VISUALIZING DATA WITH FORMAL CONCEPT ANALYSIS

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VISUALIZING DATA WITH FORMAL CONCEPT ANALYSIS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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ABSTRACT

VISUALIZING DATA WITH FORMAL CONCEPT ANALYSIS

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In this thesis, we wanted to stress the tendency to the geometry of data. This should be applicable in almost every branch of science, where data are of great importance, and also in every kind of industry, economy, medicine etc. Since machine's hard-disk capacities which is used for storing data and the amount of data you can reach through internet is increasing day by day, there should be a need to turn this information into knowledge. This is one of the reasons for studying formal concept analysis.

We wanted to point out how this application is related with algebra and logic. The beginning of the first chapter emphasis the relation between closure systems, Galois connections, lattice theory as a mathematical structure and concept analysis. Then it describes the basic step in the formalization: An elementary form of the representation of data is defined mathematically.

Second chapter explains the logic of formal concept analysis. It also shows how implications, which can be regard as special formulas on a set, between attributes can be shown by fewer implications, so called generating set for implications.

These mathematical tools are then used in the last chapter, in order to describe complex 'concept' lattices by means of decomposition methods in examples.
Keywords: Formal Concept Analysis, data analysis, lattice theory, closure systems, implications, decompositions.

ÖZ

VERİLERİN GEOMETRİSİ

ÇAĞRI DİNER

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Bu tezde verilerin geometrisine olan artan ilgiliyi vurgulamak istedik. Bu konu, verilerin olduğu her bilim ve endüstri dalında uygulama alanı bulabilir. Günümüzde bilgisayarların gitgide büyüyen hafızaları ve internet üzerinden ulaşabileceğimiz verilerin çokluğu bu verilerin bir şekilde bilgiye dönüşmesine ihtiyaç doğurmuştur. Kavram analizinin çalışılmasının sebeplerinden biri de budur.

Tüm bu uygulamaların cebir ve mantık bilimiyle nasıl ilgili olduğunu anlatmaya çalıştık. Birinci konunun başlarında kapalı sistemler, Galois bağlantıları, matematiksel bir yapı olan kafes teorisi ve kavram analizinin arasındaki bağları gösterdik. ılerleyen kısımlarda verilerin gösterilmesinde kullanılan tabloların matematiksel olarak nasıl tanımlandığını ve temel tanımları verdik.

ıkinci konu kavram analizinin mantık temelleri üzerine oldu. Veri analizinde önemli bir yere sahip olan çıkarımların daha az sayıda çıkarım kullanılarak nasıl ifade edilebileceğinin teorisini gösterdik.

En son konuda tüm bu matematiksel tanımlar ve teorileri kullanarak, karmaşık ve büyük kafes diagramlarının nasıl daha küçük parçalara ayrılabileceğini ve ifade edilebileceğini ispatladık.

Anahtar kelimeler: Formel kavram analizi, veri analizi, çıkarımlar, kapalı sistemler, kafes teorisi

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CHAPTER 1

Mathematical Background

1.1 Introduction to Lattices

In the first chapter we will introduce formal concept analysis which is mainly based on lattice theory. First we will define the elementary form of representation of data (the cross table) mathematically, which is called formal context. Then a formal concept of such a data context is explained. Next we will prove that these concepts form a mathematical structure called concept lattices. So that all concepts of a data can be represented in a diagram.
Before starting formal concept analysis, we will remind some basic definitions and results in lattice theory and closure systems.

Definition 1 *Let $(M, \leq)$ be an ordered set and A be a subset of M. A* ***lower bound*** *of A is an element s of M with $s \leq a$ for all $a \in A$. An* ***upper bound*** *of A is defined dually. If there is a largest element in the set of all lower bounds of A, it is called the* ***infimum*** *of A and is denoted by $\bigwedge A$; dually, a least upper bound is called* ***supremum*** *of A and is denoted by $\bigvee A$*

Definition 2 *An ordered set $\mathbf{V} := (V, \leq)$ is a* ***lattice****, if for any two elements x and y in $\mathbf{V}$ the supremum $x \vee y$ and the infimum $x \wedge y$ always* *exist.* $\mathbf{V}$ *is called a* ***complete lattice***, *if the supremum* $\bigvee X$ *and the infimum* $\bigwedge X$ *exist for any subset* X *of* V

Definition 3 *[5] A* ***congruence relation*** *of a complete lattice* $\mathbf{V}$ *is an equivalence relation* Θ *on* $\mathbf{V}$ *satisfying:*

$$x_t \Theta y_t \text{ for } t \in T \Rightarrow (\bigwedge_{t \in T} x_t)\Theta(\bigwedge_{t \in T} y_t) \text{ and } (\bigvee_{t \in T} x_t)\Theta(\bigvee_{t \in T} y_t)$$

, where T *is any nonempty index set.*
We define

$$[x]\Theta := \{y \in V \mid x\Theta y\}$$

which is the equivalence class of Θ *containing* x. *The* ***factor lattice***

$$\mathbf{V}/\Theta := \{[x]\Theta \mid x \in V\}$$

has the order defined by

$$[x]\Theta \leq [y]\Theta :\Leftrightarrow x\Theta(x \wedge y)$$

Definition 4 *For an element* v *of a complete lattice* $\mathbf{V}$, *we call it* $\bigvee$-***irreducible*** *if* v *cannot be represented as the supremum of strictly smaller elements. Dually, we define* $\bigwedge$-***irreducible***. *A set* $X \subseteq V$ *is called* ***supremum-dense*** *in* V, *if every element from* V *can be represented as the supremum of a subset of* X *and dually* ***infimum-dense*** *if* $v = \bigwedge\{x \in X \mid v \leq x\}$ *for all* $v \in V$.

1.2 Closure Systems and Galois Connections

Definition 5 $\mathfrak{U} \subseteq \mathfrak{P}(G)$ *is called a closure system on a set* G *if*

1. $G \in \mathfrak{U}$

2. $\mathfrak{X} \subseteq \mathfrak{U} \Rightarrow \bigcap \mathfrak{X} \in \mathfrak{U}$

The map $\psi : \mathfrak{P}(G) \to \mathfrak{P}(G)$ *is a closure operator on* G *if*

1. $X \subseteq Y \Rightarrow \psi(X) \subseteq \psi(Y)$	*(monotony)*
2. $X \subseteq \psi(X)$	*(extensity)*
3. $\psi(\psi(X) = \psi(X)$	*(idempotency)*

Examples. For many mathematical structures, the system of substructures important examples are

1. **Subspaces:** For any vector space V, the system $\mathfrak{U}$(V) of all subspaces is a closure system.

2. **Subgroups:** For any group G, the set $\mathfrak{U}$(G) of all subgroups of G is a closure system.

3. **Equivalence Relations:** For a set M, the set $\mathfrak{E}$(M) of all equivalence relations on M is a closure system on $M \times M$.

Theorem 1 *If* $\mathfrak{U}$ *is a closure system on* G *then*

$$\psi_{\mathfrak{U}}(X) := \bigcap\{A \in \mathfrak{U} \mid X \subseteq A\}$$

defines a closure operator on G. *Conversely, the set*

$$\mathfrak{U}_{\psi} := \{\psi(X) \mid X \subseteq G\}$$

of all closures of a closure operator ψ *is always a closure system, and*

$$\psi_{\mathfrak{U}_{\psi}} = \psi \text{ as well as } \mathfrak{U}_{\psi_{\mathfrak{U}}} = \mathfrak{U}$$

Proof. $\psi_{\mathfrak{U}}$ is a closure operator : First, we will prove that it satisfies monotony. Let $a \in \psi(X)$, then $a \in A$ for all $A \in \mathfrak{U}$ and $X \subseteq A$. So $a \in B$ for all $B \in \mathfrak{U}$ and $Y \subseteq B$ since $X \subseteq Y$. Hence $a \in \psi(Y)$.
Extensity is trivial.
For idempotency,

$$\psi_{\mathfrak{U}}(\psi_{\mathfrak{U}}(X)) = \bigcap\{A \in \mathfrak{U} \mid \psi_{\mathfrak{U}}(X) \subseteq A\} = \psi_{\mathfrak{U}}(X)$$

since $\psi_{\mathfrak{U}}(X) \in \mathfrak{U}$ by definition.

$\mathfrak{U}_{\psi}$ is a closure system: Firstly, $G \in \mathfrak{U}_{\psi} = \{\psi(X) \mid X \subseteq G\}$ because $G \in \psi(G)$ implies $\psi(G) = G$. To prove that set of subsets of G is closed under intersection, we will show that

$$\bigcap \mathfrak{X} = \psi(\bigcap \mathfrak{X})$$

, where $\mathfrak{X} \subseteq \mathfrak{U}_{\psi}$. Now $\bigcap \mathfrak{X} \subseteq \psi(\bigcap \mathfrak{X})$ by extensity.

$$a \in \psi(\bigcap \mathfrak{X}) \Rightarrow a \in \psi(X), \text{ for any } X \in \mathfrak{X}$$

since $\bigcap \mathfrak{X} \subseteq X \Rightarrow \psi(\bigcap \mathfrak{X}) \subseteq \psi(X)$ by monotony. This implies $a \in X$, for any $X \in \mathfrak{X}$ because $X \in \mathfrak{U}_{\psi}$ means $X = \psi(Y)$ for some $Y \subseteq G$ and $\psi(X) = \psi(\psi(Y)) = \psi(Y) = X$ by idempotency.

$$\begin{aligned} &\Rightarrow a \in \bigcap \mathfrak{X} \\ &\Rightarrow \psi(\bigcap \mathfrak{X}) \subseteq \bigcap \mathfrak{X} \end{aligned}$$

$$\Rightarrow \psi(\bigcap \mathfrak{X}) = \bigcap \mathfrak{X}$$
$$\Rightarrow \bigcap \mathfrak{X} \in \{\psi(X) \mid X \subseteq G\} = \mathfrak{U}_\psi.$$

For the last part of the theorem, we have

$$\begin{aligned} X \in \mathfrak{U} &\Leftrightarrow X = \bigcap\{A \in \mathfrak{U} \mid X \subseteq A\} \\ &\Leftrightarrow \psi_{\mathfrak{U}}(X) = X \\ &\Leftrightarrow X \in \mathfrak{U}_{\psi_{\mathfrak{U}}} \end{aligned}$$

this proves that $\psi_{\mathfrak{U}_\psi} = \psi$. For $A \in \mathfrak{U}_\psi, X \subseteq A$ is equivalent to $\psi(X) \subseteq A$. Hence

$$\begin{aligned} \psi_{\mathfrak{U}_\psi}(X) &= \bigcap\{A \in \mathfrak{U}_\psi \mid X \subseteq A\} \\ &= \bigcap\{A \in \mathfrak{U}_\psi \mid \psi(X) \subseteq A\} \\ &= \psi(X), \text{ since } \psi(X) \in \mathfrak{U}_\psi. \end{aligned}$$

•

Definition 6 *A Galois connection between the sets A and B is a pair of maps*

$$\phi : \mathfrak{P}(A) \to \mathfrak{P}(B) \text{ and } \psi : \mathfrak{P}(B) \to \mathfrak{P}(A)$$

such that for all $X, X' \subseteq A$ and all $Y, Y' \subseteq B$ the following conditions are satisfied:

1. $X \subseteq X' \Rightarrow \phi(X') \subseteq \phi(X)$ *and* $Y \subseteq Y' \Rightarrow \psi(Y') \subseteq \psi(Y)$

2. $X \subseteq \psi(\phi(X))$ *and* $Y \subseteq \phi(\psi(Y))$

The two maps then are called dually adjoint to each other.

Lemma 1 *[2] Let ϕ and ψ be a Galois connection between the sets A and B. Then*

$$\phi(\psi(\phi)) = \phi \textit{ and } \psi(\phi(\psi)) = \psi$$

Proof. Let $X \subseteq A$. By the second Galois connection property, we have $X \subseteq \psi(\phi(X))$. By applying ϕ to this gives $\phi(\psi(\phi(X))) \subseteq \phi(X)$. But we also have $\phi(X) \subseteq \phi(\psi(\phi(X)))$, by the second property of Galois connection applied to the set $\phi(X)$. This gives us $\phi(X) = \phi(\psi(\phi(X)))$. The second equality is proved similarly. •

Lemma 2 *[2] ϕ and ψ of maps form a Galois connection between A and B if and only if*

$$Y \subseteq \phi(X) \Leftrightarrow X \subseteq \psi(Y), \textit{ for all } X \subseteq A \textit{ and } Y \subseteq B$$

Proof. Suppose $Y \subseteq \phi(X)$, then $\psi(\phi(X)) \subseteq \psi(Y)$ by the first property of Galois connection. Since $X \subseteq \psi(\phi(X))$ by the second property of Galois connection, we have $X \subseteq \psi(Y)$. The other direction can be shown similarly. Conversely, from $\phi(X) \subseteq \psi(X)$ we have $X \subseteq \psi(\phi(X))$. If $X \subseteq X'$, where $X, X' \subseteq A$, we can deduce that $X \subseteq \psi(\phi(X'))$ and by the premise we get $\phi(X') \subseteq \phi(X)$. •

Proposition 1 *If the maps $\phi : \mathfrak{P}(A) \to \mathfrak{P}(B)$, $\psi : \mathfrak{P}(B) \to \mathfrak{P}(A)$ form a Galois connection*

$$\phi(\bigcup_{t \in T} A_t) = \bigcap_{t \in T} \phi(A_t), \textit{ holds for all } A_t \subseteq A$$

The same property holds for ψ.

