ORTHOGONAL POLYNOMIALS AND MOMENT PROBLEM

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ORTHOGONAL POLYNOMIALS AND MOMENT PROBLEM

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The generalized moment of order k of a mass distribution σ is given by $\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)$ for a natural number k. In extended moment problem, given a sequence $(s_k)_{k=0}^{\infty}$ of real numbers, it is required to find a mass distribution σ whose generalized moment of order k is s_k. The conditions of existence and uniqueness of the solution obtained by Hamburger are studied in this thesis by the use of orthogonal polynomials determined by a measure on $\mathbb{R}$. A chapter on the study of asymptotic behaviour of orthogonal functions on compact subsets of $\mathbb{C}$ is also included.

Keywords: Orthogonal polynomials, moment problem.

ÖZ

ORTOGONAL POLINOMLAR VE MOMENT PROBLEMI

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Bir kütle dağılımı σ için k'inci dereceden genelleştirilmiş moment $\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)$ ifadesiyle verilir. Genişletilmiş moment probleminde, bir reel sayılar dizisi $(\, s_k \,)_{k=0}^{\infty}$ verildiğinde k'inci dereceden genelleştirilmiş momenti s_k olacak şekilde bir σ kütle dağılımı bulunması gerekmektedir. Bu tezde Hamburger tarafından bulunmuş olan çözümün varlığı ve tekliği koşulları incelenmiştir. Ayrıca $\mathbb{C}$'nin kompakt alt kümeleri üzerindeki ortogonal polinomların asimptotik davranışları üzerine bir bölüm de içerilmektedir.

Anahtar Kelimeler: Moment problemi, Ortogonal polinomlar

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CHAPTER 1

THE SPACE L^2_σ AND MASS DISTRIBUTIONS

Any bounded non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$ defines a Borel measure on $\mathbb{R}$ for which measure of an interval $[a, b]$ is given by

$$\lim_{\substack{\alpha \to a^- \\ \beta \to b^+}} [\,\sigma(\beta) - \sigma(\alpha)\,].$$

Considering functions $f, g : \mathbb{R} \to \mathbb{C}$ to be equivalent if the set $\{x : f(x) \neq g(x)\}$ is of $d\sigma-$measure zero, an inner product is defined on the following linear space:

$$L^2_\sigma = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; f \text{ is d}\sigma\text{-measurable}, \int_{-\infty}^{+\infty} |\, f(\lambda)\,|^2 \; d\sigma(\lambda) < \infty \right\}$$

by

$$(\, f, g\,)_\sigma = \int_{-\infty}^{+\infty} f(\lambda)\overline{g(\lambda)}\, d\sigma(\lambda).$$

The space L^2_σ is complete with respect to the given scalar product. The proof of completeness is similar to the proof of completeness of the space L^2, which is defined as

$$L^2 = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; f \text{ is Lebesgue measurable}, \int_{-\infty}^{+\infty} |\, f(\lambda)\,|^2 \; d\lambda < \infty \right\}$$

with the scalar product

$$(\, f, g\,) = \int_{-\infty}^{+\infty} f(\lambda)\overline{g(\lambda)}\, d\lambda\ .$$

The proof may be found in standard textbooks on functional analysis (for example [1, Section I.10]). Since L^2_σ is complete, it follows that if L^2_σ is infinite dimensional then it is a Hilbert space.

Note that the function σ has at most countably many points of discontinuity since the total variation of the function σ is finite. Thus the values of the function σ at points of discontinuity does not affect the measure defined by σ. Besides, the function $\sigma + c$ yields the same measure as σ.

1.1 Mass Distributions

Let $\mathbb{C}[\lambda]$ and $\mathbb{R}[\lambda]$ denote the set of polynomials with complex and real coefficients, respectively. We will restrict our investigation to the case when $\mathbb{C}[\lambda] \subset L^2_\sigma$ and any two polynomials represent different elements of L^2_σ, i.e. $\mathbb{C}[\lambda]$ sits in L^2_σ as a vector subspace. Under these assumptions it is guaranteed that L^2_σ is a Hilbert space.

Any polynomial $p(\lambda) \in \mathbb{C}[\lambda]$ can be written uniquely in the form $p(\lambda) = p_1(\lambda) + ip_2(\lambda)$ where $p_1(\lambda), p_2(\lambda) \in \mathbb{R}[\lambda]$. Thus $\mathbb{C}[\lambda]$ sits in L^2_σ as a vector subspace if and only if $\mathbb{R}[\lambda] \subset L^2_\sigma$ and any two distinct polynomials with real coefficients represent different elements of L^2_σ.

Note that $\mathbb{R}[\lambda] \subset L^2_\sigma$ if and only if

$$\int_{-\infty}^{+\infty} [\,p(\lambda)\,]^2 \; d\sigma(\lambda) < \infty \tag{1.1}$$

for any polynomial $p(\lambda) \in \mathbb{R}[\lambda]$ and any two polynomials with real coefficients represent different elements of L^2_σ if and only if

$$\int_{-\infty}^{+\infty} [\,p(\lambda)\,]^2 \; d\sigma(\lambda) > 0 \tag{1.2}$$

for any nonzero polynomial $p(\lambda) \in \mathbb{R}[\lambda]$. By an *increase point* of σ, we mean a point t such that $\sigma(a) < \sigma(b)$ whenever $a < t < b$.

Proposition 1.1.1. $\mathbb{R}[\lambda] \subset L^2_\sigma$ *and any two distinct polynomials in* $\mathbb{R}[\lambda]$ *represent different elements of* L^2_σ *if and only if* $\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)$ *exists,* $|\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)| < \infty$ *for any* $k \in \mathbb{N}$ *and* σ *has infinite number of increase points.*

Proof. First we prove that $\mathbb{R}[\lambda] \subset L^2_\sigma$ if and only if $\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)$ exists and $|\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)| < \infty$ for any $k \in \mathbb{N}$. If $\mathbb{R}[\lambda] \subset L^2_\sigma$ then $(\lambda^k, 1)_\sigma = \int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda)$ exists and is finite for any $k \in \mathbb{N}$. The converse follows directly by linearity of the integral.

Now we claim that the inequality (1.2) is satisfied for any nonzero polynomial in $\mathbb{R}[\lambda]$ if and only if the function σ has infinite number of increase points. First assume that σ has n points $\lambda_1, \ldots, \lambda_n$ of increase, where the jumps at these points are $\mu_1, \ldots, \mu_n$ respectively. Then for any function $f(\lambda)$ which is integrable with respect to measure $d\sigma$ we have

$$\int_{-\infty}^{+\infty} f(\lambda) \, d\sigma(\lambda) = \sum_{k=1}^{n} f(\lambda_k) \mu_k \, . \tag{1.3}$$

Choosing $f(\lambda) = [\, p(\lambda) \,]^2$ for $p(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_k)$, it follows that

$$\int_{-\infty}^{+\infty} [\, p(\lambda) \,]^2 \; d\sigma(\lambda) = \int_{-\infty}^{+\infty} \prod_{k=1}^{n} (\lambda - \lambda_k)^2 \, d\sigma = 0$$

and we have proved the necessity of the condition.

To prove sufficiency, let σ have infinite number of increase points. Any polynomial $p(\lambda)$ has a finite number of zeroes so $p(\lambda)$ is nonzero at some increase point t of σ. Since $[\, p(\lambda) \,]^2$ is continuous, there is a neighborhood (a, b) of t and a real number $m > 0$ for which $[\, p(\lambda) \,]^2 > m$ for any $\lambda \in (a, b)$. Therefore

$$\begin{aligned} \int_{-\infty}^{+\infty} [\, p(\lambda) \,]^2 \; d\sigma(\lambda) &\geq \int_{a}^{b} [\, p(\lambda) \,]^2 \; d\sigma(\lambda) \\ &\geq m \, (\, \sigma(b) - \sigma(a) \,) > 0. \end{aligned} \tag{1.4}$$

□

Thus we have reached the class of functions we will deal with:

Definition 1.1.1. A non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$ is called a *mass distribution* on $\mathbb{R}$ if the integrals

$$\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda) \qquad\qquad (k \in \mathbb{N})$$

exist and are finite, if σ has infinite number of increase points and if $\lim_{u \to -\infty} \sigma(u) = 0$. Two mass distributions are said to be *equivalent* if they differ in only the values at points of discontinuity.

Note that equivalent mass distributions yield the same measure.

1.2 L^2_σ and the Linear Subspace $\mathbb{C}[\lambda]$

In this section we will investigate when the linear subspace $\mathbb{C}[\lambda]$ of polynomials is dense in L^2_σ.

Orthonormalizing the sequence $\left(\lambda^k\right)_{k=0}^{\infty}$ by Gram–Schmidt process we obtain a sequence $(P_k(\lambda))_{k=0}^{\infty}$ of polynomials with real coefficients such that degree of $P_n(\lambda)$ is n.

Given an element $(x_k)_{k=0}^{\infty}$ of the Hilbert space l^2, consider the sequence $(f_n(\lambda))_{n=0}^{\infty}$ where

$$f_n(\lambda) = \sum_{k=0}^{n} x_k P_k(\lambda).$$

For $m, n \in \mathbb{N}$, it follows by orthonormality of the polynomials $P_k(\lambda)$ and linearity of integral that

$$\begin{aligned}
\| f_n - f_m \|_\sigma^2 &= \int_{-\infty}^{+\infty} | f_n(\lambda) - f_m(\lambda) |^2 \, d\sigma(\lambda) \\
&= \sum_{i,k=m+1}^{n} x_i \bar{x}_k \int_{-\infty}^{+\infty} P_i(\lambda) P_k(\lambda) \, d\sigma(\lambda) \\
&= \sum_{k=m+1}^{n} | x_k |^2 \int_{-\infty}^{+\infty} (P_k(\lambda))^2 \, d\sigma(\lambda) = \sum_{k=m+1}^{n} | x_k |^2 .
\end{aligned}$$

The series $\sum_{k=0}^{\infty} |\, x_k \,|^2$ converges so

$$\lim_{m,n\to\infty} ||f_n(\lambda) - f_m(\lambda)||_\sigma = 0$$

and hence the sequence $(\, f_n(\lambda)\,)_{n=0}^{\infty}$ is Cauchy in L^2_σ. Then it converges to a function $f(\lambda)$ in the space L^2_σ since this space is complete. The convergence in the space L^2_σ will be denoted by

$$\operatorname*{l.i.m.}_{n\to\infty} f_n(\lambda) = f(\lambda).$$

An operator U can be defined from the Hilbert space l^2 to L^2_σ by

$$Ux = \operatorname*{l.i.m.}_{n\to\infty} \sum_{k=0}^{n} x_k P_k(\lambda) =: \sum_{k=0}^{\infty} x_k P_k(\lambda)$$

where $x = (\, x_k \,)_{k=0}^{\infty}$.

Proposition 1.2.1. *The operator U is isometric, i.e. $(\, Ux, Uy \,)_\sigma = (\, x, y \,)_{l^2}$ where x,y are elements of l^2.*

Proof. For $x = (\, x_k \,)_{k=0}^{\infty}$ and $y = (\, y_k \,)_{k=0}^{\infty}$ let $f_n(\lambda) = \sum_{k=0}^{n} x_k P_k(\lambda)$ and $g_n(\lambda) = \sum_{k=0}^{n} y_k P_k(\lambda)$. Then

$$\begin{aligned}
&|\, (\, Ux, Uy \,)_\sigma - (\, f_n, g_n \,)_\sigma \,| \\
&\quad = |\, (\, Ux - f_n, Uy \,) + (\, Ux, Uy - g_n \,) - (\, Ux - f_n, Uy - g_n \,) \,| \\
&\quad \le \|Ux - f_n\|_\sigma \|Uy\|_\sigma + \|Ux\|_\sigma \|Uy - g_n\|_\sigma + \|Ux - f_n\|_\sigma \|Uy - g_n\|_\sigma
\end{aligned}$$

which yields

$$\lim_{n\to\infty} (\, f_n, g_n \,)_\sigma = (\, Ux, Uy \,)_\sigma \;.$$

Thus

$$\begin{aligned}
(\, Ux, Uy \,)_\sigma = \lim_{n\to\infty} (\, f_n, g_n \,)_\sigma &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} x_k P_k, \sum_{k=0}^{n} y_k P_k \right)_\sigma \\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k \right)_{l^2} \\
&= (\, x, y \,)_{l^2} \;.
\end{aligned}$$

□

Since the operator U is unitary, it is one-to-one. So the inverse operator U^{-1} is defined on the image of U which will be denoted by Δ_U.

Proposition 1.2.2. *If the function $f(\lambda) \in \Delta_U$ then*

$$U^{-1}f = \left(\int_{-\infty}^{+\infty} f(\lambda) P_n(\lambda)\, d\sigma(\lambda) \right)_{n=0}^{\infty}$$

Proof. Let $f = Ux$ where $x = (\, x_n \,)_{n=0}^{\infty}$. For $m > n$

$$\left(\sum_{k=0}^{m} x_k P_k, P_n \right)_\sigma = x_n$$

by orthonormality of $\{\, P_k(\lambda) \,\}_{k=0}^{\infty}$. Then we conclude that

$$x_n = \lim_{m\to\infty} \left(\sum_{k=0}^{m} x_k P_k, P_n \right)_\sigma = (\, f, P_n \,)_\sigma = \int_{-\infty}^{+\infty} f(\lambda) P_n(\lambda)\, d\sigma(\lambda).$$

□

Δ_U is complete and it is a Hilbert subspace of L^2_σ since U is an isometric operator. It is generated by $\mathbb{C}[\lambda]$ and $\{\, P_k(\lambda) \,\}_{k=0}^{\infty}$ is a complete orthonormal system in Δ_U. Thus any element $f(\lambda) \in L^2_\sigma$ can be written uniquely in the form $f(\lambda) = f_1(\lambda) + f_2(\lambda)$ where $f_1(\lambda) \in \Delta_U$ and $f_2(\lambda) \perp \Delta_U$. The projection operator $\Pi : L^2_\sigma \to \Delta_U$ is defined for any $f(\lambda) \in L^2_\sigma$ by

$$\Pi f(\lambda) = f_1(\lambda) = \sum_{k=0}^{\infty} (\, f, P_k \,)_\sigma P_k(\lambda).$$

The function Πf is the nearest element to f among the elements of Δ_U in the metric defined by σ. We have

$$\sum_{k=0}^{\infty} |\, (\, f, P_k \,)_\sigma \,|^2 = \|f_1\|^2_\sigma \le \|f\|^2_\sigma = \int_{-\infty}^{+\infty} |\, f(\lambda) \,|^2 \; d\sigma(\lambda)$$

for any $f(\lambda) \in L^2_\sigma$. The inequality

$$\sum_{k=0}^{\infty} |\, (\, f, P_k \,)_\sigma \,|^2 \le \int_{-\infty}^{+\infty} |\, f(\lambda) \,|^2 \; d\sigma(\lambda) \tag{1.5}$$

is called *Bessel inequality*. The equality holds for only the functions $f(\lambda) = \Pi f(\lambda)$, i.e. the functions $f(\lambda) \in \Delta_U$. Thus $\Delta_U = L^2_\sigma$ if and only if the equality holds for all functions $f(\lambda) \in L^2_\sigma$. We conclude that:

Proposition 1.2.3. *Polynomials are dense in L^2_σ if and only if*

$$\sum_{k=0}^{\infty} |\,(\,f, P_k\,)_\sigma\,|^2 = \int_{-\infty}^{+\infty} |\,f(\lambda)\,|^2 \; d\sigma(\lambda)$$

for any $f(\lambda) \in L^2_\sigma$.

CHAPTER 2

ORTHOGONAL POLYNOMIALS

2.1 The Moment Problem

Mass distribution

Let a mass distribution σ be given. Then the inner product on the linear space $\mathbb{R}[\lambda]$ is determined uniquely by the sequence $(\, s_k \,)_{k=0}^{\infty}$ where

$$s_{i+k} = \left(\, \lambda^i, \lambda^k \,\right)_\sigma = \int_{-\infty}^{+\infty} \lambda^{i+k} \, d\sigma(\lambda).$$

Given a mass distribution σ, the total mass on $\mathbb{R}$ equals to the Stieltjes integral

$$\int_{-\infty}^{+\infty} d\sigma(u) = \lim_{\lambda\to\infty} \sigma(\lambda) - \lim_{\lambda\to-\infty} \sigma(\lambda).$$

The physical quantity called the static moment of a mass distribution σ with respect to point $u = 0$ is given by

$$\int_{-\infty}^{+\infty} u \, d\sigma(u)$$

and the moment of inertia with respect to point $u = 0$ is

$$\int_{-\infty}^{+\infty} u^2 \, d\sigma(u).$$

The name *generalized moment of order* k is given to the quantity

$$s_k = \int_{-\infty}^{+\infty} u^k \, d\sigma(u).$$

Any mass distribution gives a sequence $(s_k)_{k=0}^{\infty}$ of generalized moments. Moment problem is about the converse: Given a sequence $(s_k)_{k=0}^{\infty}$, can we obtain a mass distribution σ such that for any $k \in \mathbb{N}$, generalized moment of order k of σ is s_k? Apart from existence, the conditions for uniqueness, up to equivalence in the sense of definition 1.1.1, of the solution is of interest.

Note that if $s_0 = 0$ then only possible solution is $\sigma(\lambda) \equiv 0$. If $s_k = 0$ for all $k \in \mathbb{N}$, then this is the solution indeed. If $s_k \neq 0$ for some k, then we have no solution. From now on, we will not consider this trivial case and assume that $s_0 \neq 0$.

While dealing with the problem, we loose nothing if we divide each term of the sequence $(s_k)_{k=0}^{\infty}$ by a positive number c since we can recover the solution for the original problem by just multiplying the solution for $(cs_k)_{k=0}^{\infty}$ by c^{-1}. So from now on we may assume that the given sequence $(s_k)_{k=0}^{\infty}$ is *normalized*, i.e. $s_0 = 1$.

