1,1)$-gauged lower dimensional theories.

Now we will give the explicit calculations for the reduction of the scalars and the $(n-1)$-form potentials in two separate subsections.

5.1.1 Reduction of the Scalar Fields

It is easier to deal with the reduction of the scalar kinetic term $\frac{1}{4}\mathrm{tr}(d\hat{\mathcal{K}}\wedge * d\hat{\mathcal{K}}^{-1})$ when we write it in the following form:

$$\frac{1}{4}\sqrt{-\hat{g}}\mathrm{tr}\left[(\partial_{\hat{\mu}}\hat{\mathcal{K}})(\partial_{\hat{\nu}}\hat{\mathcal{K}}^{-1})\hat{g}^{\hat{\mu}\hat{\nu}}\right] \tag{5.7}$$

Here $\hat{g}^{\hat{\mu}\hat{\nu}}$ is the inverse of the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ in $D+1$ dimensions and we gave in (2.9) how it reduces to D dimensions. We have

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{-2\alpha\phi}g^{\mu\nu} & -e^{-2\alpha\phi}\mathcal{A}^{\mu} \\ -e^{-2\alpha\phi}\mathcal{A}^{\mu} & -e^{-2\alpha\phi}\mathcal{A}_{\mu}\mathcal{A}^{\mu} + e^{-2\beta\phi} \end{pmatrix} \tag{5.8}$$

If we had the standard KK ansatz the kinetic term for the scalar fields would simply reduce to $\frac{1}{4}\mathrm{tr}(d\mathcal{K}\wedge * d\mathcal{K}^{-1})$. However now $\hat{\mathcal{K}}$ depends on the internal coordinate y so the $\partial_y\hat{\mathcal{K}}$ terms also contribute. To see the form of the terms coming from the y-dependence of $\hat{\mathcal{K}}$, we substitute (5.8) in (5.7). Also using (2.17) we find

$$\begin{aligned}&\frac{1}{4}e^{2\alpha\phi}\sqrt{-g}\mathrm{tr}\Big[(\partial_\mu\hat{\mathcal{K}})(\partial_\nu\hat{\mathcal{K}}^{-1})(g^{\mu\nu}e^{-2\alpha\phi}) - (\partial_\mu\hat{\mathcal{K}})(\partial_y\hat{\mathcal{K}}^{-1})(\mathcal{A}^{\mu}e^{-2\alpha\phi}) \\ &-(\partial_y\hat{\mathcal{K}})(\partial_\nu\hat{\mathcal{K}}^{-1})(\mathcal{A}^{\nu}e^{-2\alpha\phi}) + (\partial_y\hat{\mathcal{K}})(\partial_y\hat{\mathcal{K}}^{-1})(-e^{-2\alpha\phi}\mathcal{A}_\mu\mathcal{A}^\mu + e^{-2\beta\phi})\Big]\end{aligned} \tag{5.9}$$

Using (5.3), now one can show that the y-derivative terms bring the following contributions:

$$\begin{aligned}\mathrm{tr}\left[(\partial_\mu\hat{\mathcal{K}})(\partial_y\hat{\mathcal{K}}^{-1})\mathcal{A}^{\mu}\right] &= -\mathrm{tr}\left[(\partial_\mu\mathcal{K})(M\mathcal{K}^{-1} + \mathcal{K}^{-1}M^T)\mathcal{A}^{\mu}\right] \\ \mathrm{tr}\left[(\partial_y\hat{\mathcal{K}})(\partial_\nu\hat{\mathcal{K}}^{-1})\mathcal{A}^{\nu}\right] &= \mathrm{tr}\left[(M^T\mathcal{K} + \mathcal{K}M)\mathcal{A}^{\mu}(\partial_\mu\mathcal{K}^{-1})\right] \\ \mathrm{tr}\left[(\partial_y\hat{\mathcal{K}})(\partial_y\hat{\mathcal{K}}^{-1})\right] &= -2\mathrm{tr}\left[M^2 + M^T\mathcal{K}M\mathcal{K}^{-1}\right].\end{aligned} \tag{5.10}$$

So we find that the reduction of the scalar kinetic term $\frac{1}{4}\mathrm{tr}(d\hat{\mathcal{K}}\wedge * d\hat{\mathcal{K}}^{-1})$ in $D+1$ dimensions gives a kinetic term for D-dimensional scalar fields plus a scalar potential in D dimensions, as promised in chapter 2. These terms can be read from above to be:

$$\mathcal{L}_s = \frac{1}{4}\mathrm{tr}(\mathcal{DK} \wedge *\mathcal{DK}^{-1}) + V(\phi) \tag{5.11}$$

where we have defined

$$\begin{aligned} \mathcal{DK} &= d\mathcal{K} - (M^T\mathcal{K} + \mathcal{K}M)\mathcal{A} \\ \mathcal{DK}^{-1} &= d\mathcal{K}^{-1} + (M\mathcal{K}^{-1} + \mathcal{K}^{-1}M^T)\mathcal{A} \end{aligned} \tag{5.12}$$

and the scalar potential $V(\phi)$ is

$$V(\phi) = -\frac{1}{2}e^{2(D-1)\alpha\varphi}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K}) * 1 \tag{5.13}$$

5.1.2 Reduction of the $(n-1)$-Form Potentials

We first write the ansatz (5.4) for the reduction of the $(n-1)$-form potential field in the Einstein frame as

$$\hat{A}_{n-1}(x,y) = e^{-My}[A_{n-1}(x) - A_{n-2}(x) \wedge \mathcal{A} + A_{n-2}(x) \wedge (dy + \mathcal{A})] \tag{5.14}$$

Taking the derivative of both sides we obtain

$$\begin{aligned} d\hat{A}_{n-1} &= e^{-My}[dA_{n-1} - dA_{n-2} \wedge \mathcal{A} + dA_{n-2} \wedge (dy + \mathcal{A})] \\ &\quad -Me^{-My}((-1)^{n-1}A_{n-1} \wedge dy) \end{aligned} \tag{5.15}$$

Adding and subtracting a term $(-1)^{n-1}Me^{-My}A_{n-1} \wedge \mathcal{A}$ we obtain

$$\begin{aligned} d\hat{A}_{n-1} &= e^{-My}\{dA_{n-1} - [dA_{n-2} - (-1)^{n-1}MA_{n-1}] \wedge \mathcal{A}\} \\ &\quad -Me^{-My}\{(-1)^{n-1}A_{n-1} \wedge (dy + \mathcal{A})\} + e^{-My}dA_{n-2} \wedge (dy + \mathcal{A}) \end{aligned} \tag{5.16}$$

As a result we get for $\hat{H}_n = d\hat{A}_{n-1}$ the following

$$\hat{H}_n(x,y) = e^{-My}H_n(x) + e^{-My}H_{n-1}(x)(dy + \mathcal{A}) \tag{5.17}$$

where the D-dimensional field strengths are

$$H_{n-1}(x) = dA_{n-2} - (-1)^{n-1} M A_{n-1}, \qquad H_n(x) = dA_{n-1} - H_{n-1} \wedge \mathcal{A}. \quad (5.18)$$

The reduction of the kinetic term can be done easily by using (2.23):

$$\hat{H}_n^T \hat{\mathcal{K}} \wedge \hat{*} \hat{H}_n \to [e^{-2(n-1)\alpha\varphi} H_n^T \mathcal{K} \wedge * H_n + e^{2(D-n)\alpha\varphi} H_{n-1}^T \mathcal{K} \wedge * H_{n-1}] \wedge dy. \quad (5.19)$$

The field strengths (5.18) are invariant under the following massive gauge transformations:

$$\delta A_{n-1} = d\Lambda, \qquad \delta A_{n-2} = (-1)^{n-1} M \Lambda. \quad (5.20)$$

This allows for the following gauge transformation

$$A_{n-1} \to A_{n-1} + (-1)^{n-1} M^{-1} dA_{n-2}. \quad (5.21)$$

After this gauge transformation the D-dimensional field strengths become

$$H_n = \mathcal{D} A_{n-1} = dA_{n-1} - (-1)^n M A_{n-1} \wedge \mathcal{A} \quad (5.22)$$

$$H_{n-1} = (-1)^n M A_{n-1}. \quad (5.23)$$

Then A_{n-2} disappears from the theory, and the term $H_{n-1} \wedge * H_{n-1}$ is a mass term for A_{n-1}. The degrees of freedom represented by the r fields A_{n-2} have been absorbed by the r $(n-1)$-form fields A_{n-1} which have become massive. Now $H_n = \mathcal{D} A_{n-1}$ is a gauge covariant derivative where the gauge group is the subgroup of G generated by M and the corresponding gauge field is the graviphoton $\mathcal{A}$.