Proof.

$$
\begin{aligned}
x \in \bigcap_{t \in T} \phi(A_t) &\Leftrightarrow \{x\} \subseteq \phi(A_t), \forall t \in T \\
&\Leftrightarrow A_t \subseteq \psi(\{x\}), \forall t \in T, \text{ by lemma 2} \\
&\Leftrightarrow \bigcup_{t \in T} A_t \subseteq \psi(\{x\}) \\
&\Leftrightarrow \{x\} \subseteq \phi(\bigcup_{t \in T} A_t), \text{ by lemma 2} \\
&\Leftrightarrow x \in \phi(\bigcup_{t \in T} A_t)
\end{aligned}
$$

•

Galois connections are also related to closure operators, as the following proposition shows.

Theorem 2 *Let the pair* (ϕ, ψ) *with* $\phi : \mathfrak{P}(A) \to \mathfrak{P}(B), \psi : \mathfrak{P}(B) \to \mathfrak{P}(A)$ *be a Galois connection between the sets* A *and* B*. Then* $\psi\phi$ *and* $\phi\psi$ *are closure operators on* A *and* B*, respectively.*

Proof. Monotony :

$$
\begin{aligned}
X \subseteq Y &\Rightarrow \phi(Y) \subseteq \phi(X) \\
&\Rightarrow \psi(\phi(X)) \subseteq \psi(\phi(Y)), \text{ for all } X, Y \subseteq A.
\end{aligned}
$$

Extensity: $X \subseteq \psi(\phi(X))$, which is exactly second condition for Galois connection.

Idempotency: By lemma 1 we have $\phi = \phi\psi\phi$. Applying ψ to this equation, we get idempotency. Similarly, one can prove that $\phi\psi$ is a closure operator on B. •

Now we will see the relation between Galois connections and binary relations.

Theorem 3 *For every binary relation $R \subseteq M \times N$, there is a Galois connection (ϕ_R, ψ_R) between M and N defined by*

$$\phi_R(X) := X^R(= \{y \in N \mid xRy, \textit{ for all } x \in X\})$$
$$\psi_R(Y) := Y^R(= \{x \in M \mid xRy, \textit{ for all } y \in Y\})$$

,where X and Y are subsets of M and N respectively.
If, conversely, (ϕ, ψ) is a Galois connection between M and N, then

$$\begin{aligned} R_{(\phi,\psi)} : &= \{(x, y) \in M \times N \mid x \in \psi(\{y\})\} \\ &= \{(x, y) \in M \times N \mid y \in \phi(\{x\})\} \end{aligned}$$

is a binary relation between M and N . In addition, $\phi_{R_{(\phi,\psi)}} = \phi$, $R_{(\phi_R,\psi_R)} = R$

Proof. To show that (ϕ_R, ψ_R) forms a Galois connection, we will use lemma 2. Assuming $X \subseteq \psi_R(Y)$ and $y \in Y$, we get xRy for all $x \in X$ by definition of $\psi_R(Y)$. So $Y \subseteq \phi_R(X)$.
Conversely, if (ϕ, ψ) is a Galois connection then
$\{(x, y) \in M \times N \mid x \in \psi(\{y\})\} = \{(x, y) \in M \times N \mid y \in \phi(\{x\})\}$
by lemma 2 again.
Now it remains to show that

$$\phi_{R_{(\phi,\psi)}} = \phi \ , \ \psi_{R_{(\phi,\psi)}} = \psi \text{ and } R_{(\phi_R,\psi_R)} = R$$

.

By proposition 1, we get

$$\begin{aligned} \phi(X) &= \bigcap_{x\in X} \phi\{x\} \\ &= \bigcap_{x\in X} \phi_{R_{(\phi,\psi)}}\{x\}, \text{ since } \phi_{R_{(\phi,\psi)}}\{x\} = \{y \in N \mid xRy\} = y \in \phi\{x\} \\ &= \phi_{R_{(\phi,\psi)}}(X) \end{aligned}$$

,i.e. $\phi_{R_{(\phi,\psi)}} = \phi$ and similarly $\psi_{R_{(\phi,\psi)}} = \psi$.
The last statement $R_{(\phi_R,\psi_R)} = R$ follows immediately from the equivalence $x \in \psi_R\{y\} \Leftrightarrow xRy$. •

1.3 Formal Concept Analysis

Definition 7 *A* **Formal Context** $\mathbb{K} := (G, M, I)$ *consists of two sets* G *and* M *and a relation* I *between* G *and* M*.The elements of* G *are called the* **objects** *and the elements of* M *are called the* **attributes** *of the context. The* I *relation between an object* g *and an attribute* m *is written as* gIm *or* $(g, m) \in I$ *and read as "the object* g **has** *the attribute* m*". The relation* I *is also called the* **incidence relation** *of the context.*

A small context can be represented by a cross-table, which is a rectangular in shape and the rows of which are headed by the object names and the columns headed by attribute names. A cross in row g and column m means that object g has the attribute m.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Margherita	×	×													
Napoletana	×	×	×	×	×										
Veneziana	×	×	×		×	×	×	×							
Mushroom	×	×							×						
quattro formaggi		×								×					
La reine	×	×			×				×		×				
Fiorentina	×	×			×							×	×	×	
Pizza Allo Noci		×										×		×	×
Caprina		×													
Sloppy Giuseppe	×	×				×									
Soho Pizza	×	×			×									×	
American	×	×													
Giardiniera	×	×			×				×						
Siciliana	×	×		×	×						×			×	
Capricciosa	×	×	×	×	×						×		×		
Four Seasons	×	×	×	×	×				×						
American Hot	×	×													
Cajun	×	×				×									
Neptune		×	×	×	×	×									

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	
Margherita																	
Napoletana																	
Veneziana																	
Mushroom																	
quattro formaggi																	
La reine																	
Fiorentina							×										
Pizza Allo Noci	×																
Caprina		×	×														
Sloppy Giuseppe				×	×												
Soho Pizza						×	×										
American								×									
Giardiniera									×	×	×	×					
Siciliana													×				
Capricciosa										×							
Four Seasons								×									
American Hot					×			×									
Cajun										×				×	×		
Neptune																×	

, where the objects set consists of the special pizzas in a restaurant and the attributes are the ingredients for the pizzas. So (g, m) is in the relation $I \subseteq G \times M$ if the pizza g has the ingredient m.

The numbers in the attribute set corresponds to: 1)mozzarella 2)tomato 3)capers 4)anchovies 5)olives 6)onions 7)pine kernels 8)sultanas 9)mushrooms 10)four cheese 11)ham 12) spinach 13)free range egg 14)garlic 15)walnut halves 16)gorgonzola 17)sun dried tomato 18)goat's cheese 19)hot beef 20)green peppers 21)rocket 22)parmesan 23)peperon 24)sliced tomato 25)red peppers 26)leeks 27)petits pois 28)artichokes 29)prawns 30)tobasco pepper sauce 31)tuna

Definition 8 *In a context (G, M, I), for a set $A \subseteq G$ of objects we define*

$$A' := \{m \in M \mid gIm \; for \; all \; g \in A\}$$

(the set of attributes common to the objects in A). Correspondingly, for a set B of attributes we define

$$B' := \{g \in G \mid gIm \; for \; all \; m \in B\}$$

(The set of objects which have all attributes in B) For an object $g \in G$ we write g' instead of $\{g\}'$ for the object intent $\{m \in M \mid gIm\}$ of the object g. Correspondingly, $m' := \{g \in G \mid gIm\}$ is the attribute extent of the attribute m. We call the operators ' as derivation operators.

Definition 9 *A formal concept of the context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. We call A the extent and B the intent of the concept (A, B). $\underline{\mathfrak{B}}(G, M, I)$ denotes the set of all concepts of the context (G, M, I).*

Example 1 *Some concepts in our pizza example are (Mushroom, Lareine, Giardiniera, Four Seasons; mozzarella, tomato, mushroom), (Sloppy Giuseppe, Veneziana, Cajun, Nepture; tomato, onions), (Fiorentina, Soho*

Pizza, Pizza Allo Noci, Siciliana; tomato, garlic) and (Giardiniera; mozzarella,olives,mushroom, tomato, red peppers, sliced tomato, leeks, petits pois).

Concept operators form a Galois connection between G and M.

Proposition 2 *If (G, M, I) is a context, $A, A_1, A_2 \subseteq G$ are sets of objects and $B, B_1, B_2 \subseteq M$ are sets of attributes, then*

1. $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ *and* $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$

2. $A \subseteq A''$ *and* $B \subseteq B''$

Proof.

1. Let $m \in A_2'$, then gIm , for all $g \in A_2$ by definition of A_2'. This implies gIm , for all $g \in A_1$ since $A_1 \subseteq A_2$. So $m \in A_1'$.

2. If $g \in A$ then gIm for all $m \in A'$, which implies $g \in A''$, since $A'' = \{g \in G \mid gIm \ for \ all \ m \in A'\}$.

•

This proposition shows that the two derivation operators form a Galois connection between G and M. Hence we obtain two closure systems on G and M, whose closure operators are maps assigning a closure $A'' \subseteq G$ and $B'' \subseteq M$ to each subset $A \subseteq G$ and $B \subseteq M$. By definition A is an extent if and only if $A = A''$. So extents of the concepts of the context (G, M, I) form a closure system corresponding to the closure operator. Hence A'' is the smallest extent of the concept (A'', A') containing A. Note that $A' = A'''$ by proposition 2, so (A'', A') is always a concept. The same is true for intents, i.e.

(B', B'') is always a concept. We call the concept (g'', g') as object concept and (m', m'') as attribute concept.

Since extents and also intents form a closure system on G and M, respectively intersection of any number of extents(intents) is always an extent(intent). Moreover we have :

Proposition 3 *If T is an index set and, for evert $t \in T$, $A_t \subseteq G$ is a set of objects, then*

$$(\bigcup_{t \in T} A_t)' = \bigcap_{t \in T} A'_t$$

The same holds for the sets of attributes.

Proof. Since the derivation operators form a Galois connection between the sets G and M, by proposition 1 we get the equality. •

Definition 10 *A context $\mathbb{K} := (G, M, I)$ is called clarified if for any objects $g, h \in G$ from $g' = h'$ it always follows that $g = h$ and, correspondingly, $m' = n'$ implies $m = n$ for all $m, n \in M$.*

Remark: Obviously if we merge one of the objects, with the same intent, from the context then the structure of the concept lattice remains unchanged. The same is true for merging one of the attributes with the same intent.

Example 2 *In our pizza example, the context is not clarified because $\{prawns\}' = \{tobascopeppersauce\}'$ but they are not the same thing obviously. Also $\{leeks\}' = \{petitspois\}' = \{slicedtomato\}'$, $\{sundriedtomato\}' = \{goat'scheese\}'$,$\{walnuthalves\}' = \{gorgonzola\}'$*

,$\{pinekernels\}' = \{sultanas\}'$
So the context can be clarified if when we exclude one of the attributes with the same extents. Two ingredients with the same extent means that they are always contained in the same pizzas. Note that the objects are already clarified, which means that there are no pizza's which have the same ingredients, and it is obviously an expected result.

Theorem 4 *The concepts of the context (G, M, I) form a complete lattice, denoted by $\underline{\mathfrak{B}}(G, M, I)$ in which infimum and supremum are given by :*

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)''\right)$$
$$\bigvee_{t\in T}(A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t,\right)'', \bigcap_{t\in T} B_t\right)$$

A complete lattice $\mathbf{V}$ is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\widetilde{\gamma} : G \to V$ and $\widetilde{\mu} : M \to V$ such that $\widetilde{\gamma}(G)$ is supremum-dense in $\mathbf{V}$, $\widetilde{\mu}(M)$ is infimum-dense in $\mathbf{V}$ and gIm is equivalent to $\widetilde{\gamma}g \leq \widetilde{\mu}m$ for all $g \in G$ and all $m \in M$. In particular, $\mathbf{V} \cong \underline{\mathfrak{B}}(V, V, \leq)$

Proof. We define the order of the concepts as follows

$$(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_2 \subseteq B_1$$

It is clearly reflexive, antisymmetric and transitive. And the last equivalence follows from the facts that $A_1' = B_1$, $A_2' = B_2$ and the derivation operator is a Galois connection.

Now we'll prove that infimum and supremum are well defined.

$$\left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)''\right) = \left(\left(\bigcup_{t\in T} B_t\right)', \left(\bigcup_{t\in T} B_t\right)''\right)$$

by proposition 7.

Since it has the form (X'', X'), it is a concept.

This is also the largest common subconcept of the concepts (A_t, B_t), follows from the fact that the extent of this concept is exactly the intersection of the extents of (A_t, B_t). For the supremum,

$$\begin{aligned} \bigvee_{t\in T}(A_t, B_t) &= \left(\left(\bigcup_{t\in T} A_t,\right)'', \bigcap_{t\in T} B_t\right) \\ &= \left(\left(\bigcup_{t\in T} A_t,\right)'', \bigcap_{t\in T} A_t'\right) \\ &= \left(\left(\bigcup_{t\in T} A_t,\right)'', \left(\bigcap_{t\in T} A_t\right)'\right) \end{aligned}$$

So it's a concept and also the smallest one which is greater than each of the concepts (A_t, B_t) since its extent is the closure of the union of the extents A_t. Thus we have proven that $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice.