Remark. In the moment problem above, the mass distribution is given on the whole real axis. The problem could be posed on some other subset of $\mathbb{R}$ as well by allowing existence of mass only on that set: on the half real line $[0, \infty)$, on a finite interval, on several intervals, or some other arbitrary subset. Historically, the moment problem on the half real line $[0, \infty)$ is the oldest. It is first mentioned and the related questions about solution of the problem are answered by T. Stieltjes on 1894 [13]. The moment problem on the whole real line is thus called *the extended moment problem*, which is first discussed by H. Hamburger on 1920-1921 [12], after whom the problem is usually named.

2.2 Positive Sequences and Jacobi Matrices

Positive sequences and orthogonal polynomials

Let a normalized sequence $(s_k)_{k=0}^{\infty}$ be given. Using this sequence, a linear functional Γ can be defined on $\mathbb{R}[\lambda]$, the space of polynomials with real coefficients, by

$$\Gamma(x_0 + x_1\lambda + x_2\lambda^2 + \dots x_n\lambda^n) = x_0 s_0 + x_1 s_1 + x_2 s_2 + \cdots + x_n s_n \qquad (2.1)$$

characterized by the property $\Gamma(\lambda^k) = s_k$ and linearity. We will use Γ_λ instead of Γ whenever it is necessary to indicate the variable.

Using this functional, a bilinear symmetric form can be defined on $\mathbb{R}[\lambda]$ by

$$(p(\lambda), q(\lambda)) = \Gamma(p(\lambda)q(\lambda)). \qquad (2.2)$$

For this form to be positive definite, $(p(\lambda), p(\lambda)) = \Gamma([p(\lambda)]^2)$ should be positive for any nonzero polynomial $p(\lambda)$. More explicitly, for any $n \in \mathbb{Z}$ and any $x_0, x_1, \dots, x_n \in \mathbb{R}$ at least one of which is nonzero, we should have

$$0 < \Gamma\left(\left(\sum_{k=0}^{n} x_k\lambda^k\right)^2\right) = \Gamma\left(\sum_{0\le i,k\le n} x_i x_k \lambda^{i+k}\right) = \sum_{0\le i,k\le n} x_i x_k s_{i+k}\,. \qquad (2.3)$$

By linear algebra (for example, see [11]), we have that this condition on the sequence $(s_k)_{k=0}^{\infty}$ is equivalent to the positivity of the following determinants:

$$\begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & s_3 & \dots & s_{n+1} \\ s_2 & s_3 & s_4 & \dots & s_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n} \end{vmatrix} > 0 \qquad (\forall n \in \mathbb{N}) \qquad (2.4)$$

Definition 2.2.1. A sequence $(s_k)_{k=0}^{\infty}$ is called *positive* if it satisfies the property (2.4).

Given a sequence $(s_k)_{k=0}^{\infty}$, an inner product is defined on $\mathbb{R}[\lambda]$ by (2.2) if and only if the sequence is positive. We also have the following equivalence:

Proposition 2.2.1. *The sequence* $(s_k)_{k=0}^{\infty}$ *is positive if and only if* $\Gamma(q(\lambda)) > 0$ *for any nonzero polynomial* $q(\lambda)$ *satisfying* $q(u) \ge 0$ *for all* $u \in \mathbb{R}$.

Proof. Assume Γ takes positive values for any nonzero polynomial $q(\lambda)$ such that $q(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. Then in particular $\Gamma\,(\,(p(\lambda))^2\,) = (\,p(\lambda), p(\lambda)\,) > 0$ for any nonzero $p(\lambda) \in \mathbb{R}[\lambda]$, thus $(\,s_k\,)_{k=0}^{\infty}$ is a positive sequence by equation (2.3).

To prove the converse, let $(\,s_k\,)_{k=0}^{\infty}$ be a positive sequence and let $p(\lambda)$ be a polynomial satisfying $p(\lambda) \geq 0$ for any $\lambda \in \mathbb{R}$. The polynomial $p(\lambda)$ is of the form

$$p(\lambda) = c\prod_{k=1}^{n}\left[\,(\,\lambda - \lambda_k\,)(\,\lambda - \overline{\lambda}_k\,)\,\right] = c\prod_{k=1}^{n}(\,\lambda - a_k - ib_k\,)\prod_{k=1}^{n}(\,\lambda - a_k + ib_k\,)$$

for a positive $c \in \mathbb{R}$ and for some real numbers a_k and b_k. Let $\prod_{k=1}^{n}(\,\lambda - a_k - ib_k\,) = p_1(\lambda) + ip_2(\lambda)$ where $p_1(\lambda), p_2(\lambda) \in \mathbb{R}[\lambda]$. Then

$$p(\lambda) = c\,\left(\,p_1(\lambda) + ip_2(\lambda)\,\right)\,\overline{\left(\,p_1(\lambda) + ip_2(\lambda)\,\right)} = c\left[\,(\,p_1(\lambda)\,)^2 + (\,p_2(\lambda)\,)^2\,\right].$$

So it follows that

$$\begin{aligned}\Gamma\,(\,p(\lambda)\,) &= \Gamma\left(\,c\left[\,(\,p_1(\lambda)\,)^2 + (\,p_2(\lambda)\,)^2\,\right]\,\right)\\ &= c\,\Gamma\left(\,(p_1(\lambda))^2\right) + c\,\Gamma\left(\,(p_2(\lambda))^2\right) > 0,\end{aligned}$$

which is the desired result.

□

Orthonormalizing the sequence $\left(\,\lambda^k\,\right)_{k=0}^{\infty}$ by Gram–Schmidt process and choosing the polynomials with positive leading coefficients in orthonormalization, we obtain the polynomials $P_n(\lambda)$ uniquely determined by the properties:

P1. $\deg(P_n(\lambda)) = n$.

P2. leading coefficient of $P_n(\lambda)$ is positive.

P3. $(P_i(\lambda), P_j(\lambda)) = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$.

Any polynomial $p(\lambda) \in \mathbb{R}[\lambda]$ of degree less than n can be expressed in the form

$$p(\lambda) = \sum_{i=1}^{n-1} \xi_i P_i(\lambda) \qquad\qquad (\, \xi_i \in \mathbb{R}\,)\,,$$

so by bilinearity of inner product and property P3 of polynomials $P_n(\lambda)$, it follows that

$$(P_n(\lambda), p(\lambda)) = 0 \tag{2.5}$$

whenever $\deg(p(\lambda)) < n$.

Jacobi matrices

Multiplication of polynomials $p(\lambda)$ by the indeterminate λ defines a linear operator $\mathcal{T}$ on $\mathbb{R}[\lambda]$ by

$$\mathcal{T}(p(\lambda)) = \lambda p(\lambda).$$

The matrix corresponding to $\mathcal{T}$ in the basis $\left(\lambda^k\right)_{k=0}^{\infty}$ is

$$\begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now let's consider the matrix form $\mathcal{J} = (a_{ij})$ of $\mathcal{T}$ in the basis $(\, P_n(\lambda)\,)_{n=1}^{\infty}$. We need to find the coefficients a_{kj} in the equations

$$\lambda P_k(\lambda) = \sum_{j=0}^{k+1} a_{kj} P_j(\lambda). \tag{2.6}$$

Inner product of both sides of the equation by $P_i(\lambda)$ and linearity yields

$$(\, \lambda P_k(\lambda), P_i(\lambda)\,) = \sum_{j=0}^{k+1} a_{kj}\,(\, P_j(\lambda), P_i(\lambda)\,)$$

which proves that

$$(\lambda P_k(\lambda), P_i(\lambda)) = a_{ki}.$$

by property P3 of polynomials $P_n(\lambda)$. Observing the fact

$$(\lambda P_k(\lambda), P_i(\lambda)) = \Gamma (\lambda P_k(\lambda) P_i(\lambda)) = (\lambda P_i(\lambda), P_k(\lambda)) ,$$

it follows that

$$a_{ik} = a_{ki} .$$

Also, for $i < k-1$ we have

$$a_{ki} = (\lambda P_i(\lambda), P_k(\lambda)) = 0$$

by equation (2.5) since $\deg (\lambda P_i(\lambda)) < k$.

The leading coefficients of both $P_{k+1}(\lambda)$ and $\lambda P_k(\lambda)$ are positive. So, since $\deg(\lambda P_k(\lambda) - a_{k,k+1} P_{k+1}(\lambda))$ is less than n+1, we have that $a_{k,k+1}$ is positive.

If we define

$$a_k := a_{k,k} \quad , \quad b_k := a_{k,k+1} ,$$

equation (2.6) takes the form

$$\lambda P_k(\lambda) = b_k P_{k+1}(\lambda) + a_k P_k(\lambda) + b_{k-1} P_{k-1}(\lambda). \tag{2.7}$$

Then the matrix $\mathcal{J}$ for $\mathcal{T}$ in the basis $\{ P_k(\lambda) \}_{k=1}^{\infty}$ is of the form

$$\mathcal{J} = \begin{pmatrix} a_0 & b_0 & 0 & 0 & 0 & \dots \\ b_0 & a_1 & b_1 & 0 & 0 & \dots \\ 0 & b_1 & a_2 & b_2 & 0 & \\ 0 & 0 & b_2 & a_3 & b_3 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \qquad (a_k \in \mathbb{R}, \quad b_k > 0). \tag{2.8}$$

Definition 2.2.2. Matrices of the form (2.8) are called *Jacobi matrices.*

The correspondence

We have reached the result that given any positive sequence $(s_k)_{k=0}^{\infty}$, we can define an inner product on $\mathbb{R}[\lambda]$ satisfying $(\lambda^i, \lambda^k) = s_{i+k}$ and with respect to this inner product, we obtain a unique orthogonal basis $\{ P_k(\lambda) \}_{k=0}^{\infty}$ with the properties P1, P2 & P3 and a Jacobi matrix $\mathcal{J}$ in this basis for the operator $\mathcal{T}$ of multiplication by λ.

Now the question is, when we start with a Jacobi matrix $\mathcal{J}$, can we obtain a normalized positive sequence such that $\mathcal{J}$ will correspond to the multiplication by λ operator in the basis $\{ P_k(\lambda) \}_{k=0}^{\infty}$ of orthonormal polynomials which are determined uniquely by the inner product defined by equations (2.1), (2.2) and the properties P1, P2 and P3.

For a Jacobi matrix $\mathcal{J} = (a_{ij})$, we have

$$\mathcal{J} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \\ \gamma_{k+1} \\ 0 \\ \vdots \end{pmatrix} \tag{2.9}$$

since $a_{ij} = 0$ whenever $i > j + 1$.

Define ξ_{ik} such that $\xi_{00} = 1$, $\xi_{ik} = 0$ for $i > k$ and

$$\mathcal{J}^k \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \xi_{0k} \\ \vdots \\ \xi_{kk} \\ 0 \\ \vdots \end{pmatrix} \tag{2.10}$$

for $i \leq k$.

For any $k \in \mathbb{N}$ we have $b_k > 0$ in a Jacobi matrix, so $\xi_{i-1,i-1}$ and ξ_{ii} have the same sign. This yields that $\xi_{kk} > 0$ for any k.

Let us define $s_k := \xi_{0k}$.

Now consider the triangular matrices $\mathcal{C} = (\xi_{ik})$ and its inverse $\mathcal{C}^{-1}$, which of the form

$$\mathcal{C} = \begin{pmatrix} \xi_{00} & \xi_{01} & \xi_{02} & \cdots \\ 0 & \xi_{11} & \xi_{12} & \cdots \\ 0 & 0 & \xi_{22} & \\ \vdots & \vdots & & \ddots \end{pmatrix} \quad , \quad \mathcal{C}^{-1} = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ 0 & p_{11} & p_{12} & \cdots \\ 0 & 0 & p_{22} & \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

We have that $p_{kk} > 0$ for any $k \in \mathbb{N}$ since $p_{kk}\xi_{kk} = 1$ and $\xi_{kk} > 0$. Define $P_k(\lambda) := p_{0k} + p_{1k}\lambda + p_{2k}\lambda^2 + \ldots + p_{kk}\lambda^k$ and define an inner product on $\mathbb{R}[\lambda]$ by

$$(\, P_i(\lambda), P_k(\lambda)\,) = \delta_{ij} \qquad\qquad (i, j \in \mathbb{N}).$$

Define $\mathcal{E}_i$ to be the column matrix whose all entries are zero but the i'th, which is 1. Consider the equation

$$\mathcal{C} \begin{pmatrix} p_{0k} \\ \vdots \\ p_{kk} \\ 0 \\ \vdots \end{pmatrix} = \mathcal{E}_k \,.$$

So, $\mathcal{C}$ is the transition matrix from base $\{\, \lambda^n \,\}_{n=0}^{\infty}$ to base $\{\, P_n(\lambda) \,\}_{n=0}^{\infty}$ and $\mathcal{C}^{-1}$ is the base transition matrix in the reverse direction.

By computation we get:

$$\begin{aligned} \mathcal{J}^k \mathcal{E}_0 &= \mathcal{C}\mathcal{E}_k \qquad\qquad (\text{ for any } k \in \mathbb{Z}\ ) \\ \mathcal{J}\mathcal{C}\mathcal{E}_k = \mathcal{J}^{k+1}\mathcal{E}_0 &= \mathcal{C}\mathcal{E}_{k+1} \\ \mathcal{C}^{-1}\mathcal{J}\mathcal{C}\mathcal{E}_k &= \mathcal{E}_{k+1}\,. \end{aligned} \tag{2.11}$$

By equation (2.11), the matrix $\mathcal{C}^{-1}\mathcal{J}\mathcal{C}$ is of the form

$$\mathcal{C}^{-1}\mathcal{J}\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. $\mathcal{C}^{-1}\mathcal{J}\mathcal{C}$ is matrix of multiplication by λ operator in the basis $\left\{ \lambda^k \right\}_{k=0}^{\infty}$. Since $\mathcal{C}$ is the base transition matrix it follows that $\mathcal{J}$ is the matrix of multiplication by λ in the basis $\{ P_k(\lambda) \}_{k=0}^{\infty}$.

Now we should prove that $(\lambda^i, \lambda^k) = s_{i+k}$. Indeed:

$$\begin{aligned}
(\lambda^i, \lambda^k) &= \left(\sum_{n=1}^{i} \xi_{ni} P_n(\lambda), \sum_{m=1}^{k} \xi_{mk} P_m(\lambda) \right) = \sum_{n=1}^{\min(i,k)} \xi_{ni}\xi_{nk} = (\mathcal{J}^i \mathcal{E}_0)^T \mathcal{J}^k \mathcal{E}_0 \\
&= \mathcal{E}_0^T \mathcal{J}^i \mathcal{J}^k \mathcal{E}_0 = \mathcal{E}_0^T \mathcal{J}^{i+k} \mathcal{E}_0 \\
&= \mathcal{E}_0^T \begin{pmatrix} \xi_{0,i+k} \\ \vdots \\ \xi_{i+k,i+k} \\ 0 \\ \vdots \end{pmatrix} = \xi_{0,i+k} = s_{i+k} \,.
\end{aligned}$$

Now it should be proved that $(s_k)_{k=0}^{\infty}$ is a positive sequence. Given $x_1, \dots, x_n \in \mathbb{R}$ at least one of which is nonzero, we have $\xi_1, \dots, \xi_n \in \mathbb{R}$ satisfying

$$\sum_{i=1}^{n} x_i \lambda^i = \sum_{i=1}^{n} \xi_i P_n(\lambda).$$

Computation yields

$$\Gamma\left(\left(\sum_{k=1}^{n} x_k\lambda^k\right)^2\right) = \Gamma\left(\left(\sum_{k=1}^{n} \xi_k P_k(\lambda)\right)^2\right)$$

$$\Gamma\left(\sum_{0\le i,k\le n} x_i x_k \lambda^{i+k}\right) = \Gamma\left(\sum_{0\le i,k\le n} \xi_i\xi_k P_i(\lambda)P_k(\lambda)\right)$$

$$\sum_{0\le i,k\le n} x_i x_k s_{i+k} = \sum_{0\le i,k\le n} \xi_i\xi_k (P_i(\lambda), P_k(\lambda)) = \sum_{k=1}^{n} \xi_k^2 > 0$$

which means the positivity of the sequence $(\, s_k \,)_{k=0}^{\infty}$, and also $s_0 = \xi_{00} = 1$ by definition. So the 1-1 correspondence between the Jacobi matrices and the normalized positive sequences is established.

2.3 The Recurrence Formula and the Polynomials of the Second Kind

The recurrence formula

The polynomials $(\, P_k(\lambda) \,)_{k=0}^{\infty}$ satisfy the finite difference relation

$$\lambda y_k = b_k y_{k+1} + a_k y_k + b_{k-1} y_{k-1} \tag{2.12}$$

as shown in equation (2.7), with initial conditions

$$P_0(\lambda) = 1 \qquad , \qquad P_1(\lambda) = \frac{\lambda - a_0}{b_0}\,.$$

Since the recurrence relation (2.12) is of degree 2 and the coefficient b_{k+1} of y_{k+1} is nonzero for any $k \in \mathbb{N}$, any solution $(\, y_k \,)_{k=0}^{\infty}$ of (2.12) is determined by y_0 and y_1. Thus (2.12) has two linearly independent solutions, one of which is seen to be the sequence $(\, P_k(\lambda) \,)_{k=0}^{\infty}$.

