ON THE HAMILTONIAN CIRCLE ACTIONS AND SYMPLECTIC REDUCTION

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Abstract

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Given a symplectic manifold, it is of interest how Lie group actions, their orbit spaces look like and what are some topological requirements on the existence of such actions. In this thesis we present the work of Ono, giving some sufficient conditions for non-existence of circle actions on symplectic manifolds and work of Li, describing the fundamental groups of symplectic reductions of circle actions.

Keywords: Symplectic Manifold, Circle Action, Symplectic Reduction.

Öz

HAMİLTON ÇEMBER ETKİLERİ VE SİMPLEKTİK BÖLÜM UZAYLARI

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Bir simplektik manifold üzerindeki Lie gruplarının etkisinin ve orbit uzaylarının özelliklerinin ve bu tür etkilerin var olması için gerekli topolojik şartların neler olduğunun incelenmesi önemli bir problemdir. Bu tezde Ono'nun, simplektik yapıyı koruyan çember etkisinin oluşmaması için yeterli bazı şartların verildiği, ve Li'nin simplektik bölüm uzaylarının temel grupları ile ilgili çalışmaları incelenmiştir.

Anahtar Kelimeler: Simplektik Manifoldlar, Çember etkisi, Simplektik bölüm uzayları.

To my family

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Chapter 1

INTRODUCTION

A symplectic form is a closed, non-degenerate 2-form. A symplectic manifold is a manifold equipped with a symplectic form. Due to the non-degeneracy condition all symplectic manifolds are necessarily even dimensional. Also non-degeneracy forces a symplectic manifold to be orientable. By Darboux's theorem all symplectic manifolds have locally the same symplectic structure; namely that of the Euclidian space.

Symplectic topology was first used in Hamiltonian mechanics. Its relation with mechanics is expressed as, "The phase space of a mechanical system is a symplectic manifold and the time evolution of a (conservative) dynamical system is a one-parameter family of symplectic diffeomorphisms". The role of the symplectic structure had first appeared, at least implicitly, in Lagrange's work on the variation of the orbital parameters of the planets in celestial mechanics. Its central importance emerged, however, from the work of Hamilton.

The concept of a moment map is a generalization of that of a Hamiltonian function. The notion of a moment map associated to a group action on a symplectic manifold formalizes the Noether principle which states that to every symmetry (such as a group action) in a mechanical system, there corresponds a conserved quantity. The angular momentum in $\mathbb{R}^3$ is an example of this which is the origin of the term "moment map". Therefore, the question that "when a Lie group action on a symplectic manifold admits a moment map" becomes important. Symplectic geometers mostly deal with the case when the group is S^1 (i.e. circle actions on symplectic manifolds). Hamiltonian circle actions on 4-manifolds have been classified by Karshon. However, the general structure of these actions is not yet fully understood. In this thesis, we consider the following result of Ono [11];

Theorem 1.0.1 (Ono). *Let (M, ω) be a closed symplectic manifold.*
1) If the second homotopy group $\pi_2(M)$ vanishes, then there is no circle group action on M preserving ω with non-empty fixed point set. (i.e. If $\pi_2(M) = 0$, then every S^1-action preserving ω is fixed point free)
2) If every abelian subgroup of $\pi_1(M)$ is cyclic, there is no S^1 action preserving ω. Therefore there is no compact, connected Lie Group action preserving ω.

An obvious necessary condition for a circle action to be Hamiltonian is the following: "A Hamiltonian action on a compact symplectic manifold must have fixed points which correspond to the critical points of the Hamiltonian function of this action". In [12] Ono proved that a symplectic circle action on a compact connected symplectic manifold (M, ω) such that the map

$$\wedge\omega^{n-1} : H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$$

is an isomorphism, is Hamiltonian if and only if it has fixed points. Note that this condition is satisfied for Kähler manifolds by the Hard-Lefschetz theorem.

An earlier version of the above theorem was proved by Frankel [5].

If M is a compact 4-manifold then the reduced spaces are 2-dimensional and the situation becomes quite easy to understand. In [8], McDuff , by looking at what happens to the reduced spaces as one passes a critical level, proved that a symplectic S^1 action on a closed 4-manifold is hamiltonian if and only if it has fixed points.

However, in the same paper there is an example of a symplectic but non-Hamiltonian action of S^1 on a compact 6-manifold with fixed points. Thus one needs some conditions either on M or on the action to extend the result in the preceeding theorem to higher dimensions, though it is not clear that the one given by Ono in [12] is the best possible. It is also conceivable that a symplectic circle action with isolated fixed points must be Hamiltonian. No counter example is yet known: the fixed points in the 6-dimensional example mentioned above form submanifolds of dimension 2.

The phase space of a system of n-particles is the space parametrizing the position and momenta of the particles. The mathematical model for the phase space is a symplectic manifold. Classical physicists realized that, whenever there is a symmetry group of dimension k acting on a mechanical system, then the number of degrees of freedom for the position and momenta of the particles may be reduced by 2k. Symplectic reduction formulates this feature mathematically.

The quotient of a symplectic manifold may not be a symplectic manifold. It may even fail to be even dimensional. Symplectic reduction brings a solution to this conflict. In [7], it is proved that the orbit space of a Hamiltonian G-space on which G acts freely is a symplectic manifold.

In this thesis we will stick to $G = S^1$ and investigate some topological properties of the orbit space of S^1-actions. We will analyze the work of Li in [6] which formulates the fundamental groups of symplectic reductions. In Chapter 4 we will prove

Theorem 1.0.2 (Li). *Let (M, ω) be a connected, compact symplectic manifold equipped with a Hamiltonian S^1 action. Then*

$$\pi_1(M) = \pi_1(minimum) = \pi_1(maximum) = \pi_1(M_{red}),$$

where M_{red} is the symplectic quotient at any value in the image of moment map μ and maximum and minimum are level sets of the maximum and the minimum of the moment map.

This thesis is organized as follows: In Chapter 2 we will define symplectic manifolds, Lie group actions on symplectic manifolds, and give a brief discussion of Morse Theory. We will prove Ono's theorem in Chapter 3. The last chapter is devoted to the theorem of Li.

Chapter 2

NECESSARY TOOLS

2.1 Symplectic Manifolds

A topological manifold is a Hausdorff, second countable space which is locally Euclidean. Some properties of topological manifolds are summarized in the following theorem.