Recall that a massive theory which is not gauge invariant can be made gauge invariant by the Stückelberg mechanism where one adds an auxiliary field (see section 4.3). Then the original field and the Stückelberg field are invariant under massive gauge transformations of the form (5.20) and in a fixed gauge where the auxiliary field is gauged away we recover the original theory. In the present case, the field A_{n-2} plays the role of a Stückelberg field which carries auxiliary degrees of freedom. By performing the gauge transformation (5.20) we go to a gauge in which this field is set to zero and hence the auxiliary degrees of freedom are removed. However, since the number of degrees of freedom should be invariant, now the fields A_{n-1} should carry extra degrees of freedom and indeed they do by becoming massive. We say that the fields A_{n-1} have 'eaten' A_{n-2} and become massive by analogy with the Higgs mechanism.

Note that the gauge transformation (5.21) is allowed only when M is invertible. When M is not invertible it is useful to work with flat indices $H^a = \mathcal{V}^a_{\ i} H^i$. Then $H^a = \mathcal{D}A^a = dA^a + \omega^a_{\ b} A^b$ where ω is the connection 1-form $\omega^a_{\ b} = \mathcal{V}^a_{\ i}(d\mathcal{V}^{-1})^i_{\ b}$. We also introduce $\bar{H}_a = (\Omega^{-1})_{ab} H^b$ and $M^{ab} = M^a_{\ c}\Omega^{cb}$. Here Ω is the G-invariant matrix which is symmetric if n is odd and anti-symmetric if n is even

$$\Omega^{ab} = (-1)^{n-1}\Omega^{ba}. \tag{5.24}$$

Now one has

$$H^a_{n-1} = \mathcal{D}A^a_{n-2} + (-1)^{n-1} M^{ab} \bar{A}_{(n-1)b}, \qquad \bar{H}_{(n)a} = \mathcal{D}\bar{A}_{(n-1)a} \tag{5.25}$$

where we have dropped the coupling terms to the graviphoton $\mathcal{A}$.

Note that $\mathcal{M} = e^M$ and $\mathcal{M}^T\Omega^{-1}\mathcal{M} = \Omega^{-1}$ since $\mathcal{M} \in G$ and Ω is G-invariant. (For complex representations, the condition is $\mathcal{M}^\dagger\Omega^{-1}\mathcal{M} = \Omega^{-1}$.) So the mass matrix M satisfies:

$$M^T\Omega^{-1} + \Omega^{-1}M = 0. \tag{5.26}$$

From (5.24) and (5.26) it follows that M is a symmetric matrix if n is even and antisymmetric if n is odd:

$$M^{ab} = (-1)^n M^{ba}. \tag{5.27}$$

Let the dimension of $ker(M)$ be l. Now the matrix M^{ab} can be brought into the canonical form

$$M^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & m^{\alpha'\beta'} \end{pmatrix} \tag{5.28}$$

where $m^{\alpha'\beta'}$ is an invertible, $(r-l)\times(r-l)$ matrix which is diagonal if n is even and skew-diagonal if n is odd. Here we have split the indices $a \to (\alpha, \alpha')$ where α runs from 1 to l and α' runs from $l+1$ to r. Similarly the gauge fields A can be written in the block form

$$A = \begin{pmatrix} A^0 \\ A^1 \end{pmatrix} \tag{5.29}$$

where $A^0 = (A^\alpha)$ is the projection into $ker(M)$, so that $MA = MA^1$. Now performing the gauge transformation

$$\bar{A}_{(n-1)\alpha'} \to \bar{A}_{(n-1)\alpha'} + (-1)^{n-1}(m^{-1})_{\alpha'\beta'}\mathcal{D}A^{\beta'}_{n-2} \tag{5.30}$$

one sees that the $r-l$ fields $\bar{A}_{(n-1)\alpha'}$ become massive, having eaten the $r-l$ fields $A^{\alpha'}_{n-2}$, while A^{α}_{n-2} and $\bar{A}_{(n-1)\alpha}$ both remain in the theory as massless gauge fields, with l of each. Now the field strengths for the $(n-2)$-form fields in (5.25) become

$$H^{\alpha'}_{n-1} = (-1)^n m^{\alpha'\beta'}\bar{A}_{(n-1)\beta'}, \qquad H^{\alpha}_{n-1} = \mathcal{D}A^{\alpha}_{n-2}. \tag{5.31}$$

In the $G = SL(2,\mathbb{R})$ case the parabolic mass matrix M_p is not invertible, and has a one-dimensional kernel, i.e. $r = 2, l = 1$, so that α and α' both take only one value and $A^a = (A^1, A^{1'})$. In this case the matrix $m^{\alpha'\beta'}$ in (5.28) is the 1×1 matrix $(-m)$ and from the gauge transformation (5.30) it can be seen that the $(n-1)$-form $\bar{A}_{n-1\ 1'}$ eats the $(n-2)$-form $A^{1'}_{n-2}$ and becomes massive. The remaining $n-1$ form $\bar{A}_{n-1\ 1}$ and $n-2$ form A^1_{n-2} gauge fields remain massless.

5.1.3 The D-Dimensional Lagrangian

Collecting all the results in the previous subsections we can now write down the D-dimensional Lagrangian $\mathcal{L}_D$:

$$\mathcal{L}_D = \mathcal{L}_g + \mathcal{L}_b + \mathcal{L}_s \tag{5.32}$$

where

$$\begin{aligned}
\mathcal{L}_g &= R * 1 - \frac{1}{2} d\varphi \wedge * d\varphi - \frac{1}{2} e^{-2(D-1)\alpha\varphi} \mathcal{F}_2 \wedge * \mathcal{F}_2 \\
\mathcal{L}_b &= -\frac{1}{2} e^{-2(n-1)\alpha\varphi} H_n^T \mathcal{K} \wedge * H_n - \frac{1}{2} e^{2(D-n)\alpha\varphi} H_{n-1}^T \mathcal{K} \wedge * H_{n-1} \\
\mathcal{L}_s &= \frac{1}{4} \mathrm{tr}(\mathcal{D}\mathcal{K} \wedge * \mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2} e^{2(D-1)\alpha\varphi} \mathrm{tr}(M^2 + M\mathcal{K}^{-1} M^T \mathcal{K}) * 1
\end{aligned} \tag{5.33}$$

Here the field strengths H_n and H_{n-1} are as in (5.18). When M is invertible one can perform the gauge transformation (5.21) after which they become as in (5.22) and (5.23) and the second term in $\mathcal{L}_b$ is a mass term for the gauge fields A_{n-1}. When M is not invertible, one can do the gauge transformation (5.30) and the field strengths are now as in (5.31). Then the $\mathcal{L}_b$ term in (5.33) can be replaced by the following Lagrangian:

$$\begin{aligned}
\mathcal{L}_b = & -\frac{1}{2} e^{-2(n-1)\alpha\varphi} \delta^{\alpha\beta} \bar{H}_{(n)\alpha} \wedge * \bar{H}_{(n)\beta} - \frac{1}{2} e^{-2(n-1)\alpha\varphi} \delta^{\alpha'\beta'} \bar{H}_{(n)\alpha'} \wedge * \bar{H}_{(n)\beta'} \\
& -\frac{1}{2} e^{2(D-n)\alpha\varphi} \delta_{\alpha\beta} \mathcal{D} A^{\alpha}_{n-2} \wedge * \mathcal{D} A^{\beta}_{n-2} - \frac{1}{2} e^{2(D-n)\alpha\varphi} (m^T m)^{\alpha'\beta'} \bar{A}_{(n-1)\alpha'} \wedge * \bar{A}_{(n-1)\beta'}.
\end{aligned} \tag{5.34}$$

Here we have chosen the normalisation of Ω so that $\Omega^{ac}\Omega^{bd}\delta_{cd} = \delta^{ab}$.