The recurrence formula can be rearranged as

$$b_k y_{k+1} = (\lambda - a_k) y_k - b_{k-1} y_{k-1}\,. \tag{2.13}$$

Let $(y_k)_{k=0}^{\infty}$ and $(z_k)_{k=0}^{\infty}$ be solutions of the recurrence relation (2.12) with parameters λ and μ, respectively. Then by equation (2.13)

$$b_k y_{k+1} z_k = (\lambda - a_k) y_k z_k - b_{k-1} y_{k-1} z_k \,. \tag{2.14a}$$

By symmetry it follows that

$$b_k y_k z_{k+1} = (\mu - a_k) y_k z_k - b_{k-1} y_k z_{k-1} \,. \tag{2.14b}$$

Subtracting (2.14a) from (2.14b) it follows that

$$b_k \left(y_k z_{k+1} - y_{k+1} z_k \right) = (\mu - \lambda) y_k z_k + b_{k-1} \left(y_{k-1} z_k - y_k z_{k-1} \right)$$

or equivalently;

$$b_k \left[y_k z_{k+1} - y_{k+1} z_k \right] - b_{k-1} \left[y_{k-1} z_k - y_k z_{k-1} \right] = (\mu - \lambda) y_k z_k \,. \tag{2.15}$$

Summing up equation (2.15) from $k = m$ to $n - 1$, the important formula

$$\begin{aligned} b_{n-1} \left(y_{n-1} z_n - y_n z_{n-1} \right) - b_{m-1} \left(y_{m-1} z_m - y_m z_{m-1} \right) \\ = (\mu - \lambda) \sum_{k=m}^{n-1} y_k z_k \end{aligned} \tag{2.16}$$

follows, which is an analogue of Green's formula from the theory of differential equations.

In equation (2.16), letting $y_k(\lambda) = P_k(\lambda), z_k(\mu) = P_k(\mu), m = 1$ yields the so called *Christoffel–Darboux formula*:

$$(\mu - \lambda) \sum_{k=0}^{n-1} P_k(\lambda) P_k(\mu) = b_{n-1} \left[P_{n-1}(\lambda) P_n(\mu) - P_n(\lambda) P_{n-1}(\mu) \right] . \tag{2.17}$$

The polynomials of the second kind

To make further use of analogue of Green's formula, let $y_k(\lambda) = P_k(\lambda)$, $\mu = \lambda$ and $m = 1$. Equation (2.16) transforms to

$$\begin{aligned} b_{n-1} \left[P_{n-1}(\lambda) z_n - P_n(\lambda) z_{n-1} \right] &= b_0 \left[P_0(\lambda) z_1 - P_1(\lambda) z_0 \right] \\ &= b_0 \left[z_1 - \frac{\lambda - a_0}{b_0} z_0 \right] . \end{aligned} \tag{2.18}$$

To simplify the equation, replace z_k with $Q_k(\lambda)$ which is defined by the initial conditions

$$Q_0(\lambda) = 0 \qquad , \qquad Q_1(\lambda) = \frac{1}{b_0}\,. \tag{2.19}$$

Then the equation (2.18) takes the form

$$P_{n-1}(\lambda)Q_n(\lambda) - P_n(\lambda)Q_{n-1}(\lambda) = \frac{1}{b_{n-1}}\,. \tag{2.20}$$

Proposition 2.3.1. *For any sequence* $(\,p_k(\lambda)\,)_{k=0}^{\infty}$ *of polynomials with real coefficients satisfying the recurrence relation* (2.12) *and* $p_0(\lambda)$ *is a constant, there exists unique* $c_1, c_2 \in \mathbb{R}$ *such that*

$$p_k(\lambda) = c_1 P_k(\lambda) + c_2 Q_k(\lambda)$$

for any $k \in \mathbb{R}$*.*

Proof. There exists unique c_1 and c_2 which are solutions of the linear equations

$$\begin{aligned} p_0(\lambda) &= c_1 P_0(\lambda) + c_2 Q_0(\lambda), \\ p_1(\lambda) &= c_1 P_1(\lambda) + c_2 Q_1(\lambda). \end{aligned}$$

The sequence $(\,c_1 P_k(\lambda) + c_2 Q_k(\lambda)\,)_{k=0}^{\infty}$ is a linear combination of $(\,P_k(\lambda)\,)_{k=0}^{\infty}$ and $(\,Q_k(\lambda)\,)_{k=0}^{\infty}$, and thus it is also solution of the recurrence relation (2.12). Now the result follows since the first two terms of the sequences $(\,c_1 P_k(\lambda) + c_2 Q_k(\lambda)\,)_{k=0}^{\infty}$ and $(\,p_k(\lambda)\,)_{k=0}^{\infty}$ are the same. □

The polynomials $Q_k(\lambda)$ are called *polynomials of the second kind*, and the polynomials $P_k(\lambda)$ are called *polynomials of the first kind* . Consider the polynomials

$$q_k(\lambda) = \Gamma_\mu\left(\frac{P_k(\lambda) - P_k(\mu)}{\lambda - \mu}\right) \qquad (k \in \mathbb{N}).$$

The first two terms are

$$\begin{aligned} q_0(\lambda) &= \Gamma_\mu\left(\frac{1-1}{\lambda-\mu}\right) = \Gamma_\mu(0) = 0, \\ q_1(\lambda) &= \Gamma_\mu\left(\frac{\frac{\lambda - a_0}{b_0} - \frac{\mu - a_0}{b_0}}{\lambda - \mu}\right) = \Gamma_\mu\left(\frac{1}{b_0}\right) = \frac{1}{b_0}\,. \end{aligned} \tag{2.21}$$

The sequence $(\, q_n(\lambda)\,)_{k=0}^{\infty}$ also satisfies the recurrence relation (2.12). Indeed, for $k \geq 1$ we have:

$$
\begin{aligned}
\lambda q_k(\lambda) &= \lambda \Gamma_\mu \left(\frac{P_k(\lambda) - P_k(\mu)}{\lambda - \mu} \right) = \Gamma_\mu \left(\frac{\lambda P_k(\lambda) - \lambda P_k(\mu)}{\lambda - \mu} \right) \\
&= \Gamma_\mu \left(\frac{\lambda P_k(\lambda) - \mu P_k(\mu)}{\lambda - \mu} + \frac{\mu P_k(\mu) - \lambda P_k(\mu)}{\lambda - \mu} \right) \\
&= \Gamma_\mu \Bigg(b_k \frac{P_{k+1}(\lambda) - P_{k+1}(\mu)}{\lambda - \mu} + a_k \frac{P_k(\lambda) - P_k(\mu)}{\lambda - \mu} \\
&\qquad + b_{k-1} \frac{P_{k-1}(\lambda) - P_{k-1}(\mu)}{\lambda - \mu} \Bigg) + \Gamma_\mu(\, P_k(\mu)) \\
&= b_k \Gamma_\mu \left(\frac{P_{k+1}(\lambda) - P_{k+1}(\mu)}{\lambda - \mu} \right) + a_k \Gamma_\mu \left(\frac{P_k(\lambda) - P_k(\mu)}{\lambda - \mu} \right) \\
&\qquad + b_{k-1} \Gamma_\mu \left(\frac{P_{k-1}(\lambda) - P_{k-1}(\mu)}{\lambda - \mu} \right) + (P_k(\mu), 1) \\
&= b_k q_{k+1}(\lambda) + a_k q_k(\lambda) + b_{k-1} q_{k-1}(\lambda).
\end{aligned}
\tag{2.22}
$$

The sequence $(\, q_k(\lambda)\,)_{k=0}^{\infty}$ satisfies the same initial conditions with the sequence $(\, Q_k(\lambda)\,)_{k=0}^{\infty}$ so:

$$
Q_k(\lambda) = q_k(\lambda) = \Gamma_\mu \left(\frac{P_k(\lambda) - P_k(\mu)}{\lambda - \mu} \right) \qquad (\, \forall k \in \mathbb{N}\,). \tag{2.23}
$$

2.4 Truncated Moment Problem

Non-decreasing step functions

Consider an non-decreasing step function σ with increase points $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Then we cannot define an inner product on the space $\mathbb{R}[\lambda]$ by the formula

$$
(p(\lambda), q(\lambda))_\sigma = \int_{-\infty}^{+\infty} p(\lambda) q(\lambda)\, d\sigma(\lambda) \tag{2.24}
$$

since there exists a nonzero polynomial $p(\lambda)$ of degree n such that $(p(\lambda), p(\lambda))_\sigma = \int_{-\infty}^{+\infty} (\, p(\lambda)\,)^2 \, d\sigma(\lambda) = 0$ as shown in the proof of Proposition 1.1.1 . Now let $p_{n-1}(\lambda)$ be a nonzero polynomial of degree less than n. There exists an i such that $p_{n-1}(\lambda_i) \neq 0$ since $p_{n-1}(\lambda)$ has at most $n-1$ zeroes. Let the increase of σ at λ_k be μ_k for each k. We have

$$\int_{-\infty}^{+\infty} p_{n-1}(\lambda)p_{n-1}(\lambda)\, d\sigma(\lambda) = \sum_{k=1}^{n} (p_{n-1}(\lambda_k))^2 \mu_k \geq (p_{n-1}(\lambda_i))^2 \mu_i > 0.$$

Let $\mathbb{R}_{n-1}[\lambda]$ denote the vector space of polynomials $p_{n-1}(\lambda)$ such that $\deg p_{n-1}(\lambda) < n$. Equation (2.24) defines an inner product on $\mathbb{R}_{n-1}[\lambda]$.

Note that the inner product has also the property

$$p(\lambda)q(\lambda) = \tilde{p}(\lambda)\tilde{q}(\lambda) \Rightarrow (p(\lambda), q(\lambda))_\sigma = (\tilde{p}(\lambda), \tilde{q}(\lambda))_\sigma\ . \tag{2.25}$$

Now a variant of moment problem arises: Given an inner product on $\mathbb{R}_{n-1}[\lambda]$ satisfying equation (2.25), (how) can we recover a non-decreasing step function σ_n with n points of increase such that equation (2.24) holds?

The truncated moment problem

Recall that given a positive sequence $(\, s_k\,)_{k=0}^{\infty}$, an inner product is defined on $\mathbb{R}[\lambda]$ by equation (2.2). For any $n > 0$ define an inner product $(\ ,\ )_{n-1}$ on $\mathbb{R}_{n-1}[\lambda]$ characterized by

$$\left(\, \lambda^i, \lambda^k \,\right)_{n-1} := s_{i+k} \qquad\qquad (i, k < n) \tag{2.26}$$

and linearity. These inner product spaces $\mathbb{R}_0[\lambda] \subset \mathbb{R}_1[\lambda] \subset \cdots \subset \mathbb{R}_{n-1}[\lambda] \subset \ \ldots$ form an ascending chain with compatible inner products, and all inner products are compatible with the inner product defined on $\mathbb{R}[\lambda]$ by the sequence $(\, s_k\,)_{k=0}^{\infty}$ as in (2.2):

$$(p(\lambda), q(\lambda))_{n-1} = (p(\lambda), q(\lambda)) \qquad (n > 0\, ;\, p(\lambda), q(\lambda) \in \mathbb{R}_{n-1}[\lambda]\,). \tag{2.27}$$

By equations (2.27) and (2.2) it follows that for $p(\lambda), q(\lambda), \tilde{p}(\lambda), \tilde{q}(\lambda)$ satisfying $p(\lambda)q(\lambda) = \tilde{p}(\lambda)\tilde{q}(\lambda)$ we have

$$(\, p(\lambda), q(\lambda)\,)_{n-1} = \Gamma(p(\lambda)q(\lambda)) = \Gamma(\tilde{p}(\lambda)\tilde{q}(\lambda)) = (\, \tilde{p}(\lambda), \tilde{q}(\lambda)\,)_{n-1}\ .$$

Thus the inner product $(\ ,\ )_{n-1}$ satisfies the property (2.25).

For a given sequence $(s_k)_{k=0}^{\infty}$, the variant of moment problem stated above of finding non-decreasing step function(s) σ_n with n increase points satisfying equation (2.24) for the inner product on $\mathbb{R}_{n-1}[\lambda]$ defined by the sequence $(s_k)_{k=0}^{\infty}$ via formula (2.26) takes the name *'truncated moment problem of order n'*.

Quasiorthogonal polynomials

To determine the candidates for the the solution σ_n of the truncated moment problem of order n, it is reasonable to search the eventual places of its increase points first.

Assume that we have such a solution σ_n with increase points $\lambda_1, \dots, \lambda_n$. Also let the jump of $\sigma_n(\lambda)$ at λ_i be μ_i. Then for any $p(\lambda), q(\lambda) \in \mathbb{R}_{n-1}[\lambda]$ we have

$$\int_{-\infty}^{+\infty} p(\lambda)q(\lambda)\, d\sigma_n(\lambda) = (\, p(\lambda), q(\lambda)\,)_{n-1}\ . \tag{2.28}$$

Now consider an arbitrary polynomial

$$\bar{p}(\lambda) = x_0 + x_1\lambda + \cdots + x_{2n-2}\lambda^{2n-2}$$

of degree at most $2n-2$. Then by equations (2.26), (2.28) and (2.1) it follows that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \bar{p}(\lambda)\, d\sigma_n(\lambda) &= \int_{-\infty}^{+\infty} \left(\sum_{i=0}^{n-1} x_i\lambda^i + \sum_{i=n}^{2n-2} x_i\lambda^{i-(n-1)}\lambda^{n-1} \right) d\sigma_n(\lambda) \\
&= \sum_{i=0}^{n-1} x_i \int_{-\infty}^{+\infty} \lambda^i\, d\sigma_n(\lambda) + \sum_{i=n}^{2n-2} x_i \int_{-\infty}^{+\infty} \lambda^{i-(n-1)}\lambda^{n-1}\, d\sigma_n(\lambda) \\
&= \sum_{i=0}^{n-1} x_i \left(\lambda^i, 1\right)_{n-1} + \sum_{i=n}^{2n-2} x_i \left(\lambda^{i-(n-1)}, \lambda^{n-1}\right)_{n-1} \\
&= \sum_{i=0}^{n-1} x_i s_i + \sum_{i=n}^{2n-2} x_i s_i = \sum_{i=0}^{2n-2} x_i s_i = \Gamma(\bar{p}(\lambda))
\end{aligned} \tag{2.29}$$

Define

$$p(\lambda) := \prod_{k=1}^{n} (\lambda - \lambda_i).$$

Then

$$\int_{-\infty}^{+\infty} (p(\lambda))^2 \, d\sigma_n = \sum_{i=1}^{n} (\, p(\lambda_i) \,)^2 \, \mu_i = 0.$$

Consider the orthonormal polynomials $(\, P_k(\lambda) \,)_{k=0}^{\infty}$ determined by the positive sequence $(\, s_k \,)_{k=0}^{\infty}$ as in Section 2.2 . Since $p(\lambda)$ is a polynomial of degree n, it can be written in the form

$$p(\lambda) = \sum_{k=1}^{n} a_k P_k(\lambda)$$

such that a_n is nonzero. Making use of equation (2.29) we have

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{n} a_k P_k(\lambda) \right)^2 d\sigma_n(\lambda) = \sum_{0 \le i,k \le n} \left(a_i a_k \int_{-\infty}^{+\infty} P_i(\lambda) P_k(\lambda) \, d\sigma_n(\lambda) \right) \\
&= \sum_{n-1 \le i,k \le n} \left(a_i a_k \int_{-\infty}^{+\infty} P_i(\lambda) P_k(\lambda) \, d\sigma_n(\lambda) \right) \\
&\qquad + \sum_{\substack{0 \le i,k \le n \\ min(i,k) < n-1}} \left(a_i a_k \int_{-\infty}^{+\infty} P_i(\lambda) P_k(\lambda) \, d\sigma_n(\lambda) \right) \\
&= \int_{-\infty}^{+\infty} \left[\, a_n^2 (P_n(\lambda))^2 + 2 a_{n-1} a_n P_{n-1}(\lambda) P_n(\lambda) + a_{n-1}^2 (P_{n-1}(\lambda))^2 \, \right] \, d\sigma_n(\lambda) \\
&\qquad + \sum_{\substack{0 \le i,k \le n \\ min(i,k) < n-1}} a_i a_k (P_i(\lambda), P_k(\lambda)) \\
&= \int_{-\infty}^{+\infty} \left[\, a_n P_n(\lambda) + a_{n-1} P_{n-1}(\lambda) \, \right]^2 \, d\sigma_n(\lambda) + \sum_{k=0}^{n-2} a_k^2 \, .
\end{aligned}$$

Both of the values $\int_{-\infty}^{+\infty} \left[a_n(P_n(\lambda)) + a_{n-1}(P_{n-1}(\lambda)) \right]^2 \, d\sigma_n(\lambda)$ and $\sum_{k=0}^{n-2} a_k^2$ are non-negative so they should be zero. $\sum_{k=0}^{n-2} a_k^2 = 0$ means that $a_k = 0$ for $k \le n-2$ and so $p(\lambda) = a_n P_n(\lambda) + a_{n-1} P_{n-1}(\lambda)$.

The conclusion is, points of increase of σ_n are the zeroes of a polynomial of the form $a_n P_n(\lambda) + a_{n-1} P_{n-1}(\lambda)$. This polynomial should have n zeroes for such a solution σ_n to exist, and in Section 2.5 it will be proved that it has indeed. Since the zeroes of a function is not affected if we multiply it by a nonzero number and since a_n is nonzero, we have the following result:

Proposition 2.4.1. *If $\sigma_n(\lambda)$ is a solution of the truncated moment problem, then all of its increase points are at the zeroes of the polynomial*

$$P_n(\lambda, \tau) := P_n(\lambda) - \tau P_{n-1}(\lambda) \tag{2.30}$$

for some real number τ. A polynomial of this form is named as quasiorthogonal polynomial of degree n.

Imitating the definition in equation (2.30), define

$$Q_n(\lambda, \tau) := Q_n(\lambda) - \tau Q_{n-1}(\lambda).$$

By linearity of Γ_μ and equation (2.23) it follows that

$$Q_k(\lambda, \tau) = \Gamma_\mu \left(\frac{P_k(\lambda, \tau) - P_k(\mu, \tau)}{\lambda - \mu} \right) \qquad (\, k \in \mathbb{N} \,). \tag{2.31}$$

2.5 The Quadrature Formula

In this section, the zeroes of the functions $P_k(\lambda, \tau)$ will be examined and the truncated moment problem will be solved.