Theorem 2.1.1. *A topological manifold M is locally connected, locally compact and a union of a countable collection of compact subsets; furthermore it is normal and metrizable.*

Definition 2.1.2. A smooth structure on a topological manifold M is a family $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ of coordinate neighborhoods such that:
(1) the U_α cover M and $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ a homeomorphism,
(2) for any α, β the neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^∞-compatible. In other words the change of coordinate functions are smooth.
(3) any coordinate neighborhood (V, ψ) compatible with every $(U_\alpha, \varphi_\alpha) \in \mathcal{U}$ is also in $\mathcal{U}$.

A smooth manifold is a topological manifold with a smooth structure. This form of smooth manifolds was first introduced by Weyl. Just as Riemannian Geometry deals with manifolds that have a Riemannian structure (a positive-definite, symmetric bilinear map called a Riemannian metric), Symplectic Geometry studies those manifolds carrying a so called symplectic form.

Definition 2.1.3. Let M be a smooth manifold. A symplectic structure (or a symplectic form) is a 2-form $\omega \in \Omega^2(M)$ satisfying:
i) ω is closed, i.e., $d\omega = 0$,
ii) ω is non-degenerate, in other words for $p \in M$ and $u \in T_pM$ if $\omega(u, v) = 0$ for all $v \in T_pM$, then u must be zero.
A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a symplectic structure on M.

It follows from this definition that all symplectic manifolds are necessarily even dimensional. Note that ω is non-degenerate means $\omega^n \neq 0$, where n is the half of the dimension of the manifold. So ω^n is actually a volume form on M. Another consequence of the non-degeneracy condition is that M be orientable, because the volume form determines a canonical orientation for the symplectic manifold M.

Example 2.1.4. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, ..., x_n, y_1, ..., y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is a symplectic form on M.

Example 2.1.5. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, ..., z_n$. The form

$$\omega_0 = \frac{1}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic. In fact, this form equals to that of the previous example under the identification $z_k = x_k + iy_k$.

Example 2.1.6. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at $p \in S^2$ may then be identified with vectors orthogonal to p. The standard symplectic form on S^2 induced by the inner and exterior products:

$$\omega_p(u, v) = \langle p, u \times v \rangle, \text{ for } u, v \in T_pS^2 = \{p\}^{\perp}$$

This form is closed because it is of top degree; it is non-degenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and we take for instance, $v = u \times p$.

Definition 2.1.7. Let (M, ω) be a symplectic manifold. A submanifold Y is called symplectic symplectic if $\omega|_Y$ is a symplectic form on Y.

Definition 2.1.8. Let (M_1, ω_1) and (M_2, ω_2) be 2n-dimensional symplectic manifolds, and let $g : M_1 \to M_2$ be a diffeomorphism. Then g is called a symplectomorphism if $g^*\omega_2 = \omega_1$.

Note that for any symplectomorphism g, g^{-1} is also a symplectomorphism. Therefore all symplectomorphisms on a symplectic manifold (M, ω) form a group denoted by $\mathrm{Symp}(M, \omega)$.

The following theorem implies that locally all the symplectic forms are the same provided that the manifolds have the same dimension.

Theorem 2.1.9 (Darboux). *Let (M, ω) be a 2n-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that on U*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

2.2 Complex Structures

Definition 2.2.1. Let V be a vector space. A complex structure on V is a linear map:

$$J : V \to V \text{ with } J^2 = -Id.$$

The pair (V, J) is called a complex vector space.

A complex structure J is equivalent to a stucture of a vector space over $\mathbb{C}$ if we identify the map J with multiplication by $\sqrt{-1}$.

Definition 2.2.2. Let (V, Ω) be a symplectic vector space. A complex structure J on V is said to be compatible (with Ω) if

$$G_J(u, v) = \Omega(u, Jv), \text{ for all } u, v \in V$$

is an inner product on V. That is J is Ω-compatible if and only if $\Omega(Ju, Jv) = \Omega(u, v)$ (symplectomorphism) and for all non-zero u, $\Omega(u, Ju) > 0$ (taming condition).

Compatible complex structures always exist on symplectic vector spaces:

Proposition 2.2.3. *Let (V, Ω) be a symplectic vector space. Then there is a compatible complex structure J on V.*

Definition 2.2.4. An almost complex structure J on a manifold M is a smooth field of complex structures on the tangent spaces:

$$x \mapsto (J_x : T_xM \to T_xM \text{ is linear and } J_x^2 = -Id).$$

The pair (M, J) is then called an almost complex manifold.

Definition 2.2.5. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called compatible (with ω) if the assignment

$$x(\mapsto g_x : T_xM \times T_xM \to \mathbb{R})$$

defined by $g_x(u, v) = \omega_x(u, J_xv)$ is a Riemannian metric on M.

The triple (ω, g, J) is called a compatible triple when $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.

Proposition 2.2.6. *1)Any symplectic manifold has compatible almost complex structures. The space $J(M, \omega)$ of such structures is path connected.*
2) The set $J(T_xM, \omega_x)$ is contractible for any x in M.

An almost complex structure J on a smooth manifold M is called integrable if J is naturally induced by a complex manifold structure of M. In other words, for any point $p \in M$, there is a coordinate chart (U, φ) where $\varphi : U \to \mathbb{R}^{2n}$ is a coordinate map, such that $d\varphi \circ J = J \circ d\varphi$.

Definition 2.2.7. A Kähler manifold is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure.

By definition, Kähler manifolds are both symplectic and complex. There are examples by Thurston in [14] that some symplectic manifolds are not Kähler.

2.3 Actions

Let M be a manifold and X a complete vector field on M. Let $\rho_t : M \to M$, $t \in \mathbb{R}$ be the family of diffeomorphisms generated by X. For each $p \in M$, $\rho_t(p)$ is by definition the unique integral curve of X passing through p at time 0. In other words $\rho_t(p)$ satisfies:

$$\begin{cases} \rho_0(p) &= p \\ \dfrac{d\rho_t(p)}{dt} &= X(\rho_t(p)). \end{cases}$$

The curve $\rho_t(p)$ is called the trajectory of the field X that passes through p at $t = 0$.
Note that we have $\rho_t \circ \rho_s = \rho_{t+s}$ and $\rho_t^{-1} = \rho_{-t}$. These make the following map a group homomorphism:

$$\begin{aligned} (\mathbb{R}, +) &\rightarrow \operatorname{Diff}(M) \\ t &\longmapsto \rho_t, \end{aligned}$$

where $\operatorname{Diff}(M)$ is the group of all diffeomorphisms on M. The family $\{\rho_t \mid t \in \mathbb{R}\}$ is called a one-parameter group of diffeomorphisms and satisfies

$$\rho_t = \exp tX.$$

Here the isotopy $\exp tX$ is (also) called the flow of X. These concepts are regularly used in Lie group actions on manifolds.

