HARDY SPACES ON HYPERCONVEX DOMAINS

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JULY 2003

HARDY SPACES ON HYPERCONVEX DOMAINS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

JULY 2003

Approval of the Graduate School of Natural and Applied Sciences

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

HARDY SPACES ON HYPERCONVEX DOMAINS

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July 2003, 82 pages

In this thesis, we give a new definition of Hardy Spaces on hyperconvex domains in terms of Monge-Ampère measures which unifies the Hardy spaces on polydiscs and balls. Also we survey Monge-Ampère operators and Monge-Ampère measures.

Keywords: Hardy Spaces, Monge-Ampère measures, Hyperconvex domains.

ÖZ

HİPERKONVEKS KÜMELER ÜZERİNDE HARDY UZAYLARI

Alan, Muhammed Ali

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi: Prof. Dr. Aydın Aytuna

Temmuz 2002, 82 sayfa

Bu tezde, hiperkonveks kümeler üzerinde Hardy uzaylarının, Monge-Ampère ölçümleri yordamıyla yeni bir tanımını veriyoruz. Bu yeni tanım polidiskler ve toplar üzerindeki Hardy Uzayi tanımlarını birleştirmektedir. Ayrıca Monge-Ampère operatörleri ve Monge-Ampère ölçümlerinin de bir gözden geçirilmesi yapılmaktadır.

Anahtar Kelimeler: Hardy Uzaylari , Monge-Ampère Ölçüleri, Hiperkonveks Kümeler.

To my family and her

ACKNOWLEDGMENTS

I would like to express my gratitude to my supervisor Prof. Aydın Aytuna for his supervision, tolerance and his confidence in me during this patient guidance.

I also thank to Prof. V. P. Zahariuta for his many great advices in mathematics and for the life. His motivation was so great for me.

I want to also thanks to Prof. Cemal Koç, Prof. İsmail Güloğlu, Prof. Zafer Nurlu, Prof. Hurşit Önsiper, Dr. Turgay Kaptanoğlu, and Dr. Ha Minh Dzung for their motivation and nice conversations which oriented me to mathematics.

Also I am thankful to my friends İbrahim Ünal, Mehmet Emin Özüyılmaz, Adem Çakmak, Celalettin Kaya, Selma Yıldırım, Uğur Gül, Gökhan Göğüş and Mustafa Kalafat for their motivation and geyiks.

Next, I am so thankful to my family for their support and tolerance.

Lastly, I want to thank to her. Without her support this thesis would not appear.

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CHAPTER 1

INTRODUCTION

1.1 Notations

Let Ω be an open subset of $\mathbb{C}^n$. We will use $\mathcal{C}^r(\Omega)$ for functions with continuous partial derivatives of order r on Ω. $\mathcal{C}^\infty(\Omega)$ will be used for infinitely differentiable functions on Ω. $\mathcal{C}_0^r(\Omega)$ ($\mathcal{C}_0^\infty(\Omega)$) will be used for functions with compact support which are in $\mathcal{C}^r(\Omega)$ (and in $\mathcal{C}^\infty(\Omega)$). Here support of a function ϕ is the closure of the set $\{x \in \Omega : f(z) \neq 0\}$ and will be shown by *supp*ϕ.

If $z \in \mathbb{C}^n, r > 0$, we define the *open ball* with center z and radius r

$$\boldsymbol{B}(z,r) = \{w \in \mathbb{C}^n : |z - w| < r\}$$

and if $r = (r_1, \ldots, r_n)$ then we will denote the *open polydisc* with center z and polyradius r by

$$\triangle^n(z,r) = \{w \in \mathbb{C}^n : |z_i - w_i| < r_i \ , \quad j = 1, \ldots, n\}$$

For $n = 1$, $\boldsymbol{B}(z,r)$ and $\triangle^n(z,r)$ coincide and will be shown by $\triangle(z,r)$ and will be called disc with center z and radius r.

In an open set Ω we will denote the set of all differential forms of bidegree (p,q) whose coefficients belong to $\mathcal{C}_0(\Omega,\mathbb{C})$ (respectively, $\mathcal{C}_0^\infty(\Omega,\mathbb{C})$) by $\mathcal{D}_0^{p,q}(\Omega)$ (respectively, $\mathcal{D}^{p,q}(\Omega)$) . See the Section 2.1.2.

$\mathcal{D}(\Omega)$ is the vector space of test functions.

$O(\Omega)$ is the space of all analytic functions on Ω.

$\mathcal{H}(\Omega)$ is the space of all harmonic functions on Ω.

$\mathcal{SH}(\Omega)$ is the space of all subharmonic functions on Ω.

$\mathcal{PH}(\Omega)$ is the space of all pluriharmonic functions on Ω.

$\mathcal{PSH}(\Omega)$ is the space of all plurisubharmonic functions on Ω.

$L^p(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f|^p < \infty\}$

$L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable and} \int_K |f|^p < \infty \ K \subset \Omega \text{ compact}\}$

$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable and } \sup_\Omega |f| < \infty\}$

$L^\infty_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable and } \sup_K |f| < \infty \ K \subset \Omega \text{ compact}\}$

1.2 Introduction and the Structure of Thesis

The theory of Hardy Spaces started with works of G. H. Hardy, J. E. Littlewood in 1920's. And by works of them and I. I. Privalov, F. and M. Riesz, V. Smirnov and G. Szegö the was theory improved and developed. Their work was mainly in the unit disc of $\mathbb{C}$.

Later Hardy Space theory was extended to more general classes of domains such as the ball of $\mathbb{C}^n$, the polydisc, simply connected domains in $\mathbb{C}$, Smirnov domains, pseudoconvex domains with $\mathcal{C}^2$ boundaries.

In the unit disc we have some equivalent forms of Hardy Spaces. The first form is by means of integral growth over some curves and secondly by means of harmonic majorants. The extensions of Hardy classes are done mainly by these two different ways. The first way is by some integral growth condition over some certain curves or hypersurfaces.

But the main problem arises in domains with non-smooth boundaries. For instance in several complex variables one of the basic and most important domain is the polydisc. Here the Hardy Spaces theory differ enormously from these extensions. In the polydiscs Hardy Spaces are defined by integral mean over Torus T^n which is only a very small part of the boundary. Here also the definition via majorants changes from harmonic majorants to n-harmonic majorants.

Our main goal is to unify those theories so that we do not need to give a definition for all different kinds of domains. Of course what those domains have in common is that they are all hyperconvex. For the definitions and properties of hyperconvex domains see Section 4.2.

We define Hardy Spaces in this thesis on a hyperconvex domain Ω as the set of all analytic functions on Ω such that they satisfy following integral growth condition:

$$\sup_{r<0} \int\limits_{S(r)} |f(z)|^p d\mu_{r,a} < \infty.$$

(For the details and definitions see section 5.2.)

Unfortunately these definitions depend on the point a. The first problem is to show the consistency i.e. the independency of Hardy Spaces from the point. We have some partial results on this problem. And we see that those spaces coincide with the usual Hardy Spaces on the ball and the polydisc. And we show for $n = 1$ those spaces are equivalent to having a harmonic majorant. The last chapter is devoted to those discussions.

In the first chapter we give basic notations used in the thesis and we give the structure of the thesis.

In the second chapter we give the basic preliminaries needed for the thesis. First the complex differentiation and the complex differential forms are introduced to deal with several complex variables. Next distributions and currents are introduced for later discussions. Then a short introduction to subharmonic and plurisubharmonic functions is given. Lastly classical Hardy spaces are introduced in the disc, polydisc, and the ball. And analogue theorems are given without proof.

In the third chapter we deal with complex Monge-Ampére operators. Monge-Ampére operators are important for characterizing maximal plurisubharmonic functions in Theorem 3.3.7 and are also important in complex geometry. (See [10].) Firstly, extensions of Monge-Ampére operators from $\mathcal{C}^2$

functions to L^{∞}_{loc} functions are given. Next some comparison theorems are given.

In the fourth chapter firstly, we introduce Green functions and give some important properties of Green functions. Next we give some facts about hyperconvex domains. Lastly we define Demailly-Monge-Ampére measures which is important in Intersection Theory and Pluripotential Theory. From our point of view they are very important since we will define the Hardy Classes in terms of Demailly-Monge-Ampére measures.

The last chapter is devoted to the main goal of this work; namely to extend Hardy Spaces to Hyperconvex domains.

CHAPTER 2

FOUNDATION

In this chapter firstly we will give some necessary basic definitions from the theory of several complex variables. Secondly, we will deal with complex differential forms. Next we will give some facts from the theory of distributions and currents to be able to extend the definition of Monge-Ampére measures and the Monge-Ampére operator in terms of distributions and currents. We will give the definitions and some facts about subharmonic and plurisubharmonic functions. Lastly we will give some facts about classical Hardy Spaces.

2.1 Basic Definitions and Notations in Several Complex Variables

2.1.1 The ∂ and $\bar{\partial}$ Operators and Levi form

An $\mathbb{R}$-linear map L is called $\mathbb{C}$-*linear* if $L(\lambda x) = \lambda L(x)$ for all x and for all $\lambda \in \mathbb{C}$. It is called *anti* $\mathbb{C}$-*linear* if $L(\lambda x) = \bar{\lambda} L(x)$. A matrix A is called *Hermitian* if $\bar{A}^T = A$. A bilinear form is called *Hermitian* if it is represented by a Hermitian matrix A.

We will use the following notations to deal with complex analysis:

$$dz_j \doteq dx_j + idy_j \ , \qquad d\bar{z} \doteq dx_j - idy_j,$$

$$\frac{\partial}{\partial z_j} \doteq \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

$$\frac{\partial}{\partial \bar{z}_j} \doteq \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

Let $f : \Omega \to \mathbb{C}$ be a differentiable function at $a \in \Omega \subset \mathbb{C}^n$, then the ordinary differential $d_a f : \mathbb{R}^{2n} \to \mathbb{R}^2$ can be split into $\mathbb{C}$-linear part, $\partial_a f$ and the anti $\mathbb{C}$-linear part $\bar{\partial}_a f$:

$$d_a f = \partial_a f + \bar{\partial}_a f.$$

We obtain following formulas for $d_a f$, $\partial_a f$ and $\bar{\partial}_a f$;

$$d_a f = \sum_{j=1}^{n} (\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j)$$

$$\partial_a f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j$$

$$\bar{\partial}_a f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

We will use the multi-index notation in our dealing with several variables. Recall that a *multi-index* α is an element of $(\mathbb{Z})^n$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we will write

$$z^\alpha \doteq z_1^{\alpha_1} \ldots z_n^{\alpha_n},$$

$$\bar{z}^\alpha \doteq \bar{z}_1^{\alpha_1} \ldots \bar{z}_n^{\alpha_n},$$

$$\Big(\frac{\partial}{\partial z}\Big)^\alpha \doteq \Big(\frac{\partial}{\partial z_1}\Big)^{\alpha_1} \ldots \Big(\frac{\partial}{\partial z_n}\Big)^{\alpha_n}$$

$$\Big(\frac{\partial}{\partial \bar{z}}\Big)^\alpha \doteq \Big(\frac{\partial}{\partial \bar{z}_1}\Big)^{\alpha_1} \ldots \Big(\frac{\partial}{\partial \bar{z}_n}\Big)^{\alpha_n}$$

We let $\alpha! \doteq \alpha_1! \ldots \alpha_n!$ and $|\alpha| \doteq \alpha_1 + \cdots + \alpha_n$. We will use the following multi-index notation for partial derivatives

$$D^{(\alpha)}\phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \tag{2.1}$$

If $\alpha, \beta \in (\mathbb{Z})^n$, then $\alpha < \beta$ means $\alpha_i < \beta_i$, for all $1 \leq i \leq n$ (by the same way $\alpha \leq \beta$ is defined).

Let Ω be an open subset of $\mathbb{C}^n$, and let $u \in \mathcal{C}^2(\Omega)$. Then the matrix

$$\Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(a)\Big]_{i,j} \qquad 1 \leq i \leq n \tag{2.2}$$

is called the *complex Hessian of* u at a. It is clearly an Hermitian matrix. The transpose of this matrix is shown by $\mathcal{L}u(a)$. The *Levi form* of u at a,

$\mathcal{L}u(a) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is defined by

$$\langle \mathcal{L}u(a)b, c\rangle = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(a) b_j c_k, \tag{2.3}$$

where $b = (b_1, \dots, b_n),\ c = (c_1, \dots, c_n) \in \mathbb{C}^n$; and Levi form is a Hermitian form.

2.1.2 Complex Differential Forms

In $\mathbb{C}^n$ if $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+^k,\ k \leq n$, again using multi-index notation we define

$$dz^\alpha \doteq dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_k} \ , \quad d\bar{z}^\alpha \doteq d\bar{z}^{\alpha_1} \wedge \cdots \wedge d\bar{z}^{\alpha_k} \ ,$$

and

$$\#\alpha \doteq k.$$

Let's denote the set of all alternating *r-linear* maps from $\mathbb{C}^n$ to $\mathbb{C}$ by $\bigwedge^r(\mathbb{C}^n, \mathbb{C})$ then if $p, q \in \mathbb{Z}_+$ are such that $p + q = r$ then we will denote the complex subspace of $\bigwedge^r(\mathbb{C}^n, \mathbb{C})$ generated by

$$\{dz^{(\alpha_1,\dots,\alpha_p)} \wedge d\bar{z}^{(\beta_1,\dots\beta_q)} : 1 \leq \alpha_1 < \cdots < \alpha_p \leq n,\ 1 \leq \beta_1 < \cdots < \beta_q \leq n\},$$

by $\bigwedge^{p,q}(\mathbb{C}^n, \mathbb{C})$.

Definition 2.1.1. A *differential form* ω *of bidegree* (p, q) on an open set $\Omega \subset \mathbb{C}^n$ is a map from Ω to $\bigwedge^{p,q}(\mathbb{C}^n, \mathbb{C})$.

In an open set Ω we will denote the set of all differential forms of bidegree(p,q) whose coefficients belong to $\mathcal{C}_0(\Omega,\mathbb{C})$ (respectively, $\mathcal{C}_0^\infty(\Omega,\mathbb{C})$) by $\mathcal{D}_0^{p,q}(\Omega)$ (respectively, $\mathcal{D}^{p,q}(\Omega)$) .

An element $\omega \in \mathcal{D}^{p,q}(\Omega)$ can be written as

$$\sum_{\#\alpha=p}\sum_{\#\beta=q}\omega_{\alpha\beta}dz^\alpha\wedge d\bar{z}^\beta \ . \tag{2.4}$$

The integer $p+q$ is called the *degree* of differential form ω and every differential form of degree r can be written as a sum of differential forms of bidegree (p,q), where $p+q=r$.

A form ω in $\bigwedge^{p,p}(\Omega,\mathbb{R})$ is called *strongly positive* if it is of the form $\omega=\sum_{j=1}^m\lambda_j\frac{i}{2}\eta_1\wedge\bar{\eta}_1\wedge\cdots\wedge\frac{i}{2}\eta_p\wedge\bar{\eta}_p$ for some non-negative numbers $\lambda_1,\ldots,\lambda_m$ and for some forms $\eta_1,\ldots,\eta_m$ where η_j are linearly independent $\mathbb{C}$-linear mappings from $\mathbb{C}^n$ to $\mathbb{C}$.