Now we prove firstly the special case $\mathbf{V} = \underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\widetilde{\gamma}$ *and* $\widetilde{\mu}$ with the required properties, then generalize to show that the same holds for $\mathbf{V} \cong \underline{\mathfrak{B}}(G, M, I)$. We set

$$\begin{aligned} \widetilde{\gamma} &:= (\{g\}'', \{g\}') \; for \; g \in G \\ \widetilde{\mu} &:= (\{m\}', \{m\}'') \; for \; m \in M \end{aligned}$$

$$\widetilde{\gamma} \leq \widetilde{\mu} \Leftrightarrow \{g\}'' \subseteq \{m\}' \Leftrightarrow \{m\}'' \subseteq \{g\}' \Leftrightarrow m \in \{g\}' \Leftrightarrow gIm$$

Furthermore, on account of the formulas proved above

$$\bigvee_{g\in A}(\{g\}'', \{g\}') = (A, B) = \bigwedge_{m\in B}(\{m\}', \{m\}'')$$

since

$$\begin{aligned}
\bigvee_{g\in A}(\{g\}'',\{g\}') &= \left(\left(\bigcup\{g\}''\right)'',\bigcap\{g\}'\right)\\
&= \left(\left(\bigcup\{g\}''\right)'',\left(\bigcup\{g\}\right)'\right)\\
&= \left(\left(\bigcup\{g\}''\right)'',A'\right)\\
&= \left(\left(\bigcup\{g\}''\right)'',B\right)
\end{aligned}$$

So $(\bigcup\{g\}'')'' = A$ since $B' = A$. The second equality can be shown similarly.

Hence $\widetilde{\gamma}(G)$ is supremum-dense and $\widetilde{\mu}(M)$ is infimum dense in $\underline{\mathfrak{B}}(G,M,I)$.

More generally if $\mathbf{V} \cong \underline{\mathfrak{B}}(G,M,I)$ and $\psi : \underline{\mathfrak{B}}(G,M,I) \to \mathbf{V}$ is an isomorphism, we define $\widetilde{\gamma}$ and $\widetilde{\mu}$ by

$$\begin{aligned}
\widetilde{\gamma} &:= \psi(\{g\}'',\{g\}')\ for\ g\in\ G\\
\widetilde{\mu} &:= \psi(\{m\}',\{m\}'')\ for\ m\in\ M
\end{aligned}$$

The properties claimed for these mappings are proved in a similar fashion.

If, conversely, $\mathbf{V}$ is a complete lattice and

$$\widetilde{\gamma} : G \to V \ \text{ and } \widetilde{\mu} : M \to V$$

are mappings with the properties stated in the theorem, then let's define

$$\psi : \underline{\mathfrak{B}}(G,M,I) \to \mathbf{V}$$

by

$$\psi(A,B) := \bigvee\{\widetilde{\gamma}(g) \mid g\in\ A\}$$

ψ is well-defined since $\mathbf{V}$ is a complete lattice. Now, we will show that ψ is an isomorphism.

<u>ψ is an order embedding</u>: $(A_1, B_1) \leq (A_2, B_2) \Rightarrow \psi(A_1, B_1) = \bigvee\{\widetilde{\gamma}(g) \mid g \in A_1\} \subseteq \bigvee\{\widetilde{\gamma}(g) \mid g \in A_1\} = \psi(A_2, B_2)$ since $A_1 \subseteq A_2$. It remains to show that $\psi(A_1, B_1) \leq \psi(A_2, B_2) \Rightarrow (A_1, B_1) \leq (A_2, B_2)$. Before proving this implication, we'll show that

$$\bigvee\{\widetilde{\gamma}(g) \mid g \in A\} = \bigwedge\{\widetilde{\mu}(m) \mid m \in A'\}$$

Since $\widetilde{\mu}(M)$ is dense in $\mathbf{V}$, we get

$$\begin{aligned}
& \bigvee\{\widetilde{\gamma}(g) \mid g \in A\} = \bigwedge_{m \in N} \widetilde{\mu}(m) \; for\ some\ N \subseteq M \\
\Rightarrow \; & \bigvee\{\widetilde{\gamma}(g) \mid g \in A\} \leq \widetilde{\mu}(m) \; for\ any\ m \in N \\
\Rightarrow \; & \widetilde{\gamma}(g) \leq \widetilde{\mu}(m) \; for\ any\ m \in N\ and\ g \in A \\
\Rightarrow \; & gIm, \; for\ any\ m \in N\ and\ g \in A \\
\Rightarrow \; & N = A' \\
\Rightarrow \; & \bigvee\{\widetilde{\gamma}(g) \mid g \in A\} = \bigwedge\{\widetilde{\mu}(m) \mid m \in A'\}
\end{aligned}$$

Using this fact,

$$\psi(A_1, B_1) = \bigvee\{\widetilde{\gamma}(g) \mid g \in A_1\} \leq \bigwedge\{\widetilde{\mu}(m) \mid m \in A_2'\} = \psi(A_2, B_2)$$

Now, let $a \in A_1$, then

$$\widetilde{\gamma}(a) \leq \psi(A_1, B_1) \leq \psi(A_2, B_2) \leq \widetilde{\mu}(m), \text{ for any } m \in A_2'$$

$$\begin{aligned}
\Rightarrow \; & aIm, \text{ for any } m \in A_2' \\
\Rightarrow \; & a \in A_2'' = A_2
\end{aligned}$$

Hence $(A_1, B_1) \leq (A_2, B_2)$.

ψ is a 1-1 : Let $\psi(A_1, B_1) = \psi(A_2, B_2)$. By using the fact above, that is,

$$\bigvee\{\widetilde{\gamma}(g) \mid g \in A_1\} = \bigwedge\{\widetilde{\mu}(m) \mid m \in A_1'\} = \bigwedge\{\widetilde{\mu}(m) \mid m \in A_2'\}$$

$\Rightarrow \widetilde{\gamma}(g) \leq \widetilde{\mu}(m)$ for any$m \in A_2' = B_2$ *and* $g \in A_1$

$\Rightarrow gIm,$ for any$m \in A_2' = B_2$ and$g \in A_1$

$\Rightarrow A_1' = B_2$ *and* $A_1 = B_2'$

$\Rightarrow A_1 = A_2$ *and* $B_1 = B_2$

ψ is onto: Let $v \in V$, then since $\widetilde{\gamma}(G)$ is supremum dense in $\mathbf{V}$. $v = \bigvee_{g \in A} \widetilde{\gamma}(g)$, for some $A \subseteq G$. We have to show that A is an extent. But $v = \bigvee_{g \in A} \widetilde{\gamma}(g) = \bigwedge_{m \in B} \widetilde{\mu}(m)$, for some $A \subseteq G$ and $B \subseteq M$ implies $\widetilde{\gamma}(g) \leq \widetilde{\mu}(m)$ *for any* $m \in B$ *and* $g \in A$. Hence $A' = B$ and $B' = A$. •

Definition 11 *We call* $m \in M$ *reducible attribute if there exist* $X \subseteq M$ *with* $m \notin X$ *and* $m' = X'$. *Similarly we define reducible object.*

Definition 12 *A clarified context* (G, M, I) *is called row-reduced if every object concept is* $\bigvee$*-irreducible and column reduced if every attribute concept is* $\bigwedge$*-irreducible. A context which is both row-reduced and column-reduced is called reduced.*

So if we exclude reducible attributes and reducible objects from the context, it becomes reduced.

Example 3 *The reduced attributes in the pizza context are ;*

Tomato, pine kernels, hot beef, sliced tomato, artichokes, prawns and tuna.

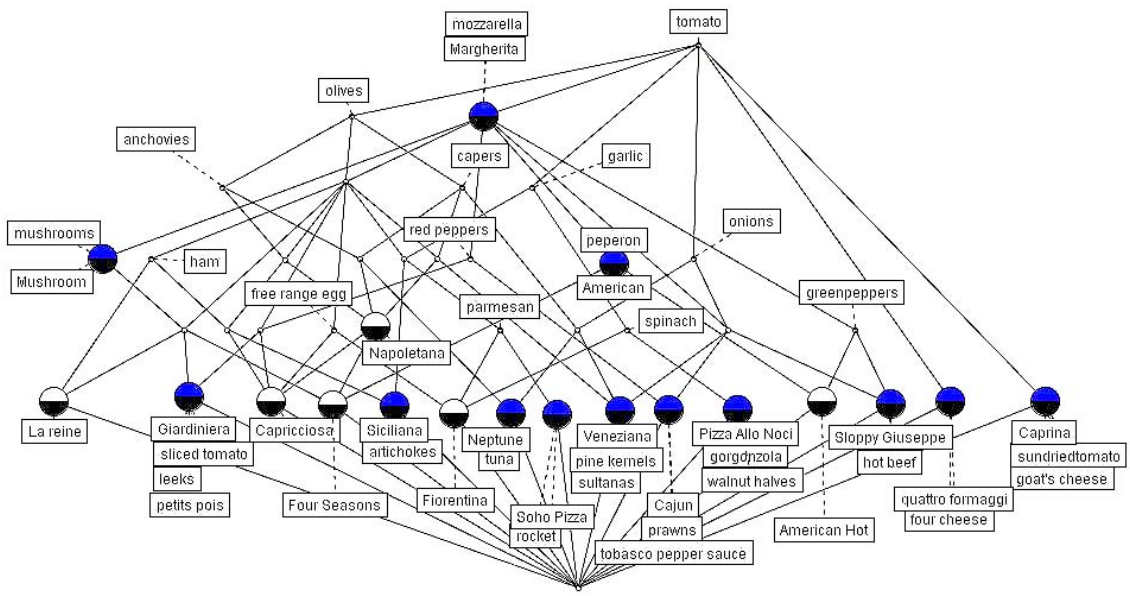

Figure 1.1: The concept lattice: Pizzas

And reduced objects are ;

Margherita, Napoletana, American.

If we take out these objects and attributes from the context then the structure of the lattice will not change.

The attribute pine kernel is reducible because

$$\{pinekernel\}' = \{onions, mozzarella, tomato, olive, capers\}'$$

Actually, it means that any pizza containing onions, mozzarella, tomato, olive and capers will also contain pine kernel.

It is even simpler to see reducible attributes in the lattice diagram.

Corollary 1 *The removal of reducible attributes and reducible objects from* *the context has no influence on the structure of the concept lattice, i.e.*

$$\underline{\mathfrak{B}}(G, M, I) \cong \underline{\mathfrak{B}}(G\backslash H, M\backslash N, I \cap (G\backslash H \times M\backslash N))$$

,where $H \neq G$ is the set of reducible objects and $N \neq M$ is the set of reducible attributes.

Proof. Let $V = \underline{\mathfrak{B}}(G, M, I)$ and $\widetilde{\gamma} : G \to V$, $\widetilde{\mu} : M \to V$ are defined as in the proof of the basic theorem. Let $\widetilde{\gamma}|_{G\backslash H}$, $\widetilde{\mu}|_{M\backslash N}$ are restriction maps of $\widetilde{\gamma}$ and $\widetilde{\mu}$ respectively.

Now, $\forall h \in H, h' = X'$, for some $X \subseteq G\backslash H$, this implies $\widetilde{\gamma}|_{G\backslash H}(X) = (X'', X') = \widetilde{\gamma}(h) = (h'', h')$. So $\widetilde{\gamma}|_{G\backslash H}(G\backslash H)$ is supremum-dense in V. Similarly one can prove that $\widetilde{\mu}|_{M\backslash N}(M\backslash N)$ is infimum-dense in V.

Clearly we have

$$gIm \Leftrightarrow \widetilde{\gamma}|_{G\backslash H}(g) \leq \widetilde{\mu}|_{M\backslash N}(m), \text{ for all } g \in G\backslash H \text{ and } m \in M\backslash N$$

since they're restriction maps.

Hence by the basic theorem $\underline{\mathfrak{B}}(G, M, I) \cong \underline{\mathfrak{B}}(G\backslash H, M\backslash N, I \cap (G\backslash H \times M\backslash N))$

•

Definition 13 *if (G, M, I) is a context, $g \in G$ an object and $m \in M$ an attribute, we write*

$$g \swarrow m :\Leftrightarrow \begin{cases} g \not{I} m \text{ and} \\ \text{if } g' \subseteq h' \text{ but } g' \neq h' \text{ implies } hIm \end{cases}$$

$$g \nearrow m :\Leftrightarrow \begin{cases} g \not{I} m \text{ and} \\ \text{if } m' \subseteq n' \text{ but } m' \neq n' \text{ implies } gIn \end{cases}$$

$$g \swarrow\!\!\!\nearrow m :\Leftrightarrow g \nearrow m \text{ and } g \swarrow m$$

Thus, $g \swarrow m$ if and only if g' is maximal among all object intents which do not contain m. In other words, the object concept $\gamma(g) = (g'', g')$ is minimal among concepts whose intent does not contain m.

Similarly $g \nearrow m$ if and only if (m', m'') is maximal among concepts whose extent does not contain g.

The significance of the arrow relations for the reduction of a context is shown by the next proposition.