5.2 Reduction with Duality Twist

The theory given by the Lagrangian (5.1) has a global symmetry G of the equations of motion which acts via duality transformations. In this section we will dimensionally reduce on a circle from $D+1 = 2n$ to D dimensions with a twist that has monodromy $\mathcal{M}$ in G. For some choices of monodromy $\mathcal{M}$ in G, this is in fact a symmetry of the action and this is a standard Scherk-Schwarz reduction, as in section 2. If it is only a symmetry of the equations of motion, then we use the doubled formalism of chapter 4 with action (4.16) supplemented by the constraint (4.29). The action (4.16) is of the same form as (5.1), so the Scherk-Schwarz reduction of the action proceeds as in the previous section. This is supplemented by the constraints arising from the dimensional reduction of (4.29). The field equations in $2n-1$ dimensions are then those from the reduced action together with the reduced constraints, and we go on to seek an action in $2n-1$ dimensions that gives both the constraints and the reduced field equations.

5.2.1 Dimensional Reduction in the Doubled Formalism

The Lagrangian (4.16) in the doubled formalism is of the same form as (5.1), but with an extra factor of 1/2 in the normalisation of the gauge field kinetic term. The Scherk-Schwarz reduction of a Lagrangian of this form was already discussed in the previous section, where we showed that it yields the Lagrangian (5.33) in D dimensions. Just as the action (4.16) should be supplemented by the $D+1$ dimensional constraint (4.29) in order to give the correct $D+1$ dimensional field equations, the D dimensional Lagrangian (5.33) should be supplemented by a constraint, which is to be obtained by the dimensional reduction of (4.29). In this section we will describe the SS reduction of the $D+1$-dimensional constraint (4.29). Note that it is G-covariant, so the y dependence of the fields in the ansatz (2.26) cancels out in the reduction.

Using the ansatz (5.3), (5.4) the $D+1$ dimensional constraint (4.29) reduces to the D-dimensional constraint:

$$H_n = e^{2(D-n)\alpha\varphi} Q * H_{n-1} \tag{5.35}$$

where Q is as in (4.30) and $\mathcal{K}$ is given by (4.31). The constraint (5.35) can be rewritten in flat indices as

$$\bar{H}_{(n)a} = e^{2(D-n)\alpha\varphi}\delta_{ab} * (\mathcal{D}A^b_{n-2} + (-1)^n M^{bc}\bar{A}_{(n-1)c}). \tag{5.36}$$

As a result, the n-form field strengths are dual to the $n-1$-form field strengths. For an untwisted reduction (i.e. one with $M = 0$, so that it is a standard reduction) this constraint can be used to eliminate the $2k$ potentials A_{n-1} so that the theory can be written in terms of the $2k$ potentials A_{n-2} (or alternatively the potentials A_{n-2} can be eliminated and the theory written in terms of the A_{n-1}, or more generally in terms of s potentials A_{n-2} and $2k-s$ potentials A_{n-1}). In the twisted case with invertible M, one can go to the gauge in which the fields A^I_{n-2} are set zero, as was discussed in section 2. In this gauge the field strengths H_n and H_{n-1} are given in (5.22) and (5.23) so that the duality condition (5.35) is:

$$DA_{n-1} = (-1)^n e^{2(D-n)\alpha\varphi}\tilde{M} * A_{n-1} \tag{5.37}$$

where $\tilde{M} = QM$. This is a massive self-duality condition for the $2k$ potentials A_{n-1}. Such self-duality conditions in odd dimensions were introduced in [32] and occur certain odd dimensional gauged supergravity theories. The self-duality constraint (5.37) implies the massive field equation (suppressing non-linear terms)

$$*D * DA_{n-1} = e^{4(D-n)\alpha\varphi}\tilde{M}^2 A_{n-1} + \ldots \tag{5.38}$$

with mass matrix proportional to $\tilde{M}^2$. However, the constraint (5.37) halves the number of degrees of freedom of a massive $n-1$ form field.

It is instructive to check the number of physical degrees of freedom. In d dimensions a massless p form gauge field A_p has c^{d-2}_p degrees of freedom, where

$$c^s_p = \frac{(s)!}{p!(s-p)!}$$

while a massive p form gauge field has c^{d-1}_p degrees of freedom. The k gauge fields A^I_{n-1} in $2n$ dimensions have kc^{2n-2}_{n-1} degrees of freedom, which can be represented by the $2k$ gauge fields A^i_{n-1} (with $2kc^{2n-2}_{n-1}$ degrees of freedom) together with k constraints (4.29) that halve the number of degrees of freedom again. In an untwisted reduction, each massless $n-1$ form gauge field in $2n$ dimensions gives rise to a massless $n-1$ form gauge field and a massless $n-2$ form gauge field, and the number of degrees of freedom is correct as

$$c^{2n-2}_{n-1} = c^{2n-3}_{n-1} + c^{2n-3}_{n-2}$$

However, the number of degrees of freedom of a massive p-form in $d-1$ dimensions is c^{d-2}_p, which is the same as the number of degrees of freedom of a massless p-form in d dimensions, and in the twisted reduction with invertible M, all the $n-1$ forms in $2n$ dimensions give rise to massive $n-1$ forms in $2n-1$ dimensions. We have $2k$ massive gauge fields A^i_{n-1} in $2n-1$ dimensions which have $2kc^{2n-2}_{n-1}$ degrees of freedom, but the self-duality constraints (5.37) remove half of the degrees of freedom, leaving kc^{2n-2}_{n-1} degrees of freedom, as required.

When M is not invertible, the field strengths are given by (5.31). Then the constraint (5.36) takes the form (dropping the coupling to the graviphoton again)[1]

$$\begin{aligned} \mathcal{D}\bar{A}_{(n-1)\alpha'} &= (-1)^n e^{2(D-n)\alpha\varphi}\delta_{\alpha'\beta'} * m^{\beta'\gamma'}\bar{A}_{(n-1)\gamma'} &(5.39)\\ \mathcal{D}\bar{A}_{(n-1)\alpha} &= e^{2(D-n)\alpha\varphi}\delta_{\alpha\beta} * \mathcal{D}A^{\beta}_{n-2} &(5.40)\end{aligned}$$

Before imposing the constraint, the $r-l$ fields $\bar{A}_{(n-1)\alpha'}$ are massive, having eaten the $r-l$ fields $A^{\alpha'}_{n-2}$, while A^{α}_{n-2} and $\bar{A}_{(n-1)\alpha}$ both remain in the theory as massless gauge fields, with l of each, as was seen in section 2. So before imposing the constraint the total number of degrees of freedom is

$$(r-l)c^{2n-2}_{n-1} + lc^{2n-3}_{n-2} + lc^{2n-3}_{n-1} = rc^{2n-2}_{n-1} = 2kc^{2n-2}_{n-1}$$

Imposing the constraint imposes self-duality on the massive fields $\bar{A}_{(n-1)\alpha'}$, halving the number of degrees of freedom, and relates $\bar{A}_{(n-1)\alpha}$ to A^{α}_{n-2}, so that half of

[1] Note that the α appearing in the exponential function is the constant in (2.8), not an index.

them can be eliminated (e.g. A^{α}_{n-2} can be eliminated leaving $\bar{A}_{(n-1)\alpha}$, or $\bar{A}_{(n-1)\alpha}$ can be eliminated leaving A^{α}_{n-2}). Thus one is left with kc^{2n-2}_{n-1} degrees of freedom, as required.

The field equations from the D-dimensional action (5.33) (noting that the factors of 1/2 in $\mathcal{L}_b$ should be replaced by 1/4, since we are reducing the doubled Lagrangian) are supplemented by the D dimensional constraint (5.35). This implies that the field strengths H_n and H_{n-1} in (5.33) are not independent but are related via the duality condition (5.35). Note that if this constraint were applied to the action, it would make the $\mathcal{L}_b$ part of (5.32) vanish. This was to be expected as the twisted self-duality condition (4.29), from which the duality condition (5.35) is obtained, implies the vanishing of the gauge kinetic term in the $D+1$-dimensional doubled Lagrangian (4.16). Thus it is important that one first varies the action corresponding to (5.33) and then imposes the constraint (5.35).

It is straightforward to verify that the field equations derived from $\mathcal{L}_b$ for the potentials A^a_{n-1} are consistent with the D dimensional constraint (5.35). This field equation is

$$d(e^{-2(n-1)\alpha\varphi}\mathcal{K}_{aj} * H^{(j)}_n) + (-1)^n(e^{-2(n-1)\alpha\varphi}M^i{}_a\mathcal{K}_{ij} \wedge *H^{(j)}_n) \wedge \mathcal{A}$$
$$+(-1)^n e^{2(D-n)\alpha\varphi}M^i{}_a\mathcal{K}_{ij} \wedge *H^{(j)}_{n-1} = 0. \tag{5.41}$$

where $a, i, j = 1, \cdots 2k$ are G indices. Now we insert the constraint (5.37) and we obtain

$$-\Omega_{ak}M^k{}_l dA^l_{n-1} - M^i{}_a\Omega_{il}[H^l_n + H^l_{n-1} \wedge \mathcal{A}] = 0. \tag{5.42}$$

Hence consistency is possible if the following is satisfied:

$$\Omega_{ak}M^k{}_l = -M^i{}_a\Omega_{il}. \tag{5.43}$$

for all a and l. This is equivalent to the equation (5.26), which we showed to hold for all mass matrices in subsection 4.1.2.