Definition 2.3.1. A Lie group is a smooth manifold G equipped with a group structure where the group operations are smooth.

Example 2.3.2. The following are well known examples of Lie groups:
1) $\mathbb{R}$ with addition.
2) S^1, regarded as unit complex numbers with multiplication.
3) $U(n)$: unitary linear transformations of $\mathbb{C}^n$.
4) $SU(n)$: unitary linear transformations of $\mathbb{C}^n$ with determinant equals to 1.
5) $O(n)$: orthogonal linear transformations of $\mathbb{R}^n$.

6) $SO(n)$: elements of $O(n)$ with with determinant equals to 1.

7) $GL(V)$: invertible linear transformations of a vector space V.

Definition 2.3.3. A representation of a Lie group G on a vector space V is a group homomorphism $G \to GL(V)$.

Actions of Lie groups play an important role in symplectic geometry.

Definition 2.3.4. An action of a Lie group G on a manifold M is a group homomorphism

$$\begin{aligned} \psi : G &\to \text{Diff}(M) \\ g &\mapsto \psi_g. \end{aligned}$$

The evaluation map associated with an action ψ is defined as follows:

$$\begin{aligned} ev_\psi : M \times G &\to M \\ (p, g) &\mapsto \psi_g(p) \end{aligned}$$

for all $p \in M, g \in G$. The action ψ is called smooth if ev_ψ is a smooth map.

Example 2.3.5. If X is a complete vector field on M, then

$$\begin{aligned} \rho : \mathbb{R} &\to \text{Diff}(M) \\ t &\mapsto \rho_t = \exp tX \end{aligned}$$

is a smooth action of $\mathbb{R}$ on M.

There is a 1-1 correspondence between complete vector fields on M and smooth actions of $\mathbb{R}$ on M.

2.3.1 Orbit Spaces

Let $\psi : G \to \text{Diff}(M)$ be any action

Definition 2.3.6. The orbit of G through $p \in M$ is the set $\{\psi_g(p) \mid g \in G\}$. The stabilizer (or isotropy) of $p \in M$ is the subgroup $G_p = \{g \in G \mid \psi_g(p) = p\}$.

Note that, if q is in the orbit of p, then G_p and G_q are conjugate subgroups.

Definition 2.3.7. We say that the action of G on M is
- **transitive** if there is just one orbit,
- **free** if all stabilizers are trivial $\{e\}$,
- **locally free** if all stabilizers are discrete.

Let $\sim$ be the orbit equivalence relation on M, i.e.,

$$p \sim q \Longleftrightarrow p \text{ and } q \text{ are in the same orbit.}$$

The space of orbits $M/_{\sim} = M/G$ is called the orbit space. Let

$$\begin{aligned} \pi : M &\to M/G \\ p &\mapsto \textit{ orbit through } p \end{aligned}$$

be the point-orbit projection. We equip M/G with the strongest topology for which π is continuous, so that $U \subseteq M/G$ is open if and only if $\pi^{-1}(U)$ is open in M. Note that his is the quotient topology.

Example 2.3.8. Let $G = \mathbb{R}$ act on $M = \mathbb{R}$ by multiplication by e^t. There are three orbits $\mathbb{R}^+, \mathbb{R}^-, \{0\}$. The point in the three-point orbit space corresponding to the orbit $\{0\}$ is not open, so the orbit space with quotient topology is not Hausdorff.

Definition 2.3.9. The Lie derivative is the operator $\mathcal{L}_v : \Omega^k(M) \to \Omega^k(M)$ defined by

$$\mathcal{L}_v\omega = \frac{d}{dt}((\exp tv)^*\omega)|_{t=0},$$

for $\omega \in \Omega^k(M)$.

When a vector field v_t is time dependent, its flow, that is, the corresponding isotopy ρ, still locally exists by Picard's Theorem. More precisely, in the neighborhood of any point p and for sufficiently small time t, there is a one-parameter family of local diffeomorphisms ρ_t satisfying:

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \text{ and } \rho_0 = id, \text{ for any } t \in \mathbb{R}.$$

Hence we say that the Lie derivative by v_t is $\mathcal{L}_{v_t}\omega = \frac{d}{dt}((\rho_t)^*\omega)|_{t=0}$. The following are the most used identities involving Lie Derivatives. See [1] [16].

Proposition 2.3.10. *1) (Cartan Magic Formula):* $\mathcal{L}_v\omega = \imath_v d\omega + d\imath_v\omega$.
2) $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega$

2.4 Morse Functions

For this section we mostly follow [10] and [2].
Let f be a real valued function on a manifold M. A point $p \in M$ is called a critical point of f if the induced map on the tangent spaces

$$f_* : T_pM \to T_{f(p)}M$$

is zero. If we choose a local coordinate system $(x_1, ..., x_n)$ in a neighbourhood U of p this means that

$$\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_n}(p) = 0.$$

Here $f(p)$ is called a critical value of f.

We denote by M^a the set of all points $x \in M$ such that $f(x) \leq a$. If a is not a critical value of f then it follows from the implicit function theorem that M^a is a smooth manifold with boundary. The boundary $f^{-1}(a)$ is a smooth submanifold of M.

A critical point p is called non-degenerate if and only if the Hessian of f at p,

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$$

is non-singular.

Definition 2.4.1. A Morse function on M is a function $h : M \to \mathbb{R}$ whose critical points are all non-degenerate.

Definition 2.4.2. A smooth function $f : M \to \mathbb{R}$ on a compact Riemannian manifold M is called a Morse-Bott function if its critical set

$$\text{Crit}(f) = \{p \in M | df(p) = 0\}$$

is a submanifold of M and for every $p \in \text{Crit}(f)$, $T_p\text{Crit}(f) = \ker\nabla^2 f(p)$ where $\nabla^2 f(p) : T_pM \to T_pM$ denotes the linear operator obtained from Hessian via the Riemannian metric.