Proposition 4

$$\gamma(g) \textit{ is } \vee\textit{-irreducible} \quad \Leftrightarrow \quad \textit{there is an } m \in M \textit{ with } g \swarrow m$$
$$\mu(g) \textit{ is } \wedge\textit{-irreducible} \quad \Leftrightarrow \quad \textit{there is a } g \in G \textit{ with } g \nearrow m$$

Furthermore, the following statements hold for every finite context.

$$\gamma(g) \textit{ is} \vee\textit{-irreducible} \quad \Leftrightarrow \quad \textit{There is an } m \in M \textit{ with } g \swarrow\!\!\!\nearrow m$$
$$\mu(g) \textit{ is} \wedge\textit{-irreducible} \quad \Leftrightarrow \quad \textit{There is a } g \in G \textit{ with } g \swarrow\!\!\!\nearrow m$$

Proof. $\gamma(g)$ is $\vee$-irreducible means $(\gamma(g))_* = \bigvee\{(A,B) \mid (A,B) < \gamma(g)\} \neq \gamma(g)$. Since $\gamma(g) > (\gamma(g))_*$, there exist m which is an element of the intent of $(\gamma(g))_*$ but not $\gamma(g)$. Also the fact that $(\gamma(g))_*$ is the largest of the concepts which are smaller than $\gamma(g)$ contains m, i.e. $g \swarrow m$.

Conversely, if $g \swarrow m$ for some $m \in M$ and $\gamma(g)$ is not $\vee$-irreducible then

$$\begin{aligned}\gamma(g) = (g'', g') &= \bigvee_{t\in T} (A_t, B_t), \text{for some concepts of}(G, M, I)\\ &= \left((\bigcup_{t\in T} A_t)'', \bigcap_{t\in T} B_t \right)\end{aligned}$$

Now, $m \in B_t$ for every $t \in T$ since $g \swarrow m$ implies m is element of every intent of the concepts which are smaller than $\gamma(g)$, and obviously $(A_t, B_t) < \gamma(g)$, for every $t \in T$. So $m \in \bigcap_{t\in T} B_t$ implies $m \in g'$ which is a contradiction.

Similarly one can prove the second statement.

In a finite context we can always find a maximal m' with $g \swarrow m$. We claim that $g \nearrow m$. Suppose there exist $m' \subseteq m_1'$ and $m_1 \not I g$ then $g' \subseteq h'$ implies hIm, so hIm_1 since $m' \subseteq m_1'$. Hence $g \swarrow m_1$. But this contradicts with the maximality of m'. So $m' \subseteq m_1'$ implies gIm_1, that is $g \nearrow m$. Therefore $g \swarrow\!\!\!\nearrow m$.

Similarly, we can prove the second statement. •

Now we will define a special context, which can be a substituted for the finite condition in proposition 4.

Definition 14 *A context (G, M, I) is called **doubly founded**, if for every object $g \in G$ and every attribute $m \in M$ with $g \not I m$, there is an object $h \in G$ and an attribute $n \in M$ with*

$$g \nearrow n \text{ and } m' \subseteq n' \text{ as well as } h \swarrow m \text{ and } g' \subseteq h'$$

CHAPTER 2

Logical Background

2.1 Introduction

In boolean algebra, we use symbols to represent things and classes of things as subjects of human conceptions, and signs of operations to combine or resolve conceptions so as to form new conceptions.

But Boole introduced a class, called "the Universe", consisting of all the individuals that exist in any class, whereas in Contextual Attribute Logic we restrict ourselves in a context. So it may be considered as a contextual version of the Boolean Logic. Tools for the analysis of such local logics may be taken from mathematical logic.

The central task of the Contextual Attribute Logic is the investigation of the "logical relations" between the attributes of the context, more generally between combinations of attributes, such as implications.

The logical relationships between formal attributes will be expressed via their extents. For example, we will say that an attribute m implies an attribute n if $m' \subseteq n'$.

Definition 15 *A compound attribute of a formal context* (G, M, I) *is inductively defined by the following rules*

1. *If m is an attribute, then we define its negation, $\neg m$, to be a compound attribute, which has the extent $G \backslash m'$.*

2. *For each set $A \subseteq M$ of attributes, we define conjunction, $\bigwedge A$, and the disjunction, $\bigvee A$, to be compound attributes, where the extent of $\bigwedge A$ is $\bigcap\{m' \mid m \in\ A\}$ and the extent of $\bigvee A$ is $\bigcup\{m' \mid m \in\ A\}$.*

3. *Iteration of the above compositions leads to the further compound attributes, the extents of which are determined in the obvious manner.*

2.2 Basic Definitions

Definition 16 *Two compound attributes are said to be extensionally equivalent in (G, M, I) if they have the same extent in (G, M, I).*

Definition 17 *Two compound attributes are said to be globally equivalent if they have the same extent in any context with attribute set M.*

Theorem 5 *Two compound attributes of an attribute set M are globally equivalent if and only if they are extensionally equivalent in $(\mathfrak{P}(M), M, \ni)$.*

Proof. Let A and B are two compound attributes and τ be any truth assignment such that $\tau(A) = 1$. We will show that $\tau(B) = 1$ holds too. This truth assignment corresponds to a characteristic subset of M which we call a model of A. Since A and B are extensionally equivalent in $(\mathfrak{P}(M), M, \in)$, M is also a model B. Hence $\tau(B) = 1$. Without loss of generality we can say that A and B are logically equivalent. Hence they're globally equivalent. •

Definition 18 *A sequent (A, S) over M is a compound attribute, represented by two sets $A, S \subseteq M$, that is globally equivalent to the compound attribute*

$$\bigvee (S \cup \{\neg m \mid m \in A\})$$

Remark : Such expressions are called clauses in Mathematical Logic. We call a sequent disjoint if $A \cap S = \emptyset$. In this paper, we will assume that all sequents are disjoints. An object g is in the extent of (A, S) if and only if $(g, m) \in I$ for at least one $m \in S$ or $(g, m) \notin I$ for at least one $m \in A$. We write

$$(A_1, S_1) \leq (A_2, S_2) :\Leftrightarrow A_1 \subseteq A_2 \text{ and } S_1 \subseteq S_2$$

A sequent is full if $A \cap S = M$.

Lemma 3 *The extent of a full sequent $(A, M \backslash A)$ in the test context $(\mathfrak{P}(M), M, \ni)$ M is $\mathfrak{P}(M) \backslash \{A\}$.*

Proof. The sequent $(A, M \backslash A)$ is globally equivalent to

$$\bigvee (M \backslash A \cup \{\neg m \mid m \in A\})$$

So any element, say $\mathfrak{K}$, of $\mathfrak{P}(M)$, which does not contain A, is clearly an extent of $(A, M \backslash A)$ since there exist $a \in A$ such that $a \in \mathfrak{K}$ which is equivalent to saying $\mathfrak{K}$ is an extent of $\neg a$. Hence it is also an extent of $\bigvee (M \backslash A \cup \{\neg m \mid m \in A\})$. But any subset of M, which contains A, is an extent of one of the elements of $M \backslash A$ unless it's equal to A. •

Lemma 4 *The extent of a conjection of a full sequents $(A_t, M \backslash A_t), t \in T$ is $\mathfrak{P}(M) \backslash \{A_t \mid t \in T\}$.*

Two compound attributes are extensionally equivalent in the context $(\mathfrak{P}(M), M, \ni)$ means that they've the same models.

Theorem 6 *Every compound attribute set is globally equivalent to a unique conjunction of full disjoints sequents.*

Proof. For any compound attribute take the subsets of M, say A_t's, which are not extents of the attribute in $(\mathfrak{P}(M), M, \ni)$. Then we can write the extent of the attribute in the form $\mathfrak{P}(M)\backslash\{A_t \mid t \in T\}$ which is exactly the extent of conjunction of full sequents $(A_t, M\backslash A_t), t \in T$. •

Definition 19 *A compound attribute is all-extensional in (G, M, I) if its extent is the set G of all objects. A sequent (A, S) may be interpreted as an implication*

$$\bigwedge A \to \bigvee S$$

The compound attribute $\bigwedge A$ implies $\bigvee S$ in (G, M, I) if and only if the sequent (A, S) is all-extensional.

Definition 20 *A clause set over M is a set of sequents over M. The clause logic of a formal context (G, M, I) is the set of all sequents that are all-extensional in (G, M, I).*

Definition 21 *A clause set $\mathfrak{C}$ is regular if it satisfies the following conditions*

1. *If $(A, S) \in \mathfrak{C}$ and $(A, S) \leq (B, T)$ then $(B, T) \in \mathfrak{C}$*
2. *If for each sequent (B, T) satisfying $(A, S) \leq (B, T)$ there is some sequent $(C, U) \in \mathfrak{C}$ with $(B, T) \leq (C, U)$ then $(A, S) \in \mathfrak{C}$*

Proposition 5 *Two conditions of regularity are equivalent to the condition :*

$$(A,S) \in \mathfrak{C} \Leftrightarrow \forall X \subseteq M\ ((A,S) \leq (X, M\backslash X) \Rightarrow (X, M\backslash X) \in \mathfrak{C})$$

Proof. Suppose $(A,S) \in \mathfrak{C}$ and $(A,S) \leq (X, M\backslash X)$, for any $X \subseteq M$. Then by condition 1, obviously $(X, M\backslash X) \in \mathfrak{C}$. Assuming $(X, M\backslash X) \in \mathfrak{C}$ and $(A,S) \leq (X, M\backslash X)$ we get $(A,S) \in \mathfrak{C}$ by condition 2.

For the converse, assume $(A,S) \in \mathfrak{C}$ and$(A,S) \leq (B,T)$. Then choose $X \subseteq M$ such that $B \subseteq X$ and $T \cap X = \emptyset$. We can choose such an X since we assume B and T are disjoint. Then $(X, M\backslash X) \in \mathfrak{C}$ will imply $(B,T) \in \mathfrak{C}$ because $(B,T) \leq (X, M\backslash X)$.

Similarly, for condition 2, if there is some sequence $(C,U) \in \mathfrak{C}$ with $(B,T) \leq (C,U)$ then we can find $(X, M\backslash X)$ such that $(C,U) \leq (X, M\backslash X)$ and hence $(X, M\backslash X) \in \mathfrak{C}$. But this will also imply that $(A,S) \in \mathfrak{C}$ since $(A,S) \leq (X, M\backslash X)$. •

Proposition 6 *Two regular clause sets containing the same full sequents are equal.*

Proof. By the previous proposition we can say that (A,S) belongs to $\mathfrak{C}$. Hence if full sequents are equal in two regular clause set then all of the sequences will be the same. •

Theorem 7 *[4] A clause set is regular if and only if it is the clause logic of some formal context.*

Proof. It is easy to verify that the clause logic of any formal context satisfies the two conditions of regularity. To construct a context for a given regular clause set $\mathfrak{C}$ let $\{A_t \mid t \in T\}$ denote the set of first components of full sequents $(A_t, M\backslash A_t)$ in $\mathfrak{C}$ and let

$$\mathfrak{G} := \mathfrak{P}(M)\backslash\{A_t \mid t \in T\}$$

$\mathfrak{G}$ is the extent of the conjunction of these full sequents in the test context. Therefore these full sequents are exactly the ones which are all-extensional in $(\mathfrak{G}, M, \ni)$. By the preceding proposition, the clause logic of this context is equal to $\mathfrak{C}$. •

Actually this theorem is not new. In the language of propositional logic, the elements of $\mathfrak{G}$ would correspond to those truth assignments that make all clauses in $\mathfrak{G}$ true.

Proposition 7 *A compound attribute is all-extensional in (G, M, I) if and only if the object intents of (G, M, I) are the extents of the compound attribute in the test context.*

Proof. Let c be a compound attribute which is all-extensional in (G, M, I), then by theorem 1 we can represent c as a conjunction of full disjoint sequents, say c is globally equivalent to $(A_1, S_1) \bigwedge (A_2, S_2) \bigwedge ... \bigwedge (A_n, S_n)$. So it's extensionally equivalent to this conjunction in (G, M, I).

Let g be any element of G. Since c is all-extensional, g is in the extent of the compound attribute. The extent of c in the test context are the sets which satisfies the sentence :

If $X \subseteq \mathfrak{P}(m)$ contains A_i then X contains at least one of the elements of S_i, for all $i = 1..n$

Since g is an extent of c in (G, M, I), g' satisfies the above sentence. •

Definition 22 *We call the extent of a compound attribute c (or of the conjunction of a clause set $\mathfrak{C}$) in the test context as **free extent** of c(or of $\mathfrak{C}$).*

The elements of the free extent of c(or of $\mathfrak{C}$) are the smallest subsets **consistent** with the attribute c(or of the clause set $\mathfrak{C}$).
So from above propositions, we can deduce that $A \subseteq M$ is consistent with the clause set $\mathfrak{C}$ if and only if the sequent $(A, M/A)$ is not contained in the regular closure of $\mathfrak{C}$, since $\mathfrak{G} := \mathfrak{P}(M) \backslash \{A_t \mid t \in T\}$, where $\mathfrak{G}$ is the extent of the conjunction of full sequents $(A_t, M/A_t)$ in the test context.