Position of zeroes of $P_k(\lambda)$

Proposition 2.5.1. *The zeroes of $P_n(\lambda, \tau)$ are real and simple.*

Proof. Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m$$

denote the real and simple zeroes of $P_n(\lambda, \tau)$. Consider the polynomial $p(\lambda) = \prod_{k=1}^{m} (\lambda - \lambda_k)$. Then $P_n(\lambda, \tau) p(\lambda) \geq 0$ for any $\lambda \in \mathbb{R}$, thus

$$(P_n(\lambda, \tau), p(\lambda)) = \Gamma(P_n(\lambda, \tau) p(\lambda)) > 0$$

by Proposition 2.2.1 . But if we had $m < n - 1$ then we would also have $\deg(p(\lambda)) < n - 1$ and by equation (2.5) it would follow that

$$(P_n(\lambda, \tau), p(\lambda)) = 0.$$

Therefore $m \geq n - 1$ and $P_n(\lambda, \tau)$ has at least $n - 1$ real and simple zeroes. Then all of its n zeroes are simple; and being a polynomial with real coefficients, all of its zeroes are real.

□

Proposition 2.5.2. *The zeroes of $P_n(\lambda)$ and $P_{n-1}(\lambda)$ alternate.*

Proof. The Christoffel–Darboux formula (equation (2.17)) can be written in the form

$$\sum_{k=0}^{n-1} P_k(\lambda) P_k(\mu) = b_{k-1} \left(P_{n-1}(\lambda) \frac{P_n(\mu) - P_n(\lambda)}{\mu - \lambda} - P_n(\lambda) \frac{P_{n-1}(\mu) - P_{n-1}(\lambda)}{\mu - \lambda} \right).$$

Taking limit of both sides of the equation as $\mu \to \lambda$ yields

$$\sum_{k=0}^{n-1} [P_k(\lambda)]^2 = b_{n-1} [P_{n-1}(\lambda) P_n'(\lambda) - P_n(\lambda) P_{n-1}'(\lambda)], \tag{2.32}$$

so $P_{n-1}(\lambda) P_n'(\lambda) - P_n(\lambda) P_{n-1}'(\lambda) > 0$ for any λ . Let λ_i and λ_{i+1} be two consecutive zeroes of $P_n(\lambda)$. Then

$$P_{n-1}(\lambda_i) P_n'(\lambda_i) > 0 \quad \text{and} \quad P_{n-1}(\lambda_{i+1}) P_n'(\lambda_{i+1}) > 0.$$

Since all the zeroes of $P_n(\lambda) = P_n(\lambda, 0)$ are simple, the signs of $P_n'(\lambda_i)$ and $P_n'(\lambda_{i+1})$ are opposite. So the signs of $P_{n-1}(\lambda_i)$ and $P_{n-1}(\lambda_{i+1})$ are opposite.

Then the polynomial $P_{n-1}(\lambda)$ has at least one zero in the interval $(\lambda_i, \lambda_{i+1})$. The polynomial $P_n(\lambda)$ has n distinct real zeroes so $P_{n-1}(\lambda)$ has exactly one zero between two consecutive zeroes of $P_n(\lambda)$.

□

The quadrature formula

Consider the truncated moment problem of order n. Let a normalized positive sequence $(s_k)_{k=0}^{\infty}$ be given and let $\lambda_1 < \cdots < \lambda_n$ be the zeroes of $P_n(\lambda, \tau)$ for some $\tau \in \mathbb{R}$. We are to find positive $\mu_1, \ldots, \mu_n$ such that

$$\Gamma(p_{2n-2}(\lambda)) = \sum_{k=1}^{n} p_{2n-2}(\lambda_k)\, \mu_k \tag{2.33}$$

for any polynomial $p_{2n-2}(\lambda)$ of degree at most $2n-2$. By division algorithm, $p_{2n-2}(\lambda)$ can be represented as

$$p_{2n-2}(\lambda) = P_n(\lambda, \tau) q_{n-2}(\lambda) + r_{n-1}(\lambda)$$

where $q_{n-2}(\lambda)$ and $r_{n-1}(\lambda)$ are polynomials of degrees $n-2$ and $n-1$, respectively. Since $P_n(\lambda, \tau)$ is orthogonal to any polynomial of degree less than $n-1$, we have $\Gamma(P_n(\lambda, \tau) q_{n-2}(\lambda)) = (P_n(\lambda, \tau), q_{n-2}(\lambda)) = 0$ and thus

$$\Gamma(p_{2n-2}(\lambda)) = \Gamma(r_{n-1}(\lambda)). \tag{2.34}$$

$P_n(\lambda, \tau) = c_n \prod_{k=1}^{n} (\lambda - \lambda_k)$ for some constant c_n. So its derivative is

$$P_n'(\lambda, \tau) = c_n \sum_{j=1}^{n} \left(\prod_{i \neq j} (\lambda - \lambda_i) \right)$$

and at zeroes of $P_n(\lambda, \tau)$ it takes the form

$$P_n'(\lambda_k, \tau) = c_n \prod_{i \neq k} (\lambda_k - \lambda_i)$$

The Lagrange interpolation formula and the equality $p_{2n-2}(\lambda_k) = r_{n-1}(\lambda_k)$ at zeroes of $P_n(\lambda,\tau)$ yields

$$\begin{aligned} r_{n-1}(\lambda) &= \sum_{k=1}^{n} r_{n-1}(\lambda_k)\frac{\prod_{i\neq k}(\lambda-\lambda_i)}{\prod_{i\neq k}(\lambda_k-\lambda_i)} \\ &= \sum_{k=1}^{n} p_{2n-2}(\lambda_k)\frac{P_n(\lambda,\tau)}{P'_n(\lambda_k,\tau)(\lambda-\lambda_k)}\,. \end{aligned}$$

Applying Γ to both sides and by $P_n(\lambda_k,\tau)=0$ we have

$$\Gamma(r_{n-1}(\lambda)) = \sum_{k=1}^{n} p_{2n-2}(\lambda_k)\frac{\Gamma\left(\frac{P_n(\lambda,\tau)-P_n(\lambda_k,\tau)}{\lambda-\lambda_k}\right)}{P'_n(\lambda_k,\tau)}\,.$$

Using equations (2.31) and (2.34) yields

$$\Gamma(p_{2n-2}(\lambda)) = \sum_{k=1}^{n} p_{2n-2}(\lambda_k)\frac{Q_n(\lambda_k,\tau)}{P'_n(\lambda_k,\tau)}\,. \tag{2.35}$$

The equation (2.35) is called *the quadrature formula*. By the quadrature formula, the required μ_k's in equation (2.33) are obtained if quantities $Q_n(\lambda_k,\tau)/P'_n(\lambda_k,\tau)$ are positive. To prove that they are positive, arrange $P_n(\lambda_k) - \tau P_{n-1}(\lambda_k) = 0$ as

$$\tau = \frac{P_n(\lambda_k)}{P_{n-1}(\lambda_k)} \qquad\qquad (1 \leq k \leq n).$$

This yields

$$\frac{Q_n(\lambda_k,\tau)}{P'_n(\lambda_k,\tau)} = \frac{P_{n-1}(\lambda_k)Q_n(\lambda) - P_n(\lambda_k)Q_{n-1}(\lambda)}{P_{n-1}(\lambda_k)P'_n(\lambda) - P_n(\lambda_k)P'_{n-1}(\lambda)} > 0 \tag{2.36}$$

since the numerator is positive by equation (2.20) and the denominator is positive by equation (2.32). Thus we can choose

$$\mu_k = \frac{Q_n(\lambda_k,\tau)}{P'_n(\lambda_k,\tau)}\,. \tag{2.37}$$

Substituting equations (2.20) and (2.32) into equation (2.37), we have

$$\mu_k = \frac{1}{\sum_{i=0}^{n-1} [P_i(\lambda_k)]^2} .$$

Now let $\sigma'(\lambda)$ be any solution to the truncated moment problem of order n with the same increase points $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let the increase of $\sigma'(\lambda)$ at λ_k be c_k. By equation (2.33), the numbers $c_k, 1 < k < n$ satisfy the system of n linear equations

$$\sum_{k=1}^{n} c_k \lambda_k^i = \Gamma(\lambda^i) = s_i \qquad (0 < i < n-1), \tag{2.38}$$

where $\lambda_k^0 = 1$ for any k. The matrix

$$\begin{pmatrix} {\lambda_1}^0 & {\lambda_2}^0 & \ldots & {\lambda_n}^0 \\ {\lambda_1}^1 & {\lambda_2}^1 & \ldots & {\lambda_n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ {\lambda_1}^{n-1} & {\lambda_2}^{n-1} & \ldots & {\lambda_n}^{n-1} \end{pmatrix}$$

has nonzero determinant and the system of equations (2.38) has unique solution. Thus $c_k = \mu_k$ for any $k, 1 \leq k \leq n$. Combining these results with Proposition 2.4.1, we obtain the following proposition:

Proposition 2.5.3. *Given a positive sequence* $(s_k)_{k=0}^{\infty}$ *and* $\tau \in \mathbb{R}$, *there exists a unique solution* $\sigma_n^\tau(\lambda)$ *of the truncated moment problem of order* n *whose increase points* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ *are the zeroes of the quasiorthogonal polynomial* $P_n(\lambda, \tau)$ *of order* n *and the increase of* $\sigma_n^\tau(\lambda)$ *at* λ_k *is*

$$\mu_k = \frac{Q_n(\lambda_k, \tau)}{P_n'(\lambda_k, \tau)} = \frac{1}{\sum_{i=0}^{n-1} [P_i(\lambda_k)]^2} \tag{2.39}$$

for each $k, 1 \leq k \leq n$. *Moreover, any solution of the truncated moment problem of order* n *is of this form.*

2.6 Solvability Criteria for the Extended Moment Problem

To obtain a solution of the extended moment problem, two theorems of E. Helly will be used:

Theorem 2.6.1 (Helly's Choice Theorem). *Let* $(\sigma_n)_{n=0}^{\infty}$ *be a sequence of uniformly bounded non-decreasing functions on* $\mathbb{R}$*. Then it has a subsequence* $(\sigma_{n_i})_{i=0}^{\infty}$ *such that for some non-decreasing function* $\sigma : \mathbb{R} \to \mathbb{R}$*,*

$$\lim_{i\to\infty} \sigma_{n_i}(\lambda) = \sigma(\lambda)$$

for any $\lambda \in \mathbb{R}$ *.*

Proof. Choose a sequence $(a_k)_{k=0}^{\infty}$ of real numbers whose terms are dense in $\mathbb{R}$. The sequence $(\sigma_n(a_1))_{n=0}^{\infty}$ is bounded so it has a convergent subsequence $(\sigma_{n1}(a_1))_{n=0}^{\infty}$. Similarly if we have a sequence $(\sigma_{ni}(\lambda))_{n=0}^{\infty}$ convergent for $\lambda = a_1, \ldots, a_i$ then it has a subsequence $(\sigma_{n,i+1}(\lambda))_{n=0}^{\infty}$ convergent for $\lambda = a_1, \ldots, a_i, a_{i+1}$. So the diagonal sequence $(\sigma_{nn}(\lambda))_{n=0}^{\infty}$ converges for any $\lambda = a_k$, $k \in \mathbb{N}$. Since the sequence $(a_k)_{k=0}^{\infty}$ is dense in $\mathbb{R}$, the sequence $(\sigma_{nn}(\lambda))_{n=0}^{\infty}$ converges to a non-decreasing function σ' for all but countable number of real numbers. Repeating the diagonal construction on this countable set, the desired subsequence $(\sigma_{n_i})_{i=0}^{\infty}$ and the non-decreasing function σ are obtained.

□

Theorem 2.6.2 (Helly's Convergence Theorem). *Let* $(\sigma_n)_{n=0}^{\infty}$ *be a sequence of uniformly bounded non-decreasing functions on a finite interval* $[a, b]$ *such that* $\lim_{n\to\infty} \sigma_n(\lambda) = \sigma(\lambda)$ *for any* $\lambda \in [a, b]$ *and let* $f(\lambda)$ *be a continuous function on* $[a, b]$*. Then we have*

$$\lim_{n\to\infty} \int_a^b f(\lambda)\, d\sigma_n(\lambda) = \int_a^b f(\lambda)\, d\sigma(\lambda).$$

Proof. The function $f(\lambda)$ is uniformly continuous on the compact set $[a,b]$, so given $\varepsilon > 0$ there exists δ such that if $|\, t - t' \,| < \delta$ then $|\, f(t) - f(t') \,| < \varepsilon$. For any partition $a = a_0 < a_1 < a_2 < \cdots < a_m = b$ of $[a,b]$ with length of subintervals less than δ, consider the Riemann–Stieltjes sums

$$S_n = \sum_{i=1}^{m} f(a_i)[\sigma_n(a_i) - \sigma_n(a_{i-1})] \quad , \quad S = \sum_{i=1}^{m} f(a_i)[\sigma(a_i) - \sigma(a_{i-1})]$$

and the integrals

$$I_n = \int_a^b f(\lambda)\, d\sigma_n(\lambda) = \sum_{i=1}^{m} f(\xi_{n_i})[\sigma_n(a_i) - \sigma_n(a_{i-1})]$$
$$I = \int_a^b f(\lambda)\, d\sigma(\lambda) = \sum_{i=1}^{m} f(\xi_i)[\sigma(a_i) - \sigma(a_{i-1})]$$

for any n, where the second equalities hold for some choice $\xi_i, \xi_{n_i} \in (a_{i-1}, a_i)$ by Mean Value Theorem. It follows that $|S_n - I_n|$, $|S - I| < 2M\varepsilon$ where M is the common bound for the functions σ_n, and hence also for their limit σ. Since $\sigma_n(a_i) \to \sigma(a_i)$, we have that there exists N_ε such that $|\, S_n - S \,| < M\varepsilon$ for any $n > N_\varepsilon$. Thus $n > N_\varepsilon$ implies $|I - I_n| < |I - S| + |S - S_n| + |S_n - I_n| < 5M\varepsilon$, which completes the proof.

□

Now we can turn back to the extended moment problem.

Theorem 2.6.3. (Hamburger) *Given a sequence* $(\, s_k \,)_{k=0}^{\infty}$ *of real numbers, there exists a mass distribution* σ *such that*

$$\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda) = s_k \qquad (k \in \mathbb{N})$$

if and only if the sequence $(\, s_k \,)_{k=0}^{\infty}$ *is positive.*

Proof. To prove necessity, let a mass distribution σ be given satisfying $\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda) = s_k$ and choose any nonzero $p(\lambda) = \sum_{k=0}^{n} x_k \lambda^k \in \mathbb{R}[\lambda]$. Then by the expression (1.4) we have

$$0 < \int_{-\infty}^{+\infty} [p(\lambda)^2]\, d\sigma(\lambda) = \int_{-\infty}^{+\infty} \left(\sum_{k=0}^{n} x_k \lambda^k \right)^2 d\sigma(\lambda)$$
$$= \sum_{0 \le i,k \le n} x_i x_k \int_{-\infty}^{+\infty} \lambda^{i+k}\, d\sigma(\lambda) = \sum_{0 \le i,k \le n} x_i x_k s_{i+k}\,.$$

This inequality is valid for any $n \in \mathbb{N}$ and any choice of $x_1, \ldots, x_n$ not all zero. Thus positivity of the sequence $(\, s_k \,)_{k=0}^{\infty}$ follows.

Now let a positive sequence $(\, s_k \,)_{k=0}^{\infty}$ be given. Take a sequence $(\, \sigma_n \,)_{n=0}^{\infty}$ of functions such that $\sigma_n(\lambda)$ is a solution of the truncated moment problem of order n. Then by Helly's choice theorem it has a subsequence $(\, \sigma_{n_i} \,)_{i=0}^{\infty}$ converging to a non-decreasing function $\sigma(\lambda)$ pointwise. For any $k \in \mathbb{N}$ Helly's convergence theorem yields the equality

$$\int_a^b \lambda^k\, d\sigma(\lambda) = \lim_{i \to \infty} \int_a^b \lambda^k\, d\sigma_{n_i}(\lambda)$$

for any finite interval $[a, b]$. If the integrals $\int_a^b \lambda^k\, d\sigma_{n_i}(\lambda)$ converge uniformly with respect to the variable i then the equality

$$\int_{-\infty}^{+\infty} \lambda^k\, d\sigma(\lambda) = \lim_{i \to \infty} \int_{-\infty}^{+\infty} \lambda^k\, d\sigma_{n_i}(\lambda) = s_k \tag{2.40}$$

follows. To prove the uniform convergence, choose a and b such that $a < -1$, $b > 1$ and let $2r$ be an even number greater than k.

$$\begin{aligned}
\left| s_k - \int_a^b \lambda^k \, d\sigma_{n_i} \right| &= \left| \int_{-\infty}^{+\infty} \lambda^k \, d\sigma_{n_i}(\lambda) - \int_a^b \lambda^k \, d\sigma_{n_i}(\lambda) \right| \\
&\le \int_{-\infty}^{a} |\, \lambda^k \,| \, d\sigma_{n_i}(\lambda) + \int_b^{+\infty} |\, \lambda^k \,| \, d\sigma_{n_i}(\lambda) \\
&\le \int_{-\infty}^{a} \frac{\lambda^{2r}}{-\lambda} \, d\sigma_{n_i}(\lambda) + \int_b^{+\infty} \frac{\lambda^{2r}}{\lambda} \, d\sigma_{n_i}(\lambda) \\
&\le \frac{1}{|\, a \,|} \int_{-\infty}^{a} \lambda^{2r} \, d\sigma_{n_i}(\lambda) + \frac{1}{|\, b \,|} \int_b^{+\infty} \lambda^{2r} \, d\sigma_{n_i}(\lambda) \\
&\le \left(\frac{1}{|\, a \,|} + \frac{1}{|\, b \,|} \right) \int_{-\infty}^{+\infty} \lambda^{2r} \, d\sigma_{n_i} = \frac{s_{2r}}{|\, a \,| + |\, b \,|} \, .
\end{aligned}$$

Thus the equation (2.40) is proved. For $\sigma(\lambda)$ to be a mass distribution, it should have infinite number of increase points. Assume that its only increase points are $\lambda_1, \ldots, \lambda_m$. Consider the polynomial

$$p(\lambda) = \prod_{k=1}^{m} (\, \lambda - \lambda_k \,) = \sum_{k=1}^{m} a_k \lambda^k.$$

But then we have

$$0 = \int_{-\infty}^{+\infty} (\, p(\lambda) \,)^2 \; d\sigma(\lambda) = \int_{-\infty}^{+\infty} \sum_{i,k=1}^{m} a_i a_k \lambda^{i+k} \, d\sigma(\lambda) = \sum_{i,k=1}^{m} a_i a_k s_{i+k} \, ,$$

contradicting the positivity of the sequence $(\, s_k \,)_{k=0}^{\infty}$. □

CHAPTER 3

UNIQUENESS OF MOMENT PROBLEM SOLUTION

3.1 Investigations for a Condition

Whenever we have a mass distribution σ, and hence a measure on $\mathbb{R}$, the integral of a function $f : \mathbb{R} \to \mathbb{C}$ of the form $f(\lambda) = f_1(\lambda) + i f_2(\lambda)$ with respect to that measure is defined as

$$\int_{-\infty}^{+\infty} f(\lambda)\, d\sigma(\lambda) := \int_{-\infty}^{+\infty} f_1(\lambda)\, d\sigma(\lambda) + i \int_{-\infty}^{+\infty} f_2(\lambda)\, d\sigma(\lambda)$$

whenever both $f_1(\lambda)$ and $f_2(\lambda)$ are integrable for the given measure.