This is the natural generalization of the notion of Morse function to the case where the critical set is not just isolated points. If f is a Morse-Bott function, then $\text{Crit}(f)$ decomposes into finitely many connected critical manifolds C. The tangent space T_pM at $p \in C$ decompose as a direct sum

$$T_pM = T_pC \oplus E_p^+ \oplus E_p^-,$$

where E_p^+ and E_p^- are spanned by the positive and negative eigenspaces of $\nabla^2 f(p)$. The index of a connected critical submanifolds C is defined to be the integer $n_C^- = \dim E_p^-$, for any $p \in C$, whereas the coindex of C is defined as $n_C^+ = \dim E_p^+$.

2.5 Some Algebraic Topology

The followings can be found in any algebraic topology book. We follow [3].

Definition 2.5.1. The n^{th} homotopy group of a space Y at a point $y_0 \in Y$ is the set of homotopy classes of continuous maps $(S^n, p_0) \to (Y, y_0)$ and denoted by $\pi_n(Y)$. If $n = 1$, this group is called the fundamental group of Y.

Definition 2.5.2. A map $p : X \to Y$ is called a covering map (and X is a covering space of Y) if each $y \in Y$ has an arcwise connected neighborhood U such that $p^{-1}(U)$ is a non-empty disjoint union of sets U_α (which are the arc components of $p^{-1}(U)$) on each of which the restriction $p|_{U_\alpha}$ of p to $U_\alpha$is a homeomorphism of U_α onto U.

Consider the exponential map $p : \mathbb{R} \to S^1$ defined by $p(t) = e^{2\pi it}$, which is a covering map. Let $f : I \to S^1$ be any loop at $1 \in S^1$. Let $\tilde{f} : I \to \mathbb{R}$ be a lifting of

f such that $\tilde{f}(0) = 0$. Then $\tilde{f}(1) \in p^{-1}(\{1\}) = \mathbb{Z}$. Let $n = \tilde{f}(1)$, which depends only on the homotopy class $[f] \in \pi_1(S^1)$. This integer n is called the degree of f and we write $\deg(f) = n$.

Theorem 2.5.3 (Van Kampen). *Let $X = U \cup V$ with U, V and $U \cap V$ all open, non-empty and arcwise connected. Let the base point of all these be some point $x_0 \in U \cap V$. Then the canonical maps of the fundamental groups of U, V and $U \cap V$ into that of X induce an isomorphism:*

$$\theta : \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \xrightarrow{\approx} \pi_1(X),$$

where $*$ is the free product of groups with amalgamation.

We finish this section with Hurewicz Theorem.

Proposition 2.5.4 (Hurewicz). *Let $n \geq 2$ be an integer. Then, if X is (n-1)-connected, the map $h_n : \pi_n(X) \to H_n(X)$ is an isomorphism.*

Chapter 3

OBSTRUCTION TO CIRCLE GROUP ACTIONS

3.1 Preliminaries

Let (M, ω) be a symplectic manifold and G be a Lie group.
Let $\psi : G \longrightarrow \text{Diff}(M)$ be a (smooth) action. In other words, G acts on M by diffeomorphisms.

Definition 3.1.1. The action ψ is called a *symplectic* action if

$$\psi : G \longrightarrow \text{Symp}(M, \omega) \subset \text{Diff}(M).$$

i.e., G acts by symplectomorphisms.

Definition 3.1.2. An action ψ of S^1 or $\mathbb{R}$ on (M, ω) is Hamiltonian if there is a function,

$$H : M \longrightarrow \mathbb{R}$$

with $dH = \imath_{\mathcal{X}}\omega$, where $\mathcal{X}$ is the vector field generated by the action. Here H is called a Hamiltonian function for the action.

Now using the Cartan Magic Formula (Proposition 2.3.10)

$$\mathcal{L}_X\omega = \imath_X d\omega + d\imath_X\omega$$

we see that $\mathcal{L}_X\omega = d\imath_X\omega$ since ω is closed. So, the action is symplectic if and only if $\mathcal{L}_X\omega = 0$, and hence if and only if $\imath_X\omega$ is closed. Moreover, if the closed 1-form $\imath_X\omega$ is exact i.e. $\imath_X\omega = dH$ then the action is Hamiltonian.

More generally, let (M,ω) be a symplectic manifold, G a Lie Group, $\mathbf{g}$ Lie algebra of G, $\mathbf{g}^*$ the dual vector space of $\mathbf{g}$ and $\psi : G \longrightarrow \mathrm{Symp}(M,\omega)$ a symplectic action.

Definition 3.1.3. The action ψ is a Hamiltonian action if there exists a map

$$\mu : M \longrightarrow \mathbf{g}^*$$

satisfying;
1. For each $X \in \mathbf{g}$, let
$*$ $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X\rangle$ be the component of μ along X,
$*$ $X^\sharp$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$. Then

$$d\mu^X = \imath_{X^\sharp}\omega.$$

In other words, μ^X is a Hamiltonian function for the vector field $X^\sharp$.
2. μ is equivariant with respect to the given action ψ of G on M and the coadjoint action $\mathbf{Ad}^*$ of G on $\mathbf{g}^*$:

$$\mu \circ \psi_g = \mathbf{Ad}^*_{\mathbf{g}} \circ \mu, \quad \text{for all } g \in G.$$

The vector (M,ω,G,μ) is then called a *Hamiltonian G-space* and μ is a *moment map*.