Proposition 8 *The regular clause sets form a closure system.*

Proof. Let $\mathfrak{C}_t$ be a family of regular clause sets, $(A, S) \in \bigcap_{t \in T} \mathfrak{C}_t$, and $(X, M/X)$ be a full sequent that contains (A, S). Since $(A, S) \in \mathfrak{C}_t$, for every $t \in T$, $(X, M/X)$ also belongs to every $\mathfrak{C}_t$ since they are regular clause sets. So arbitrary intersection of regular clause sets is regular. Obviously, the set of all sequents is also a regular clause set. Actually it is the clause logic of the formal context $(\emptyset, M, \emptyset)$. Hence the regular clause sets form a closure system. • So each clause set is contained in a smallest regular one. We call this regular clause set generated by given one, or its regular closure. For reasons of simplicity it's often of interest to describe a regular clause set by few clauses.
Our main theorem will introduce this generating set. It consists of cumulated clauses, which are compound attributes of the form

$$\bigwedge A \to \bigvee_{t \in T} \bigwedge A_t$$

, where T is some index set and A, A_t are subsets of M. On the first glance these expressions may seem complicated but it's easy to describe their extents: An object $g \in G$ is in the extent if and only if it satisfies the following condition:

If g has all the attributes from A, then g has all the attributes from at least one $A_t, t \in T$.

This generating set has some disadvantages:

- It may be difficult to find
- There may be smaller generating sets

But it has also some advantages

- It is always irreduntant
- Because of its recursive nature it can be used for knowledge acquisition algorithms.

The theory **generated** by a set $\mathfrak{L}$ of formulas is $Th(Mod(\mathfrak{L}))$. $\mathfrak{L}$ is called **irreduntant** if it does not contain a smaller generating set,i.e. if for each $\lambda \in \mathfrak{L}$ we have $Mod(\mathfrak{L}/\{\lambda\}) \neq Mod(\mathfrak{L})$.

To each set $A \subseteq M$ of attributes we can associate a cumulated clause c_A in (G, M, I), namely

$$\bigwedge A \rightarrow \bigvee (\bigwedge \{g'/A \mid A \subseteq g'\})$$

It's easy to observe that c_A is all-extensional in (G, M, I). Because for any $g \in G, gIa$ for all $a \in A$ that is $A \subseteq g'$ then obviously gIx for all $x \in g'/A$. Hence g is an extent of c_A.

Definition 23 *Let (G, M, I) be a formal context with finite attribute set M. A pseudo object intent of (G, M, I) is a subset $P \subseteq M$ having the following properties:*

1. *P is not an object intent of (G, M, I)*
2. *For each pseudo object intent $Q \subseteq P$ with $Q \neq P$ there exist some object $g \in G$ with $Q \subseteq g' \subseteq P$.*

Theorem 8 *The set of cumulated clauses*

$$\mathfrak{B} := \{c_P \mid P \subseteq M \text{ pseudo object intent }\}$$

is an irreduntant generating set for the clause logic of (G, M, I).

Lemma 5 *The object intents of a context (G, M, I) are precisely the subsets $P \subseteq M$ satisfying the sentence: For each pseudo object intent $Q \subseteq P$ there is some object intent g' with $Q \subseteq g' \subseteq P$.*

Proof. If P is an object intent then take $g' = P$ so it satisfies our sentence. Observe that the sets, which satisfies the second condition of pseudo object intent definition, are either object intents or pseudo object intents. Because But pseudo objects intent cannot satisfy our sentence because if we choose $P = Q$ then we cannot find an object intent with $Q \subseteq g' \subseteq P$. So the subsets which satisfies the sentence are objects intent of (G, M, I). •

Lemma 6 *If P and Q are pseudo object intents of (G, M, I) then P is a model of c_Q if and only if $P \neq Q$.*

Proof. P is a model of c_Q obviously implies $P \neq Q$. If $P \neq Q$ then assuming that P and$Q$ are pseudo object intents of (G, M, I) we get either $Q \subseteq P$ or not. If not then it is a model of c_Q. If P contains Q then since there exist an object intent $g \in G$ with $Q \subseteq g' \subseteq P$. Hence P is a model of c_Q. •

Proof.(of the theorem) By proposition 7 the free extent of the clause logic of (G, M, I) are the object intents of the formal context. Let $\mathfrak{F} \subseteq \mathfrak{P}(M)$ be the object intents of the context (G, M, I). $\mathfrak{B}$ generates the clause logic of (G, M, I) means $Th(Mod(\mathfrak{B})) = Th(\mathfrak{F})$.
So we have to show that every model of $\mathfrak{B}$ is an object intent of the context (G, M, I). Let X be a model of $\mathfrak{B}$, and let $P \subseteq X$ be a pseudo object intent of (G, M, I). Then X is a model of $c_P \in \mathfrak{B}$. So there must be an object intent g' with $P \subseteq g' \subseteq X$. Thus X fulfills the sentence in the lemma1, and consequently $X \in \mathfrak{F}$.
To see that the set $\mathfrak{B}$ is irreduntant according to lemma2 each pseudo object intent P is a model of $\mathfrak{B}/c_P$. • This generating system is not minimal, in the sense that, there may be other generating set for the clause logic of (G, M, I) whose order is less than the set $\mathfrak{B}$.

However, in the case of implicational logic, that is, instead of considering the sequents in (G, M, I), if we take the set of implications of a context (G, M, I) then it turns out that the generating set is minimal.

2.3 Implications

Implications are natural to describe classifications because objects are usually grouped according to their common attribute. In such situation, implications encode expressions of the form "each object with attributes $a_1, a_2, .., a_n$ also has the attributes $b_1, b_2, .., b_n$."

However, the system of all implications between attributes which hold in a context tend to be very large and contain many trivial implications. In this section, we try to find subsystem which suffice to describe the system of all implications.

Let's start with basic definitions.

Definition 24 *An implication is a pair of sets, denoted by $A \to B$. In the language of propositional logic it is denoted by*

$$\bigwedge A \to \bigwedge B$$

For example, in our pizza menu we may classify pizzas whether they contain garlic or onions. We can deduce from the lattice that there is no pizza which contains both. The pizzas that has spinach in it would be a classification inside the garlic pizzas, because there is an implication *spinach* $\to$ *garlic* which means that every pizza containing spinach also contain garlic.

Definition 25 *A subset $T \subseteq M$ **respects** an implication $A \to B$ if $A \nsubseteq T$ or $B \subseteq T$. T **respects a set** $\mathfrak{L}$ of implications if T respects every single implication in $\mathfrak{L}$.*

*$A \to B$ **holds** in a set $\{T_1, T_2, ...\}$ of subsets if each of the subsets T_i respects*

the implication $A \to B$

$A \to B$ ***holds in a context*** (G, M, I) *if it holds in the system of objects intents. We also say* $A \to B$ *is* ***an implication of the context*** (G, M, I).

Proposition 9 *An implication* $A \to B$ *holds in a context in* (G, M, I) *if and only if* $B \subseteq A''$. *It then automatically holds in the set of all concept intents as well.*

Proof. Suppose $A \to B$ holds in (G, M, I) and $m \in B$. Let $g \in G$ and $A \subseteq g'$, then $B \subseteq g'$, which is equivalent to say $g \in A'$ implies $g \in B'$, because $A \subseteq g' \Rightarrow g \in g'' \subseteq A'$. Since $m \in B$ we get $g \in m'$. Hence $A' \subseteq m'$. Therefore $\{m\} \subseteq m'' \subseteq A''$, which says $m \in A''$.

For the second sentence of the proposition, we'll use the fact that every concept intent is the intersection of its object intents, that is,

$$X' = \bigcap_{g \in X} g', \text{ where X is a concept extent}$$

So, if $A \subseteq X' = \bigcap_{g \in X} g'$, then $A \subseteq g'$, for every $g \in X$. By the previous observation $B \subseteq g'$, for every $g \in X$. Hence $B \subseteq X'$. •

Definition 26 *An implication* $A \to B$ ***follows*** *from a set* $\mathfrak{L}$ *of implications between attributes if each subset of* M *respecting* $\mathfrak{L}$ *also respects* $A \to B$. *A family of implications* $\mathfrak{L}$ *is called* ***closed*** *if every implication following from* $\mathfrak{L}$ *is already contained in* $\mathfrak{L}$.

A set $\mathfrak{L}$ *of implications of a context* (G, M, I) *is called* ***complete*** *if every implication of* (G, M, I) *follows from* $\mathfrak{L}$.

The closed sets of implications lend themselves to a syntactic characterization.

Theorem 9 *A set $\mathfrak{L}$ of implications on M is closed if and only if the following conditions are satisfied for all $W, X, Y, Z \subseteq M$:*

1. $X \to X \in \mathfrak{L}$
2. $X \to Y \in \mathfrak{L}$, *then* $X \bigcup Z \to Y \in \mathfrak{L}$
3. $X \to Y \in \mathfrak{L}$ *and* $Y \bigcup Z \to W \in \mathfrak{L}$, *then* $X \bigcup Z \to W \in \mathfrak{L}$.

Lemma 7 *If we assume that $\mathfrak{L}$ satisfies these three conditions then* $(X_1 \to P_1 \in \mathfrak{L}$ *and* $X_2 \to P_2 \in \mathfrak{L}) \Rightarrow X_1 \bigcup X_2 \to P_1 \bigcup P_2 \in \mathfrak{L}$

Proof. $X_1 \to P_1 \in \mathfrak{L} \Rightarrow X_1 \bigcup X_2 \to P_1 \in \mathfrak{L}$ by 2. Similarly, $X_1 \bigcup X_2 \to P_2 \in \mathfrak{L}$ and $P_1 \bigcup P_2 \to P_1 \bigcup P_2 \in \mathfrak{L}$ by 1. So by 3 $X_1 \bigcup X_2 \bigcup P_1 \to P_1 \bigcup P_2 \in \mathfrak{L}$.
Now,$X_1 \bigcup X_2 \to P_1 \in \mathfrak{L}$ and $X_1 \bigcup X_2 \bigcup P_1 \to P_1 \bigcup P_2 \in \mathfrak{L}$ implies $X_1 \bigcup X_2 \to P_1 \bigcup P_2 \in \mathfrak{L}$ by 3, where $X_1 \bigcup X_2 = X, P_1 = Y, Z = X_1 \bigcup X_2, W = P_1 \bigcup P_2$. •

Lemma 8 $X \to Y \in \mathfrak{L} \Rightarrow X \to Y' \in \mathfrak{L}$, *where* $Y' \subseteq Y$.

Proof. $Y' \to Y' \in \mathfrak{L} \Rightarrow Y \to Y' \in \mathfrak{L}$ by 1 and 2. $X \to Y \in \mathfrak{L}$ and $Y \to Y' \in \mathfrak{L} \Rightarrow X \to Y' \in \mathfrak{L}$ by 3. •

Lemma 9 *Let $\mathfrak{L}$ be a set of implications on M, satisfying conditions 1,2,3. Define* $\mathfrak{L}(A) = \bigcup\{Y \mid X \to Y \in \mathfrak{L}, X \subseteq A\}$ *if* $P \subseteq \mathfrak{L}(A)$ *then there exist* $X \subseteq A$ *such that* $X \to P \in \mathfrak{L}$.

Proof. Since $P \subseteq \mathfrak{L}(A)$, there exist X_i's such that $X_i \subseteq A$ and $X_i \to P_i$, where $P \subseteq \bigcup_{i=1}^{n} P_i$, for some n. By lemma 7, we have $\bigcup_{i=1}^{n} X_i \to$

$\bigcup_{i=1}^{n} P_i \in \mathfrak{L}$ and by lemma 8 $\bigcup_{i=1}^{n} X_i \to P \in \mathfrak{L}$, where $\bigcup_{i=1}^{n} X_i \subseteq A$. •

Proof.(of the theorem) Let $A \to B$ follows from $\mathfrak{L}$. We will show that $A \to B \in \mathfrak{L}$. Now, suppose $P \to Q \in \mathfrak{L}$ and $P \subseteq \mathfrak{L}(A)$. Then $X \to P \in \mathfrak{L}$, where $X \subseteq A$, by lemma 9. So $P \to Q \in \mathfrak{L}$ and $X \to P \in \mathfrak{L}$ implies $X \to Q \in \mathfrak{L}$ by condition 3. Therefore $Q \subseteq \mathfrak{L}(A)$. This implies $\mathfrak{L}(A)$ respects $\mathfrak{L}$.

Hence it respects $A \to B$. Since $A \subseteq \mathfrak{L}(A)$, we have $B \subseteq \mathfrak{L}(A)$. So $X \to B \in \mathfrak{L}$, where $X \subseteq A$. By condition 1 $A \to B \in \mathfrak{L}$. •

Definition 27 *A set $\mathfrak{L}$ of implications of a context is called **non-redundant** if none of the implications follows from the others.*

Definition 28 *$P \subseteq M$ is called the **pseudo-intent** of (G, M, I) if and only if $P \neq P''$ and $Q'' \subseteq P$ holds for every pseudo-intent $Q \subseteq P, Q \neq P$.*

Theorem 10 *[1] The set of implications*

$$\mathfrak{L} := \{P \to P'' \mid P pseudo - intent\}$$

is non-redundant and complete.

Proof. Evidently, $\mathfrak{L}$ holds in (G, M, I). In order to show that $\mathfrak{L}$ is complete, we have to show that every set $T \subseteq M$ respecting $\mathfrak{L}$ is an intent. Each such set in particular in particular respects all implications $Q \to Q''$, where Q is a pseudo-intent and $Q \subseteq T$. If we assume that $T \neq T'', T$ itself satisfies the definition of a pseudo-intent and the implication $T \to T''$ is in $\mathfrak{L}$ but is not respected by T, a contradiction.

In order to show that $\mathfrak{L}$ is non-redundant, we consider an arbitrary pseudo-intent P and show that P respects the set $\mathfrak{L}/\{P \to P''\}$. In fact, if $Q \to Q''$

is an implication in $\mathfrak{L}/\{P \to P''\}$ with $Q \subseteq P$, then $Q'' \subseteq P$ must hold, since P is a pseudo-intent. •

Now we will state a very simple but useful proposition.