If the function $f : \mathbb{R} \to \mathbb{C}$ is continuous and bounded then $f_1(\lambda)$ and $f_2(\lambda)$ are continuous and bounded. Therefore both functions $f_1(\lambda)$ and $f_2(\lambda)$ are $d\sigma$-integrable and hence $f(\lambda)$ is $d\sigma$-integrable.

Let a positive sequence $(s_k)_{k=0}^{\infty}$ be given. We examine the values of the integral

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma(\lambda)$$

for complex numbers z such that $\operatorname{Im} z \neq 0$, where σ is a solution of the moment problem. This integral exists for any such z since the function $f(\lambda) = \frac{1}{\lambda - z}$ is continuous and bounded.

Recall that if two mass distributions are equivalent in the sense of definition 1.1.1 then they yield the same measure. In particular, given a positive sequence $(s_k)_{k=0}^{\infty}$, if the related moment problem has a unique solution then for two solutions σ_1 and σ_2 of the moment problem and for any $z \notin \mathbb{R}$ we have

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_1(\lambda) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_2(\lambda).$$

The following theorem asserts that the converse is also true.

Theorem 3.1.1. *Given two mass distributions σ_1 and σ_2, let*

$$\int_{-\infty}^{+\infty} \lambda^k\, d\sigma_1(\lambda) = \int_{-\infty}^{+\infty} \lambda^k\, d\sigma_2(\lambda) \tag{3.1}$$

for any $k \in \mathbb{N}$. Then σ_1 and σ_2 are equivalent if and only if they satisfy

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_1(\lambda) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_2(\lambda).$$

for any non-real complex number z. In particular, the moment problem for a positive sequence has a unique solution if and only if for any $z \notin \mathbb{R}$, the value of the integral

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma(\lambda)$$

is the same for any solution σ of the moment problem.

Proof. Let

$$\int_{-\infty}^{+\infty} \frac{d\sigma_1(\lambda)}{\lambda - z} = \int_{-\infty}^{+\infty} \frac{d\sigma_2(\lambda)}{\lambda - z} \qquad (\forall\, z \notin \mathbb{R})$$
$$\int_{-\infty}^{+\infty} \lambda^k\, d\sigma_1(\lambda) = \int_{-\infty}^{+\infty} \lambda^k\, d\sigma_2(\lambda) \qquad (\forall\, k \in \mathbb{N}).$$

Define the charge $\omega\colon \mathbb{R} \to \mathbb{R}$ by

$$\omega(\lambda) = \sigma_1(\lambda) - \sigma_2(\lambda).$$

For ω we get:

$$\int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\omega(\lambda) = 0, \qquad \int_{-\infty}^{+\infty} \lambda^k\, d\omega(\lambda) = 0 \qquad (\forall\, z \notin \mathbb{R},\ \forall\, k \in \mathbb{N}).$$

The complex numbers z are of the form $z = x + iy$ where $x, y \in \mathbb{R}$, $y \notin \mathbb{R}$. Then

$$\frac{1}{\lambda - z} = \frac{1}{(\lambda - x) - iy} = \frac{\lambda - x}{(\lambda - x)^2 + y^2} + i\frac{y}{(\lambda - x)^2 + y^2}\,.$$

Thus the integral $\int_{-\infty}^{+\infty} \frac{d\omega(\lambda)}{\lambda - z}$ takes the form

$$\int_{-\infty}^{+\infty} \frac{d\omega(\lambda)}{\lambda - z} = \int_{-\infty}^{+\infty} \frac{\lambda - x}{(\lambda - x)^2 + y^2}\, d\omega(\lambda) + i \int_{-\infty}^{+\infty} \frac{y}{(\lambda - x)^2 + y^2}\, d\omega(\lambda).$$

Since the value of this integral is zero, its complex part is zero either; i.e.

$$\int_{-\infty}^{+\infty} \frac{y}{(\lambda - x)^2 + y^2}\, d\omega(\lambda) = 0.$$

Thus for any points of continuity a, b of $\omega(\lambda)$

$$\begin{aligned} 0 &= \int_a^b \left(\int_{-\infty}^{+\infty} \frac{y\, d\omega(\lambda)}{(\lambda - x)^2 + y^2} \right) dx \\ &= \int_{-\infty}^{+\infty} \left(\int_a^b \frac{y\, dx}{(\lambda - x)^2 + y^2} \right) d\omega(\lambda). \end{aligned}$$

By the substitution $\xi = \frac{\lambda - x}{y}$, we obtain

$$0 = \int_{-\infty}^{+\infty} \left(\arctan \frac{\lambda - a}{y} - \arctan \frac{\lambda - b}{y} \right) d\omega(\lambda),$$

and applying integration by parts we get

$$\begin{aligned} 0 = &\left(\omega(\lambda) \arctan \frac{\lambda - a}{y} - \omega(\lambda) \arctan \frac{\lambda - b}{y} \right) \Bigg|_{\lambda = -\infty}^{\infty} \\ &+ \int_{-\infty}^{+\infty} \left(\frac{y\omega(\lambda)}{(\lambda - b)^2 + y^2} \right) d\lambda - \int_{-\infty}^{+\infty} \left(\frac{y\omega(\lambda)}{(\lambda - a)^2 + y^2} \right) d\lambda. \end{aligned}$$

The function $\omega(\lambda)$ is bounded and $\lim_{\lambda \to \mp\infty} \arctan(\lambda) = 0$ so the equation takes the form

$$y \int_{-\infty}^{+\infty} \left(\frac{\omega(\lambda)}{(\lambda - b)^2 + y^2} \right) d\lambda = y \int_{-\infty}^{+\infty} \left(\frac{\omega(\lambda)}{(\lambda - a)^2 + y^2} \right) d\lambda. \tag{3.2}$$

The function

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\omega(\lambda)}{(\lambda - x)^2 + y^2} \right) d\lambda$$

is the solution of the Dirichlet problem on half plane for the function ω , i.e. $u(x,y)$ is a harmonic function such that

$$\lim_{(x,y)\to(\lambda,0)} u(x,y) = \omega(\lambda) \tag{3.3}$$

whenever ω is continuous at λ [14, Section 4.5]. The equation (3.2) yields

$$u(a,y) = u(b,y)$$

for any non-zero $y \in \mathbb{R}$. Then by property (3.3) of the function $u(x,y)$ we get:

$$\omega(a) = \lim_{y\to 0} u(a,y) = \lim_{y\to 0} u(b,y) = \omega(b)$$

Thus the function ω attains the same value at its points of continuity. Since $\omega(\lambda)$ has at most countably many points of discontinuity and $\lim_{\lambda\to-\infty} \omega(\lambda) = \lim_{\lambda\to-\infty} (\sigma_1(\lambda) - \sigma_2(\lambda)) = 0$, it follows that $\omega(\lambda) = \sigma_1(\lambda) - \sigma_2(\lambda) \neq 0$ for only a countable number of points. Therefore σ_1 and σ_2 are equivalent. □

3.2 Construction of Moment Problem Solutions

Let a positive sequence $(s_k)_{k=0}^{\infty}$ be given. For any $\tau \in \mathbb{R}$, the corresponding truncated moment problem of order n has a solution $\sigma_n(\lambda) = \sigma_n^{\tau}(\lambda)$ whose increase points are the zeroes of $P_n(\lambda, \tau)$ by Proposition 2.5.3 . Degree of the polynomial $Q_n(\lambda, \tau)$ is $n-1$, so by Lagrange interpolation formula we have

$$Q_n(z,\tau) = P_n(z,\tau) \sum_{k=1}^{n} \frac{Q_n(\lambda_k, \tau)}{P_n'(\lambda_k,\tau)(z - \lambda_k)} \,.$$

Let the increase of $\sigma_n(\lambda)$ at λ_k be μ_k for each $k, 1 \leq k \leq n$. By equation (2.39) it follows that

$$\begin{aligned} -\frac{Q_n(z,\tau)}{P_n(z,\tau)} &= \sum_{k=1}^{n} \frac{\mu_k}{\lambda_k - z} \\ &= \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_n^\tau(\lambda). \end{aligned} \tag{3.4}$$

Now define the function $w_n(z,\tau)$ by

$$w_n(z,\tau) = -\frac{Q_n(z,\tau)}{P_n(z,\tau)} = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_n^\tau(\lambda).$$

Consider $w_n(z,\tau)$ for a fixed z and let τ to vary. By the integral expression of $w_n(z,\tau)$ it follows that imaginary parts of z and $w_n(z,\tau)$ have the same sign. Since

$$\frac{Q_n(z,\tau)}{P_n(z,\tau)} = \frac{Q_n(\lambda) - \tau Q_{n-1}(\lambda)}{P_n(\lambda) - \tau P_{n-1}(\lambda)}$$

is a linear fraction with respect to the variable τ, it follows that for a fixed z, $w_n(z,\tau)$ describes a circle which lies completely on the same side of the real axis with z. We will denote this circle by $C_n(z)$, and the union of $C_n(z)$ and the circular region bounded by it by $K_n(z)$. From the equality

$$\frac{a - \tau b}{c - \tau d} = \frac{a\bar{d} - b\bar{c}}{c\bar{d} - d\bar{c}} - \frac{ad - bc}{c\bar{d} - d\bar{c}} \cdot \frac{\bar{c} - \tau\bar{d}}{c - \tau d}$$

it follows that radius of $C_n(z)$ is

$$r_n(z) = \left| \frac{Q_n(z)P_{n-1}(z) - Q_{n-1}(z)P_n(z)}{P_n(z)\overline{P_{n-1}(z)} - P_{n-1}(z)\overline{P_n(z)}} \right| .$$

If we let $\mu = \overline{\lambda}$ in Christoffel–Darboux formula (2.17) for denominator and use the equation (2.20) for numerator we obtain

$$r_n(z) = \frac{1}{|\, z - \bar{z}\,| \sum_{k=0}^{n-1} |\, P_k(z)\,|^2} . \tag{3.5}$$

For arbitrary $w \in \mathbb{C}$, in equation (2.16) let $m = 1$, $\mu = \bar{\lambda}$, $y_k = wP_k(\lambda) + Q_k(\lambda)$ and $z_k = \overline{y_k}$. The equation takes the form

$$\sum_{k=0}^{n-1} |\, wP_k(\lambda) + Q_k(\lambda)\,|^2 - \frac{w - \bar{w}}{\lambda - \overline{\lambda}}$$
$$= b_{n-1}\,|\, wP_{n-1}(\lambda) + Q_{n-1}(\lambda)\,|^2 \, \frac{\operatorname{Im} \frac{wP_n(\lambda)+Q_n(\lambda)}{wP_{n-1}(\lambda)+Q_{n-1}(\lambda)}}{\operatorname{Im}\lambda} . \quad (3.6)$$

Note that $w \in C_n(z)$ if and only if there exists $\tau \in \mathbb{R}$ such that $w = w_n(z, \tau)$. From the definition of the function $w_n(z, \tau)$ it follows that

$$\tau = \frac{w_n(z,\tau)P_n(z) + Q_n(z)}{w_n(z,\tau)P_{n-1}(z) + Q_{n-1}(z)} \in \mathbb{R}.$$

Thus $w \in C_n(z)$ if and only if

$$\sum_{k=0}^{n-1} |\, wP_k(z) + Q_k(z)\,|^2 - \frac{w - \overline{w}}{z - \overline{z}} = 0$$

since $P_{n-1}(\lambda)$ and $Q_{n-1}(\lambda)$ does now vanish together and therefore

$$b_{n-1}\,|\, wP_{n-1}(z) + Q_{n-1}(z)\,|^2 \, \frac{\operatorname{Im} \frac{wP_n(z)+Q_n(z)}{wP_{n-1}(z)+Q_{n-1}(z)}}{\operatorname{Im} z} = 0$$

if and only if $\frac{wP_n(z)+Q_n(z)}{wP_{n-1}(z)+Q_{n-1}(z)}$ is real.

Now fix z and consider the function

$$\varphi_n(w) = \sum_{k=0}^{n-1} |\, wP_k(z) + Q_k(z)\,|^2 - \frac{w - \overline{w}}{z - \overline{z}} .$$

Expanding the function in the form

$$\varphi_n(w) = A\,|\, w\,|^2 + Bw + C\bar{w} + D,$$

it follows that

$$A = \sum_{k=0}^{n-1} |\, P_k(z)\,|^2$$

so $\lim_{w\to\infty}\varphi_n(w)=\infty$. Since φ_n is a continuous function on the half plane at the side of the real axis which contains z, and since $\varphi_n(w)=0$ if and only if $w\in C_n(z)$ on that half plane, it follows that

$$\varphi_n(w)=\sum_{k=0}^{n-1}|\,wP_k(z)+Q_k(z)\,|^2-\frac{w-\overline{w}}{z-\overline{z}}>0 \tag{3.7}$$

for any w such that $w\notin K_n(z)$ and $\operatorname{Im}z=\operatorname{Im}w$. Now let $w\in C_{n-1}(z)$. Then

$$\varphi_{n-1}(w)=\sum_{k=0}^{n-2}|\,wP_k(z)+Q_k(z)\,|^2-\frac{w-\overline{w}}{z-\overline{z}}=0.$$

Thus $\varphi_n(w)=|\,wP_{n-1}(z)+Q_{n-1}(z)\,|>0$ and so $w\in K_n(z)$. Therefore we have a chain

$$K_0(z)\supseteq K_1(z)\supseteq K_2(z)\supseteq\dots$$

and so we have a limiting circle $C_\infty(z)$ and a limiting circular region $K_\infty(z)$, both of which may degenerate into a point. We also have that the circles $C_n(z)$ and $C_{n-1}(z)$ intersect since

$$w_n(z,0)=w_{n-1}(z,\infty)$$

by definition of the function $w_n(z,\tau)$. Radius of the circle $C_\infty(z)$ is the limit of the radii of $C_n(z)$, so it is equal to

$$r_\infty(z)=\frac{1}{|\,z-\bar z\,|\sum_{k=0}^{\infty}|\,P_k(z)\,|^2} \tag{3.8}$$

by equation (3.5). Thus $C_n(z)$ degenerates into a point if and only if $\sum_{k=0}^{\infty}|\,P_k(z)\,|^2=\infty$.

Proposition 3.2.1. *For any $w\in K_\infty(z)$ we have a solution σ of the moment problem such that*

$$\int_{-\infty}^{+\infty}\frac{1}{\lambda-z}\,d\sigma(\lambda)=w.$$

Proof. First, choose a point w on $C_\infty(z)$. Since $C_\infty(z)$ is the limit of circles $C_n(z)$, we can choose a sequence $(\tau_n)_{n=0}^\infty$ of real numbers so that $\lim_{n\to\infty} w_n(z,\tau_n) = w$. By Proposition 2.5.3 and equation (3.4), for each τ_n we have a solution $\sigma_n(\lambda)$ to the truncated moment problem of order n such that

$$w_n(z,\tau_n) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_n(\lambda).$$

By the proof of Theorem 2.6.3, $(\sigma_n)_{n=0}^\infty$ has a subsequence $(\sigma_{n_i})_{i=0}^\infty$ such that $\lim_{i\to\infty} \sigma_{n_i}(\lambda) = \sigma(\lambda)$ for some solution $\sigma(\lambda)$ of moment problem. Since $\frac{1}{\lambda - z}$ is continuous, we can apply Helly's Convergence Theorem 2.6.2 and get

$$\lim_{i\to\infty} \int_a^b \frac{1}{\lambda - z}\, d\sigma_{n_i}(\lambda) = \int_a^b \frac{1}{\lambda - z}\, d\sigma(\lambda)$$

for any $a, b \in \mathbb{R}$. If $a < -1$ and $1 < b$ then

$$\begin{aligned} &\left| \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_n(\lambda) - \int_a^b \frac{1}{\lambda - z}\, d\sigma_n(\lambda) \right| \\ &\qquad = \left| \int_{-\infty}^{a} \frac{1}{\lambda - z}\, d\sigma_n(\lambda) + \int_b^{+\infty} \frac{1}{\lambda - z}\, d\sigma_n(\lambda) \right| \\ &\qquad \le \int_{-\infty}^{a} \left| \frac{1}{\lambda - z} \right| d\sigma_n(\lambda) + \int_b^{-\infty} \left| \frac{1}{\lambda - z} \right| d\sigma_n(\lambda) \\ &\qquad \le \frac{1}{|a|} \int_{-\infty}^{a} \left| \frac{\lambda}{\lambda - z} \right| d\sigma_n(\lambda) + \frac{1}{|b|} \int_b^{-\infty} \left| \frac{\lambda}{\lambda - z} \right| d\sigma_n(\lambda) \\ &\qquad \le \frac{1}{|a| + |b|} \int_{-\infty}^{+\infty} \left| \frac{\lambda}{\lambda - z} \right| d\sigma_n(\lambda). \end{aligned}$$