We will work mostly with the group $G = S^1$. So $\mathbf{g} \backsimeq \mathbf{g}^* \backsimeq \mathbb{R}$ and the moment map $\mu : M \longrightarrow \mathbb{R}$ satisfies:
1) For the generator $X = 1$ of $\mathbf{g}$, we have $\mu^X = \mu(p)\cdot 1$, i.e., $\mu^X = \mu$, and $X^\sharp$ is the vector field on M generated by S^1. Then $d\mu = \imath_{X^\sharp}\omega$.
2) μ is invariant: $\mathcal{L}_{X^\sharp}\mu = \imath_{X^\sharp}d\mu = 0$

Definition 3.1.4. Let (M,ω) be a symplectic manifold such that the cohomology class $[\omega] \in H^2(M,\mathbb{R})$ lies in the image of the canonical map $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$:

$$[\omega] \in \mathbf{Im}\{H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{R})\}.$$

If the S^1 action on M is only symplectic but not Hamiltonian, then there is, by [15], a so called generalised moment map $\mu : M \longrightarrow S^1$ such that

$$\imath_X \omega + \mu^* \frac{d\theta}{2\pi} = 0.$$

Lemma 3.1.5 ([8]). *Let ω be an S^1-invariant symplectic form on M such that $\imath_X \omega$ is nonzero. Then there is an S^1-invariant symplectic form which admits a generalized moment map μ.*

Proof. First of all note that if $[\omega]$ is rational then so is $[\imath_X \omega]$. Let ϕ_t be the flow of X. Then $[\imath_X \omega]$ on a loop λ equals the value of $[\omega]$ on the $2-cycle$ $[\phi_t(\lambda) : 0 \leq t \leq 1]$. If $[\omega]$ is not rational, there is always a symplectic form whose cohomology class is rational and which is so close to ω that its average $\hat{\omega}$ over S^1 is symplectic and satisfies $[\imath_X \hat{\omega}] \neq 0$. Thus a multiple of $\hat{\omega}$ admits a generalized moment map.

□

Remark 3.1.6. Since any S^1-invariant Riemannian metric $\hat{g}$ is related to ω by the identity $\hat{g}(\cdot,\cdot) = \omega(\cdot, A\cdot)$ for a unique A, which is non-singular, skew-symmetric and S^1-invariant, there is always an S^1-invariant metric g on M which is compatible with ω. (i.e. $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, where J is an S^1-invariant almost complex structure on M.) Now using $\Lambda = -A^2$ which is positive-definite and S^1-invariant and we see that

$$g(\cdot,\cdot) = \hat{g}(\cdot, \Lambda^{\frac{-1}{2}} A\cdot)$$

has the desired property.

Remark 3.1.7. If we identify S^1 with $\mathbb{R}/\mathbb{Z}$ in the usual way, we may define the gradient vector field of μ with respect to g. It is easy to check that this is just JX so that it commutes with X. Observe that

$$[X, JX] = \mathcal{L}_X(JX) = \mathcal{L}_X(J)X + J\mathcal{L}_X X = 0$$

Lemma 3.1.8. *Assume that the generalized moment map $\mu : M \longrightarrow S^1$ can not be lifted to a continuous map $M \longrightarrow \mathbb{R}$. Then given any point $p \in M$, there exists a homologically non-trivial loop $\gamma : S^1 \longrightarrow M$ passing through p such that $d\theta(\mu_* \dot{\gamma}) > 0$ everywhere except at fixed points.*

Remark 3.1.9. μ lifts to a map $M \longrightarrow \mathbb{R}$ if and only if the map

$$\mu_* : \pi_1(M) \longrightarrow \pi_1(S^1)$$

is trivial.

Remark 3.1.10. Let S^1 act on $\mathbb{C}^n$ in the following way,

$$S^1 \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$(\omega, z_1, ..., z_n) \longmapsto (\omega^{k_1} z_1, ..., \omega^{k_n} z_n)$$

where $k_1, ..., k_n$ are fixed integers. Then the moment map of this action is of the form

$$\mu = -\tfrac{1}{2} \sum k_i |z_i|^2 = -\tfrac{1}{2} \sum k_i (x_i^2 + y_i^2)$$

For example if n=2,

$$\mu = -\tfrac{1}{2} k_1 (x_1^2 + y_1^2) - \tfrac{1}{2} k_2 (x_2^2 + y_2^2)$$

The Hessian of μ will be the diagonal matrix:

$$H(\mu) = \begin{bmatrix} -k_1 & & & \\ & -k_1 & & \\ & & -k_2 & \\ & & & -k_2 \end{bmatrix}$$

Note that the index of the Hessian will always be even for such an action. Since we have an S^1-invariant almost complex structure on M, at a fixed point $p \in M$ of the action identifying (T_pM, J_p) with $\mathbb{C}^n$, we get an action of S^1 on $\mathbb{C}^n$. Thus the index of the Hessian of a moment map at any critical point will be even.

Proof. of the Lemma. By Remark 3.1.6 we can always choose an S^1-invariant Rimannian metric and an S^1-invariant almost complex structure compatible with ω. (Actually since g and ω determines J, it should be S^1-invariant if g and ω are.)
Now, if we regard a generalized moment map locally as a function, we can define the Hessian at critical points, their indices (number of negative eigenvalues) and the gradient flow of μ.
If M has a critical point at which the Hessian is positive or negative definite (i.e. that point is maximum or minimum) the moment map induces a trivial map in the fundamental groups. By Remark 3.1.9 μ lifts to a map $M \to \mathbb{R}$, which is not the case by the assumption of the lemma. So by the hypothesis we may assume that all critical points are indefinite.
Let X be the quotient space of M by the equivalence relation:

$$x \sim y \Longleftrightarrow \exists t \in S^1 \text{ such that x and y belong to the same connected component of } \mu^{-1}(t).$$

By Remark 3.1.10, since the index of the Hessian of the moment map at a critical point is even, X has no branch point. Also since the Hessian at critical points are indefinite, X has no boundary points, and thus homeomorphic to S^1. Thus we can deform the trajectories of the gradient flow of μ to X which has the properties of a loop as in the lemma. □

Now we are ready to state and prove Ono's main results.

Theorem 3.1.11. *Let (M, ω) be a closed symplectic manifold,*
1) If the second homotopy group $\pi_2(M)$ vanishes, there is no circle group action on M preserving ω with non-empty fixed point set.
(i.e. If $\pi_2(M) = 0$, every S^1-action preserving ω is fixed point free.)
2) If every abelian subgroup of $\pi_1(M)$ is cyclic, there is no S^1 action preserving ω. Therefore there is no compact, connected Lie Group action preserving ω.