Proposition 10 *[1] If P and Q are concept or pseudo-intents with $P \nsubseteq Q$ and $Q \nsubseteq P$, then $P \bigcap Q$ is an intent.*

Proof. P as well as Q and thus $P \bigcap Q$ respect all implications in $\mathfrak{L}$ with the possible exception of $P \to P''$ and $Q \to Q''$. If $P \neq P \bigcap Q \neq Q$, then $P \bigcap Q$ also respects these implications, i.e., it is an intent. •

The following proposition shows among other things that there can be no complete set which contains fewer implications than pseudo-intents.

Proposition 11 *[1] Every complete set $\sum$ of implications contains an implication $A \to B$ with $A'' = P''$ for every pseudo-intent P.*

Proof. A pseudo-intent P is always not equal P''. Therefore, provided that $\sum$ is complete, there must be at least one implication $A \to B$ in $\sum$ which leads out P,i.e., with $A \subseteq P$ and $B \nsubseteq P$. On account of $B \subseteq A''$, we get $A'' \nsubseteq P$, and thus $A'' \bigcap P$ cannot be a concept intent. By the previous proposition this yields $P \subseteq A''$ and thus $P'' = A''$. •

CHAPTER 3

Decompositions of Concept Lattices

3.1 Subdirect Decomposition

In this chapter, our aim is to split up a lattice into simpler parts so that the new diagrams reflect the structure of the lattice better. To do so, we'll try to apply some well-known mathematical theories in algebra and express these results also in contextual language. The relation between lattices as an algebra and the contexts leads to powerful algorithmic tools. Because it is easier to have an algorithm in the context structure.

In this section, we'll state specialized version, actually lattice version, of G.Birkhoff's fundamental theorem: "Every non-trivial algebra is isomorphic to a subdirect product of subdirectly irreducible algebras." which is "Every doubly founded complete lattice has a subdirect decomposition into subdirectly irreducible factors." Then we will try to separate our lattice into its factors. Now, let's give definitions of subdirect decompositions and subdirectly irreducible lattices.

Definition 29 *Let T be an arbitrary index set. For a family $(V_t)_{t\in\ T}$ of a complete lattices, the product is defined to be*

$$\times_{t\in\ T}\mathbf{V}_t :=\ (\times_{t\in\ T}V_t, \leq)$$

with

$$(x_t)_{t\in\ T} \leq (y_t)_{t\in\ T} :\Leftrightarrow x_t \leq y_t,\ \textit{for all}\ t \in\ T$$

The lattices $\mathbf{V}_t, t \in\ T$ *are the factors of the product, and the maps*

$$\pi_s : \times_{t\in\ T}\mathbf{V}_t \rightarrow \mathbf{V}_s$$

with

$$\pi_s((x_t)_{t\in\ T}) :=\ x_s$$

defined for $s \in\ T$ *are the* **canonical projections**.

Without difficulty one can prove that

Theorem 11 *[1] Every product of complete lattices is a complete lattice. The infimum and the supremum can be formed componentwise. The canonical projections are surjective homomorphism.*

Definition 30 *A subdirect product of complete lattice is a complete sublattice of the direct product for which the canonical projection maps onto the factors are all surjective.*

Definition 31 *A subdirect decomposition of a complete lattice* $\mathbf{V}$ *is a family* $\Theta_t, t \in\ T$, *of complete congruence relations of* $\mathbf{V}$ *with*

$$\bigcap_{t\in\ T} \Theta_t\ =\ \triangle$$

,where $\triangle$ *denotes the trivial congruence* $\triangle :=\ \{(x,x)\ |\ x \in\ V\}$. *The lattices* $\mathbf{V}/\Theta_t, t \in\ T$, *are called the factors of the subdirect decomposition.*

Definition 32 *A complete lattice* $\mathbf{V}$ *is called subdirectly irreducible if* $\mathbf{V}$ *is isomorphic to a subdirect product of lattices* $\mathbf{V}_t, t \in T$ *then* $\mathbf{V}$ *is canonically isomorphic to one of the factors* $\mathbf{V}_t$

Fortunately, the examination of subdirect decomposition and subdirectly irreducibility can be carried out directly on the context. So that by considering the context we will be able to understand whether the given lattice is subdirectly irreducible or not and whether the given congruence relations are subdirect decompositions. But before stating these theorems we have to introduce some special subcontexts.

Definition 33 *If* (G, M, I) *is a context and if* $H \subseteq G$ *and* $N \subseteq M$*, then* $(H, N, I \cap H \times N)$ *is called a subcontext of* (G, M, I)*.*

Definition 34 *A subcontext* $(H, N, I \cap H \times N)$ *is called compatible if the pair* $(A \cap H, B \cap N)$ *is a concept of the subcontext for every* $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$*.*

Definition 35 *A subcontext* $(H, N, I \cap H \times N)$ *of a clarified context* (G, M, I) *is arrow closed if the following holds*

- $h \nearrow m$ *and* $h \in H \Rightarrow m \in N$
- $g \swarrow n$ *and* $n \in N \Rightarrow g \in H$

Proposition 12 *[1] Every compatible subcontext is arrow-closed.*
Every arrow-closed subcontext of a doubly founded context is compatible.

(The proof of this proposition contains lots of technicalities so we will not go into details.)
The following proposition shows the relation between compatible subcontext and homomorphism.

Proposition 13 *[1] A subcontext* $(H, N, I \cap H \times N)$ *of* (G, M, I) *is compatible if and only if*

$$\pi_{H,N}(A, B) := (A \cap H, B \cap N), \text{ for all } (A, B) \in \underline{\mathfrak{B}}(G, M, I)$$

defines a surjective complete homomorphism

$$\pi_{H,N} : \underline{\mathfrak{B}}(G, M, I) \to \underline{\mathfrak{B}}(H, N, I \cap H \times N)$$

Proof. According to the definition of compatibility, $(H, N, I \cap H \times N)$ is compatible if and only if $\pi_{H,N}$ is a map. The fact that this map must necessarily be infimum-preserving can be recognized by examining the extents: The map $A \to (A \cap H)$ is evidently $\cap - preserving$, and the infimum of concepts is defined in terms of the intersection of their extents. Dually, we infer that $\pi_{H,N}$ is supremum-preserving. The surjectivity can be seen as follows: if $(C, C \cap N)$ is a concept of $(H, N, I \cap H \times N)$, then $\pi_{H,N}(C'', C') = (C'' \cap H, C' \cap N)$ is a concept with the same intent, i.e., the same concept. •

The homomorphism theorem is like a bridge between congruence relations and compatible subcontext with the above proposition.

Theorem 12 *(Homomorphism Theorem) [5] If* Θ *is a complete congruence relation of a complete lattice* $\mathbf{V}$*, then* $x \to [x]\Theta$ *is a complete homomorphism of* $\mathbf{V}$ *onto* $\mathbf{V}/\Theta$*. If, conversely,* $\phi : \mathbf{V_1} \to \mathbf{V_2}$ *is a surjective complete homomorphism between complete lattices, then*

$$ker\phi := \{(x, y) \in \mathbf{V_1} \times \mathbf{V_2} \mid \phi(x) = \phi(y)\}$$

is a complete congruence relation of $\mathbf{V_1}$*; besides,*

$$[x]ker\phi \to \phi(x)$$

describes an isomorphism of $\mathbf{V_1}/ker\phi$ *onto* $\mathbf{V_2}$*.*

By the proposition, the compatible subcontext defines a homomorphism, and homomorphism theorem says that with the kernel of this homomorphism, $\Theta_{H,N}$, we get

$$\underline{\mathfrak{B}}(H, N, I \cap H \times N) \cong \underline{\mathfrak{B}}(G, M, I)/\Theta_{H,N}$$

with

$$(A_1, B_1)\Theta_{H,N}(A_2, B_2) \Leftrightarrow A_1 \cap H = A_2 \cap H \Leftrightarrow B_1 \cap N = B_2 \cap N$$

Remark : We can deduce that every compatible subcontext induces a congruence, but it's still unanswered that when a congruence Θ is induced by a subcontext.

Definition 36 *We say that a complete congruence Θ is induced by a subcontext if a compatible subcontext $(H, N, I \cap H \times N)$ with $\Theta = \Theta_{H,N}$.*

Now we will state a theorem without proof.

Theorem 13 *[1] If $\underline{\mathfrak{B}}(G, M, I)$ is doubly founded, then every complete congruence relation is induced by a subcontext.*
Moreover, if (G, M, I) is reduced then this subcontext is uniquely determined by the congruence.

Thus, in this case the arrow-closed subcontexts correspond bijectively to the complete congruences whenever the context (G, M, I) is reduced and doubly founded.
Coming back to the relation between subdirect decomposition and arrow-closed subcontext :

Proposition 14 *If (G, M, I) is reduced context of a doubly founded concept lattice, then the subdirect decompositions of $\underline{\mathfrak{B}}(G, M, I)$ correspond bijectively to the families of arrow-closed subcontext $(G_t, M_t, I \cap G_t \times M_t)$ with $\bigcup_{t\in T} G_t = G$ and $\bigcup_{t\in T} M_t = M$.*

Proof. The order of the congruence relations is also reflected by the arrow-closed subcontext.If Θ and ψ are two congruences of $\mathbf{V}$, then

$$\begin{aligned}
\Theta \subseteq \psi \;\Leftrightarrow\;& (A, B)\Theta(C, D) \Rightarrow (A, B)\psi(C, D), \text{ for all } (A, B), (C, D) \in \mathbf{V} \\
\Leftrightarrow\;& A \cap G_\Theta = C \cap G_\Theta \Rightarrow A \cap G_\psi = C \cap G_\psi \text{ and} \\
& B \cap M_\Theta = D \cap M_\Theta \Rightarrow B \cap M_\psi = D \cap M_\psi \\
& \text{for all } (A, B), (C, D) \in \mathbf{V} \\
\Leftrightarrow\;& G_\psi G_\Theta \subseteq G_\Theta \text{ and } M_\psi \subseteq M_\Theta
\end{aligned}$$

Hence, if we order the subcontext by

$$(H_1, N_1, I \cap H_1 \times N_1) \leq (H_2, N_2, I \cap H_2 \times N_2)$$

$$:\Leftrightarrow H_1 \subseteq H_2 \text{ and } N_1 \subseteq N_2,$$

under the conditions specified, the ordered set of the arrow-closed subcontext is dually isomorphic to the lattice of congruences. So, $\bigcap_{t\in T} \Theta_t = \triangle$ holds for a family $\Theta_t, t \in T$ of congruences if and only if $\bigcup_{t\in T} G_t = G$ and $\bigcup_{t\in T} M_t = M$ holds for the corresponding arrow-closed subcontext $(G_t, M_t, I \cap G_t \times M_t)$. •

Definition 37 *A context is called 1-generated if there is an object g such that the smallest arrow-closed subcontext containing g is the context itself.*

Proposition 15 *A doubly founded reduced context is 1-generated if and only if* $\underline{\mathfrak{B}}(G, M, I)$ *is subdirectly irreducible.*

Proof. For every subdirect decomposition of $\underline{\mathfrak{B}}(G, M, I)$, there is corresponding arrow-closed subcontexts $(G_t, M_t, I \cap G_t \times M_t)$ with $\bigcup_{t\in\ T} G_t = G$ and $\bigcup_{t\in\ T} M_t = M$. Since (G, M, I) is arrow-closed there exist $t \in T$ with $G_t = G$ and $M_t = M$, which is equivalent to say that $\underline{\mathfrak{B}}(G, M, I)$ is subdirectly irreducible. The converse is similar. •

Theorem 14 *[1] Every doubly founded complete lattice has a subdirect decomposition into subdirectly irreducible factors.*

Proof. Without loss of generality we may assume that $\mathbf{V}$ is the concept lattice of a reduced context (G, M, I). We may then assume that $\mathbf{V} = \underline{\mathfrak{B}}(G, M, I)$. For $g \in G$ let $(G_g, M_g, I \cap G_g \times M_g)$ denote the smallest arrow-closed subcontext of (G, M, I) containing g. Since $\bigcup_{g\in\ G} G_g = G$ and $\bigcup_{g\in\ G} M_g = M$ the corresponding congruence relations is a subdirect decomposition and the corresponding lattice of the arrow-closed subcontexts are subdirectly irreducible because the contexts are 1-generated. •

3.2 An Example

Suppose we have a context:

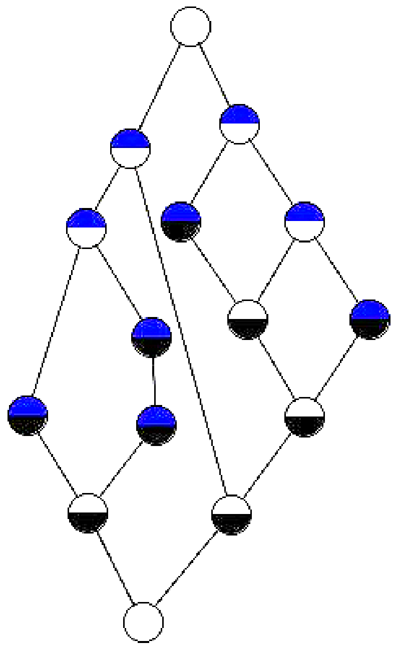

Figure 3.1: Concept Lattice

	B	C	D	E	F	G	H	I	J
1	×						×		×
2			×			×		×	
3	×	×	×			×		×	
4		×				×			
5	×			×	×		×		×
6	×				×				×
7		×	×			×		×	
8	×			×			×		×
9		×				×		×	

, which is reduced and doubly founded(since finite).