The *volume form* on $\mathbb{C}^n$ is defined as follows:

$$dV(z)=\left(\frac{i}{2}\right)^n dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge\cdots\wedge dz_n\wedge d\bar{z}_n. \tag{2.5}$$

In real notation,

$$\begin{aligned}dV(z)&=(\frac{i}{2}\Big)^n(2idx_1\wedge dy_1)\wedge\cdots\wedge(2idx_n\wedge dy_n)\\&=dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n,\end{aligned}$$

which is the standard volume form in $\mathbb{R}^{2n}$. We remark that the volume form on $\mathbb{C}^n$ is a strongly positive form.

If ω is a differential form of bidegree (p,q) as in (2.4), we will use the well-known operators d, ∂ and $\bar{\partial}$ defined on differential forms of bidegree (p,q). Recall that for $0 \le p, q \le n$,

$$\begin{aligned} d\omega &= \textstyle\sum_{\alpha,\beta} d\omega_{\alpha\beta} \wedge dz^{\alpha} \wedge d\bar{z}^{\beta}, \\ \partial\omega &= \textstyle\sum_{\alpha,\beta} \partial\omega_{\alpha\beta} \wedge dz^{\alpha} \wedge d\bar{z}^{\beta}, \\ \bar{\partial}\omega &= \textstyle\sum_{\alpha,\beta} \bar{\partial}\omega_{\alpha\beta} \wedge dz^{\alpha} \wedge d\bar{z}^{\beta}. \end{aligned} \tag{2.6}$$

The forms $\partial\omega$ and $\bar{\partial}\omega$ are of bidegree$(p+1,q)$ and $(p,q+1)$, respectively. We remark that

$$d = \partial + \bar{\partial}.$$

Now we will define another very important operator d^c

$$d^c \doteq i(\bar{\partial} - \partial). \tag{2.7}$$

This definition of d^c is not standard. Some authors such as [10] use

$$d^c = \frac{i}{2\pi}(\bar{\partial} - \partial). \tag{2.8}$$

We will use the first definition. Note that

$$dd^c = 2i\partial\bar{\partial}, \tag{2.9}$$

and, if $u \in \mathcal{C}^2(\Omega)$, then

$$dd^c u = 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k. \tag{2.10}$$

From this formula and the fact that for any $b, c \in \mathbb{C}^n$,

$$dz_j \wedge d\bar{z}_k(b,c) = b_j \bar{c}_k - \bar{b}_k c_j,$$

we deduce that

$$(dd^c u)(a)(b,c) = -4\mathrm{Im} < \mathcal{L}u(a)b, c > \tag{2.11}$$

for $a \in \Omega$ and $b, c \in \mathbb{C}^n$.

2.2 Distributions and Currents

In classical differential calculus historically there were some difficulties due to existence of functions which are not differentiable. In 1945 L. Schwartz introduced the theory of distributions in his paper "Généralisation de la notion, de dérivation, de transformation de Fourier et applications mathématiques et physiques " appeared in Annale de l'Université de Gronoble. By his great work, theory of distributions allowed us to extend differentiability properties to a more general class of functions. Currents play a similar role for differential forms. The notion of currents in its primitive form was first introduced by de Rham in his papers in 1929 "Intégrales Multiples et Analysis Situs" appearing in Comptes Rendus des Séances de l'Académie des Sciences and in 1931 "Sur l'Analysis Situs des Variétés à n Dimensions" in Journal de Mathématiques Pures et Appliquées. Note that this is earlier then the appearance of the theory of distributions. After Schwartz introduced distributions de Rham adopted this concept to get a more general and best suiting theory of currents in his paper in 1950 "Intégrales harmoniques et théorie des intersections" in Proceedings of the international Congress of Mathematicians and in his lecture "Harmonic Integrals" delivered at the Institute for Advanced Study in Princeton. Later while dealing with complex *Monge-Ampére operator* we will use his theory of currents. Therefore, here we want to give a short review of test functions, distributions and currents.

Definition 2.2.1. Let Ω be a open subset of $\mathbb{C}^n$ (or $\mathbb{R}^n$), then the space of *test functions* $\mathcal{D}(\Omega)$ is the vector space of functions ϕ of class $\mathcal{C}^\infty$, each of which has compact support.

We give a topolgy to the space $\mathcal{D}(\Omega)$ which gives following notion of convergence of sequences: A sequence $\phi^m \in \mathcal{D}(\Omega)$ *converges in* $\mathcal{D}(\Omega)$ *to the function* $\phi \in \mathcal{D}(\Omega)$ if there exists some fixed compact set $K \subset \Omega$ such that the supports of $\phi^m - \phi$ are in K for all m and, for each choice of $\alpha = (\alpha_1, \ldots, \alpha_n)$

$$\frac{\partial^{|\alpha|}\phi^m}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \longrightarrow \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \tag{2.12}$$

as $m \to \infty$, *uniformly on* K.(See [20, p. 128].)

Definition 2.2.2. A *distribution* T is a continuous linear functional on $\mathcal{D}(\Omega)$, and whenever $\phi^n \in \mathcal{D}(\Omega)$ and $\phi^n \to \phi$ in $\mathcal{D}(\Omega)$ then $T(\phi^n) \to T(\phi)$. The space of *distributions* is the topological dual of $\mathcal{D}(\Omega)$ equipped with the $w*$-topology, and shown by $\mathcal{D}'(\Omega)$. In other words a sequence of distributions $T^j \in \mathcal{D}'(\Omega)$ *converges in* $\mathcal{D}'(\Omega)$ to $T \in \mathcal{D}'(\Omega)$ if, for every $\phi \in \mathcal{D}(\Omega)$, $T^j(\phi)$ converge to $T(\phi)$.

Example 2.2.3. Let f be a function in $L^1_{loc}(\Omega)$ any $\phi \in \mathcal{D}(\Omega)$ and define

$$T_f(\phi) := \int_\Omega f\phi dV,$$

For $\phi^m \to \phi$, suppose K is the compact set which contains the supports of $\phi^m - \phi$ then we have

$$\begin{aligned}\Big|T_f(\phi) - T_f(\phi^m)\Big| &= \Big|\int_\Omega (\phi(x) - \phi^m(x))f(x)dx\Big| \\ &\leq \sup_{x\in K} |\phi(x) - \phi^m(x)| \int_K |f(x)|dx,\end{aligned}$$

which tends to zero as $m \to \infty$. Moreover clearly T_f is linear hence, T_f defines a distribution.

If a distribution T is given by $T_f(\phi) := \int_\Omega f\phi dx$ for some $f \in L^1_{loc}(\Omega)$, then we will identify T_f with f. This identification makes sense since $T_f = T_g$ if and only if $f = g$ a.e.

Definition 2.2.4. A distribution T is said to be *of order* r if for every sequence ϕ_n which satisfies

- the supports of ϕ_n are all contained in a fixed compact set $K \in \Omega$
- $\sup_x |D^{(\alpha)}\phi_n(x)| =\longrightarrow 0$, as $n \to \infty$ for all indices (α) such that $|\alpha| \leq r$.

We have $T(\phi_n) \to 0$. (See [18, p. 1].)

If T is finite order r, then T can be extended to the space $\mathcal{C}_0^r(\Omega)$ of functions of class $\mathcal{C}^r$ with compact support in Ω ; when $\mathcal{C}_0^r(\Omega)$ is given the topology of uniform convergence on compact subsets of Ω with all the derivatives of order less then or equal to r.

A *linear operator* is positive if $l(\phi) \geq 0$ for all $\phi \in L$ such that $\phi \geq 0$. Similarly a distribution T is called positive if $T(\phi) \geq 0$ for all $\phi \in \mathcal{D}(\Omega)$ such that $\phi(x) \geq 0$ for all $x \in \Omega$.

Consider a positive distribution T then we can consider it as a positive linear operator on the space $\mathcal{C}_0^0(\Omega)$ can be regarded as a positive measure by Riesz Representation Theorem.(For Riesz Representation Theorem see [20, p. 55])

Definition 2.2.5. Let T be in $\mathcal{D}'(\Omega)$ and let $\alpha_0, \ldots, \alpha_n$ be positive integers such that $\alpha = (\alpha_1, \ldots, \alpha_n)$. *Distributional* or *weak derivative* of order α of $T, D^{(\alpha)}T$, is defined by its action on each test function $\phi \in \mathcal{D}(\Omega)$ as follows :

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi) \tag{2.13}$$

(See [20, p. 131])

Remark 2.2.1. Let f is a $\mathcal{C}^{|\alpha|}(\Omega)-$function, then the weak derivative agrees with the classical derivative. Since if $G_i := \frac{\partial f}{\partial x_i}$ be distributional derivative of f, then

$$G_i(\phi) = (\frac{\partial f}{\partial x_i})(\phi) = (-1)^{|i|}\int_{\Omega}(\frac{\partial \phi}{\partial x_i})f = (-1)^{|i|}\int_{\Omega}\phi(\frac{\partial f}{\partial x_i})$$

where the first and second equalities are by definition of distributional derivative and the third is due to integration by parts formula and ϕ has compact support and f has continuous partial derivatives.

After this short introduction to distributions, we will now define currents.

Definition 2.2.6. A *current* t of bidegree (p, q) on an open subset Ω of $\mathbb{C}^n$ is a linear functional t which is defined and continuous on the space of differential $(n-p, n-q)$-forms with infinitely differentiable coefficients and compact support. Here continuity means: If ϕ_n are differential $(n-p, n-q)$-forms with smooth coefficients and $\phi_n \to 0$ i.e.

- supports $supp(\phi_n)$ are contained in a fixed compact set $K \subset \Omega$
- for each coefficient $\phi_{n,(i)}$ of ϕ_n, and each multi-index (α), $D^{(\alpha)}\phi_{n,(i)} \to 0$ uniformly on K.

then $t(\phi_n) \to 0$.

Namely $t(\phi_n) \to 0$ if the coefficients of ϕ_n's converge to 0 in $\mathcal{D}'(\Omega)$.

A current is a differential $(p, q)-$form with distribution coefficients. In this case we call the current is of *bidegree* (p, q). (See [15, p. 107].)

Example 2.2.7. Let α be a homogenous differential form of bidegree (s, r) with continuous coefficients. Then

$$\alpha(\phi) \doteq \int_{\Omega} \alpha \wedge \phi$$

for smooth differential form ϕ of degree $(n-s, n-r)$ with compact support defines a current of bidegree (s, r).

Example 2.2.8. Let S be a k dimensional compact submanifold of Ω. Then current of integration $[S]$ is defined by

$$[S](\phi) \doteq \int_{S} \phi$$

for smooth differential form ϕ of degree k with compact support defines a current of degree k.

A current t is said to be *of order* r if it has the special property that $t(\phi_n) \to 0$ for every sequence ϕ_n which satisfies,

- the supports of ϕ_n are all contained in a fixed compact set $K \in \Omega$
- $\sup_x |D^{(\alpha)}\phi_n(x)| = m_n^{(\alpha)} \to 0$, as $n \to +\infty$ for all indices (α) such that $|\alpha| \leq r$.

A current t of bidegree (p, p) is said to be *(weakly) positive* if for every choice of smooth $(1, 0)$-forms $\alpha_1, \dots, \alpha_p$ on Ω

$$t \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \bar{\alpha}_p \tag{2.14}$$

is a positive measure, i.e. positive multiple of dV.

A *Radon measure* μ on Ω is by definition a Borel measure such that for any compact set $K \subset \Omega$ we have $\mu(K) < \infty$.

In particular a distribution of order zero on Ω can be identified by a Radon measure.

Let us denote all continuous functions on Ω with compact support by $\mathcal{C}_0(\Omega)$, and let ϕ be continuous linear functional on Ω. Then by Riesz Representation theorem there exists a unique Borel measure μ on Ω, such that

$$\phi(\varphi) = \int_\Omega \varphi d\mu.$$

Since φ has compact support; μ becomes a Radon measure.

For any Radon measure μ on Ω we can associate a positive linear functional Λ on $\mathcal{C}_0(\Omega)$, by

$$\Lambda(\varphi) = \int_\Omega \varphi d\mu.$$

Hence we can identify all Radon measures by the positive linear functionals on $\mathcal{C}_0(\Omega)$.

We remark that (n, n)-currents are just distributions on Ω, and a $(0, 0)$-currents of order 0 are complex measures.

We endow $(\mathcal{C}_0(\Omega))'$ with *weak**-topology: in this topology, $\mu_j \to \mu$ as $j \to \infty$ if $\mu_j(\phi) \to \mu(\phi)$ for each $\phi \in \mathcal{C}_0(\Omega)$.

We will denote the class of currents on Ω of bidegree (m, m) and order 0 by $M_m(\Omega)$. Those are differential forms of bidegree (m, m) with Borel measure coefficients endowed with weak topology.

2.3 Plurisubharmonic Functions

2.3.1 Basic Definitions and Some Basic Facts

We want to start with some basic definitions from classical potential theory.

Definition 2.3.1. Let $h \in \mathcal{C}^2(\Omega)$ defined on an open subset of $\mathbb{R}^n$ is called a *harmonic function* if it satisfies the homogeneous Laplace equation:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \equiv 0 \quad in\ \Omega. \tag{2.15}$$

The space of harmonic functions on a domain Ω form a vector space since the Laplace operator is linear. This space is denoted by $H(\Omega)$.

In particular if $\Omega \subset \mathbb{C}^n$ then,

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} &= \sum_{i=1}^{n} \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\Big)\Big(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\Big) f \\ &= \frac{1}{4}\sum_{i=1}^{n}\Big(\frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial y_i^2}\Big) = \frac{1}{4}\Delta f\end{aligned}$$

As a consequence we have the following important example.

Example 2.3.2. Let f be analytic function in a domain $\Omega \subset \mathbb{C}^n$ then $\mathrm{Re} f(z)$ and $\mathrm{Im} f(z)$ are harmonic.

Observe that $n = 1 \quad 4dd^c u = \Delta u dz \wedge d\bar{z}$ for $u \in \mathcal{C}^2(\Omega)$

Theorem 2.3.3. *A continuous function f is harmonic in Ω if and only if*

$$u(y) = \frac{1}{\sigma(\partial B(y,R))} \int_{B(y,R)} u(x) d\sigma(x) \tag{2.16}$$

for any $B(y,R)$ such that $\overline{B(y,R)} \subset \Omega$, where σ is the usual Lebesgue measure on the sphere.