Lemma 3.1.12. *If $\pi_2(M) = 0$ then there is no S^1 action admitting a moment map.*

Proof. Suppose $\mu : M \longrightarrow \mathbb{R}$ is a moment map. Now consider the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M, \mathbb{Z})$. Let $p \in M$ and set

$$\overline{M(p)} = \bigcup_{t \in S^1} t(\gamma_p),$$

where γ_p is the trajectory of the gradient flow of the moment map passing through the generic point p. Note that

$$\overline{M(p)} = \bigcup_{t \in S^1} t(\gamma_p) \simeq S^2$$

which is the image of a continuous map $S^2 \to M$. But we have

$$\int_{\overline{M(p)}} \omega > 0$$

which implies that $[\overline{M(p)}] \neq 0$ in $H_2(M, \mathbb{Z})$. Hence, $[\overline{M(p)}] \neq 0$ in $\pi_2(M)$. But this contradicts with $\pi_2(M) = 0$. □

Proof. (of theorem 3.1.11) Assume that S^1 acts symplectically on M and $\pi_2(M) = 0$. By Lemma 3.1.5, we may assume that there is a generalized moment map $\mu : M \to S^1$. By the above lemma this map can not be lifted to a map $M \to \mathbb{R}$. We must show that this map does not have any fixed points. Suppose p is a fixed point of μ. Then by Lemma 3.1.8 there is a loop γ passing through p such that $d\theta(\mu_* \dot{\gamma}) > 0$ everywhere except at fixed points. Let $f : S^2 \to M$ defined by $(t, \gamma) \mapsto t \cdot \gamma$ which induces $f_* : \pi_2(S^2) \to H_2(M, \mathbb{Z})$. Then $C = \bigcup_{t \in S^1} (t \cdot (\gamma_p))$ is the image of a continuous map from S^2 and since $\pi_2(M) = 0$, $[C]$ is zero in $H_2(M, \mathbb{Z})$. But this is a contradiction to the fact that $\int_C \omega > 0$.
To see the last inequality, let $\dot{\gamma}$ be the velocity vector field of γ. Then

$$\omega(\dot{\gamma}, v) = \imath_v \omega(\dot{\gamma}) = d\mu(\dot{\gamma}) = \mu_*(d\theta)(\dot{\gamma}) = d\theta(\mu_*(\dot{\gamma})) > 0,$$

where v is the symplectic vector field generated by the action and $d\mu$ is the pull back of $d\theta$.

For the second part of theorem, since we have shown that there is no fixed points,

$$C = \bigcup_{t \in S^1} t \cdot (\gamma)$$

is the image of a continuous map

$$F : T^2 \to M$$

which induces a homomorphism

$$F_* : \pi_1(T^2) \to \pi_1(M)$$

The induced map can not be injective because a cyclic group can not contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$. Therefore, there exists a homologically non-trivial loop on T^2 which is mapped to a null-homotopic loop. This implies that F is homotopic to a map which maps this loop to a point. Thus C is the image of the composition of a continuous map from S^2 to T^2 (which identifies the poles of the sphere) with a map to C as in Figure 3.1 .

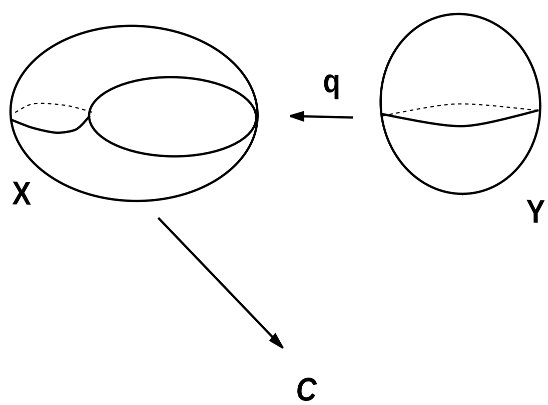

Figure 3.1: C as image of a sphere

Therefore $[C]$ is represented by a continuous map from a 2-sphere. But

$\pi_2(M) = 0$ implies $[C] = 0$ in $H_2(M, \mathbb{Z})$. On the other hand we have $\int_C \omega > 0$ which is a contradiction again . Therefore there is no S^1 action preserving ω. $\square$

Chapter 4

π_1 of Hamiltonian S^1 manifolds

4.1 Preliminaries

Definition 4.1.1. Let (M, ω) be a connected,compact, symplectic manifold such that S^1 acts on M in a Hamiltonian way. Let $\mu : M \to \mathbb{R}$ be a corresponding moment map, which is a Perfect Morse-Bott function. For $a \in im(\mu)$, and $\mu^{-1}(a) = \{x \in M \mid \mu(x) = a\}$ define

$$M_a = \mu^{-1}(a)/S^1$$

to be the symplectic quotient or reduced space of M.

More generally we have the following theorem:

Theorem 4.1.2 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $\imath : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *The orbit space $M_{red} = \mu^{-1}(0)/G$ is a smooth manifold*
- *$\pi : \mu^{-1} \to M_{red}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

Here, the pair (M_{red}, ω_{red}) is called the **reduction**, or the **symplectic quotient** of (M, ω) with respect to $G, \mu,$.

Remark 4.1.3. If a is a regular value of μ, and if the circle action on $\mu^{-1}(a)$ is not free, then M_a is an orbifold, and we have an orbi-bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & \mu^{-1}(a) \\ & & \downarrow \\ & & M_a. \end{array}$$

If a is a critical value of μ, then M_a is a stratified space [13].

Example 4.1.4. Let

$$\omega = \frac{i}{2}\sum dz_j \wedge \overline{dz_j} = \sum dx_j \wedge dy_j = \sum r_j dr_j \wedge d\theta_j$$

and assume S^1 acts on $(\mathbb{C}^n, \omega)$ with $t \in S^1 \longmapsto \psi_t$, multiplication by e^{it}. Then the action ψ is hamiltonian with moment map

$$\begin{array}{rcl} \mu : \mathbb{C}^n & \rightarrow & \mathbb{R} \\ z & \mapsto & \frac{-|z|^2}{2} + constant. \end{array}$$

Since $d\mu = -\frac{1}{2}d(\sum r_j^2)$ we have

$$X^\sharp = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + ... + \frac{\partial}{\partial\theta_n}$$

so that

$$\imath_{X^\sharp}\omega = -\sum r_j dr_j = -\frac{1}{2}\sum dr_j^2.$$

Now if we choose the constant to be 1/2, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}.$$

Hence, $\mathbb{CP}^{n-1}$ is a reduced space.

4.2 Some Morse Theory

The aim of this chapter is to prove Li's theorem which states that the fundamental group of the symplectic quotient is the same as that of the original manifold. Namely we will prove:

Theorem 4.2.1. *Let (M, ω) be a connected, compact symplectic manifold equipped with a Hamiltonian S^1 action. Then*

$$\pi_1(M) = \pi_1(minimum) = \pi_1(maximum) = \pi_1(M_{red}),$$

where M_{red} is the symplectic quotient at any value in the image of moment map μ.