The concept lattice of the context is shown above:

In this example, our aim is to find a subdirect decomposition of this lattice such that every factor of the congruences will be subdirectly irreducible. To find this subdirect decomposition we'll use proposition 14 and proposition 15. Proposition 14 says that arrow-closed subcontexts $(G_t, M_t, I \cap G_t \times M_t)$ with $\bigcup_{t\in T} G_t = G$ and $\bigcup_{t\in T} M_t = M$ corresponds to subdirect decomposition, and proposition 15 guarantees that they are subdirectly irreducible if 1-generated.

So let's start with computing 1-generated arrow-closed subcontext. Now, applying the definition of arrow-closed subcontext

$$3 \nearrow E, 3 \in H_3 \Rightarrow E \in N_3,$$

$$6 \swarrow E, E \in N_3 \Rightarrow 6 \in H_3,$$

$$6 \nearrow J, 6 \in H_3 \Rightarrow J \in N_3,$$

$$3 \nearrow F, 3 \in H_3 \Rightarrow F \in N_3,$$

$$F \swarrow\!\!\!\!\nearrow 13, F \in N_3 \Rightarrow 13 \in H_3$$

, continuing in this way, we obtain

$$H_3 = \{3, 6, 8, 9, 13\}$$

$$N_3 = \{E, F, G, J, H\}$$

Observing that $4 \notin H_3$, let's start generating a context by 4. We get

$$H_4 = \{4, 8, 12\}$$

$$N_4 = \{B, C, G\}$$

and also 6,7,8,9,12,13,15 generates subcontexts:

$$\begin{aligned} H_6 &= \{6\} \\ N_6 &= \{J\}, \\ H_7 &= \{7,8,12\} \\ N_7 &= \{B,I,G\}, \\ H_8 &= \{8\} \\ N_8 &= \{G\}, \\ H_9 &= \{9,8,6\} \\ N_9 &= \{H,J,G\}, \\ H_{12} &= \{12\} \\ N_{12} &= \{B\}, \\ H_{13} &= \{13,8,6\} \\ N_{13} &= \{J,F,G\}, \\ H_{15} &= \{15,8,12\} \\ N_{15} &= \{B,D,G\}, \end{aligned}$$

But the subcontexts $(H_9, N_9, I \cap H_9 \times N_9), (H_6, N_6, I \cap H_6 \times N_6), (H_8, N_8, I \cap H_8 \times N_8), (H_{13}, N_{13}, I \cap H_{13} \times N_{13})$ are subsets of the context $(H_3, N_3, I \cap H_3 \times N_3)$ and $(H_{12}, N_{12}, I \cap H_{12} \times N_{12})$ is subset of $(H_7, N_7, I \cap H_7 \times N_7)$ So we do not need to consider them.

The corresponding congruences for the arrow-closed 1-generated subcontexts look like:

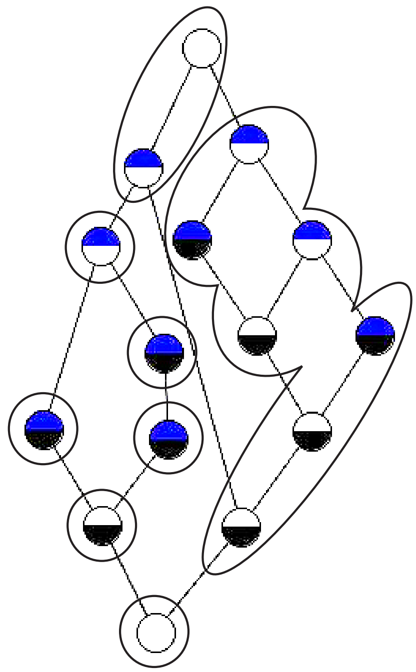

Figure 3.2: Congruence relation generated by 3

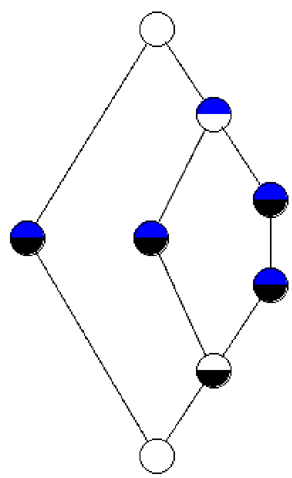

Figure 3.3: Concept lattice of the context generated by 3

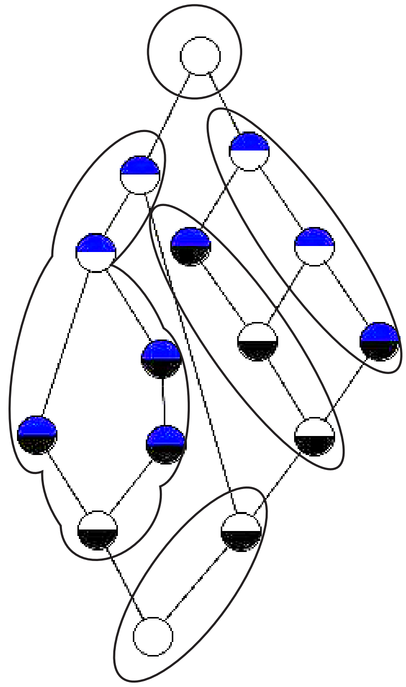

Figure 3.4: Congruence relation generated by 4

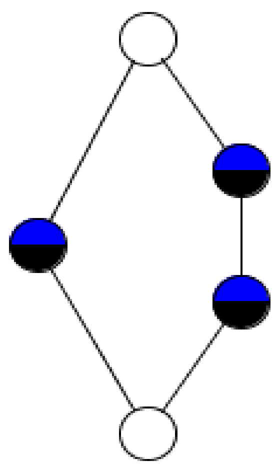

Figure 3.5: Concept lattice of the context generated by 4

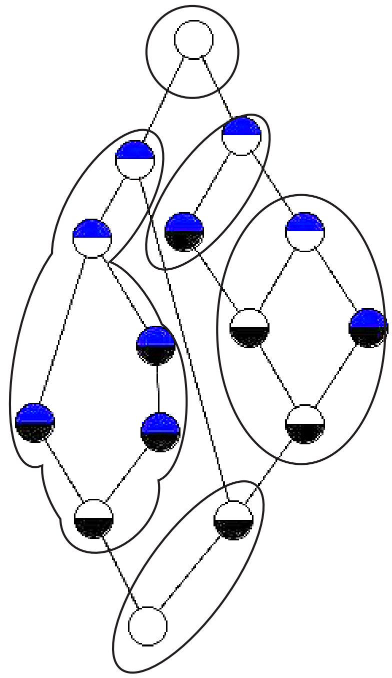

Figure 3.6: Congruence relation generated by 7

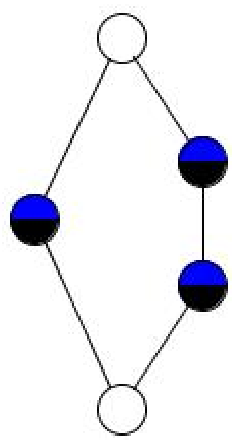

Figure 3.7: Concept lattice of the context generated by 7

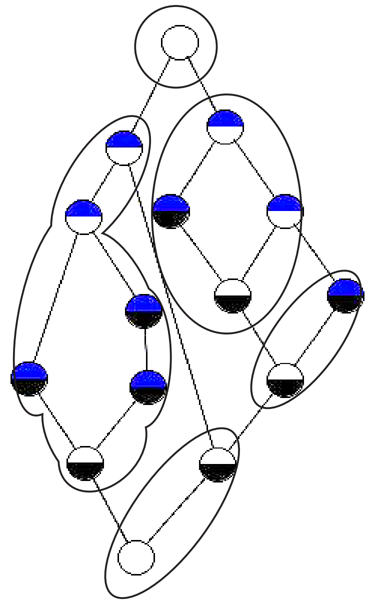

Figure 3.8: Congruence relation generated by 15

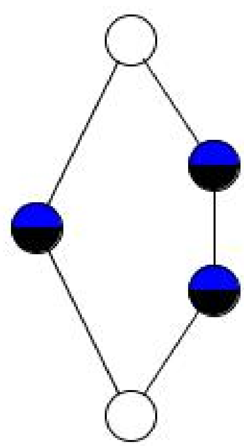

Figure 3.9: Concept lattice of the context generated by 15

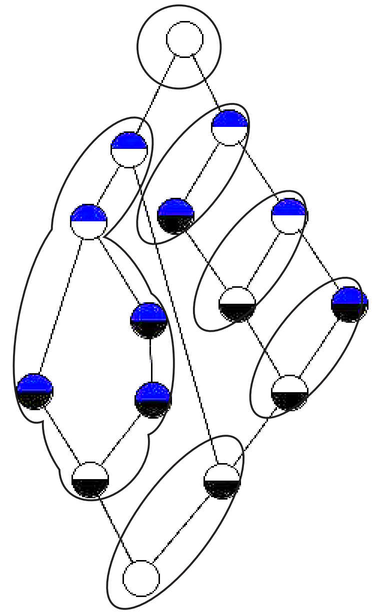

Figure 3.10: Concept lattice of the context generated by 7 and 15

So, our lattice is subdirect product of the lattices in figures 3.3, 3.5, 3.7, 3.7. ,where each factor is subdirectly irreducible.
We can also decompose the lattice into two factors, but at least one of the factors will not be subdirectly irreducible.
The subcontext

$$H_{7,15} = \{7, 8, 12, 15\}$$

$$N_{7,15} = \{B, D, G, I\},$$

corresponds to the congruence $\Theta_{7,15}$

and $\Theta_{7,15} \bigcap \Theta_3 = \triangle$, so they are subdirect decomposition. Hence we can deduce that the lattice $\mathbf{L}$ is subdirect product of $\mathbf{L_3}$ and $\mathbf{L_{7,15}}$

3.3 Atlas Decomposition

In the previous section, we used congruence relations to decompose the lattice into smaller parts. In this section, we will use tolerance relations, which is reflexive, symmetric, but not necessarily transitive and compatible with lattice operations, for an analogous procedure for large unwieldily lattices.

Definition 38 *A tolerance relation on* $\mathbf{V}$ *is a relation* $\Theta \leq \mathbf{V} \times \mathbf{V}$ *which is reflexive, symmetric, and compatible with suprema and infima i.e., for which the following equation holds*

$$x_t \Theta y_t, \text{ for all } t \in T \Rightarrow (\bigcap_{t \in T} x_t)\Theta(\bigcap_{t \in T} y_t) \text{ and } (\bigcup_{t \in T} x_t)\Theta(\bigcup_{t \in T} y_t)$$

Definition 39 *If* Θ *is a tolerance relation on* $\mathbf{V}$ *and* $a \in \mathbf{V}$*, we define*

$$a_\Theta := \bigcap \{x \in V \mid a \Theta x\}$$

and

$$a^\Theta := \bigcup \{x \in V \mid a \Theta x\}$$

The intervals $[a]_\Theta := [a_\Theta, (a_\Theta)^\Theta]$, $a \in \mathbf{V}$ *are called the blocks of* Θ.

Definition 40 *The set of all blocks of a tolerance relation of* $\mathbf{V}$ *is denoted by* $\mathbf{V}/\Theta$ *and ordered by*

$$B_1 \leq B_2 :\Leftrightarrow \bigcap B_1 \leq \bigcap B_2 (\Leftrightarrow \bigcup B_1 \leq \bigcup B_2)$$

Theorem 15 *[1] With the order described above,* $\mathbf{V}/\Theta$ *is a complete lattice called the factor lattice of* $\mathbf{V}$ *of* Θ*. The followings are the supremum and infimum of this lattice.*

$$\bigcup_{t \in T} B_t = [\bigcup_{t \in T} \bigcap B_t]^\Theta$$

$$\bigcap_{t \in T} B_t = [\bigcap_{t \in T} \bigcup B_t]_\Theta$$

Definition 41 *By a block relation of a context (G, M, I) we mean a relation $J \subseteq G \times M$ which satisfies the following conditions*

1. $I \subseteq J$

2. $\forall g \in G, g^J$ *is an intent of* (G, M, I)

3. $\forall m \in M, m^J$ *is an extent of* (G, M, I)

The intersection of any number of block relations of (G, M, I) is again a block relation since $g^{\cap J_t} = \cap g^{J_t}$, and the intersection of intents is always an intent. Hence the block relations of (G, M, I) form a closure system and thus a complete lattice.

Theorem 16 *[1] The lattice of all block relations of (G, M, I) is isomorphic to the lattice of all complete tolerance relations of $\underline{\mathfrak{B}}(G, M, I)$. The map β assigning to any tolerance relation Θ to the block relation is defined by*

$$g\beta(\Theta)m :\Leftrightarrow \gamma(g)\Theta(\gamma(g) \wedge \mu(m))$$

is an isomorphism. Conversely

$$(A, B)\beta^{-1}(J)(C, D) \Leftrightarrow A \times D \cup C \times B \subseteq J$$

yields the tolerance corresponding to a block relation.