Proof. See [15, p. 30]. □

In particular for $\Omega \subset \mathbb{C}$ then 2.16 reduces to

$$f(y) = \frac{1}{2\pi} \int_0^{2\pi} f(y + re^{i\theta}) d\theta \tag{2.17}$$

Theorem 2.3.4. *Let u be a harmonic function in Ω and continuous in $\bar{\Omega}$, where Ω is a bounded domain in $\mathbb{C}^n$ (in $\mathbb{R}^n$). Then either u is constant or*

$$u(x) < \sup_{y \in \partial\Omega} u(y) \qquad (x \in \Omega). \tag{2.18}$$

Proof. Let $\alpha = \sup_{y \in \partial\Omega} u(y)$. Now we define $A = u^{-1}(\alpha)$. A is closed in Ω since u is continuous. If A is not empty then we will show that A is open. Since Ω is connected then A will be whole Ω. Let $a \in A$, $r > 0$, and $\overline{B(a,r)} \subset \Omega$. If there exists $b \in B(a,r) \setminus A$, then the function u is strictly less then $u(a) = \alpha$ in a neighborhood of b. Then let $| b - a |= s$ then

$$\alpha = u(a) = \frac{1}{\sigma(\partial B(y,s))} \int\limits_{\partial B(y,s)} u(x) d\sigma(x) < \frac{1}{\sigma(\partial B(y,s))} \int\limits_{\partial B(y,s)} \alpha d\sigma(x) = \alpha$$

which gives a contradiction. Therefore $B(a,r) \subset \Omega$. Hence A is open, and so $A = \Omega$. □

The *classical Dirichlet problem* for a given continuous function on the boundary of a bounded domain $\phi : \partial\Omega \to \mathbb{R}$ is finding a function $u \in H(\Omega) \cap \mathcal{C}(\partial\Omega)$ such that $\lim_{z\to\zeta, z\in\Omega} \phi(z) = \phi(\zeta)$ for all $\zeta \in \partial\Omega$.

If Ω is the unit disc in the complex plane we can solve the Dirichlet problem for any continuous function on the boundary. In fact we have more, we have solution for any integrable function on the boundary:

Theorem 2.3.5. *[21, p. 88] Let $\phi : \partial\Delta \to \mathbb{R}$ is a Lebesgue integrable function, then its Poisson integral defined by*

$$P\phi(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2rcos(\theta - t) + r^2}\phi(z + e^{i\theta})d\theta. \tag{2.19}$$

is harmonic in Ω, and if ϕ is continuous then $\lim_{z\to\zeta} P\phi(z) = \phi(\zeta)$ for all $\zeta \in \partial\Delta$.

Definition 2.3.6. Let X be a topological space. A function $g : X \to [-\infty, \infty)$ is called *upper semicontinuous* if the set $\{x \in X : g(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$.

If X is a metric space then g is upper semicontinuous if and only if

$$\limsup_{y\to x} g(y) = g(x), \quad (x \in X)$$

where $\limsup_{y\to x} g(y)$ is defined as follows:

$$\limsup_{y\to x} g(y) = \inf_{\delta>0} \sup\{f(y) : y \in (B(x,\delta)\cap\Omega)\}.$$

Definition 2.3.7. Let X be a topological space, let $g : X \to [-\infty,\infty)$ be a function which is locally bounded above on X. Its *upper semicontinuous -regularization* $g^* : X \to [-\infty,\infty)$ is defined by;

$$g^* := \limsup_{y\to x} g(y) = \inf_N (\sup_{y\in N} g(y)) \quad (x \in X)$$

the infimum being taken over all neighborhoods N of x.

It is obvious that g^* is an upper semicontinuous function on X such that $g^* \geq g$, and also it is the smallest such function.

In particular, a function $g : X \to [-\infty,\infty)$ is upper semicontinuous if and only if it coincides with its upper semicontinuous regularization.

Definition 2.3.8. An upper semicontinuous function $u : \Omega \to [-\infty,\infty)$ which is not identically $-\infty$ is called *subharmonic* if

$$u(y) \leq \frac{1}{\sigma(\partial B(y,R))} \int_{\partial B(y,R)} u(x)d\sigma(x)$$

for any $B(y,R)$ such that $\overline{B(y,R)} \subset \Omega$, where σ is the usual Lebesgue measure on the sphere.

Definition 2.3.9. An upper semicontinuous function g on a domain D

which is not identically equal to $-\infty$ is called *plurisubharmonic* if $a \in D$ and $b \in \mathbb{C}^n$, the function $\lambda \to g(a+\lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in D\}$. Namely g is plurisubharmonic if and only if restriction of g to any complex line is subharmonic. The set of all plurisubharmonic functions on D is denoted by $\mathcal{PSH}(D)$.

Definition 2.3.10. A function h is called *pluriharmonic* if both h and $-h$ are plurisubharmonic.

Theorem 2.3.11. *Let $g : \Omega \to \mathbb{R}$ be a $\mathcal{C}^2$ function on an open set Ω. Then g is plurisubharmonic if and only if $\langle \mathcal{L}g(a)b, b\rangle \geq 0$ for all $a \in \Omega, b \in \mathbb{C}^n$.*

Proof. $\Rightarrow$:Fix $a \in \Omega$ and $b \in \mathbb{C}^n$. Consider the function $h(\lambda) = g(a + \lambda b)$. Since g is plurisubharmonic, h is subharmonic. Therefore $\Delta h(\lambda)|_{\lambda=0} = \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial g^2 u}{\partial z_j \partial \bar{z}_j}(a) b_j \bar{b}_k \geq 0$

$\Leftarrow$: Conversely assume $\langle \mathcal{L}u(a)b, b\rangle \geq 0$ for all $a \in \Omega, b \in \mathbb{C}^n$, then let $h(\lambda) = g(a + \lambda b)$. Then $\Delta h(\lambda)|_{\lambda=0} = \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial g^2 u}{\partial z_j \partial \bar{z}_j}(a) b_j \bar{b}_k$ which is nonnegative by the assumption. Hence, h is subharmonic. Thus g is plurisubharmonic. □

Corollary 2.3.12. *A function f is pluriharmonic if and only if for any $b \in \mathbb{C}^n$ we have $\langle \mathcal{L}f(a)b, b\rangle = 0$.*

Proof. Since f and $-f$ are plurisubharmonic we have the conclusion by 2.3.11. □

In fact we will also show that this characterization is still valid for non-smooth plurisubharmonic functions where the derivatives are taken in the sense of distributions in Theorem 2.3.16 below. Now we have another characterization in terms of integral means.

Proposition 2.3.13. *Let $g : \Omega \to [\infty, \infty)$ be upper semicontinuous and not identically $-\infty$ on any connected component of $\Omega \subseteq \mathbb{C}^n$. Then $g \in \mathcal{PSH}(\Omega)$ if and only if for each $a \in \Omega$ and $b \in \mathbb{C}^n$ such that*

$$\{a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subseteq \Omega$$

we have

$$g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{it}b)dt. \tag{2.20}$$

Proof. $\Rightarrow$:Let g be plurisubharmonic . We define $h(\lambda) = g(a + \lambda b)$ which is subharmonic since g is plurisubharmonic .

$$h(a) \leq \frac{1}{2\pi} \int_0^{2\pi} h(a + e^{it}b)dt = \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{it}b)dt$$

where the second inequality is due to h subharmonic. Hence g is plurisubharmonic.

$\Leftarrow$: Let $g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{it}b)dt$. Then for all $a \in \Omega$ and $b \in \mathbb{C}^n$ such that $\{a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subseteq \Omega$ we define $h(\lambda) = g(a + \lambda b)$ and

$$h(a) = g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{it}b)dt = \frac{1}{2\pi} \int_0^{2\pi} h(a + e^{it}b)dt.$$

Hence h is subharmonic, and g is plurisubharmonic. ∎

Now we want to give some theorems about smoothing of plurisubharmonic functions. If $u, v \in L^1(\mathbb{R}^n)$, then the *convolution* u^*v of u and v is defined by formula

$$(u^*v)(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dV(y).$$

It is easy to see that $u^*v = v^*u$ by a change of variable formula of advanced calculus. Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by the formula

$$h(t) = \begin{cases} exp(-1/t) & (t > 0), \\ 0 & (t \leq 0). \end{cases} \tag{2.21}$$

It is elementary fact that $h \in \mathcal{C}^\infty(\mathbb{R})$. Now we define

$$\chi(x) = \ h(1 - \|x\|^2)/K \quad (x \in \mathbb{R}^n)$$

where

$$K = \left(\int_{B(0,1)} h(1 - \|x\|^2)dV(x) \right).$$

It is obvious that $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $supp\chi = \overline{B}(0,1)$, and $\int_{\mathbb{R}^n} \chi(x)dV(x) = 1$. Since $\chi(x)$ depends only on $r = \|x\|$, we will use $\chi(r)$ instead of $\chi(x)$.

For $\varepsilon > 0$ we define

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi(\frac{x}{\varepsilon})$$

The functions χ_ε are often called *standard smoothing kernels.*

Let $\Omega \subset \mathbb{R}^n$ be open. If $\Omega \neq \mathbb{R}^n$, we set

$$\Omega_\varepsilon = \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$$

for $\varepsilon > 0$. If $\Omega = \mathbb{R}^n$, we set $\Omega_\varepsilon = \mathbb{R}^n$ for $\varepsilon > 0$.

Proposition 2.3.14. *Let Ω be an open subset of $\mathbb{C}^n$, and $u \in L^1_{loc}(\Omega)$. Suppose that $a \in \Omega, b \in \mathbb{C}^n$, and $\{a + \lambda b : \lambda \in \mathbb{C}, \mid \lambda \mid \leq 1\} \subset \Omega$. Then*

$$\left((\frac{1}{2\pi}\int_0^{2\pi} g(x + e^{it}b)dt)^* \chi_\varepsilon)\right|_{x=a} = \frac{1}{2\pi}\int_0^{2\pi}(g(a + e^{it}b)^*\chi_\varepsilon)dt \tag{2.22}$$

Proof.

$$\left((\frac{1}{2\pi}\int_0^{2\pi} g(x + e^{it}b)dt)^* \chi_\varepsilon)\right|_{x=a} = \int_{\mathbb{C}^n}\left(\frac{1}{2\pi}\int_0^{2\pi} g(a + e^{it}b - w)dt\right)\chi_\varepsilon(w)dV(w)$$

which equals to

$$\int_0^{2\pi}\left(\frac{1}{2\pi}\int_{\mathbb{C}^n} g(a + e^{it}b - w)dt\right)\chi_\varepsilon(w)dV(w)$$

by Fubini Theorem and this equals to

$$\frac{1}{2\pi}\int_0^{2\pi}(g(a + e^{it}b)^*\chi_\varepsilon)dt.$$

□

Theorem 2.3.15. *Let $\Omega \subseteq \mathbb{C}^n$ be an open set, let $g \in \mathcal{PSH}(\Omega)$. If $\varepsilon > 0$ is* *such that $\Omega_\varepsilon \neq \emptyset$, then $u^*\chi_\varepsilon \in \mathcal{C}^\infty \cap \mathcal{PSH}(\Omega_\varepsilon)$. Moreover, $u^*\chi_\varepsilon$ monotonically decreases with decreasing ε, and $\lim_{n\to 0} u^*\chi_\varepsilon(z) = u(z)$ for each $z \in \Omega$.*

Proof. Observe that $u^*\chi_\varepsilon$ is smooth since

$$u^*\chi_\varepsilon = \chi_\varepsilon^* u = \int_{\mathbb{C}^n} \chi_\varepsilon(x-y)u(y)dV(y)$$

and $\chi_\varepsilon(x)$ is smooth and differentiation under integral sign. The function $u^*\chi_\varepsilon$ is plurisubharmonic since by Proposition 2.3.14 we have:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u^*\chi_\varepsilon(a+e^{it}b)dt &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_{\mathbb{C}^n} u(a+e^{it}b-w)\chi_\varepsilon(w)dw\Big)dt \\
&= \int_{\mathbb{C}^n}\Big(\frac{1}{2\pi}\int_0^{2\pi} u(a+e^{it}b-w)\chi_\varepsilon(w)dt\Big)dw \\
&\geq \int_{\mathbb{C}^n} u(a-w)\chi_\varepsilon(w)dw \\
&= u^*\chi_\varepsilon(a).
\end{aligned}$$

Third inequality comes from the fact that u is plurisubharmonic, others are directly from definitions. Hence $u^*\chi_\varepsilon$ is plurisubharmonic. For the rest of the proof we refer to [15, p. 44]. □

Theorem 2.3.16. *If $\Omega \subset \mathbb{C}^n$ is open and $u \in \mathcal{PSH}(\Omega)$, then for each $b = (b_1, \dots, b_n) \in \mathbb{C}^n$, we have*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} b_j \bar{b}_k \geq 0 \tag{2.23}$$

in Ω, in the sense of distributions , i.e.

$$\int_\Omega u(z)\langle \mathcal{L}\phi(z)b, b\rangle dV(z) \geq 0 \tag{2.24}$$

for any non-negative test function $\phi \in \mathcal{C}_0^\infty(\Omega)$. Conversely, if $v \in L^1_{loc}(\Omega)$ is such that for each $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \bar{z}_k} b_j \bar{b}_k \geq 0 \tag{2.25}$$

in Ω, in the sense of distributions, then the function $u = \lim_{\varepsilon\to 0}(v^*\chi_\varepsilon)$ exists, plurisubharmonic in Ω, and is equal to v almost everywhere in Ω.

Proof. Let $u \in \mathcal{PSH}(\Omega)$, and let $u_\varepsilon = u^*\chi_\varepsilon$ for $\varepsilon > 0$. For any non-negative test function $\phi \in \mathcal{C}_0^\infty$ and $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ we have,

$$\begin{aligned}\int_\Omega u(z)\langle \mathcal{L}\phi(z)b, b\rangle dV(z) &= \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(z)\langle \mathcal{L}\phi(z)b, b\rangle dV(z)\\ &= \lim_{\varepsilon\to 0}\int_\Omega (z)\phi(z)\langle \mathcal{L}u_\varepsilon b, b\rangle dV(z) \geq 0\end{aligned}$$

where the first equality is by Lebesgue's dominated convergence theorem and the second equation is obtained by using integration by parts formula twice for smooth functions and ϕ has compact support. It is positive since u_ε is plurisubharmonic and smooth by 2.3.11.

Conversely assume $v \in L^1_{loc}(\Omega)$, and $\int_\Omega v(z)\langle \mathcal{L}\phi(z)b, b\rangle dV(z) \geq 0$. Let $v_\varepsilon = v^*\chi_\varepsilon$ for $\varepsilon > 0$. Then $\int_\Omega v_\varepsilon(z)\langle \mathcal{L}\phi(z)b, b\rangle dV(z) \geq 0$. Therefore, v_ε is plurisubharmonic in the sense of distributions. Moreover v_ε is smooth hence, it is plurisubharmonic in usual sense. For $\varepsilon_2 > \varepsilon_1 > 0$ and $x \in \Omega$, we have

$$\begin{aligned} v_{\varepsilon_2}(x) &= \lim_{\delta\to 0}(v^*\chi_{\varepsilon_2})^*\chi_\delta(x) = \lim_{\delta\to 0}(v^*\chi_\delta)\chi_{\varepsilon_2}(x) \\ &\geq \lim_{\delta\to 0}(v^*\chi_\delta)\chi_{\varepsilon_1}(x) = \lim_{\delta\to 0}(v^*\chi_{\varepsilon_1})^*\chi_\delta(x) = v_{\varepsilon_1}(x). \end{aligned}$$

Remark that v_{ε_n} are plurisubharmonic. □

Definition 2.3.17. A set $E \subset \Omega$ is called *polar* set if for each point $a \in E$ there is a neighborhood V of a and a function $u \in \mathcal{SH}(\Omega)$ such that $E \cap V \subset \{z \in V : u(z) = -\infty\}$.

Definition 2.3.18. A set $E \subset \Omega$ is called *pluripolar* set if for each point $a \in E$ there is a neighborhood V of a and a function $u \in \mathcal{PSH}(\Omega)$ such that $E \cap V \subset \{z \in V : u(z) = -\infty\}$.