5.2.2 Lagrangian for Reduced Theory

The odd dimensional massive self-duality condition can be obtained from a Chern-Simons action of the form (1.5). In this section we will seek an action of this form from which the D-dimensional constraint (5.37) and the D dimensional field equations can be obtained. In the case in which M is invertible, the D-dimensional constraint (5.37) follows from the following Lagrangian:

$$\mathcal{L}'_b = \frac{1}{2}(\Omega^{-1}M)_{ij}[(-1)^{n-1}A^{(i)}_{n-1} \wedge DA^{(j)}_{n-1} + e^{2(D-n)\alpha\varphi}\tilde{M}^j{}_k A^{(i)}_{n-1} \wedge *A^{(k)}_{n-1}]. \quad (5.44)$$

We can see this as follows. Consider the variation of (5.44) with respect to the field A^a_{n-1}:

$$\begin{aligned} &\tfrac{1}{2}(-1)^{n-1}[(\Omega^{-1}M)_{aj} + (-1)^n(\Omega^{-1}M)_{ja}]dA^j_{n-1} \\ &+\tfrac{1}{2}[(\Omega^{-1}M^2)_{aj} + (-1)^{n-1}(\Omega^{-1}M^2)_{ja}]A^j \wedge \mathcal{A} \\ &+\tfrac{1}{2}e^{2(D-n)\alpha\varphi}[(\Omega^{-1}M\tilde{M})_{aj} + (\Omega^{-1}M\tilde{M})_{ja}] * A^j_{n-1} = 0. \end{aligned} \quad (5.45)$$

So the conditions that we should have are:

1. $(\Omega^{-1}M)^T = (-1)^n(\Omega^{-1}M)$
2. $(\Omega^{-1}M^2)^T = (-1)^{n-1}(\Omega^{-1}M^2)$
3. $(\Omega^{-1}M\tilde{M})^T = (\Omega^{-1}M\tilde{M})$.

The first and the second conditions can be shown to hold easily by using (5.26) and (5.24). For the third condition, one uses the facts that $\Omega^{-1}M\tilde{M} = -M^T\Omega^{-1}\tilde{M} = -M^T\Omega^{-1}\Omega\mathcal{K}M = -M^T\mathcal{K}M$ and the matrix $\mathcal{K}$ is symmetric. As a result we have

$$\delta_A\mathcal{L}'_b = (\Omega^{-1}M)_{aj}DA^j + (-1)^{n-1}e^{2(D-n)\alpha\varphi}(\Omega^{-1}M\tilde{M})_{ak} * A^k = 0. \quad (5.46)$$

Multiplying both sides with $(\Omega^{-1}M)^{la}$, which is possible when M is invertible, we obtain the constraint (5.37).

Now consider the Lagrangian

$$\mathcal{L}'_D = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}'_b \tag{5.47}$$

where $\mathcal{L}_g$ and $\mathcal{L}_s$ are as in (5.33). Our claim is that this Lagrangian $\mathcal{L}'_D$ is equivalent to the Lagrangian $\mathcal{L}_D$ in that they yield the same field equations. The field equations for the potential fields A_{n-1} have already been discussed. Below we will discuss the field equations for the other fields, that is, for the metric and the scalar fields.

First consider the field equations for the metric. We have to show that

$$\delta_g \mathcal{L}_b = \delta_g \mathcal{L}'_b \tag{5.48}$$

when the reduced equation of constraint (5.35) is imposed. First note that in a given coordinate chart we have

$$\begin{aligned} H_n^T \mathcal{K} \wedge *H_n &= \mathcal{K}_{ij}\sqrt{-g} H^{(i)}_{\mu_1\cdots\mu_n} H^{(j)\mu_1\cdots\mu_n} \\ &= \mathcal{K}_{ij}\sqrt{-g}\; g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n} H^{(i)}_{\mu_1\cdots\mu_n} H^{(j)}{}_{\nu_1\cdots\nu_n}. \end{aligned} \tag{5.49}$$

Using the relations

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu} \tag{5.50}$$

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \tag{5.51}$$

one can show that

$$\begin{aligned} \frac{\delta}{\delta g^{\alpha\beta}}(H_n^T \mathcal{K} \wedge *H_n) &= \mathcal{K}_{ij}\sqrt{-g}(-n H_n^{(i)\alpha\mu_1\cdots\mu_{n-1}}\; H_n^{(j)\beta}{}_{\mu_1\cdots\mu_{n-1}} \\ &\quad + \frac{1}{2} g^{\alpha\beta} H_n^{(i)\mu_1\cdots\mu_n} H^{(j)}_{n\;\mu_1\cdots\mu_n} \end{aligned} \tag{5.52}$$

Now it easily follows that the variation of the Lagrangian $\mathcal{L}_b$ with respect to the metric consists of the two terms below:

$$\frac{1}{4n!} e^{-2(n-1)\alpha\varphi} \mathcal{K}_{ij}\sqrt{-g}\left(n H_n^{(i)\alpha\mu_1\cdots\mu_{n-1}}\; H_n^{(j)\beta}{}_{\mu_1\cdots\mu_{n-1}} - \frac{1}{2} g^{\alpha\beta} H_n^{(i)\mu_1\cdots\mu_n} H^{(j)}_{n\mu_1\cdots\mu_n}\right) \tag{5.53}$$

$$+\frac{e^{2(D-n)\alpha\varphi}}{4(n-1)!}\mathcal{K}_{ij}\sqrt{-g}\left((n-1)H_{n-1}^{(i)\alpha\mu_1\cdots\mu_{n-2}}\ H_{n-1\ \mu_1\cdots\mu_{n-2}}^{(j)\beta}-\frac{1}{2}g^{\alpha\beta}H_{n-1}^{(i)\mu_1\cdots\mu_{n-1}}H_{n-1\mu_1\cdots\mu_{n-1}}^{(j)}\right) \tag{5.54}$$

Now we will insert the constraint (5.35), which can be written in the component form as

$$H_{n\ \mu_1\cdots\mu_n}^{(i)}=(-1)^n e^{2(D-n)\alpha\varphi}\frac{1}{(n-1)!}\tilde{M}^i{}_k\epsilon^{\lambda_1\cdots\lambda_{n-1}}{}_{\mu_1\cdots\mu_n}A_{n-1\ \lambda_1\cdots\lambda_{n-1}}^{(k)} \tag{5.55}$$

Here we have used (5.23). Using the relation[2]

$$\frac{1}{(N-m)!m!}\ \epsilon^{a_1\cdots a_m r_1\cdots r_{N-m}}\epsilon_{b_1\cdots b_m r_1\cdots r_{N-m}}=(-1)^t\ \delta_{[b_1\cdots b_m]}^{a_1\cdots a_m} \tag{5.56}$$

one can show that

$$\begin{aligned}\frac{\delta\mathcal{L}_b}{\delta g^{\alpha\beta}} &= \frac{e^{2(D-n)\alpha\varphi}}{2(n-1)!}\sqrt{-g}\mathcal{K}_{ij}M^i{}_kM^j{}_l\left((n-1)A_{(n-1)}^{(k)\alpha\sigma_1\cdots\sigma_{n-2}}A_{(n-1)\sigma_1\cdots\sigma_{n-2}}^{(l)\beta}\right.\\ &\qquad\left.-\frac{1}{2}g^{\alpha\beta}A_{(n-1)\sigma_1\cdots\sigma_{n-1}}^{(k)}A_{(n-1)}^{(l)\sigma_1\cdots\sigma_{n-1}}\right) \qquad (5.57)\\ &= -\frac{1}{2}e^{2(D-n)\alpha\varphi}\frac{\delta}{\delta g^{\alpha\beta}}\left(\mathcal{K}_{ij}M_k^iM_l^jA_{n-1}^{(k)}\wedge *A_{n-1}^{(l)}\right)=\frac{\delta\mathcal{L}_b'}{\delta g^{\alpha\beta}}.\end{aligned}$$

In the last line we have used the equality $\Omega^{-1}M\tilde{M}=-M^T\mathcal{K}M$, as we discussed before.