Since $\lim_{\lambda\to\mp\infty} \left| \frac{\lambda}{\lambda - z} \right| = 1$ and $\left| \frac{\lambda}{\lambda - z} \right|$ is continuous, we have $\sup_{\lambda\in\mathbb{R}} \left| \frac{\lambda}{\lambda - z} \right| = M < \infty$. Hence

$$\int_{-\infty}^{+\infty} \left| \frac{\lambda}{\lambda - z} \right| d\sigma_n(\lambda) \le \int_{-\infty}^{+\infty} M\, d\sigma_n(\lambda) = M s_0$$

and we conclude that

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma(\lambda) &= \lim_{i\to\infty} \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma_{n_i}(\lambda) \\ &= \lim_{i\to\infty} w_{n_i}(\tau_{n_i}, z) = w. \end{aligned}$$

Now let w be an arbitrary element of $K_\infty(z)$. Then w is on a line segment joining two points $w_1, w_2 \in C_\infty(z)$ and so there exists $t \in [0,1]$ such that $w = tw_1 + (1-t)w_2$. Let $\sigma_1(\lambda)$ and $\sigma_2(\lambda)$ be the solutions of the moment problem corresponding to the points w_1 and w_2, respectively. Then $\sigma = t\sigma_1 + (1-t)\sigma_2$ is a solution of the moment problem. Indeed, for any $k \in \mathbb{N}$

$$\begin{aligned}\int_{-\infty}^{+\infty} \lambda^k \, d\sigma(\lambda) &= t \int_{-\infty}^{+\infty} \lambda^k \, d\sigma_1(\lambda) + (1-t) \int_{-\infty}^{+\infty} \lambda^k \, d\sigma_2(\lambda) \\ &= ts_k + (1-t)s_k \\ &= s_k.\end{aligned}$$

We also have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z} &= t \int_{-\infty}^{+\infty} \frac{d\sigma_1(\lambda)}{\lambda - z} + (1-t) \int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z} \\ &= tw_1 + (1-t)w_2 \\ &= w.\end{aligned}$$

□

3.3 Conditions for Uniqueness

Proposition 3.3.1. *Let a positive sequence* $(\, s_k \,)_{k=0}^{\infty}$ *be given. For any non-real* z*, consider the closed disc* $K_\infty(z)$ *defined in previous section. Then we have*

$$\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z} \in K_\infty(z)$$

for any solution σ *of the moment problem.*

Proof. Let $w = \int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z}$. To prove $w \in K_\infty(z)$, by equation (3.7) it is enough to prove that

$$\sum_{k=0}^{\infty} |\, wP_k(z) + Q_k(z) \,|^2 - \frac{w - \overline{w}}{z - \overline{z}} \leq 0.$$

By the Bessel inequality ((1.5) of Chapter 1) it follows that

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{|\,\lambda - z\,|^2} &= \int_{-\infty}^{+\infty} \left| \frac{1}{\lambda - z} \right|^2 d\sigma(\lambda) \\ &\geq \sum_{k=0}^{\infty} \left| \left(\frac{1}{\lambda - z}, P_k(\lambda) \right)_\sigma \right|^2 \\ &= \sum_{k=0}^{\infty} \left| \int_{-\infty}^{+\infty} \frac{P_k(\lambda)}{\lambda - z}\, d\sigma(\lambda) \right|^2 \\ &\geq \left| \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \frac{P_k(\lambda)}{\lambda - z}\, d\sigma(\lambda) \right|^2 . \end{aligned} \tag{3.9}$$

The expression $\int_{-\infty}^{+\infty} \frac{P_k(\lambda)}{\lambda - z}\, d\sigma(\lambda)$ can be computed as

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{P_k(\lambda)}{\lambda - z}\, d\sigma(\lambda) &= \int_{-\infty}^{+\infty} \frac{P_k(\lambda) - P_k(z)}{\lambda - z}\, d\sigma(\lambda) + P_k(z) \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\sigma(\lambda) \\ &= Q_k(\lambda) + wP_k(\lambda).\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{1}{|\,\lambda - z\,|^2} &= \frac{1}{(\,\lambda - z\,)\,(\,\lambda - \bar{z}\,)} \\ &= \frac{1}{z - \bar{z}} \left(\frac{1}{\lambda - z} - \frac{1}{\lambda - \bar{z}} \right)\end{aligned}$$

and thus for $\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{|\,\lambda - z\,|^2}$ we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{|\,\lambda - z\,|^2} &= \frac{1}{z - \bar{z}} \left(\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z} - \int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - \bar{z}} \right) \\ &= \frac{w - \bar{w}}{z - \bar{z}} .\end{aligned}$$

Together with these results, the inequality (3.9) transforms into

$$\frac{w - \bar{w}}{z - \bar{z}} \geq \sum_{k=0}^{\infty} |\, Q_k(z) + wP_k(z)\,|^2 , \tag{3.10}$$

which is the desired result. □

Proposition 3.3.2. *For a positive sequence* $(s_k)_{k=0}^{\infty}$*, the related moment problem has a unique solution if and only if associated orthonormal polynomials satisfy*

$$\sum_{k=0}^{\infty} |P_k(z)|^2 = \infty \tag{3.11}$$

for any $z \notin \mathbb{R}$*.*

Proof. Let condition (3.11) hold for any $z \notin \mathbb{R}$. Then each $K_\infty(z)$ deforms into a point. Thus by Proposition 3.3.1 above, the value of the integral $\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z}$ is the same for any solution σ of the moment problem. By Theorem 3.1.1 this yields that the moment problem has a unique solution.

Now assume that the moment problem has a unique solution. Then by Proposition 3.1.1 it follows that $\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z}$ is the same for any solution σ of the moment problem. Using Proposition 3.2.1, this implies that $K_\infty(z)$ consists of a single point, thus

$$r_\infty(z) = \frac{1}{|z - \bar{z}| \sum_{k=0}^{\infty} |P_k(z)|^2} = 0$$

and hence $\sum_{k=0}^{\infty} |P_k(z)|^2 = \infty$. □

In fact a much stronger form of Proposition 3.3.2 also holds. To prove it we need some preparation.

Lemma 3.3.3. *Let* $(p_k(z))_{k=0}^{\infty}$ *be a sequence of functions on* $\mathbb{C}$*. Let* $\sum_{n=0}^{\infty} |p_k(z_0)|^2 < \infty$ *for some* $z_0 \in \mathbb{C}$*. If there are numbers* a_{nk}, $0 \le k < n$*, satisfying*

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |a_{nk}|^2 < \infty$$

such that

$$p_n(z_1) = p_k(z_0) + (z_1 - z_0) \sum_{k=0}^{n-1} a_{nk} p_k(z_1) \tag{3.12}$$

for some $z_1 \in \mathbb{C}$ *then* $\sum_{n=0}^{\infty} |p_k(z_1)|^2 < \infty$*.*

Proof. Let $r = |\, z_1 - z_0 \,|$. Choose some $\varepsilon < 1$. Then there exists $m \in \mathbb{N}$ such that

$$\left(\sum_{n=m}^{\infty} \sum_{k=0}^{n-1} |\, a_{nk} \,|^2 \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{r} \quad , \quad \left(\sum_{n=m}^{\infty} |\, p_k(z_0) \,|^2 \right)^{\frac{1}{2}} \leq \varepsilon.$$

Let $N > m$. By triangle inequality in l^2, equation (3.12) yields that

$$\left(\sum_{n=m}^{N} |\, p_n(z_1) \,|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{n=m}^{N} |\, p_k(z_0) \,|^2 \right)^{\frac{1}{2}} + |\, z_1 - z_0 \,| \left(\sum_{n=m}^{N} \left| \sum_{k=0}^{n-1} a_{nk} p_k(z_1) \right|^2 \right)^{\frac{1}{2}}.$$

Keep ε and m fixed. We have

$$\begin{aligned}
\left(\sum_{n=m}^{N} |\, p_n(z_1) \,|^2 \right)^{\frac{1}{2}} &\leq \varepsilon + r \left(\sum_{n=m}^{N} \left| \sum_{k=0}^{n-1} a_{nk} p_k(z_1) \right|^2 \right)^{\frac{1}{2}} \\
&\leq \varepsilon + r \left(\sum_{n=m}^{N} \left(\sum_{k=0}^{n-1} |\, a_{nk} \,|^2 \sum_{i=0}^{n-1} |\, p_i(z_1) \,|^2 \right) \right)^{\frac{1}{2}} \\
&\leq \varepsilon + r \left(\left(\sum_{n=m}^{N} \sum_{k=0}^{n-1} |\, a_{nk} \,|^2 \right) \left(\sum_{i=0}^{N} |\, p_i(z_1) \,|^2 \right) \right)^{\frac{1}{2}} \\
&\leq \varepsilon + r \frac{\varepsilon}{r} \left(\sum_{i=0}^{N} |\, p_i(z_1)^2 \,| \right)^{\frac{1}{2}} = \varepsilon + \varepsilon \left(\sum_{i=0}^{N} |\, p_i(z_1) \,|^2 \right)^{\frac{1}{2}} \\
&\leq \varepsilon + \varepsilon \left(\sum_{i=0}^{N} |\, p_i(z_1) \,|^2 \right)^{\frac{1}{2}} + \varepsilon \left(\sum_{i=m}^{N} |\, p_i(z_1) \,|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus we have

$$\left(\sum_{n=m}^{N} |\, p_n(z_1) \,|^2 \right)^{\frac{1}{2}} \leq \frac{\varepsilon + \varepsilon \left(\sum_{i=0}^{m-1} |\, p_i(z_1)^2 \,| \right)^{\frac{1}{2}}}{1 - \varepsilon}$$

for any $N > m$. Hence we have proved the convergence of $\sum_{n=m}^{\infty} |\, p_n(z_1)\,|^2$. □

Lemma 3.3.4. *If the series* $\sum_{k=0}^{\infty} |\, P_k(z_0)\,|^2$ *converges for some* $z_0 \in \mathbb{C}$ *then the series* $\sum_{k=0}^{\infty} |\, Q_k(z_0)\,|^2$ *also converges.*

Proof. If $\sum_{k=0}^{\infty} |\, P_k(z_0)\,|^2 < \infty$ then

$$r_\infty(z_0) = \frac{1}{|\, z_0 - \bar{z}_0\,| \sum_{k=0}^{\infty} |\, P_k(z_0)\,|^2} > 0$$

and thus $K_\infty(z_0)$ is a circle. Let $w \in K_\infty(z_0)$. By equation (3.10) we have

$$\sum_{k=0}^{\infty} |\, Q_k(z_0) + wP_k(z_0)\,|^2 \leq \frac{w - \bar{w}}{z_0 - \bar{z}_0} < \infty$$

and thus $(\, Q_k(z_0) + wP_k(z_0)\,)_{k=0}^{\infty} \in l_2$. Since also $(\, P_k(z_0)\,)_{k=0}^{\infty} \in l_2$, we have

$$(\, Q_k(z_0)\,)_{k=0}^{\infty} = (\, Q_k(z_0) + wP_k(z_0)\,)_{k=0}^{\infty} - w\,(\, P_k(z_0)\,)_{k=0}^{\infty} \in l_2$$

i.e. $\sum_{k=0}^{\infty} |\, Q_k(z_0)\,|^2$ converges. □

Proposition 3.3.5. *If* $\sum_{k=0}^{\infty} |\, P_k(z_0)\,|^2$ *converges for some* $z_0 \in \mathbb{C}$ *then the series* $\sum_{k=0}^{\infty} |\, P_k(z)\,|^2$ *converges for any* $z \in \mathbb{C}$.

Proof. Note that $\frac{P_n(z) - P_n(z_0)}{z - z_0}$ is a polynomial of degree $n-1$, thus there exists $a_{nk} \in \mathbb{N}$ for $0 \leq k < n$ such that

$$\frac{P_n(z) - P_n(z_0)}{z - z_0} = \sum_{i=0}^{n-1} a_{ni} P_i(z). \tag{3.13}$$

Using orthonormality of the polynomials $P_k(\lambda)$, we can compute each a_{nk} as

$$\begin{aligned}
a_{nk} &= \left(P_k(z), \sum_{i=0}^{n-1} a_{ni} P_i(z) \right) = \left(P_k(z) \, , \frac{P_n(z) - P_n(z_0)}{z - z_0} \right) \\
&= \Gamma_z \left(P_k(z) \frac{P_n(z) - P_n(z_0)}{z - z_0} \right) \\
&= \Gamma_z \left([\, P_k(z) - P_k(z_0) \,] \, \frac{P_n(z) - P_n(z_0)}{z - z_0} \; + P_k(z_0) \frac{P_n(z) - P_n(z_0)}{z - z_0} \right) \\
&= \left(\frac{P_k(z) - P_k(z_0)}{z - z_0} \; , P_n(z) \right) - P_n(z_0) \Gamma_z \left(\frac{P_k(z) - P_k(z_0)}{z - z_0} \right) \\
&\qquad + P_k(z_0) \Gamma_z \left(\frac{P_n(z) - P_n(z_0)}{z - z_0} \right).
\end{aligned}$$

Since deg $\frac{P_k(z) - P_k(z_0)}{z - z_0} = k - 1 < n$, the first term vanishes. Also using equation (2.23) for other two terms we get

$$a_{nk} = P_n(z_0) Q_k(z_0) - P_k(z_0) Q_n(z_0).$$

Then the representation (3.13) of $\frac{P_n(z) - P_n(z_0)}{z - z_0}$ yields

$$P_n(z) = P_n(z_0) + (\, z - z_0 \,) \sum_{k=0}^{n-1} [\, P_n(z_0) Q_k(z_0) - P_k(z_0) Q_n(z_0) \,] \, P_k(z).$$

Now observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |\, a_{nk} \,|^2 \\
&\quad = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |\, P_n(z_0) Q_k(z_0) - P_k(z_0) Q_n(z_0) \,|^2 \\
&\quad \leq 2 \left(\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |\, P_n(z_0) Q_k(z_0) \,|^2 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |\, P_k(z_0) Q_n(z_0) \,|^2 \right) \\
&\quad \leq 2 \left(\sum_{n=1}^{\infty} |\, P_n(z_0) \,|^2 \sum_{k=0}^{n-1} |\, Q_k(z_0) \,|^2 + \sum_{n=1}^{\infty} |\, P_k(z_0) \,|^2 \sum_{k=0}^{n-1} |\, Q_n(z_0) \,|^2 \right) \\
&\quad \leq 4 \sum_{n=1}^{\infty} |\, P_n(z_0) \,|^2 \sum_{k=0}^{\infty} |\, Q_k(z_0) \,|^2 .
\end{aligned}$$

The series $|\, Q_k(z_0)\, |^2$ converges by Lemma 3.3.4 since $\sum_{n=1}^{\infty} |\, P_n(z_0)\, |^2$ converges, and thus $\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} |\, a_{nk}\, |^2 < \infty$. Therefore we can apply Lemma 3.3.3 for $p_k(z) = P_k(z)$ and conclude that $\sum_{n=1}^{\infty} |\, P_n(z)\, |^2$ converges. □

Now the advanced version of Proposition 3.3.2 follows:

Theorem 3.3.6. *For a positive sequence $(\, s_k\,)_{k=0}^{\infty}$, the related moment problem has a unique solution if and only if*

$$\sum_{k=0}^{\infty} |\, P_k(z)\, |^2 = \infty \tag{3.14}$$

holds for some (and hence all) $z \in \mathbb{C}$.

In fact, the Jacobi matrix corresponding to the moment problem determines whether the solution of a moment problem is unique. The relation is formulated in the following theorem.

Theorem 3.3.7. *Given a positive sequence $(\, s_k\,)_{k=0}^{\infty}$, let*

$$\mathcal{J} = \begin{pmatrix} a_0 & b_0 & 0 & \dots \\ b_0 & a_1 & b_1 & \\ 0 & b_1 & a_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} \tag{3.15}$$

be the corresponding Jacobi matrix and let z be a non-real complex number. Then all solutions $(\, y_k\,)_{k=0}^{\infty}$ of the recurrence relation

$$b_{k-1}y_{k-1} + a_k y_k + b_{k+1}y_{k+1} = z\, y_k$$

are in l^2 , i.e. $\sum_{k=0}^{\infty} |\, y_k\, |^2 < \infty$, if and only if the solution of the related moment problem is not unique.

Proof. If every solution of the recurrence relation belongs to l^2 then

$$\sum_{k=0}^{\infty} |\, P_k(z)\, |^2 < \infty$$

and thus by Theorem 3.3.6 the solution is not unique.

Now assume that the solution is not unique and thus $\sum_{k=0}^{\infty} |\, P_k(z) \,|^2 < \infty$, i.e. $(\, P_k(z) \,)_{k=0}^{\infty} \in l^2$. Then by Lemma 3.3.4 we have $(\, Q_k(z) \,)_{k=0}^{\infty} \in l^2$. Also $P_0(z) = 1, Q_0(z)$ and $Q_1(z) \neq 0$ implies that $(\, P_k(z) \,)_{k=0}^{\infty}$ and $(\, Q_k(z) \,)_{k=0}^{\infty} = 0$ are linearly independent solutions of the recurrence relation. Consequently, any solution $(\, y_k \,)_{k=0}^{\infty}$ of the recurrence relation is of the form $(\, y_k \,)_{k=0}^{\infty} = c_1 \,(\, P_k(z) \,)_{k=0}^{\infty} + c_2 \,(\, Q_k(z) \,)_{k=0}^{\infty}$ and hence is in l^2. $\square$

CHAPTER 4

ASYMPTOTIC BEHAVIOUR OF ORTHONORMAL POLYNOMIALS

In this chapter, measures on $\mathbb{C}$ with compact support will be investigated by use of orthonormal polynomials $\{ P_k(z) \}_{k=0}^{\infty}$ determined by the given measure. The final results of the chapter will be on bounds for the asymptotic behaviour of the orthonormal polynomials, obtained by J. Ullman. First we start with a restriction on the kind of measures we will work on:

We call a Borel measure μ on $\mathbb{C}$ to be a *finite measure* if $\mu(\mathbb{C}) < \infty$. Also μ is said to be a *unit measure* if $\mu(\mathbb{C}) = 1$.