Remark that polar and pluripolar sets have Lebesgue measure zero [15, p. 41].

2.3.2 Maximal Plurisubharmonic Functions

Definition 2.3.19. A *maximal plurisubharmonic* function is a plurisubharmonic function f on an open set Ω such that for any relatively compact open subset G of Ω and any upper semicontinuous function g defined on $\bar{G}$, and plurisubharmonic on G such that $g \leq f$ on the boundary of G , then $g \leq f$ in G.

We will use the symbol $\mathcal{MPSH}(\Omega)$ to denote the space of maximal plurisubharmonic functions on Ω.

Proposition 2.3.20. *Let Ω be an open subset $\mathbb{C}^n$, and let $f : \Omega \longrightarrow \mathbb{R}$ be a plurisubharmonic function. Then the following are equivalent:*

1. *for every relatively compact open subset G of Ω and every function $g \in \mathcal{PSH}(G)$, if $\liminf_{z\to\xi}(f(z) - g(z)) \geq 0$ for all $\xi \in \partial G$, then $f \geq g$ in G;*

2. *for $g \in \mathcal{PSH}(\Omega)$ and for each $\varepsilon \geq 0$ there exists a compact set $K \subset \Omega$ such that $f - g \geq -\varepsilon$ in $\Omega \setminus K$, then $f \geq g$ in Ω;*

3. *for $g \in \mathcal{PSH}(\Omega)$, G is a relatively compact open subset of Ω, and $f \geq g$ on ∂G, then $f \geq g$ on G;*

4. *for $g \in \mathcal{PSH}(\Omega)$, G is a relatively compact open subset of Ω, and $f \geq g$ on ∂G, and for each $\xi \in \partial G$,*

$$\liminf_{z\to\xi z\in G}(f(z) - g(z)) \geq 0,$$

   then $f \geq g$ in G;

5. *f is maximal*

Proof. See [15, p. 88]. □

Proposition 2.3.21. *[15, p. 92] Let Ω be a bounded open subset of $\mathbb{C}^n$, and let u, v be $\mathcal{C}^2-$plurisubharmonic functions in a neighborhood of $\overline{\Omega}$. If $v \leq u$ on $\partial\Omega$ and*

$$\det\Big[\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}\Big]_{1\leq j,k\leq n} \geq \det\Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\Big]_{1\leq j,k\leq n} \qquad in \quad \Omega,$$

then $v \leq u$ in Ω.

Proof. For $\varepsilon > 0$, we define

$$v_\varepsilon(z) = v(z) + \varepsilon(\|z\|^2 - \sup_{w\in\partial\Omega}\|w\|^2).$$

Now v_ε is plurisubharmonic since $\langle \mathcal{L}v_\varepsilon(a)b, b\rangle \geq 0$ since v is plurisubharmonic and $(\varepsilon(\|z\|^2 - \sup_{w\in\partial\Omega}\|w\|^2))$ is positive. Then

$$0 < \det\Big[\frac{\partial^2 v_\varepsilon}{\partial z_j \partial \bar{z}_k}\Big] - \Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\Big] = \int_0^1 \frac{d}{dt}\det\Big[\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(tv_\varepsilon + (1-t)u)\Big]dt$$

$$= \int_0^{2\pi}\Big(\sum_{j,k=1}^{n} A_t^{jk}\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(v_\varepsilon - u)\Big)dt = \sum_{j,k=1}^{n} B^{jk}\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(v_\varepsilon - u),$$

where $[A_t^{jk}]$ is the cofactor matrix of

$$\Big[\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(tv_\varepsilon + (1-t)u)\Big]_{1\leq j,k\leq n}$$

and $[B^{jk}]$ is its integral with respect to t. Hence $[B^{jk}]$ is positive definite, and hence $v_\varepsilon - u$ has no local maximum in Ω. Thus $v_\varepsilon \leq u$ in Ω and, as ε tends to 0, we get the result. $\square$

Theorem 2.3.22. *[15, p. 93] Let $g \in \mathcal{C}^2(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is open. Then $g \in \mathcal{MPSH}(\Omega)$ if and only if*

$$\det \left[\frac{\partial^2 g}{\partial z_j \partial \bar{z}_k}\right]_{1\leq j,k\leq n} \equiv 0 \quad \textit{in}\ \Omega$$

Proof. $\Rightarrow$: Assume that u is maximal and $\det \left[\frac{\partial^2 g}{\partial z_j \partial \bar{z}_k}\right]_{j,k}$ is not identically 0 in Ω. Then there exists $w \in \Omega$ such that for every $b \in \mathbb{C}^n \backslash \{0\}$, $\langle \mathcal{L}u(w)b, b\rangle > 0$. Since it is twice continuously differentiable, there exists a closed ball $\overline{B(a,r)} \subset \Omega$ such that for every $z \in \overline{B(a,r)}$ and $\langle \mathcal{L}u(z)b, b\rangle > 0$. Thus for some $c > 0, \langle \mathcal{L}u(z)b, b\rangle \geq c\mid b\mid^2$ for every $z \in \overline{B(a,r)}$ and $b \in \mathbb{C}^n \backslash 0$. Now if we define

$$v(z) = \begin{cases} u(z) & (z \in \Omega \backslash \overline{B(a,r)}), \\ u(z) + c(r^2 - \mid z-a\mid^2) & (z \in \overline{B(a,r)}) \end{cases}$$

Because of our choice of c, v is plurisubharmonic $\langle \mathcal{L}(u(z) + c(r^2 - \mid z-a\mid^2))(z)b, b\rangle \geq 0$ then we have $u = v$ on $\partial B(a,r)$ and $v(a) > u(a)$, which contradicts maximality of u.

$\Leftarrow$: Let G be a relatively compact open subset of Ω, and let $v \in \mathcal{PSH}(\Omega)$ such that $v \leq u$ on $\partial\Omega$. We apply Proposition 2.3.21 to $(v-\delta)^*\chi_\varepsilon$ instead of v (where $\delta > 0$, $\varepsilon > 0$ are sufficiently small that we can apply the proposition i.e. since G is relatively compact there exists $\varepsilon > 0$ such that $G \subset \Omega_\varepsilon$), u and G. We conclude that $(v-\delta)^*\chi_\varepsilon \leq u$ in G. We get $v \leq u$ as $\delta, \varepsilon \to 0$. □

2.4 Classical Hardy Spaces

We will begin by the definitions of Hardy Spaces on the unit disc and collect some results from this theory.

2.4.1 Hardy Spaces on the Unit Disc

$\Delta = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C}$ is the unit disc in $\mathbb{C}$. For $1 \leq p < \infty$ then Hardy Spaces are defined as follows :

$$H^p(\Delta) = \{f \in O(\Delta) : \sup_{0<r<1} (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}} < \infty\}. \tag{2.26}$$

One also can define

$$H^\infty(\Delta) = \{f \in O(\Delta) : \sup_{z\in\Delta} |f(z)| < \infty\} \text{ for } p = \infty \ . \tag{2.27}$$

For $1 \leq p < \infty$ we equip $H^p(\Delta)$ spaces with the following norms :

$$\|f\|_p := \sup_{0<r<1} (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}} \tag{2.28}$$

and for $H^\infty(\Delta)$, as usual, we define:

$$\|f\|_\infty := \sup_{z\in\triangle} |f(z)| \tag{2.29}$$

Remark 2.4.1. We will point out the following facts concerning H^p spaces :

- $\| \ \|$ is a norm on $H^p(\Delta)$ making it a Banach Space
- The inclusion map $i : H^p(\Delta) \to L^p(\Delta, \mu)$ is a continuous imbedding. This follows from:

$$\left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right) r dr\right)^{\frac{1}{p}} \leq \frac{1}{2^p}\|f\|_p$$

- $H^2(\Delta)$ is a Hilbert space with inner product

$$\langle f, g\rangle \doteq \sup_r \frac{1}{2\pi}\left(\int_0^\pi f(re^{i\theta}) g(re^{-i\theta}) d\theta\right)^{\frac{1}{2}}$$

  it is not difficult to express this inner product in terms of the Taylor series expansions of f and g:
  $\langle f, g\rangle = \sum_{i=1}^{\infty} a_n \bar{b}_n$ where $f(z) = \sum_{i=1}^{\infty} a_n z^n$, $g(z) = \sum_{i=1}^{\infty} b_n z^n$

Definition 2.4.1. A function g is said to be a *convex function of* $\log r$ if

$$\log r = \alpha \log r_1 + (1-\alpha) \log r_2 \qquad (0 < r_1 < r_2 < 1; 0 < \alpha < 1)$$

then

$$\log g(r) \le \alpha \log g(r_1) + (1-\alpha)\log g(r_2),$$

or

$$g(r) \le [g(r_1)]^{\alpha}[g(r_2)]^{1-\alpha}.$$

Theorem 2.4.2. ***(Hardy's Convexity Theorem)*** *Let $f(z)$ be an analytic function in the unit disc, then (i)$M_p(r,f)$ is a nondecreasing function of r; (ii)$\log M_p(r,f)$ is a convex function of $\log r$. where*

$$M_p(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}, \qquad (0 < p < \infty) \tag{2.30}$$

$$M_\infty(r,f) = \max_{0\le\theta<2\pi} |f(re^{i\theta})| \tag{2.31}$$

Proof. See [11, p. 9]. □

For $1 \le p \le \infty$ then for $f \in H^p(\Delta)$ define

$$f_r(\theta) = f(re^{i\theta})$$

Theorem 2.4.3. ***(Fatou)*** *Let $f \in H^p(\Delta)$, $1 \le p \le \infty$ then for almost all θ radial limits of f exists, i.e.*

$$f^*(\theta) \doteq \lim_{r\to 1} f_r(\theta) = \lim_{r\to 1} f(re^{i\theta})$$

exists a.e. and moreover f^* is in $L^p(\partial\Delta, \frac{d\theta}{2\pi})$.

Proof. See [17, p. 289]. □

Theorem 2.4.4. *If $f \in H^p(\Delta)$, $p > 0$, then*

$$\log |\, f(re^{i\theta})\,| \leq \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t)\log |\, f(e^{i\theta})\,| dt$$

Proof. See [11, p. 23]. □

Theorem 2.4.5. *Let f be an analytic function in the Δ, then $f \in H^p(\Delta)$ if and only if $|f|^p$ has a harmonic majorant in the disc.*

Proof. Assume $u(z)$ be a harmonic majorant for $|\, f(z)\,|^p$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |\, f(re^{i\theta})\,|^p d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u d\theta = [u(0)]. \tag{2.32}$$

Conversely assume f is in $H^p(\Delta)$, then by Theorem 2.4.4,

$$|\, f(re^{i\theta})\,|^p \leq \exp\Big\{\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t)\log |\, f(e^{i\theta})\,|^p \,dt\Big\} \tag{2.33}$$

$$\leq \int_0^{2\pi} P(r,\theta - t)|\, f(e^{i\theta})\,|^p dt \tag{2.34}$$

where the last inequality is by Jensen's inequality. Therefore, $|\, f(z)\,|^p$ is dominated by Poisson integral of its boundary function which is harmonic. □

It can be shown that there exists a smallest harmonic majorant U for $|\, f(z)\,|^p$ for $f \in H^p(\Delta)$. By means of these harmonic majorants we can define a new equivalent norm on $H^p(\Delta)$ by defining:

$$\| f \|_p := U(0).$$

They are equivalent by 2.32.

Yet another equivalent norm can be given to $H^p(\Delta)$ by defining

$$\| f \|_p := \| f^* \|_p$$

where $\| f^* \|_p$ is the norm of boundary function f^* of f in $L^p(\partial\Delta)$.

2.4.2 Hardy Spaces on the Polydiscs

Recall that an *open polydisc* with center z and polyradius r is

$$\triangle^n(z,r) = \{w \in \mathbb{C}^n : |z_i - w_i| < r_i \ , \quad j = 1, \ldots, n\}.$$

We will deal with the unit polydisc centered at the origin and simply show it by Δ^n. Hardy spaces on the polydiscs are defined as follows :

$$H^p(\Delta^n) = \big\{ f \in O(\Delta^n) : \sup_{0<r<1} \big(\frac{1}{(2\pi)^n} \int_{T^n} | f(rz) |^p \, d\mu \big)^{\frac{1}{p}} < \infty \big\}$$

where

$$T^n = \{ z \in \partial\Omega : | z_i | = 1 \ for \ all \ 1 \le i \le n \}$$

and $\mu = \theta_1 \ldots \theta_n$ is the usual Lebesgue measure on the torus. As usual we let:

$$H^\infty(\Delta^n) = \{f \in O(\Delta^n) : \sup_{z\in\Delta^n} |f(z)| \equiv \|f\|_\infty < \infty\} \ .$$

We equip these spaces with the following norms:

$$\|f\|_p \doteq \sup_{0<r<1} \Big(\frac{1}{(2\pi)^n}\int_{T^n} \mid f(rz) \mid^p \, d\mu\Big)^{\frac{1}{p}}$$

and

$$\|f\|_\infty \doteq \sup_{z\in\Delta^n} |f(z)|$$

We set for $f \in O(\Delta^n)$, $w \in T^n$

$$f_r(w) = f(rw) \qquad 0 < r < 1, \tag{2.35}$$

and define

$$f^*(w) \doteq \lim_{r\to 1} f(rw) \tag{2.36}$$

for every $w \in T^n$ at which this radial limit exists.

Theorem 2.4.6. *Let* f *be in* $H^p(\Delta^n)$ *then* $f^*(w)$ *exists for almost all* $w \in T^n$.

Proof. See [22, p. 51]. □

Theorem 2.4.7. *For* $0 < p < \infty$ *and* $f \in H^p(\Delta^n)$, *then*

$$\lim_{r\to 1} \int_{T^n} \mid f_r - f^* \mid^p d\mu = 0.$$

Proof. See [22, p. 51]. □

Theorem 2.4.8. *$f \in H^p(\Delta^n)$ if and only if $|f(z)|^p$ has an n-harmonic majorant where n-harmonic means harmonic in each variable separately.*

Proof. First, assume $|f(z)|^p$ has an n-harmonic majorant u then we have

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} |\, f(re^{i\theta_1},\ldots,re^{i\theta_n})\,|^p d\theta_1\ldots d\theta_1$$
$$\leq \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} u d\theta_1\ldots d\theta_1 = [u(0)]$$

which is finite hence $f \in H^p(\Delta^n)$.