Now we check the field equations for the scalar fields. Let κ represent any of the scalar fields in the theory except for the Kaluza-Klein field φ. Then

$$\begin{aligned}\delta_\kappa\mathcal{L}_b &= -\frac{1}{4}e^{-2(n-1)\alpha\varphi}H_n^T\frac{\delta\mathcal{K}}{\delta\kappa}\wedge *H_n-\frac{1}{4}e^{2(D-n)\alpha\varphi}H_{n-1}^T\frac{\delta\mathcal{K}}{\delta\kappa}\wedge *H_{n-1}\\ &= -\frac{e^{2(D-n)\alpha\varphi}}{4}A_{n-1}^T\tilde{M}^T\frac{\delta\mathcal{K}}{\delta\kappa}\tilde{M}\wedge *A_{n-1}-\frac{e^{-2(n-1)\alpha\varphi}}{4}A_{n-1}^TM^T\frac{\delta\mathcal{K}}{\delta\kappa}M\wedge *A_{n-1}\end{aligned}$$

In the second line we have imposed the constraint (5.35). The first term is equal to the second term. One can see this as follows:

$$\begin{aligned}\tilde{M}^T\frac{\delta\mathcal{K}}{\delta\kappa}\tilde{M} &= M^T\mathcal{K}^T\Omega^T\frac{\delta\mathcal{K}}{\delta\kappa}\Omega\mathcal{K}M=M^T\mathcal{K}^T\Omega\mathcal{K}\Omega\frac{\delta\mathcal{K}}{\delta\kappa}M\\ &= M^T\mathcal{K}^T\mathcal{K}^{-1}\Omega\Omega^{-1}\frac{\delta\mathcal{K}}{\delta\kappa}M=M^T\frac{\delta\mathcal{K}}{\delta\kappa}M. \qquad (5.58)\end{aligned}$$

[2] Here t is the number of timelike coordinates and $t=1$ in our case.

Above we have used the following: a)$\mathcal{K}^T\Omega\mathcal{K} = \Omega$, b)$\mathcal{K}^T = \mathcal{K}$, c)$\frac{\delta\mathcal{K}}{\delta\kappa}\ \Omega\mathcal{K} = (-1)^{n-1}\mathcal{K}\Omega\ \frac{\delta\mathcal{K}}{\delta\kappa}$, d)$\Omega^T = (-1)^{n-1}\Omega$ and e)$Q^2 = (\Omega\mathcal{K})^2 = (-1)^{n-1}\mathbb{I}$

As a result we get

$$\delta_\kappa\mathcal{L}_b = -\frac{1}{2}e^{2(D-n)\alpha\varphi}A_{n-1}^T M^T \frac{\delta\mathcal{K}}{\delta\kappa} M \wedge *A_{n-1} = \delta_\kappa\mathcal{L}_b' \tag{5.59}$$

where the last equality follows from the fact that $\Omega^{-1}M\tilde{M} = -M^T\mathcal{K}M$.

The last thing to be checked is the equality of the field equations of $\mathcal{L}_D$ and $\mathcal{L}_D'$ for the Kaluza-Klein field φ:

$$\begin{aligned}\delta_\varphi\mathcal{L}_b &= \frac{1}{2}(n-1)\alpha[e^{-2(n-1)\alpha\varphi}H_n^T\mathcal{K}\wedge *H_n - e^{2(D-n)\alpha\varphi}H_{n-1}^T\mathcal{K}\wedge *H_{n-1}] \\ &= -(n-1)\alpha e^{2(D-n)\alpha\varphi}H_{n-1}^T\mathcal{K}\wedge *H_{n-1} \\ &= -(n-1)\alpha e^{2(D-n)\alpha\varphi}A_{n-1}^T M^T\mathcal{K}M\wedge *A_{n-1} = \delta_\varphi\mathcal{L}_b'.\end{aligned} \tag{5.60}$$

In the second line we imposed the constraint (5.35) and used the fact that $\tilde{M}^T\mathcal{K}\tilde{M} = M^T\mathcal{K}M$. In the third line we used (5.23). Finally in the forth line we used the fact that $D - n = n - 1$.

As a result we have a new D-dimensional Lagrangian which yields the D-dimensional field equations and also the constraint:

$$\begin{aligned}\mathcal{L}_D &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 \\ &\quad +\frac{1}{2}(\Omega^{-1}M)_{ij}[(-1)^{n-1}A_{n-1}^{(i)}\wedge DA_{n-1}^{(j)} + e^{2(D-n)\alpha\varphi}\tilde{M}^j{}_k A_{n-1}^{(i)}\wedge *A_{n-1}^{(k)}] \\ &\quad +\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{2(D-1)\alpha\varphi}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K})*1.\end{aligned} \tag{5.61}$$

If M is not invertible, there is a similar action with a Chern-Simons action for the massive $n-1$ form gauge fields, and a standard action for the massless gauge fields. First note that the Lagrangian (5.44) can be written in flat indices as

$$\mathcal{L}_b' = \frac{1}{2}P_{ab}[(-1)^{n-1}A_{n-1}^a\wedge\tilde{\mathcal{D}}A_{n-1}^b + e^{2(D-n)\alpha\varphi}\tilde{M}^b{}_c A_{n-1}^a\wedge *A_{n-1}^c], \tag{5.62}$$

where $P_{ab} = P_{ij}(\mathcal{V}^{-1})^i{}_a(\mathcal{V}^{-1})^j{}_b = (\Omega^{-1})_{ac}M^c{}_b$ and $\tilde{M}^a{}_b = \tilde{M}^i{}_j\mathcal{V}^a_i(\mathcal{V}^{-1})^j{}_b$. Note that one has $P^{ab} = P_{cd}\Omega^{ca}\Omega^{db} = (-1)^{n-1}M^{ab}$. When M is not invertible, A^α_{n-1} drops out from this Lagrangian, which is now just a Lagrangian for $\bar{A}_{(n-1)\alpha'}$:

$$\mathcal{L}'_{b1} = \frac{1}{2}m^{\alpha'\beta'}[\bar{A}_{(n-1)\alpha'}\wedge\tilde{\mathcal{D}}\bar{A}_{(n-1)\beta'}+(-1)^{n-1}e^{2(D-n)\alpha\varphi}\delta_{\beta'\gamma'}m^{\gamma'\rho'}\bar{A}_{(n-1)\alpha'}\wedge*\bar{A}_{(n-1)\rho'}]. \tag{5.63}$$

Here we have used that $\tilde{M} = QM = \Omega\mathcal{K}M$ so that $\tilde{M}^a{}_b = \Omega^{ac}\delta_{cd}M^d{}_b$. Note that $m^{\alpha'\beta'} = (-1)^n m^{\beta'\alpha'}$ because of (5.27) and $m^{\alpha'\beta'}\delta_{\beta'2(D-n)\alpha\varphi'}m^{2(D-n)\alpha\varphi'\rho'}$ is always symmetric, as it should be. It is easy to see that the field equations of (5.63) for the gauge fields $\bar{A}_{(n-1)\alpha'}$ does indeed give the constraint (5.39). The Lagrangian for A^α arises from (5.34) (with an extra factor of 1/2):

$$\mathcal{L}_{b2} = -\frac{1}{4}e^{-2(D-n)\alpha\varphi}\delta^{\alpha\beta}\bar{H}_{(n)\alpha}\wedge*\bar{H}_{(n)\beta} - \frac{1}{4}e^{2(D-n)\alpha\varphi}\delta_{\alpha\beta}\mathcal{D}A^\alpha_{n-2}\wedge*\mathcal{D}A^\beta_{n-2} \tag{5.64}$$

subject to the constraint (5.40), which can be used to eliminate either A^α_{n-2} or A^α_{n-1}. Choosing the first, the Lagrangian for A^α_{n-1} is

$$\mathcal{L}'_{b2} = -\frac{1}{2}e^{2(D-n)\alpha\varphi}\delta^{\alpha\beta}\mathcal{D}\bar{A}_{(n-1)\alpha}\wedge*\mathcal{D}\bar{A}_{(n-1)\beta}. \tag{5.65}$$

Then the total Lagrangian is

$$\mathcal{L}'_D = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}'_{b1} + \mathcal{L}'_{b2} \tag{5.66}$$

where $\mathcal{L}_g$ and $\mathcal{L}_s$ are as in (5.33). It is straightforward to show that these give the right field equations, by an argument similar to that in the invertible case above.