The support of a Borel measure μ is the set of points whose any neighborhood has positive measure, denoted by $\mathrm{supp}(\mu)$; i.e.

$$\mathrm{supp}(\mu) = \{ z \in \mathbb{C} : U \text{ open in } \mathbb{C} \text{ and } z \in U \implies \mu(U) \neq 0 \} .$$

The support of a measure is always closed [3, Thorem A.1.2] thus if $\mathrm{supp}(\mu)$ is bounded then it is compact.

4.1 Potential Theoretic Preliminaries

The main tool in the subject will be potential theory so we begin with some potential theoretic definitions and facts.

Definition 4.1.1. Let a finite measure μ be given. Then its *potential* is the function $p_\mu(z)$ on $\mathbb{C}$ defined as

$$p_\mu(z) := \int \log |\, z - w \,|\; d\mu(w).$$

A function $f\colon X \to \mathbb{R}$ is called *upper semicontinuous* if $\{x \in X \mid f(x) < \alpha\}$ is open for any $\alpha \in \mathbb{R}$. Let $\mathbf{D}[z;r]$ denote the closed disc $\{w \in \mathbb{C}\colon |z-w| \le r\}$.

Definition 4.1.2. Let U be an open subset of $\mathbb{C}$. Then:

(i) An upper semicontinuous function $f\colon U \to [-\infty, \infty)$ is called *subharmonic* if for any $z \in D$ there exists $\mathbf{D}[z;r] \subset D$ such that

$$f(z) \le \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt.$$

(ii) A function $f\colon U \to \mathbb{R}$ is called *harmonic* if it is continuous and for any $z \in D$ there exists $\mathbf{D}[z;r] \subset D$ such that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt.$$

Harmonic functions could equivalently be defined as the solutions of the Laplace equation; i.e. a function $f(z)$ on an open subset U of $\mathbb{C}$ is harmonic if and only if $f(z) \in C^2(U)$ and

$$f_{xx} + f_{yy} = 0,$$

where we consider $z = x + iy$ and the subscripts stand for partial derivatives [3, Theorems 1.1.6 and 1.2.7]. It is obvious from definition that if a function $f(z)$ is harmonic then both $f(z)$ and $-f(z)$ are subharmonic. In fact, the converse also holds [3, Corollary 2.4.2].

Proposition 4.1.1. *Potential* $p_\mu(z)$ *of a measure* μ *is subharmonic on* $\mathbb{C}$ *and harmonic on* $\mathbb{C}\backslash\mathrm{supp}(\mu)$.

Proof. See [3, Theorem 3.1.2]. □

Definition 4.1.3. Let a finite measure μ on $\mathbb{C}$ be given. *Energy* of μ is defined as

$$I(\mu) := \int p_\mu(z)\,d\mu(z) = \int\int \log|z-w|\,d\mu(w)\,d\mu(z).$$

Proposition 4.1.2. *Let μ and ν be unit measures with compact support. For any $c \in \mathbb{R}$, if*

$$p_\nu(z) + c \leq p_\mu(z)$$

holds μ-almost everywhere then it holds for all $z \in \mathbb{C}$.

Proof. See [4, Theorem 1.27]. $\square$

Definition 4.1.4. Let E be a subset of $\overline{\mathbb{C}}$. The *capacity* of E is given by

$$\operatorname{cap}(E) = \sup \left\{ e^{I(\mu)} : \mu \text{ is a unit measure}, \ \operatorname{supp}(\mu) \subseteq E \right\}.$$

A set E is said to be of *capacity zero* if $\operatorname{cap}(E) = 0$. A property is said to hold *quasi everywhere* (in short, *q.e.*) on a set S if the set of points in S for which the property is not satisfied is of capacity zero.

Proposition 4.1.3. *(a) If $\operatorname{cap}(S) = 0$ then S is of Lebesgue measure zero, i.e. quasi everywhere implies almost everywhere. Also $\mu(S) = 0$ for any finite measure μ with $I(\mu) > -\infty$.*

(b) If $\operatorname{cap}(S_n) = 0$ for any $n \in \mathbb{N}$ then $\operatorname{cap}(\bigcup_{k=0}^{\infty} S_k) = 0$.

(c) Let B_1 and B_2 be Borel sets such that $B_1 \subseteq B_2$. Then $\operatorname{cap}(B_1) \leq \operatorname{cap}(B_2)$.

Proof. See [3, Theorems 3.2.3 and 5.1.2(a), Corrolaries 3.2.4 and 3.2.5]. $\square$

Proposition 4.1.4. *Let B be a Borel set and let $\operatorname{cap}(Z) = 0$. Then*

$$\operatorname{cap}(B \cup Z) = \operatorname{cap}(B) = \operatorname{cap}(B \backslash Z).$$

Proof. If $\operatorname{cap}(B \cup Z) = 0$ then the statement follows immediately by Proposition 4.1.3(c) and nonnegativity of capacity. Now let $\operatorname{cap}(B \cup Z) > 0$. Let μ be any unit measure such that $I(\mu) > -\infty$ and $\operatorname{supp}(\mu) \subseteq B \cup Z$. By the second part of Proposition 4.1.3(a) it follows that $\mu(Z) = 0$ since Z is of capacity zero. Thus $I(\mu) = I(\mu|_{B\backslash Z})$ and hence $\operatorname{cap}(B \cup Z) = \operatorname{cap}(B \backslash Z)$. $\square$

Proposition 4.1.5. *Let μ be a finite measure with compact support. Then* $\operatorname{cap}(Z) = 0$ *where*

$$Z := \{ z \in \mathbb{C} : p_\mu(z) = -\infty \}.$$

Proof. See [3, Theorem 3.5.1]. □

Proposition 4.1.6. *Given a Borel set E there exists a unique unit measure ω_E satisfying*

$$p_{\omega_E}(z) \geq \log(\operatorname{cap}(E)) \quad \text{on } \mathbb{C},$$
$$p_{\omega_E}(z) = \log(\operatorname{cap}(E)) \quad \text{q.e. on } E.$$

We also have that $\operatorname{supp}(E) \subseteq \bar{E}$. *The measure ω_E is called the* equilibrium measure *of E.*

Proof. See [5, Appendix IV]. □

Proposition 4.1.7. *Given a compact subset K of $\mathbb{C}$, equilibrium measure ω_K of K is the unique unit measure with* $\operatorname{supp}(\mu) \subseteq K$ *satisfying $I(\omega_K) \geq I(\mu)$ for any unit measure μ. In other words, $I(\omega_K) = \log(\operatorname{cap}(K))$.*

Proof. See [3, Theorems 3.3.2 and 3.3.4]. □

Proposition 4.1.8. *Let B be a proper Borel subset of $\overline{\mathbb{C}}$ with bounded complement. Let $E = \overline{\mathbb{C}} \backslash B$ be of positive capacity. Then the function g_B^∞ defined as*

$$g_B^\infty(z) = p_{\omega_E}(z) - \log(\operatorname{cap}(E)) \tag{4.1}$$

is the unique function satisfying

i. $g_B^\infty(z)$ *is non-negative and* $g_B^\infty(z) = 0$ *q.e. on* $\mathbb{C} \backslash B$.

ii. $g_B^\infty(z)$ *is harmonic on* $\operatorname{int}(B \backslash \{\infty\})$ *and subharmonic on* $\mathbb{C}$.

iii. $\lim_{z \to \infty} (\log|z| - g_B^\infty(z)) = \log(\operatorname{cap}(E))$.

We also have that if $B' \subseteq B$ then $g_{B'}^\infty(z) \leq g_B^\infty(z)$.

Proof. See [5, Theorem A2]. □

Definition 4.1.5. Let B be a Borel set in $\overline{\mathbb{C}}$ with bounded complement. Then the unique function $g_B^\infty(z)$ defined in Proposition 4.1.8 is called the *Green function of B at infinity.* If complement of B is of capacity zero, then set $g_B^\infty(z) \equiv \infty$.

Definition 4.1.6. The sequence of measures $(\mu_n)_{n=0}^\infty$ is said to *converge weakly* to the measure μ if for every continuous function f we have

$$\lim_{n\to\infty} \int f(z)\, d\mu_n(z) = \int f(z)\, d\mu(z).$$

Weak convergence is denoted as $\mu_n \xrightarrow{*} \mu$.

Proposition 4.1.9. *Let B be a Borel set. Then there exists a sequence $(K_n)_{n=0}^\infty$ of compact subsets of $\mathbb{C}$ such that*

$$\begin{aligned} &\lim_{n\to\infty} \operatorname{cap}(K_n) = \operatorname{cap}(B), \\ &\lim_{n\to\infty} g_{\mathbb{C}\setminus K_n}^\infty(z) = g_{\mathbb{C}\setminus C}^\infty(z) \qquad (\forall z \in \mathbb{C}), \\ &\omega_{K_n} \xrightarrow{*} \omega_B. \end{aligned}$$

Proof. See [5, page 8]. □

The following theorem is the version of Helly's Choice Theorem (Theorem 2.6.1) for $\mathbb{C}$:

Theorem 4.1.10. *Let $(\mu_k)_{k=0}^\infty$ be a sequence of unit measures with uniformly bounded supports. Then it has a subsequence $(\mu_{k_i})_{i=0}^\infty$ such that $\mu_{k_i} \xrightarrow{*} \mu$ for some measure μ with compact support.*

Proof. See [3, Theorem A.4.2]. □

We have the following relation between weak convergence and the potentials of measures:

Theorem 4.1.11. *Let* $(\mu_k)_{k=0}^{\infty}$ *be a sequence of unit measures with uniformly bounded supports such that* $\mu_k \xrightarrow{*} \mu$. *Then*

$$p_\mu(z) = \limsup_{k\to\infty} p_{\mu_k}(z) \quad q.e. \text{ on } \mathbb{C}.$$

Proof. See [4, Theorem 3.8]. □

4.2 Weight Measures and Carriers

Given a measure μ on $\mathbb{C}$, a scalar product is defined on the linear space

$$L^2_\mu := \left\{ f(z) : \mathbb{C} \to \mathbb{C} \;\middle|\; f(z) \text{ is } \mu\text{-measurable and } \int |f(z)|^2 \, d\mu(z) < \infty \right\}$$

by

$$(f(z), g(z))_\mu = \int f(z)\overline{g(z)}\, d\mu(z).$$

The space L^2_μ is complete, which can be proved similarly as the completeness of L^2_σ, introduced in Section 1.1 . To force the linear space L^2_μ to be a Hilbert space, we introduce further restriction on the cardinality of supp(μ) to ensure that L^2_μ is infinite dimensional:

Definition 4.2.1. A unit measure on $\mathbb{C}$ is said to be a *weight measure* if supp(μ) is compact (or equivalently, bounded) and has infinite number of points.

With a proof similar to the one of Proposition 1.1.1 (only the intervals should be replaced by appropriate discs), it follows that $\mathbb{C}[\lambda]$ is a subspace of L^2_μ and thus the space L^2_μ is a Hilbert space with the given scalar product.

Definition 4.2.2. Let μ be a weight measure. A bounded Borel subset E of supp(μ) is called a *carrier* of μ if

$$\mu(\mathbb{C}\backslash E) = \mu(\text{supp}(\mu)\backslash E) = 0.$$

Obviously $\mathrm{supp}(\mu)$ is a carrier of μ itself.

Definition 4.2.3. For a weight measure μ, the *minimal carrier capacity* c_μ of μ is

$$c_\mu := \inf \{ \mathrm{cap}(E) : E \text{ is a carrier of } \mu \}$$

and the *minimal carrier Green function* of μ is

$$g_\mu^\infty(z) := \sup \left\{ g_{\mathbb{C}\backslash E}^\infty(z) : E \text{ is a carrier of } \mu \right\}.$$

A carrier $C \subseteq \mathrm{supp}(\mu)$ of μ is called a *minimal carrier* if $\mathrm{cap}(C) = c_\mu$.

Proposition 4.2.1. *For any carrier E of μ there exists a minimal carrier C of μ such that $C \subseteq E$. In particular, there exists at least one minimal carrier of μ.*

Proof. Since c_μ is the infimum of capacities of carriers, there exists a sequence $(E_n)_{n=1}^\infty$ of carriers such that $\lim_{n\to\infty} \mathrm{cap}(E_n) = c_\mu$. Given a carrier E of μ, set $E_0 := E$. Then $C = \bigcap_{n=0}^\infty E_n$ is a carrier of μ since $\mathbb{C}\backslash C = \bigcup_{n=0}^\infty \mathbb{C}\backslash E_n$ is a union of μ-measure zero sets so it is also of μ-measure zero. Thus $\mathrm{cap}(C) \geq c_\mu$. Besides, $C \subseteq E_n$ for any n so $\mathrm{cap}(C) \leq c_\mu$. Therefore C is a minimal carrier of μ such that $C \subseteq E$. □

Proposition 4.2.2. *For any minimal carrier C of a weight measure μ we have*

$$g_{\mathbb{C}\backslash C}^\infty(z) \equiv g_\mu^\infty(z).$$

Also

$$g_\mu^\infty(z) = 0 \quad \textit{q.e. on } C.$$

Proof. If $c_\mu = 0$ then for any minimal carrier C' of μ we have $\mathrm{cap}(C') = 0$ and thus $g_{\mathbb{C}\backslash C'}^\infty(z) \equiv \infty$. Hence $g_{\mathbb{C}\backslash C}^\infty(z) \equiv g_\mu^\infty(z) \equiv \infty$. Second equation is automatically satisfied since $\mathrm{cap}(C) = 0$.

Now assume $c_\mu > 0$. Let C_1 and C_2 be two minimal carriers. Let $C_0 := C_1 \cap C_2$. Then C_0 is a carrier of μ, and so $c_\mu = \mathrm{cap}(C_1) \geq \mathrm{cap}(C_0) \geq c_\mu$ implies that C_0 is a minimal carrier of μ. Since both functions $g^\infty_{\overline{\mathbb{C}}\setminus C_1}(z)$ and $g^\infty_{\overline{\mathbb{C}}\setminus C_2}(z)$ satisfy all three defining properties of the Green function of C_0, from uniqueness it follows that

$$g^\infty_{\overline{\mathbb{C}}\setminus C_1}(z) \equiv g^\infty_{\overline{\mathbb{C}}\setminus C_2}(z)$$

and thus all Green functions for the complements of carriers are identical. Since every carrier includes a minimal carrier by Proposition 4.2.1 and since $E \supseteq C'$ implies $g^\infty_{\overline{\mathbb{C}}\setminus E}(z) \leq g^\infty_{\overline{\mathbb{C}}\setminus C'}(z)$, by definition of $g^\infty_\mu(z)$ it follows that $g^\infty_\mu(z) = g^\infty_{\overline{\mathbb{C}}\setminus C}(z)$ for any minimal carrier C. Now, $g^\infty_\mu(z) = 0$ q.e. follows from $g^\infty_\mu(z) = g^\infty_{\overline{\mathbb{C}}\setminus C}(z)$ and Proposition 4.1.8(i). □

By the previous theorem and the uniqueness of equilibrium measure in representation (4.1) of $g^\infty_\mu(z)$, we have the following result:

Corollary 4.2.3. *For a weight measure μ, all minimal carriers have the same equilibrium measure, which will be denoted by ω_μ. The minimal carrier Green function is of the form*

$$g^\infty_\mu(z) = p_{\omega_\mu}(z) - \log c_\mu\,. \tag{4.2}$$

Proposition 4.2.4. *Let μ and ν be unit measures and let $c \in \mathbb{R}$. If $p_\nu(z) \leq c$ q.e. on a minimal carrier C of μ then*

$$p_\nu(z) - c \leq g^\infty_\mu(z) \qquad (\forall z \in \mathbb{C}),$$
$$c \geq -\log c_\mu.$$

Proof. If $c_\mu = 0$ then $g^\infty_\mu(z) \equiv \infty$ and $-\log c_\mu = \infty$, hence the result follows. Let $c_\mu > 0$ and let $Z \subset C$ denote the set of capacity zero for which $p_\nu(z) \leq c$ does not hold. Take an ascending sequence $(\,K_n\,)_{n=0}^\infty$ of compact sets in C as in Proposition 4.1.9 such that $\mathrm{cap}(K_n) > 0$ and limit relations hold for the Borel set C. Denote the equilibrium measure of K_n as ω_n. For any n we have $\omega_n(Z) = 0$ since the measures ω_n are all unit. Then for $z \in C\backslash Z$ we have

$$p_\nu(z) - c \le 0 \le g^{\infty}_{\mathbb{C}\backslash K_n}(z) = p_{\omega_n}(z) - \log \operatorname{cap}(K_n)$$

and thus

$$p_\nu(z) - (\, c - \log \operatorname{cap}(K_n)\,) \le p_{\omega_n}(z).$$

But then by Proposition 4.1.2 it follows that the inequality holds everywhere, and passing to the limits we obtain the first inequality stated in the proposition. To obtain $c \le -\log c_\mu$, consider the representation (4.2) of $g^{\infty}_{\mu}(z)$ and let $z \to \infty$. □

4.3 Orthogonal Polynomials in L^2_μ

Throughout the section, let $\{\, P_k(z)\,\}_{k=0}^{\infty}$ denote a sequence of polynomials with positive leading coefficients which are orthonormal in the space L^2_μ and $\deg P_n(z) = n$. By the same considerations as in Section 1.2, if μ is a weight measure then there exists such a sequence. Let γ_n be the leading coefficient of $P_n(z)$ and define

$$\tilde{P}_n(z) := \frac{P_n(z)}{\gamma_n}$$

to be the corresponding monic polynomials. Define

$$N_n := \|\tilde{P}_n(z)\|$$

where the norm comes from the Hilbert space structure of L^2_μ. Since norm of $P_n(z)$ is 1, we have the relation

$$N_n = \frac{1}{\gamma_n}\,.$$

Let $p(z)$ be any monic polynomial of degree n other than $\tilde{P}(z)$. Then $p(z)$ can be represented as

$$p(z) = \tilde{P}_n(z) + \sum_{k=0}^{n-1} c_n P_k(z)$$

for some $c_0, \ldots, c_{n-1} \in \mathbb{R}$ at least one of which is nonzero. Then

$$\begin{aligned}\|p(z)\|^2 &= (\, p(z), p(z)\,) \\ &= \Big(\tilde{P}_n(z), \tilde{P}_n(z)\Big) + \sum_{k=0}^{n-1} c_k^2\,(\, P_k(z), P_k(z)\,) \\ &= \|\tilde{P}_n(z)\|^2 + \sum_{k=0}^{n-1} c_k^2 \\ &> \|\tilde{P}_n(z)\|^2.\end{aligned}$$

Thus we have that the polynomial $\tilde{P}_n(z)$ is characterized by the property that $\|\tilde{P}_n(z)\| \leq \|p(z)\|$ for any monic polynomial $p(z)$ of degree n. Using this fact, we will prove:

Proposition 4.3.1. *Zeroes of the all polynomials $\tilde{P}_n(z)$, and hence $P_n(z)$, are in convex hull of* $\mathrm{supp}(\mu)$.