Conversely assume $f \in H^p(\Delta^n)$ then by Theorem 2.4.4, we have

$$|f(z)|^p \leq \exp\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta_1-t_1)\log|\, f(re^{i\theta_1},z_2\ldots,z_n)\,|^p d\theta_1$$

we repeat same argument for z_2 and we get:

$$|f(z)|^p \leq \exp\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta_1-t_1)\log\exp\int_0^{2\pi} P(r,\theta_2-t_2)$$
$$|\, f(re^{i\theta_1},z_2\ldots,z_n)\,|^p d\theta_2 d\theta_1$$

then repeating this procedure n times we have a bound for $|f(z)|^p$:

$$\exp\Big(\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta_1-t_1)\int_0^{2\pi} P(r,\theta_n-t_n)$$
$$|\, f(re^{i\theta_1},re^{i\theta_2}\ldots,re^{i\theta_n})\,|^p)d\theta_1\ldots d\theta_n\Big)$$

now using Jensen inequalty n times we get that $|f(z)|^p$ is dominated by

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\dots\int_0^{2\pi} P(r,\theta_1-t_1)\dots P(r,\theta_n-t_n)|\ f(re^{i\theta_1},\dots,re^{i\theta_n})\ |^p d\theta_1\dots d\theta_1$$

which is n-harmonic. □

2.4.3 Hardy Spaces in the Unit Ball of $\mathbb{C}^n$

We will define the Hardy Spaces in the unit ball of $\mathbb{C}^n$ for $1 \leq \infty$:

$$H^p(\mathbf{B}) \doteq \{f \in O(\mathbf{B}) : \sup_{0<r<1} \int_{S(r)} |f(z)|^p d\mu < \infty\}$$

where $S(r)$ is the sphere with center 0 and radius r and μ is the usual Lebesgue measure on the sphere. We will use S for $S(1)$. As usual we define

$$H^\infty(B) = \{f \in O(B) : \sup_{z\in B} |f(z)| < \infty\}\ .$$

We give following norms to Hardy Spaces on the ball:

$$\|f\|_p \doteq \sup_{0<r<1} (\int_{S(r)} |f(z)|^p d\mu)^{1/p}$$

and

$$\|f\|_\infty \doteq \sup_{z\in B} |f(z)|.$$

As in the case of polydiscs we define f_r as $f_r(z) = f(rz)$ for $0 < r < 1$.

Also we define

$$f^*(w) = \lim_{r \to 1} u(rw)$$

for every $w \in S$ at which this radial limit exists.

Theorem 2.4.9. *If f is in $H^p(B)$, for $1 \leq \infty$ then, for almost all $w \in S$, $f^*(w)$ exists.*

Proof. See [23, p. 85]. □

Theorem 2.4.10. *If f is in $H^p(B)$, for $1 \leq \infty$ then $\lim_{r \to 1} \int\limits_S \mid f^* - f_r \mid^p d\sigma = 0$.*

Proof. See [23, p. 85]. □

Theorem 2.4.11. *An analytic function f is in $H^p(B)$ if and only if $\mid f(z) \mid^p$ has an harmonic majorant.*

Proof. See [17, p. 291]. □

CHAPTER 3

THE COMPLEX MONGE-AMPÉRE OPERATOR

In this chapter we will discuss complex Monge-Ampére operators $(dd^c)^n$. Firstly, we will define complex Monge-Ampére operator for functions of class $\mathcal{C}^2$ and extend this definition for continuous plurisubharmonic functions. Then, we will define complex Monge-Ampére operator for plurisubharmonic functions which are class L^∞_{loc}. This extension is the most important result of the chapter. Lastly we will give some comparison theorems which enable us to carry the inequalities on the boundary to the domain and to compare the complex Monge-Ampére operators of two plurisubharmonic functions. Using these comparison theorems we will give a characterization of maximal plurisubharmonic functions. In this chapter we will basically follow the works of Bedford and Taylor [2, 3, 4] and we will sometimes refer to [15].

3.1 The Dirichlet Problem

In classical potential theory the *Generalized Dirichlet problem* on an open subset Ω of $\mathbb{R}^n$ is to find a harmonic function u such that $u\big|_{\partial\Omega} = f$ for a given extended real valued function f defined in $\partial\Omega$. In particular for given $f \in \mathcal{C}^{\infty}(\partial\Omega)$. Here extended means a function taking values in $[-\infty, \infty]$.

In several variables the Pluricomplex Dirichlet problem asks to find an upper semi-continuous function u on $\overline{\Omega}$, $u : \overline{\Omega} \longrightarrow \mathbb{R}$ for a given extended real valued function f defined in $\partial\Omega$ such that $(u\big|_{\Omega}) \in \mathcal{MPSH}(\Omega)$ and $u\big|_{\partial\Omega} \equiv f$.

Recall that the *complex Monge-Ampére operator* in $\mathbb{C}^n$ is defined as the n-th exterior power of dd^c, namely

$$(dd^c)^n = \underbrace{dd^c \wedge \cdots \wedge dd^c}_{n-times}.$$

If $u \in \mathcal{C}^2(\Omega)$, then an easy calculation allow that

$$(dd^c u)^n = 4^n n! det\Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\Big] dV, \tag{3.1}$$

where

$$dV = \Big(\frac{i}{2}\Big)^n dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

is the usual volume form in $\mathbb{C}^n$.

In particular for $n = 1$ $(dd^c u)^n$ becomes usual Laplacian times the area form in $\mathbb{R}^2$. Namely we can regard $(dd^c u)^n$ as usual Laplacian Δu.

Recall that real Monge-Ampére equations which can be formulated as :

$$\det\Big(\frac{\partial^2 u}{\partial x_j \partial x_k}\Big) = f(x_1, \ldots, x_n) \tag{3.2}$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$. These equations are studied extensively in Differential Geometry.

Complex version of the Monge-Ampére equation is

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = f(z_1, \ldots, z_n)$$

and can be formulated also as

$$(dd^c u)^n = \left(\frac{i}{2}\right)^n f dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

One of the important features of the Monge-Ampére operators is the fact that the maximality of plurisubharmonic functions can be characterized in terms of these equations.

Corollary 3.1.1. *Let Ω be an open subset of $\mathbb{C}^n$, and let $u \in \mathcal{C}^2 \cap \mathcal{PSH}(\Omega)$. Then u is maximal if and only if $(dd^c u)^n = 0$ in Ω.*

Proof. Let u be maximal plurisubharmonic function then by Theorem 2.3.22 we have $\det\left[\frac{\partial^2 g}{\partial z_j \partial \bar{z}_k}\right]_{1 \le j,k \le n} \equiv 0$, hence we have $(dd^c u)^n = 0$ by 3.1.
Conversely assume $(dd^c u)^n = 0$ then by 3.1 we have $\det\left[\frac{\partial^2 g}{\partial z_j \partial \bar{z}_k}\right]_{1 \le j,k \le n} \equiv 0$ and by Theorem 2.3.22 we have u maximal. $\square$

3.2 The Complex Monge-Ampére Operator

In the previous subsection we defined the *Monge-Ampére Operator* for twice differentiable functions. We know that $dd^c u$ can be defined as a positive $(1,1)$ current for non-differentiable L^1_{loc} cases by 2.3.22. But it is well known that higher powers of $dd^c u$ cannot be defined for non-differentiable case in the same manner. We will give an example of a plurisubharmonic function for which $(dd^c u)^n$ cannot be defined as a positive current. This example is due to Shifmann and Taylor.

Example 3.2.1. Let B be open unit ball of $\mathbb{C}^n$ where $n > 1$. Let $Z = \{z_2 = \cdots = z_n = 0\}$. We will construct a plurisubharmonic function u in Ω which is in $\mathcal{C}^\infty(B \setminus Z)$ such that $\int\limits_{B(r)\setminus Z} (dd^c u)^n$ is infinite for $0 < r < 1$.

Let k be a positive integer and A be a positive number. Let

$$f_{k,A} = \left|\frac{z_1^k}{A}\right| + \sum_{i=1}^{n} |z_i|^2.$$

For $\nu \geq 1$ we define

$$D_\nu = B\Big(1 - \frac{1}{\nu}\Big) \cap \Big\{ \sum_{i=q}^{n} |z_i|^2 > \frac{1}{\nu^2} \Big\} \ .$$

Now for $\mu, \nu \geqq 1$ there exists a positive integer $C_{\mu,\nu}$ such that for $k \geqq 1$ there exists a positive integer $A(k)$ such that every partial derivative of $\log f_{k,A(k)}$ up to order μ is bounded by $C_{\mu,\nu}$ on D_ν. Fix $0 < r < 1$. For $\nu \geqq 2$, let $k_\nu = (2^\nu C_{\nu,\nu})^n$. Since $\log f_{k,A(k)}$ is plurisubharmonic function on $\mathbb{C}^n \setminus 0$, there exists a $\mathcal{C}^\infty$ plurisubharmonic function g_ν on $\mathbb{C}^n$ such that every partial derivative of g_ν up to order ν is bounded by $2C_{\nu,\nu}$ on $D_{\nu-1}$. Now we define:

$$u = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu C_{\nu,\nu}} g_\nu \ .$$

then

$$\int\limits_{B(r)\setminus Z} (dd^c u)^n = \infty.$$

Now $\int\limits_{B(r)} (dd^c u)^n = \infty$ and if as $r \to 0$ we get infinite mass at the origin. If we take a smooth function ϕ with compact support which is not identically zero in a neighborhood of 0 then $\int \phi (dd^c u)^n = \infty$. Hence $(dd^c u)^n$ cannot be defined as a (n, n) current. For details see [24].

We will now extend the definition of *Monge-Ampére operator* to functions in $\mathcal{C}(\Omega)$. Later we will extend the definition to slightly more general families of plurisubharmonic functions namely, $\mathcal{L}^\infty_{loc}(\Omega)$.

In order to extend $(dd^c)^n$ to $L^\infty_{loc}(\Omega)$, we will use some norm estimates due to Chern, Levine and Nirenberg [25]. Using Chern, Levine and Nirenberg estimate and modified version of the estimate of Bedford and Taylor we will first define $(dd^c u)^n$ for continuous plurisubharmonic functions. Now we refer to some theorems from Bedford and Taylor [2] in order to construct this extension.

Proposition 3.2.2. *Let $u_1, \ldots, u_m \in \mathcal{C}^2(\Omega)$, and ϕ be differential form of* *type* $(n-m, n-m)$ *with coefficients from* $\mathcal{C}_0^\infty(\Omega)$. *Then for* $2 \leq m \leq n$,

$$\int_\Omega \phi \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_m = -\int_\Omega dd^c\phi \wedge du_1 \wedge d^c u_2 \wedge dd^c u_3 \wedge \ldots dd^c u_m \quad (3.3)$$

and

$$\int_\Omega \phi \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_m = \int_\Omega u_1 dd^c\phi \wedge dd^c u_2 \wedge \ldots dd^c u_m \quad (3.4)$$

Proof. Since ϕ has compact support and by using Stoke's theorem we deduce that

$$\int_\Omega \phi \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_m = -\int_\Omega d\phi \wedge d^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_m$$

however $(n-1+1, n-m+1)$ parts of $d\phi \wedge d^c u$ and $du \wedge d^c \phi$ are the same, so the last integral equals to

$$-\int_\Omega du_1 \wedge d^c\phi \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_m = -\int_\Omega d^c u_2 \wedge d(du_1 \wedge d^c\phi \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_m)$$

$$= -\int_\Omega dd^c\phi \wedge du_1 \wedge d^c u_2 \wedge dd^c u_3 \wedge \ldots dd^c u_m$$

which gives 3.3. To show 3.4 we will use Stokes Theorem to get

$$-\int_\Omega du_1 \wedge d^c\phi \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_m = -\int_\Omega dd^c\phi \wedge du_1 \wedge d^c u_2 \wedge dd^c u_3 \wedge \ldots dd^c u_m.$$

□

From this proposition we see some conditions under which $(dd^cu)^n$ can be defined. For example, when $m = n = 2$, and $u_1 = u_2$, $\int \phi(dd^cu)^2$ 3.3 tells us that $(dd^cu)^n(\phi)$ should be equal to $-\int du \wedge d^cu \wedge dd^c\phi$ as a current.

Next proposition is a slightly modified version of Chern, Levine and Nirenberg's inequality and will be referred as the Chern-Levine-Nirenberg estimate.

Proposition 3.2.3. *Let K be a compact set in $\mathbb{C}^n$ and Ω is an open neighborhood of K. There exists a constant $C > 0$ and a compact set $L \subset \Omega \setminus K$, which depend on K and Ω, such that for all $u_1, \ldots, u_n \in \mathcal{PSH} \cap \mathcal{C}^2(\Omega)$,*

$$\int_K dd^cu_1 \wedge \ldots \wedge dd^cu_n \leq C \parallel u_1 \parallel_L \cdot \ldots \cdot \parallel u_n \parallel_L \tag{3.5}$$

where $\parallel u \parallel_L$ is the sup norm on L.

Proof. See [15, p. 111]. □

We will denote the class of currents on Ω of bidegree (m, m) and order 0 by $M_m(\Omega)$.

Proposition 3.2.4. *Let $T_m(u) = ((dd^cu), (dd^cu)^2, \ldots, (dd^cu)^m)$ be an defined operator defined from $\mathcal{C}^2(\Omega)$ into $\prod_{k=1}^{n} M_k(\Omega)$ for $1 \leq m \leq n$. Then*

If $u_j, v_j \in \mathcal{C}(\Omega)$, and $\lim_{j \longrightarrow \infty} u_j = \lim_{j \longrightarrow \infty} v_j = u$ uniformly on compact subsets of Ω, such that $u \in \mathcal{C}^2(\Omega)$ and if both the limits $\lim_{j \longrightarrow \infty} T_m u_j$ and

$\lim_{j\longrightarrow\infty} T_m v_j$ *exist, then they are equal. Consequently,* T_m *has a unique extension to a continuous operator on all of* $\mathcal{C}(\Omega)\cap\mathcal{PSH}(\Omega)$ *in view of the fact that any continuous plurisubharmonic function can be approximated by smooth plurisubharmonic functions.*

Proof. See [2]. □

By second part of this proposition we can extend $(dd^c)^n$ to continuous plurisubharmonic functions by taking a sequence of smooth plurisubharmonic functions u_j of class $\mathcal{C}$ functions (for instance take $u_\varepsilon = u^*\chi_\varepsilon$) then we can define

$$(dd^c u)^n \doteq \lim_{j\to\infty}(dd^c u_j)^n. \tag{3.6}$$

By second part of the Proposition 3.2.4 this limit is independent of the sequence u_j.

Next proposition assures us $(dd^c u)^n$ coincides with the classical definition whenever u is in $\mathcal{C}^2(\Omega)$.

Proposition 3.2.5. *Let* $u\in\mathcal{PSH}\cap\mathcal{C}^2(\Omega)$ *and suppose* $dd^c u = 2i\sum_{j,k=1}^{n} h_{j\bar{k}}dz_j\wedge d\overline{z}_k$, *where the* $h_{j\bar{k}}\in[L^1_{loc}(\Omega)]^m$. *Then* $(dd^c u)^n \in M_m(\Omega)$ *has locally integrable coefficients and is given by* $[2i\sum_{j,k=1}^{n} h_{j\bar{k}}dz_j\wedge d\overline{z}_k]^m$.

Proof. Let u be a continuous plurisubharmonic function. Consider smooth plurisubharmonic u_ε where $u_\varepsilon = u^*\chi_\varepsilon$ converging to u. Then by weak continuity of $(dd^c u)^n$ given by part (2) of Proposition 3.2.4 we have the result. □

Now we will extend the operator $(dd^c)^n$ to L^∞ plurisubharmonic functions as positive (k,k) current by induction. We follow Bedford and Taylor [2].