Remark 4.2.2. Since the action is Hamiltonian, the moment map $\mu : M \to \mathbb{R}$ is a perfect Morse-Bott function. Its critical sets are precisely the fixed point sets M^{S^1} of the S^1 action, and M^{S^1} is a disjoint union of symplectic submanifolds. Each fixed point set has even index. By Atiyah, μ has a unique local minimum and a local maximum. The spaces *minimum* and *maximum* in Theorem 4.2.1 are the level sets of these points.

In order to prove Theorem 4.2.1, we need some lemmas from Morse Theory. We begin with a definition.

Definition 4.2.3. For any real number $a \in \mathbb{R}$, define

$$M^a = \{x \in M | \mu(x) \leq a\}.$$

Lemma 4.2.4 ([10]). *Assume $[a, b] \subseteq im(\mu)$ is an interval consisting of regular values, then M^a is diffeomorphic to M^b.*

Lemma 4.2.5 ([10]). *If $c \in (a, b)$ is the only critical value of μ in $[a, b]$, let $F \subset \mu^{-1}(c)$ be the fixed point set component, D^- be the negative disk bundle of F and $S(D^-)$ its sphere bundle. Then M^b is homotopy equivalent to $M^a \cup_{S(D^-)} D^-$.*

Milnor states the above lemma in a different way:

Lemma 4.2.6. *Let $f : M \to \mathbb{R}$ be a smooth function and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$ suppose that $F^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical points of f other than p, for some $\varepsilon > 0$. Then for sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.*

Proof. (Sketch) The idea of the proof is as follows. Introduce a new function $F : M \to \mathbb{R}$ which coincides with the height function f except that $F < f$ in a small neighborhood of p. Thus the region $F^{-1}(-\infty, c-\varepsilon)$ will consist of $M^{c-\varepsilon}$ together with a region H near p. Choosing a suitable cell $e^{\lambda} \subset H$, a direct argument (i.e. pushing in along the horizontal lines) will show that $M^{c-\varepsilon} \cup e^{\lambda}$ is a deformation retract of $M^{c-\varepsilon} \cup H$. Finally, by applying previous lemma to the function and the region $F^{-1}[c-\varepsilon, c+\varepsilon]$ we will see that $M^{c-\varepsilon} \cup H$ is a deformation retract of $M^{c+\varepsilon}$. □

These ideas can also be used in proving,

Lemma 4.2.7. *Under the same hypothesis of Lemma 4.2.5 $\mu^{-1}(a) \cup_{S(D^-)} D^-$ has the homotopy type of $\mu^{-1}(c)$.*

Also the quotients of the spaces by S^1 in Lemma 4.2.7 will give,

Lemma 4.2.8. *Under the same hypothesis of Lemma 4.2.5, M^c has the same homotopy type of $M_a \cup_{S(D^-)/S^1} D^-/S^1$.*

Finally we will need the following lemma to prove the main result of this chapter.

Lemma 4.2.9. *Assume F is a critical set, $\mu(F) \in (a, b)$ and there are no other critical sets in $\mu^{-1}([a, b])$. If index$(F) = 2$, then there is an emdedding i, from F to M_a, such that $S(D^-)$ can be identified with the restriction of $\mu^{-1}(a)$ to F. i.e., we have the following bundle identification*

$$\begin{array}{ccccc} S^1 \hookrightarrow & S(D^-) & \xrightarrow{g} & \mu^{-1}(a) \\ & \downarrow & & \downarrow \\ & F & \xrightarrow{i} & M_a. \end{array}$$

Proof. Since μ is a perfect Morse-Bott function its critical sets are even dimensional manifolds of even index and coindex. As Atiyah noted, T_pM splits as $T_pM = T_pF \oplus D^+ \oplus D^-$, where D^+ and D^- are spanned by positive and negative eigenspaces of $\nabla^2(\mu)$. Dimension of D^- gives the index of F. Assume the positive normal bundle D^+ of F has rank m. We may arrange (by adding some constant) μ so that, $\mu(F) = 0$. Since there are no other critical sets in $\mu^{-1}([a, b])$, we can assume $a = -\epsilon$ and $b = \epsilon$ for ϵ small.

By the equivariant symplectic embedding theorem, a tubular neighborhood of F is diffeomorphic to $P \times_G (\mathbb{C} \times \mathbb{C}^m)$, where $G = S^1 \times U(m)$ and P is a principle G bundle over F. The moment map can be written $\mu = -p_0|z_0|^2 + p_1|z_1|^2 + \cdots + p_m|z_m|^2$, where $p_0, p_1, ..., p_m$ are positive integers. Then $\mu = -\epsilon$ gives

$$p_0|z_0|^2 = \epsilon + p_1|z_1|^2 + \cdots + p_m|z_m|^2 > 0.$$

Therefore $z_0 \neq 0$ and $\mu^{-1}(-\epsilon) = P \times_G (S^1 \times \mathbb{C}^m)$ and

$$\mu^{-1}(-\epsilon)/S^1 = P \times_G (S^1 \times \mathbb{C}^m)/S^1.$$

To project the tubular neighborhood around F we must have $z_1 = z_2 = \cdots = z_m = 0$. Therefore F is diffeomorphic to $P \times_G (S^1 \times 0)/S^1 \subset M_{-\epsilon}$ and the negative sphere bundle of F, $S(D^-)$ is $P \times_G S^1$ which is the restriction of $\mu^{-1}(-\epsilon)$ to F as required. □

4.3 Proof of Li's Theorem

Now we are ready to prove Theorem 4.2.1.