Proof. Assume that z_0 is a zero of $\tilde{P}_n(z)$ such that z_0 is not in convex hull of $\mathrm{supp}(\mu)$. Then there is a straight line $l \subset \mathbb{C}$ separating $\mathrm{supp}(\mu)$ and z_0. Let U denote the part of plane, separated by l, which includes $\mathrm{supp}(\mu)$. Let z_0' be the nearest point to z_0 on line l. For any $z \in U$ we have

$$\left| \frac{z - z_0'}{z - z_0} \right| < 1.$$

Also we have that

$$p(z) = \frac{z - z_0'}{z - z_0} \tilde{P}_n(z)$$

is a monic polynomial of degree n since z_0 is a zero of $\tilde{P}_n(z)$. We have $|\, p(z)\,| \leq \left| \tilde{P}_n(z) \right|$ for any $z \in U$ and equality holds on U only at zeroes of $\tilde{P}_n(z)$ in U. Since $\mathrm{supp}(\mu)$ is infinite and $\tilde{P}_n(z)$ has only a finite number of zeroes, inequality holds on a subset of $\mathrm{supp}(\mu)$ of positive measure. But then

$$\begin{aligned}\|p(z)\|^2 = \int \left| \frac{z - z_0'}{z - z_0} \tilde{P}_n(z) \right|^2 d\mu(z) &= \int_U \left| \frac{z - z_0'}{z - z_0} \tilde{P}_n(z) \right|^2 d\mu(z) \\ < \int_U \left| \tilde{P}_n(z) \right|^2 d\mu(z) &= \int \tilde{P}_n(z)^2 \, d\mu(z) = \|P_n(z)\|^2,\end{aligned}$$

which contradicts with the minimality of the norm of $\tilde{P}_n(z)$ among monic polynomials of degree n. $\square$

Let $\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_{n_j}^{(n)}$ be the distinct zeroes of $P_n(z)$. With the use of orthogonal polynomials, define the sequence of measures $(\,\nu_k\,)_{k=0}^{\infty}$ such that weight of ν_n is gathered at the zeroes of $P_n(z)$, and the weight of ν_n at $\lambda_i^{(n)}$ is $m_i^{(n)}/n$, where $m_i^{(n)}$ is the multiplicity of the zero $\lambda_i^{(n)}$ of $P_n(z)$. From the definition it follows that ν_n is a unit measure for any $n \in \mathbb{N}$. Integral of a function $f(z)$ with respect to the measure ν_n is

$$\int f(z)\, d\nu_n(z) = \sum_{i=1}^{n_j} \frac{m_i^{(n)}}{n} f(\lambda_i).$$

Since $\operatorname{supp}(\nu_n)$ consists of zeroes of $P_n(z)$, by Proposition 4.3.1 we have that the sets $\operatorname{supp}(\nu_n)$ are uniformly bounded.

Lemma 4.3.2. *Let μ be a weight measure. Then* $\limsup_{k\to\infty} |\, P_k(z)\,|^{\frac{1}{k}} \le 1$ *on some carrier E of μ.*

Proof. Let $M = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. Then for any $n \in \mathbb{N}$

$$\int \sum_{k=1}^{n} \frac{|\, P_k(z)\,|^2}{k^2} d\mu(z) = \sum_{k=1}^{n} \frac{1}{k^2} \int |\, P_k(z)\,|^2\, d\mu(z) = \sum_{k=1}^{n} \frac{1}{k^2} \le M.$$

Hence by Monotone Convergence Theorem it follows that the integral $\int \sum_{k=1}^{\infty} |\, P_k(z)\,|^2 / k^2\, d\mu(z)$ converges. Therefore the sum $\sum_{k=0}^{\infty} |\, P_k(z)\,|^2 / k^2$ converges on a complement of a set of μ-measure zero, and thus on a carrier E of μ. Thus there exists m such that $|\, P_k(z)\,|^2 / k^2 \le 1$ and hence $|\, P_k(z)\,| / k \le 1$ for $k > m$. Therefore for $z \in E$ we have

$$1 > \limsup_{n\to\infty} \left| \frac{P_n(z)}{n} \right|^{\frac{1}{n}} = \frac{\limsup\limits_{n\to\infty} |\, P_n(z)\,|^{\frac{1}{n}}}{\lim\limits_{n\to\infty} n^{\frac{1}{n}}} = \limsup_{n\to\infty} |\, P_n(z)\,|^{\frac{1}{n}}.$$

$\square$

Lemma 4.3.3. *Let the polynomials $\tilde{P}_n(z)$ and the unit measures ν_n be defined relative to a sequence $(\, P_k(z)\,)_{k=0}^{\infty}$ of orthonormal polynomials as on page 59. Then for any $n \in \mathbb{N}$*

$$p_{\nu_n}(z) = \log |\tilde{P}_n(z)|^{\frac{1}{n}}.$$

Proof.

$$\begin{aligned} p_{\nu_n}(z) &= \int \log |\, z - w \,| \; dw(z) = \sum_{i=1}^{n_j} \frac{m_i^{(n)}}{n} \log \left| \, z - \lambda_i^{(n)} \, \right| \\ &= \log \prod_{i=1}^{n_j} \left| \, z - \lambda_i^{(n)} \, \right|^{\frac{m_i^{(n)}}{n}} = \log \left| \, \prod_{i=1}^{n_j} \left(\, z - \lambda_i^{(n)} \, \right)^{m_i^{(n)}} \, \right|^{\frac{1}{n}} \\ &= \log |\tilde{P}_n(z)|^{\frac{1}{n}}. \end{aligned}$$

□

Proposition 4.3.4. *Let μ be a weight measure. Let $(\, P_{k_i}(z)\,)_{i=0}^{\infty}$ be a subsequence of $(\, P_k(z)\,)_{k=0}^{\infty}$ such that $\lim_{i\to\infty} (\, N_{k_i}\,)^{\frac{1}{k_i}} = L$ and $\nu_{k_i} \xrightarrow{*} \nu$. Then $p_\nu(z) \le \log L$ q.e. on some carrier E of μ. Moreover, if $\operatorname{cap}(E) > 0$ then $L > 0$.*

Proof. By Lemma 4.3.2 it follows that

$$\limsup_{k\to\infty} |\tilde{P}_{k_i}(z)|^{\frac{1}{k_i}} = \limsup_{k\to\infty} N_k^{\frac{1}{k_i}} |P_{k_i}(z)|^{\frac{1}{k_i}} = L \, \limsup_{k\to\infty} |P_{k_i}(z)|^{\frac{1}{k_i}} \le L \quad (4.3)$$

on a carrier E of μ. Hence for any $z \in E$ it follows by Lemma 4.3.3 that

$$\limsup_{i\to\infty} p_{\nu_{k_i}}(z) = \log \left(\limsup_{i\to\infty} |\tilde{P}_{k_i}(z)|^{\frac{1}{k_i}} \right) \le \log L.$$

Therefore by Theorem 4.1.11 we conclude that

$$p_\nu(z) \le \log L$$

q.e. on E.

Now assume $\operatorname{cap}(E) > 0$. Let Z_1 be the capacity-zero set such that $p_\nu(z) \le \log L$ on $E \backslash Z_1$. Let $Z_2 = \{\, z \in \mathbb{C} : p_\mu(z) = -\infty \,\}$. By Theorem

4.1.5 we have $\operatorname{cap}(Z_2) = 0$. Thus $\operatorname{cap}(Z_1 \cup Z_2) = 0$ and hence $\operatorname{cap}(E \backslash (Z_1 \cup Z_2)) = \operatorname{cap}(E) > 0$ by Proposition 4.1.4. Therefore $E \backslash (Z_1 \cup Z_2)$ is nonempty and there exists a point z_0 such $p_\nu(z_0) \leq \log L$ and $p_\nu(z_0) > -\infty$. We conclude that

$$\log L \geq p_\nu(z_0) > -\infty$$

and hence $L > 0$. □

4.4 Asymptotic Behaviour of $P_n(z)$ and Its Leading Coefficients

The following theorem which gives an upper bound for the asymptotic behaviour of the orthonormal polynomials is proved by J.L. Ullman in 1984 [8]:

Theorem 4.4.1. *Let μ be a weight measure and let $(\, P_k(z) \,)_{k=0}^{\infty}$ be a sequence of orthonormal polynomials such that $\deg P_n(z) = n$. Then*

$$\limsup_{k \to \infty} |P_k(z)|^{\frac{1}{k}} \leq e^{g_\mu^\infty(z)}. \tag{4.4}$$

Proof. First of all, we may assume without loss of generality that the leading coefficients of the orthogonal polynomials $P_k(z)$ are positive real numbers since we can obtain such a sequence by multiplying the polynomials with complex scalars of norm one, which does not affect the absolute value of the polynomials at any z and we still obtain a sequence of orthonormal polynomials.

If $c_\mu = 0$ then $g_\mu^\infty(z) = \infty$ and the result follows trivially. Consider the nontrivial case $c_\mu > 0$. Let $\{\, \nu_k \,\}_{k=0}^{\infty}$ be the measures defined on page 59 determined by the polynomials $\{\, P_k(z) \,\}_{k=0}^{\infty}$. Since supports of the measures ν_k are uniformly bounded, by Theorem 4.1.10 the sequence $(\, \nu_k \,)_{k=0}^{\infty}$ has a subsequence $(\, \nu_{k_i} \,)_{i=0}^{\infty}$ such that $\nu_{k_i} \xrightarrow{*} \nu$ for some unit measure ν with compact support. We may assume that the values $(\, N_{k_i} \,)^{\frac{1}{k_i}}$ converge to a value

L (allow $L = \infty$) since if not, we may choose such a subsequence of $(\,\nu_{k_i}\,)_{i=0}^{\infty}$. Then by Proposition 4.3.4 it follows that

$$p_\nu(z) \leq \log L \tag{4.5}$$

holds on $E \backslash Z$ where E is a carrier of ν and $\operatorname{cap}(Z) = 0$. We also have $L > 0$ since $\operatorname{cap}(E) \geq c_\mu > 0$. Let C be a minimal carrier such that $C \subseteq E$. Then equation (4.5) holds on $C \backslash Z$. Then by Proposition 4.2.4 and representation (4.2) of $g_\mu^\infty(z)$ it follows that

$$p_\nu(z) - \log L \leq g_\mu^\infty(z) = p_{\omega_\mu}(z) - \log c_\mu \tag{4.6}$$

for all $z \in \mathbb{C}$.

Now fix $z \in \mathbb{C}$. If $\limsup_{k\to\infty} |P_k(z)|^{\frac{1}{k}} = 0$ then the claim of the theorem follows immediately. Assume $M := \limsup_{k\to\infty} |P_k(z)|^{\frac{1}{k}} > 0$. Choose a subsequence $(\nu_{k_{i_j}})_{j=0}^{\infty}$ (which will be denoted as $\left(\,\nu_{n_j}\,\right)_{j=0}^{\infty}$ for short) such that $\lim_{k\to\infty} \left|\, P_{n_j}(z)\,\right|^{\frac{1}{n_j}} = M$. We still have $\nu_{n_j} \xrightarrow{*} \nu$ and $\lim_{j\to\infty} \left(\, N_{n_j}\,\right)^{\frac{1}{n_j}} = L$. Using Lemma 4.3.3 it follows that

$$\begin{aligned}
\log M &= \lim_{j\to\infty} \log \left|\, P_{n_j}(z)\,\right|^{\frac{1}{n_j}} = \lim_{j\to\infty} \log \frac{\left|\, \tilde{P}_{n_j}(z)\,\right|^{\frac{1}{n_j}}}{\left(\, N_{n_j}\,\right)^{\frac{1}{n_j}}} \\
&= \lim_{j\to\infty} \left(\log |\tilde{P}_{n_j}(z)|^{\frac{1}{n_j}} - \log \left(\, N_{n_j}\,\right)^{\frac{1}{n_j}} \right) \\
&= \lim_{j\to\infty} p_{\nu_{n_j}}(z) - \log L.
\end{aligned}$$

Also applying inequality (4.6) and Theorem 4.1.11 we have

$$\lim_{j\to\infty} p_{\nu_{n_j}}(z) - \log L \leq p_\nu(z) + g_\mu^\infty(z) - p_\nu(z) = g_\mu^\infty(z)$$

and we conclude that

$$\log M \leq g_\mu^\infty(z). \qquad \square$$

Given a compact subset S of $\mathbb{C}$, let $\Omega(S)$ denote the unbounded component of $\mathbb{C}\backslash S$. For a measure μ on $\mathbb{C}$, let $\Omega_\mu = \Omega(\text{supp}(\mu))$ and let $\text{conv}(\mu)$ denote the convex hull of $\text{supp}(\mu)$. The following theorem gives a lower asymptotic bound for the values $|P_n(z)|^{\frac{1}{n}}$ [5, Th. 1.1.4, eq. (1.7)]. (H. Widom had proved this lower bound for a special kind of measures which he called *'admissible measures'* in 1967 [10]) :

Theorem 4.4.2. *Let μ be a weight measure and let $(\, P_k(z)\,)_{k=1}^{\infty}$ be a sequence of related orthonormal polynomials such that* $\deg P_n(z) = n$. *Then*

$$\liminf_{k\to\infty} |P_k(z)|^{\frac{1}{k}} \geq e^{g^{\infty}_{\Omega_\mu}(z)} \tag{4.7}$$

for $z \in \mathbb{C}\backslash\text{conv}(\mu)$.

For a compact subset S of $\mathbb{C}$, let $S^* := \mathbb{C}\backslash\Omega(S)$ and let ∂S denote the boundary of the set S. Note that S^* is the smallest simply connected set including S as a subset. Since $S \subseteq S^*$ we have $\text{cap}(S) \leq \text{cap}(S^*)$. Besides, $\text{supp}(\omega_{S^*}) \subseteq \partial S^* \subseteq S$ yields that $\text{cap}(S) = e^{I(\omega_S)} \geq e^{I(\omega_{S^*})} = \text{cap}(S^*)$ by Proposition 4.1.7. Thus we conclude that $\text{cap}(S) = \text{cap}(S^*)$. We also have that $\omega_{S^*} = \omega_S$ since $\text{supp}(\omega_{S^*}) \subseteq S$ and $I(\omega_{S^*}) = \log(\,\text{cap}(S)\,)$. So, by definition of Green function, we have

$$\begin{aligned} g^{\infty}_{\Omega_\mu}(z) &= p_{\omega_{S^*}}(z) - \log(\,\text{cap}(S^*)\,) \\ &= p_{\omega_S}(z) - \log(\,\text{cap}(S)\,)\,, \end{aligned}$$

where $S = \text{supp}(\mu)$.

If we let $z \to \infty$ for the lower bound given by expression (4.4), for the leading coefficients γ_n of $P_n(z)$ we get

$$\begin{aligned} \limsup_{k\to\infty} (\,\gamma_n\,)^{\frac{1}{n}} &= \lim_{z\to\infty} \limsup_{k\to\infty} |P_k(z)|^{\frac{1}{k}} \\ &\leq \lim_{z\to\infty} e^{g^{\infty}_{\mu}(z)} = \lim_{z\to\infty} \left(\frac{e^{p_\mu(z)}}{c_\mu} \right) \\ &\leq \frac{1}{c_\mu}\,. \end{aligned}$$

Let $S = \text{supp}(\mu)$. Letting $z \to \infty$ in expression (4.7) yields

$$\begin{aligned}\liminf_{k\to\infty} (\,\gamma_n\,)^{\frac{1}{n}} &= \lim_{z\to\infty} \liminf_{k\to\infty} |P_k(z)|^{\frac{1}{k}} \\ &\geq \lim_{z\to\infty} e^{g^{\infty}_{\Omega_\mu}(z)} = \lim_{z\to\infty} \left(\frac{e^{p_{\omega_S}(z)}}{\text{cap}(S)} \right) \\ &\geq \frac{1}{\text{cap}(S)}\,.\end{aligned}$$

If we put these two facts together and recall that $\int \tilde{P}_n(z)\, d\mu(z) = \frac{1}{\gamma_n}$, we get that

$$c_\mu \leq \liminf_{k\to\infty} \left(\int \tilde{P}_n(z)\, d\mu(z) \right)^{\frac{1}{n}} \leq \limsup_{k\to\infty} \left(\int \tilde{P}_n(z)\, d\mu(z) \right)^{\frac{1}{n}} \leq \text{cap}(S),$$

from which we have the following result:

Corollary 4.4.3. *If* $c_\mu = \text{cap}(\text{supp}(\mu))$ *then*

$$\lim_{k\to\infty} (\,\gamma_n\,)^{\frac{1}{n}} = \frac{1}{\text{cap}(\text{supp}(\mu))}$$

and equivalently

$$\lim_{k\to\infty} \left(\int \tilde{P}_n(z)\, d\mu(z) \right)^{\frac{1}{n}} = \text{cap}(\text{supp}(\mu)).$$

The measures which satisfy the condition $c_\mu = \text{cap}(\text{supp}(\mu))$ are named as *determined measures* by J.L. Ullman [8].

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