Definition 3.2.6. Let $u \in L^\infty(\Omega) \cap \mathcal{PSH}(\Omega)$. We define $(dd^c u)^k$ inductively for $1 \le k \le n$. For a $(n-k, n-k)$ form χ with smooth coefficients with compact support in Ω, then

$$\int_\Omega (dd^c u)^k \wedge \chi = \int_\Omega u(dd^c u)^{k-1} \wedge dd^c \chi. \tag{3.7}$$

Proposition 3.2.7. *The operator $(dd^c)^k$ defined by 3.7 for plurisubharmonic functions in $L^\infty(\Omega)$ is a positive (k,k) current.*

Proof. By the Proposition 3.3.5 of [15], we know that for any plurisubharmonic function u $(dd^c u)^1$ is a positive current. Now assume that $(dd^c u)^{k-1}$ is defined as a positive $(k-1, k-1)$ current. Since u is upper semicontinuous and locally bounded, $u(dd^c u)^{k-1}$ again has measure coefficients, and thus $(dd^c u)^k$ is a (k,k) current.

For positivity take a strongly positive test form of bidegree $(n-k, n-k)$ whose support is contained in Ω. Let G be a relatively compact subset of Ω that contains $supp\chi$. By Theorem 2.3.15, there exists a decreasing sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH} \cap \mathcal{C}^\infty(G)$ converging to u in G. For each j, the form $dd^c u_j \wedge \chi$ is a strongly positive current. Therefore by the induction assumption,

$$\int_\Omega (dd^c u)^{k-1} \wedge (dd^c u \wedge \chi) \geq 0.$$

Now by the dominated convergence theorem, we have,

$$\begin{aligned}
\int_\Omega (dd^c u)^k \wedge \chi &= \int_\Omega u(dd^c u)^{k-1} \wedge dd^c \chi \\
&= \lim_{j\to\infty} \int_\Omega u_j (dd^c u)^{k-1} \wedge dd^c \chi \\
&= \lim_{j\to\infty} \int_\Omega dd^c u_j (dd^c u)^{k-1} \wedge \chi \geq 0.
\end{aligned}$$

□

For $k = 1$ we know that $(dd^c)^k$ is continuous. But the higher orders of dd^c need not be continuous in general. Next theorem, due to Bedford and Taylor, shows that $(dd^c u)^n$ is continuous under decreasing sequences of plurisubharmonic functions which are locally bounded.

Theorem 3.2.8. *Let* $\{v_j^1\}, \ldots, \{v_j^k\}$ *be decreasing sequences of functions in* $\mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ *and assume that for all* $z \in \Omega$,

$$\lim_{j\to\infty} v_j^i = v^i \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega), \qquad 1 \leq i \leq k.$$

Then

$$\lim_{j\to\infty} dd^c v_j^1 \wedge \ldots \wedge dd^c v_j^k = dd^c v^1 \wedge \ldots \wedge dd^c v^k \tag{3.8}$$

where the limit is in the weak topology on $M_k(\Omega)$.

Proof. See [3]. □

Next corollary is a direct consequence of above theorem:

Corollary 3.2.9. *The map* $(v^1, \ldots, v^k) \mapsto dd^c v^1 \wedge \ldots \wedge dd^c v^k$ *is a symmetric, multi-linear map from* $\mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ *into the cone of nonnegative closed currents of bidegree* (k, k).

Proposition 3.2.10. *Let* $u_1, \ldots, u_k$ *be a continuous (finite) plurisubharmonic functions and let* $u_1^j, \ldots, u_k^j$ *be sequences of plurisubharmonic functions converging locally uniformly to* $u_1, \ldots, u_k$. *If* T_j *is a sequence of closed positive currents converging weakly to* T, *then*

$$u_1^j dd^c u_2^j \wedge \cdots \wedge dd^c u_k^j \wedge T_j \longrightarrow u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_k \wedge T \qquad \textit{weakly and}$$

$$dd^c u_1^j \wedge \cdots \wedge dd^c u_k^j \wedge T_j \longrightarrow dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge T \qquad \textit{weakly.}$$

Proof. See [10]. □

Now we will define a class of measures usingthe complex Monge-Ampére operator. We start with an open connected subset $\Omega \subset \mathbb{C}^n$ let $a \in \Omega$. And we define

$$\begin{aligned} \mathcal{PSH}(\Omega; a) &= \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega \setminus \{a\}), \\ \mathcal{C}_0^\infty(\Omega; a) &= \{\varphi \in \mathcal{C}_0^\infty(\Omega) : supp(d\varphi) \subset \Omega \setminus \{a\}\}. \end{aligned} \tag{3.9}$$

Lemma 3.2.11. *The space* $\mathcal{C}_0^\infty(\Omega; a)$ *is dense in* $\mathcal{C}_0^0(\Omega)$ *in the topolgy of uniform convergence on compact subsets.*

Proof. See [15, p. 228]‘. □

Proposition 3.2.12. *Let Ω be an open set in $\mathbb{C}^n$, and let $u \in \mathcal{PSH}(\Omega; a)$. Then there exists a positive Borel measure μ on Ω such that, for any decreasing sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH} \cap L^\infty_{loc}(\Omega)$ convergent to u at each point in Ω, the sequence $\{(dd^c u_j)^n\}_{j\in\mathbb{N}}$ is weak*−convergent to μ.*

Proof. If $\varphi \in \mathcal{C}_0^\infty(\Omega; a)$, then $supp(dd^c\varphi) \subset \Omega \setminus \{a\}$, and thus

$$\int_\Omega \varphi(dd^c u_j)^n = \int_\Omega u_j(dd^c u_j)^{n-1} \wedge (dd^c\varphi) \to \int_\Omega u(dd^c u)^{n-1} \wedge (dd^c\varphi) \quad (3.10)$$

as $j \to \infty$, by Theorem 3.2.8. By the Chern-Levine-Nirenberg estimates, the set $\{(dd^c u)^n\}_{j\in\mathbb{N}}$ is relatively sequentially compact in the *weak**-topology. By Lemma 3.2.11 and 3.10, it is convergent on a dense subspace of $\mathcal{C}_0^\infty(\Omega)$. Consequently it converges to a measure μ. □

By this proposition we can define $(dd^c u)^n$ as a positive Borel measure on Ω.

Corollary 3.2.13. *Let $a \in \mathbb{C}^n$, and let $R > 0$. If $u(z) = \log(\|z-a\|/R)$ for all $z \in \mathbb{C}^n$, then $(dd^c u)^n = (2\pi)^n\delta_a$, where δ_a is the Dirac delta function at a.*

Proof. See [15, p. 229]. □

3.3 Comparison Theorems

We continue investigating the properties of complex Monge-Ampére operator. In this part we will give some comparison theorems which enable us to extend the inequalities on the boundary to the domain. We will follow the works of Bedford and Taylor closely [3]. Following theorems are due to Bedford and Taylor [3].

Theorem 3.3.1. ***(Comparison Theorem)*** *Let* Ω *be an open bounded subset of* $\mathbb{C}^n$. *Let* $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ *and suppose that* $\liminf_{\zeta \to \partial\Omega} u(\zeta) - v(\zeta) \geq 0$ *(e.g.* $u \geq v$ *on* $\partial\Omega$*). Then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Proof. See [3]. □

Theorem 3.3.2. *Let* Ω *be an open bounded subset of* $\mathbb{C}^n$. *Let* $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ *and suppose that* $\liminf_{\zeta \to \partial\Omega} u(\zeta) - v(\zeta) \geq 0$ *(i.e.* $u \geq v$ *on* $\partial\Omega$*). Then*

$$\int_{\{u\leq v\}} (dd^c v)^n \leq \int_{\{u\leq v\}} (dd^c u)^n.$$

Proof. Let $u_\varepsilon = u - \varepsilon$, and $S_\varepsilon = \{u_\varepsilon < v\}$. Clearly, S_ε decreases to $\{u \leq v\}$ as ε decreases to 0. For some ε small enough $S_\varepsilon \subset \omega \subset \Omega$ for some relatively compact set ω in Ω. Then, by Theorem 3.3.1, we have

$$\int_{S_\varepsilon} (dd^c v)^n \leq \int_{S_\varepsilon} (dd^c u)^n.$$

Letting $\varepsilon \to 0$, we obtain the desired inequality. ☐

Corollary 3.3.3. *Let* Ω *be an open bounded subset of* $\mathbb{C}^n$. *Let* $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ *satisfying (i)*$\lim_{\zeta\to\partial\Omega} u(\zeta) = \lim_{\zeta\to\partial\Omega} v(\zeta) = 0$, *(ii)* $u \leq v \in \Omega$ *Then*

$$\int\limits_\Omega (dd^c v)^n \leq \int\limits_\Omega (dd^c u)^n.$$

Proof. By theorem 3.3.1 we have

$$\int_\Omega (dd^c v)^n \leq \int_\Omega [dd^c(1+\varepsilon)u]^n = (1+\varepsilon)\int_\Omega (dd^c u)^n$$

for $\varepsilon > 0$, which give the result. ☐

Proposition 3.3.4. *Let* Ω *be an open bounded subset of* $\mathbb{C}^n$. *Let* $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $\limsup_{\zeta\to\partial\Omega} \mid u(\zeta) - v(\zeta) \mid = 0$, *and* $(dd^c u)^n = (dd^c v)^n$ *in* Ω, *then* $u \equiv v$ *in* Ω.

Proof. We will show that the set $\{u < v\}$ is empty. Assume that it is not empty. Let $\psi < 0$ be a strongly positive plurisubharmonic function on $\overline{\Omega}$. If the set $\{u < v\}$ is not empty, then $S = \{u < v + \varepsilon\psi\}$ is not empty for some $\varepsilon > 0$. Since u and $v + \varepsilon\psi$ are plurisubharmonic, S has positive measure else they are identical. Now by Theorem 3.3.1 ,

$$\int_S (dd^c u)^n \geq \int_S (dd^c[v + \varepsilon\psi])^n \geq \int_S (dd^c v)^n + \varepsilon^n \int_S (dd^c\psi)^n,$$

which is a contradiction since the last integral over S is strictly positive. □

Corollary 3.3.5. ***(Domination Principle)*** *Let Ω be an open bounded subset of $\mathbb{C}^n$. Let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, such that (i) $\limsup_{\zeta\to\partial\Omega} \mid u(\zeta) - v(\zeta) \mid = 0$, (ii) $\int\limits_{u<v} (dd^c u)^n = 0$ Then $u \geq v$ in Ω.*

Proof. Let $\tilde{v} = v - \varepsilon + \delta \mid z \mid$ where ε, δ are chosen so that $\widetilde{v} < v$ on $\overline{\Omega}$, then

$$0 < \int_{u<\tilde{v}} (dd^c\tilde{v})^n \leq \int_{u<\tilde{v}} (dd^c u)^n \leq \int_{u<v} (dd^c u)^n$$

which is a contradiction unless $\{u < v\}$ is empty. □

Corollary 3.3.6. *Let Ω be an open bounded subset of $\mathbb{C}^n$. Let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $\limsup_{\zeta\to\partial\Omega}(u(\zeta) - v(\zeta)) \geq 0$, and $(dd^c u)^n = 0$ in Ω, then $u \geq v$ in Ω.*

Proof. Let $p(z) = \frac{\|z\|^2}{4}$, $z \in \mathbb{C}^n$; then $(dd^c p)^n = n!dV$, where dV is the volume form in $\mathbb{C}^n$. Now, define $v_{\varepsilon,\delta} = v + \varepsilon p - \delta$ for $\varepsilon > 0$, $\delta > 0$. We will choose ε and δ so that $v_{\varepsilon,\delta} < v$ on $\overline{\Omega}$. Then u and $v_{\varepsilon,\delta}$, satisfies the assumptions of the corollary. If the set $\{u < v\}$ is non-empty, then, for some ε and δ, the set $\{u < v_{\varepsilon,\delta}\}$ is non-empty. The set $\{u < v_{\varepsilon,\delta}\}$ must have positive Lebesgue measure.

$$\int_{\{u<v_{\varepsilon,\delta}\}}(dd^c v)^n+\int_{\{u<v_{\varepsilon,\delta}\}}(dd^c(\varepsilon p-\delta))^n\le\int_{\{u<v_{\varepsilon,\delta}\}}(dd^c v_{\varepsilon,\delta})^n$$
$$\le\int_{\{u<v_{\varepsilon,\delta}\}}(dd^c u)^n\le\int_{\{u<v\}}(dd^c u)^n=0$$

which is impossible, as

$$\int_{\{u<v_{\varepsilon,\delta}\}}(dd^c(\varepsilon p-\delta))^n=\varepsilon^n\int_{\{u<v_{\varepsilon,\delta}\}}dV>0.$$

□

A direct consequences of the above comparison theorems is the following theorem characterizing maximal functions for plurisubharmonic functions which are maximal.

Theorem 3.3.7. *Let Ω be an open subset of $\mathbb{C}^n$ and $u\in\mathcal{PSH}(\Omega)\cap L^\infty_{loc}(\Omega)$ then it satisfies the Monge-Ampère equation $(dd^cu)^n=0$, if and only if u is maximal.*

Proof. Let $v\in\mathcal{PSH}(\Omega)$ and for each $\varepsilon>0$ there is compact set $K\subset\Omega$ such that $u-v\ge-\varepsilon$ in $\Omega\setminus K$, then $\limsup_{\zeta\to\partial\Omega}(u(\zeta)-v(\zeta))\ge 0$, then by Corollary 3.3.6 we have $u\ge v$. Hence by second part of Proposition 2.3.20, we have u maximal. □

CHAPTER 4

HYPERCONVEX DOMAINS AND MONGE-AMPÉRE MEASURES

In this chapter we want to introduce hyperconvex domains and some important properties of hyperconvex domains from pluripotential theoretic viewpoint. Secondly we want to discuss *Monge-Ampére measure* over hyperconvex domains, following J.P.Demailly, using Bedford and Taylor's methods. Here our most important result is what Demailly called "Lelong-Jensen formula". We will call this result as "Demailly-Lelong-Jensen formula".

4.1 Pluricomplex Green Functions

Let Ω be an open bounded subset of $\mathbb{C}$, and $a \in \Omega$. Let $G_\Omega(z,a)$ be a function from Ω to $[0,+\infty]$ such that:

(i) $G_\Omega(z,a)$ is harmonic in $\Omega\backslash\{a\}$.

(ii) $G_\Omega(z,a) \to 0$ as $z \to w$, for each $w \in \partial\Omega$.

(iii) $G_\Omega(z,a) + \log \mid z-a \mid$ extends to a harmonic function on Ω.This function is called *the classical Green function with pole at a.*

Indeed, by maximum principle if Green function exists then it is unique.

Note that $-G_\Omega(z,w)$ is subharmonic.

Now we will give some properties of classical Green functions on domains in $\mathbb{C}$. We will refer to [21] for the proofs.

Theorem 4.1.1. *If boundary of* Ω *is non-polar, then it has a unique Green function for* Ω.