Theorem 17 *[1] If Θ is a tolerance relation on $\underline{\mathfrak{B}}(G, M, I)$ and $J = \beta(\Theta)$ is the corresponding block relation, then*

$$\underline{\mathfrak{B}}(G, M, I)/\Theta \cong \underline{\mathfrak{B}}(G, M, J)$$

and if (C, B) is a concept of (G, M, J) then

$$[(B^I, B), (C, C^I)] \;=\; \underline{\mathfrak{B}}(C, B, I \cap C \times B)$$

for the corresponding block of Θ.

With the theorem above, we obtained a strong and a close relation between tolerances and block relations. Now our aim is to decompose $\mathbf{V}$ into blocks of a tolerance relation and draw the lattice diagram of these blocks by the help of the block relation J and the corollary.

Definition 42 *A tolerance relation Θ of a lattice $\mathbf{V}$ has overlapping neighborhoods if*

$$B_1 < B_2 \text{ in } \mathbf{V}/\Theta \text{ implies } B_1 \cap B_2 \neq \emptyset$$

Let $\Sigma(\mathbf{V})$ denote the smallest tolerance relation comprising all pairs (x, y) with $x \prec y$ in $\mathbf{V}$.In the case of doubly founded lattices, $\Sigma(\mathbf{V})$ is called the skeleton tolerance.

Theorem 18 *[1] Let (G, M, I) be a doubly founded context and let*

$$\Sigma :=\; \Sigma(\underline{\mathfrak{B}}(G, M, I))$$

be the skeleton tolerance. then the following statements hold for the corresponding block relation $J :=\; \beta(\Sigma)$:

1. *J is the smallest block relation of (G, M, I) containing all pairs (g, m) with $g \swarrow\!\!\!\nearrow m$.*

2. *J contains all pairs (g, m) with $g \swarrow m$ or $g \nearrow m$.*

Definition 43 *Let* $\mathbf{V}_q, q \in Q$ *be a family of doubly founded complete lattices. Let the index set* Q *be a lattice of finite length. We call* $(\mathbf{V}_q \mid q \in Q)$ *a* Q*-atlas with overlapping neighbour maps, if for each two elements* $q, r \in Q$ *the following conditions are satisfied:*

1. $\mathbf{V}_q \subseteq \mathbf{V}_r \Rightarrow q = r$
2. *if* $q \leq r$*, then* $\mathbf{V}_q \cap \mathbf{V}_r$ *is an order filter in* $\mathbf{V}_q$ *and an order ideal in* $\mathbf{V}_r$*.*
3. *if* q *is a lower neighbour of* r*, then* $\mathbf{V}_q \cap \mathbf{V}_r \neq \emptyset$*.*
4. *The orders of* $\mathbf{V}_q$ *and* $\mathbf{V}_r$ *coincide on the intersection* $\mathbf{V}_q \cap \mathbf{V}_r$*.*
5. $\mathbf{V}_q \cap \mathbf{V}_r \subseteq \mathbf{V}_{q \wedge r} \cap \mathbf{V}_{q \vee r}$*.*
6. $q \leq r \leq s \Rightarrow \mathbf{V}_q \cap \mathbf{V}_s \subseteq \mathbf{V}_r$

Theorem 19 *(main theorem) [1] The sum of a* Q*-atlas with overlapping neighbour maps is a complete lattice* $\mathbf{V}$ *where the summands* $\mathbf{V}_q, q \in Q$ *are precisely the blocks of a complete tolerance relation* Θ *and where* $q \mapsto \mathbf{V}_q$ *describes an isomorphism of* Q *onto* $\mathbf{V}/\Theta$*.*
Conversely, in a complete lattice $\mathbf{V}$ *the blocks of a tolerance* Θ *with overlapping neighborhoods, for which* $Q := \mathbf{V}/\Theta$ *is of finite length, always form a* Q*-atlas with overlapping neighbour maps whose sum is* $\mathbf{V}$*.*

3.4 An Example

The concept lattice of the context

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	×	×			×	×	×	×						×	×
2							×								
3								×							
4					×										
5	×							×				×	×		×
6		×	×				×								×
7													×	×	×
8						×	×	×							
9	×	×												×	
10								×							
11					×										
12	×	×			×	×	×	×	×			×		×	×
13	×							×							
14				×		×									×
15	×	×	×	×	×	×	×	×			×	×	×	×	×

is shown in the below figure:

In this section we will decompose this lattice to smaller and manageable parts by using atlas decomposition method. Also this method contains the additional information which show how the individual parts are related. The main theorem of the last section says that the blocks of a tolerance relation Θ with overlapping neighbour form a Q-atlas, where $Q := \mathbf{V}/\Theta$.

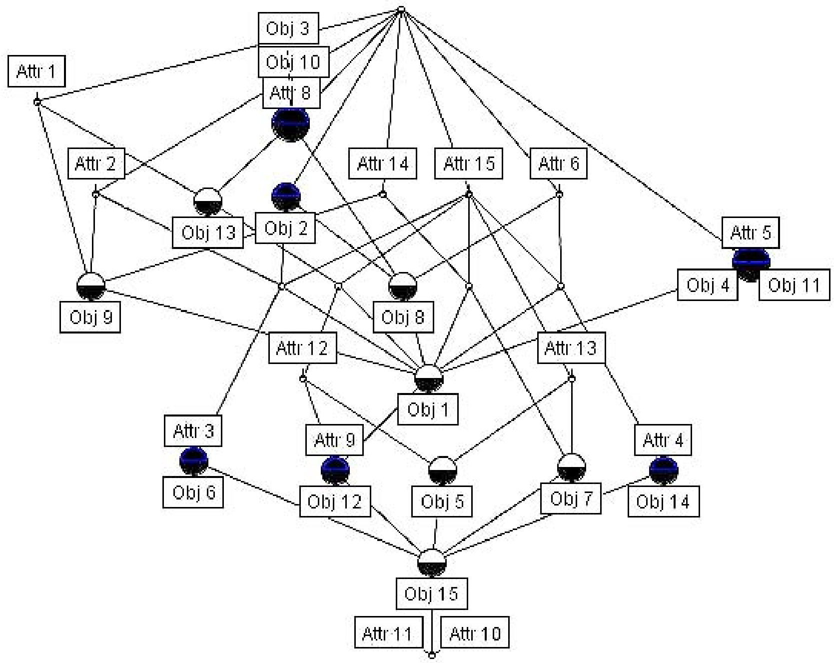

Figure 3.11: Concept lattice of the above context

But by theorem 17, we do not need to find this relation to get the blocks of Θ, instead it is enough to consider the corresponding block relation $J = \beta(\Theta)$ since

$$\underline{\mathfrak{B}}(G, M, I)/\Theta \cong \underline{\mathfrak{B}}(G, M, J)$$

and every concept (G, M, J) corresponds to the block of Θ by the equation

$$[(B^I, B), (C, C^I)] = \underline{\mathfrak{B}}(C, B, I \cap C \times B)$$

,where (C, B) is a concept of (G, M, J).

To find the corresponding block relation, we will use theorem 18. So we enter the arrow relations into the context and enrich the relation

$$J := I \cup \swarrow \cup \nearrow$$

Then check that whether this relation is a block relation or not. Because theorem 18 does not guarantee that it is a block relation. The context of the enriched relation is shown below:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	×	×	↗	↗	×	×	×	×	↙			⤢	↗	×	×
2	↗	↗			↗	↗	×	↗						↗	↗
3	↗	↗			↗	↗	↗	×						↗	↗
4	⤢	⤢			×	⤢	⤢	⤢						⤢	⤢
5	×	⤢	↙	↙	⤢	⤢	⤢	×	↙			×	×	⤢	×
6	⤢	×	×	↙	⤢	⤢	×	⤢	↙			↙	⤢	⤢	×
7	⤢	⤢	↙	↙	⤢	⤢	⤢	⤢	↙			↙	×	×	×
8	⤢	⤢			⤢	×	×	×						⤢	⤢
9	×	×			⤢	⤢	⤢	⤢						×	⤢
10	↗	↗			↗	↗	↗	×						↗	↗
11	⤢	⤢			×	⤢	⤢	⤢						⤢	⤢
12	×	×	⤢	⤢	×	×	×	×	×			×	⤢	×	×
13	×	↗			↗	↗	↗	×						↗	⤢
14	⤢	⤢	↙	×	⤢	×	⤢	⤢	↙			↙	⤢	⤢	×
15	×	×	×	×	×	×	×	×	⤢	⤢	×	×	×	×	×

So J becomes

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	×	×	×	×	×	×	×	×	×			×	×	×	×
2	×	×			×	×	×	×						×	×
3	×	×			×	×	×	×						×	×
4	×	×			×	×	×	×						×	×
5	×	×	×	×	×	×	×	×	×			×	×	×	×
6	×	×	×	×	×	×	×	×	×			×	×	×	×
7	×	×	×	×	×	×	×	×	×			×	×	×	×
8	×	×			×	×	×	×						×	×
9	×	×			×	×	×	×						×	×
10	×	×			×	×	×	×						×	×
11	×	×			×	×	×	×						×	×
12	×	×	×	×	×	×	×	×	×			×	×	×	×
13	×	×			×	×	×	×						×	×
14	×	×	×	×	×	×	×	×	×			×	×	×	×
15	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×

Now, it can be easily checked that J satisfies the definition of block relation. Hence it is the corresponding block relation of the skeleton tolerance.

According to theorem 17, we obtain the blocks as concept lattices of subcontext.

There are 3 concepts of the context (G, M, I). The contexts and their diagram is shown below.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
15	×	×	×	×	×	×	×	×			×	×	×	×	×

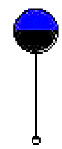

Figure 3.12: First Block

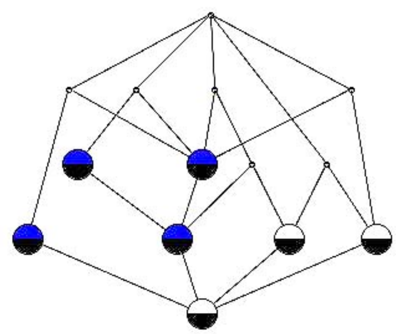

Figure 3.13: Second Block

	1	2	3	4	5	6	7	8	9	12	13	14	15	
1	×	×			×	×	×	×				×	×	
5	×							×		×	×		×	
6		×	×				×						×	
7											×	×	×	
12	×	×			×	×	×	×	×	×		×	×	
14				×		×							×	
15	×	×	×	×	×	×	×	×	×	×	×	×	×	

	1	2	5	6	7	8	14	15	
1	×	×	×	×	×	×	×	×	
2					×				
3						×			
4			×						
5	×					×		×	
6		×			×			×	
7							×	×	
8				×	×	×			
9	×	×					×		
10						×			
11			×						
12	×	×	×	×	×	×	×	×	
13	×					×			
14		×		×				×	
15	×	×	×	×	×	×	×	×	

These methods provide us to navigate through lattice and make it more manageable so that creating knowledge and obtaining results from the lattice of data would be easier. But we should admit that these tools do not always work practically. So as a future work, we can study some other decomposition methods by using algebraic facts.

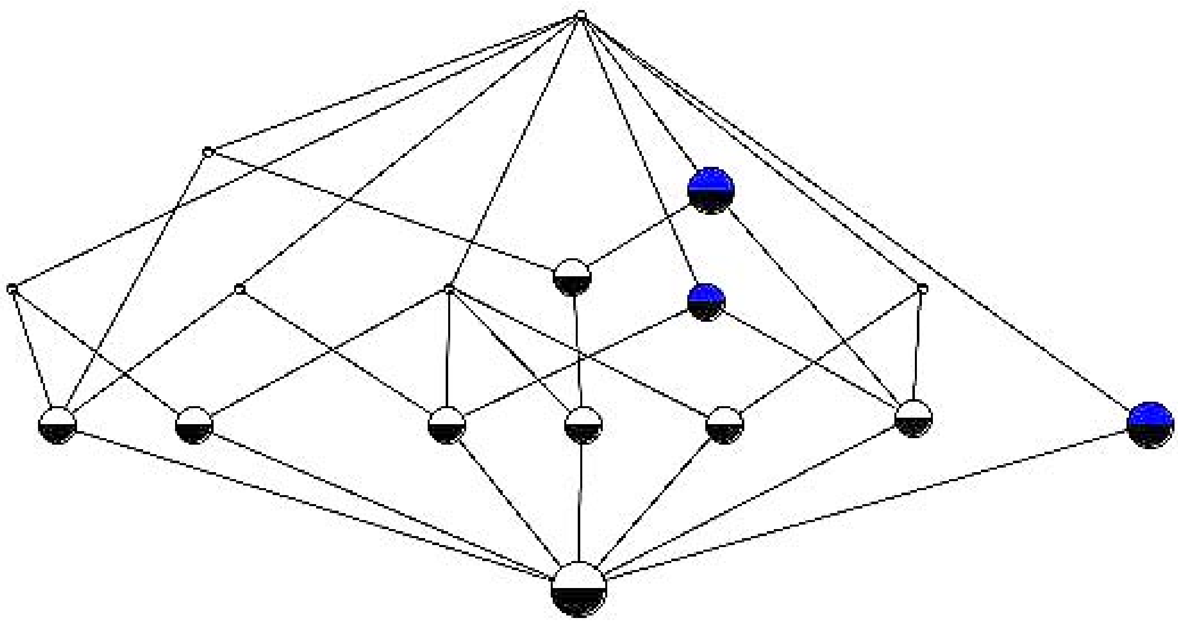

Figure 3.14: Third Block

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