5.2.3 G = SL(2, ℝ) Case

In this subsection we will consider the case $G = SL(2,\mathbb{R})$. In this case the matrices $\mathcal{K}$ and Ω are as in (4.4) and (4.20). There are three distinct reductions corresponding to the three conjugacy classes of $SL(2,\mathbb{R})$ as discussed in section

2. The mass matrices representing the three conjugacy classes are given in (5.6). Now we will give the reduced Lagrangians for each mass matrix M_e, M_h and M_p.

- M_e:

There are two massive, $(n-1)$-forms in the theory which we will call A^1 and A^2. This is a $SO(2)$-gauged theory since M_e generates the $SO(2)$ subgroup of $SL(2,\mathbb{R})$. This is the only case when the theory has a stable minimum of the potential [21]. The global minimum of the potential is at $\chi = \phi = 0$. The Lagrangian is:

$$\begin{aligned}\mathcal{L}_D &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 \\ &+\frac{1}{2}m\{(-1)^{n-1}A^1\wedge DA^1 + (-1)^{n-1}A^2\wedge DA^2 - me^{2(D-n)\alpha\varphi}e^{\phi}[A^1\wedge *A^1 \\ &+(e^{-2\phi}+\chi^2)A^2\wedge *A^2 + 2\chi A^1\wedge *A^2]\} \\ &+\frac{1}{4}\mathrm{tr}(\mathcal{DK}\wedge *\mathcal{DK}^{-1}) - 2e^{2(D-1)\alpha\varphi}m^2[\sinh^2\phi + \chi^2(2+e^{2\phi}(2+\chi^2))]*1.\end{aligned} \tag{5.67}$$

- M_h:

There are two massive, $(n-1)$-forms in the theory which we will call A^1 and A^2, as before. The gauge group is $SO(1,1)$ in this case. The Lagrangian is:

$$\begin{aligned}\mathcal{L}_D &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 \\ &+\frac{1}{2}m\{(-1)^{n-1}2A^1\wedge DA^2 - me^{2(D-n)\alpha\varphi}e^{\phi}[\chi A^1\wedge *A^1 \\ &+\chi A^2\wedge *A^2 + (e^{-2\phi}+\chi^2+1)A^1\wedge *A^2]\} \\ &+\frac{1}{4}\mathrm{tr}(\mathcal{DK}\wedge *\mathcal{DK}^{-1}) - 2e^{2(D-1)\alpha\varphi}m^2[1+\chi^2e^{2\phi}]*1.\end{aligned} \tag{5.68}$$

- M_p:

There is one massive $(n-1)$-form field $\bar{A}_1$, one massless $(n-1)$-form field $\bar{A}_2$ and one massless $(n-2)$-form field B^2. However one can eliminate B^2 by using the reduced constraint (5.40), as was discussed in the previous subsection. The gauge group is $SO(1,1)$ in this case.

$$\begin{aligned}
\mathcal{L}_D &= R*1-\frac{1}{2}d\varphi\wedge *d\varphi-\frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 \\
&+\frac{1}{2}m[\bar{A}_1\wedge\mathcal{D}\bar{A}_1+(-1)^{n-1}e^{2(D-n)\alpha\varphi}m\bar{A}_1\wedge *\bar{A}_1]-\frac{1}{2}e^{2(D-n)\alpha\varphi}\mathcal{D}\bar{A}_2\wedge *\mathcal{D}\bar{A}_2 \\
&+\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1})-\frac{1}{2}e^{2(D-1)\alpha\varphi}m^2(e^{-\phi}+e^{\phi}\chi^2)^2*1.
\end{aligned} \tag{5.69}$$

5.3 Supergravity Applications

In this section, we will apply our results to the twisted reduction of supergravity theories in $d=4,6,8$ dimensions to $D=3,5,7$.

5.3.1 Reduction of $d=8$ Maximal Supergravity

The $N=2$ $D=8$ maximal supergravity [45] can be obtained from 11-dimensional supergravity by toroidal compactification and has field equations invariant under the duality group $SL(2,\mathbb{R})\times SL(3,\mathbb{R})$. The bosonic fields consist of a metric, a 3-form gauge potential A_3, 6 vector fields in the **(2,3)** representation of $SL(2,\mathbb{R})\times SL(3,\mathbb{R})$, 3 2-form gauge potentials in the **(1,3)** representation of $SL(2,\mathbb{R})\times SL(3,\mathbb{R})$, and scalars taking values in the coset space $SL(3,\mathbb{R})/SO(3)\times SL(2,\mathbb{R})/SO(2)$. The gauge potential A_3 combines with the dual gauge potential $\tilde{A}_3$ to form a doublet under $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$ is a symmetry of the action whereas $SL(2,\mathbb{R})$ is a symmetry of the field equations only, as it acts through electromagnetic duality on the 3-form gauge fields.

There is a consistent truncation of this theory where only the $SL(3,\mathbb{R})$ singlets are kept and all the other fields are set to zero [77]. Then the truncated theory consists of a metric, a 3-form gauge potential and scalars taking values in $SL(2,\mathbb{R})/SO(2)$, with an $SL(2,\mathbb{R})$ S-duality symmetry. This truncated theory is precisely of the form (4.1) with $n = 4$ and the twisted reduction with an $SL(2,\mathbb{R})$ twist gives three distinct reduced theories corresponding to the three conjugacy classes, with Lagrangians (5.67), (5.68) or (5.69).

This can be extended to the full theory, as the reduction of the fields that are not $SL(3,\mathbb{R})$ singlets is a standard Scherk-Schwarz reduction. However, there are some complications resulting from the Chern-Simons interactions of the $d = 8$ theory [78]. There are three distinct classical theories, while the distinct quantum theories correspond to the distinct $SL(2,Z)$ conjugacy classes.

5.3.2 Reduction of $d = 4, N = 4$ Supergravity

The Lagrangian for $N = 4$ supergravity coupled to p vector multiplets has an $O(6,p)$ symmetry of the action and an $SL(2,\mathbb{R})$ S-duality symmetry of the equations of motion. The vector potentials A_1^I $(I = 1,2,...,6{+}p)$ are in the fundamental **6+p** representation of $O(6,p)$ and combine with dual potentials $\tilde{A}_1^I$ to form 6+p doublets A_1^{mI} transforming in the **(2,6+p)** of $SL(2,\mathbb{R})\times O(6,p)$, where $m = 1,2$. The scalars take values in the coset $SL(2,\mathbb{R})/SO(2)\times O(6,22)/O(6)\times O(22)$. The scalars in $O(6,22)/O(6)\times O(22)$ can be represented by a metric $\mathcal{N}_{IJ}$ on the coset space while the 2 scalars ϕ, χ in $SL(2,\mathbb{R})/SO(2)$ can be represented by a metric on the coset space $\mathcal{K}_{mn}$ which is of the same form as (5.87).

The Lagrangian can be written as [28, 79, 80]:

$$\begin{aligned}\mathcal{L} &= R*1+\frac{1}{4}\mathrm{tr}(d\mathcal{K}\wedge *d\mathcal{K}^{-1})+\frac{1}{4}\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1})-\frac{1}{2}e^{-\phi}F_2^I\mathcal{N}_{IJ}\wedge *F_2^J\\ &\quad -\frac{1}{2}\chi F_2^I L_{IJ}\wedge F_2^J \qquad (5.70)\end{aligned}$$

where L is the $O(6,p)$ invariant metric and the matrices $\mathcal{N}$ and L satisfy

$$\mathcal{N}^T=\mathcal{N}, \qquad \mathcal{N}^T L\mathcal{N}=L. \qquad (5.71)$$

Now the field equation can be written as $dG_2 = 0$ where

$$G_2^I = (L^{-1})^{IJ} \frac{\delta \mathcal{L}}{\delta F_2^J} = -e^{-\phi} \mathcal{R}^I{}_J * F_2^J - \chi F_2^I \tag{5.72}$$

and the matrix $\mathcal{R}$ is defined as

$$L_{PI} \mathcal{R}^I{}_J = \mathcal{N}_{PJ}. \tag{5.73}$$

Note that $\mathcal{R}^2 = 1$. Now we can write

$$\begin{aligned} \mathcal{L}' &= R * 1 + \frac{1}{4}\mathrm{tr}(d\mathcal{K} \wedge * d\mathcal{K}^{-1}) + \frac{1}{4}\mathrm{tr}(d\mathcal{N} \wedge * d\mathcal{N}^{-1}) \\ &\quad + \frac{1}{2} F_2^I L_{IJ} \wedge G_2^J \end{aligned} \tag{5.74}$$