Proof. See [21, p. 106]. ☐

Theorem 4.1.2. *Let* Ω_1, Ω_2 *be domains with non-polar boundaries and* $f : \Omega_1 \longrightarrow \Omega_2$ *an analytic function. Then* $G_{\Omega_2}(f(z), f(w)) \geq G_{\Omega_1}(z,w)$, *equality holds if and only if f is biholomorphic.*

Proof. See [21, p. 107]. ☐

Just a direct corollary of this theorem is as follows:

Corollary 4.1.3. *Let* Ω_1, Ω_2, *be domains with non-polar boundaries such that* $\Omega_1 \subseteq \Omega_2$ *then* $G_{\Omega_1}(z,w) \leq G_{\Omega_2}(z,w) \quad (z,w \in \Omega_1)$

Theorem 4.1.4. *Let* Ω *be a bounded domain in* $\mathbb{C}$ *whose boundary is non-polar , and let* $(\Omega_n)_{n\geq 1}$ *be subdomains of* Ω *such that (i)* Ω_n *has non-polar boundary, (ii)* $\Omega_n \subseteq \Omega_{n+1}$, *(iii)* $\cup_n \Omega_n = \Omega$. *Then*

$$\lim_{n\to\infty} G_{\Omega_n}(z,w) = G_\Omega(z,w) \quad (z,w \in \Omega).$$

Proof. See [21, p. 108]. ☐

Theorem 4.1.5. *Let Ω be a bounded domain in $\mathbb{C}$ whose boundary is non-polar, then $G_\Omega(z,w) = G_\Omega(w,z)$.*

Proof. See [21, p. 110]. □

Now we want to introduce Perron function and the harmonic measures.

Definition 4.1.6. Let Ω be a bounded domain in $\mathbb{C}$, and let $\phi : \partial\Omega \to \mathbb{R}$ be a bounded function. The *Perron function* $H_\Omega\phi : \Omega \to \mathbb{R}$ is defined by

$$H_\Omega\phi = \sup_{u \in \mathcal{U}} u, \tag{4.1}$$

where $\mathcal{U}$ denotes the family of all subharmonic functions u on Ω such that $\limsup_{z\to\zeta} u(z) \le \phi(z)$ for each $\zeta \in \partial\Omega$.

The importance of Perron function associated with the function ϕ is if the Dirichlet problem has a solution, then it is clearly $H_\Omega\phi$.

Definition 4.1.7. Let Ω be a bounded domain in $\mathbb{C}$, we will denote the $\sigma-$algebra of Borel subsets of $\partial\Omega$ by $\mathcal{B}(\partial\Omega)$. A *harmonic measure* for Ω is a function $\omega_\Omega : D \times \mathcal{B}(\partial\Omega) \to [0,1]$ such that:

1. for each $z \in \Omega$, the map $B \to \omega_\Omega(z,B)$ is a Borel probability measure on $\partial\Omega$;

2. if $\phi : \partial\Omega \to \mathbb{R}$ is a continuous function, then $H_\Omega\phi = P_\Omega\phi$ on Ω, where $P_\Omega\phi$ is the *generalized Poisson integral* of ϕ on Ω, given by

$$P_\Omega\phi \doteq \int_{\partial\Omega} \phi(\zeta) d\omega(z,\zeta) \quad (z \in \Omega). \tag{4.2}$$

Example 4.1.8. Let Ω be a smoothly bounded open bounded subset of $\mathbb{C}$ then consider the Green identity:

$$\int_{\Omega} u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - u\frac{\partial v}{\partial n}. \tag{4.3}$$

Let u be a harmonic function and suppose that $G(;z)$ is the Green function on Ω. Then equation 4.3 becomes

$$\int_{\Omega} u\Delta G = \int_{\partial\Omega} u\frac{\partial G}{\partial n}$$

and it is well known that $\Delta G = 2\pi\delta(z)$ that we have $u(z)2\pi = \int_{\partial\Omega} u\frac{\partial G}{\partial n}$. Namely $\frac{\partial G}{\partial n}$ gives the harmonic measure.

Theorem 4.1.9. *([21, p, 117])**(Poisson-Jensen Formula)** Let Ω be a bounded regular domain in $\mathbb{C}$, and let u be a subharmonic function on a neighborhood of $\bar{\Omega}$. Then*

$$u(z) = \int_{\partial\Omega} u(\zeta)d\omega_{\Omega}(z,\zeta) - \frac{1}{2\pi}\int_{\Omega} G_{\Omega}(z,w)\Delta u(w) \qquad (z \in \Omega) \tag{4.4}$$

Theorem 4.1.10. *Let Ω be a bounded domain of $\mathbb{C}$, such that $\partial\Omega$ is non-polar, and u be a subharmonic function on Ω with u is not identically $-\infty$.*
(i) If u has a harmonic majorant on Ω, then it has a least harmonic majorant

h and we have

$$u(z) = h(z) - \frac{1}{2\pi} \int_{\Omega} G_{\Omega}(z,w)\Delta u(w) \qquad (z \in \Omega).$$

(ii) If u has no harmonic majorant on Ω*, then*

$$\frac{1}{2\pi} \int_{\Omega} G_{\Omega}(z,w)\Delta u(w) = \infty \qquad (z \in \Omega) \tag{4.5}$$

Proof. See [21, p, 118]. □

In 1981 Lempert constructed a function Φ_w for each strictly convex bounded domain $D \subset \mathbb{C}^n$ and for each $w \in D$ such that if we set $u(z) = \log \Phi_w(z)$ then we have:

$$\begin{cases} \det \mid \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} \mid = 0 & \text{for} \quad z \in D \backslash w, \\ u \in \mathcal{PSH}(D) & \\ \lim_{z \to \xi z \in D} u(z) = 0 & \xi \in \partial D \\ u(z) - \log \mid z - w \mid = O(1) & for\ z \to w. \end{cases} \tag{4.6}$$

For $n = 1$, the function $-u$ is just the classical Green function for D with pole at w. Due to analogy between Laplacian and the *complex Monge-Ampére operator* in $\mathbb{C}^n$, we can regard u as Pluricomplex version of classical Green function. In 1985 Klimek introduced a Pluricomplex Green function in [14]. His definition is

$$g_\Omega(z,w) = \sup\{u(z)\}, \tag{4.7}$$

where the supremum is taken over all non-positive plurisubharmonic functions on Ω (including $u \equiv -\infty$) such that the function $t \to u(t) - \log|t-w|$ is bounded from above in a neighborhood of w,

In 1987 Demailly showed that Klimek's definition gives a solution to the generalized Pluricomplex Dirichlet problem in any hyperconvex domain. For details see 4.2.9.

4.2 Hyperconvex Domains

In this section we will give the definition of hyperconvex domains and some important properties of hyperconvex domains from different points of view.

Definition 4.2.1. [13] A connected open subset Ω of $\mathbb{C}^n$ is called hyperconvex if there exists a plurisubharmonic function $g : \Omega \longrightarrow [-\infty, 0)$ such that $\{z \in \Omega : g(z) < c\}$ is relatively compact for each $c < 0$. Here g is called a *plurisubharmonic exhaustion* or a *defining function* for Ω.

Now we will present some examples

Example 4.2.2. $\Delta = \Delta(0,1) \subset \Omega$ is a hyperconvex domain. $g : \Delta \to [-\infty, 0)$ defined by $g(z) = \log(|\ z\ |)$. This is the most important example and it will be a motivating tool for us. The level sets are just discs with radius e^r so they are relatively compact.

Example 4.2.3. $B(0,1) \subset \mathbb{C}^n$ is a hyperconvex domain with defining function $g : B \to [-\infty, 0)$ given by $g(z) = \log(\|z\|)$.

Another important example in this context is the polydisc $\Delta^n \subset \mathbb{C}^n$ with plurisubharmonic exhaustion function $g : \Delta^n \to [-\infty, 0)$ defined by $g(z) = \log(\max | z_1 |, \ldots, | z_n |)$.

It is clear from the definition that every hyperconvex domain is a pseudoconvex set (i.e. there exists a plurisubharmonic exhaustion function). Moreover every pseudoconvex set can be written as increasing union of hyperconvex sets. Since for any pseudoconvex domain there exists a smooth exhaustive plurisubharmonic function φ and consider the level sets of φ which are of the form $\{\varphi < c\}$ then any level set is hyperconvex by negative continuous plurisubharmonic function $\psi = \varphi - c$ which is exhaustive.

Now we will present some properties of hyperconvex domains from different perspectives.

Next theorem gives a good characterization of hyperconvex domains in terms of Frechet Spaces of analytic functions on them.

Theorem 4.2.4. *Let Ω be a bounded subset of $\mathbb{C}^n$. Then $O(\Omega)$, the Frechet Space of analytic functions on Ω is isomorphic to $A(\Delta^n)$ if and only if Ω is a hyperconvex domain.*

Proof. See [26]. □

Theorem 4.2.5. *Let Ω be a bounded domain of holomorphy in $\mathbb{C}^n$ which is complete with respect to the Carathéodory metric. Then Ω is hyperconvex.*

Proof. See [1]. □

Lemma 4.2.6. *([15, p. 225]) Let Ω be a hyperconvex domain, then for each $a \in \Omega$ and $w \in \partial\Omega$*

$$\lim_{z \to w, z \in \Omega} g_\Omega(z, a) = 0. \tag{4.8}$$

Proof. Let ρ be a defining function for Ω. Let $a \in \Omega$,and choose $r, R > 0$ such that $\overline{B(a,r)} \subset \Omega \subset \bar{\Omega} \subset B(a, R)$. Define

$$v(z) = \begin{cases} \max\{C\rho(z), \log(\|z-a\|/R)\} & z \in \Omega \setminus B(a,r)), \\ \log(\|z-a\|/R) & (z \in B(a,r)), \end{cases}$$

where the constant $C > 0$ is chosen such that $C\rho < \log(r/R)$ on the unit sphere $\partial B(a, r)$. Clearly, v is a plurisubharmonic function and hence $v(z) \leq g_\Omega(z, a)$ in Ω. Since $v(z) = C\rho(z)$ when z is sufficiently close to the boundary of Ω, the result follows. □

Lemma 4.2.7. *([15, p. 227]) Let $\Omega \subset \mathbb{C}^n$ be hyperconvex, and let $a \in \Omega$, Then for each $\varepsilon > 0$ and for each neighborhood $U \subset \Omega$ of a there exists a neighborhood V of a such that V is relatively compact subset of U and*

$$(1+\varepsilon)^{-1} \leq \frac{g_\Omega(z,x)}{g_\Omega(z,y)} \leq (1+\varepsilon), \tag{4.9}$$

for all $(z, x), (z, y) \in (\Omega \setminus U) \times V$.

Proof. Let $\varepsilon > 0$. Choose $s > 0$ such that $\overline{B(a,s)} \subset \Omega$, and we define $U = B(a,s)$. Let $r := \inf\{\|z-w\| : z \in U, w \in \partial\Omega\}$ and $R := \sup\{\|z-w\| : z \in U, w \in \partial\Omega\}$. Then

$$\log(\|z-x\|/R) \leq g_\Omega(z,x) \leq \log(\|z-x\|/r) \qquad ((z,x) \in \Omega \times U).$$

Now we find a $\lambda \in (0,s)$ such that $(1+\varepsilon)\log(3\lambda/2r) < \log(\lambda/2R)$, and we define $V = B(a,\lambda/2)$. For any two points $x, y \in V$, define

$$v(x) = \begin{cases} \log(\|z-y\|/R) & (z \in \overline{B(a,s)}) \\ \max\{(1+\varepsilon)g_\Omega(z,x), \log(\|z-y\|/R)\} & (z \in \Omega \setminus \overline{B(a,s)}). \end{cases}$$

If $\|z-a\| = s$, then

$$\begin{aligned}(1+\varepsilon)g_\Omega(z,x) &\leq (1+\varepsilon)\log(\|z-y\|/R) \leq (1+\varepsilon)\log(3\lambda/2r) \\ &< \log(\lambda/2R) < \log(\|z-y\|/R).\end{aligned}$$

Therefore $v \in \mathcal{PSH}(\Omega)$. Also, we have $v < 0$ by the maximum principle. Hence, $v \leq g_\Omega(\cdot,y)$. So if $x, y \in V$ and $z \in \Omega U$, then we have

$$(1+\varepsilon)g_\Omega(z,x) \leq v(z) \leq g_\Omega(z,x),$$

and we have

$$\frac{g_\Omega(z,x)}{g_\Omega(z,y)} \leq (1+\varepsilon).$$

By a change of x and y, we have also

$$\frac{g_\Omega(z,y)}{g_\Omega(z,x)} \leq (1+\varepsilon)^{-1}.$$

□

Theorem 4.2.8. *if $\Omega \subset \mathbb{C}^n$ is hyperconvex, then the pluricomplex Green function $g_\Omega : \bar{\Omega} \times \Omega \to [-\infty, 0]$ is continuous (where $g_\Omega\big|_{\partial\Omega\times\Omega} \equiv 0$).*

Proof. See [8] □

Theorem 4.2.9. *If Ω is hyperconvex domain. Then there exists a unique continuous plurisubharmonic function $g_\Omega(\cdot, w)$ which is a solution to the problem 4.6 and $g(\cdot, w) : \bar{\Omega} \to [-\infty, 0]$ is plurisubharmonic on Ω, and continuous on $\bar{\Omega}$ satisfying 4.6 and $(dd^c u)^n = (2\pi)^n \delta_w$.*

Proof. See [9]. □

4.3 Monge-Ampére Measures and Lelong-Jensen Formula

This section is devoted to Demailly measures (or Monge-Ampère measures) which will be used in the construction of Hardy Spaces on hyperconvex domains. Hence we want to overview Demailly measures. We will follow basically Demailly. All terminology and theory belongs to Jean-Pierre Demailly [8, 9, 10]

Now let Ω be a hyperconvex domain $\in \mathbb{C}^n$ and $\varphi : \Omega \to [-\infty, 0)$ be a negative continuous plurisubharmonic exhaustion for Ω. Now we define pseudoball:

$$B(r) = \{z \in \Omega : \varphi(z) < r\}, \qquad r \in [-\infty, 0), \tag{4.10}$$

and the pseudosphere :

$$S(r) = \{z \in \Omega : \varphi(z) = r\}, \qquad r \in [-\infty, 0) \tag{4.11}$$

and we set

$$\varphi_r = \max\{\varphi, r\}, \qquad r \in (-\infty, 0) \tag{4.12}$$

For every $r \in (-\infty, 0)$ the measures $(dd^c\varphi_r)^n$ are well defined since the functions φ_r's are continuous.

In 1985 by means of *Monge-Ampére operator* Jean-Pierre Demailly introduced the measures

$$\mu_r = (dd^c\varphi_r)^n - \chi_{\Omega\setminus B(r)}(dd^c\varphi)^n \qquad r \in (-\infty, 0) \tag{4.13}$$

where χ_ω is the characteristic function of the $\omega \subset \Omega$. Demailly [9] calls these measures as *Monge-Ampère measures*. We shall call those measures as Demailly-Monge-Ampère measures. In fact in his paper [9] he defined this measures for any Stein space with an exhaustion function φ but for our purposes we restrict ourselves to only hyperconvex domains.

$\varphi_r \equiv r$ on $B(r)$ so we have $(dd^c\varphi_r)^n = 0$ on $B(r)$. Moreover on $\Omega \setminus \{z : \varphi(z) \leq r\}$ we have $(dd^c\varphi_r)^n = (dd^c\varphi)^n$. Thus those measures are supported on the pseudospheres $S(r)$.

Remark that when φ is maximal off a compact set K contained in $B(r)$, (i.e.$(dd^c\varphi)^n = 0$) then μ_r reduces to $\mu_r = (dd^c\varphi_r)^n$.