Proof. First we will prove $\pi_1(min) = \pi_1(M_{red}) = \pi_1(max)$.
Let's put the critical values of μ in the order

$$minimal = 0 < a_1 < a_2 < \cdots < a_k = maximal.$$

For $a \in (0, a_1)$, by the equivariant symplectic embedding theorem, $\mu^{-1}(a)$ is a

sphere bundle over the minimum. Assume the fiber of this sphere bundle is S^{2l+1}, then M_a is diffeomorphic to a *weighted* $\mathbb{CP}^l$ bundle over the minimum (possibly an orbifold). Note that disk bundles are even dimensional so their sphere bundles must be odd dimensional.
Setting $F = \mu^{-1}(0)$, a tubular neighborhood $v(F)$ of F is diffeomorphic to

$$P \times_G \mathbb{C}^{l+1}.$$

Here G is just $U(l+1)$ but not $S^1 \times U(\cdot)$ because bundle is over the minimum. P is a principle $U(l+1)$ bundle over F.

$$\begin{array}{ccc} U(l+1) & \rightarrow & P \\ & & \downarrow \\ & & F. \end{array}$$

S^1 acts on $v(F)$ by $z(w_1, ..., w_{l+1}) = (z^{p_1} w_1, ..., z^{p_{l+1}} w_{l+1})$. If we had $p_1 = ... = p_{l+1} = 1$ then

$$M_a = P \times_{U(L+1)} S^{2l+1}/S^1 = S^{2l+1}/S^1 = \mathbb{CP}^l.$$

In this case the action may not be free and thus the quotient may not be a smooth manifold but an orbifold. Therefore M_a is diffeomorphic to a weighted projective space denoted $w\mathbb{CP}^l$(which is an orbifold). This gives a fibration

$$\begin{array}{ccc} w\mathbb{CP}^l & \rightarrow & M_a \\ & & \downarrow \\ & & F. \end{array}$$

This fibration induces a sequence in homotopy groups

$$\cdots \rightarrow \pi_2(F) \rightarrow \pi_1(w\mathbb{CP}^l) \rightarrow \pi_1(M_a) \rightarrow \pi_1(F) \rightarrow 0.$$

We claim that $\pi_1(w\mathbb{CP}^l) = 0$. To prove the claim we use the following fact: If

$$\begin{array}{ccc} K & \rightarrow & Y \\ & & \downarrow \\ & & X \end{array}$$

is a fibration such that K is connected then $\pi_1(Y) \rightarrow \pi_1(X)$ is onto.
To see this, note that by the homotopy lifting property any loop in X can be lifted which is not necessarily a path. But since fibers are connected the lifts of these paths may be completed to a loop. Now using the fact in the following fibration

$$\begin{array}{ccc} S^1 & \rightarrow & S^{2l+1} \\ & & \downarrow \\ & & w\mathbb{CP}^l \end{array}$$

where S^1 is clearly connected and $\pi_1(S^{2l+1}) = 0$, we have proved result of the claim. Therefore, we proved that $\pi_1(M_a) = \pi_1(F) = \pi_1(minimum)$.

Next let $b \in (a_1, a_2)$ and $F \subset \mu^{-1}(a_1)$ be the critical set. By Lemma 4.2.8, M_{a_1} is homotopy equivalent to

$$M_a \cup_{S(D^-)/S^1} D^-/S^1$$

where $M_{a_1} = \mu^{-1}(a_1)/S^1$. By Van-Kampen theorem we have

$$\pi_1(M_{a_1}) = \pi_1(M_a) *_{\pi_1(S(D^-)/S^1)} \pi_1(D^-/S^1).$$

Since $S(D^-)/S^1$ is a weighted projectivized bundle over F and D^-/S^1 is homotopy equivalent to F we have $\pi_1(M_{a_1}) = \pi_1(M_a)$. This can be seen by analyzing the induced homotopy sequence of the following bundle:

$$\begin{array}{ccc} w\mathbb{CP}^l & \rightarrow & S(D^-)/S^1 \\ & & \downarrow \\ & & F. \end{array}$$

We have $\pi_1(w\mathbb{CP}^l) = 0$ in the sequence

$$\cdots \to \pi_2(F) \to \pi_1(w\mathbb{CP}^l) \to \pi_1(S(D^-)/S^1) \to \pi_1(F) \to 0$$

which implies the above result.
Now using $-\mu$ as above we can obtain $\pi_1(M_b) = \pi_1(M_{a_1})$. By induction on the critical values, repeating the argument each time μ crosses a critical level, we see that if $a' \in (a_{k-1}, a_k)$, then $\pi_1(M_{a'}) = \pi_1(minimum)$. Similar to the proof of $\pi_1(M_a) = \pi_1(minimum)$ when $a \in (0, a_1)$, we have $\pi_1(M_{a'}) = \pi_1(maximum)$. Therefore we proved that $\pi_1(M_{red}) = \pi_1(minimum) = \pi_1(maximum)$.

Next we prove $\pi_1(minimum) = \pi_1(M)$.
If $a \in (0, a_1)$ then M^a is a complex disk bundle over the minimum i.e.,

$$\begin{array}{ccc} D^- & \to & M^a \\ & & \downarrow \\ & & F. \end{array}$$

Since D^- is contractible $\pi_1(M^a) = \pi_1(F) = \pi_1(minimum)$.
Consider $b \in (a_1, a_2)$ and let $F \subset \mu^{-1}(a_1)$ be the critical set. We have two cases depending on the index of F.
If $index(F) = 2$, by Lemma 4.2.5 we have $M^b = M^a \cup_{S(D^-)} D^-$ so that

$$\cdots \to \pi_2(M_a) \to \pi_1(S^1) \xrightarrow{j} \pi_1(\mu^{-1}(a)) \xrightarrow{f} \pi_1(M_a) \to 0.$$

Since f is surjective the image of $ker(f) = im(j)$ in $\pi_1(M_a)$ is 0. But we have found that $\pi_1(M_a) = \pi_1(M^a)$. Therefore, the image of $ker(f)$ is also 0 in $\pi_1(M^a)$. This implies that there is no new generator in the free product

$$\pi_1(M^a) *_{\pi_1(S(D^-))} \pi_1(F)$$

and hence we have $\pi_1(M^b) = \pi_1(M^a) = \pi_1(minimum)$.
The other case is when $index(F) > 2$ which is almost obvious. Here we have the

following fibration

$$\begin{array}{ccc} S^i & \to & S(D^-) \\ & & \downarrow \\ & & F \end{array}$$

where $i \geq 3$ and its homotopy exact sequence is

$$\pi_2(F) \to \pi_1(S^i) \to \pi_1(S(D^-)) \to \pi_1(F) \to 0$$

in which $\pi_1(S^i) = 0$. Therefore $\pi_1(S(D^-)) = \pi_1(F)$ and we have $\pi_1(M^b) = \pi_1(M^a)$.

By induction we get $\pi_1(M) = \pi_1(minimum)$. $\square$

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