As before, the field equations $dG_2 = 0$ imply the existence of dual potentials $\tilde{A}_1$, with $G_2^I = d\tilde{A}_1^I$. Then the full set of vector fields A_1^i in the doubled formalism is $A_1^{mI} = (A_1^I, \tilde{A}_1^I)$ where $i = 1, ..., 2(6+p)$ becomes the composite index mI. The field strengths are the $6+p$ $SL(2, \mathbb{R})$-doublets:

$$H_2^I = \begin{pmatrix} dA_1^I \\ d\tilde{A}_1^I \end{pmatrix}. \tag{5.75}$$

We also impose the twisted self-duality constraint

$$H_2^{mI} = J^m{}_n \mathcal{R}^I{}_J * H_2^{nJ}, \tag{5.76}$$

where $J^m{}_n$ is as in (4.19). So the matrix Q in (4.29) is now the $12+2p \times 12+2p$ matrix

$$Q = J \otimes \mathcal{R}, \tag{5.77}$$

which satisfies $Q^2 = -1$ since $J^2 = -1$ and $R^2 = +1$. One can show that the doubled Lagrangian

$$\begin{aligned} \mathcal{L} &= R * 1 + \frac{1}{4}\mathrm{tr}(d\mathcal{K} \wedge * d\mathcal{K}^{-1}) + \frac{1}{4}\mathrm{tr}(d\mathcal{N} \wedge * d\mathcal{N}^{-1}) \\ &\quad - \frac{1}{4} \mathcal{N}_{IJ} H_2^{mI} \mathcal{K}_{mn} \wedge * H_2^{nJ}. \end{aligned} \tag{5.78}$$

gives the same field equations as those of (5.70) when the constraint equation (5.76) is imposed. This Lagrangian is of the same form as (4.16), with $\mathcal{K}_{ij}$ given by

$$\mathcal{K}_{mI\ nJ} = \mathcal{K}_{mn}\mathcal{N}_{IJ}.$$

So the Scherk-Schwarz reduction of (5.78) can be performed as before and the three dimensional Lagrangian that one obtains is:

$$\begin{aligned}\mathcal{L}_3' \quad = \quad & R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \frac{1}{4}\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1}) \\ & -\frac{1}{4}e^{-\varphi}\mathcal{N}_{IJ}H_2^I\mathcal{K}\wedge *H_2^J - \frac{1}{4}e^{\varphi}\mathcal{N}_{IJ}H_1^I\mathcal{K}\wedge *H_1^J \\ & +\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{2\varphi}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K}) * 1.\end{aligned} \tag{5.79}$$

This Lagrangian is to be supplemented by the reduced constraint

$$H_2^{mI} = e^{\varphi}J^m{}_n\mathcal{R}^I{}_J * H_1^{nJ}. \tag{5.80}$$

When M is invertible, this becomes

$$DA_1^{mI} = e^{\varphi}J^m{}_n\mathcal{R}^I{}_J M^n{}_p * A_1^{pJ}, \tag{5.81}$$

after gauging the Stückelberg fields away, as in subsection 5.1.2.

As before one can find a three dimensional Lagrangian from which the field equations and the constraint can be derived. This Lagrangian is (for invertible M):

$$\begin{aligned}\mathcal{L}_3 \quad = \quad & R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \frac{1}{4}\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1}) \\ & +\frac{1}{2}(\Omega^{-1}M)_{mn}(-L_{IJ}A_1^{mI}\wedge DA_1^{nJ} + \mathcal{N}_{IJ}e^{\varphi}\tilde{M}^n{}_p A_1^{mI}\wedge *A_1^{pJ}) \\ & +\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{2\varphi}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K}) * 1.\end{aligned} \tag{5.82}$$

5.3.3 Reduction of $d = 4, N = 8$ Supergravity

The $d = 4, N = 8$ theory has E_7 duality symmetry of the equations of motion. There are 70 scalars taking values in the coset $E_7/SU(8)$, and 28 vector fields A^I which combine with their duals to give A^i transforming as a **56** of E_7. The bosonic action can be written as (4.28) with the constraint (4.29) where Q is as in (4.30) and Ω^{ij} is the symplectic invariant of E_7 [30]. Now $\mathcal{K}$ is the matrix which parametrizes the scalar coset $E_7/SU(8)$. The theory can be reduced to 3-dimensions using any mass matrix M in the Lie algebra of E_7. Naively, this introduces 133 mass parameters, but these theories are not all independent and the independent theories correspond to the distinct conjugacy classes; the classification of conjugacy classes in this case is not known. The matrix $M^{ab} = M^a{}_c \Omega^{cb}$ introduced in subsection 5.1.2 is a symmetric matrix since $n = 2$ in (5.27), so by choosing a suitable basis, it can be brought into the diagonal form:

$$M^{ab} = \begin{pmatrix} m_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & m_{56} \end{pmatrix} \tag{5.83}$$

For example, consider performing the Scherk-Schwarz reduction with the Lie algebra element M^{ab} of the form:

$$M^{ab} = m \begin{pmatrix} 0_l & & \bigcirc \\ & 1_p & \\ \bigcirc & & -1_q \end{pmatrix} \tag{5.84}$$

where $l + p + q = 56$. Then one obtains a 3-dimensional theory with one mass parameter with p massive, self-dual vector fields, q massive, anti-self-dual vector fields and l massless vector fields which are dual to the l massless scalar fields coming from the reduction of the vector field in the 4-dimensional theory.

5.3.4 Reduction of $d = 6$ Supergravity

The $D = 6$ theory of [81], obtained from the compactification of a truncation of type IIB supergravity, has an

$$SO(2,2) \equiv SL(2,\mathbb{R})_{EM} \times SL(2,\mathbb{R})_{IIB} \tag{5.85}$$

symmetry of the equations of motion. The $SL(2,\mathbb{R})_{IIB}$ is inherited from the $SL(2,\mathbb{R})$ symmetry of type IIB in ten dimensions and is a symmetry of the action in the six dimensional theory. However $SL(2,\mathbb{R})_{EM}$ is a symmetry of the field equations only. The Lagrangian is:

$$\begin{aligned} \mathcal{L} &= R * 1 + \frac{1}{4}\mathrm{tr}(d\mathcal{K} \wedge *d\mathcal{K}^{-1}) + \frac{1}{4}\mathrm{tr}(d\mathcal{N} \wedge *d\mathcal{N}^{-1}) \\ &\quad -\frac{1}{2}\kappa_2 F^I \mathcal{N}_{IJ} \wedge *F^J - \frac{1}{2}\kappa_1 F^I \Omega_{IJ} \wedge F^J. \end{aligned} \tag{5.86}$$

Here $F^I = dA_2^I$ are the two 3-form field strengths and $I, J = 1, 2$ are $SL(2,\mathbb{R})_{IIB}$ indices. We also introduce $SL(2,\mathbb{R})_{EM}$ indices $m, n = 1, 2$. There are two $SL(2,\mathbb{R})/SO(2)$ scalar cosets in the theory. κ_1 and κ_2 parametrize the scalar coset $SL(2,\mathbb{R})_{EM}/SO(2)$, represented by the matrix $\mathcal{K}$:

$$\mathcal{K}_{mn} = \frac{1}{\kappa_2}\begin{pmatrix} |\kappa|^2 & -\kappa_1 \\ -\kappa_2 & 1 \end{pmatrix} \tag{5.87}$$

where

$$\kappa = \kappa_1 + i\kappa_2 = \frac{1}{4}D + \frac{3}{2}ie^{2G},$$

D and G are as defined in [81]. There are two more scalars in the theory, l and ϕ, parametrizing the scalar coset $SL(2,\mathbb{R})_{IIB}/SO(2)$ represented by the matrix

$$\mathcal{N}_{IJ} = \frac{1}{\lambda_2}\begin{pmatrix} |\lambda|^2 & -\lambda_1 \\ -\lambda_2 & 1 \end{pmatrix} \tag{5.88}$$

where

$$\lambda = \lambda_1 + i\lambda_2 = l + ie^{-\phi}.$$

The invariant matrices are Ω^{mn}, Ω^{IJ}.