Proposition 4.3.1. *Assume that φ is smooth near $S(r)$ and $d\varphi \neq 0$ on $S(r)$. Then the Demailly-Monge-Ampère measure μ_r reduces to $(dd^c\varphi)^{n-1} \wedge d^c\varphi\big|_{S(r)}$.*

Proof. We start by taking decreasing sequence of smooth convex functions with $\psi_k(t) = r$ for $t \leq r - 1/k$ and $\psi_k(t) = t$ for $t \geq r + 1/k$. Here $\lim_{k\to\infty} \psi_k(t) = \max\{t, r\}$ and $\lim_{k\to\infty} \psi' \circ \varphi = \chi_{\Omega\setminus B(r)}$ a.e., and by Proposition 3.2.10 we have $(dd^c\psi_k \circ \varphi)^n$ converges to $(dd^c\varphi_r)^n$. Let h be a smooth function with compact support near $S(r)$.

$$\begin{aligned}
\int_\Omega h(dd^c\varphi_r)^n &= \lim_{k\to\infty} \int_\Omega h(dd^c\psi_k \circ \varphi)^n \\
&= \lim_{k\to\infty} \int_\Omega -dh(dd^c\psi_k \circ \varphi)^{n-1} \wedge d^c(\psi_k \circ \varphi) \\
&= \lim_{k\to\infty} \int_\Omega -(\psi' \circ \varphi)dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\
&= \int_{\Omega\setminus B(r)} -dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\
&= \int_{S(r)} h(dd^c\varphi)^{n-1} \wedge d^c\varphi + \int_{\Omega\setminus B(r)} h(dd^c\varphi)^{n-1} \wedge d^c\varphi.
\end{aligned}$$

Hence we have

$$(dd^c\varphi_r)^n = (dd^c\varphi)^{n-1} \wedge d^c\varphi\Big|_{S(r)} + \chi_{\Omega\setminus B(r)}(dd^c\varphi)^n$$

□

Example 4.3.2. In the case of the unit disc we have $g_\Delta = \log|z|$. Now by using the exhaustion function $g_\Delta(z) = \log|z|$. The Demailly measure μ_r is obtained by evaluating just $d^c g\Big|_{S(r)}$

$$d^c g = \frac{-y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$$

now we put $x = r\cos\theta$ and $y = r\sin\theta$ we get

$$d^c g\Big|_{S(r)} = d\theta.$$

Namely it gives usual Lebesgue measure on the circle with radius r.

Example 4.3.3. Now let Ω be the unit ball of $\mathbb{C}^n$ and let $\phi(z) = \log\|z\|$ be the defining function of $B(0,1)$ We want to evaluate the Demailly measure on the level sets $B(0,r), \quad$ for $0 < r < 1$. Now we directly calculate by using Proposition 4.3.1. Now on $S(r)$

$$d^c \log\|z\| = i\sum_{i=1}^{n} \frac{z_i}{\|z\|^2}d\bar{z} - \frac{\bar{z}_i}{\|z\|^2}dz = \frac{1}{2r^2}d^c\|z\|^2$$

By the same way

$$dd^c \log \|z\| = \frac{1}{2r^2} dd^c \|z\|^2$$

$$d^c\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1} = 2^{2n-1}(n-1)! r d\sigma_r$$

where σ is the usual Lebesgue measure on $S(r)$, hence we have

$$\frac{1}{(2\pi)^n}(d^c \log \|z\|) \wedge (dd^c \log \|z\|)^{n-1}\Big|_{S(r)} = \frac{1}{\sigma(S(r))} d\sigma_r \tag{4.14}$$

which is the normalized Lebesgue measure on the Sphere with radius r.

Example 4.3.4. Let Ω be the polydisc $\Delta^n = \{z : \varphi(z) = \log(\max |z_j|) < 1\}$ then the pseudoballs $B(r)$ are the polydisc with polyradius $(r, \ldots, r)$. By Corollary 5.4 of [10] we have

$$(dd^c\varphi)^n = \frac{1}{(2\pi)^n}\delta_0. \tag{4.15}$$

Now we want to find measures μ_r. If z is a point that all terms z_j are not equal, then we can omit the smallest term in a neighborhood of z without changing φ. Now since φ depends on only $(n-1)$ variables we have $(dd^c\varphi_r)^n = 0$ Hence $\mu_r = 0$ near z. Hence μ_r is supported on distinguished boundary of polydisc $B(r)$. Since φ is invariant under the rotations $z_j \longrightarrow e^{i\theta_j} z_j$, the measure μ_r is also invariant and we see that μ_r is a constant multiple of $d\theta_1 \dots d\theta_n$ by 4.15 and by Theorem 4.2.9 we have

$$\mu_r = \frac{1}{(2\pi)^n} d\theta_1 \dots d\theta_n \tag{4.16}$$

So we get the usual Lebesgue measure on the Torus.

In 1985 Demailly found a useful formula in his paper [9], what he calls Lelong-Jensen formula. We will call it Demailly-Lelong-Jensen formula.

Theorem 4.3.5. *Let V be a plurisubharmonic function in Ω. Then V is μ_r−integrable for every $r \in (-\infty, 0)$ and*

$$\int_\Omega V d\mu_r - \int_{B(r)} V (dd^c\varphi_r)^n = \int_{-\infty}^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} \tag{4.17}$$

which is equivalent to

$$\int_\Omega V d\mu_r - \int_{B(r)} V (dd^c\varphi_r)^n = \int_{B(r)} (r - \varphi) dd^cV \wedge (dd^c\varphi)^{n-1}. \tag{4.18}$$

Proof. See [9]. □

Now by Demailly-Lelong-Jensen formula we have

$$\mu_r(V) - \mu_{r_0}(V) + \int_{B(r_0)\setminus B(r)} V (dd^c\varphi)^n = \int_{r_0}^r dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1} \tag{4.19}$$

where $\mu(V) = \int_\Omega V d\mu$.

Corollary 4.3.6. *Assume that* $(dd^c\varphi)^n = 0$ *on* $\Omega \setminus S(-\infty)$ *and* V *is positive. Then the function* $r \longmapsto \mu_r(V)$ *is a convex and increasing function of* r.

Proof. By 4.19 we have

$$\mu_r(V) = \mu_{r_0}(V) + \int_{r_0}^{r} dt \int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1}.$$

Since $\int_{B(t)} dd^cV \wedge (dd^c\varphi)^{n-1}$ are increasing and non-negative, we have $r \longmapsto \mu_r(V)$ is a convex and increasing function of r. □

Another result of Demailly-Lelong-Jensen Formula is total mass of μ_r is:

$$\|\mu_r\| = \mu_r(1) = \int_{B(r)} (dd^c\varphi)^n. \tag{4.20}$$

Another important theorem due to Jean-Pierre Demailly is the following theorem.

Theorem 4.3.7. *Let* $\varphi : \Omega \longrightarrow [-\infty, 0)$ *be a continuous plurisubharmonic exhaustion function for* Ω. *And suppose that the total Monge-Ampère mass is finite, i.e.*

$$\int_{\Omega} (dd^c\varphi)^n < \infty. \tag{4.21}$$

Then as r *tends to* 0, μ_r *converge to a positive measure* μ $weak^* - ly$ *on* Ω *with total mass* $\int_{\Omega} (dd^c\varphi)^n$ *and supported on* $\partial\Omega$. *We associate this limit measure* μ *on the boundary with the exhaustion function* φ.

Proof. The measures μ_r are uniformly bounded by $\int_\Omega (dd^c\varphi)^n$ for $r < 0$. It suffices to show that for any function $h \in \mathcal{C}^2(\Omega)$ the limit of $r \to \mu_r(h)$ exists. If γ is a strictly plurisubharmonic function in $\mathcal{C}^2(\Omega)$ We can choose a constant $C > 0$ such that $V = h + C\gamma$ is plurisubharmonic and nonnegative on $\bar{\Omega}$. By corollary 4.3.6 $\mu_r(\gamma), \mu_r(V)$ are increasing function of r. Hence the limits $\mu_r(\gamma)$ and $\mu_r(V)$ exist as $r \to 0$. Hence $\mu_r(h)$ exists as $r \to 0$. □

Another application of Demailly-Lelong-Jensen formula is next theorem.

Theorem 4.3.8. *Let $\varphi : \Omega \longrightarrow [-\infty, 0)$ is a continuous plurisubharmonic exhaustion function for Ω and $V \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Then*

$$\mu(V) = \int_\Omega V(dd^c\varphi)^n + \int_\Omega dd^cV \wedge |\varphi|(dd^c\varphi)^{n-1} \qquad \textit{for } n \geq 1,$$

$$\mu(V) = \int_\Omega V(dd^c\varphi)^n + \int_\Omega dd^cV \wedge (dd^c\varphi)^{n-2} \wedge d\varphi \wedge d^c\varphi \qquad \textit{for } n \geq 2.$$

Proof. For the first equation consider Demailly-Lelong-Jensen formula 4.17, and we let $r \to 0$ then we have the first equation. Second equation is directly obtained from the first one by writing the integral $\int_\Omega dd^cV \wedge |\varphi|(dd^c\varphi)^{n-1}$ as $\int_\Omega (dd^c\varphi)^n + \int_\Omega dd^cV \wedge (dd^c\varphi)^{n-2} \wedge d\varphi \wedge d^c\varphi$ by Proposition 3.2.2. □

From now on we want to restrict ourselves to only Pluricomplex Green function with pole at a. Recall that $(dd^cg_\Omega)^n = (2\pi)^n\delta_a$, then Theorem 4.3.8 reduces to

$$V(a) = \frac{1}{(2\pi)^n}\int_{\partial\Omega} V d\mu - \frac{1}{(2\pi)^n}\int_\Omega dd^cV \wedge |g_\Omega|(dd^cg_\Omega)^{n-1} \tag{4.22}$$

For $n = 1$ it reduces to the Poisson-Jensen formula 4.4. A direct consequence is the next corollary

Corollary 4.3.9. *Let* $\varphi : \Omega \longrightarrow [-\infty, 0)$ *is a continuous plurisubharmonic exhaustion function for* Ω *and* $V \in \mathcal{C}(\bar{\Omega})$ *be a pluriharmonic function in* Ω*. Then we have*

$$V(a) = \frac{1}{(2\pi)^n} \int_{\partial\Omega} V d\mu \tag{4.23}$$

Proof. For a pluriharmonic function V we have $dd^c V = 0$. Hence we get the result. □

For $n = 1$ for $\Omega = \Delta$ then we have the usual Poisson formula. We can consider it as the Pluricomplex version of Poisson formula. (See [9].)

CHAPTER 5

HARDY SPACES OVER HYPERCONVEX DOMAINS OF $\mathbb{C}^n$

5.1 Introduction

This chapter is the most important part of this thesis because main goal of this thesis is to give a definition of Hardy spaces over hyperconvex domains. In this chapter we want to extend the theory of Hardy Spaces to hyperconvex domains in terms of integral mean growth of analytic functions . In this chapter we will give a definition of Hardy Spaces on hyperconvex domains that unifies the theories of Hardy Spaces on polydiscs and the unit ball of $\mathbb{C}^n$(or more generally pseudoconvex domains with $\mathcal{C}^1$ boundary).

5.2 Hardy Spaces over Hyperconvex Domains

Let Ω be a hyperconvex domain in $\mathbb{C}^n$. Fix a point a and let $g(z) = g_\Omega(z, a)$ be the Pluricomplex Green function which is a continuous exhaustion.

And now we define Hardy Spaces over hyperconvex domains in terms of Demailly-*Monge-Ampére measure* related to g. Define *Hardy Spaces* over Ω

as :

$$H^p_a \doteq \{ \ f \in O(\Omega) : \sup_{r<0} \int\limits_{S(r)} |f(z)|^p d\mu_{r,a} < \infty \ \}.$$

On H^p_a we define norm as:

$$\|f\|_p \doteq \left[\sup_{r<0} \int\limits_{S(r)} |f(z)|^p d\mu_{r,a} \right]^{1/p}$$

Remark 5.2.1. Some obvious facts are:

1. If $\Omega = \Delta,$ then H^p_a are usual Hardy Spaces by Example 4.3.2.

2. If $\Omega = \Delta^n,$ then H^p_a are usual Hardy Spaces. by Example 4.3.4.

3. If Ω is a ball in $\mathbb{C}^n$ then H^p_a are usual Hardy Spaces by Example 4.3.3.

Next theorem is an analogous theorem for Hardy's Convexity Theorem 2.4.2 for Hardy Spaces in the unit disc in $\mathbb{C}$.

Theorem 5.2.1. *Let $f \in H^p_a$ then the function $r \longmapsto \int_{S(r)} |f(z)|^p d\mu_{r,a}$ is an increasing and convex function of r.*

Proof. Follows directly from Corollary 4.3.6 since $|f(z)|^p$ is plurisubharmonic.

□

Theorem 5.2.2. *Let Ω be a hyperconvex domain in $\mathbb{C}$. Then $f \in H^p_a(\Omega)$ if and only if $|f|^p$ has a harmonic majorant.*

Proof. Let f has a harmonic majorant u. Then

$$\int\limits_{S(r)} |f(z)|^p d\mu_{r,a} \leq \int\limits_{S(r)} u d\mu_{r,a} = u(a) < \infty$$

by Corollary 4.3.9.

Conversely, let f be in $H_a^p(\Omega)$ and assume that f has no harmonic majorant. Then by Theorem 4.1.10 we have

$$\frac{1}{2\pi} \int_\Omega G_D(z,w) \Delta |\ f(z)\ |^p = \infty \tag{5.1}$$

On the other hand we have

$$\frac{1}{2\pi} \int_{S_r} |\ f(z)\ |^p d\mu_{r,a} - |\ f(a)\ |^p \leq C < \infty \tag{5.2}$$

where C is independent of r, we have

$$\frac{1}{2\pi} \int_{S_r} |\ f(z)\ |^p d\mu_{r,a} - |\ f(a)\ |^p = \frac{1}{2\pi} \int_{B_r} G_{D_r}(z,w) dd^c |\ f(z)\ |^p \tag{5.3}$$

by Demailly-Lelong-Jensen formula 4.22. Since G_{D_r} are increasing by Corollary 4.1.3 and $dd^c|\ f(z)\ |^p$ and $G_{B_r}(z,w)$ are positive, and $\lim_{r\to 0} G_{B_r}(z,w) = G_\Omega(z,w)$, by 4.1.4. Now since $G_{B(r)}(z,w)\chi_{B(r)} \uparrow G_\Omega(z,w)$ on Ω, by using

Monotone Convergence Theorem

$$\begin{aligned}\lim_{r\to 0}\frac{1}{2\pi}\int_{B_r} G_{B_r}(z,w)dd^c|\ f(z)\ |^p &= \lim_{r\to 0}\frac{1}{2\pi}\int_{\Omega} G_{B_r}(z,w)dd^c|\ f(z)\ |^p\chi_{B(r)}\\ &= \frac{1}{2\pi}\int_{\Omega} g_{\Omega}(z,w)dd^c|\ f(z)\ |^p\end{aligned}$$

which is a contradiction since left-hand side stays bounded by 5.2, whereas the right hand side is ∞ in view of 5.1. Hence it has a harmonic majorant. This contradiction proves the theorem. □

Corollary 5.2.3. *For $n = 1$, the class $H_a^p(\Omega)$ is independent of the base point a.*

Proof. Let f is in $H_a^p(\Omega)$. Then $|f|^p$ has a harmonic majorant u by 5.2.2. Therefore $f \in H_b^p(\Omega)$ for any $b \in \Omega$ by 5.2.2. □

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