The Lagrangian (5.86) is of the same form as (5.70) where now I ranges from 1 to 2 and the $O(6,p)$ invariant L_{IJ} has been replaced by the $SL(2,\mathbb{R})_{IIB}$ invariant matrix Ω_{IJ}. So (5.86) is equivalent to the doubled Lagrangian (5.78) (now with 3-form field strengths H_3) when supplemented by the constraint (5.76). Note that the matrix Q in (5.77) now satisfies $Q^2 = +1$ as it should in 6 dimensions, since now $\mathcal{R}^2 = -1$, whereas $\mathcal{R}^2 = +1$ and hence $Q^2 = -1$ in the 4-dimensional case.

By performing the Scherk-Schwarz reduction of the doubled Lagrangian one obtains the following auxiliary five-dimensional Lagrangian:

$$\begin{aligned} \mathcal{L}'_5 &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-4\varphi/\sqrt{6}}\mathcal{F}_2\wedge *\mathcal{F}_2 + \frac{1}{4}\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1}) \\ &\quad -\frac{1}{4}e^{-2\varphi/\sqrt{6}}\mathcal{N}_{IJ}H_3^I\mathcal{K}\wedge *H_3^J - \frac{1}{4}e^{2\varphi/\sqrt{6}}\mathcal{N}_{IJ}H_2^I\mathcal{K}\wedge *H_2^J \\ &\quad +\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{4\varphi/\sqrt{6}}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K})*1 \end{aligned} \tag{5.89}$$

This is to be supplemented by the five-dimensional reduced constraint

$$H_3^{mI} = e^{2\varphi/\sqrt{6}}J^m{}_n\mathcal{R}^I{}_J * H_2^{nJ}. \tag{5.90}$$

When M is invertible one can gauge the Stückelberg fields away and in this gauge the constraint in (5.90) takes the form

$$DA_2^{mI} = -e^{2\varphi/\sqrt{6}}J^m{}_n\mathcal{R}^I{}_J M^n{}_p * A_2^{pJ}. \tag{5.91}$$

The five dimensional reduced Lagrangian from which the reduced constraint (5.91) and the field equations of (5.89) can be derived is obtained by using the techniques of the previous sections:

$$\begin{aligned} \mathcal{L}_5 &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-4\varphi/\sqrt{6}}\mathcal{F}_2\wedge *\mathcal{F}_2 + \frac{1}{4}\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1}) \\ &\quad +\frac{1}{2}(\Omega^{-1}M)_{mn}(\Omega_{IJ}A_2^{mI}\wedge DA_2^{nJ} + \mathcal{N}_{IJ}e^{2\varphi/\sqrt{6}}\tilde{M}^n{}_p A_2^{nI}\wedge *A_2^{pJ}) \\ &\quad +\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{4\varphi/\sqrt{6}}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K})*1. \end{aligned} \tag{5.92}$$

5.3.5 Reduction of $d = 6, N = 8$ Supergravity

The maximal supergravity in six dimensions has noncompact global symmetry $SO(5,5)$ which can be realized at the level of field equations only [31]. There are five 3-form field strengths which split into five self-dual ones and five anti-self dual ones, and these ten transform as a **10** of $SO(5,5)$. There are 25 scalar fields in the theory and they parametrize the coset space $SO(5,5)/SO(5) \times SO(5)$. The bosonic Lagrangian can be written as (4.28), plus terms which we will not give explicitly here involving the vector fields, with the constraint (4.29) where Q is as in (4.30) and Ω^{ij} is the symplectic invariant of $SO(5,5)$ [29]. Now $\mathcal{K}$ is the matrix which parametrizes the scalar coset $SO(5,5)/SO(5) \times SO(5)$. The theory can be reduced to 5-dimensions using any mass matrix M in the Lie algebra of $SO(5,5)$. The number of distinct reductions is given by the number of conjugacy classes of $SO(5,5)$.

Consider the matrix $M^{ab} = M^a{}_c \Omega^{cb}$ introduced in subsection 5.1.2. It is an anti-symmetric matrix since $n = 3$ in (5.27). So in a particular basis it can be brought into the skew-diagonal form:

$$M^{ab} = \begin{pmatrix} 0 & m_1 & & \bigcirc & \\ -m_1 & 0 & & & \\ & & \ddots & & \\ \bigcirc & & & 0 & m_5 \\ & & & -m_5 & 0 \end{pmatrix} \tag{5.93}$$

Consider a mass matrix of the form

$$M^{ab} = m \begin{pmatrix} \mathbf{0}_l & & & & & \\ & 0 & 1 & & & \\ & -1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} \qquad (5.94)$$

where there are l zero eigenvalues and the number of skew-diagonal blocks is $(10-l)/2$. On reduction, one obtains, in five dimensions, a gauged theory with one mass parameter including $10-l$ massive self-dual 2-form fields and l massless 2-form fields, which could be dualised to l massless 1-form fields.

CHAPTER 6

CONCLUSIONS AND FUTURE DIRECTIONS

In this thesis we have considered twisted reductions of even dimensional supergravity theories, where the twist is given by a duality symmetry which is a symmetry of the field equations only. In this way we have obtained new gauged and massive supergravity theories in 3, 5 and 7 dimensions. All of these theories are such that the fields which are in non-trivial representations of the duality group G acquire masses. Also the scalar fields obtain a potential, which leads to supersymmetry breaking or as has been considered in [21], to moduli stabilisation. We have also seen that all the new supergravity theories we obtain in odd dimensions include gauge fields which exhibit self-duality in the generalized sense of [32].

This work can be generalized in several directions. Recall that in the 8 and 6 dimensional cases we considered truncated versions of the supergravity theories rather than the full theory itself. This is mainly because in the full version there are Chern-Simons type interactions which make the realization of the doubled formalism difficult. If these difficulties can be sorted out then one can perform the Scherk-Schwarz reduction of the full theory and obtain new gauged supergravity theories in 7, 5 dimensions with more gauge fields and couplings than present.

In this thesis we mainly focused on reductions with $SL(2,\mathbb{R})$ twists. Exploring the twisted reductions with larger duality groups such as $E_{n(n)}$ or $SO(p,q)$ would be interesting since such duality groups, having more conjugacy classes, are likely to give rise to richer structures. However, the classification of the conjugacy classes of these groups are not known. A more tractable project would be the consideration of the $SL(3,\mathbb{R})$ conjugacy classes. $d = 8$ maximal supergravity has an $SL(3,\mathbb{R})$ duality symmetry of the action and its Scherk-Schwarz reduction with a $SL(3,\mathbb{R})$ twist would give rise to new gauged supergravity theories in 7 dimensions.

In chapter 3 we have seen the relation between the twisted reductions and flux compactifications. Twists with larger duality groups may correspond to compactifications with more flux. On the other hand, we have the experience from string theory that appearance of background fields with non-trivial flux might turn on noncommutativity [82]. It would be very interesting to see if the manifold structure of the twisted torus can be understood in terms of noncommutative geometry. In fact the structure that we described in subsection 2.3.3 is similar to the manifold structure that was considered in [5]. There we perform Kaluza-Klein reduction of a noncommutative type, where the compact space corresponds to the C^* algebra $M_3(C)$. Pursuing this similarity, it might be possible to understand the structure of the twisted torus and the relation with flux compactifications better.

It is interesting to note that the Lagrangian (5.33) maintains G-invariance under the transformations (5.2) when the mass matrix M transforms as

$$\delta M \to L^{-1} M L. \tag{6.1}$$

The mass matrix M defines not only the mass parameters but also the structure constants of the non-abelian gauge group of the lower dimensional theory. Hence the transformation (6.1) should be regarded as a pseudo-duality [83], i.e. the duality transformation of the coupling constants (see [22] for a discussion). Assuming this transformation of the mass parameters, one can now perform a second S^1 reduction of (5.33) with a G-twist. In [15], the SS reduction of type IIB string theory with an $SL(2,\mathbb{R})$ twist had been interpreted as the compactification of the 12 dimensional F-theory on the total space B, where B is defined as in subsection 2.3.3. A second twisted reduction on S^1 would give an F-theory origin for the resulting 8 dimensional theory. Now the total space is C, with fiber space B and the base space $S^1(y)$ where y is the parameter of the circle on which the first twisted reduction had been performed. This means

$$\begin{array}{ccc} T^2 \longrightarrow & B & \\ & \downarrow & \\ & S^1 & \end{array} \qquad\qquad \begin{array}{ccc} B \longrightarrow & C & \\ & \downarrow & \\ & S^1(y) & \end{array}$$

For two subsequent twisted S^1 reductions the mass matrices M_1 and M_2 should commute in order to assure the independence of the resulting theory of the compactified coordinates y.

We believe that all these ideas deserve further investigation.

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